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► **To cite this version:**

Mingjian Ding, Jiang Zeng. Some Identities Involving q -Stirling Numbers of the Second Kind in Type B. The Electronic Journal of Combinatorics, 2024, 31 (1), 10.37236/12147 . hal-04450373

HAL Id: hal-04450373

<https://hal.science/hal-04450373>

Submitted on 15 Mar 2024

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Some identities involving q -Stirling numbers of the second kind in type B

Ming-Jian Ding^a Jiang Zeng^b

Submitted: June 19, 2023; Accepted: Jan 4, 2024; Published: Feb 9, 2024

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Abstract

The recent interest in type B q -Stirling numbers of the second kind prompted us to give a type B analogue of a classical identity connecting the q -Stirling numbers of the second kind and Carlitz's major q -Eulerian numbers, which turns out to be a q -analogue of an identity due to Bagno, Biagioli and Garber. We provide a combinatorial proof of this identity and an algebraic proof of a more general identity for colored permutations. In addition, we prove some q -identities about the q -Stirling numbers of the second kind in types A, B and D.

Mathematics Subject Classifications: 05A05, 05A18, 05A19

1 Introduction

The Stirling number of the second kind, denoted $S(n, k)$, is the number of ways to partition n distinct objects into k nonempty subsets. It satisfies the well-known triangular recurrence

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)$$

with the initial conditions $S(0, k) = \delta_{0k}$, where δ_{ij} is the Kronecker delta. Carlitz [7] introduced the type A q -Stirling numbers of the second kind $S[n, k]$ by

$$S[n, k] := S[n - 1, k - 1] + [k]_q S[n - 1, k], \quad (1.1)$$

where $[k]_q := 1 + q + q^2 + \dots + q^{k-1}$ for $k \geq 1$ and $[0]_q := 0$, and $S[0, k] = \delta_{0k}$.

Let \mathfrak{S}_n be the symmetric group on the set $[n] = \{1, 2, \dots, n\}$. An element $\pi \in \mathfrak{S}_n$ is written as $\pi = \pi_1\pi_2 \cdots \pi_n$. The descent set of $\pi \in \mathfrak{S}_n$ is defined by

$$\text{Des}(\pi) := \{i \in [n - 1] \mid \pi_i > \pi_{i+1}\}$$

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and the cardinality of $\text{Des}(\pi)$ is called the number of descents of π , denoted $\text{des}(\pi)$. The Eulerian number $A_{n,k}$ is the number of $\pi \in \mathfrak{S}_n$ with k descents. There exists a well-known identity connecting the Stirling numbers of the second kind and Eulerian numbers as follows:

$$k!S(n, k) = \sum_{\ell=1}^k A_{n,\ell-1} \binom{n-\ell}{k-\ell} \quad (1.2)$$

for all nonnegative integers $0 \leq k \leq n$. A combinatorial proof of identity (1.2) in terms of the ordered set partitions and permutations is quite easy and well known, see [5, Theorem 1.17], for example.

The q -binomial coefficients are defined for $n, k \in \mathbb{N}$ by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad \text{for } 0 \leq k \leq n,$$

where $[n]_q! := [1]_q [2]_q \cdots [n]_q$ is the q -factorial of n . To give a q -analogue of identity (1.2) we need to find a suitable *Mahonian statistic* over permutations, that is, a statistic whose generating function over \mathfrak{S}_n is $[n]_q!$. It turns out that MacMahon's *major index* [17] is a good fit for our q -analogue. Recall that the *major index* (maj) of $\pi \in \mathfrak{S}_n$ is defined by

$$\text{maj}(\pi) := \sum_{i \in \text{Des}(\pi)} i.$$

We define the corresponding q -analogue of Eulerian polynomial (of type A) by

$$A_n(t, q) := \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)} = \sum_{k=0}^n A_{n,k}(q) t^k. \quad (1.3)$$

The reader is referred to [11, 18] and references therein for further q -Eulerian polynomials.

Using analytic method, Zeng and Zhang [27, Proposition 4.5] proved the following q -analogue of identity (1.2)¹

$$q^{\binom{k}{2}} [k]_q! S[n, k] = \sum_{\ell=1}^k q^{k(k-\ell)} A_{n,\ell-1}(q) \begin{bmatrix} n-\ell \\ k-\ell \end{bmatrix}_q \quad (1.4)$$

for nonnegative integers $0 \leq k \leq n$. In 1997, in order to give a combinatorial proof of (1.4), Steingrímsson [23] proposed several statistics on ordered set partitions and conjectured that their generating functions were given by either side of (1.4). In the following years, Zeng et al. [16, 13, 14] confirmed all his conjectures, and finally Remmel and Wilson [20, Section 5.1] found a combinatorial proof of (1.4) using the major index on the starred permutations.

¹Proposition 4.5 in [27] is actually a *fractional version* of (1.4) and valid for $n \in \mathbb{C}$.

This paper arose from the desire to give a type B analogue of (1.4). In analogy with the usual (type A) Stirling numbers of the second kind (see [26, 10, 3, 21]), the *type B Stirling numbers of the second kind* $S_B(n, k)$ can be defined by

$$S_B(n, k) := S_B(n - 1, k - 1) + (2k + 1)S_B(n - 1, k)$$

with the initial conditions $S_B(0, k) = \delta_{0k}$.

For integer $i \in \mathbb{Z}$ we denote its opposite integer $-i$ by \bar{i} . Let \mathcal{B}_n be the group of signed permutations of $[n]$, i.e., the set of all permutations on the set $[\pm n] := \{\bar{n}, \dots, \bar{1}, 1, \dots, n\}$ such that $\pi(\bar{i}) = \overline{\pi(i)}$. In what follows, we write $\pi(i)$ as π_i for $i \in [\pm n]$ and use the *natural order* on $\langle n \rangle := \{\bar{n}, \dots, \bar{1}, 0, 1, \dots, n\}$, namely,

$$\bar{n} < \dots < \bar{1} < 0 < 1 < \dots < n.$$

The type B descent set of $\pi \in \mathcal{B}_n$ [18, Section 11.5.2] is defined by

$$\text{Des}_B(\pi) = \{i \in \{0\} \cup [n - 1] \mid \pi_i > \pi_{i+1}\},$$

with $\pi_0 = 0$, and the cardinality of $\text{Des}_B(\pi)$ is called the number of type B descents of π , denoted $\text{des}_B(\pi)$.

Let $B_{n,k}$ be the number of permutations in \mathcal{B}_n with k descents. By a bijection between the set of ordered type B set partitions and the set of signed permutations with separators, Bagno, Biagioli and Garber [3] combinatorially proved the following type B analogue of (1.2):

$$2^k k! S_B(n, k) = \sum_{\ell=0}^k B_{n,\ell} \binom{n-\ell}{k-\ell} \quad (1.5)$$

for all nonnegative integers $0 \leq k \leq n$.

Recently Sagan and Swanson [21] studied the *type B q -Stirling numbers of the second kind* $S_B[n, k]$, which are defined by the recurrence relation

$$S_B[n, k] := S_B[n - 1, k - 1] + [2k + 1]_q S_B[n - 1, k] \quad (1.6)$$

with the initial conditions $S_B[0, k] = \delta_{0k}$, see [25, Section 1.10] and [4] for related works.

Remark 1.1. Chow-Gessel [8, Eq. (18) and Proposition 4.2] defined a kind of type B q -Stirling numbers of the second kind $S_{n,k}(q)$ by the following recurrence relation

$$S_{n,k}(q) := q^{2k-1}(1+q)S_{n-1,k-1}(q) + [2k+1]_q S_{n-1,k}(q)$$

with the initial conditions $S_{n,0}(q) = 1$ for $n \geq 0$. It is routine to verify that the above two types B q -Stirling numbers of the second kind are related as follows

$$S_{n,k}(q) = (1+q)^k q^{k^2} S_B[n, k]. \quad (1.7)$$

Adin and Roichman [1] defined the *flag-major index* of $\pi \in \mathcal{B}_n$ as follows

$$\text{fmaj}(\pi) := \sum_{i \in \text{Des}_B(\pi)} 2i + \text{neg}(\pi), \quad (1.8)$$

where $\text{neg}(\pi)$ is the number of negative elements in π , i.e., $|\{i \in [n] : \pi_i < 0\}|$. Then, as a q -analogue of Eulerian polynomial of type B, Chow and Gessel [8] studied the enumerative polynomials of statistic $(\text{des}_B, \text{fmaj})$ over \mathcal{B}_n ,

$$B_n(t, q) := \sum_{\pi \in \mathcal{B}_n} t^{\text{des}_B(\pi)} q^{\text{fmaj}(\pi)} = \sum_{k=0}^n B_{n,k}(q) t^k. \quad (1.9)$$

In this paper, using Sagan and Swanson's q -Stirling numbers of the second kind in type B [21] and Chow and Gessel's q -Eulerian numbers of type B, we prove a q -analogue of Bagno et al.'s identity (1.5). The following is our first main result.

Theorem 1.2. For $0 \leq k \leq n$ we have

$$[2]^k [k]_{q^2}! S_B[n, k] = \sum_{\ell=0}^k q^{k(k-2\ell)} B_{n,\ell}(q) \begin{bmatrix} n-\ell \\ k-\ell \end{bmatrix}_{q^2}. \quad (1.10)$$

We shall provide a combinatorial proof for Theorem 1.2 in Section 2. In Section 3, we define a q -Stirling numbers of the second kind in type D and give q -analogues of some known identities connecting the Stirling numbers of the second kind in types A, B and D. Next, we prove algebraically a general identity (see Theorem 4.2) between the r -colored q -Stirling numbers of the second kind and q -Eulerian numbers of colored permutations in Section 4. Note that the proof of Theorem 4.2 yields another proof of Theorem 1.2.

2 Combinatorial proof of Theorem 1.2

In this section, we give a combinatorial proof of (1.10) by generalizing Remmel and Wilson's proof of identity (1.4) in [20]. Our strategy is to study the polynomial

$$\sum_{\pi \in \mathcal{B}_n} q^{\text{fmaj}(\pi)} \prod_{i=1}^{\text{des}_B(\pi)} \left(1 + \frac{z}{q^{2i-1}} \right) \quad (2.1)$$

in $\mathbb{R}[q][z]$ and interpret the coefficient of z^k combinatorially in two different ways.

2.1 Permutations of type B

For any $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathcal{B}_n$, we say that an index $i \in [n-1]$ has π -*sign type* $++$ (resp., $--$, $+-$, $-+$) if the sign of π_i is positive (resp., negative, positive, negative) and that of π_{i+1} is positive (resp., negative, negative, positive).

In the rest of this section, we denote by Π_1 (resp., Π_2, Π_3) the set of descents of π with π -sign type $++$ (resp., $--, +-)$ and by Π'_1 (resp., Π'_2, Π'_3) the set of ascents of π with π -sign type $++$ (resp., $--, -+)$.

For any $\pi \in \mathcal{B}_n$, define the mapping $\psi : \pi \rightarrow \tilde{\pi}$ on \mathcal{B}_n by

$$\tilde{\pi}_i := \begin{cases} \pi_{n+1-i} - n - 1, & \text{if } \pi_{n+1-i} > 0; \\ \pi_{n+1-i} + n + 1, & \text{if } \pi_{n+1-i} < 0. \end{cases}$$

For example, if $\pi = 1\bar{5}\bar{3}4\bar{6}\bar{2}$, then $\tilde{\pi} = 5\bar{1}\bar{3}4\bar{2}\bar{6}$.

Remark 2.1. Let $r : \pi \mapsto \pi^r$ be the *reversing operator* on \mathcal{B}_n defined by $\pi_i^r = \pi_{n+1-i}$ and $c : \pi \mapsto \pi^c$ the type B *completion operator* on \mathcal{B}_n defined by $\pi_i^c = \varepsilon_i \cdot (n + 1 - |\pi_i|)$, where $\varepsilon_i = 1$ if $\pi_i < 0$ and -1 if $\pi_i > 0$ for $i \in [n]$. It is easy to verify that $\tilde{\pi} = (\pi^r)^c$.

Clearly, if i is a descent (resp., an ascent) position in $\pi \in \mathcal{B}_n$ and the product of π_i and π_{i+1} is positive, then $n - i$ is an ascent (resp., a descent) position in $\tilde{\pi}$; if i is a descent (resp., an ascent) position in $\pi \in \mathcal{B}_n$ and the product of π_i and π_{i+1} is negative, then $n - i$ is a descent (resp., an ascent) position in $\tilde{\pi}$.

In fact, the mapping ψ is a bijection between all permutations in \mathcal{B}_n with k descents and all permutations in \mathcal{B}_n with $n - k$ descents by the following result.

Lemma 2.2. The mapping ψ is a bijection on \mathcal{B}_n such that for any $\pi \in \mathcal{B}_n$, we have $\text{des}_B(\tilde{\pi}) = n - \text{des}_B(\pi)$.

Proof. It is convenient to associate a permutation in \mathcal{B}_n with a character string in $\{+, -\}^n$ by replacing each positive (resp., negative) element with $+$ (resp., $-$). For example, the string for permutation $1\bar{5}\bar{3}4\bar{6}\bar{2}$ is $+-+ +-$. Let $\pi \in \mathcal{B}_n$ with $\text{des}_B(\pi) = k$. We consider the following four cases in terms of the signs of π_1 and π_n .

(i) If $\pi_1 > 0$ and $\pi_n > 0$, then

$$|\Pi_1| + |\Pi_2| + |\Pi_3| = k \text{ and } |\Pi'_1| + |\Pi'_2| + |\Pi'_3| = n - k - 1.$$

In addition, $|\Pi_3|$ (resp., $|\Pi'_3|$) is the number of $+-$ (resp., $-+$) occurring in the character string of π . Obviously, we have $|\Pi_3| = |\Pi'_3|$ since $\pi_1 > 0$ and $\pi_n > 0$. For the permutation $\tilde{\pi} = \psi(\pi)$, it is easy to see that $(n - \Pi'_1) \cup (n - \Pi'_2) \cup (n - \Pi_3)$ is a subset of descent positions in $\tilde{\pi}$, where $n - \Pi$ denotes the set $\{n - i \mid i \in \Pi\}$. Note that 0 is also a descent position in $\tilde{\pi}$ since $\tilde{\pi}_1 = \pi_n - n - 1 < 0$, hence

$$\text{des}_B(\tilde{\pi}) = 1 + |n - \Pi'_1| + |n - \Pi'_2| + |n - \Pi_3| = 1 + |\Pi'_1| + |\Pi'_2| + |\Pi'_3| = n - k.$$

(ii) If $\pi_1 > 0$ and $\pi_n < 0$, then

$$|\Pi_1| + |\Pi_2| + |\Pi_3| = k, |\Pi'_1| + |\Pi'_2| + |\Pi'_3| = n - k - 1 \text{ and } |\Pi_3| = |\Pi'_3| + 1.$$

Hence, we have

$$\text{des}_B(\tilde{\pi}) = |n - \Pi'_1| + |n - \Pi'_2| + |n - \Pi_3| = |\Pi'_1| + |\Pi'_2| + |\Pi'_3| + 1 = n - k.$$

(iii) If $\pi_1 < 0$ and $\pi_n > 0$, then

$$|\Pi_1| + |\Pi_2| + |\Pi_3| = k - 1, |\Pi'_1| + |\Pi'_2| + |\Pi'_3| = n - k \text{ and } |\Pi_3| = |\Pi'_3| - 1.$$

Note that 0 is a descent position since $\tilde{\pi}_1 = \pi_n - n - 1 < 0$, hence

$$\text{des}_B(\tilde{\pi}) = 1 + |n - \Pi'_1| + |n - \Pi'_2| + |n - \Pi_3| = |\Pi'_1| + |\Pi'_2| + |\Pi'_3| = n - k.$$

(iv) If $\pi_1 < 0$ and $\pi_n < 0$, then

$$|\Pi_1| + |\Pi_2| + |\Pi_3| = k - 1, |\Pi'_1| + |\Pi'_2| + |\Pi'_3| = n - k \text{ and } |\Pi_3| = |\Pi'_3|,$$

which implies that

$$\text{des}_B(\tilde{\pi}) = |n - \Pi'_1| + |n - \Pi'_2| + |n - \Pi_3| = |\Pi'_1| + |\Pi'_2| + |\Pi'_3| = n - k.$$

Summarising the above four cases we are done. □

Lemma 2.3. Let $\pi \in \mathcal{B}_n$ and $\text{neg}(\pi) = m$.

(a) If $\pi_n < 0$, then $\sum_{i \in \Pi_3} i + m = \sum_{i \in \Pi'_3} i + n$;

(b) If $\pi_n > 0$, then $\sum_{i \in \Pi_3} i + m = \sum_{i \in \Pi'_3} i$.

Proof. Let $\pi \in \mathcal{B}_n$ with $\Pi_3 = \{i_1, i_2, \dots, i_\ell\}$ and $\Pi'_3 = \{j_1, j_2, \dots, j_r\}$ for some integers $\ell, r \geq 1$. As the proof of (b) is similar, we only prove (a) by considering two cases.

(i) $\pi_1 > 0$ and $\pi_n < 0$, we have $\ell = r + 1$. It is easy to see that $i_{k+1} - j_k$ is the number of positive elements between the k th ascent position and the $(k + 1)$ th descent position from left to right. Note that $|\Pi_3| = |\Pi'_3| + 1$ in this case. Therefore, we have $i_1 + \sum_{k=1}^r (i_{k+1} - j_k) = n - m$.

(ii) $\pi_1 < 0$ and $\pi_n < 0$, we have $\ell = r$. Similarly, $i_k - j_k$ is the number of positive elements between the k th ascent position and the k th descent position from left to right. Then we have $\sum_{k=1}^r (i_k - j_k) = n - m$.

Combining the above two cases completes the proof of (a). □

The following q -symmetry of $B_{n,k}(q)$ is crucial for our combinatorial proof of identity (1.10).

Proposition 2.4. For each fixed nonnegative integer n and the polynomial $B_{n,k}(q)$ defined in (1.9), we have

$$B_{n,k}(q) = q^{2nk-n^2} B_{n,n-k}(q) \tag{2.2}$$

for $0 \leq k \leq n$.

Proof. For any $\pi \in \mathcal{B}_n$ with k descents and m negative elements, then $\text{des}_B(\psi(\pi)) = n - k$ by Lemma 2.2. Hence, it suffices to show that

$$\text{fmaj}(\pi) = 2nk - n^2 + \text{fmaj}(\psi(\pi)).$$

Let $\tilde{\pi} = \psi(\pi)$, we consider the proof in terms of the signs of π_1 and π_n . We only give the proof for this case $\pi_1 > 0$ and $\pi_n > 0$ and omit similar discussions for other three cases for the brevity.

If $\pi_1 > 0$ and $\pi_n > 0$, by the definition of the mapping ψ , then the set of descents in $\tilde{\pi}$ is the disjoint union

$$\{0\} \cup (n - \Pi'_1) \cup (n - \Pi'_2) \cup (n - \Pi_3)$$

and $\tilde{\pi}$ has $n - m$ negative elements. Then,

$$\text{fmaj}(\tilde{\pi}) = 2 \left(\sum_{i \in n - \Pi'_1} i + \sum_{i \in n - \Pi'_2} i + \sum_{i \in n - \Pi_3} i \right) + n - m. \quad (2.3)$$

By Case (i) in the proof of Proposition 2.2, we have $|\Pi'_1| + |\Pi'_2| + |\Pi_3| = n - k - 1$. Hence identity (2.3) is equivalent to

$$\begin{aligned} \text{fmaj}(\tilde{\pi}) &= 2n(n - k - 1) - 2 \left(\sum_{i \in \Pi'_1} i + \sum_{i \in \Pi'_2} i + \sum_{i \in \Pi_3} i \right) + n - m \\ &= 2n(n - k - 1) - 2 \left(\binom{n}{2} - \sum_{i \in \Pi_1} i - \sum_{i \in \Pi_2} i - \sum_{i \in \Pi'_3} i \right) + n - m \\ &= n^2 - 2nk + 2 \sum_{i \in \Pi_1} i + 2 \sum_{i \in \Pi_2} i + 2 \sum_{i \in \Pi'_3} i - m, \end{aligned}$$

where the second equality uses the fact that the sum of all descent and ascent indexes is $\binom{n}{2}$. By statement (b) of Proposition 2.3, the above identity equals

$$\begin{aligned} \text{fmaj}(\tilde{\pi}) &= n^2 - 2nk + 2 \sum_{i \in \Pi_1} i + 2 \sum_{i \in \Pi_2} i + 2 \sum_{i \in \Pi_3} i + m \\ &= n^2 - 2nk + \text{fmaj}(\pi). \end{aligned}$$

This is the desired result. □

2.2 Ordered set partitions of type B

Recall that $\langle n \rangle = \{\bar{n}, \dots, \bar{1}, 0, 1, \dots, n\}$. There are at least two equivalent definitions of type B set partition. We say that a set partition of $\langle n \rangle$ is a *type B partition* if it satisfies the following properties

- (1) there exactly is one zero block T such that $0 \in T$ and $-T = T$;
- (2) if T appears as a block then $-T$ is also a block.

It is known [3, 21] that $S_B(n, k)$ is the number of type B partitions of $\langle n \rangle$ with $2k + 1$ blocks. An *ordered signed partition* of $\langle n \rangle$ is a sequence $(T_0, T_1, T_2, \dots, T_{2k})$ of disjoint subsets (blocks) T_i of $\langle n \rangle$ satisfying

- (1) $0 \in T_0$ and $T_0 = \overline{T}_0$, and
- (2) $T_{2i} = \overline{T}_{2i-1}$ for $i \in [k]$,

where $\overline{T} = \{\bar{t} : t \in T\}$. The blocks T_{2i} and T_{2i-1} are called *paired*. Clearly the number of all ordered signed partitions of $\langle n \rangle$ with $2k + 1$ blocks is $2^k k! S_B(n, k)$.

For our purpose, it is convenient to use the following equivalent definition of ordered signed partition. An *ordered set partition with sign* of $S = \{0, 1, \dots, n\}$ is a sequence (S_0, S_1, \dots, S_k) such that

- (1) $S_0 = \{t \in T_0 : t \leq 0\}$, and
- (2) $S_i = T_{2i-1}$ for $i \in [k]$.

For example, the sequence $(\{0, \bar{3}, \bar{1}, \bar{4}\}, \{\bar{2}, 7\}, \{\bar{6}\}, \{8, \bar{5}\})$ is an ordered set partition with sign of $\{0, 1, \dots, 8\}$.

On the other hand, as in [20], we can consider an ordered set partition with sign as a descent-starred signed permutation, i.e, for any $\pi \in \mathcal{B}_n$, the space following element π_i , satisfying $\pi_i > \pi_{i+1}$ for some $0 \leq i \leq n - 1$, is starred or unstarred. That is to say, instead of using brackets to signify separations between blocks, the spaces between elements sharing a block can be marked with stars and all blocks are written in decreasing order. Note that we require that the block including element 0 always stands first on the list.

For example, the ordered set partition with sign $(\{0, \bar{3}, \bar{1}, \bar{4}\}, \{\bar{2}, 7\}, \{\bar{6}\}, \{8, \bar{5}\})$ can be written as $0_* \bar{1}_* \bar{3}_* \bar{4} 7_* \bar{2} \bar{6} 8_* \bar{5}$. The above discussion shows that there is a bijection between all ordered set partitions with sign of the set $\{0, 1, \dots, n\}$ and all descent-starred signed permutations in \mathcal{B}_n . For $0 \leq k \leq n$ define the set

$$\mathcal{B}_{n,k}^> := \{(\pi, S) : \pi \in \mathcal{B}_n, S \subseteq \text{Des}_B(\pi), |S| = k\}, \tag{2.4}$$

where S is the set of the starred descent positions.

For $(\pi, S) \in \mathcal{B}_{n,k}^>$ define the statistic

$$\text{fmaj}((\pi, S)) := \text{fmaj}(\pi) - \sum_{j \in S} (2|\text{Des}_B(\pi) \cap \{j, \dots, n - 1\}| - 1)$$

and the polynomial

$$B_{n,k}^{\text{fmaj}}(q) := \sum_{(\pi, S) \in \mathcal{B}_{n,k}^>} q^{\text{fmaj}((\pi, S))}. \tag{2.5}$$

By the definition of the statistic $\text{fmaj}((\pi, S))$, we attach the i th descent position of π (from right to left) with the weight 1 if this descent position is unstarred and the weight z/q^{2i-1} if this descent position is starred. Therefore, the following identity holds

$$\sum_{k=0}^n B_{n,k}^{\text{fmaj}}(q)z^k = \sum_{\pi \in \mathcal{B}_n} q^{\text{fmaj}(\pi)} \prod_{i=1}^{\text{des}_B(\pi)} \left(1 + \frac{z}{q^{2i-1}}\right). \quad (2.6)$$

For convenience, we recall two known q -identities (see [2, Theorem 3.3])

$$\prod_{i=1}^N (1 - zq^{i-1}) = \sum_{j=0}^N \begin{bmatrix} N \\ j \end{bmatrix}_q (-1)^j z^j q^{j(j-1)/2}, \quad (2.7)$$

$$\frac{1}{\prod_{i=1}^N (1 - zq^{i-1})} = \sum_{j=0}^{\infty} \begin{bmatrix} N + j - 1 \\ j \end{bmatrix}_q z^j. \quad (2.8)$$

We first establish the following result for polynomials $B_{n,n-\ell}(q)$ and $B_{n,n-k}^{\text{fmaj}}(q)$ defined by (1.9) and (2.5).

Proposition 2.5. For $0 \leq k \leq n$ we have

$$B_{n,n-k}^{\text{fmaj}}(q) = \sum_{\ell=0}^k q^{(n-k)(2\ell-n-k)} B_{n,n-\ell}(q) \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_{q^2}.$$

Proof. Let $(\pi, S) \in \mathcal{B}_{n,n-k}^>$, then there are $n - k$ starred descents in (π, S) , this means that the number of ascents is in $\{0\} \cup [k]$. Suppose that the signed permutation π has ℓ ascents, where $\ell \in \{0\} \cup [k]$, then the signed permutation π can be any permutation in \mathcal{B}_n with $n - \ell$ descents. Therefore, the sum of q -counting about the flag-major statistic for all possible signed permutations with $n - \ell$ descents is the polynomial $B_{n,n-\ell}(q)$.

In addition, for a signed permutation π with $n - \ell$ descents, we can choose $n - k$ descents from $n - \ell$ descents in π and mark them with stars. By the definition of the statistic $\text{fmaj}((\pi, S))$ and identities (2.6) and (2.7), we have

$$[z^{n-k}] \prod_{i=1}^{n-\ell} \left(1 + \frac{zq}{q^{2i}}\right) = q^{(n-k)(2\ell-n-k)} \begin{bmatrix} n - \ell \\ n - k \end{bmatrix}_{q^2},$$

where $[z^k]f(z)$ denotes the coefficient of z^k in the polynomial $f(z)$. Using the symmetry of q -binomial coefficients

$$\begin{bmatrix} n - \ell \\ n - k \end{bmatrix}_{q^2} = \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_{q^2},$$

we complete the proof. □

To derive a recurrence relation for the polynomials $B_{n,n-k}^{\text{fmaj}}(q)$, we introduce some notations. For other unstarred positions, we label the rightmost position in our descent-starred

signed permutation with 0, and then label its unlabelled descent positions from right to left with $1, 2, \dots$. Next, all other unlabelled positions from left to right are labelled with increasing labels starting from the next number. We call the above labelling as *fmaj-labelling*. For example, if $(\pi, S) = 4_*3_*\bar{1}7_*\bar{2}\bar{6}8_*\bar{5}$, then the fmaj-labelling for (π, S) is

$${}_24_*3_*\bar{1}_37_*\bar{2}_1\bar{6}_48_*\bar{5}_0.$$

For $\alpha = n$ or \bar{n} , we define the mapping

$$\phi_{\alpha,k}^\perp : \{0, 1, \dots, n - k - 1\} \times \mathcal{B}_{n-1,k}^\gt \rightarrow \mathcal{B}_{n,k}^\gt \quad (2.9)$$

by sending $(i, (\pi, S))$ to the descent-starred signed permutation obtained from (π, S) by

- (1) inserting α at the fmaj-labelling i , and then
- (2) moving each star on the right of α one descent to its left.

Clearly, the rightmost descent will be unstarred when the letter n is not inserted after π_{n-1} . Thus, we have the following relation between these labels and insertion mappings.

Lemma 2.6. For $0 \leq k \leq n - 1$ we have

- (a) if $(\pi, S) \in \mathcal{B}_{n-1,k}^\gt$, then $\text{fmaj}(\phi_{n,k}^\perp(i, (\pi, S))) = \text{fmaj}((\pi, S)) + 2i$ for $i \in \{0\} \cup [n - k - 1]$;
- (b) if $(\pi, S) \in \mathcal{B}_{n-1,k}^\gt$, then $\text{fmaj}(\phi_{\bar{n},k}^\perp(i, (\pi, S))) = \text{fmaj}((\pi, S)) + 2i - 1$ for $i \in [n - k - 1]$;
- (c) if $(\pi, S) \in \mathcal{B}_{n-1,k}^\gt$, then $\text{fmaj}(\phi_{\bar{n},k}^\perp(0, (\pi, S))) = \text{fmaj}((\pi, S)) + 2n - 2k - 1$.

Proof. We will discuss the change of the statistic $\text{fmaj}((\pi, S))$ in terms of the insertion position of n or \bar{n} . Suppose that the space labelled i under the fmaj-labelling of (π, S) is the space immediately following π_p . Moreover, we suppose that there are a starred descents and b unstarred descents to the left of π_p and c unstarred descents and d starred descents to the right of π_{p+1} in (π, S) .

For (a), inserting n into the space labelled i . Let $(\tau, T) = \phi_{n,k}^\perp(i, (\pi, S))$. If $i = 0$, that is to say we insert n at the end, then the insertion of n does not affect $\text{fmaj}((\pi, S))$, thus $\text{fmaj}((\tau, T)) = \text{fmaj}((\pi, S))$. For $i \neq 0$, there will exist two cases in terms of the values of π_p and π_{p+1} .

Case (i): If $\pi_p > \pi_{p+1}$, then $i = c + 1$. By inserting n after π_p , which preserves each descent position before π_p and increases each descent position after π_p by one. Thus, the statistic $\text{fmaj}(\tau) = \text{fmaj}(\pi) + 2c + 2d + 2$. In addition, the insertion of n does not affect the starred descents before π_p to the corresponding sum $\sum_{j \in S} (2|\text{Des}_B(\pi) \cap \{j, \dots, n - 2\}| - 1)$. Moving each star after π_{p+1} one descent to its left that increases the sum $\sum_{j \in S} (2|\text{Des}_B(\pi) \cap \{j, \dots, n - 2\}| - 1)$ by two. Therefore, we have

$$\sum_{j \in T} (2|\text{Des}_B(\tau) \cap \{j, \dots, n - 1\}| - 1) = \sum_{j \in S} (2|\text{Des}_B(\pi) \cap \{j, \dots, n - 2\}| - 1) + 2d$$

since there are d stars after π_{p+1} . Hence,

$$\begin{aligned} \text{fmaj}((\tau, T)) &= \text{fmaj}(\tau) - \sum_{j \in T} (2|\text{Des}_B(\tau) \cap \{j, \dots, n-1\}| - 1) \\ &= \text{fmaj}(\pi) + 2c + 2d + 2 - \sum_{j \in S} (2|\text{Des}_B(\pi) \cap \{j, \dots, n-2\}| - 1) - 2d \\ &= \text{fmaj}((\pi, S)) + 2c + 2 \\ &= \text{fmaj}((\pi, S)) + 2i \end{aligned}$$

for $i \in [n - k - 1]$.

Case (ii): If $\pi_p < \pi_{p+1}$, then $i = p + 1 - a + c$. By inserting n after π_p , which preserves each descent position before π_p and increases each descent position after π_p by one. Besides, note that there is a new descent, $p + 1 \in \text{Des}_B(\tau)$ while inserting n after π_p . Thus, the statistic $\text{fmaj}(\tau) = \text{fmaj}(\pi) + 2p + 2 + 2c + 2d$. In addition, the insertion of n increases each starred descent before π_p to the corresponding sum $\sum_{j \in S} (2|\text{Des}_B(\pi) \cap \{j, \dots, n-2\}| - 1)$ by two. Moving each star after π_{p+1} one descent to its left that increases the sum $\sum_{j \in S} (2|\text{Des}_B(\pi) \cap \{j, \dots, n-2\}| - 1)$ by two. Therefore,

$$\sum_{j \in T} (2|\text{Des}_B(\tau) \cap \{j, \dots, n-1\}| - 1) = \sum_{j \in S} (2|\text{Des}_B(\pi) \cap \{j, \dots, n-2\}| - 1) + 2a + 2d$$

since there are a stars before π_p and d stars after π_{p+1} . Hence,

$$\begin{aligned} \text{fmaj}((\tau, T)) &= \text{fmaj}(\tau) - \sum_{j \in T} (2|\text{Des}_B(\tau) \cap \{j, \dots, n-1\}| - 1) \\ &= \text{fmaj}(\pi) + 2p + 2 + 2c + 2d \\ &\quad - \sum_{j \in S} (2|\text{Des}_B(\pi) \cap \{j, \dots, n-2\}| - 1) - 2a - 2d \\ &= \text{fmaj}((\pi, S)) + 2p + 2 - 2a + 2c \\ &= \text{fmaj}((\pi, S)) + 2i \end{aligned}$$

for $i \in [n - k - 1]$.

For (b), inserting \bar{n} into the space labelled i . Let $(\mu, R) = \phi_{\bar{n}, k}^{\perp}(i, (\pi, S))$. For $i \neq 0$, all changes for $\text{fmaj}((\mu, R))$ are the same to (a) except that for the statistic $\text{fmaj}(\pi)$ when the new descent position generated by \bar{n} . In this case, there always exists one descent between π_p and \bar{n} . The descent generated by \bar{n} increases the statistic $\text{fmaj}(\pi)$ by 1 when $\pi_p > \pi_{p+1}$ and $2p + 1$ when $\pi_p < \pi_{p+1}$, respectively. For the insertion of n at same position, the changes separately are 2 and $2p + 2$ for those two cases. Following the discussion of (a), it is easy to know that

$$\text{fmaj}((\mu, R)) = \text{fmaj}((\pi, S)) + 2i - 1$$

for $i \in [n - k - 1]$.

For (c), if $i = 0$, inserting \bar{n} after π_{n-1} , then the only change is the new descent $\pi_{n-1} > \bar{n}$. That is to say, the insertion of \bar{n} increases $\text{fmaj}(\pi)$ and $\sum_{j \in S} (2|\text{Des}_B(\pi) \cap \{j, \dots, n-2\}| - 1)$ by $2n-1$ and $2k$, respectively. Thus,

$$\text{fmaj}((\mu, R)) = \text{fmaj}((\pi, S)) + 2n - 2k - 1.$$

Summarising the above cases we have completed the proof. □

As mentioned before, the mapping $\phi_{\alpha,k}^{\downarrow}$ preserves the number of stars in the mapping process. Similarly, we need to define some mappings that increase the number of stars by one as follows:

$$\phi_{n,k}^* : \{1, 2, \dots, n-k\} \times \mathcal{B}_{n-1,k-1}^> \rightarrow \mathcal{B}_{n,k}^> \quad (2.10)$$

and

$$\phi_{\bar{n},k}^* : \{0, 1, \dots, n-k\} \times \mathcal{B}_{n-1,k-1}^> \rightarrow \mathcal{B}_{n,k}^>, \quad (2.11)$$

which send $(i, (\pi, S))$ to the descent-starred signed permutation obtained from (π, S) by

- (1) inserting n (resp., \bar{n}) at the fmaj -labelling i , then
- (2) moving each star on the right of n (resp., \bar{n}) one descent to its left, and then
- (3) placing a star at the rightmost descent of the resulting descent-starred signed permutation.

In analogy with the discussion in the proof of Lemma 2.6, let $\alpha = n$ or \bar{n} and $(\tau, T) = \phi_{\alpha,k}^*(i, (\pi, S))$. The first step and second one from the mapping $\phi_{\alpha,k}^*$ have same effect with $\phi_{\alpha,k}^{\downarrow}$ to the statistics $\text{fmaj}(\pi)$ and $\sum_{j \in S} (2|\text{Des}_B(\pi) \cap \{j, \dots, n-2\}| - 1)$. The last step from the mapping $\phi_{\alpha,k}^*$, placing a star at the rightmost of resulting descent-starred signed permutation, which increases the sum $\sum_{j \in S} (2|\text{Des}_B(\pi) \cap \{j, \dots, n-2\}| - 1)$ by one. Therefore, we have the following results, of which the proof is omitted for the brevity.

Lemma 2.7. For $1 \leq k \leq n$ we have

- (a) if $(\pi, S) \in \mathcal{B}_{n-1,k-1}^>$, then $\text{fmaj}(\phi_{n,k}^*(i, (\pi, S))) = \text{fmaj}((\pi, S)) + 2i - 1$ for $i \in [n-k]$;
- (b) if $(\pi, S) \in \mathcal{B}_{n-1,k-1}^>$, then $\text{fmaj}(\phi_{\bar{n},k}^*(i, (\pi, S))) = \text{fmaj}((\pi, S)) + 2i - 2$ for $i \in [n-k]$;
- (c) if $(\pi, S) \in \mathcal{B}_{n-1,k-1}^>$, then $\text{fmaj}(\phi_{\bar{n},k}^*(0, (\pi, S))) = \text{fmaj}((\pi, S)) + 2n - 2k$.

By definitions (2.9) and (2.10), the mappings $\phi_{\alpha,k}^{\downarrow}$ and $\phi_{\alpha,k}^*$ ($\alpha = n$ or \bar{n}) have their images $\mathcal{I}_0 \cup \mathcal{I}_1$ and \mathcal{I}_2 , respectively, where

$$\begin{aligned} \mathcal{I}_0 &= \{(\pi, S) \in \mathcal{B}_{n,k}^> : \pi_n = n\}; \\ \mathcal{I}_1 &= \{(\pi, S) \in \mathcal{B}_{n,k}^> : \text{rightmost descent is unstarred in } (\pi, S) \text{ and } \pi_n \neq n\}; \\ \mathcal{I}_2 &= \{(\pi, S) \in \mathcal{B}_{n,k}^> : \text{rightmost descent is starred in } (\pi, S) \text{ and } \pi_n \neq n\}. \end{aligned}$$

Obviously, the disjoint union of those three sets is $\mathcal{B}_{n,k}^>$. Now, we are ready to prove the following recurrence relation for the polynomial $B_{n,k}^{\text{fmaj}}(q)$ defined in (2.5).

Proposition 2.8. For $n \geq 1$ we have the recurrence relation

$$B_{n,k}^{\text{fmaj}}(q) = [2n - 2k]_q B_{n-1,k}^{\text{fmaj}}(q) + [2n - 2k + 1]_q B_{n-1,k-1}^{\text{fmaj}}(q),$$

where $B_{n,k}^{\text{fmaj}}(q)$ is 1 when $k = n$ and is 0 when $k < 0$ or $k > n$.

Proof. Since $\mathcal{B}_{n,k}^>$ is the disjoint union of the images of mappings $\phi_{\alpha,k}^{\downarrow}$ and $\phi_{\alpha,k}^*$, we have

$$B_{n,k}^{\text{fmaj}}(q) = \sum_{(\pi,S) \in \mathcal{I}_0} q^{\text{fmaj}((\pi,S))} + \sum_{(\pi,S) \in \mathcal{I}_1} q^{\text{fmaj}((\pi,S))} + \sum_{(\pi,S) \in \mathcal{I}_2} q^{\text{fmaj}((\pi,S))}. \quad (2.12)$$

By the definition of mapping $\phi_{\alpha,k}^{\downarrow}$ and Lemma 2.6, the first two summations of identity (2.12) is

$$\begin{aligned} & \sum_{(\pi,S) \in \mathcal{I}_0} q^{\text{fmaj}((\pi,S))} + \sum_{(\pi,S) \in \mathcal{I}_1} q^{\text{fmaj}((\pi,S))} \\ &= \sum_{i=0}^{n-k-1} \sum_{(\pi,S) \in \mathcal{B}_{n-1,k}^>} q^{\text{fmaj}(\phi_{n,k}^{\downarrow}(i,(\pi,S)))} + \sum_{i=0}^{n-k-1} \sum_{(\pi,S) \in \mathcal{B}_{n-1,k}^>} q^{\text{fmaj}(\phi_{n,k}^{\downarrow}(i,(\pi,S)))} \\ &= \sum_{(\pi,S) \in \mathcal{B}_{n-1,k}^>} q^{\text{fmaj}((\pi,S))} \left(\sum_{i=0}^{n-k-1} q^{2i} + \sum_{i=1}^{n-k-1} q^{2i-1} + q^{2n-2k-1} \right) \\ &= [2n - 2k]_q B_{n-1,k}^{\text{fmaj}}(q). \end{aligned} \quad (2.13)$$

Similarly, by the definition of mapping $\phi_{\alpha,k}^*$ and Lemma 2.7, the last summation of identity (2.12) is

$$\begin{aligned} \sum_{(\pi,S) \in \mathcal{I}_2} q^{\text{fmaj}((\pi,S))} &= \sum_{i=1}^{n-k} \sum_{(\pi,S) \in \mathcal{B}_{n-1,k-1}^>} q^{\text{fmaj}(\phi_{n,k}^*(i,(\pi,S)))} + \sum_{i=0}^{n-k} \sum_{(\pi,S) \in \mathcal{B}_{n-1,k-1}^>} q^{\text{fmaj}(\phi_{n,k}^*(i,(\pi,S)))} \\ &= \sum_{(\pi,S) \in \mathcal{B}_{n-1,k-1}^>} q^{\text{fmaj}((\pi,S))} \left(\sum_{i=1}^{n-k} q^{2i-1} + \sum_{i=1}^{n-k} q^{2i-2} + q^{2n-2k} \right) \\ &= [2n - 2k + 1]_q B_{n-1,k-1}^{\text{fmaj}}(q). \end{aligned} \quad (2.14)$$

Combining (2.12)-(2.14) completes the proof. \square

Proof of Theorem 1.2. By Proposition 2.4 we can rewrite identity (1.10) as

$$[2]_q^k [k]_{q^2}! S_B[n, k] = \sum_{\ell=0}^k q^{(n-k)(2\ell-n-k)} B_{n,n-\ell}(q) \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_{q^2}. \quad (2.15)$$

Let $S_B^{\circ}[n, k]$ be the left-hand side of (2.15). It follows from Eq. (1.6) that the sequence $(S_B^{\circ}[n, k])_{0 \leq k \leq n}$ is determined by the recurrence relation

$$S_B^{\circ}[n, k] := [2k]_q S_B^{\circ}[n - 1, k - 1] + [2k + 1]_q S_B^{\circ}[n - 1, k] \quad (2.16)$$

with $S_B^o[0, k] = \delta_{0k}$. Invoking Proposition 2.8 we see that the polynomials $B_{n, n-k}^{\text{fmaj}}(q)$ satisfy recurrence relation (2.16), namely

$$B_{n, n-k}^{\text{fmaj}}(q) = S_B^o[n, k].$$

Combining with Proposition 2.5, we have a combinatorial proof of (2.15). \square

3 q -Stirling numbers of the second kind in type D

Recently, Bagno et al. [3] studied some identities about the type D Stirling numbers of the second kind $S_D(n, k)$. As far as we know, there is no q -Stirling numbers of the second kind in type D in the literature. In this section, we first define a q -Stirling numbers of the second kind in type D and prove q -analogues of two known results about the Stirling numbers of the second kind in types A, B and D, see Proposition 3.6. Then, we establish a q -identity connecting the q -falling factorials of type D and the q -Stirling numbers of the second kind in type D, see Proposition 3.8.

3.1 Two q -identities about the q -Stirling numbers of the second kind

Using the definitions and notations of ordered signed partition in Subsection 2.2, we say that the set $\{T_0, T_1, T_2, \dots, T_{2k}\}$ is a *signed partition* of $\langle n \rangle$ if $(T_0, T_1, T_2, \dots, T_{2k})$ is an ordered signed partition. A signed partition $\pi = \{T_0, T_1, T_2, \dots, T_{2k}\}$ of $\langle n \rangle$ is called *type D* if $\#T_0 \neq 3$, where $\#T$ denotes the cardinality of a finite set T , in other words, the block T_0 contains at least two positive elements or only contains 0. Let $S_D(n, k)$ be the number of all type D signed partitions of $\langle n \rangle$ with $2k + 1$ blocks, see an equivalent definition of $S_D(n, k)$ in [3]. The numbers $S_D(n, k)$ are called the *Stirling numbers of the second kind in type D*.

For $0 \leq k \leq n$, the following two identities about the Stirling numbers of the second kind in types A, B and D were implicitly given in [26, Corollary 12], [8, Eq. (19)] and [24, Proposition 3]:

$$S_B(n, k) = \sum_{j=k}^n 2^{j-k} \binom{n}{j} S(j, k); \tag{3.1}$$

$$S_B(n, k) = S_D(n, k) + n \cdot 2^{n-k-1} S(n-1, k). \tag{3.2}$$

In this subsection, we define a kind of type D q -Stirling numbers of the second kind $S_D[n, k]$, and give q -analogues of identities (3.1) and (3.2).

Definition 3.1. For any $S \subset \mathbb{Z} \setminus \{0\}$ let $\bar{S} = \{\bar{i} : i \in S\}$. A *standard signed partition* (SSP for short) of S is a sequence $\pi = (S_1, S_2, \dots, S_k)$ of disjoint nonempty subsets of $S \cup \bar{S}$ such that

- (1) $\{S_1, \dots, S_k, \bar{S}_1, \dots, \bar{S}_k\}$ is a partition of $S \cup \bar{S}$;
- (2) $\min |S_1| \leq \min |S_2| \leq \dots \leq \min |S_k|$, where $|S_i| = \{|j| : j \in S_i\}$ for $i \in [k]$.

The sets S_1, S_2, \dots, S_k are the *blocks* of π (so π has k blocks). A *partial standard signed partition* (PSSP for short) of S is a standard signed partition of a subset of S .

Let $B(S, k)$ (resp., $B_{\subseteq}(S, k)$) be the set of all SSP (resp., PSSP) of S with k blocks. Let $D_{\subseteq}([n], k)$ denote the set of all PSSP of $[n]$ that excludes all SSP of $[n] \setminus \{i\}$ with k blocks for $i \in [n]$, namely,

$$D_{\subseteq}([n], k) = B_{\subseteq}([n], k) \setminus \bigcup_{i=1}^n B([n] \setminus \{i\}, k).$$

Lemma 3.2. For $0 \leq k \leq n$ we have

$$2^k S_D(n, k) = \#D_{\subseteq}([n], k).$$

Proof. For any PSSP $\pi = (T_1, T_2, \dots, T_k) \in D_{\subseteq}([n], k)$, it is clear that the sequence $(\langle n \rangle \setminus \{T \cup \bar{T}\}, T_1, \bar{T}_1, \dots, T_k, \bar{T}_k)$ is an ordered signed partition of the set $\langle n \rangle$, where $T = \cup_{i=1}^k T_i$. Thus, the set

$$\Pi = \{\langle n \rangle \setminus \{T \cup \bar{T}\}, T_1, \bar{T}_1, \dots, T_k, \bar{T}_k\}$$

is a type D signed partition of $\langle n \rangle$. Due to the choice of T_i and \bar{T}_i , both PSSP $\pi = (T_1, \dots, T_i, \dots, T_k)$ and $\pi' = (T_1, \dots, \bar{T}_i, \dots, T_k)$ correspond to the type D signed partition Π , which implies the desired result. \square

Definition 3.3. For $\pi = (S_1, S_2, \dots, S_k) \in B_{\subseteq}(S, k)$, define the statistics

$$\text{pos}(\pi) := \# \left\{ x \in \bigcup_{i=1}^k S_i : x > 0 \right\}$$

and

$$m(\pi) := 2 \sum_{i=1}^k i \cdot \#S_i - \text{pos}(\pi). \tag{3.3}$$

The following result was incorrectly stated in [8, Proposition 4.2] with $m(\pi) = 2 \sum_{i=1}^k (i-1) \#S_i + n + 1 - \text{pos}(\pi)$. For completeness, we reproduce their proof with correction.

Proposition 3.4. Let $m(\pi)$ be defined by (3.3). Then we have

$$q^{k^2} [2]_q^k S_B[n, k] = \sum_{\pi \in B_{\subseteq}([n], k)} q^{m(\pi)} \tag{3.4}$$

for $0 \leq k \leq n$.

Proof. Let

$$S_B(n, k, q) = \sum_{\pi \in B_{\subseteq}([n], k)} q^{m(\pi)}.$$

By recurrence (1.6) of $S_B[n, k]$, it suffices to show that $S_B(n, k, q)$ satisfies

$$S_B(n, k, q) = q^{2k-1}(1+q)S_B(n-1, k-1, q) + [2k+1]_q S_B(n-1, k, q)$$

with the initial conditions $S_B(0, k, q) = \delta_{0k}$ for $q \neq 0$. The case $n = 0$ is trivial. Suppose that $n > 0$ and $\pi = (T_1, \dots, T_k) \in B_{\subseteq}([n], k)$. If $\{n\}$ (resp., $\{-n\}$) is a block of π , then $\{n\} = T_k$ (resp., $\{-n\} = T_k$) and removing it from π yields a PSSP τ of $[n-1]$ into $k-1$ blocks, such that $\text{pos}(\tau) = \text{pos}(\pi) - 1$ (resp., $\text{pos}(\tau) = \text{pos}(\pi)$) and $m(\pi) = m(\tau) + 2k - 1$ (resp., $m(\pi) = m(\tau) + 2k$)².

If n is an element of T_i for some $i \in [k]$, then removing it from T_i yields a PSSP τ' of $[n-1]$ into k blocks such that $\text{pos}(\tau') = \text{pos}(\pi) - 1$ and $m(\pi) = m(\tau') + 2i - 1$. Similarly, if $-n$ is an element of T_i for some $i \in [k]$, then removing it from T_i yields a PSSP τ' of $[n-1]$ into k blocks such that $\text{pos}(\tau') = \text{pos}(\pi)$ and $m(\pi) = m(\tau') + 2i$. If neither n nor $-n$ is in any block of π , then $\pi \in B_{\subseteq}([n-1], k)$.

Thus

$$\begin{aligned} S_B(n, k, q) &= q^{2k-1}(1+q) \sum_{\pi \in B_{\subseteq}([n-1], k-1)} q^{m(\pi)} + (1+q) \sum_{i=1}^k q^{2i-1} \sum_{\tau' \in B_{\subseteq}([n-1], k)} q^{m(\tau')} \\ &\quad + \sum_{\pi \in B_{\subseteq}([n-1], k)} q^{m(\pi)} \\ &= [2k+1]_q S_B(n-1, k, q) + q^{2k-1}(1+q) S_B(n-1, k-1, q). \end{aligned}$$

This finishes the proof. \square

Definition 3.5. We define the q -Stirling numbers of the second kind in type D by

$$S_D[n, k] := \frac{1}{q^{k^2} [2]_q^k} \sum_{\pi \in D_{\subseteq}([n], k)} q^{m(\pi)}. \quad (3.5)$$

The following results are q -analogues of identities (3.1) and (3.2), which also show that $S_D[n, k]$ is a polynomial in q . Let $S[n, k]_{q^2}$ denote $S[n, k]$ with q replaced by q^2 , i.e.,

$$S[n, k]_{q^2} := S[n, k] \Big|_{q \leftarrow q^2}.$$

Proposition 3.6. Let $S_D[n, k]$ be defined by (3.5). Then the identities

$$S_B[n, k] = \sum_{j=k}^n \binom{n}{j} [2]_q^{j-k} q^{j-k} S[j, k]_{q^2}, \quad (3.6)$$

$$S_B[n, k] = S_D[n, k] + n \cdot [2]_q^{n-k-1} q^{n-k-1} S[n-1, k]_{q^2} \quad (3.7)$$

hold for $0 \leq k \leq n$.

²In the proof of [8, Proposition 4.2] with $m(\pi) = 2 \sum_{i=1}^k (i-1) \#S_i + n + 1 - \text{pos}(\pi)$ the equation $m(\pi) = m(\tau) + 2k - 1$ (resp., $m(\pi) = m(\tau) + 2k$) does not hold.

Proof. We first prove identity (3.6). Define the polynomial $\tilde{B}_{n,k}(q)$ by

$$\tilde{B}_{n,k}(q) := \sum_{\pi \in B([n],k)} q^{m(\pi)}.$$

Let π and π' be SSPs with k blocks in two different nonempty subsets $\{i_1, i_2, \dots, i_\ell\}$ and $\{j_1, j_2, \dots, j_\ell\}$ of $[n]$, respectively. Obviously, the set of all SSPs π of $\{i_1, i_2, \dots, i_\ell\}$ and that of SSPs π' of $\{j_1, j_2, \dots, j_\ell\}$ are equivalent regardless of the letters. Then we can rewrite identity (3.4) as

$$q^{k^2} [2]_q^k S_B[n, k] = \sum_{j=k}^n \binom{n}{j} \tilde{B}_{j,k}(q).$$

Thus, to prove identity (3.6), it is sufficient to show that

$$[2]_q^n q^{k(k-1)+n} S[n, k]_{q^2} = \tilde{B}_{n,k}(q).$$

Next, we will prove that both sides of the above identity have the same recurrence relation and initial condition. By the definition of SSP, there exist two ways to get a SSP of $[n]$ by inserting n or \bar{n} in one of $[n-1]$.

- (i) The letter n or \bar{n} inserts a SSP in $B([n-1], k-1)$ and forms a new block listing the last position, which increases the statistic $m(\pi)$ by $2k-1$ and $2k$, respectively.
- (ii) The letter n or \bar{n} inserts the i th block of a SSP in $B([n-1], k)$, which increases the statistic $m(\pi)$ by $2i-1$ and $2i$, respectively.

From those, we have the recurrence relation

$$\tilde{B}_{n,k}(q) = [2]_q q^{2k-1} \tilde{B}_{n-1,k-1}(q) + q \cdot [2]_q [k]_{q^2} \tilde{B}_{n-1,k}(q),$$

with the initial condition $\tilde{B}_{0,0}(q) = 1$. Due to the recurrence relation (1.1) of q -Stirling numbers of the second kind in type A, the desired result is obtained.

For identity (3.7), by identity (3.4) and the definition of $S_D[n, k]$, it suffices to show that

$$n \cdot [2]_q^{n-1} q^{k(k-1)+n-1} S[n-1, k]_{q^2} = \sum_{i=1}^n \sum_{\pi \in B([n] \setminus \{i\}, k)} q^{m(\pi)},$$

which is immediate by the above discussion. \square

Remark 3.7. For nonnegative integers $n \geq k$ with $n \neq 1$, Bagno et al. [3] proved the following identity:

$$S_D(n, k) = \frac{1}{2^k k!} \left[\sum_{\ell=0}^k D(n, \ell) \binom{n-\ell}{k-\ell} + n \cdot 2^{n-1} (k-1)! S(n-1, k-1) \right], \quad (3.8)$$

where $D(n, \ell)$ is the number of permutations in \mathcal{D}_n , which is the set of all signed permutations with even signs in \mathcal{B}_n , with ℓ descents, see [18, Section 11.5.4] for more details. As for the type D q -Stirling numbers of the second kind $S_D[n, k]$ defined by (3.5), we leave it as an open problem to find a q -analogue of identity (3.8) in the spirit of identities (1.4) and (1.10) for types A and B.

3.2 Falling factorials and q -Stirling numbers of the second kind in type D

For the Stirling numbers of the second kind $S(n, k)$, a well-known identity involving the connection between the standard basis of the polynomial ring $\mathbb{R}_n[t]$ and the basis consisting of falling factorials is that, for $n \in \mathbb{N}$ and $t \in \mathbb{C}$, we have

$$t^n = \sum_{k=0}^n S(n, k)(t)_k, \quad (3.9)$$

where $(t)_k = t(t-1)\cdots(t-(k-1))$ and $(t)_0 := 1$.

A classical combinatorial interpretation for (3.9) pointed out that t^n is the number of all mappings from the set $[n]$ to the set $[t]$ ($t \in \mathbb{N}^+$) and $S(n, k)(t)_k$ is the number of surjections that map the set $[n]$ to all k -subsets of $[t]$, see [22, Eq. (1.96)] for more details. Similarly, for the Stirling numbers of the second kind in types B and D, Bagno et al. [3, Theorems 5.1 and 5.4] used a geometric method to obtain the following identities:

$$t^n = \sum_{k=0}^n S_B(n, k)(t)_k^B, \quad (3.10)$$

where $(t)_k^B = (t-1)(t-3)\cdots(t-(2k-1))$ and $(t)_0^B := 1$, and

$$t^n = \sum_{k=0}^n S_D(n, k)(t)_k^D + n((t-1)^{n-1} - (t)_{n-1}^D), \quad (3.11)$$

where $(t)_k^D$ is defined as

$$(t)_k^D := \begin{cases} 1, & k = 0; \\ (t-1)(t-3)\cdots(t-(2k-1)), & 1 \leq k < n; \\ (t-1)(t-3)\cdots(t-(2n-3))(t-(n-1)), & k = n. \end{cases}$$

Naturally, those q -analogues for identities (3.9) and (3.10) were also given as

$$t^n = \sum_{k=0}^n S[n, k](t)_{k,q}, \quad (3.12)$$

where $(t)_{k,q} = t(t-[1]_q)\cdots(t-[k-1]_q)$ and $(t)_{0,q} := 1$ (see Carlitz [7, Eq. (3.1)]), and

$$t^n = \sum_{k=0}^n S_B[n, k](t)_{k,q}^B, \quad (3.13)$$

where $(t)_{k,q}^B = (t-[1]_q)(t-[3]_q)\cdots(t-[2k-1]_q)$ and $(t)_{0,q}^B := 1$ (see Sagan and Swanson [21, Corollary 2.4] and Komatsu et al. [15, Theorem 2.2]).

Define a q -falling factorial of type D by

$$(t)_{k,q}^D := \begin{cases} 1, & k = 0; \\ (t-[1]_q)(t-[3]_q)\cdots(t-[2k-1]_q), & 1 \leq k < n; \\ (t-[1]_q)(t-[3]_q)\cdots(t-[2n-3]_q)(t-[n-1]_q), & k = n. \end{cases}$$

We have a q -analogue of identity (3.11) as follows.

Proposition 3.8. Let $S_D[n, k]$ be defined by (3.5). Then

$$t^n = \sum_{k=0}^n S_D[n, k](t)_{k,q}^D + n(t-1)^{n-1} - [n]_q q^{n-1}(t)_{n-1,q}^D$$

for $n \in \mathbb{N}$ and $t \in \mathbb{C}$.

Proof. From equation (3.7) we derive the identity

$$S_B[n, k] = S_D[n, k] + n \cdot [2]_q^{n-k-1} q^{n-k-1} S[n-1, k]_{q^2}. \quad (3.14)$$

Thus, multiplying both sides of (3.14) by $(t)_{k,q}^D$ and summing over $0 \leq k \leq n$, we have

$$\sum_{k=0}^n S_B[n, k](t)_{k,q}^D = \sum_{k=0}^n S_D[n, k](t)_{k,q}^D + \sum_{k=0}^{n-1} n \cdot [2]_q^{n-k-1} q^{n-k-1} S[n-1, k]_{q^2}(t)_{k,q}^D. \quad (3.15)$$

First, for the left-hand side of (3.15), we have

$$\begin{aligned} \sum_{k=0}^n S_B[n, k](t)_{k,q}^D &= \sum_{k=0}^{n-1} S_B[n, k](t)_{k,q}^D + S_B[n, n](t)_{n,q}^D - [n]_q q^{n-1}(t)_{n-1,q}^D + [n]_q q^{n-1}(t)_{n-1,q}^D \\ &= \sum_{k=0}^n S_B[n, k](t)_{k,q}^B + [n]_q q^{n-1}(t)_{n-1,q}^D \\ &= t^n + [n]_q q^{n-1}(t)_{n-1,q}^D, \end{aligned} \quad (3.16)$$

where the second equality and last one use the facts $S_B[n, n] = 1$ and

$$(t)_{n,q}^B = (t)_{n,q}^D - [n]_q q^{n-1}(t)_{n-1,q}^D,$$

and identity (3.13), respectively. In addition, for the second summation in the right-hand side of (3.15), we have

$$\begin{aligned} &\sum_{k=0}^{n-1} n \cdot [2]_q^{n-k-1} q^{n-k-1} S[n-1, k]_{q^2}(t)_{k,q}^D \\ &= n \cdot [2]_q^{n-1} q^{n-1} \sum_{k=0}^{n-1} S[n-1, k]_{q^2} \left(\frac{t-1}{[2]_q q} \right) \left(\frac{t-1}{[2]_q q} - [1]_{q^2} \right) \cdots \left(\frac{t-1}{[2]_q q} - [k-1]_{q^2} \right) \\ &= n \cdot [2]_q^{n-1} q^{n-1} \left(\frac{t-1}{[2]_q q} \right)^{n-1} \\ &= n(t-1)^n, \end{aligned} \quad (3.17)$$

where the second equality uses identity (3.12). Combining (3.15), (3.16) and (3.17), we complete the proof. \square

4 Generalization to colored permutations

In this section, instead of proving Theorem 1.2 by an algebraic proof, we shall prove a more general identity. Define the r -colored q -Stirling numbers of the second kind $S_r[n, k]$ by the recurrence relation

$$S_r[n, k] := S_r[n - 1, k - 1] + [rk + 1]_q S_r[n - 1, k] \quad (4.1)$$

with the initial conditions $S_r[0, k] = \delta_{0k}$.

It is not difficult to verify (see [19, Theorem 1] for a more general result) that

$$t^n = \sum_{k=0}^n S_r[n, k] (t)_{k,q}^r, \quad (4.2)$$

where $(t)_{k,q}^r = (t - [1]_q)(t - [r + 1]_q) \cdots (t - [r(k - 1) + 1]_q)$ and $(t)_{0,q}^r := 1$. Using Rook theory, Remmel and Wachs gave a combinatorial interpretation of identity (4.2) in [19, Theorem 7].

Substituting t by $[rm + 1]_q$ in (4.2) yields

$$[rm + 1]_q^n = \sum_{k=0}^n q^{r \binom{k+1}{2} + (1-r)k} [r]_q^k [k]_{q^r}! S_r[n, k] \begin{bmatrix} m \\ k \end{bmatrix}_{q^r},$$

which, by (2.8), is equivalent to the generating function identity,

$$\sum_{k=0}^n \frac{q^{r \binom{k+1}{2} + (1-r)k} [r]_q^k [k]_{q^r}! S_r[n, k] t^k}{\prod_{i=0}^k (1 - tq^{ri})} = \sum_{m=0}^{\infty} [rm + 1]_q^n t^m. \quad (4.3)$$

The colored permutations group of n letters with r colors can be looked as the wreath product group

$$\mathbb{Z}_r \wr \mathfrak{S}_n = \mathbb{Z}^r \times \mathfrak{S}_n,$$

which consists of all permutations $\pi \in [0, r - 1] \times [n]$. Namely, the element in $\mathbb{Z}_r \wr \mathfrak{S}_n$ is thought of as $\pi = \pi_1^{z_1} \pi_2^{z_2} \cdots \pi_n^{z_n}$, where $z_i \in [0, r - 1]$ and $\pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_n$. Define the following total order relation on the elements of $\mathbb{Z}_r \wr \mathfrak{S}_n$:

$$n^{r-1} < \cdots < n^1 < \cdots < 1^{r-1} < \cdots < 1^1 < 0 < 1 < \cdots < n,$$

where k^0 is replaced with k for $k \in [n]$.

An integer $i \in \{0\} \cup [n - 1]$ is called a *descent* of $\pi \in \mathbb{Z}_r \wr \mathfrak{S}_n$ if $\pi_i^{z_i} > \pi_{i+1}^{z_{i+1}}$, where $\pi_0^{z_0} = 0$. Let $\text{Des}_r(\pi)$ denote the descent set of $\pi \in \mathbb{Z}_r \wr \mathfrak{S}_n$ and $\text{des}_r(\pi)$ the number of descents of π , i.e., $|\text{Des}_r(\pi)|$. The r -colored Eulerian number $A_{n,k}^r$ is the number of all colored permutations in $\mathbb{Z}_r \wr \mathfrak{S}_n$ with k descents. For each $\pi \in \mathbb{Z}_r \wr \mathfrak{S}_n$, as in [1], define the r -flag-major index of π by

$$\text{fmaj}_r(\pi) := r \sum_{i \in \text{Des}_r(\pi)} i + \sum_{i=1}^n z_i. \quad (4.4)$$

A q -analogue of the r -colored Eulerian polynomial $A_n^r(t, q)$ is defined by

$$A_n^r(t, q) := \sum_{\pi \in \mathbb{Z}_r \wr \mathfrak{S}_n} t^{\text{des}_r(\pi)} q^{\text{fmaj}_r(\pi)} = \sum_{k=0}^n A_{n,k}^r(q) t^k. \quad (4.5)$$

When r takes 1 and 2, (4.5) reduces to (1.3) and (1.9), respectively. The following Carlitz's identity for $\mathbb{Z}_r \wr \mathfrak{S}_n$ was proved in [6, Proposition 8.1] and [9, Theorem 9]

$$\frac{A_n^r(t, q)}{\prod_{i=0}^n (1 - tq^{ri})} = \sum_{m=0}^{\infty} [rm + 1]_q^n t^m. \quad (4.6)$$

Combining (4.3) and (4.6) we obtain the following identity.

Proposition 4.1. For the polynomials $S_r[n, k]$ in (4.1) and $A_n^r(t, q)$ in (4.5), the q -Frobenius formula holds

$$\frac{A_n^r(t, q)}{\prod_{i=0}^n (1 - tq^{ri})} = \sum_{k=0}^n \frac{q^{r\binom{k+1}{2} + (1-r)k} [r]_q^k [k]_{q^r}! S_r[n, k] t^k}{\prod_{i=0}^k (1 - tq^{ri})}.$$

The following result is a q -analogue of Theorem 6.6 in [3] about an identity between the r -colored Stirling numbers of the second kind $S_r(n, k)$ (the sequence defined by (4.1) when $q = 1$, see also [3, Section 6.1]) and r -colored Eulerian numbers $A_{n,k}^r$.

Theorem 4.2. For the r -colored q -Stirling numbers of the second kind $S_r[n, k]$ in (4.1) and q -Eulerian numbers $A_{n,k}^r(q)$ in (4.5), we have the identity

$$q^{r\binom{k+1}{2} + (1-r)k} [r]_q^k [k]_{q^r}! S_r[n, k] = \sum_{\ell=0}^k q^{rk(k-\ell)} A_{n,\ell}^r(q) \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_{q^r} \quad (4.7)$$

for $0 \leq k \leq n$.

Proof. Summing for both sides of (4.7) multiplying by $t^k / \prod_{i=0}^k (1 - tq^{ri})$ over all k , it is clear that (4.7) is equivalent to

$$\sum_{k=0}^n \frac{q^{r\binom{k+1}{2} + (1-r)k} [r]_q^k [k]_{q^r}! S_r[n, k] t^k}{\prod_{i=0}^k (1 - tq^{ri})} = \sum_{k=0}^n \sum_{\ell=0}^k \frac{q^{rk(k-\ell)} A_{n,\ell}^r(q) t^k}{\prod_{i=0}^k (1 - tq^{ri})} \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_{q^r}.$$

By Proposition 4.1, it is sufficient to show that

$$\frac{A_n^r(t, q)}{\prod_{i=0}^n (1 - tq^{ri})} = \sum_{k=0}^n \sum_{\ell=0}^k \frac{q^{rk(k-\ell)} A_{n,\ell}^r(q) t^k}{\prod_{i=0}^k (1 - tq^{ri})} \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_{q^r},$$

or equivalently,

$$\frac{A_n^r(t, q)}{\prod_{i=0}^n (1 - tq^{ri})} = \sum_{\ell=0}^n A_{n,\ell}^r(q) t^\ell \sum_{k=\ell}^n \frac{(tq^{r \cdot k})^{k-\ell}}{\prod_{i=0}^k (1 - tq^{ri})} \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_{q^r},$$

which will follow from the following identity

$$\frac{1}{\prod_{i=0}^n (1 - tq^{ri})} = \sum_{k=\ell}^n \frac{(tq^{r \cdot k})^{k-\ell}}{\prod_{i=0}^k (1 - tq^{ri})} \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_{q^r} \quad (4.8)$$

for $0 \leq \ell \leq n$. That is to say, the index ℓ does not affect the summation in the right-hand side of (4.8). Substituting $q^r \rightarrow q$ and applying (2.8) to extract the coefficients of t^m on both sides of (4.8) we obtain

$$\begin{aligned} \begin{bmatrix} n + m \\ m \end{bmatrix}_q &= \sum_{k=\ell}^n \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_q \begin{bmatrix} k + m - (k - \ell) \\ m - (k - \ell) \end{bmatrix}_q q^{k(k-\ell)} \\ &= \sum_{k=0}^{n-\ell} \begin{bmatrix} n - \ell \\ k \end{bmatrix}_q \begin{bmatrix} m + \ell \\ m - k \end{bmatrix}_q q^{k(k+\ell)}, \end{aligned}$$

which is a q -analogue of Chu-Vandermonde summation [2, Eq. (3.3.10)]. \square

Following the recurrence (4.1), we have $S_1[n, k] = S[n + 1, k + 1]$ and $S_2[n, k] = S_B[n, k]$. When $r = 1$ and $r = 2$, identity (4.7) (Theorem 4.2) reduces to (1.4) and (1.10), respectively. Indeed, the case $r = 2$ is obvious, i.e., Theorem 1.2 is a special case of Theorem 4.2. For $r = 1$, Theorem 4.2 reduces to

$$q^{\binom{k+1}{2}} [k]_q! S[n + 1, k + 1] = \sum_{\ell=0}^k q^{k(k-\ell)} A_{n,\ell}(q) \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_q, \quad (4.9)$$

which is equivalent to identity (1.4). By (1.4), the right-hand side of (4.9) equals

$$\begin{aligned} &\sum_{\ell=0}^k q^{k(k-\ell)} A_{n,\ell}(q) \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_q \\ &= \sum_{\ell=1}^k q^{k(k-(\ell-1))} A_{n,\ell-1}(q) \begin{bmatrix} n - (\ell - 1) \\ k - (\ell - 1) \end{bmatrix}_q + A_{n,k}(q) \\ &= \sum_{\ell=1}^k q^{k(k+1-\ell)} A_{n,\ell-1}(q) \left(q^{k+1-\ell} \begin{bmatrix} n - \ell \\ k + 1 - \ell \end{bmatrix}_q + \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_q \right) + A_{n,k}(q) \\ &= \sum_{\ell=1}^{k+1} q^{\binom{k+1}{2} + (k+1-\ell)} A_{n,\ell-1}(q) \begin{bmatrix} n - \ell \\ k + 1 - \ell \end{bmatrix}_q + q^k \sum_{\ell=1}^k q^{k(k-\ell)} A_{n,\ell-1}(q) \begin{bmatrix} n - \ell \\ k - \ell \end{bmatrix}_q \\ &= q^{\binom{k+1}{2}} [k + 1]_q! S[n, k + 1] + q^k q^{\binom{k}{2}} [k]_q! S[n, k], \end{aligned}$$

which yields (4.9) by recurrence relation (1.1) of $S[n, k]$. Inversely, starting from (4.9), the above last equality shows that (1.4) follows from (4.9) by induction on k for fixed n .

In addition, by (1.4) and (4.8), we have the following q -Frobenius formula [12, Eq. (4.1)] related to q -Stirling numbers of the second kind and q -Eulerian polynomials of type A:

$$\frac{tA_n(t, q)}{\prod_{i=0}^n (1 - tq^i)} = \sum_{k=0}^n \frac{q^{\binom{k}{2}} [k]_q! S[n, k] t^k}{\prod_{i=0}^k (1 - tq^i)}. \quad (4.10)$$

Following the above discussion, it is clear that identity (4.10) is a special case of Proposition 4.1 for $r = 1$.

Remark 4.3. Similar to the combinatorial proofs of (1.4) and (1.10), it is natural to ask for a combinatorial proof of identity (4.7). One difficulty for such a proof is that a counterpart of the q -symmetry (2.2) is missing for $A_{n,k}^r(q)$. Note that the r -colored Eulerian polynomials $A_n^r(t, 1)$ are not symmetric for $r \geq 3$. We leave it as an open problem to give a combinatorial proof of identity (4.7).

Acknowledgements

The first author was supported by the *China Scholarship Council*. This work was done during his visit at Université Claude Bernard Lyon 1 in 2022-2023.

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