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Sliding Mode Observers for Set-valued Lur'e Systems with Uncertainties Beyond Observational Range

Samir Adly* Jun Huang[†] Ba Khiet Le [‡]

Abstract

In this paper, we introduce a new approximate sliding mode observer for Lur'e set-valued dynamical systems, particularly addressing challenges posed by uncertainties not within the standard range of observation. Traditionally, Luenberger-like observers and sliding mode observer have been designed only for uncertainties in the range of observation. Central to our approach is the treatment of the uncertainty term which we decompose into two components: the first part in the observation subspace and the second part in its complemented subspace. We establish that when the second part converges to zero, an exact sliding mode observer for the system can be obtained. In scenarios where this convergence does not occur, our methodology allows for the estimation of errors between the actual state and the observer state. This leads to a practical interval estimation technique, valuable in situations where part of the uncertainty lies outside the observable range. Finally, we show that our observer is also an H^∞ observer.

Keywords. Sliding mode observer; set-valued Lur'e systems; uncertainties outside the observation subspace

AMS Subject Classification. 28B05, 34A36, 34A60, 49J52, 49J53, 93D20

1 Introduction

Set-valued Lur'e dynamical systems is a general and widely studied framework in applied mathematics and control theory which covers many other models usually used to study nonsmooth dynamical systems such as evolution variational inequalities, projected dynamical systems, relay systems, complementarity systems ... Especially it has been extensively studied in the recent decades [1, 2, 3, 4, 7, 8, 9, 11, 12, 14, 15, 16, 19, 20, 21, 22, 27]. Despite advancements in the understanding of the system's existence and uniqueness, stability and asymptotic analysis, the design of control and observers still poses many interesting open questions. Specifically, most observers for set-valued Lur'e systems follow the Luenberger-like design, which has limitations when the system is subjected to uncertainty, as discussed in references such as [4, 5, 10, 11, 12, 15, 16, 21, 27]. To our

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understanding, the concept of Luenberger-like observers for set-valued Lur'e systems was initially explored by Brogliato-Heemels in [11], and later extended to incorporate uncertainty by Huang et al in [15]. The key idea of Huang et al. [15] is to approximate the unknown by using an additional ODE, aiming to achieve observer convergence with an unknown rate. On the other hand, by using a sliding mode observer, under the same or improved assumptions, Adly-Le [4] obtain the exponential convergence without solving an additional ODE. Furthermore, the authors proposed a new efficient smoothing approximation using the time guiding functions and confirms that sliding mode technique is robust to sensitive data. However, the essential restrictness of the existing results using Luenberger-like observers or sliding mode observers (see, e.g. [4, 15, 16, 21, 24, 26, 27, 28]) is the requirement of the range condition for the uncertainties. One of the other approaches is the construction of H^∞ observers. Nonetheless, H^∞ observers only provides the estimation for the total process under the zero initial condition rather than the time instant estimation (see, e.g. [16]). This fact motivates us to provide an approximate sliding mode observer to obtain a time instant estimation for the Lur'e system subject to an uncertainty not in the range of observation. The idea is to decompose ξ into 2 parts: $\xi = \xi_1 + \xi_2$ where ξ_1 is the projection of ξ onto the observation subspace and ξ_2 is in the complement of the observation subspace. We show that if ξ_2 is bounded, we can have a useful estimation for the state of the original system. Interval estimation is practical and unavoidable if the uncertainty ξ is general and not in the range of observation. We refer here some interval observers results for some specific systems [17, 23]. In addition, if ξ_2 tends to vanish when the time is large, we obtain the exact sliding mode observer, i.e., the observer state converges to the original state asymptotically. Finally, we show that our proposed observer is also an H^∞ observer.

The paper is structured as follows: In Section 2, we revisit established definitions and results essential for our subsequent analyses. Section 3 presents our proposal for an approximate sliding mode observer for the original system, which simultaneously functions as an H^∞ observer. In Section 4, we give several numerical examples to validate the theoretical findings. The paper concludes in Section 5, where we offer our closing remarks and explore potential future directions.

2 Notations and mathematical background

We denote the scalar product and the corresponding norm of Euclidean spaces by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. A matrix $P \in \mathbb{R}^{n \times n}$ is called positive definite, written $P > 0$, if there exists $a > 0$ such that

$$\langle Px, x \rangle \geq a\|x\|^2, \quad \forall x \in \mathbb{R}^n.$$

The Sign functions in \mathbb{R}^n is defined by

$$\text{Sign}(x) = \begin{cases} \frac{x}{\|x\|} & \text{if } x \neq 0 \\ \mathbb{B} & \text{if } x = 0, \end{cases}$$

where \mathbb{B} denotes the unit ball in \mathbb{R}^n .

A set-valued mapping $\mathcal{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is called *monotone* if for all $x, y \in \mathbb{R}^m$ and $x^* \in \mathcal{F}(x), y^* \in \mathcal{F}(y)$, one has

$$\langle x^* - y^*, x - y \rangle \geq 0.$$

Furthermore, \mathcal{F} is called *maximal monotone* if there is no monotone operator \mathcal{G} such that the graph of \mathcal{F} is contained strictly in the graph of \mathcal{G} (see, e.g. [6, 13]).

Let us recall a general version of Gronwall's inequality [25].

Lemma 1. *Let $T > 0$ be given and $a(\cdot), b(\cdot) \in L^1([0, T]; \mathbb{R})$ with $b(t) \geq 0$ for almost all $t \in [0, T]$. Let the absolutely continuous function $w : [0, T] \rightarrow \mathbb{R}_+$ satisfy:*

$$(1 - \alpha)w'(t) \leq a(t)w(t) + b(t)w^\alpha(t), \quad a.e. \ t \in [0, T], \quad (1)$$

where $0 \leq \alpha < 1$. Then for all $t \in [0, T]$:

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(0)\exp\left(\int_0^t a(\tau)d\tau\right) + \int_0^t \exp\left(\int_s^t a(\tau)d\tau\right)b(s)ds. \quad (2)$$

3 Approximate sliding mode observer and the convergence analysis

In this section, we consider the Lur'e system subject to an uncertainty not in the range of observation as follows

$$\begin{cases} \dot{x} = Ax + B\omega + f(x, u) + \xi(t), \\ \omega \in -\mathcal{F}(Cx), \\ y = Fx, \end{cases} \quad (3)$$

where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, F \in \mathbb{R}^{p \times n}$ are given matrix, $x \in \mathbb{R}^n$ is the state, $\mathcal{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a maximal monotone operator, $u \in \mathbb{R}^r$ is the control input and $y \in \mathbb{R}^p$ is the measurable output. The nonlinear functions f is Lipschitz continuous and $\xi \in \mathbb{R}^l$ is unknown. The model in (3) is quite general since in the right-hand side of the ODE, one has the linear Ax , the nonlinear $f(x, u)$, the set-valued $B\mathcal{F}(Cx)$ and the unknown ξ . To our knowledge, within the literature for this general framework, it is common for the term ξ to typically fulfill the range condition: $\xi \in \text{Im}(P^{-1}F^T)$ where P is a positive definite matrix which appeared in some constructed Lyapunov function (see, e.g. [4, 15, 21]). This means that the uncertainty lies within the range of observation, which poses a limitation. In our paper, we do not impose the range condition. Our approach involves decomposing ξ into two components: $\xi = \xi_1 + \xi_2$, where $\xi_1 = P^{-1}F^T\theta$ represents the projection of ξ onto the subspace $\text{Im}(P^{-1}F^T)$ and ξ_2 is in the complement of the observation subspace.

Next, we propose an approximate sliding mode observer for the system (3) under the following assumptions:

Assumption 1: The set-valued operator $\mathcal{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a maximal monotone operator.

Assumption 2: The functions $f : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^n$, $(x, u) \mapsto f(x, u)$ is L_f - Lipschitz continuous w.r.t x , i.e.:

$$\|f(x_1, u) - f(x_2, u)\| \leq L_f \|x_1 - x_2\|.$$

Assumption 3: There exist $\epsilon > 0$, $P \in \mathbb{R}^{n \times n} > 0$, $L \in \mathbb{R}^{n \times p}$ and $K \in \mathbb{R}^{m \times p}$ such that

$$P(A - LF) + (A - LF)^T P + 2L_f \|P\| I + 2\epsilon I \leq 0, \quad (4)$$

$$B^T P = C - KF. \quad (5)$$

Assumption 4: The unknown ξ is continuous and can be decomposed as $\xi(t) = P^{-1}F^T \xi_1(t) + \xi_2(t)$, where $P^{-1}F^T \xi_1(t)$ is the projection of ξ onto $\text{Im}(P^{-1}F^T)$. The terms ξ_1 and ξ_2 are unknown, bounded by known continuous positive functions $\kappa_1(t)$ and $\kappa_2(t)$ respectively, i.e., for all $t \geq 0$, we have

$$|\xi_1(t)| \leq \kappa_1(t), \quad |\xi_2(t)| \leq \kappa_2(t).$$

Remark 1. i) Assumption 4 is quite general, since it does not require the uncertainty in the range of $P^{-1}F^T$, i.e., the range of observation.

ii) Suppose that Assumption 4 holds, then we have $FP^{-1}\xi_2 = 0$, i.e., ξ_2 is in the kernel of FP^{-1} .

The proposed approximate sliding mode observer for (3) is

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + B\tilde{\omega} - Le_y + f_1(\tilde{x}, u) - P^{-1}F^T(\kappa_1 \text{Sign}(e_y) + \frac{\kappa_3 e_y}{\|e_y\|^2 + \delta}), \\ \tilde{\omega} \in -\mathcal{F}(C\tilde{x} - Ke_y), \\ \tilde{y} = F\tilde{x}, \end{cases} \quad (6)$$

where

$$e := \tilde{x} - x, \quad e_y := Fe, \quad (7)$$

for some given small $\delta > 0$. Since the nonlinear f and the unknown ξ are continuous, we can obtain the existence and uniqueness of the original system (3) and the approximate observer (6) under some additional mild conditions (see, e.g., [4, 11, 15]). In the following, we show that if ξ_2 is bounded, we obtain a useful estimation for the error $e(t)$. On the other hand, if ξ_2 tends to vanish when the time is large, we have an exact observer indeed.

Theorem 2. *Suppose that Assumptions 1–4 hold. Let $V(t) = \langle Pe(t), e(t) \rangle$. Then*

$$\sqrt{V(t)} \leq \sqrt{V(0)} \exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) + \frac{\|P\|}{\sqrt{\lambda_{min}}} \int_0^t \exp\left(\frac{\epsilon(s-t)}{\lambda_{max}}\right) \|\kappa_2(t)\| ds, \quad (8)$$

where λ_{max} and λ_{min} are the largest and smallest eigenvalues of P respectively. In addition

a) If $\kappa_2(\cdot)$ is bounded by some $k > 0$, then

$$\|e(t)\| \leq \frac{\lambda_{max}\|P\|k}{\lambda_{min}\epsilon} + \exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) \left(\sqrt{\frac{V(0)}{\lambda_{min}}} - \frac{\lambda_{max}\|P\|k}{\epsilon\lambda_{min}} \right). \quad (9)$$

Additionally, we introduce the attractive set $\Omega := [0, \frac{k\|P\|}{\epsilon}]$. Consequently, the error norm $\|e(t)\|$ converges to any neighborhood of Ω in finite time and remains within that vicinity.

b) If $\kappa_2(t) \rightarrow 0$ as $t \rightarrow \infty$, then $\|e(t)\| \rightarrow 0$ as $t \rightarrow \infty$, i.e., (6) is indeed an observer of (3).

c) If $\|\kappa_2(t)\| \leq ke^{-at}$, then $e(t)$ converges exponentially to zero.

d) If $k_2 \in L^2(0, \infty)$, then

$$\begin{aligned}\sqrt{V(t)} &\leq \sqrt{V(0)}\exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) + \frac{\|P\|}{\sqrt{\lambda_{min}}} \sqrt{\int_0^t \kappa_2^2(s) ds} \sqrt{\frac{\lambda_{max}}{2\epsilon} (1 - \exp(\frac{-2\epsilon t}{\lambda_{max}}))} \\ &\leq \sqrt{V(0)}\exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) + \frac{\|P\| \|\kappa_2\|_{L^2(0, \infty)}}{\sqrt{\lambda_{min}}} \sqrt{\frac{\lambda_{max}}{2\epsilon} (1 - \exp(\frac{-2\epsilon t}{\lambda_{max}}))}.\end{aligned}$$

Proof. From (3) and (6), we have

$$\begin{aligned}\dot{e} &\in (A - LF)e - B(\tilde{\omega} - \omega) + f(\tilde{x}, u) - f(x, u) \\ &\quad - P^{-1}F^T(\kappa_1 \text{Sign}(e_y) + \frac{\kappa_3 e_y}{\|e_y\|^2 + \delta}) - (P^{-1}F^T \xi_1(t) + \xi_2(t)).\end{aligned}\tag{10}$$

With $V(t) = \langle Pe(t), e(t) \rangle$, one has

$$\begin{aligned}\frac{1}{2} \frac{dV}{dt} &= \langle P\dot{e}, e \rangle = \langle P(A - LF)e + PB(\tilde{\omega} - \omega), e \rangle \\ &\quad + \langle P(f_1(\tilde{x}, u) - f_1(x, u)), e \rangle - (\kappa_1(t) - \|\xi_1(t)\|) \|e_y\| - \langle P\xi_2(t), e \rangle \\ &\quad - \frac{\kappa_3 \|e_y\|^2}{\|e_y\|^2 + \delta}.\end{aligned}\tag{11}$$

From (5) and the monotonicity of \mathcal{F} , we have

$$\langle PB(\tilde{\omega} - \omega), e \rangle = \langle \tilde{\omega} - \omega, B^T Pe \rangle = \langle \tilde{\omega} - \omega, (C - KF)e \rangle \leq 0.\tag{12}$$

On the other hand, using Assumption 2, we obtain

$$\langle P(f_1(\tilde{x}, u) - f_1(x, u)), e \rangle \leq L_1 \|P\| \|e\|^2.\tag{13}$$

Thus from (4), (10)-(13), we deduce that

$$\frac{1}{2} \frac{dV}{dt} \leq -\epsilon \|e\|^2 + \|P\| \|\kappa_2(t)\| \|e(t)\| \leq -\frac{\epsilon}{\lambda_{max}} V + \frac{\|P\| \|\kappa_2(t)\|}{\sqrt{\lambda_{min}}} \sqrt{V}.\tag{14}$$

Using Lemma 1 with $\alpha = 1/2$, we have

$$\sqrt{V(t)} \leq \sqrt{V(0)} \exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) + \frac{\|P\|}{\sqrt{\lambda_{min}}} \int_0^t \exp\left(\frac{\epsilon(s-t)}{\lambda_{max}}\right) \|\kappa_2(s)\| ds.$$

a) If $\kappa_2(\cdot)$ is bounded by some $k > 0$, then

$$\begin{aligned}\sqrt{\lambda_{min}} \|e(t)\| \leq \sqrt{V(t)} &\leq \sqrt{V(0)} \exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) + \frac{\|P\| k}{\sqrt{\lambda_{min}}} \int_0^t \exp\left(\frac{\epsilon(s-t)}{\lambda_{max}}\right) ds \\ &= \sqrt{V(0)} \exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) + \frac{\lambda_{max} \|P\| k}{\epsilon \sqrt{\lambda_{min}}} (1 - \exp(\frac{-\epsilon t}{\lambda_{max}})) \\ &= \frac{\lambda_{max} \|P\| k}{\epsilon \sqrt{\lambda_{min}}} + \exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) (\sqrt{V(0)} - \frac{\lambda_{max} \|P\| k}{\epsilon \sqrt{\lambda_{min}}}).\end{aligned}$$

Thus, one obtains (9). On the other hand, from (14) we have

$$\frac{dV}{dt} \leq -\epsilon \|e\|^2 + k \|P\| \|e\|.$$

If $\|e\| \geq \frac{k\|P\|}{\epsilon} + \rho$ for some $\rho > 0$, one has $\frac{dV}{dt} \leq -\rho\epsilon \|e\|$. Hence classically $\|e\|$ converges to any neighborhood of Ω in finite time and stay there.

b) In particular, $\kappa_2(\cdot)$ is bounded by some $k > 0$. Note that from (9), we imply that

$$\|e(t)\| \leq \frac{\lambda_{max}\|P\|k}{\lambda_{min}\epsilon} + \exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) \sqrt{\frac{V(0)}{\lambda_{min}}}, \quad \forall t \geq 0. \quad (15)$$

In particular, $e(t)$ is bounded and hence $V(e(t))$ is bounded by some V^* . Let $T > 0$ be given, $T_n = nT$ and $c_n := \sup_{t \in [T_n, +\infty)} |k_2(t)|$, $n \geq 1$, then (k_n) converges to zero since $k(t) \rightarrow 0$ as $t \rightarrow \infty$. Using (15), by replacing 0 by T_n , for all $t \geq T_n$, we obtain

$$\|e(t)\| \leq \frac{\lambda_{max}\|P\|c_n}{\lambda_{min}\epsilon} + \exp\left(\frac{-\epsilon(t - T_n)}{\lambda_{max}}\right) \sqrt{\frac{V^*}{\lambda_{min}}}. \quad (16)$$

For given $\delta > 0$, we can choose n_0 such that $\frac{\lambda_{max}\|P\|c_{n_0}}{\lambda_{min}\epsilon} \leq \delta$. Then

$$\|e(t)\| \leq \delta + \exp\left(\frac{-\epsilon(t - T_{n_0})}{\lambda_{max}}\right) \sqrt{\frac{V^*}{\lambda_{min}}}, \quad \forall t \geq T_{n_0}. \quad (17)$$

Thus

$$\limsup_{t \rightarrow \infty} \|e(t)\| \leq \delta.$$

Since δ is arbitrary, we must have

$$\limsup_{t \rightarrow \infty} \|e(t)\| \leq 0,$$

and thus

$$\lim_{t \rightarrow \infty} \|e(t)\| = 0.$$

c) If $\|\kappa_2(t)\| \leq ke^{-at}$, then

$$\sqrt{V(t)} \leq \sqrt{V(0)} \exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) + \frac{\|P\|k}{\sqrt{\lambda_{min}}} \int_0^t \exp\left(\frac{\epsilon(s-t)}{\lambda_{max}} - as\right) ds \quad (18)$$

$$= \sqrt{V(0)} \exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) + \frac{\lambda_{max}\|P\|k}{\sqrt{\lambda_{min}}(\epsilon - a\lambda_{max})} (e^{-at} - \exp\left(\frac{-\epsilon t}{\lambda_{max}}\right)), \quad (19)$$

if $\epsilon \neq a\lambda_{max}$. Thus the error $e(t)$ converges to zero with exponential rate. The case $\epsilon = a\lambda_{max}$ is trivial.

d) If $k_2 \in L^2(0, \infty)$, then from (8), one has

$$\begin{aligned}
\sqrt{V(t)} &\leq \sqrt{V(0)}\exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) + \frac{\|P\|}{\sqrt{\lambda_{min}}} \sqrt{\int_0^t \kappa_2^2(s) ds} \sqrt{\int_0^t \exp\left(\frac{2\epsilon(s-t)}{\lambda_{max}}\right) ds} \\
&= \sqrt{V(0)}\exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) + \frac{\|P\|}{\sqrt{\lambda_{min}}} \sqrt{\int_0^t \kappa_2^2(s) ds} \sqrt{\frac{\lambda_{max}}{2\epsilon} (1 - \exp\left(\frac{-2\epsilon t}{\lambda_{max}}\right))} \\
&\leq \sqrt{V(0)}\exp\left(\frac{-\epsilon t}{\lambda_{max}}\right) + \frac{\|P\| \|\kappa_2\|_{L^2(0,\infty)}}{\sqrt{\lambda_{min}}} \sqrt{\frac{\lambda_{max}}{2\epsilon} (1 - \exp\left(\frac{-2\epsilon t}{\lambda_{max}}\right))}. \tag{20}
\end{aligned}$$

The proof is thereby completed. \square

Remark 2. i) In the first case, we have the interval estimation for the error, i.e., the original state $x(t)$ belongs to some known interval centered at the approximate observer $\tilde{x}(t)$. In the remaining cases, one obtains the exact observer. It covers the case where the unknown ξ vanishes after some time instant $T_0 > 0$ as considered in some examples in [16]. Note that with our observer, we only require that the unknown part ξ_2 (in the complement of the observation space) tends to vanish (see Figure 3, Example 2).

ii) The term $\frac{\kappa_3 e_y}{\|e_y\|^2 + \delta}$ makes e_y small very fast and remain small. Note that if $\|e_y\|^2 \geq \delta$, then $\frac{e_y^2}{\|e_y\|^2 + \delta} \geq \frac{1}{2}$. Thus, if we choose $\kappa_3(t) > \frac{(\|P\| \|\kappa_2(t)\| + \rho)^2}{2\epsilon}$ for some $\rho > 0$ and $\|e_y\|^2 \geq \delta$, then

$$\begin{aligned}
\frac{1}{2} \frac{dV}{dt} &= \langle P\dot{e}, e \rangle = \langle P(A - LF)e + PB(\tilde{\omega} - \omega), e \rangle + \langle P(f_1(\tilde{x}, u) - f_1(x, u)), e \rangle \\
&\quad - (\kappa_1(t) - \|\xi_1(t)\|) \|e_y\| - \frac{\kappa_3(t) e_y^2}{\|e_y\|^2 + \delta} - \langle P\xi_2(t), e \rangle \\
&\leq -\epsilon \|e\|^2 - \frac{\kappa_3(t)}{2} + \|P\| \|\kappa_2(t)\| \|e\| \leq -\rho \|e\|. \tag{21}
\end{aligned}$$

Then $\|e(t)\|$ decreases very fast in finite time such that $\|e_y(t)\|^2 < \delta$.

iii) Suppose that $Pk_2(\cdot)$ is bounded by k' , then we can improve the set Ω by the new attractive set $\Omega' = [0, \frac{k'}{\epsilon}]$.

iv) The same result is obtained if the unknown $\xi(t)$ is replaced by $\xi(t, x, u)$.

v) Theorem 2 can be also used to obtain an H^∞ observer for the original system.

Let us recall the definition of H^∞ observer (see, e.g., [16]).

Definition 1. If the error $e(t)$ is asymptotically stable for $\xi \equiv 0$ and $\|e\|_{L^2(0,\infty)} \leq \mu \|\xi\|_{L^2(0,\infty)}$ for some $\mu > 0$ under zero initial condition for non-zero $\xi \in L^2(0, \infty)$, then (6) is said an H^∞ observer for the system (3).

Assumption 5: There exists some $\mu > 0$ such that

$$\begin{pmatrix} \Omega & -P \\ -P & -2\mu\epsilon I \end{pmatrix} \leq 0 \tag{22}$$

where $\Omega := P(A - LF) + (A - LF)^T P + 2L_f \|P\| I + 2\epsilon I$ and $\epsilon > 0$ is in Assumption 4.

Theorem 3. *Suppose that Assumptions 1–5 hold. Then (6) is an H^∞ observer for the system (3).*

Proof. From Theorem 2, if the uncertainty $\xi \equiv 0$, then the error $e(t)$ converges to zero exponentially. For non-zero $\xi \in L^2(0, \infty)$, from (11)–(13), with the same $V(t) = \langle Pe(t), e(t) \rangle$, we have

$$\begin{aligned}
\frac{1}{2} \frac{dV}{dt} &= \langle P\dot{e}, e \rangle \\
&= \langle P(A - LF)e + PB(\tilde{\omega} - \omega), e \rangle \\
&+ \langle P(f_1(\tilde{x}, u) - f_1(x, u)), e \rangle - \kappa_1(t) \|e_y\| - \frac{\kappa_3 \|e_y\|^2}{\|e_y\|^2 + \delta} - \langle P\xi, e \rangle \\
&\leq \langle P(A - LF)e, e \rangle + L_f \|P\| \|e\|^2 - \langle P\xi, e \rangle \\
&\leq \epsilon (-\|e\|^2 + \mu \|\xi\|^2)
\end{aligned} \tag{23}$$

where the last inequality is deduced by using Assumption 5. Integrating both side from 0 to ∞ , using $V(0) = 0$ and note that V is non-negative, we obtain that $\|e\|_{L^2(0, \infty)} \leq \mu \|\xi\|_{L^2(0, \infty)}$. It means that (6) is an H^∞ observer for the system (3). \square

Remark 3. i) Assumption 5 implies that $\Omega = P(A - LF) + (A - LF)^T P + 2L_f \|P\| I + 2\epsilon I \leq 0$, which is consistent to Assumption 4. In Ω , we use the term ϵI instead of I as in [16], which is easier to have the non-positiveness of Ω .

ii) The positive number μ exists, for example, if $\Omega \leq -2\epsilon I$, we can choose $\mu > \frac{\|P\|^2}{4\epsilon^2}$.

4 Numerical examples

Example 1. Next we consider the system (3) with

$$A = \begin{pmatrix} -6 & 4 & 0 \\ 7 & -8 & 0 \\ 0 & 0 & -7 \end{pmatrix}, B = \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}, f(x, u) = \begin{pmatrix} u + 2 \sin x_2 \\ 2u + 3 \cos x_1 \\ -u + 4 \sin x_3 \end{pmatrix}$$

$$C = (8 \quad 6 \quad -3), F = (1 \quad 0 \quad 0), \quad u = 5 \sin t.$$

Suppose that the unknown $\xi(t, x, u) = \begin{pmatrix} 2 \\ 5 \cos x_1 \\ 4 \sin t \end{pmatrix}$ and

$$\mathcal{F}(x) = \begin{cases} \text{sign}(x)(3|x| + 6) & \text{if } x \neq 0, \\ [-6, 6] & \text{if } x = 0. \end{cases}$$

Then $L = 4$ and $k = \sqrt{41}$. Then Assumptions 1–4 are satisfied with

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 0 \\ 11 \\ 0 \end{pmatrix}, \quad \epsilon = 2, \quad K = 4.$$

One can observe that the total error $\|e\|$ converges to any neighborhood of the set $\Omega = [0, \sqrt{41}/2]$ in finite time while the observed error e_1 tends to zero very fast. Consequently, the observer state \tilde{x} using (6) provides a good estimation for the original state x .

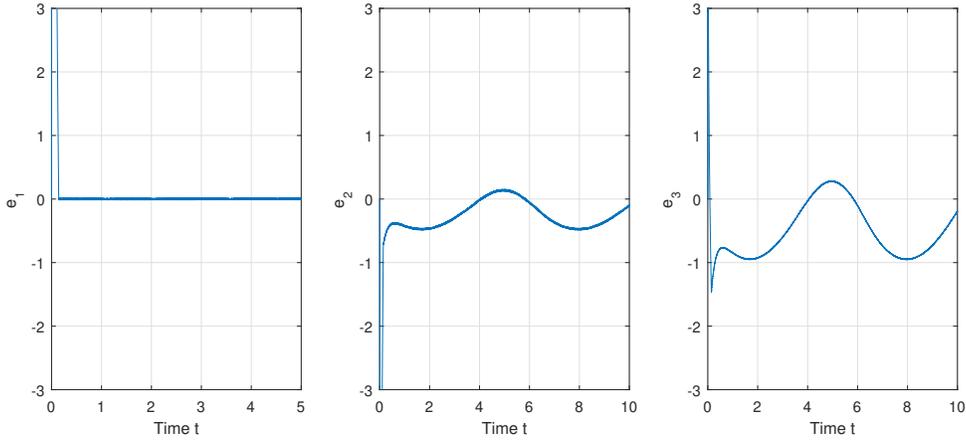


Figure 1: Errors using the approximate sliding mode observer (6)

Example 2. Next let us consider the rotor system with friction as in [10, 18]:

$$\begin{aligned} J_u \ddot{\theta}_u + k_\theta(\theta_u - \theta_l) + T_{fu}(\dot{\theta}_u) &= k_m u \\ J_l \ddot{\theta}_l - k_\theta(\theta_u - \theta_l) + T_{fl}(\dot{\theta}_l) &= 0, \end{aligned} \quad (24)$$

where θ_u and θ_l are the angular positions of the upper and lower discs, respectively. Let $x_1 = \theta_u - \theta_l$, $x_2 = \dot{\theta}_u$ and $x_3 = \dot{\theta}_l$, we have

$$\begin{aligned} \dot{x}_1 &= x_2 - x_3 \\ \dot{x}_2 &= \frac{k_m}{J_u} u - \frac{k_\theta}{J_u} x_1 - \frac{1}{J_u} T_{fu}(x_2) \\ \dot{x}_3 &= \frac{k_\theta}{J_l} x_1 - \frac{1}{J_l} T_{fl}(x_3), \end{aligned} \quad (25)$$

where

$$T_{fu}(x_2) = \begin{cases} T_{cu}(x_2) \text{sign}(x_2), & x_2 \neq 0 \\ [-T_{su} + \Delta T_{su}, T_{su} + \Delta T_{su}], & x_2 = 0 \end{cases} \quad (26)$$

and

$$T_{fl}(x_3) = \begin{cases} [T_{sl} + T_1(1 - \frac{2}{1+e^{\omega_1|x_3|}}) \\ + T_2(1 - \frac{2}{1+e^{\omega_2|x_3|}})]\text{sign}(x_3) + b_l x_3, & x_3 \neq 0 \\ [-T_{sl}, T_{sl}], & x_3 = 0. \end{cases} \quad (27)$$

For simplicity, we take $T_{fu}(x_2) = b_{up}x_2$. Then (25) can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 - x_3 \\ \dot{x}_2 &= \frac{k_m}{J_u}u - \frac{k_\theta}{J_u}x_1 - \frac{b_{up}}{J_u}x_2 \\ \dot{x}_3 &= \frac{k_\theta}{J_l}x_1 - \frac{1}{J_l}T_{fl}(x_3). \end{aligned} \quad (28)$$

The estimation of the parameters are given in the following tablet:

Parameter	Estimated value
J_u [kg m^2 / rad]	0.4765
k_m [Nm/ V]	3.9950
b_{up} [Nms/ rad]	2.2247
k_θ [Nm/ rad]	0.0727
J_l [kg m^2 / rad]	0.0326
T_{sl} [Nm]	0.1642
T_1 [Nm]	0.0603
T_2 [Nm]	-0.2267
ω_1 [s/rad]	5.7468
ω_2 [s/rad]	0.2941
b_l [Nms/ rad]	0.0109

Substituting the values into the function, we have

$$\begin{aligned} T_{fu}(x_2) &= 2.2247x_2, \\ T_{fl}(x_3) &= \begin{cases} [0.1642 + 0.0603(1 - \frac{2}{1+e^{5.7468|x_3|}}) \\ -0.2267(1 - \frac{2}{1+e^{0.2941|x_3|}})]\text{sign}(x_3) + 0.0109x_3, & x_3 \neq 0 \\ [-0.1642, 0.1642], & x_3 = 0. \end{cases} \end{aligned}$$

In order to analyse the real system, we have to use the mathematical model in a more abstract form:

$$\begin{cases} \dot{x} = Ax + B\omega + Gu \\ \omega \in -T_{fl}(Cx) \\ y = Fx, \end{cases} \quad (29)$$

where $A = \begin{bmatrix} 0 & 1 & -1 \\ -\frac{k_\theta}{J_u} & -\frac{b_{up}}{J_u} & 0 \\ \frac{k_\theta}{J_l} & 0 & 0 \end{bmatrix}$, $G = \begin{bmatrix} 0 \\ \frac{k_m}{J_u} \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{J_l} \end{bmatrix}$, $C = [0 \ 0 \ 1]$, $F = [1 \ 0 \ 0]$. Substituting the estimated values of the parameters, we have

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -0.1526 & -4.6688 & 0 \\ 2.2301 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 8.3841 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 30.6748 \end{bmatrix}.$$

Since $T_{fl}(x_3)$ is not monotone, we use the loop transformation $\tilde{T}_{fl}(Cx) = T_{fl}(Cx) - mCx$, where $m = -0.021$. Then

$$\tilde{T}_{fl}(x_3) = \begin{cases} [0.1642 + 0.0603(1 - \frac{2}{1+e^{5.7468|x_3|}}) \\ -0.2267(1 - \frac{2}{1+e^{0.2941|x_3|}})]\text{sign}(x_3) + 0.0319x_3, & x_3 \neq 0 \\ [-0.1642, 0.1642], & x_3 = 0. \end{cases}$$

Under the transformation, we have a new system

$$\begin{cases} \dot{x} = \bar{A}x + B\omega + Gu \\ \omega \in -\mathcal{F}(Cx) \\ y = Cx, \end{cases} \quad (30)$$

where

$$\bar{A} = A - mBC = \begin{bmatrix} 0 & 1 & -1 \\ -0.1526 & -4.6688 & 0 \\ 2.2301 & 0 & 0.6442 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 8.3841 \\ 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 30.6748 \end{bmatrix}, \quad C = [0 \ 0 \ 1], \quad F = [1 \ 0 \ 0].$$

$$\mathcal{F}(\lambda) = \begin{cases} [0.1642 + 0.0603(1 - \frac{2}{1+e^{5.7468|\lambda|}}) \\ -0.2267(1 - \frac{2}{1+e^{0.2941|\lambda|}})]\text{sign}(\lambda) + 0.0319\lambda, & \lambda \neq 0 \\ [-0.1642, 0.1642], & \lambda = 0. \end{cases}$$

From [18], $\lambda \mapsto \mathcal{F}(\lambda)$ is monotone.

The input u is the input voltage to the power amplifier of the motor, and u is chosen as 2. Solving the following LMIs:

$$\begin{aligned} (\bar{A} - LF)^T P + P(\bar{A} - LF) &= -Q \\ B^T P &= C - KF, \end{aligned} \quad (31)$$

we obtain

$$P = \begin{bmatrix} 0.2958 & 0.0417 & 0.0600 \\ 0.0417 & 0.0286 & 0 \\ 0.0600 & 0 & 0.0326 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.1327 & -0.0064 & -0.0158 \\ -0.0064 & -0.1837 & 0.0183 \\ 0.0158 & 0.0183 & -0.0780 \end{bmatrix},$$

$$L = \begin{bmatrix} 3.3069 \\ -1.2140 \\ -12.2290 \end{bmatrix}, \quad K = -1.8392, \quad \epsilon = 0.0714.$$

Now, let's assume that the system (30) is influenced by an uncertainty $\xi(t, x)$, which is described as follows:

$$\begin{cases} \dot{x} = \bar{A}x + B\omega + Gu + \xi(t, x) \\ \omega \in -\mathcal{F}(Cx) \\ y = Cx. \end{cases} \quad (32)$$

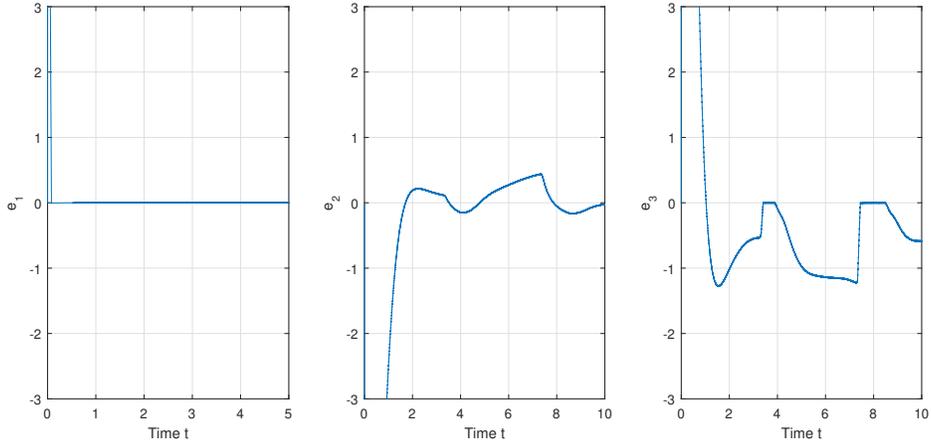


Figure 2: Errors using (6) applied to Example 2 with the uncertainty ξ_1

Suppose that the unknown $\xi(t, x) = \xi_1(t, x) = \begin{pmatrix} 1 \\ 4 \sin x_2 \\ \cos t \end{pmatrix}$. Using the approximate sliding mode

observer (6), we can see that observation error e_1 converges to zero very fast and the total error module $\|e(t)\|$ belongs to the attractive set $\Omega' = [0, 7.75]$ in finite time (Figure 2). The numerical result is consistent with Theorem 2 when the uncertainty is chosen randomly. Next we consider the unknown

$$\xi(t, x) = \xi_2(t, x) = \begin{pmatrix} 16.0552 \\ -23.4092 \\ -29.5495 \end{pmatrix} + \begin{pmatrix} e^{-t} \\ e^{-2t} \\ e^{-1.5t} \end{pmatrix},$$

which has 2 parts: the first part in $\text{Im}(P^{-1}F^T)$ and the second part vanishes when the time is large. Then the approximate sliding mode observer (6) is indeed the observer of the original system (Figure 3).

5 Conclusions

In this paper, we introduce an approximate sliding mode observer, which also functions as an H^∞ observer, designed for a specific category of set-valued Lur'e dynamical systems. Importantly, our approach accommodates scenarios where the uncertainty does not fall within the range of observation. Consequently, we are able to obtain accurate state estimations for the original systems. Furthermore, in cases where the unobservable portion of uncertainty tends to vanish when the time is large, we achieve exact observers. It would be interesting to investigate strategies that enhance the robustness of our proposed observer, enabling it to handle increasingly complex and dynamic

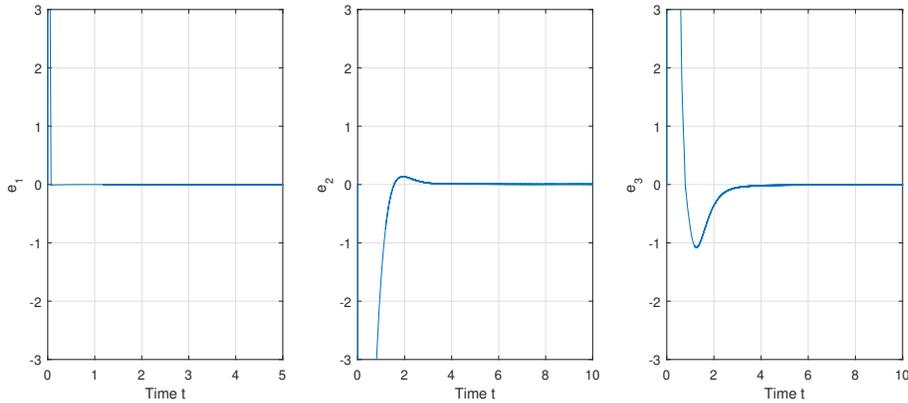


Figure 3: Errors using (6) applied to Example 2 with the uncertainty ξ_2

uncertainties or disturbances. Exploring ways to incorporate adaptive control strategies into our observer framework would be a promising direction, facilitating its adaptation to evolving system conditions. Optimizing and reducing the size of the attractive set could be a valuable pursuit, potentially resulting in quicker convergence times. The practical application and validation of our observer in real-world systems, such as autonomous vehicles, robotics, or industrial processes, present good opportunities for future exploration. Extending our observer framework to address the challenges posed by nonlinear systems with set-valued uncertainties would further broaden its applicability and impact. These open questions require further investigation and are beyond the scope of the current paper. They will be the focus of a new research project.

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