

Supplementary Material for Universal Correlation Profiles in Single-File Systems

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I. ANNEALED VS QUENCHED CORRELATION FUNCTIONS

We consider the general observable

$$\mathcal{O}_t[f, g] = \sum_i [f(x_i(t)) - g(x_i(0))] , \quad (\text{S1})$$

with $f(x) \underset{x \rightarrow \pm\infty}{\simeq} g(x)$ to ensure convergence in the limit $N \rightarrow \infty$. We also define the density of particles

$$\rho(x, t) = \sum_i \delta(x - x_i(t)) . \quad (\text{S2})$$

The joint statistical properties of these two observables are encoded in the joint cumulant generating functions. In the annealed case, it takes the form

$$\log \overline{\langle e^{\lambda \mathcal{O}_t + \chi \rho(x, t)} \rangle} = \log \overline{\langle e^{\lambda \mathcal{O}_t} \rangle} + \chi \frac{\overline{\langle \rho(x, t) e^{\lambda \mathcal{O}_t} \rangle}}{\overline{\langle e^{\lambda \mathcal{O}_t} \rangle}} + O(\chi^2) . \quad (\text{S3})$$

Similarly, we consider the quenched joint cumulant generating function

$$\overline{\log \langle e^{\lambda \mathcal{O}_t + \chi \rho(x,t)} \rangle} = \overline{\log \langle e^{\lambda \mathcal{O}_t} \rangle} + \chi \frac{\overline{\langle \rho(x,t) e^{\lambda \mathcal{O}_t} \rangle}}{\overline{\langle e^{\lambda \mathcal{O}_t} \rangle}} + O(\chi^2). \quad (\text{S4})$$

This motivates the definition of the correlation profiles w_A and w_Q in the main text.

II. DERIVATION OF THE CUMULANTS AND CORRELATIONS

A. Average over the time evolution

We begin by computing the moment generating function for a given number N of particles. We will denote $\{y_i = x_i(0)\}$ the positions of the particles at initial time, which are at the moment treated as parameters. By definition,

$$\begin{aligned} \langle e^{\lambda \mathcal{O}_t} \rangle &= \int_{-\infty}^{\infty} dx_1 \cdots dx_N K_t(\{x_i\}|\{y_i\}) \prod_{i=1}^N e^{\lambda(f(x_i) - g(y_i))} \\ &= \frac{1}{N!} \sum_{\sigma} \prod_{i=1}^N e^{-\lambda g(y_i)} \int_{-\infty}^{\infty} G_t(x|y_{\sigma(i)}) e^{\lambda f(x)} dx \\ &= \prod_{i=1}^N e^{-\lambda g(y_i)} \int_{-\infty}^{\infty} G_t(x|y_i) e^{\lambda f(x)} dx, \end{aligned} \quad (\text{S5})$$

since all the integrals are independent and equivalent. Similarly, we can compute

$$\begin{aligned} \langle \rho(x,t) e^{\lambda \mathcal{O}_t} \rangle &= \sum_n \int_{-\infty}^{\infty} dx_1 \cdots dx_N \delta(x - x_n) K_t(\{x_i\}|\{y_i\}) \prod_{i=1}^N e^{\lambda(f(x_i) - g(y_i))} \\ &= \frac{1}{N!} \sum_n \sum_{\sigma} G_t(x|y_{\sigma(n)}) e^{\lambda f(x) - \lambda g(y_n)} \prod_{i \neq n} e^{-\lambda g(y_i)} \int_{-\infty}^{\infty} G_t(x'|y_{\sigma(i)}) e^{\lambda f(x')} dx' \\ &= \frac{1}{N!} \sum_n \sum_q \sum_{\sigma, \sigma(n)=q} G_t(x|y_q) e^{\lambda f(x) - \lambda \sum_i g(y_i)} \prod_{j \neq q} \int_{-\infty}^{\infty} G_t(x'|y_j) e^{\lambda f(x')} dx' \\ &= \sum_q G_t(x|y_q) e^{\lambda f(x) - \lambda g(y_q)} \prod_{j \neq q} \int_{-\infty}^{\infty} G_t(x'|y_j) e^{\lambda f(x') - \lambda g(y_j)} dx'. \end{aligned} \quad (\text{S6})$$

The higher order correlations with the density are also accessible in the same way, for instance,

$$\begin{aligned} \langle \rho(x,t) \rho(y,t) e^{\lambda \mathcal{O}_t} \rangle &= \sum_{n,m} \int_{-\infty}^{\infty} dx_1 \cdots dx_N \delta(x - x_n) \delta(y - x_m) K_t(\{x_i\}|\{y_i\}) \prod_{i=1}^N e^{\lambda(f(x_i) - g(y_i))} \\ &= \frac{1}{N!} \sum_{n \neq m} \sum_{\sigma} \tilde{G}_t(x|y_{\sigma(n)}) \tilde{G}_t(y|y_{\sigma(m)}) \prod_{i \neq n,m} \int_{-\infty}^{\infty} \tilde{G}_t(x'|y_{\sigma(i)}) dx' \\ &\quad + \frac{1}{N!} \sum_n \sum_{\sigma} \tilde{G}_t(x|y_{\sigma(n)}) \delta(x - y) \prod_{i \neq n} \int_{-\infty}^{\infty} \tilde{G}_t(x'|y_{\sigma(i)}) dx' \\ &= \sum_{q \neq p} \tilde{G}_t(x|y_q) \tilde{G}_t(y|y_p) \prod_{j \neq p,q} \int_{-\infty}^{\infty} \tilde{G}_t(x'|y_j) dx' \\ &\quad + \delta(x - y) \sum_q \tilde{G}_t(x|y_q) \prod_{j \neq q} \int_{-\infty}^{\infty} \tilde{G}_t(x'|y_j) dx', \end{aligned} \quad (\text{S7})$$

where we have denoted $\tilde{G}_t(x|y) = e^{\lambda f(x)} G_t(x|y) e^{-\lambda g(y)}$. To take the average over the initial condition, we need to treat separately the annealed and quenched cases.

B. Average over the initial positions in the annealed case

In the annealed case, we compute

$$\overline{\langle e^{\lambda \mathcal{O}_t} \rangle} = \left(\frac{1}{N} \int dy \rho_0(y) e^{-\lambda g(y)} \int G_t(x|y) e^{\lambda f(x)} dx \right)^N. \quad (\text{S8})$$

For finite N , the condition $\int \rho_0 = N$ ensures that the above integrals are convergent. But in the limit $N \rightarrow \infty$ this would not be the case. Take for instance a constant density profile $\rho_0(x) = \rho$, then the integral would diverge in the domains ($x > 0, y > 0$) and ($x < 0, y < 0$). We can solve this issue by writing the above expression as

$$\overline{\langle e^{\lambda \mathcal{O}_t} \rangle} = \left(1 + \frac{1}{N} \int dy \rho_0(y) \int G_t(x|y) (e^{\lambda f(x) - \lambda g(y)} - 1) dx \right)^N, \quad (\text{S9})$$

where we have used that $\int G_t(x|y) dx = 1$. In this expression, if x and y are large, with the same sign, the integrand vanishes. And if they are both large with opposite signs, $G_t(x|y)$ is small. Therefore, the integral is always well defined, and thus

$$\psi_A(\lambda, t) \equiv \lim_{N \rightarrow \infty} \ln \overline{\langle e^{\lambda \mathcal{O}_t} \rangle} = \int dy \rho_0(y) \int G_t(x|y) (e^{\lambda f(x) - \lambda g(y)} - 1) dx. \quad (\text{S10})$$

To compute the density profiles, we need

$$\overline{\langle \rho(x, t) e^{\lambda \mathcal{O}_t} \rangle} = \left(\int dy \rho_0(y) G_t(x|y) e^{\lambda f(x) - \lambda g(y)} \right) \left(\frac{1}{N} \int dy \rho_0(y) e^{-\lambda g(y)} \int G_t(x'|y) e^{\lambda f(x')} dx' \right)^{N-1}. \quad (\text{S11})$$

Taking the ratio with (S9) and using the same rewriting as before, we get

$$\frac{\overline{\langle \rho(x, t) e^{\lambda \mathcal{O}_t} \rangle}}{\overline{\langle e^{\lambda \mathcal{O}_t} \rangle}} = \frac{e^{\lambda f(x)} \int dy \rho_0(y) G_t(x|y) e^{-\lambda g(y)}}{1 + \frac{1}{N} \int dy \rho_0(y) \int G_t(x'|y) (e^{\lambda f(x') - \lambda g(y)} - 1) dx'}. \quad (\text{S12})$$

In the limit $N \rightarrow \infty$, this becomes

$$\psi_A(x, \lambda, t) \equiv \lim_{N \rightarrow \infty} \frac{\overline{\langle \rho(x, t) e^{\lambda \mathcal{O}_t} \rangle}}{\overline{\langle e^{\lambda \mathcal{O}_t} \rangle}} = e^{\lambda f(x)} \int dy \rho_0(y) G_t(x|y) e^{-\lambda g(y)}. \quad (\text{S13})$$

For the two points correlation function, we need to compute

$$\begin{aligned} \overline{\langle \rho(x, t) \rho(y, t) e^{\lambda \mathcal{O}_t} \rangle} = & \sum_{q \neq p} \left(\int_{-\infty}^{\infty} dz \frac{\rho_0(z)}{N} \tilde{G}_t(x|z) \right) \left(\int_{-\infty}^{\infty} dz \frac{\rho_0(z)}{N} \tilde{G}_t(y|z) \right) \left(\int_{-\infty}^{\infty} dz \frac{\rho_0(z)}{N} \int_{-\infty}^{\infty} dx' \tilde{G}_t(x'|z) \right)^{N-2} \\ & + \delta(x - y) \sum_q \left(\int_{-\infty}^{\infty} dz \frac{\rho_0(z)}{N} \tilde{G}_t(x|z) \right) \left(\int_{-\infty}^{\infty} dz \frac{\rho_0(z)}{N} \int_{-\infty}^{\infty} dx' \tilde{G}_t(x'|z) \right)^{N-1}. \end{aligned} \quad (\text{S14})$$

The sums no longer depend on p and q , and thus,

$$\begin{aligned} \overline{\langle \rho(x, t) \rho(y, t) e^{\lambda \mathcal{O}_t} \rangle} = & \frac{N-1}{N} \left(\int_{-\infty}^{\infty} dz \rho_0(z) \tilde{G}_t(x|z) \right) \left(\int_{-\infty}^{\infty} dz \rho_0(z) \tilde{G}_t(y|z) \right) \left(\int_{-\infty}^{\infty} dz \frac{\rho_0(z)}{N} \int_{-\infty}^{\infty} dx' \tilde{G}_t(x'|z) \right)^{N-2} \\ & + \delta(x - y) \left(\int_{-\infty}^{\infty} dz \rho_0(z) \tilde{G}_t(x|z) \right) \left(\int_{-\infty}^{\infty} dz \frac{\rho_0(z)}{N} \int_{-\infty}^{\infty} dx' \tilde{G}_t(x'|z) \right)^{N-1}. \end{aligned} \quad (\text{S15})$$

Taking the ratio with (S9), and using again the same rewriting for the denominator, we get for $N \rightarrow \infty$,

$$\lim_{N \rightarrow \infty} \frac{\overline{\langle \rho(x, t) \rho(y, t) e^{\lambda \mathcal{O}_t} \rangle}}{\langle e^{\lambda \mathcal{O}_t} \rangle} = \delta(x - y) \left(\int_{-\infty}^{\infty} dz \rho_0(z) G_t(x|z) e^{\lambda f(x) - \lambda g(z)} \right) + \left(\int_{-\infty}^{\infty} dz \rho_0(z) G_t(x|z) e^{\lambda f(x) - \lambda g(z)} \right) \left(\int_{-\infty}^{\infty} dz \rho_0(z) G_t(y|z) e^{\lambda f(x) - \lambda g(z)} \right). \quad (\text{S16})$$

In particular, we have

$$\lim_{N \rightarrow \infty} \left[\frac{\overline{\langle \rho(x, t) \rho(y, t) e^{\lambda \mathcal{O}_t} \rangle}}{\langle e^{\lambda \mathcal{O}_t} \rangle} - \frac{\overline{\langle \rho(x, t) e^{\lambda \mathcal{O}_t} \rangle}}{\langle e^{\lambda \mathcal{O}_t} \rangle} \frac{\overline{\langle \rho(y, t) e^{\lambda \mathcal{O}_t} \rangle}}{\langle e^{\lambda \mathcal{O}_t} \rangle} \right] = \delta(x - y) w_A(x, \lambda, t). \quad (\text{S17})$$

C. Average over the initial positions in the quenched case

In the quenched case, the cumulant generating function is given by

$$\ln \langle e^{\lambda \mathcal{O}_t} \rangle = \sum_i \frac{1}{N} \int dy \rho_0(y) \ln \left[\int_{-\infty}^{\infty} G_t(x|y) e^{\lambda f(x) - \lambda g(y)} dx \right]. \quad (\text{S18})$$

The terms in the sum are independent of i , and the integral converges in the limit $N \rightarrow \infty$, thus

$$\psi_Q(\lambda, t) \equiv \lim_{N \rightarrow \infty} \overline{\ln \langle e^{\lambda \mathcal{O}_t} \rangle} = \int dy \rho_0(y) \ln \left[\int_{-\infty}^{\infty} G_t(x|y) e^{\lambda f(x) - \lambda g(y)} dx \right]. \quad (\text{S19})$$

Similarly, for the profiles we compute

$$\frac{\overline{\langle \rho(x, t) e^{\lambda \mathcal{O}_t} \rangle}}{\langle e^{\lambda \mathcal{O}_t} \rangle} = \sum_q \frac{G_t(x|y_q) e^{\lambda f(x) - \lambda g(y_q)}}{\int_{-\infty}^{\infty} G_t(x'|y_q) e^{\lambda f(x') - \lambda g(y_q)} dx'}. \quad (\text{S20})$$

We can easily take the average over the initial positions y_i , which gives

$$w_Q(x, \lambda, t) \equiv \lim_{N \rightarrow \infty} \frac{\overline{\langle \rho(x, t) e^{\lambda \mathcal{O}_t} \rangle}}{\langle e^{\lambda \mathcal{O}_t} \rangle} = \int dy \rho_0(y) \frac{G_t(x|y) e^{\lambda f(x) - \lambda g(y)}}{\int_{-\infty}^{\infty} G_t(x'|y) e^{\lambda f(x') - \lambda g(y)} dx'}. \quad (\text{S21})$$

Finally, for the two points correlation functions, we have

$$\frac{\overline{\langle \rho(x, t) \rho(y, t) e^{\lambda \mathcal{O}_t} \rangle}}{\langle e^{\lambda \mathcal{O}_t} \rangle} = \sum_{q \neq p} \frac{\tilde{G}_t(x|y_p)}{\int_{-\infty}^{\infty} \tilde{G}_t(x'|y_p) dx'} \frac{\tilde{G}_t(y|y_q)}{\int_{-\infty}^{\infty} \tilde{G}_t(x'|y_q) dx'} + \delta(x - y) \sum_q \frac{\tilde{G}_t(x|y_q)}{\int_{-\infty}^{\infty} \tilde{G}_t(x'|y_q) dx'}. \quad (\text{S22})$$

Averaging over the initial condition yields

$$\frac{\overline{\langle \rho(x, t) \rho(y, t) e^{\lambda \mathcal{O}_t} \rangle}}{\langle e^{\lambda \mathcal{O}_t} \rangle} = \delta(x - y) \int_{-\infty}^{\infty} dz \rho_0(z) \frac{\tilde{G}_t(x|z)}{\int_{-\infty}^{\infty} \tilde{G}_t(x'|z) dx'} + \frac{N-1}{N} \left(\int_{-\infty}^{\infty} dz \rho_0(z) \frac{\tilde{G}_t(x|z)}{\int_{-\infty}^{\infty} \tilde{G}_t(x'|z) dx'} \right) \left(\int_{-\infty}^{\infty} dz \rho_0(z) \frac{\tilde{G}_t(y|z)}{\int_{-\infty}^{\infty} \tilde{G}_t(x'|z) dx'} \right). \quad (\text{S23})$$

In the large N limit, this gives

$$\lim_{N \rightarrow \infty} \left[\frac{\overline{\langle \rho(x, t) \rho(y, t) e^{\lambda \mathcal{O}_t} \rangle}}{\langle e^{\lambda \mathcal{O}_t} \rangle} - \frac{\overline{\langle \rho(x, t) e^{\lambda \mathcal{O}_t} \rangle}}{\langle e^{\lambda \mathcal{O}_t} \rangle} \frac{\overline{\langle \rho(y, t) e^{\lambda \mathcal{O}_t} \rangle}}{\langle e^{\lambda \mathcal{O}_t} \rangle} \right] = \delta(x - y) w_Q(x, \lambda, t), \quad (\text{S24})$$

as in the annealed case.

D. Applications

In all the following applications, we consider an initial density of particles $\rho_0(x) = \rho_+\Theta(x) + \rho_-\Theta(-x)$ for simplicity, and consider the case of the current Q_t , corresponding to $f(x) = g(x) = \Theta(x)$.

1. Brownian particles

As a first application, we revisit the well-studied case of reflecting Brownian particles on the real line. The propagator of an individual particle with diffusion coefficient D is $G_t(x|y) = e^{-\frac{(x-y)^2}{4Dt}}/\sqrt{4\pi Dt}$. Using this propagator in the general expressions (S10,S13), we obtain $\psi_A(\lambda, t) = \omega\sqrt{\frac{Dt}{\pi}}$ and

$$w_A(x > 0, \lambda, t) = \rho_+ + \frac{\rho_- e^\lambda - \rho_+}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{4Dt}}\right), \quad (\text{S25})$$

where we have denoted $\omega = \rho_+(e^{-\lambda} - 1) + \rho_-(e^\lambda - 1)$, by analogy with the single parameter identified in the simple exclusion process (SEP) [S1]. Indeed, these expressions coincide with the low density limit of the ones obtained for the SEP [S1–S3]. The profile for $x < 0$ can be deduced from the symmetry $x \rightarrow -x$, $\lambda \rightarrow -\lambda$, $\rho_\pm \rightarrow \rho_\mp$. Similarly, in the quenched case, we recover the expression found in [S4] for the cumulants, and we additionally get the correlation profiles,

$$w_Q(x, \lambda, t) = e^{\lambda\Theta(x)} \int_{-\infty}^{\infty} \rho_0(y) \frac{e^{-\left(\frac{x}{\sqrt{4Dt}} - y\right)^2}}{1 + \frac{e^\lambda - 1}{2} \operatorname{erfc}(-y)} \frac{dy}{\sqrt{\pi}}. \quad (\text{S26})$$

A related expression for constant ρ_0 can be found in [S5]. The two profiles w_A and w_Q are represented in Fig. S1 (solid black lines).

2. Run and tumble particles

The Laplace transform of the propagator of an individual particle takes a simple form [S6],

$$\int_0^\infty e^{-st} G_t(x|y) dt = \frac{\Lambda(s)}{2s} e^{-\Lambda(s)|x-y|}, \quad (\text{S27})$$

where $\Lambda(s) = \sqrt{s(s+2\gamma)}/v_0$. We can easily obtain the annealed profile and cumulant generating function in the Laplace domain since the expressions (S10,S13) are linear in the propagator. The inverse Laplace transform can be computed explicitly using the expressions given in [S7], and we get $\psi_A(\lambda, t) = \frac{\omega}{2} v_0 t e^{-\gamma t} (I_0(\gamma t) + I_1(\gamma t))$, where we have again denoted $\omega = \rho_+(e^{-\lambda} - 1) + \rho_-(e^\lambda - 1)$, by analogy with the single parameter identified in the simple exclusion process (SEP) [S1], and I_ν is a modified Bessel function. Similarly, the correlation profile reads

$$w_A(x > 0, \lambda, t) = \rho_+ + \frac{\rho_- e^\lambda - \rho_+}{2} \Theta(v_0 t - x) \left(e^{-\frac{\gamma x}{v_0}} + \frac{\gamma x}{v_0} \int_1^{\frac{v_0 t}{x}} \frac{e^{-\frac{\gamma x T}{v_0}} I_1\left(\frac{\gamma x}{v_0} \sqrt{T^2 - 1}\right)}{\sqrt{T^2 - 1}} dT \right). \quad (\text{S28})$$

The expression for $x < 0$ can again be obtained by symmetry. Finally, in the large time limit, ψ_A and w_A (S28) reduce to the Brownian case, with a diffusion coefficient $D = v_0^2/(2\gamma)$.

3. Lévy flights

Next, as an illustration of non diffusive dynamics at long times, we consider particles performing Lévy flights, corresponding to an individual propagator

$$G_t(x|y) = \frac{1}{t^{1/\alpha}} g\left(\frac{x-y}{t^{1/\alpha}}\right), \quad g(x) = \int_{-\infty}^{\infty} e^{-ikx - |k|^\alpha} \frac{dk}{2\pi}, \quad (\text{S29})$$

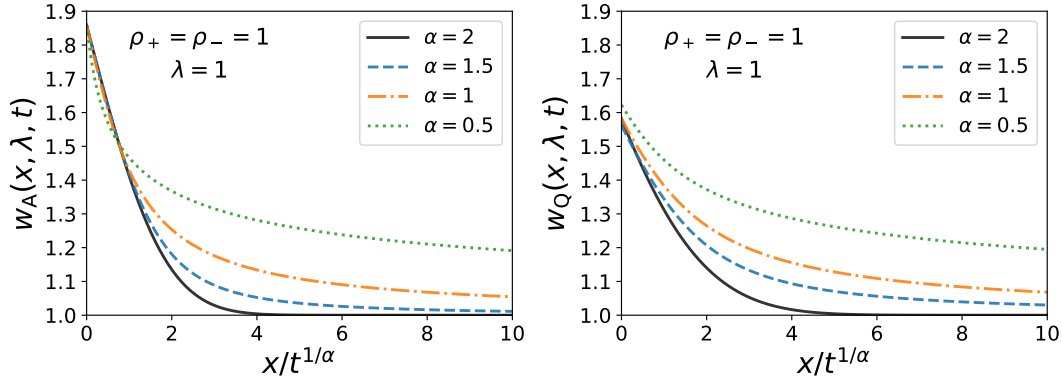


FIG. S1. Annealed correlation profile w_A (left) and quenched correlation profile w_Q (right) for Lévy particles, for different values of the Lévy exponent α . They are compared to the one of the Brownian particles with $D = 1$, corresponding to $\alpha = 2$ (black solid line). For $0 < \alpha < 2$, the correlation profiles decay as a power law, as $x^{-\alpha}$, while for the Brownian particles, the decay is much faster, as $e^{-\frac{x^2}{4Dt}}/x$.

with the exponent $\alpha \in]0, 2]$. Plugging this expression into the formulas (S13) for w_A and (S21) for w_Q , we obtain that these two profiles are functions of $x/t^{1/\alpha}$ only. The annealed profile reads

$$w_A(x = zt^{1/\alpha}, \lambda, t) = \rho_+ + (\rho_- e^\lambda - \rho_+) \int_z^\infty g(y) dy, \quad (\text{S30})$$

for $x > 0$. Similarly, the quenched profile is

$$w_Q(x = zt^{1/\alpha}, \lambda, t) = e^{\lambda\Theta(z)} \left[\rho_+ \int_0^\infty dy \frac{g(z-y)}{1 + (e^\lambda - 1) \int_{-y}^\infty g(u) du} + \rho_- \int_{-\infty}^0 dy \frac{g(z-y)}{1 + (e^\lambda - 1) \int_{-y}^\infty g(u) du} \right]. \quad (\text{S31})$$

They are represented for different values of α in Fig. S1. The correlations present a power law behaviour, $w_A(x, \lambda, t) - \rho_+ \underset{x \rightarrow \infty}{\sim} x^{-\alpha}$ for $0 < \alpha < 2$, due to the fact that the particles can perform large jumps in this case.

III. THE COMB GEOMETRY

A. Propagator in the continuous limit

The comb lattice is a two dimensional lattice in which all the links parallel to the x -axis have been removed, except those on the axis itself, called the backbone. The propagator $P_t(\vec{r}|\vec{s})$ of a particle performing a random walk in discrete time on this lattice is given in [S8]. Its expressions depends is different if the arrival point $\vec{r} = (r_x, r_y)$ and the starting point $\vec{s} = (r'_x, r'_y)$ are on the same teeth of the comb ($r_x = r'_x$ and $r_y r'_y > 0$) or not. The (discrete) Laplace transform of the propagator reads [S8]

$$\hat{P}_\xi(\vec{r}|\vec{s}) \equiv \sum_{t=0}^\infty \xi^t P_t(\vec{r}|\vec{s}) = \frac{1 + \delta_{r_y, 0}}{2} G_2(\xi) f_2(\xi)^{|r_x - r'_x|} f_1(\xi)^{|r_y| + |r'_y|}, \quad \text{if } r_x \neq r'_x \text{ or } r_y r'_y < 0, \quad (\text{S32})$$

$$\hat{P}_\xi(\vec{r}|\vec{s}) = f_1(\xi)^{|r_y - r'_y|} \left\{ \frac{1}{2} G_2(\xi) f_1(\xi)^{2\tilde{r}} + \frac{2}{\xi} f_1(\xi)^{2\tilde{r}-1} + G_1(\xi) \left[1 - f_1(\xi)^{2(\tilde{r}-1)} \right] \right\}, \quad \text{if } r_x = r'_x \text{ and } r_y r'_y > 0, \quad (\text{S33})$$

where $\tilde{r} = \min(|r_y|, |r'_y|)$, and

$$f_1(\xi) = \frac{1 - \sqrt{1 - \xi^2}}{\xi}, \quad f_2(\xi) = \frac{1 + \sqrt{1 - \xi^2} - \sqrt{2}\sqrt{1 - \xi^2 + \sqrt{1 - \xi^2}}}{\xi}, \quad (\text{S34})$$

$$G_1(\xi) = \frac{1}{\sqrt{1-\xi^2}}, \quad G_2(\xi) = \sqrt{\frac{2}{1-\xi^2 + \sqrt{1-\xi^2}}}. \quad (\text{S35})$$

One can define a continuous version of this random walk in different ways. A convenient one is to define it as the long time limit of the discrete one. We introduce a scaling parameter T and appropriately rescale space and time with T and let $T \rightarrow \infty$. We obtain a nontrivial limit with the following scalings,

$$r_x = x T^{1/4}, \quad r_y = y \sqrt{T}, \quad t = \tau T, \quad (\text{S36})$$

with x , y and τ fixed. This implies $t \rightarrow \infty$ when $T \rightarrow \infty$, corresponding to $\xi \rightarrow 1$, such that $1 - \xi = u/T$ in the Laplace domain. Inserting these scaling into the propagator (S32), we get,

$$\hat{P}_\xi(\vec{r}|\vec{r}') \underset{T \rightarrow \infty}{\simeq} \frac{1}{2} \left(\frac{2T}{u} \right)^{1/4} (1 - 2^{3/4} u^{1/4} T^{-1/4}) |x-x'| T^{1/4} (1 - \sqrt{2u}/\sqrt{T})^{(|y|+|y'|)\sqrt{T}}. \quad (\text{S37})$$

We can define the propagator in the continuous limit as

$$\tilde{P}_u(x, y|x', y') \equiv \lim_{T \rightarrow \infty} T^{-1/4} \hat{P}_\xi(\vec{r}|\vec{r}') = \frac{1}{2} \left(\frac{2}{u} \right)^{1/4} e^{-2^{3/4} u^{1/4} |x-x'| - \sqrt{2u}(|y|+|y'|)}, \quad (\text{S38})$$

where the prefactor $T^{-1/4} = T^{1/4} \times \sqrt{T} \times (T)^{-1}$ comes from the rescaling of both space directions and time (and thus ξ). This expression holds for different teeth of the comb, thus $x \neq x'$. Taking the continuous limit of (S33) for the case of $x = x'$ (same teeth), we get

$$\tilde{P}_u(x, y|x' = x, y') \equiv \lim_{T \rightarrow \infty} T^{-1/2} \hat{P}_\xi(\vec{r}|\vec{r}') = \frac{1}{\sqrt{2u}} e^{-\sqrt{2u}|y-y'|} (1 - e^{-2\sqrt{2u} \min(|y|, |y'|)}), \quad (\text{S39})$$

where this time the prefactor $T^{-1/2} = \sqrt{T} \times (T)^{-1}$ comes from the rescaling of y and t (and not x since it is fixed). We can combine these two expression into

$$\tilde{P}_u(x, y|x', y') = \frac{1}{2^{3/4} u^{1/4}} e^{-2^{3/4} u^{1/4} |x-x'| - \sqrt{2u}(|y|+|y'|)} + \frac{1}{\sqrt{2u}} \left(e^{-\sqrt{2u}|y-y'|} - e^{-\sqrt{2u}|y+y'|} \right) \delta(x-x') \Theta(y y'), \quad (\text{S40})$$

where Θ is the Heaviside step function. Note that the second term is the Laplace transform of a propagator of a random walk on y with reflecting boundary condition at the origin. One can check that this propagator is indeed normalised by integrating over x and y ,

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \tilde{P}_u(x, y|x', y') = \frac{1}{u}, \quad (\text{S41})$$

which is indeed the Laplace transform of 1.

B. Current through the bond $(0,0) - (1,0)$

As in the 1D case, the current through the horizontal bond $(0,0) - (1,0)$ can be expressed in terms of the number of particles on the right half space (this is true because the lattice is a tree). In the continuous limit, Q_t is thus again given by (S1), with $f(x) = g(x) = \Theta(x)$ as in the main text. The derivation of Section II can be straightforwardly adapted to the case of the comb by replacing the 1D integrals by two-dimensional integrals. For instance, the annealed correlation profile reads

$$w_A(x, y, \lambda, \tau) = e^{\lambda \Theta(x)} \int dx' \int dy' \rho_0(x', y') P_\tau(x, y|x', y') e^{-\lambda \Theta(x')}. \quad (\text{S42})$$

Since this expression is linear in the propagator, one can take its Laplace transform in time,

$$\tilde{w}_A(x, y, \lambda, u) \equiv \int_0^\infty w_A(x, y, \lambda, \tau) e^{-u\tau} d\tau = e^{\lambda \Theta(x)} \int dx' \int dy' \rho_0(x', y') \tilde{P}_u(x, y|x', y') e^{-\lambda \Theta(x')}. \quad (\text{S43})$$

Computing these integrals with the propagator (S40), with a constant density $\rho_0(x, y) = \rho$, we obtain,

$$\tilde{w}_A(x, y, \lambda, u) = \frac{\rho}{u} \left(1 + \frac{e^{\lambda \text{sign}(x)} - 1}{2} e^{-2^{3/4} u^{1/4} |x| - \sqrt{2u}|y|} \right). \quad (\text{S44})$$

This expression can be inverted to the time domain numerically, and is plotted in the main text. The fact that \tilde{w}_A is a function of $u^{1/4}x$ and \sqrt{uy} in the Laplace domain implies that w_A is a function of $x/\tau^{1/4}$ and $y/\sqrt{\tau}$ in the time domain.

IV. DERIVATION OF THE CUMULANTS AND CORRELATIONS FOR THE TRACER

A. Averaging over the time evolution

We consider that we have initially $2N + 1$ particles, at positions y_n , $n \in [-N, N]$. We take the middle particle to be the tracer, initially at position $y_0 = 0$. Therefore, $y_{-n} < 0$ for $n > 0$ and $y_n > 0$ for $n > 0$. The distribution $P_t(X|\{y_i\})$ of the tracer at time t is obtained by imposing that N particles remain on the left of the tracer, and N on the right,

$$P_t(X|\{y_i\}) \equiv \langle \delta(X - x_0(t)) \rangle = (2N + 1) \binom{2N}{N} \prod_{n=1}^N \int_{-\infty}^X dx_{-n} \int_X^{\infty} dx_n K_t(\{x_i\}|\{y_i\}), \quad (\text{S45})$$

where K_t is now the propagator of the $2N + 1$ particles, and we set $x_0 = X$. The combinatorial factor comes from the fact that the tracer can be any of the $(2N+1)$ particles, and then we choose N particles among the remaining $2N$ to be on the left of X . This gives,

$$P_t(x|\{y_i\}) = \frac{1}{(N!)^2} \sum_{\sigma} G_t(X|y_{\sigma(0)}) \prod_{n=1}^N \left(\int_{-\infty}^X dz G_t(z|y_{\sigma(-n)}) \right) \left(\int_X^{\infty} dz G_t(z|y_{\sigma(n)}) \right) \quad (\text{S46})$$

$$= \frac{1}{(N!)^2} \sum_{\sigma} G_t(X|y_{\sigma(0)}) \prod_{n=1}^N \left(\int_{-X}^{\infty} dz G_t(-z|y_{\sigma(-n)}) \right) \left(\int_X^{\infty} dz G_t(z|y_{\sigma(n)}) \right). \quad (\text{S47})$$

Partitioning the summations over the value of $\sigma(0)$, we get,

$$P_t(X|\{y_i\}) = \frac{1}{(N!)^2} \sum_{q=-N}^N \sum_{\sigma, \sigma(0)=q} G_t(X|y_q) \prod_{n=1}^N \left(\int_0^{\infty} dz G_t(-z|y_{\sigma(-n)} - X) \right) \left(\int_0^{\infty} dz G_t(z|y_{\sigma(n)} - X) \right). \quad (\text{S48})$$

In the product, all the terms are present, except y_q . The signs of z and n are distributed so that there are N positive and N negative ones (representing the particles on the right and on the left of the tracer respectively). Introducing $\varepsilon_n = \pm 1$ to represent these sign variables, we obtain

$$P_t(X|\{y_i\}) = \sum_{n=-N}^N G_t(X|y_n) \sum_{\varepsilon_j = \pm 1} \delta_{\sum_{j \neq n} \varepsilon_j, 0} \prod_{i \neq n} \int_0^{\infty} G_t(\varepsilon_i z | y_i - X) dz. \quad (\text{S49})$$

with the Kroenecker δ enforcing that there are N minus signs, and N plus signs. The $(N!)^2$ term cancelled with the fact that (S48) is invariant under the permutation of the N particles on the left and the N particles on the right. Representing the Kroenecker δ as

$$\delta_{x,y} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{i(x-y)\theta}, \quad (\text{S50})$$

we get

$$P_t(X|\{y_i\}) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \sum_{n=-N}^N G_t(X|y_n) \prod_{j \neq n} \int_0^{\infty} (e^{i\theta} G_t(z|y_i - X) + e^{-i\theta} G_t(-z|y_i - X)) dz. \quad (\text{S51})$$

We can proceed similarly to study the average density of particles with a tracer at position X :

$$\langle \rho(x, t) \delta(x_0 - X) \rangle = \sum_{q \neq 0} \langle \delta(x - x_q) \delta(X - x_0) \rangle = \frac{(2N + 1)!}{(N!)^2} \sum_{q \neq 0} \prod_{n=1}^N \int_{-\infty}^X dx_{-n} \int_X^{\infty} dx_n K_t(\{x_i\}|\{y_i\}) \delta(x - x_q). \quad (\text{S52})$$

Splitting the sum over $q > 0$ and $q < 0$, we get

$$\begin{aligned} \langle \rho(x, t) \delta(x_0 - X) \rangle_E = & \\ & \frac{\Theta(X - x)}{(N!)^2} \sum_{q < 0} \sum_{\sigma} G_t(X|y_{\sigma(0)}) G_t(x|y_{\sigma(q)}) \prod_{n=-N, n \neq q}^{-1} \left(\int_{-\infty}^X dz G_t(z|y_{\sigma(n)}) \right) \prod_{n=1}^N \left(\int_X^{\infty} dz G_t(z|y_{\sigma(n)}) \right) \\ & + \frac{\Theta(x - X)}{(N!)^2} \sum_{q > 0} \sum_{\sigma} G_t(X|y_{\sigma(0)}) G_t(x|y_{\sigma(q)}) \prod_{n=-N}^{-1} \left(\int_{-\infty}^X dz G_t(z|y_{\sigma(n)}) \right) \prod_{n=1, n \neq q}^N \left(\int_X^{\infty} dz G_t(z|y_{\sigma(n)}) \right). \quad (\text{S53}) \end{aligned}$$

As we did for the tracer only, we can get rid of the sum over the permutations by labelling the permuted indices $\sigma(n) = j$, and introducing auxilliary variables $\varepsilon_j = \pm 1$ to keep track of the signs which originate from the integrals to the left of X or to the right of X . This gives

$$\begin{aligned} \langle \rho(x, t) \delta(x_0 - X) \rangle = & \\ & \Theta(X - x) \sum_{n=-N}^N \sum_{p \neq n} G_t(X|y_n) G_t(x|y_p) \sum_{\{\varepsilon_j\}, \varepsilon_p = -1} \delta_{\sum_{j \neq n} \varepsilon_j, 0} \prod_{i \neq n, p} \int_0^\infty G_t(\varepsilon_i z | y_i - X) dz \\ & + \Theta(x - X) \sum_{n=-N}^N \sum_{p \neq n} G_t(X|y_n) G_t(x|y_p) \sum_{\{\varepsilon_j\}, \varepsilon_p = +1} \delta_{\sum_{j \neq n} \varepsilon_j, 0} \prod_{i \neq n, p} \int_0^\infty G_t(\varepsilon_i z | y_i - X) dz. \end{aligned} \quad (\text{S54})$$

Introducing again the integral representation of the Kroenecker delta, we obtain

$$\begin{aligned} \langle \rho(x, t) \delta(x_0 - X) \rangle = & \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} (e^{-i\theta} \Theta(X - x) + e^{i\theta} \Theta(x - X)) \sum_{n=-N}^N G_t(X|y_n) \sum_{p \neq n} G_t(x|y_p) \\ & \times \prod_{j \neq n, p} \int_0^\infty (e^{i\theta} G_t(z|y_i - X) + e^{-i\theta} G_t(-z|y_i - X)) dz. \end{aligned} \quad (\text{S55})$$

We can easily check that, by integrating over x , we recover $2N$ times (S51), as it should.

B. Average over the initial positions

We now average over the initial positions of the particles. We consider that the particles are distributed according to a density ρ_0 , such that

$$\int_0^\infty \rho_0(y) dy = N, \quad \int_{-\infty}^0 \rho_0(y) dy = N. \quad (\text{S56})$$

On each side of the tracer, the particles are indistinguishable, so we write the average over the positions y_n as

$$\overline{(\dots)} = \int_{-\infty}^0 \prod_{n=1}^N \frac{\rho_0(y_{-n}) dy_{-n}}{N} \int_0^\infty \prod_{n=1}^N \frac{\rho_0(y_n) dy_n}{N} (\dots). \quad (\text{S57})$$

We can thus compute

$$\begin{aligned} \overline{P_t(X|\{y_i\})} = & \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \left\{ G_t(X, 0) \left[\int_{-\infty}^0 \frac{dy}{N} \rho_0(y) \tilde{G}_t(X, y, \theta) \right]^N \left[\int_0^\infty \frac{dy}{N} \rho_0(y) \tilde{G}_t(X, y, \theta) \right]^N \right. \\ & \sum_{n=-N}^{-1} \int_{-\infty}^0 \frac{dy_{-n}}{N} \rho_0(y_{-n}) G_t(X|y_{-n}) \left[\int_{-\infty}^0 \frac{dy}{N} \rho_0(y) \tilde{G}_t(X, y, \theta) \right]^{N-1} \tilde{G}_t(X, 0, \theta) \left[\int_0^\infty \frac{dy}{N} \rho_0(y) \tilde{G}_t(X, y, \theta) \right]^N \\ & \left. + \sum_{n=1}^N \int_0^\infty \frac{dy_n}{N} \rho_0(y_n) G_t(X|y_n) \left[\int_{-\infty}^0 \frac{dy}{N} \rho_0(y) \tilde{G}_t(X, y, \theta) \right]^N \left[\int_0^\infty \frac{dy}{N} \rho_0(y) \tilde{G}_t(X, y, \theta) \right]^{N-1} \tilde{G}_t(X, 0, \theta) \right\}, \end{aligned} \quad (\text{S58})$$

with

$$\tilde{G}_t(X, y, \theta) = \int_0^\infty (e^{i\theta} G_t(z|y - X) + e^{-i\theta} G_t(-z|y - X)) dz. \quad (\text{S59})$$

We can write this expression in a more compact form, as

$$\begin{aligned} \overline{P_t(X|\{y_i\})} = & \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \left[\int_{-\infty}^0 \frac{dy}{N} \rho_0(y) \tilde{G}_t(X, y, \theta) \right]^N \left[\int_0^\infty \frac{dy}{N} \rho_0(y) \tilde{G}_t(X, y, \theta) \right]^N \left\{ G_t(X, 0) \right. \\ & \left. + N \tilde{G}_t(X, 0, \theta) \frac{\int_{-\infty}^0 dy \rho_0(y) G_t(X|y)}{\int_{-\infty}^0 dy \rho_0(y) \tilde{G}_t(X, y, \theta)} + N \tilde{G}_t(X, 0, \theta) \frac{\int_0^\infty dy \rho_0(y) G_t(X|y)}{\int_0^\infty dy \rho_0(y) \tilde{G}_t(X, y, \theta)} \right\}. \end{aligned} \quad (\text{S60})$$

In order to take the limit $N \rightarrow \infty$, we need to ensure the convergence of the integrals. We thus rewrite

$$\int_{-\infty}^0 \frac{dy}{N} \rho_0(y) \tilde{G}_t(X, y, \theta) = e^{-i\theta} + \frac{e^{i\theta} - e^{-i\theta}}{N} \int_{-\infty}^0 dy \rho_0(y) \int_0^{\infty} dz G_t(z|y - X), \quad (\text{S61})$$

$$\int_0^{\infty} \frac{dy}{N} \rho_0(y) \tilde{G}_t(X, y, \theta) = e^{i\theta} + \frac{e^{-i\theta} - e^{i\theta}}{N} \int_0^{\infty} dy \rho_0(y) \int_0^{\infty} dz G_t(-z|y - X), \quad (\text{S62})$$

which are convergent for $N \rightarrow \infty$. We thus get,

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle P_t(X|\{y_i\}) \rangle_I &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \left\{ G_t(X, 0) \right. \\ &\quad \left. + \tilde{G}_t(X, 0, \theta) \left[e^{i\theta} \int_{-\infty}^0 dy \rho_0(y) G_t(X|y) + e^{-i\theta} \int_0^{\infty} dy \rho_0(y) G_t(X|y) \right] \right\} \\ &\quad \exp \left[(e^{2i\theta} - 1) \int_{-\infty}^0 dy \rho_0(y) \int_0^{\infty} dz G_t(z|y - X) + (e^{-2i\theta} - 1) \int_0^{\infty} dy \rho_0(y) \int_0^{\infty} dz G_t(-z|y - X) \right]. \quad (\text{S63}) \end{aligned}$$

Let us write this expression in a more compact form, as

$$\boxed{P_t(X) = \lim_{N \rightarrow \infty} \overline{P_t(X|\{y_i\})} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} f_{X,t}(\theta) e^{\phi_{X,t}(\theta)},} \quad (\text{S64})$$

with

$$\phi_{X,t} = (e^{2i\theta} - 1) \int_{-\infty}^0 dy \rho_0(y) \int_0^{\infty} dz G_t(z|y - X) + (e^{-2i\theta} - 1) \int_0^{\infty} dy \rho_0(y) \int_0^{\infty} dz G_t(-z|y - X), \quad (\text{S65})$$

$$f_{X,t}(\theta) = G_t(X, 0) + \tilde{G}_t(X, 0, \theta) \left[e^{i\theta} \int_{-\infty}^0 dy \rho_0(y) G_t(X|y) + e^{-i\theta} \int_0^{\infty} dy \rho_0(y) G_t(X|y) \right]. \quad (\text{S66})$$

This coincides with the expression given in [S9] for constant initial density ρ_0 . Proceeding similarly from (S55), we get

$$\begin{aligned} \overline{\langle \rho(x, t) \delta(x_0 - X) \rangle} &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} (e^{-i\theta} \Theta(X - x) + e^{i\theta} \Theta(x - X)) \\ &\quad \times \left[\int_{-\infty}^0 \frac{dy}{N} \rho_0(y) \tilde{G}_t(X, y, \theta) \right]^N \left[\int_0^{\infty} \frac{dy}{N} \rho_0(y) \tilde{G}_t(X, y, \theta) \right]^N \\ &\quad \left\{ G_t(X, 0) \left[N \frac{\int_{-\infty}^0 dy \rho_0(y) G_t(x|y)}{\int_{-\infty}^0 dy \rho_0(y) \tilde{G}_t(X, y, \theta)} + N \frac{\int_0^{\infty} dy \rho_0(y) G_t(x|y)}{\int_0^{\infty} dy \rho_0(y) \tilde{G}_t(X, y, \theta)} \right] \right. \\ &\quad \left. + G_t(x, 0) \left[N \frac{\int_{-\infty}^0 dy \rho_0(y) G_t(X|y)}{\int_{-\infty}^0 dy \rho_0(y) \tilde{G}_t(X, y, \theta)} + N \frac{\int_0^{\infty} dy \rho_0(y) G_t(X|y)}{\int_0^{\infty} dy \rho_0(y) \tilde{G}_t(X, y, \theta)} \right] \right. \\ &\quad \left. + N \tilde{G}_t(X, 0, \theta) \frac{\int_{-\infty}^0 dy \rho_0(y) G_t(X|y)}{\int_{-\infty}^0 dy \rho_0(y) \tilde{G}_t(X, y, \theta)} \left[(N-1) \frac{\int_{-\infty}^0 dy \rho_0(y) G_t(x|y)}{\int_{-\infty}^0 dy \rho_0(y) \tilde{G}_t(X, y, \theta)} + N \frac{\int_0^{\infty} dy \rho_0(y) G_t(x|y)}{\int_0^{\infty} dy \rho_0(y) \tilde{G}_t(X, y, \theta)} \right] \right. \\ &\quad \left. + N \tilde{G}_t(X, 0, \theta) \frac{\int_0^{\infty} dy \rho_0(y) G_t(X|y)}{\int_0^{\infty} dy \rho_0(y) \tilde{G}_t(X, y, \theta)} \left[N \frac{\int_{-\infty}^0 dy \rho_0(y) G_t(x|y)}{\int_{-\infty}^0 dy \rho_0(y) \tilde{G}_t(X, y, \theta)} + (N-1) \frac{\int_0^{\infty} dy \rho_0(y) G_t(x|y)}{\int_0^{\infty} dy \rho_0(y) \tilde{G}_t(X, y, \theta)} \right] \right\}. \quad (\text{S67}) \end{aligned}$$

Using the same large N analysis as before, we get

$$\begin{aligned} \lim_{N \rightarrow \infty} \overline{\langle \rho(x, t) \delta(x_0 - X) \rangle} &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} (e^{-i\theta} \Theta(X - x) + e^{i\theta} \Theta(x - X)) \\ &\exp \left[(e^{2i\theta} - 1) \int_{-\infty}^0 dy \rho_0(y) \int_0^{\infty} dz G_t(z|y - X) + (e^{-2i\theta} - 1) \int_0^{\infty} dy \rho_0(y) \int_0^{\infty} dz G_t(-z|y - X) \right] \\ &\left\{ \left[G_t(X, 0) + \tilde{G}_t(X, 0, \theta) \left(e^{i\theta} \int_{-\infty}^0 dy \rho_0(y) G_t(X|y) + e^{-i\theta} \int_0^{\infty} dy \rho_0(y) G_t(X|y) \right) \right] \right. \\ &\quad \times \left[e^{i\theta} \int_{-\infty}^0 dy \rho_0(y) G_t(x|y) + e^{-i\theta} \int_0^{\infty} dy \rho_0(y) G_t(x|y) \right] \\ &\quad \left. + G_t(x, 0) \left[e^{i\theta} \int_{-\infty}^0 dy \rho_0(y) G_t(X|y) + e^{-i\theta} \int_0^{\infty} dy \rho_0(y) G_t(X|y) \right] \right\}. \end{aligned} \quad (\text{S68})$$

Isolating the dependence in x , we can write the conditional profile in a compact form as

$$\boxed{\lim_{N \rightarrow \infty} \frac{\overline{\langle \rho(x, t) \delta(x_0 - X) \rangle}}{\overline{\langle \delta(x_0 - X) \rangle}} = \alpha_t^{\pm}(X) G_t(x|0) + \beta_t^{\pm}(X) \int_{\mathbb{R}^{\mp}} \rho_0(y) G_t(x|y) dy + \int_{\mathbb{R}^{\pm}} \rho_0(y) G_t(x|y) dy,} \quad (\text{S69})$$

with the \pm signs corresponding to $x \gtrless X$

$$\alpha_t^{\pm}(X) = \frac{\int_{-\pi}^{\pi} d\theta e^{\pm i\theta} g_{X,t}(\theta) e^{\phi_{X,t}(\theta)}}{\int_{-\pi}^{\pi} d\theta f_{X,t}(\theta) e^{\phi_{X,t}(\theta)}}, \quad \beta_t^{\pm}(X) = \frac{\int_{-\pi}^{\pi} d\theta e^{\pm 2i\theta} f_{X,t}(\theta) e^{\phi_{X,t}(\theta)}}{\int_{-\pi}^{\pi} d\theta f_{X,t}(\theta) e^{\phi_{X,t}(\theta)}}, \quad (\text{S70})$$

with

$$g_{X,t}(\theta) = e^{i\theta} \int_{-\infty}^0 dy \rho_0(y) G_t(X|y) + e^{-i\theta} \int_0^{\infty} dy \rho_0(y) G_t(X|y). \quad (\text{S71})$$

For a step initial density profile, G_t invariant by translation and symmetric, these expressions reduce to the ones given in the main text.

V. THE CASE OF TWO TRACERS

Let us consider the joint distribution of the positions of two tracers, denoted $x_0(t)$ and $x_K(t)$, with $x_0(t) < x_K(t)$ without loss of generality. We assume that there are initially $K - 1$ particles between the two tracers, at positions $x_n(t)$, $1 \leq n < K$. Initially, we have $x_0(0) = 0$, and $x_K(0) = Y_0$. As in the previous case, we place N particles on the left of x_0 , denoted x_{-n} for $0 < n \leq N$, and N particles on the right of x_K , denoted x_n with $K < n \leq N + K$. We consider that these particles are initially distributed according to a density ρ_0 such that

$$\int_{-\infty}^0 \rho_0 = N, \quad \int_0^{Y_0} \rho_0 = K - 1, \quad \int_{Y_0}^{\infty} \rho_0 = N. \quad (\text{S72})$$

We will compute the averages over the time evolution of the particles, and their initial positions for finite N , and then let $N \rightarrow \infty$.

A. Averaging over the time evolution

The computations are similar to the case of one tracer, except that we must now enforce two constraints: the number N of particles on the left of x_0 is fixed, and the number N of particles to the right of x_K is also fixed. The total number of particles being conserved by the joint propagator, this automatically enforces that there are $K - 1$ particles between x_0 and x_K . We again denote $\{y_i\}$ the initial positions of the particles. We get,

$$\begin{aligned} P_t(X, Y | \{y_i\}) &\equiv \langle \delta(X - x_0(t)) \delta(Y - x_K(t)) \rangle \\ &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-iN\theta} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} e^{-iN\varphi} \sum_{n=-N}^{N+K} G_t(X|y_n) \sum_{m \neq n} G_t(Y|y_m) \prod_{j \neq n, m} (I_L(y_j) e^{i\theta} + I_C(y_j) + I_R(y_j) e^{i\varphi}), \end{aligned} \quad (\text{S73})$$

where we have defined

$$I_L(y) = \int_{-\infty}^X G_t(z|y)dz, \quad I_C(y) = \int_X^Y G_t(z|y)dz, \quad I_R(y) = \int_Y^{\infty} G_t(z|y)dz. \quad (\text{S74})$$

The integrals over the two phases in (S73) enforce the two conservation constraints. Eq. (S73) is the analogous to Eq. (S51) obtained for one tracer. This reasoning can be extended to an arbitrary number of tracers. The same procedure can be used to obtain

$$\begin{aligned} \langle \delta(X - x_0(t))\delta(Y - x_K(t))\rho(x, t) \rangle &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-iN\theta} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} e^{-iN\varphi} (e^{i\theta}\Theta(X-x) + \Theta(x-X)\Theta(Y-x) + e^{i\varphi}\Theta(x-Y)) \\ &\quad \sum_p G_t(x|y_p) \sum_{n \neq p} G_t(X|y_n) \sum_{m \neq n, p} G_t(Y|y_m) \prod_{j \neq p, n, m} (I_L(y_j)e^{i\theta} + I_C(y_j) + I_R(y_j)e^{i\varphi}). \end{aligned} \quad (\text{S75})$$

B. Averaging over the initial condition

We must now compute the average over the initial positions $\{y_i\}$, defined as

$$\overline{f(\{y_i\})} = \int_{-\infty}^0 \prod_{n=1}^N \frac{dy_{-n}}{N} \int_0^{Y_0} \prod_{n=1}^K \frac{dy_n}{K-1} \int_{Y_0}^{\infty} \prod_{n=K+1}^{K+N} \frac{dy_n}{N} f(y_{-N}, \dots, y_{-1}, 0, y_1, \dots, y_{K-1}, Y_0, y_{K+1}, \dots, y_{N+K}). \quad (\text{S76})$$

Performing this averaging for the probability (S73), we obtain,

$$\begin{aligned} P_t(X, Y) &\equiv \overline{P_t(X, Y|\{y_i\})} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-iN\theta} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} e^{-iN\varphi} [J_L]^N [J_C]^{K-1} [J_R]^N \left\{ G_t(X|0)G_t(Y|Y_0) + G_t(X|Y_0)G_t(Y|0) \right. \\ &+ G_t(X|0) \left[\frac{\int_{-\infty}^0 \rho_0(y)G_t(Y|y)dy}{J_L} + \frac{\int_0^{Y_0} \rho_0(y)G_t(Y|y)dy}{J_C} + \frac{\int_{Y_0}^{\infty} \rho_0(y)G_t(Y|y)dy}{J_R} \right] [I_L(Y_0)e^{i\theta} + I_C(Y_0) + I_R(Y_0)e^{i\varphi}] \\ &+ G_t(X|Y_0) \left[\frac{\int_{-\infty}^0 \rho_0(y)G_t(Y|y)dy}{J_L} + \frac{\int_0^{Y_0} \rho_0(y)G_t(Y|y)dy}{J_C} + \frac{\int_{Y_0}^{\infty} \rho_0(y)G_t(Y|y)dy}{J_R} \right] [I_L(0)e^{i\theta} + I_C(0) + I_R(0)e^{i\varphi}] \\ &+ G_t(Y|0) \left[\frac{\int_{-\infty}^0 \rho_0(y)G_t(Y|y)dy}{J_L} + \frac{\int_0^{Y_0} \rho_0(y)G_t(Y|y)dy}{J_C} + \frac{\int_{Y_0}^{\infty} \rho_0(y)G_t(Y|y)dy}{J_R} \right] [I_L(Y_0)e^{i\theta} + I_C(Y_0) + I_R(Y_0)e^{i\varphi}] \\ &+ G_t(Y|Y_0) \left[\frac{\int_{-\infty}^0 \rho_0(y)G_t(Y|y)dy}{J_L} + \frac{\int_0^{Y_0} \rho_0(y)G_t(Y|y)dy}{J_C} + \frac{\int_{Y_0}^{\infty} \rho_0(y)G_t(Y|y)dy}{J_R} \right] [I_L(0)e^{i\theta} + I_C(0) + I_R(0)e^{i\varphi}] \\ &\quad + [I_L(0)e^{i\theta} + I_C(0) + I_R(0)e^{i\varphi}] [I_L(Y_0)e^{i\theta} + I_C(Y_0) + I_R(Y_0)e^{i\varphi}] \times \left[\right. \\ &\quad \frac{\int_{-\infty}^0 \rho_0(y)G_t(X|y)dy}{J_L} \left(\frac{N-1}{N} \frac{\int_{-\infty}^0 \rho_0(y)G_t(Y|y)dy}{J_L} + \frac{\int_0^{Y_0} \rho_0(y)G_t(Y|y)dy}{J_C} + \frac{\int_{Y_0}^{\infty} \rho_0(y)G_t(Y|y)dy}{J_R} \right) \\ &\quad + \frac{\int_0^{Y_0} \rho_0(y)G_t(Y|y)dy}{J_C} \left(\frac{\int_{-\infty}^0 \rho_0(y)G_t(Y|y)dy}{J_L} + \frac{K-2}{K-1} \frac{\int_0^{Y_0} \rho_0(y)G_t(Y|y)dy}{J_C} + \frac{\int_{Y_0}^{\infty} \rho_0(y)G_t(Y|y)dy}{J_R} \right) \\ &\quad \left. + \frac{\int_{Y_0}^{\infty} \rho_0(y)G_t(Y|y)dy}{J_R} \left(\frac{\int_{-\infty}^0 \rho_0(y)G_t(Y|y)dy}{J_L} + \frac{\int_0^{Y_0} \rho_0(y)G_t(Y|y)dy}{J_C} + \frac{N-1}{N} \frac{\int_{Y_0}^{\infty} \rho_0(y)G_t(Y|y)dy}{J_R} \right) \right] \Big\} \quad (\text{S77}) \end{aligned}$$

where

$$J_L = \int_{-\infty}^0 \frac{dy}{N} \rho_0(y) (I_L(y)e^{i\theta} + I_C(y) + I_R(y)e^{i\varphi}), \quad J_R = \int_{Y_0}^{\infty} \frac{dy}{N} \rho_0(y) (I_L(y)e^{i\theta} + I_C(y) + I_R(y)e^{i\varphi}), \quad (\text{S78})$$

$$J_C = \int_0^{Y_0} \frac{dy}{K-1} \rho_0(y) (I_L(y)e^{i\theta} + I_C(y) + I_R(y)e^{i\varphi}). \quad (\text{S79})$$

To take the thermodynamic limit $N \rightarrow \infty$, we must regularise the expressions of J_L and J_R as

$$J_L = e^{i\theta} + \int_{-\infty}^0 \frac{dy}{N} \rho_0(y) [(I_L(y) - 1)e^{i\theta} + I_C(y) + I_R(y)e^{i\varphi}] , \quad (\text{S80})$$

$$J_R = e^{i\varphi} + \int_{Y_0}^{\infty} \frac{dy}{N} \rho_0(y) [I_L(y)e^{i\theta} + I_C(y) + (I_R(y) - 1)e^{i\varphi}] . \quad (\text{S81})$$

Taking the limit $N \rightarrow \infty$ yields

$$\boxed{\lim_{N \rightarrow \infty} P_t(X, Y) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} f_t(X, Y, Y_0, \theta, \varphi) e^{\phi_t(X, Y, Y_0, \theta, \varphi)} ,} \quad (\text{S82})$$

where

$$\begin{aligned} \phi_t(X, Y, \theta, \varphi) = & \int_{-\infty}^0 dy \rho_0(y) [(I_L(y) - 1) + I_C(y)e^{-i\theta} + I_R(y)e^{i(\varphi - \theta)}] \\ & + \int_{Y_0}^{\infty} dy \rho_0(y) [I_L(y)e^{i(\theta - \varphi)} + I_C(y)e^{-i\varphi} + (I_R(y) - 1)] , \end{aligned} \quad (\text{S83})$$

$$\begin{aligned} f_t(X, Y, Y_0, \theta, \varphi) = & [J_C]^{K-1} \left\{ G_t(X|0)G_t(Y|Y_0) + G_t(X|Y_0)G_t(Y|0) \right. \\ & + \left(e^{-i\theta} \int_{-\infty}^0 \rho_0(y)G_t(Y|y)dy + \frac{\int_0^{Y_0} \rho_0(y)G_t(Y|y)dy}{J_C} + e^{-i\varphi} \int_{Y_0}^{\infty} \rho_0(y)G_t(Y|y)dy \right) \\ & \times \left[G_t(X|0) [I_L(Y_0)e^{i\theta} + I_C(Y_0) + I_R(Y_0)e^{i\varphi}] + G_t(X|Y_0) [I_L(0)e^{i\theta} + I_C(0) + I_R(0)e^{i\varphi}] \right. \\ & \left. + G_t(Y|0) [I_L(Y_0)e^{i\theta} + I_C(Y_0) + I_R(Y_0)e^{i\varphi}] + G_t(Y|Y_0) [I_L(0)e^{i\theta} + I_C(0) + I_R(0)e^{i\varphi}] \right] \\ & + [I_L(0)e^{i\theta} + I_C(0) + I_R(0)e^{i\varphi}] [I_L(Y_0)e^{i\theta} + I_C(Y_0) + I_R(Y_0)e^{i\varphi}] \times \left[\right. \\ & \left. \left(e^{-i\theta} \int_{-\infty}^0 \rho_0(y)G_t(Y|y)dy + \frac{\int_0^{Y_0} \rho_0(y)G_t(Y|y)dy}{J_C} + e^{-i\varphi} \int_{Y_0}^{\infty} \rho_0(y)G_t(Y|y)dy \right) \right. \\ & \left. \times \left(e^{-i\theta} \int_{-\infty}^0 \rho_0(y)G_t(X|y)dy + e^{-i\varphi} \int_{Y_0}^{\infty} \rho_0(y)G_t(Y|y)dy \right) \right. \\ & \left. + \frac{\int_0^{Y_0} \rho_0(y)G_t(Y|y)dy}{J_C} \left(e^{-i\theta} \int_{-\infty}^0 \rho_0(y)G_t(Y|y)dy + \frac{K-2}{K-1} \frac{\int_0^{Y_0} \rho_0(y)G_t(Y|y)dy}{J_C} + e^{-i\varphi} \int_{Y_0}^{\infty} \rho_0(y)G_t(Y|y)dy \right) \right] \left. \right\} . \end{aligned} \quad (\text{S84})$$

Similarly, one can compute from (S75) the conditional profile, which takes the form

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\langle \rho(x, t) \delta(x_0(t) - X) \delta(x_K(t) - Y) \rangle}{\langle \delta(x_0(t) - X) \delta(x_K(t) - Y) \rangle} = & a_t^{(i)}(X, Y|Y_0)G_t(x|0) + b_t^{(i)}(X, Y|Y_0)G_t(x|Y_0) \\ & + c_t^{(i)}(X, Y|Y_0) \int_{-\infty}^0 \rho_0(y)G_t(x|y)dy + d_t^{(i)}(X, Y|Y_0) \int_0^{Y_0} \rho_0(y)G_t(x|y)dy + e_t^{(i)}(X, Y|Y_0) \int_{Y_0}^{\infty} \rho_0(y)G_t(x|y)dy , \end{aligned} \quad (\text{S85})$$

where the superscripts $i = L, C, R$ respectively stand for $x < X$, $X < x < Y$, $Y < x$. The coefficients a_t , b_t , c_t , d_t and e_t are given by ratios of integrals of functions of the form (S84), which are rather cumbersome, so we do not write them explicitly here. The important point is that these coefficients do not depend on x , so that the dependence of the conditional profile in x is simply given by the propagator or its integral with the initial density.

C. Long time limit: alternative derivation

The joint distribution of the two tracers simplifies in the long time limit, if the propagator has, a scaling form

$$G_t(x|y) = \frac{1}{\sigma_t} g\left(\frac{x-y}{\sigma_t}\right), \quad (\text{S86})$$

with $\sigma_t \rightarrow \infty$ when $t \rightarrow \infty$. Let us also rescale the different parameters by σ_t ,

$$X = \xi \sigma_t, \quad Y = \xi' \sigma_t, \quad Y_0 = z \sigma_t. \quad (\text{S87})$$

One could use these scalings into (S82) and evaluate the integrals with a saddle point method. This is however quite tricky to do in practice. Instead, one could use an alternative approach, used in [S10, S11] with one tracer. Here, we extend this method to two tracers. The main idea is to consider the generalised current

$$J_t(X|X_0) = \sum_n [\Theta(x_i(t) - X) - \Theta(x_i(0) - X_0)] \quad (\text{S88})$$

This observable measures the difference between the number of particles at the right of X at time t and the number of particles to the right of X_0 at $t = 0$. We define the tracer to be the first particle at the right of X_0 at $t = 0$. Its position random, but this effect will be negligible in the long time limit because of the rescaling of space (S87). Since the order of the particles is conserved, the position of the tracer at time t can be found by finding X such that $J_t(X|X_0) = 0$. The distribution of the position of the tracer is obtained, in the long time limit, by $P_t(X) = \mathbb{P}[J_t(X|X_0) = 0]$ [S10, S11].

In the case of two tracers, we need the joint distribution of two currents,

$$P_t(X, Y|X_0, Y_0) = \mathbb{P}[J_t(X|X_0) = 0 \text{ and } J_t(Y|Y_0) = 0]. \quad (\text{S89})$$

In the following, we will set $X_0 = 0$, and assume $Y_0 > 0$, and thus $Y > X$.

The first step is to consider the joint cumulant generating function of the two currents,

$$\psi(\lambda, \mu) \equiv \lim_{N \rightarrow \infty} \ln \overline{\langle e^{\lambda J_t(X|0) + \mu J_t(Y|Y_0)} \rangle} = \int dy \rho_0(y) \int G_t(x|y) (e^{\lambda(\Theta(x-X) - \Theta(y)) + \mu(\Theta(x-Y) - \Theta(y-Y_0))} - 1) dx, \quad (\text{S90})$$

which follows directly from (S10) with $f(x) = \lambda\Theta(x-X) + \mu\Theta(x-Y)$ and $g(x) = \lambda\Theta(x) + \mu\Theta(x-Y_0)$. Using the scaling form (S86) for the propagator, with the definitions (S87), we get, in the case of a constant density $\rho_0(x) = \rho$,

$$\begin{aligned} \psi(\lambda, \mu) = \rho \sigma_t \{ & (e^{-\lambda} - 1)[h(-\xi) - h(z - \xi)] + (e^{-\lambda - \mu} - 1)h(z - \xi) + (e^\lambda - 1)[h(\xi) - h(\xi')] \\ & + (e^{-\mu} - 1)[h(z - \xi') - h(z - \xi)] + (e^{\lambda + \mu} - 1)h(\xi') + (e^\mu - 1)[h(\xi' - z) - h(\xi')] \}, \end{aligned} \quad (\text{S91})$$

where we have introduced the double primitive of the propagator

$$h(x) = \int_x^\infty du \int_u^\infty dv g(v). \quad (\text{S92})$$

The joint distribution of the two currents can be obtained by a double Laplace inversion (in λ and μ) of $e^{\sigma_t \psi(\lambda, \mu)}$. In the long time limit, the integration can be performed by a saddle point method, which reduces to a Legendre transform,

$$\mathbb{P}[J_t(X|0) = \sigma_t q \text{ and } J_t(Y|Y_0) = \sigma_t q'] \simeq e^{-\sigma_t \phi(q, q')}, \quad \phi(q, q') = -\psi(\lambda^*, \mu^*) + \lambda^* q + \mu^* q', \quad (\text{S93})$$

where λ^* and μ^* are given by

$$\partial_\lambda \psi(\lambda, \mu)|_{\lambda^*, \mu^*} = q, \quad \partial_\mu \psi(\lambda, \mu)|_{\lambda^*, \mu^*} = q'. \quad (\text{S94})$$

The joint distribution of the two tracers is obtained by setting $q = q' = 0$, and takes the large deviation form,

$$P_t(X, Y|0, Y_0) \simeq e^{-\sigma_t \phi(0, 0)}. \quad (\text{S95})$$

We can obtain the behaviour of the distribution near around the mean values $X = 0$ and $Y = Y_0$ by expanding λ^* and μ^* in powers of ξ and $\xi' - z$. This gives at leading order a quadratic behaviour for ϕ ,

$$\phi(0, 0) \simeq \frac{1}{2} (\xi \quad \xi' - z) \begin{pmatrix} \frac{h(0)}{2h'(0)^2} & \frac{h(0)}{2h'(z)^2} \\ \frac{h(0)}{2h'(z)^2} & \frac{h(0)}{2h'(0)^2} \end{pmatrix}^{-1} \begin{pmatrix} \xi \\ \xi' - z \end{pmatrix}. \quad (\text{S96})$$

This gives a Gaussian behaviour around the average values, with the fluctuations directly obtained from the covariance matrix,

$$\boxed{\text{Var}(X) = \text{Var}(Y) = \frac{h(0)}{2h'(0)^2}\sigma_t, \quad \text{Cov}(X, Y) = \frac{h(z)}{2h'(0)^2}\sigma_t.} \quad (\text{S97})$$

VI. CONDITIONAL PROFILES VS CORRELATION PROFILES

We have introduced two equivalent ways to quantify the statistical properties of the position $x_0(t)$ of the tracer, and its correlations with the density $\rho(x, t)$ of particles:

(i) the cumulant generating function and the correlation profile

$$\psi_A(\lambda, t) = \ln \overline{\langle e^{\lambda x_0(t)} \rangle}, \quad w_A(x, \lambda, t) = \frac{\overline{\langle \rho(x, t) e^{\lambda x_0(t)} \rangle}}{\overline{\langle e^{\lambda x_0(t)} \rangle}}, \quad (\text{S98})$$

(ii) the distribution and the conditional profile

$$P_t(X) = \overline{\langle \delta(X - x_0(t)) \rangle}, \quad \tilde{w}_A(x, X, t) = \frac{\overline{\langle \rho(x, t) \delta(X - x_0(t)) \rangle}}{\overline{\langle \delta(X - x_0(t)) \rangle}}. \quad (\text{S99})$$

The two formulations are related by Laplace transforms,

$$e^{\psi_A(\lambda, t)} = \int_{-\infty}^{\infty} e^{\lambda X} P_t(X) dX, \quad w_A(x, \lambda, t) = \frac{\int_{-\infty}^{\infty} e^{\lambda X} \tilde{w}_A(x, X, t) P_t(X) dX}{\int_{-\infty}^{\infty} e^{\lambda X} P_t(X) dX}. \quad (\text{S100})$$

The correlation profiles can therefore be obtained from the conditional profile and vice versa. We will now see that in the long time limit, this relation greatly simplifies.

We start from the scaling form of the propagator (S86), with a step initial density $\rho_0(x) = \rho_- \Theta(-x) + \rho_+ \Theta(x)$. Plugging these into the distribution (S64), we obtain that the term $\phi_{X,t}(\theta)$ in the exponential is proportional to σ_t . The integral over θ can thus be evaluated in the limit $t \rightarrow \infty$ by a saddle point method. This gives that $P_t(X)$ takes a large deviation form,

$$P_t(X) \underset{t \rightarrow \infty}{\simeq} e^{-\sigma_t \chi(\xi = X/\sigma_t)}, \quad \chi(\xi) = \left(\sqrt{\rho_+ h(-\xi)} - \sqrt{\rho_- h(\xi)} \right)^2, \quad (\text{S101})$$

where h is the double primitive of the propagator (S92). This generalises the formula given in [S12] for Brownian particles, and in [S9] for a flat density $\rho_+ = \rho_-$. Importantly, this large deviation form allows to also evaluate the integrals in (S100) by a saddle point, which gives,

$$\psi_A(\lambda, t) \underset{t \rightarrow \infty}{\simeq} \sigma_t [\lambda \xi^*(\lambda) - \chi(\xi^*(\lambda))], \quad \text{and} \quad w_A(x, \lambda, t) \underset{t \rightarrow \infty}{\simeq} \tilde{w}_A(x, \sigma_t \xi^*(\lambda), t) \quad \text{where} \quad \chi'(\xi^*(\lambda)) = \lambda. \quad (\text{S102})$$

In this limit, the conditional profile and the correlation profile are thus equal. But this is not the case at arbitrary time. It turns out that the correlation profiles are easier to compute for observables of the form (S1), while the conditional profiles are more directly obtained for the position of a tracer.

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