



HAL
open science

Highly singular (frequently sparse) steady solutions for the 2D Navier-Stokes equations on the torus

Pierre Gilles Lemarié-Rieusset

► To cite this version:

Pierre Gilles Lemarié-Rieusset. Highly singular (frequently sparse) steady solutions for the 2D Navier-Stokes equations on the torus. 2024. <hal-04451680>

HAL Id: hal-04451680

<https://hal.science/hal-04451680v1>

Preprint submitted on 11 Feb 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Highly singular (frequently sparse) steady solutions for the 2D Navier–Stokes equations on the torus

Pierre Gilles Lemarié-Rieusset*

Abstract

We construct non-trivial steady solutions in H^{-1} for the 2D Navier–Stokes equations on the torus. In particular, the solutions are not square integrable, so that we have to redefine the notion of solutions.

Keywords : Navier–Stokes equations, steady solutions, lacunary Fourier series, nonuniqueness, Koch and Tataru theorem.

AMS classification : 35K55, 35Q30, 76D05.

Introduction

In this paper, we are looking for steady solutions \vec{u} of the 2D Navier–Stokes equations on the torus $\mathbb{T}^d = \mathbb{R}^2/2\pi\mathbb{Z}^2$, i.e. for solutions of the equations

$$\begin{cases} \Delta \vec{u} - \mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u}) = 0 \\ \operatorname{div} \vec{u} = 0 \end{cases} \quad (1)$$

where \vec{u} is a periodical distribution vector field, with mean value 0:

$$\int_{\mathbb{T}^2} \vec{u}(x) dx = 0.$$

*LaMME, Univ Evry, CNRS, Université Paris-Saclay, 91025, Evry, France; e-mail : pierre.gilles.lemarierieusset@univ-evry.fr

Such a periodical distribution vector field \vec{u} can be written as a Fourier series

$$\vec{u}(x) = \sum_{k \in \mathbb{Z}_+^2} \cos(k \cdot x) \vec{v}_k + \sin(k \cdot x) \vec{w}_k$$

where $k \in \mathbb{Z}_+^2$ if and only if $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ and $\arg(k_1 + ik_2) \in (-\pi/2, \pi/2]$. The convergence in \mathcal{D}' is given by a slow growth of the coefficients:

$$|\vec{v}_k| + |\vec{w}_k| \leq C|k|^N$$

for some constants C and N ; in particular, $\vec{u} \in H^s(\mathbb{T}^2)$ for $s < -N - 1$. If $k \in \mathbb{Z}^2$, $k \neq (0, 0)$ and $k \notin \mathbb{Z}_+^2$, then we may replace k with $-k$ (with $-k \in \mathbb{Z}_+^2$) and write $\cos(k \cdot x) \vec{v}_k + \sin(k \cdot x) \vec{w}_k = \cos((-k) \cdot x) \vec{v}_k + \sin((-k) \cdot x) (-\vec{w}_k)$, hence the condition $k \in \mathbb{Z}_+^2$ is not essential.

In our equations, \mathbb{P} is the Leray projection operator on solenoidal vector fields, defined by

$$\mathbb{P}\left(\sum_{k \in \mathbb{Z}_+^2} \cos(k \cdot x) \vec{v}_k + \sin(k \cdot x) \vec{w}_k\right) = \sum_{k \in \mathbb{Z}_+^2} \rho_k \cos(k \cdot x + \theta_k) k^\perp \quad (2)$$

with

$$\rho_k \cos(k \cdot x + \theta_k) = \cos(k \cdot x) \frac{\vec{v}_k \cdot k^\perp}{|k|^2} + \sin(k \cdot x) \frac{\vec{w}_k \cdot k^\perp}{|k|^2}$$

where

$$(k_1, k_2)^\perp = (-k_2, k_1), |(k_1, k_2)|^2 = k_1^2 + k_2^2 \text{ and } \rho_k = \frac{\sqrt{(\vec{v}_k \cdot k^\perp)^2 + (\vec{w}_k \cdot k^\perp)^2}}{|k|^2}.$$

It is easy to check that, if the solution \vec{u} satisfies $\vec{u} \in H^1(\mathbb{T}^2)$, then $\vec{u} = 0$. Thus, we shall not require (weak) differentiability for \vec{u} . Usually, it is customary to rewrite $\vec{u} \cdot \vec{\nabla} \vec{u}$ in the equations as $\partial_1(u_1 \vec{u}) + \partial_2(u_2 \vec{u})$ (since $\text{div} \vec{u} = 0$), where the derivatives are taken in the sense of distributions. In order to define $u_i \vec{u}$, one then usually requires that \vec{u} be square integrable.

As we shall see it, it is easy to check that, if the solution \vec{u} satisfies $\vec{u} \in L^p(\mathbb{T}^2)$ for some $p > 2$, then $\vec{u} = 0$. This is even true when \vec{u} belongs to the Lorentz space $L^{2,1}(\mathbb{T}^2)$. This is still probably the case for $\vec{u} \in L^2(\mathbb{T}^2)$, thus we will search for some solution \vec{u} which is not square integrable. We need however to be able to define $\mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u})$ when \vec{u} is no longer square integrable.

Definition 1 (Admissible vector fields). *A divergence free periodical distribution vector field is admissible for the steady problem on \mathbb{T}^2 if it is of the form*

$$\vec{u} = \sum_{k \in \mathbb{Z}_+^2} \rho_k \cos(k \cdot x + \theta_k) k^\perp \quad (3)$$

with

$$\sum_{k_1 \in \mathbb{Z}_+^2} \sum_{k_2 \in \mathbb{Z}_+^2} \rho_{k_1} \rho_{k_2} \|\mathbb{P}(\cos(k_1 \cdot x + \theta_{k_1}) \sin(k_2 \cdot x + \theta_{k_2}) (k_1^\perp \cdot k_2) k_2^\perp)\|_{H^{-N}} < +\infty \quad (4)$$

for some N .

If $\vec{u} = \sum_{k \in \mathbb{Z}_+^2} \rho_k \cos(k \cdot x + \theta_k) k^\perp = \sum_{k \in \mathbb{Z}_+^2} \vec{u}_k$ is an admissible vector field, we will then define $\mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u})$ as

$$\mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u}) = \sum_{k_1 \in \mathbb{Z}_+^2} \sum_{k_2 \in \mathbb{Z}_+^2} \mathbb{P}(\vec{u}_{k_1} \cdot \vec{\nabla} \vec{u}_{k_2}). \quad (5)$$

The main result in this paper is then the following one:

Theorem 1. *There exists non-trivial solutions to the equations*

$$\begin{cases} \Delta \vec{u} - \mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u}) = 0 \\ \operatorname{div} \vec{u} = 0 \end{cases} \quad (6)$$

where \vec{u} is an admissible vector field (with mean value 0) with $\vec{u} \in H^{-1}(\mathbb{T}^2) \cap BMO^{-1}$.

The paper is organized in the following manner: in Section 1, we recall classical results on steady solutions on \mathbb{T}^d ; in section 2, we describe some examples of admissible vector fields; in section 3, we prove Theorem 1; in section 4, we comment on the Koch and Tataru theorem.

1 Steady solutions for the Navier-Stokes problem on \mathbb{T}^d : known results.

In this section, we recall known results on steady solutions for the Navier-Stokes problem in $L^2(\mathbb{T}^d)$, for $d \geq 2$.

Case $\vec{u} \in H^1(\mathbb{T}^d) \cap L^4(\mathbb{T}^d)$.

If $\vec{u} \in H^1(\mathbb{T}^d) \cap L^4(\mathbb{T}^d)$ (recall that, when $d \leq 4$, $H^1(\mathbb{T}^d) \subset L^4(\mathbb{T}^d)$), we can compute

$$- \int_{\mathbb{T}^d} |\vec{\nabla} \otimes \vec{u}|^2 dx = \int_{\mathbb{T}^d} \vec{u} \cdot \mathbb{P}(\vec{u} \cdot \nabla \vec{u}) dx = \int_{\mathbb{T}^d} \vec{u} \cdot (\vec{u} \cdot \nabla \vec{u}) dx = \int_{\mathbb{T}^d} \operatorname{div} \left(\frac{|\vec{u}|^2}{2} \vec{u} \right) dx = 0.$$

Thus $\vec{u} = 0$ (we are interested in vector fields with null mean value).

Case $\vec{u} \in L^p(\mathbb{T}^d)$, $p > d$.

As p is subcritical when $p > d$, the initial value problem

$$\begin{cases} \partial_t \vec{v} = \Delta \vec{v} - \mathbb{P}(\vec{v} \cdot \nabla \vec{v}) \\ \operatorname{div} \vec{v} = 0 \\ \vec{v}(0, x) = \vec{u}_0(x) \end{cases} \quad (7)$$

with $\vec{u}_0 \in L^p$ has a unique solution in $\mathcal{C}([0, T], L^p)$ for some time T . This solution \vec{v} is smooth on $(0, T) \times \mathbb{T}^d$.

If \vec{u} is a steady solution of (1), then $\vec{v}(t, x) = \vec{u}(x)$ defines a solution of the Cauchy problem (7) with initial value $\vec{u}_0 = \vec{u}$. Hence, if $\vec{u} \in L^p$, we find that $\vec{u} \in H^1 \cap L^\infty$ and finally $\vec{u} = 0$.

Case $\vec{u} \in L^d(\mathbb{T}^d)$, $d \geq 3$.

The value $p = d$ is critical for the Cauchy problem (7). When $\vec{u}_0 \in L^d$, the problem has a solution \vec{v} in $\mathcal{C}([0, T], L^d)$ for some time T and this solution \vec{v} is smooth on $(0, T) \times \mathbb{T}^d$. But uniqueness of solutions in $\mathcal{C}([0, T], L^d)$ is known only for $d \geq 3$. Thus, if \vec{u} is a steady solution of (1) with $\vec{u} \in L^d(\mathbb{T}^d)$ and if $d \geq 3$, then $\vec{u} = 0$.

Case $\vec{u} \in L^2(\mathbb{T}^d)$, $d \geq 4$.

Recently, Luo [7] constructed non trivial steady solutions in $L^2(\mathbb{T}^d)$, $d \geq 4$ (this solution belongs to $L^p(\mathbb{T}^d)$ for some $p \in (2, d)$). His proof was following the scheme of convex integration developed by De Lellis and Székelyhidi [2] in the case of non-steady solutions for the Euler equations and by Buckmaster and Vicol [1] in the case of non-steady solutions for the Navier–Stokes equations. However, his proof requires the spatial dimension d to be no lesser than 4.

Case $\vec{u} \in L^2(\mathbb{T}^2)$.

Uniqueness of solutions of the Cauchy problem (7) in $\mathcal{C}([0, T], L^2(\mathbb{T}^2))$ is not known. Proofs of uniqueness in $\mathcal{C}([0, T], L^3(\mathbb{T}^3))$ are based on maximal regularity properties which are no longer true in the 2D case [3, 6, 9, 10, 4, 8]. On the other hand, the proof of non-uniqueness in $\mathcal{C}([0, T], L^2(\mathbb{T}^3))$ is based on convex integration methods which cannot be applied in the 2D case [1].

However, we have uniqueness in $\mathcal{C}([0, T], L^{2,1}(\mathbb{T}^2))$, where $L^{2,1}$ is a Lorentz space:

Proposition 1.

If \vec{v}_1 and \vec{v}_2 are two solutions of the Cauchy problem (7) with $\vec{v}_1, \vec{v}_2 \in \mathcal{C}([0, T], L^{2,1}(\mathbb{T}^2))$ and $\vec{v}_1(0, \cdot) = \vec{v}_2(0, \cdot) = \vec{u}_0$, then $\vec{v}_1 = \vec{v}_2$.

Proof. We follow the lines of [4] and [9]. If

$$T^* = \sup\{S \geq 0 / \vec{v}_1 = \vec{v}_2 \text{ on } [0, S)\}$$

and if $T^* < T$, then $\vec{v}_1(T^*, \cdot) = \vec{v}_2(T^*, \cdot)$: it is obvious if $T^* = 0$, and is a consequence of continuity if $0 < T^* < T$. Moreover, we can write the integral formulation of the Navier–Stokes equations with initial time T^* : for $T^* \leq t < T$ and $j = 1, 2$

$$\vec{v}_j(t, \cdot) = e^{(t-T^*)\Delta} \vec{v}_j(T^*, \cdot) + \int_{T^*}^t \int_{\mathbb{R}^2} K(t-s, \cdot - y) (\vec{v}_j(s, y) \otimes \vec{v}_j(s, y)) dy ds$$

with

$$|K(t, x)| \leq C \frac{1}{(\sqrt{t} + |x|)^3} \leq C(\mathbf{1}_{|x| < \pi} \frac{1}{t^{3/2}} + \mathbf{1}_{|x| \geq \pi} \frac{1}{|x|^3}).$$

Let

$$K_{\text{per}}(t, x) = \sum_{k \in \mathbb{Z}^2} K(t, x - 2\pi k).$$

Then $K_{\text{per}}(t, \cdot) \in L^1(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$, with

$$\|K_{\text{per}}(t, \cdot)\|_1 \leq C \frac{1}{\sqrt{t}} \text{ and } \|K_{\text{per}}(t, \cdot)\|_\infty \leq C(1 + \frac{1}{t^{3/2}}).$$

We write $\vec{v}_j = \vec{w}_0 + \vec{w}_j$, with $\vec{w}_0 = e^{(t-T^*)\Delta} \vec{v}_j(T^*, \cdot)$. By density of $L^\infty(\mathbb{T}^2)$ in $L^{2,1}(\mathbb{T}^2)$, we have

$$\lim_{\delta \rightarrow 0^+} \sup_{T^* < t < T^* + \delta} \sqrt{t - T^*} \|\vec{w}_0(t, \cdot)\|_\infty = 0,$$

while, by continuity of \vec{v}_j and \vec{w}_0 , we have, for $j = 1, 2$,

$$\lim_{\delta \rightarrow 0^+} \sup_{T^* < t < T^* + \delta} \|\vec{w}_j(t, \cdot)\|_{L^{2,1}(\mathbb{T}^2)} = 0.$$

We now write, for $\vec{w} = \vec{v}_1 - \vec{v}_2$ and $T^* \leq t < T$,

$$\begin{aligned} \vec{w}(t, x) &= \int_{T^*}^t \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}(s, y) \otimes \vec{w}_0(s, y)) dy ds \\ &\quad + \int_{T^*}^t \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}(s, y) \otimes \vec{w}_1(s, y)) dy ds \\ &\quad + \int_{T^*}^t \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}_0(s, y) \otimes \vec{w}(s, y)) dy ds \\ &\quad + \int_{T^*}^t \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}_2(s, y) \otimes \vec{w}(s, y)) dy ds \\ &= \vec{z}_1(t, x) + \vec{z}_2(t, x) + \vec{z}_3(t, x) + \vec{z}_4(t, x) \end{aligned}$$

and we estimate $\|\vec{w}(t, \cdot)\|_{L^{2,\infty}(\mathbb{T}^2)}$.

We first write, for $\vec{w} = \vec{v}_1 - \vec{v}_2$ and $T^* \leq t < T$,

$$\begin{aligned} \|\vec{z}_1(t, \cdot)\|_{L^{2,\infty}} + \|\vec{z}_3(t, \cdot)\|_{L^{2,\infty}} &\leq C \int_{T^*}^t \|K_{\text{per}}(t-s, \cdot)\|_1 \|\vec{w}_0(s, \cdot)\|_{\infty} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}} ds \\ &\leq C' \int_{T^*}^t \frac{1}{\sqrt{t-s}\sqrt{s-T^*}} ds \sup_{T^* < s < t} \sqrt{s-T^*} \|\vec{w}_0(s, \cdot)\|_{\infty} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}} \\ &= \pi C' \sup_{T^* < s < t} \sqrt{s-T^*} \|\vec{w}_0(s, \cdot)\|_{\infty} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}. \end{aligned}$$

For $A > 0$, we write

$$\begin{aligned} \vec{z}_2(t, x) + \vec{z}_4(t, x) &= \int_{\sup(t-A, T^*)}^t \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}(s, y) \otimes \vec{w}_1(s, y)) dy ds \\ &\quad + \int_{T^*}^{\sup(t-A, T^*)} \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}(s, y) \otimes \vec{w}_1(s, y)) dy ds \\ &\quad + \int_{\sup(t-A, T^*)}^t \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}_2(s, y) \otimes \vec{w}(s, y)) dy ds \\ &\quad + \int_{T^*}^{\sup(t-A, T^*)} \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}_2(s, y) \otimes \vec{w}(s, y)) dy ds \\ &= \vec{z}_{5,A}(t, x) + \vec{z}_{6,A}(t, x) + \vec{z}_{7,A}(t, x) + \vec{z}_{8,A}(t, x). \end{aligned}$$

Since the pointwise product is bounded from $L^{2,1} \times L^{2,\infty}$ to L^1 , we have

$$\begin{aligned} \|\vec{z}_{5,A}(t, \cdot)\|_1 &\leq C \int_{\sup(t-A, T^*)}^t \|K_{\text{per}}(t-s, \cdot)\|_1 \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}} ds \\ &\leq C' \int_{\sup(t-A, T^*)}^t \frac{1}{\sqrt{t-s}} ds \sup_{T^* < s < t} \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}} \\ &\leq 2C' \sqrt{A} \sup_{T^* < s < t} \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}. \end{aligned}$$

Similarly

$$\|\vec{z}_{7,A}(t, \cdot)\|_1 \leq C \sqrt{A} \sup_{T^* < s < t} \|\vec{w}_2(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}.$$

On the other hand, we have (for $T^* \leq t < \min(T, T^* + 1)$)

$$\begin{aligned} \|\vec{z}_{6,A}(t, \cdot)\|_\infty &\leq C \int_{T^*}^{\sup(t-A, T^*)} \|K_{\text{per}}(t-s, \cdot)\|_\infty \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}} ds \\ &\leq C' \int_{T^*}^{\sup(t-A, T^*)} \frac{1}{(t-s)^{3/2}} ds \sup_{T^* < s < t} \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}} \\ &\leq 2C' \frac{1}{\sqrt{A}} \sup_{T^* < s < t} \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}. \end{aligned}$$

Similarly

$$\|\vec{z}_{8,A}(t, \cdot)\|_\infty \leq C \frac{1}{\sqrt{A}} \sup_{T^* < s < t} \|\vec{w}_2(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}.$$

As $L^{2,\infty} = [L^1, L^\infty]_{\frac{1}{2}, \infty}$, we find that, for $T^* \leq t < \min(T, T^* + 1)$,

$$\|\vec{z}_2(t, \cdot)\|_{L^{2,\infty}} \leq C \sup_{T^* < s < t} \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}$$

and

$$\|\vec{z}_4(t, \cdot)\|_{L^{2,\infty}} \leq C \sup_{T^* < s < t} \|\vec{w}_2(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}.$$

Putting together those estimates, we get that, for $0 < \delta < \min(1, T - T^*)$,

$$\sup_{T^* \leq t \leq T^* + \delta} \|\vec{w}(t, \cdot)\|_{L^{2,\infty}} \leq CA(\delta) \sup_{T^* \leq t \leq T^* + \delta} \|\vec{w}(t, \cdot)\|_{L^{2,\infty}}$$

with

$$A(\delta) = \sup_{T^* < t < T^* + \delta} \sqrt{t - T^*} \|\vec{w}_0(t, \cdot)\|_\infty + \|\vec{w}_1(t, \cdot)\|_{L^{2,1}(\mathbb{T}^2)} + \|\vec{w}_2(t, \cdot)\|_{L^{2,1}(\mathbb{T}^2)}.$$

As

$$\lim_{\delta \rightarrow 0^+} A(\delta) = 0,$$

we get that $\vec{w} = \vec{v}_1 - \vec{v}_2$ is equal to 0 on $[0, T^* + \delta]$ for δ small enough, in contradiction with the definition of T^* . Thus $T^* = T$, and $\vec{v}_1 = \vec{v}_2$. \square

Corollary 1.

If \vec{u} is a steady solution of (1) with $\vec{u} \in L^{2,1}(\mathbb{T}^2)$, then $\vec{u} = 0$.

Proof. We consider the Cauchy problem (7) where the initial value \vec{u}_0 is equal to our steady solution \vec{u} . We can construct a mild solution \vec{v}_1 on a small time interval $[0, T]$ such that $\vec{v}_1 \in \mathcal{C}([0, T], L^{2,1})$, $\sup_{0 < t < T} \sqrt{t} \|\vec{v}_1(t, \cdot)\|_\infty < +\infty$ and $\lim_{t \rightarrow 0^+} \sqrt{t} \|\vec{v}_1(t, \cdot)\|_\infty = 0$. We have another solution in $\mathcal{C}([0, T], L^{2,1})$, namely $\vec{v}_2(t, \cdot) = \vec{u}$. By uniqueness, we find that $\vec{u} = \vec{v}_1(\frac{T}{2}, \cdot) \in L^\infty$, and thus $\vec{u} = 0$. \square

2 Admissible vector fields.

In this section, we describe some examples of admissible divergence free periodical distribution vector fields

$$\vec{u} = \sum_{k \in \mathbb{Z}_+^2} \rho_k \cos(k \cdot x + \theta_k) k^\perp = \sum_{k \in \mathbb{Z}_+^2} \vec{u}_k. \quad (8)$$

Square integrable vector fields

The most obvious example is the case $\vec{u} \in L^2$, i.e. $\sum_{k \in \mathbb{Z}_+^2} \|\vec{u}_k\|_2^2 < +\infty$. We have $\vec{u}_{k_1} \otimes \vec{u}_{k_2} \in L^1$ while the frequencies appearing in $\vec{u}_{k_1} \otimes \vec{u}_{k_2}$ are $k_1 + k_2$ and $k_1 - k_2$ (if $k_1 \neq k_2$, since $\vec{u}_k \cdot \vec{\nabla} \vec{u}_k = 0$). Thus,

$$\|\mathbb{P}(\vec{u}_{k_1} \cdot \vec{\nabla} \vec{u}_{k_2})\|_{H^{-N}} \leq C_N \|\vec{u}_{k_1}\|_2 \|\vec{u}_{k_2}\|_2 (|k_1 + k_2|^{-N-1} + |k_1 - k_2|^{-N-1}).$$

If $N > 1$, we have

$$\sum_{j \in \mathbb{Z}^2 \setminus \{(0,0)\}} |j|^{-N-1} < +\infty,$$

hence

$$\sum_{j \in \mathbb{Z}^2 \setminus \{(0,0)\}} |j|^{-N-1} \|\vec{u}_{k_2+j}\|_2 \in l^2$$

and thus

$$\sum_{k_1 \in \mathbb{Z}_+^2} \sum_{k_2 \in \mathbb{Z}_+^2} \|\mathbb{P}(\vec{u}_{k_1} \cdot \vec{\nabla} \vec{u}_{k_2})\|_{H^{-N}} < +\infty.$$

Lacunary Fourier series

Let us consider a lacunary Fourier series

$$\vec{u} = \sum_{j=0}^{+\infty} \rho_{k_j} \cos(k_j \cdot x + \theta_{k_j}) k_j^\perp = \sum_{j=0}^{+\infty} \vec{u}_{k_j}$$

with

$$|k_{j+1}| > 8|k_j|$$

and

$$\sum_{j=0}^{+\infty} \rho_{k_j}^2 |k_j|^{-2N} < +\infty$$

where $N \geq 0$ (so that $\vec{u} \in H^{-N-1}$). We have.

$$\begin{aligned} \|\mathbb{P}(\vec{u}_{k_j} \cdot \vec{\nabla} \vec{u}_{k_p})\|_{H^{-2N-3}} &\leq C \rho_{k_j} \rho_{k_p} |k_j| |k_p| (|k_j| + |k_p|)^{-2N-2} \\ &\leq C' |k_j|^{-N} \rho^{k_j} |k_p|^{-N} \rho^{k_p} \left(\frac{\min(|k_j|, |k_p|)}{\max(|k_j|, |k_p|)} \right)^{N+1}. \end{aligned}$$

Noticing that

$$\sum_{j=1}^{+\infty} \sum_{0 \leq p \leq j-1} \rho_{k_j} \rho_{k_p} \frac{|k_j| |k_p|}{(|k_j| + |k_p|)^{2N+2}} \leq C \sum_{j=1}^{+\infty} \sum_{0 \leq p \leq j-1} \frac{\rho_{k_j}}{|k_j|^N} \frac{\rho_{k_p}}{|k_p|^N} 8^{-(j-p)(N+1)},$$

we find that

$$\sum_{p=0}^{+\infty} \sum_{j=0}^{+\infty} \|\mathbb{P}(\vec{u}_{k_p} \cdot \vec{\nabla} \vec{u}_{k_j})\|_{H^{-2N-3}} < +\infty.$$

Remark: We could have proved that $\mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u}) \in H^{-2N-3}$ in another way: we have $\vec{u} \in H^{-N-1} \cap B_{\infty, \infty}^{-N-1}$; using paradifferential calculus and decomposing the product $\vec{u} \cdot \vec{\nabla} \vec{u}$ in two paraproducts and a remainder, we see that the paraproducts are controlled in H^{-2N-3} by $\|\vec{u}\|_{H^{-N-1}} \|\vec{u}\|_{B_{\infty, \infty}^{-N-1}}$, while the remainder is equal to 0.

Lacunary resonant Fourier modes

$$\vec{u} = \sum_{j=0}^{+\infty} \rho_{k_j} (\cos(k_j \cdot x + \theta_{k_j}) k_j^\perp + \cos((k_j + \omega_j) \cdot x + \eta_{k_j}) (k_j + \omega_j)^\perp) = \sum_{j=0}^{+\infty} \vec{u}_{k_j}$$

with

$$|k_{j+1}| > 8|k_j|, \quad |k_j| > 8|\omega_j|, \quad \omega_j \cdot k_j = 0$$

and

$$\sum_{j=0}^{+\infty} \rho_{k_j}^2 \frac{|k_j|}{|\omega_j|} < +\infty.$$

We write

$$\vec{v}_{k_j} = \rho_{k_j} \cos(k_j \cdot x + \theta_{k_j}) k_j^\perp \quad \text{and} \quad \vec{w}_{k_j} = \rho_{k_j} \cos((k_j + \omega_j) \cdot x + \eta_{k_j}) (k_j + \omega_j)^\perp.$$

In particular, we have that $\sum_{j=0}^{+\infty} \rho_{k_j}^2 < +\infty$, so that $\vec{u} \in H^{-1} \cap B_{\infty, \infty}^{-1}$. Following the computations of the case of lacunary solutions, we find that

$$\begin{aligned} \sum_{p=0}^{+\infty} \sum_{0 \leq j, j \neq p} \|\mathbb{P}(\vec{v}_{k_p} \cdot \vec{\nabla} \vec{v}_{k_j})\|_{H^{-3}} &< +\infty, \quad \sum_{p=0}^{+\infty} \sum_{0 \leq j, j \neq p} \|\mathbb{P}(\vec{v}_{k_p} \cdot \vec{\nabla} \vec{w}_{k_j})\|_{H^{-3}} < +\infty, \\ \sum_{p=0}^{+\infty} \sum_{0 \leq j, j \neq p} \|\mathbb{P}(\vec{w}_{k_p} \cdot \vec{\nabla} \vec{v}_{k_j})\|_{H^{-3}} &< +\infty, \quad \sum_{p=0}^{+\infty} \sum_{0 \leq j, j \neq p} \|\mathbb{P}(\vec{w}_{k_p} \cdot \vec{\nabla} \vec{w}_{k_j})\|_{H^{-3}} < +\infty. \end{aligned}$$

We now estimate the diagonal terms $\mathbb{P}(\vec{u}_{k_j} \cdot \vec{\nabla} \vec{u}_{k_j})$. We have

$$\vec{v}_{k_j} \cdot \vec{\nabla} \vec{v}_{k_j} = \vec{w}_{k_j} \cdot \vec{\nabla} \vec{w}_{k_j} = 0,$$

while

$$\begin{aligned} \vec{v}_{k_j} \cdot \vec{\nabla} \vec{w}_{k_j} &= \rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos(k_j \cdot x + \theta_{k_j}) \cos((k_j + \omega_j) \cdot x + \eta_{k_j} + \frac{\pi}{2}) (k_j + \omega_j)^\perp \\ &= \frac{1}{2} \rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos((2k_j + \omega_j) \cdot x + \theta_{k_j} + \eta_{k_j} + \frac{\pi}{2}) (k_j + \omega_j)^\perp \\ &\quad + \frac{1}{2} \rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos(\omega_j \cdot x - \theta_{k_j} + \eta_{k_j} - \frac{\pi}{2}) (k_j + \omega_j)^\perp \end{aligned}$$

and

$$\begin{aligned}\vec{w}_{k_j} \cdot \vec{\nabla} \vec{v}_{k_j} &= \rho_{k_j}^2 (k_j \cdot \omega_j^\perp) \cos(k_j \cdot x + \theta_{k_j} + \frac{\pi}{2}) \cos((k_j + \omega_j) \cdot x + \eta_{k_j}) k_j^\perp \\ &= \frac{1}{2} \rho_{k_j}^2 (k_j \cdot \omega_j^\perp) \cos((2k_j + \omega_j) \cdot x + \theta_{k_j} + \eta_{k_j} + \frac{\pi}{2}) k_j^\perp \\ &\quad + \frac{1}{2} \rho_{k_j}^2 (k_j \cdot \omega_j^\perp) \cos(\omega_j \cdot x - \theta_{k_j} + \eta_{k_j} - \frac{\pi}{2}) k_j^\perp.\end{aligned}$$

We have

$$\|\mathbb{P}(\rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos((2k_j + \omega_j) \cdot x + \theta_{k_j} + \eta_{k_j} + \frac{\pi}{2}) (k_j + \omega_j)^\perp)\|_{H^{-3}} \leq C \rho_{k_j}^2 \frac{|\omega_j|}{|k_j|}$$

and

$$\|\mathbb{P}(\rho_{k_j}^2 (k_j \cdot \omega_j^\perp) \cos((2k_j + \omega_j) \cdot x + \theta_{k_j} + \eta_{k_j} - \frac{\pi}{2}) k_j^\perp)\|_{H^{-3}} \leq C \rho_{k_j}^2 \frac{|\omega_j|}{|k_j|}.$$

On the other hand, we have

$$\mathbb{P}(\rho_{k_j}^2 (k_j \cdot \omega_j^\perp) \cos(\omega_j \cdot x - \theta_{k_j} + \eta_{k_j} + \frac{\pi}{2}) k_j^\perp) = 0$$

and

$$\mathbb{P}(\rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos(\omega_j \cdot x - \theta_{k_j} + \eta_{k_j} - \frac{\pi}{2}) (\omega_j + k_j)^\perp) = \rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos(\omega_j \cdot x - \theta_{k_j} + \eta_{k_j} - \frac{\pi}{2}) \omega_j^\perp$$

so that

$$\|\mathbb{P}(\rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos(\omega_j \cdot x - \theta_{k_j} + \eta_{k_j} - \frac{\pi}{2}) (\omega_j + k_j)^\perp)\|_{H^{-3}} \leq C \rho_{k_j}^2 \frac{|k_j|}{|\omega_j|}.$$

Thus, we get

$$\sum_{j=0}^{+\infty} \|\mathbb{P}(\vec{v}_{k_j} \cdot \vec{\nabla} \vec{w}_{k_j})\|_{H^{-3}} < +\infty, \quad \sum_{j=0}^{+\infty} \|\mathbb{P}(\vec{w}_{k_j} \cdot \vec{\nabla} \vec{v}_{k_j})\|_{H^{-3}} < +\infty,$$

and $\mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u})$ is well defined in H^{-3} .

3 2D steady solutions.

We are going to prove Theorem 1 following the lines of [1] and [7], i.e. applying the convex integration scheme by using intermittencies in the Fourier spectrum of the solution. In our case, however, computations will be much more simple than in the ones in [1] and [7], as we don't bother on convergence in L^2 .

We shall look for a solution

$$\vec{u} = \sum_{j=0}^{+\infty} \vec{u}_j = \vec{u}_0 + \sum_{j=1}^{+\infty} \vec{v}_j + \vec{w}_j$$

where

- $\vec{u}_0 = \rho_0 \cos(k_0 \cdot x) k_0^\perp$ with $0 < \rho_0 < 1$ and $k_0 \in \mathbb{Z}^2 \setminus \{(0, 0)\}$,
- $\vec{v}_j = \rho_j \cos(k_j \cdot x) k_j^\perp$ and $\vec{w}_j = \rho_j \cos((k_j + \omega_j) \cdot x + \eta_j) (k_j + \omega_j)^\perp$ with $\rho_j > 0$ and $k_j, \omega_j \in \mathbb{Z}^2 \setminus \{(0, 0)\}$,
- for $j \geq 1$, $|k_j| > 8|k_{j-1}|$, $|k_j| > 8|\omega_j|$, $\omega_j \cdot k_j = 0$.

k_j, ω_j and η_j will be constructed by induction and we'll check that

$$\sum_{j=1}^{+\infty} \rho_j^2 \frac{|k_j|}{|\omega_j|} < +\infty,$$

so that \vec{u} is an admissible vector field such that $\vec{u} \in H^{-1}$ (hence $\Delta \vec{u} \in H^{-3}$) and $\mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u}) \in H^{-3}$.

Defining $\vec{U}_n = \sum_{j=0}^n \vec{u}_j$, we have the convergence of $\Delta \vec{U}_n - \mathbb{P}(\vec{U}_n \cdot \vec{\nabla} \vec{U}_n)$ to $\Delta \vec{u} - \mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u})$ in H^{-3} . We write, for $n \geq 1$,

$$\Delta \vec{U}_n - \mathbb{P}(\vec{U}_n \cdot \vec{\nabla} \vec{U}_n) = \vec{V}_0 + \sum_{j=1}^n \vec{V}_j + \vec{W}_j$$

- $\vec{V}_0 = \Delta \vec{u}_0 (= \Delta \vec{u}_0 - \mathbb{P}(\vec{u}_0 \cdot \vec{\nabla} \vec{u}_0)) = -\rho_0 |k_0|^2 \cos(k_0 \cdot x) k_0^\perp$
- for $n \geq 1$,

$$\begin{aligned} \vec{V}_n &= \Delta \vec{U}_n - \mathbb{P}(\vec{u}_n \cdot \vec{\nabla} \vec{U}_{n-1}) - \mathbb{P}(\vec{U}_{n-1} \cdot \vec{\nabla} \vec{u}_n) \\ &\quad - \frac{1}{2} (k_n^\perp \cdot \omega_n) \rho_n^2 \mathbb{P}(\cos((2k_n + \omega_n) \cdot x + \eta_n + \frac{\pi}{2}) (k_n + \omega_n)^\perp) \\ &\quad - \frac{1}{2} (k_n \cdot \omega_n^\perp) \rho_n^2 \mathbb{P}(\cos((2k_n + \omega_n) \cdot x + \eta_n - \frac{\pi}{2}) k_n^\perp) \end{aligned}$$

- for $n \geq 1$, $\vec{W}_n = -\frac{1}{2}\rho_n^2(k_n^\perp \cdot \omega_n) \cos(\omega_n \cdot x + \eta_n - \frac{\pi}{2})\omega_n^\perp$

Let us write A_n for the set of frequencies involved in the expansion of \vec{V}_n :

$$\vec{V}_n = \sum_{k \in A_n} \cos(k \cdot x + \alpha_{n,k}) \vec{v}_{n,k} = \sum_{k \in A_n} \lambda_{n,k} \cos(k \cdot x + \alpha_{n,k}) k^\perp,$$

with $\lambda_{n,k} = \frac{\vec{v}_{n,k} \cdot k^\perp}{|k|^2}$. Using the formula

$$\begin{aligned} & \mathbb{P}(\cos(\alpha \cdot x + \theta) \alpha^\perp \cdot \vec{\nabla}(\cos(\beta \cdot x + \eta) \beta^\perp) + \cos(\beta \cdot x + \eta) \beta^\perp \cdot \vec{\nabla}(\cos(\alpha \cdot x + \theta) \alpha^\perp)) \\ &= -\mathbb{P}((\alpha^\perp \cdot \beta) \cos(\alpha \cdot x + \theta) \sin(\beta \cdot x + \eta) \beta^\perp) \\ & \quad - \mathbb{P}((\beta^\perp \cdot \alpha) \sin(\alpha \cdot x + \theta) \cos(\beta \cdot x + \eta) \alpha^\perp) \\ &= -\frac{1}{2} \mathbb{P}((\cos((\alpha + \beta) \cdot x + \theta + \eta - \frac{\pi}{2}))((\alpha^\perp \cdot \beta) \beta^\perp + (\beta^\perp \cdot \alpha) \alpha^\perp)) \\ & \quad - \frac{1}{2} \mathbb{P}((\cos((\alpha - \beta) \cdot x + \theta - \eta - \frac{\pi}{2}))(-(\alpha^\perp \cdot \beta) \beta^\perp + (\beta^\perp \cdot \alpha) \alpha^\perp)) \\ &= -\frac{1}{2} \cos((\alpha + \beta) \cdot x + \theta + \eta + \frac{\pi}{2}) (\beta^\perp \cdot \alpha) \frac{|\beta|^2 - |\alpha|^2}{|\alpha + \beta|^2} (\alpha + \beta)^\perp \\ & \quad - \frac{1}{2} \cos((\alpha - \beta) \cdot x + \theta - \eta + \frac{\pi}{2}) (\alpha^\perp \cdot \beta) \frac{|\beta|^2 - |\alpha|^2}{|\alpha + \beta|^2} (\alpha - \beta)^\perp \end{aligned}$$

we see that we have more precisely $8n - 1$ frequencies in A_n for $n \geq 1$:

- $k = k_n$ with $\lambda_{n,k} = -\rho_n |k_n|^2$ and $\eta_{n,k} = 0$
- $k = k_n + \omega_n$ with $\lambda_{n,k} = -\rho_n |k_n + \omega_n|^2$ and $\eta_{n,k} = \eta_n$
- $k = 2k_n + \omega_n$ with $\lambda_{n,k} = -\frac{1}{2} \frac{(k_n^\perp \cdot \omega_n)^2}{|2k_n + \omega_n|^2} \rho_n^2$ and $\eta_{n,k} = \eta_n - \frac{\pi}{2}$
- for $j = 0, \dots, n-1$, $k = k_n + k_j$ with $\lambda_{n,k} = \frac{1}{2} \rho_n \rho_j (k_j^\perp \cdot k_n) \frac{|k_j|^2 - |k_n|^2}{|k_n + k_j|^2}$ and $\eta_{n,k} = \frac{\pi}{2}$
- for $j = 0, \dots, n-1$, $k = k_n - k_j$ with $\lambda_{n,k} = \frac{1}{2} \rho_n \rho_j (k_n^\perp \cdot k_j) \frac{|k_j|^2 - |k_n|^2}{|k_n - k_j|^2}$ and $\eta_{n,k} = \frac{\pi}{2}$
- for $j = 1, \dots, n-1$, $k = k_n + k_j + \omega_j$ with $\eta_{n,k} = \eta_j + \frac{\pi}{2}$ and $\lambda_{n,k} = \frac{1}{2} \rho_n \rho_j ((k_j + \omega_j)^\perp \cdot k_n) \frac{|k_j + \omega_j|^2 - |k_n|^2}{|k_n + k_j + \omega_j|^2}$

- for $j = 1, \dots, n-1$, $k = k_n - k_j - \omega_j$ with $\eta_{n,k} = -\eta_j + \frac{\pi}{2}$ and $\lambda_{n,k} = \frac{1}{2}\rho_n\rho_j(k_n^\perp \cdot (k_j + \omega_j)) \frac{|k_j + \omega_j|^2 - |\vec{k}_n|^2}{|k_n - k_j - \omega_j|^2}$
- for $j = 0, \dots, n-1$, $k = k_n + \omega_n + k_j$ with $\eta_{n,k} = \eta_n + \frac{\pi}{2}$ and $\lambda_{n,k} = \frac{1}{2}\rho_n\rho_j(k_j^\perp \cdot (k_n + \omega_n)) \frac{|k_j|^2 - |k_n + \omega_n|^2}{|k_n + \omega_n + k_j|^2}$
- for $j = 0, \dots, n-1$, $k = k_n + \omega_n - k_j$ with $\eta_{n,k} = \eta_n + \frac{\pi}{2}$ and $\lambda_{n,k} = \frac{1}{2}\rho_n\rho_j((k_n + \omega_n)^\perp \cdot k_j) \frac{|k_j|^2 - |\vec{k}_n + \omega_n|^2}{|k_n + \omega_n - k_j|^2}$
- for $j = 1, \dots, n-1$, $k = k_n + \omega_n + k_j + \omega_j$ with $\eta_{n,k} = \eta_n + \eta_j + \frac{\pi}{2}$ and $\lambda_{n,k} = \frac{1}{2}\rho_n\rho_j((k_j + \omega_j)^\perp \cdot (k_n + \omega_n)) \frac{|k_j + \omega_j|^2 - |k_n + \omega_n|^2}{|k_n + \omega_n + k_j + \omega_j|^2}$
- for $j = 1, \dots, n-1$, $k = k_n + \omega_n - k_j - \omega_j$ with $\eta_{n,k} = \eta_n + -\eta_j + \frac{\pi}{2}$ and $\lambda_{n,k} = \frac{1}{2}\rho_n\rho_j((k_n + \omega_n)^\perp \cdot (k_j + \omega_j)) \frac{|k_j + \omega_j|^2 - |\vec{k}_n + \omega_n|^2}{|k_n + \omega_n - k_j - \omega_j|^2}$

For $k \in A_n$, we find that $\frac{5}{8}|k_n| \leq |k| \leq \frac{11}{8}|k_n|$, with

$$\frac{11}{8}|k_n| \leq \frac{11}{64}|k_{n+1}| \leq \frac{3}{8}\left(\frac{5}{8}|k_{n+1}|\right)$$

and the frequencies occuring in A_{n+1} are greater than those occuring in A_n .

We then write

$$A_0 = \{\gamma_1\}, A_1 = \{\gamma_2, \dots, \gamma_8\}, \dots, A_n = \{\gamma_{4n^2-5n+3}, \dots, \gamma_{4n^2+3n+1}\}, \dots$$

We write, for $j \geq 0$,

$$\vec{V}_j = \sum_{\gamma_p \in A_j} \lambda_p \cos(\gamma_p \cdot x + \alpha_p) \gamma_p^\perp = \sum_{\gamma_p \in A_j} |\lambda_p| \cos(\gamma_p \cdot x + \alpha_p + \epsilon_p \pi) \gamma_p^\perp$$

with $\epsilon_p \in \{0, 1\}$. Thus we have

$$\begin{aligned} \Delta \vec{U}_n - \mathbb{P}(\vec{U}_n \cdot \vec{\nabla} \vec{U}_n) &= \sum_{j=0}^n \sum_{\gamma_p \in A_j} |\lambda_p| \cos(\gamma_p \cdot x + \alpha_p + \epsilon_p \pi) \gamma_p^\perp \\ &\quad - \sum_{j=1}^n \frac{1}{2} \rho_j^2 (k_j^\perp \cdot \omega_j) \cos(\omega_j \cdot x + \eta_j - \frac{\pi}{2}) \omega_j^\perp. \end{aligned}$$

We know the values of ρ_0, k_0 , hence of $\gamma_1 = k_0$, $|\lambda_1| = \rho_0 |k_0|^2$ and $\alpha_0 + \epsilon_0 \pi = \pi$. We shall define by induction ω_n, k_n, ρ_n and η_n for $n \geq 1$: we remark that

$\gamma_n \in A_{j(n)}$ for some $j(n) < n$ (as $n < 4n^2 - 5n + 3$). Thus, if we already know ω_j , k_j , ρ_j and η_j for $0 \leq j \leq n-1$, we already know γ_n , $|\lambda_n|$ and $\alpha_n + \epsilon_n\pi$. The main idea is then to require that

$$|\lambda_n| \cos(\gamma_n \cdot x + \alpha_n + \epsilon_n\pi) \gamma_n^\perp = \frac{1}{2} \rho_n^2 (k_n^\perp \cdot \omega_n) \cos(\omega_n \cdot x + \eta_n - \frac{\pi}{2}) \omega_n^\perp.$$

We thus make the following choices:

- We take $\omega_n = \gamma_n$.
- We take $k_n = N_n \omega_n^\perp$, where the integer N_n will fulfill some requirements. Our first requirement will be that $N_n \in \mathbb{N}$ is large enough to grant that $N_n > 8$ and $|k_n| > 8|k_{n-1}|$.

- We then have

$$\frac{1}{2} \rho_n^2 (k_n^\perp \cdot \omega_n) \cos(\omega_n \cdot x + \eta_n - \frac{\pi}{2}) \omega_n^\perp = \frac{1}{2} \rho_n^2 N_n |\omega_n|^2 \cos(\omega_n \cdot x + \eta_n + \frac{\pi}{2}) \omega_n^\perp.$$

Thus, we take

$$\rho_n = \sqrt{\frac{2|\lambda_n|}{N_n |\omega_n|^2}} \text{ and } \eta_n = \alpha_n + \epsilon_n\pi - \frac{\pi}{2}.$$

- We shall add another requirement on N_n in order to grant that

$$\sum_{j=1}^{+\infty} \rho_j^2 \frac{|k_j|}{|\omega_j|} < +\infty,$$

Recall that $0 < \rho_0 < 1$. Take $N_0 = 1$. We first check by induction that $\rho_n \leq \rho_0 N_n^{-1/4} (\leq 1)$. Indeed, there is a constant C_0 such that $|\omega_n| \geq \frac{1}{C_0} |k_{j(n)}|$ and

$$|\lambda_n| \leq C_0 |k_{j(n)}|^2 \rho_{j(n)} \sup(1, \rho_0, \dots, \rho_{j(n)})$$

so that, by induction, $|\lambda_n| \leq C_0 |k_{j(n)}|^2 \frac{\rho_0}{N_{j(n)}^{1/4}}$ and $\rho_n \leq \sqrt{\frac{2C_0^3}{N_n}} \leq \rho_0 N_n^{-1/4}$ (if we take $N_n \geq 4C_0^6 \rho_0^{-4}$). We have

$$\rho_n^2 \frac{|k_n|}{|\omega_n|} = N_n \rho_n^2 = \frac{2|\lambda_n|}{|\omega_n|^2} \leq 2C_0^3 \rho_0 N_{j(n)}^{-1/4}$$

Thus,

$$\sum_{j=1}^{+\infty} \rho_j^2 \frac{|k_j|}{|\omega_j|} \leq 2C_0^3 \rho_0 \sum_{n=0}^{+\infty} \sum_{j \in A_n} N_n^{-1/4} \leq 2C_0^3 \rho_0 (1 + \sum_{n=1}^{+\infty} (8n-1) N_n^{-1/4}).$$

Hence, our last requirement on N_n will be that $N_n \geq (8n-1)^{12}$.

Theorem 1 is proved.

4 A remark on the Koch–Tataru theorem.

In our construction, we have $\vec{u} \in H^{-1} \cap BMO^{-1}$ with

$$\|\vec{u}\|_{H^{-1}} + \|\vec{u}\|_{BMO^{-1}} < C\rho_0.$$

Moreover,

$$\|\vec{u} - \vec{U}_n\|_{BMO^{-1}} \leq C\rho_0 j(n)^{-3} \rightarrow_{n \rightarrow +\infty} 0.$$

By the Koch–Tataru theorem [5], for ρ_0 small enough, the evolutionary problem

$$\begin{cases} \partial_t \vec{v} = \Delta \vec{v} - \mathbb{P}(\vec{v} \cdot \nabla \vec{v}) \\ \operatorname{div} \vec{v} = 0 \\ \vec{v}(0, x) = \vec{u}(x) \end{cases} \quad (9)$$

will have a smooth solution on $(0, +\infty) \times \mathbb{T}^2$ such that

- $\sup_{t>0} \sqrt{t} \|\vec{v}(t, \cdot)\|_\infty < +\infty$
- $\sup_{t>0} t \|\vec{\nabla} \otimes \vec{v}(t, \cdot)\|_\infty < +\infty$
- $\vec{v} \in \mathcal{C}([0, +\infty), BMO^{-1})$.

The steady solution \vec{u} is another solution of the evolutionary problem (9), with $\vec{u} \in \mathcal{C}([0, +\infty), BMO^{-1})$. Of course, $\vec{v} \neq \vec{u}$ as $\lim_{t \rightarrow +\infty} \|\vec{v}(t, \cdot)\|_{H^{-1}} = 0$.

References

- [1] Buckmaster, T. and Vicol, V.: Nonuniqueness of weak solutions to the Navier–Stokes equation. *Ann. Math.* 189, 101–144, 2019.

- [2] De Lellis, C. and Székelyhidi, L. Jr.: The Euler equations as a differential inclusion. *Ann. Math. (2)*, 170(3), 1417–1436, 2009.
- [3] Furioli, G., Lemarié-Rieusset, P.G. and Terraneo, E.: Sur l’unicité dans $L^3(\mathbb{R}^3)$ des solutions “mild” de l’équation de Navier–Stokes. *C. R. Acad. Sci. Paris, Série I* 325, 1253–1256, 1997.
- [4] Furioli, G., Lemarié-Rieusset, P.G. and Terraneo, E.: Unicité dans $L^3(\mathbb{R}^3)$ et d’autres espaces limites pour Navier–Stokes. *Revista Mat. Iberoamericana* 16, 605–667, 2000.
- [5] Koch, H. and Tataru, D.: Well-posedness for the Navier–Stokes equations. *Adv. Math.* 157, 22–35, 2001.
- [6] Lions, P.-L. and Masmoudi, N.: Unicité des solutions faibles de Navier–Stokes dans $L^N(\Omega)$. *C. R. Acad. Sci. Paris, Série I* 327, 491–496, 1998.
- [7] Luo, X.: Stationary Solutions and Nonuniqueness of Weak Solutions for the Navier–Stokes Equations in High Dimensions *Arch. Rational Mech. Anal.* 233, 701–747, 2019.
- [8] May, R.: Extension d’une classe d’unicité pour les équations de Navier–Stokes. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 27, 705–718, 2010.
- [9] Meyer, Y.: *Wavelets, paraproducts and Navier–Stokes equations*. Current developments in mathematics 1996, International Press, PO Box 38-2872, Cambridge, MA 02238-2872, 1999.
- [10] Monniaux, S.: Uniqueness of mild solutions of the Navier–Stokes equation and maximal L^p -regularity. *C. R. Acad. Sci. Paris, Série I* 328, 663–668, 1999.