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Decomposition, local Purity and Perverse variation of MHS

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Preface

To Pierre Deligne and to the memory of Jean Louis Verdier

The purpose of this volume is to prove the existence of *special structures* on the cohomology of a family of complex projective varieties, namely *Lefschetz and Hodge decompositions*. We mention:

1. The construction of *perverse extension of degenerating variation of graded polarized mixed Hodge structures (VMHS)* along a normal crossing divisor (NCD).
2. The *decomposition theorem* of the derived image of a perverse extension of degenerating polarized variation Hodge structures (VHS) by a projective morphism on complex algebraic varieties by methods of algebraic topology.
3. The proof of *local purity* on complex varieties adapted from the proof on varieties of characteristic $p > 0$, and an other new proof.
4. The construction of *perverse extension of degenerating variation of polarized Hodge structures (VHS)* on compact complex varieties by application of the decomposition theorem to a desingularization.
5. The definition of perverse variation of mixed Hodge structures (VMHS) on complex algebraic varieties.

Actually, there exist two proofs of the *decomposition theorem*. The first is on algebraic varieties over fields of characteristic $p > 0$ which extends to polarized variation of Hodge structures (VHS) of geometric origin on complex varieties. The second proof is for Hodge modules and equivalently for polarized VHS on complex algebraic varieties.

We give two new proofs by methods of algebraic topology of complex algebraic varieties. The first is based on a generalization of the construction of mixed Hodge complex on the nearby cycles complex.

The second is based on a description of the weight filtration on the logarithmic complex $\Omega^*\mathcal{L} := \Omega^*_X(LogY) \otimes_{\mathcal{O}_X} \mathcal{L}_X$ with coefficients in Deligne's extension $\mathcal{L}_X$ of a VHS $\mathbf{L}$ on the complement of a NCD $Y \subset X$.

The *local purity theorem*, states conditions on the weights of a NCD contracting into an isolated stratum in a Thom-Whitney stratification. As an application of the decomposition theorem, we adapt the original unpublished proof by Deligne and Gabber to complex algebraic varieties.

In particular, we prove the *local and global invariant cycle theorems*. We give also an other direct proof of local purity.

To figure the interest in these results, let us recall the developments after 1950 in this field of algebraic geometry by the schools influenced the work of *J.P. Serre, A. Grothendieck, P. Deligne and Ph. Griffiths*.

1. Weight and Hodge filtrations. The cohomology groups of a complex algebraic variety X, constructed with the transcendental topology, are enriched by the definition of filtrations by sub-spaces reflecting the geometry and the topology of the variety and its singularities.

In the case of a compact non singular algebraic variety X, a Hodge filtration F is defined on the de Rham cohomology groups $H^i(X, \mathbb{C})$.

Using the conjugate $\bar{F}$ of F with respect to the $\mathbb{Q}-$structure of the cohomology defined by the embedding $H^i(X, \mathbb{Q}) \subset H^i(X, \mathbb{C})$, a Hodge decomposition of the cohomology is defined by the finite direct sum of subspaces $H^{p,q} = F^p \cap \bar{F}^q$ for $p + q = i$.

The Hodge filtration F is already defined on the de Rham complex but the proof of the induced decomposition on the cohomology is based on advanced analysis on varieties. Two homeomorphic varieties which are not isomorphic, give rise to two different decompositions in general, after identification of the cohomology spaces.

In the case of non singular but non compact algebraic varieties, the Hodge filtration on the de Rham complex does not give a satisfactory structure on the cohomology. Instead, it is necessary to embed a non singular X into a non singular compact algebraic variety with a normal crossing divisor (NCD) as complement (at infinity), which is possible thanks to Hironaka's desingularization theorem. The presence of the NCD at infinity leads to an additional filtration W on the cohomology with $\mathbb{Q}-$coefficients called the weight filtration.

For a first reading, the definition of all terms can be found in the following book

Claire Voisin: Hodge theory and complex algebraic geometry I, II. *Cambridge University Press*, 76, 77, 2007.

For an advanced reading, we refer to the foundational work of

Pierre Deligne: Théorie de Hodge II. *Publ. Math. IHES, Paris* 40:5–57, 1972.
Théorie de Hodge III. *Publ. Math. IHES, Paris* 44:6–77, 1975.
Conjecture de Weil II. *Publ. Math. IHES, Paris* 52, 1980.

As well, other basic references are cited in the text.

2. Mixed Hodge structure (MHS) In the case of non compact non singular varieties, Deligne introduced a new complex called the logarithmic complex on which both the weight and Hodge filtrations are defined and induce the correct filtrations on the cohomology spaces. We remark that this correct Hodge filtration contains additional information at infinity.

Both weight W and Hodge F filtrations figure in a precise positions in the cohomology space and satisfy strong linear algebraic conditions called mixed Hodge structure (MHS).

Such MHS are also defined on the cohomology of singular algebraic varieties. Thus, the cohomology of any complex algebraic variety has a naturally defined MHS. It is surprising that the theory of MHS is motivated by the concept of purity and weight on $\ell-$adic cohomology of varieties over fields of positive characteristic. In fact, the theory of MHS is motivated by Deligne's results on the Weil conjecture. A " heuristic dictionary " developed by Deligne helps to translate results from one language into the other:

Pierre Deligne: Théorie de Hodge I. *Cong. intern. math.Nice, Tome 1,* 425–430, 1970.

3. Polarization and Lefschetz decomposition of the cohomology of complex non singular projective varieties. The cohomology class of an hyperplane section of a non singular projective variety X, defines Lefschetz isomorphisms on the cohomology by iteration of the cup-product with its cohomology class. From which we deduce, with Poincaré's duality, a scalar product on the cohomology $H^i(X, \mathbb{C})$ and a Lefschetz decomposition into a direct sum of *polarized subspaces*.

Reciprocally, Deligne's proof of Lefschetz isomorphisms by induction is based on the existence of the polarization (Weil II, 4.1, Lemma 4.1.4).

4. Variation of mixed Hodge structures (VMHS). In the case of a family of algebraic varieties defined by an algebraic morphism $f : X \to V$, the cohomology of the fibres form a constructible complex: there exists a stratification $\mathcal{S}$ of the complex variety V such that the restriction of the i-th cohomology sheaf of the derived image by f of the constant sheaf $\mathbb{Q}_X$ is locally constant on each stratum S. The locally constant sheaf on S, is determined by a monodromy representation of the fundamental group of S on the i-th cohomology of the fibres.

To go beyond topology, in the case of a proper smooth family, the Hodge filtration F of the cohomology space $H^i(X_v, \mathbb{C})$ of non singular complex compact algebraic fibers X_v at $v \in V$, defines a variation of Hodge structures (VHS) on V. Griffith gave sense to the fact that the variation of the Hodge filtration is analytic, or in other terms the filtration is defined by analytic subbundles of the flat bundle $R^i f_* \mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_V$. The VHS has an additional property

called Griffith's transversality with respect to the connection determined by the local system of the i-th cohomology $R^i f_* \mathbb{C}$ on V.

In the case of a projective family, we obtain a VMHS on each stratum S of a stratification $\mathcal{S}$ of V.

In this volume we describe results on the degeneration and the extension of VMHS on the boundary of a stratum S by algebraic topology techniques in order to give a proof of the decomposition and purity theorems.

In a family, singular fibers occur in general. For example, any pencil of curves on a surface contains singular special curves. The behavior of the family near the singular fiber is useful (for example to the classification of the families).

For this reason, we introduce the concept of *degeneration of the MHS* near a singularity on the parameter space V. At a singularity $v \in V$, the degeneration reflects the properties of the family on a punctured neighborhood of v. In particular we can blow up v to study the degeneration.

For this reason, the study of the degeneration is reduced in general to the complement of a normal crossing divisor (NCD) in a non singular variety.

The singularities appear also when it is not possible to extend the local system or the Hodge filtration of the VHS on a non singular parameter variety V. For example, a singularity of the connection defined by the local system may exists on a non singular V. The problems of the degeneration of the filtration F are of analytic nature.

5. Decomposition of derived image of polarized VHS. The polarization on a VHS leads to a decomposition into irreducible polarized sub-VHS.

Moreover Lefschetz and Hodge structures on the fibers of a smooth projective morphism $f : X \to V$, lead to the degeneration of Leray's spectral sequence of f.
This degeneration may be interpreted as a decomposition (in the derived filtered category $D^+F(V, \mathbb{C})$ of abelian sheaves on V), of the complex $K := f_* \mathbb{C}_X$ on V defined by the derived image of the constant sheaf $\mathbb{C}$ on X, into its cohomology sheaves. We refer to:

P. Deligne, Théorèmes de Lefschetz et critères de dégénérescence de suites spectrales, *Publ. Math. IHES*, 35, 107–126, 1968.

6. Degeneration of VMHS. In the case of an abstract analytic polarized VHS on a punctured complex disc D^*, the main result is the existence of Schmid's limit Hodge filtration F on the space of multiple sections of the local system $\mathbf{L}$ underlying the VHS on D^*. We refer to:

Philip Griffiths and Wilfried Schmid: Recent developments in Hodge theory. *Bombay Colloquium on: Discrete subgroups of Lie Groups, Oxford University Press* 1973,
for a first reading and to the foundational work of

Wilfried Schmid: Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.* 22:211–319, 1973.

Moreover, the monodromy filtration $W(N)$ defined by the logarithm of the monodromy and the limit filtration F form a MHS. The VHS is said to degenerate into a limit MHS.

In the text Weil II by Deligne, as well in the article of Steenbrink, the theory appears for an algebraic family over curves.

Graded polarized VMHS. In the case of an abstract VMHS on D^* with filtrations W and F, there is no natural limit Hodge filtration F. The existence of a limit Hodge filtration is added by definition of a class of VMHS called admissible.

In this case Deligne defined a relative monodromy filtration $M := M(N, W)$ and asked for precise properties which must be satisfied by M and the limit F, in particular (M, F) form a MHS on the limit vector space defined by the local system underlying the VMHS on D^*.

The properties required of an admissible VMHS are inspired by the case of algebraic families and all algebraic families over a punctured disc define admissible VMHS on D^* called geometric.

We refer to Weil II on varieties over fields of characteristic $p > 0$, where many problem originated.

In the case of higher dimensional variations, the degeneration of a VHS (resp VMHS) over a product of punctured discs $(D^*)^n$ is far more difficult to study and the results present compatibility relations between the various degenerations along the components of the NCD at "infinity" in D^n.

In this text, degeneration results are recalled but not proved since there exist many references cited in the text.

7. Extension of VHS. In the case of curves, in Weil II by Deligne and in the corresponding work by Zucker on a complex curve V, an answer to the problem of extension is given along a singularity $v \in V$ of a polarized VHS on a locally constant sheaf $\mathbf{L}$.

Let $j_*\mathbf{L}$ denotes the sheaf extending $\mathbf{L}$ such that the section at v of $j_*\mathbf{L}$ are defined by the invariant subspace under the action of monodromy near v, with its induced limit MHS at v including the limit Hodge filtration F.

The main result is that the cohomology $(H^i(V, j_*\mathbf{L}), F)$ with the induced filtration F is a pure HS. This is the first example of a Hodge structure not deduced from HS defined by compact non singular varieties.

In the case where $\dim V > 1$, the main problem is the extension of a VHS defined on the big stratum S of V. For example on a surface S, the critical case is an isolated stratum on the boundary of S.

No progress has been made until the work by Goresky and Mac-Pherson in 1980 on Intersection cohomology satisfying Poincaré duality on topological spaces eventually singular, appeared.

8. Perverse sheaves. The interpretation by Deligne of Intersection cohomology in the language of derived category and Poincaré-Verdier duality, leads to the theory of intermediate extensions of a local system and more

generally to the definition of perverse sheaves developed in 1983 on varieties over fields of strictly positive characteristic. We refer to:

A. Beilinson, J. Bernstein, P. Deligne, O. Gabber: Analyse et Topologie sur les espaces singuliers Vol.I, *Astérisque, France* 100, 1983.

9. Purity. The long awaited theory of purity, well understood in the case of curves in Weil II, makes sense in all dimensions as follows:

In the case of an abstract analytic polarized VHS underlying a local system $\mathbf{L}$ on a product of punctured complex discs $(D^*)^n$, the correct extension is a complex of sheaves called the intermediate extension of $\mathbf{L}$ with induced filtration defined by the limit Hodge filtration F.

Globally, if a polarized Hodge filtration $(\mathbf{L}, F)$ is given on the complementary of a NCD $Y \subset X$, the intermediate extension is defined in this text as a sub-complex $IC^*\mathcal{L}$ of the logarithmic complex $\Omega^*\mathcal{L} := \Omega^*_X(LogY) \otimes_{\mathcal{O}_X} \mathcal{L}_X$ with coefficients in Deligne's extension $\mathcal{L}_X$ with induced filtration F deduced from the degenerate Hodge filtration.

The hypercohomology of the sub-complex $(IC^*\mathcal{L}, F)$ on a projective variety X with its induced Hodge filtration carry a pure HS.

After the development of the theory of perverse sheaves the interest shifted to the property of decomposition of pure perverse sheaves.

The first purity result in the domain is proved over fields of strictly positive characteristic by Deligne and Gabber in 1981 and has been a motivation of subsequent work on Hodge theory on complex varieties. We cite among the consequences, the proof of local and global invariant cycle theorems and local purity.

In the case of a normal crossing divisor Y in a complex algebraic variety X contracting to an isolated stratum $v \in V$ in a stratified algebraic variety V, local purity has consequences on the classical cohomology with support in Y in terms of restrictions on the weights, related to the contractibility of the NCD to a point.

Although the proofs are on varieties over fields of strictly positive characteristic, it is possible to deduce the result in the case where the coefficients are VHS of geometric origin on complex varieties.

In parallel, the relation to $D-$modules appears in the thesis of Brylinski in 1982, the work of Kashiwara in 1983 and the work of Saito who introduced Hodge modules in 1988 and proved the property of decomposition of such modules.

The theory of differential modules is a natural domain to study the degeneration and leads to the fact that Hodge modules are direct sums of variations of intermediate extensions of Hodge structures.

It is surprising that perverse sheaves have been defined and studied first in the language of derived category.

10. Perverse extension of VMHS. *A central problem* is the definition of the extension of a polarized VHS $(\mathbf{L}, F)$ on a stratum S of a stratified complex

variety V (S is non singular but V may be singular along S). We give in this text a definition of the extension of the filtration F by desingularization using algebraic topology constructions. We apply the result to develop the theory of perverse VMHS on a singular variety.

A general and useful study of extensions of perverse sheaves along a locally principal divisor by considering the nearby and vanishing cycles figures in the work of J. L. Verdier. The application to Hodge theory is written by M. Saito in the language of $D-$modules.

We develop in the text the results on the extension of graded polarized VMHS in terms of perverse sheaves and uilding on results of Kashiwara in the case of NCD, we give an alternative definition in terms of algebraic topology.

11. Proof of the decomposition and purity theorems. The main simplification in the exposition of the text consists in the proof of the decomposition and local purity theorems for a projective morphism by reduction to the case of a NCD contracting to an isolated stratum v in V (not merely an isolated singularity) and its application to define the perverse extension of a VHS in general.

For this reason, we insist in this text to state the decomposition theorem in terms of Thom stratification of a morphism $f : X \to V$ of algebraic varieties and we express the decomposition theorem in terms of the stratification. The reduction to isolated strata is by induction and restriction to a general hyperplane section.

In the case of an isolated stratum v, the proof is reduced to the case of existence of a projection of V to the projective line defined by a general hyperplane section through v. The spectral sequence of the nearby cycles functor with respect to the monodromy weight filtration is relatively simple to study in this case.

This allows the simple definition of the category of intermediate extension of variations of Hodge structures and the category of variations of mixed Hodge structures over singular varieties.

We recall, that even if we start with constant coefficients, the derived direct image consists of a complex with constructible cohomology sheaves with additional structure of intermediate extensions of polarized VHS.

Thus, instead of constant coefficients, the general theory is carried with cohomology coefficients in polarized VHS and more generally VMHS.

Organization of the book. The text is divided in three sections.

1. The first section covers three chapters.
   Chapter 1 gives the statements of the local purity theorem, the decomposition theorem in terms of Tom-Whitney stratification and the extension of a polarized VHS.
   Chapter 2 covers the preliminaries on perverse sheaves and on the degeneration of VMHS.

Chapter 3 centers on perverse sub-sheaves of the logarithmic complex in the case of a normal crossing divisor (NCD) at infinity.

2. The second section covers chapters $(4-8)$.
   In chapter 4, we introduce in the case of a unipotent VHS: $(\mathbf{L}, F)$, the nearby cycles complex $\Psi^* I\mathcal{L}$, to define the limit Hodge filtration F and transfer the filtration to the unipotent part of the nearby cycles complex $\psi_f^u j_{!*}{}^p\mathbf{L}$ with coefficients in the intermediate extension of $\mathbf{L}$.
   We prove in chapter 5 the existence of a natural structure of mixed Hodge complex on the nearby and vanishing cycles complex with coefficients in an intermediate extension $j_{!*}{}^p\mathbf{L}$ of a (shifted) polarized variation of Hodge structure $(\mathbf{L}, F)$.
   We give a proof of the decomposition and purity theorems in chapter 6 and deduce in chapter 7 the Hodge filtration on perverse intermediate extensions on singular varieties by desingularization, the local and global invariant cycle theorems.
   This allows us to adapt in chapter 8 the proof of the local purity theorem by Deligne and Gabber to the case of complex varieties.
3. The third section covers chapters $(9-12)$.
   In chapter 9, the weight filtration on the logarithmic complex in the case of a NCD at infinity is defined directly along the NCD.
   In chapter 10 a combined simple proof of the decomposition and purity theorems is given in terms of Intersection morphism and a relative interpretation of local purity.

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Chapter 1
Introduction

An overview of the domain of research is presented in the preface. This volume is divided in three sections. The first section covers chapters $1-3$.
In this chapter, the **statement of the local purity and of the decomposition theorems as well the extension of a polarized VHS** on a complex algebraic variety, are given.

In chapter 2, preliminaries on perverse sheaves and variation of mixed Hodge structures may be useful for a first reading.

We develop in chapter three the normal crossing divisor case in details. It is an example of perverse sheaves.

The Abbreviations and Conventions figure in §2.4.0.2 and the induced filtrations on perverse cohomology in §2.4.

1.1 Local purity

In an unpublished note [DeG 81], Deligne and Gabber announced the following result, known as the purity theorem on a projective variety over a field of positive characteristic:

Theorem. *Let $\mathbb{F}_q$ be a finite field, X_0 a smooth scheme of finite type over $\mathbb{F}_q$, $j : V_0 \hookrightarrow X_0$ an open subset of X_0, and K_0 a perverse sheaf on V_0. If K_0 is pure of weight w, the complex $j_{!*}K_0$ is pure of same weight.*

Let $j : V \to X$ denote the extension of $j : V_0 \to X_0$ over an algebraic closure of $\mathbb{F}_q$ and K the extension of K_0 on V. We may also say that $j_{!*}K$ is pure of same weight as K.

The proof by induction on an hyperplane section leads to the following local purity statement:

Local purity theorem. *Let E_0 be a finite set of zero dimensional points in X_0, $V_0 := X_0 \setminus E_0$, $E \subset X$ the extensions over an algebraic closure of $\mathbb{F}_q$ and K the extension on V of a pure complex K_0 of weight ${}^p w$ on V_0.*

Let B_v be a henselization of X at a zero dimensional point v of E. The weight w of the cohomology $\mathbb{H}^i(B_v \setminus \{v\}, K)$, satisfies the inequalities:

$$w \leq {}^p w + i \text{ if } i \leq -1 \text{ and } w > {}^p w + i \text{ if } i \geq 0.$$

Next, we give the interpretation and application of this theorem on complex varieties. It is a central result in the development of Hodge theory. Moreover, it is related to the proof of the decomposition theorem.

1.1.0.1 Deligne's dictionary

According to Deligne [De 71], the henselization of a complex algebraic variety V at a point v corresponds to a ball B_v, defined by a local embedding of V in $\mathbb{C}^N$ at v, as the intersection with V of a ball of $\mathbb{C}^N$ centered at v of small radius, such that B_v is a cone over a topological space called the Link L of V at v which is a topological invariant independent of the embedding in $\mathbb{C}^N$.

Purity in positive characteristic corresponds to Hodge structure (HS) on cohomology [De 72] and the theory of weight is translated into a weight filtration local purity theorem.

In particular the conditions on the weights above can be translated in the transcendental case into conditions on the weights of a natural filtration W mentioned in the text as the local purity theorem.

1.1.1 Local purity on complex varieties

The Intersection complex on a complex algebraic variety V, defined with coefficients in a local system on a non singular locally closed Zariski subset, has been introduced first by a topological construction [GMacP 83].

Later, Deligne extended the definition to algebraic varieties over finite fields (see [Br 82]). The Intersection complex appears as the Intermediate extension in the general theory of perverse sheaves ([BBD 83] Definition 2.1.5 and Proposition 2.1.11).

Let V^* denote a smooth algebraic open dense subset of a complex algebraic variety V of pure dimension, and $j : V^* \hookrightarrow V$ the open embedding. Let $\mathbf{L}$ denote a polarized variation of Hodge structures (VHS) of weight w' on V^*.

We attach to $\mathbf{L}$ the perverse sheaf ${}^p\mathbf{L} = \mathbf{L}[\dim V]$ on V^*. Due to the shift in degrees, the perverse sheaf ${}^p\mathbf{L}$ is concentrated in degree $-\dim V$ and has weight ${}^p w = w' + \dim V$. Let $j_{!*}{}^p\mathbf{L}$ denote the Intermediate extension of ${}^p\mathbf{L}$ (see §2.2), v a zero dimensional point of V, $i_v : v \to V$ and $k : (V \setminus v) \to X$.

In the text, we prove the local purity theorem in terms of the mixed Hodge structure (MHS) on the cohomology space $H^*(i_v^* k_* k^* j_{!*}{}^p\mathbf{L})$ (a derived functor RF in the text will be denoted simply by F, in particular k_* stands for Rk_*).

We find it more suggestive to say that we put a MHS on the hypercohomology of a *punctured small ball* $B_v^* := B_v \setminus v$ by fixing an isomorphism $H^*(i_v^* k_* k^* j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^*(B_v \setminus v, j_{!*}{}^p\mathbf{L})$ for B_v small enough. The local purity theorem is stated in terms of the weight w of a natural MHS as follows:

Theorem 1.1 (Local purity). *Let $j_{!*}{}^p\mathbf{L}$ be the intermediate extension of weight ${}^pw$, v a zero dimensional point of V, $i_v : v \to V$ and $k : (V \setminus v) \to X$.*

The weight w of the space $H^i(i_v^* k_* k^* j_{!*}{}^p\mathbf{L})$, isomorphic to the Intersection cohomology $\mathbb{H}^i(B_v \setminus \{v\}, j_{!*}{}^p\mathbf{L})$ of a small punctured ball B_v at v, satisfies the relations:

$$w \leq {}^pw + i \text{ if } i \leq -1 \text{ and } w > {}^pw + i \text{ if } i \geq 0$$

The proof in [DeG 81] is adapted to complex variety in chapter 8. A new proof is given in chapters 10 and 10.3. Already, in the simplest case of an isolated singularity $v \in V$, the theorem has important consequences on the weights of the cohomology with support $H_v^i(V, \mathbb{Q})$. A direct proof of these consequences appears in the work of Navarro Aznar [Na 85], where the results are treated in terms of the exceptional NCD in a desingularization and the MHS of the isolated singularity v.

1.2 Decomposition theorem and Perverse cohomology

The introduction of the category of perverse sheaves and perverse cohomology lead (see §2.2 for a summary) to vast developments in the theory and to the unification of the language and the structure of proofs in positive characteristic as well on complex algebraic varieties.

To state the decomposition theorem in ([BBD 83]) we need to introduce the perverse cohomology sheaves ${}^p\mathcal{H}^i(K)$ of a complex K on V (cf §2.2).

Let X_0 be a separated scheme of finite type over a finite field $\mathbb{F}_q$ of q elements of characteristic p, and $X := X_0 \otimes_{\mathbb{F}_q} \mathbb{F}$ its extension over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$.

Let $l \neq p$ and let $j : U \to X$ be the inclusion of a smooth connected open subset of pure dimension d and a $\overline{\mathbb{Q}}_l$-smooth irreducible sheaf ${}^p\mathbf{L}$ on U.

Let $f : X_0 \to V_0$ be a projective morphism. The **purity theorem** states the decomposition of ${}^p\mathcal{H}^j(f_* j_{!*}{}^p\mathbf{L})$ into a finite direct sum of intermediate extension on the projective variety V non necessarily smooth, of $\overline{\mathbb{Q}}_l$-smooth irreducible sheaves ${}^p\mathbf{L}_i$ on locally closed Zariski subsets $j_i : U_i \to V$ ([BBD 83], théorème 5.3.8):

$$ {}^p\mathcal{H}^j(f_* j_{!*}{}^p\mathbf{L}) \simeq \oplus_{i \in I} j_{i!*}{}^p\mathbf{L}_i \tag{1.1}$$

The **relative Hard Lefshetz theorem** ([BBD 83], théorème 5.4.10) states that the class $\ell \in H^2(X, \mathbb{Q}_l(1))$ of an hyperplane section (or a relatively ample bundle) induces isomorphisms for $i \geq 0$:

$$\ell^i : {}^p\mathcal{H}^i(f_* j_{!*} {}^p\mathbf{L}) \xrightarrow{\sim} {}^p\mathcal{H}^i(f_* j_{!*} {}^p\mathbf{L}) \tag{1.2}$$

Both equations 1.2 and 1.1 are known as the **Decomposition theorem**.

1.2.0.1 Geometric statements

To deduce results concerning *geometric* statements on schemes of finite type over $\mathbb{C}$ from corresponding statements in characteristic $p > 0$, a general procedure, described in ([BBD 83], §6), applies to perverse sheaves of geometric origin ([BBD 83], §6.2.4).

Let $f : X \to Y$ be a proper morphism and P a simple perverse sheaf over $X(\mathbb{C})$ of geometric origin, there exists a decomposition ([BBD 83], théorème 6.2.5)

$$f_* P \simeq \oplus_i {}^p\mathcal{H}^i(f_* P)[-i] \in D^b_c(V(\mathbb{C}), \mathbb{C}) \tag{1.3}$$

in the derived category [Ve 77]. Moreover, each perverse cohomology ${}^p\mathcal{H}^i(f_* P)$ decomposes into a direct sum of simple perverse sheaves of geometric origin.

Example 1.1. Let $g : Z \to Y$ be a proper morphism, by definition any simple component of ${}^p\mathcal{H}^i(g_* \mathbb{C})$ is of geometric origin.

We recommend strongly to read the full proof in chapter 6 of [BBD 83].

Indeed, the argument consists of general conditions and steps to deduce theorems on algebraic complex varieties from theorems on varieties over fields of characteristic $p > 0$, as an application of Thom-Whitney stratifications.

1.2.0.2 Analytic statements

On complex varieties, Intermediate extensions of polarized VHS correspond to pure Intermediate extensions in positive characteristic.

Purity. In the case of a polarized VHS $\mathbf{L}$ of weight w' on a Zariski open subset $j : U \to X$ complement of a normal crossing divisor (NCD) Y in a non singular complex projective variety X, the Intersection cohomology $\mathbb{H}^i(X, j_{!*}\mathbf{L})$ has an induced pure HS of weight $w' + i$ ([CaKSc 87] and [KaK 87]).

Decomposition. Let $f : X \to V$ be a projective morphism of complex algebraic varieties. There exists a decomposition of the perverse cohomology into a finite direct sum of intermediate extensions of polarized VHS ${}^p\mathbf{L}_i$ on locally closed Zariski subsets $j_i : U_i \to V$:

$${}^p\mathcal{H}^i(f_* j_{!*} {}^p\mathbf{L}) \xrightarrow{\sim} \oplus_{i \in I} \; j_{i!*} {}^p\mathbf{L}_i \tag{1.4}$$

and there exists a decomposition in the category $D^b_c(V,\mathbb{Q})$

$$f_*j_{!*}\mathbf{L} \simeq \oplus_{i\in\mathbb{Z}}{}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L})[-i]. \tag{1.5}$$

The decomposition is proved for polarized Hodge modules [Sa 88] and [Sa 90] in terms of the theory of differential modules ($\mathcal{D}_X$-modules). We give a proof inspired by [DeG 81].

1.2.1 Decomposition and Intersection morphisms

In the text we give a proof of the decomposition theorem in terms of stratifications and Intersection morphisms, by induction on the dimension of X, using Hodge theory on the complex of nearby and vanishing cycles in chapter 6. In the last chapters 10 and 11 we give a combined simple proof using the weight filtration on the logarithmic complex $\Omega^*\mathcal{L}$.

Intersection morphisms. Let $f : X \to V$ be a projective morphism of algebraic varieties. We express the results in terms of a Thom-Whitney stratification of f (Definition 2.3) adapted to $j_{!*}{}^p\mathbf{L}$ in the sense that the restriction of the cohomology groups of $j_{!*}{}^p\mathbf{L}$, to the various strata of X, are locally constant. The cohomology groups of $f_*(j_{!*}{}^p\mathbf{L})$ are locally constant when restricted to the strata of V.

Let $D^b_c(V,\mathbb{Q})$ denote the derived category of complexes with constructible cohomology sheaves on V, $X_S := f^{-1}(S)$ the inverse image of a stratum $S \in V$, $f_S : X_S \to S$ the restriction of f, $i_{X_S} : X_S \to X$ and $i_S : S \to V$ the embeddings.

The intersection morphism in the derived category $D^b_c(V,\mathbb{Q})$

$$I : i^!_{X_S}j_{!*}{}^p\mathbf{L} \to i^*_{X_S}j_{!*}{}^p\mathbf{L} \tag{1.6}$$

is defined by the composition of $i_{X_S*}i^!_{X_S}j_{!*}{}^p\mathbf{L} \to j_{!*}{}^p\mathbf{L}$ with the restriction morphism $j_{!*}{}^p\mathbf{L} \to i_{X_S*}i^*_{X_S}j_{!*}{}^p\mathbf{L}$ ([BBD 83] subsection 1.4.6 and Definition 2.1.2).

Definition 1.1. For each stratum S, the local system $\mathbf{L}^i_S$ on S is the image of the induced morphism by I:

$$\mathbf{L}^i_S := \mathrm{Im}\left\{R^{-\dim S+i}f_{S*}(i^!_{X_S}j_{!*}{}^p\mathbf{L}) \xrightarrow{I_S} R^{-\dim S+i}f_{S*}(i^*_{X_S}j_{!*}{}^p\mathbf{L})\right\} \tag{1.7}$$

such that ${}^p\mathbf{L}^i_S := \mathbf{L}^i_S[\dim S]$ is perverse.

The local systems $\mathbf{L}^i_S$ are necessarily components of any decomposition. The shift by $-\dim S$ in $R^{-\dim S+i}f_{S*}$ is motivated by the conventions on perverse sheaves. The local system $\mathbf{L}^i_S = i^*_S H^{-\ell}({}^p\mathcal{H}^i(K))$ is uniquely defined by the

decomposition. On a stratum $S = U$ open subset of V of dimension m, $\mathbf{L}_U^i := R^{-m+i} f_{S*} j_{!*} {}^p\mathbf{L}$.

For example if X is smooth of dimension n and ${}^p\mathbf{L} = \mathbb{Q}[n]$. At a point $v \in U$: $(\mathbf{L}_U^i)_v = \mathbb{H}^{n-m+i}(X_v, \mathbb{Q})$, which leads to a Hard Lefschetz formula centered at 0 on the fiber X_v of dimension $n-m$.
Since $X_S \to S$ is a topological fibration $R^i f_{S*}(R i^!_{X_S} j_{!*} {}^p\mathbf{L})$ and $R^i f_{S*}(i^*_{X_S} j_{!*} {}^p\mathbf{L})$ are local systems on S. For a general normal section N_s at $s \in S$ such that $X_{N_s} := f^{-1}(N_s)$ is transversal to the strata of X, we have

1. $R^i f_{S*}(R i^!_{X_S} j_{!*} {}^p\mathbf{L})_s \simeq H^i_{X_s}(X_{N_s}, j_{!*} {}^p\mathbf{L})$ is a MHS of weight $w \geq {}^pw + i$
2. $R^i f_{S*}(R i^!_{X_S} j_{!*} {}^p\mathbf{L})_s \simeq H^i_{X_s}(X_{N_s}, j_{!*} {}^p\mathbf{L})$ is a MHS of weight $w \geq {}^pw + i$.

Then, $R^i i^!_{X_S} j_{!*} {}^p\mathbf{L}$ and $R^i f_{S*}(i^*_{X_S} j_{!*} {}^p\mathbf{L})$ endowed with induced filtrations W and F are variations of MHS ($VMHS$) (§2.3) and ${}^p\mathbf{L}_S^i$ is a shifted VHS of weight ${}^pw + i$ on each stratum S.

1.2.1.1 Decomposition theorem on complex varieties

Let $K := f_* j_{!*} {}^p\mathbf{L} \in D^b_c(V, \mathbb{Q})$ and $i_S : S \to V$ the embedding of a stratum S in V. The cup product with the class $c_1 \in \mathbb{H}^2(X, \mathbb{Z})$ of an hyperplane section of X induces morphisms ([BBD 83], Théorème 5.4.10):

$$\eta : K \to K[2],\ \eta : i_{S!*} {}^p\mathbf{L}_S^i \to i_{S!*} {}^p\mathbf{L}_S^{i+2} \text{ and } \eta : {}^p\mathcal{H}^i(K) \to {}^p\mathcal{H}^{i+2}(K).$$

The proof in chapter 6 of the following statement of decomposition is based on the use of nearby and vanishing cycles complexes in [DeG 81], whose construction in chapters 4 and 5 extends Steenbrink's construction.

We do not use the theory of Hodge modules as in [Sa 88] and [Sa 90] but a parallel method by induction on the dimension of X. Instead we rely on the abelian category of IMHS ([Ka 86] Prop. 5.2.6). The theorem will be proved by a different method in chapters 10 and 11.

Theorem 1.2. *Let $f : X \to V$ be a projective morphism of complex algebraic varieties, ${}^p\mathbf{L}$ a shifted polarized variation of Hodge structure of weight ${}^pw$, $\mathcal{S}$ a Thom-Whitney stratification of V adapted to f, $j_{!*} {}^p\mathbf{L}$ and $K = f_* j_{!*} {}^p\mathbf{L}$.*

i) There exists a decomposition of the perverse cohomology on V for all $i \in \mathbb{Z}$

$$ {}^p\mathcal{H}^i(K) \xrightarrow{\sim} \oplus_{S \in \mathcal{S}} (i_S)_{!*} {}^p\mathbf{L}_S^i \tag{1.8}$$

into a direct sum of intermediate extensions of shifted polarized VHS ${}^p\mathbf{L}_S^i$ (Equation 1.7) of weight ${}^pw + i$ on the strata S.

ii) Hard Lefschetz: The iterated cup-product with the class of a relatively ample line bundle on X

$$\eta^i : {}^p\mathcal{H}^{-i}(K) \xrightarrow{\sim} {}^p\mathcal{H}^i(K)$$

is an isomorphism over V for all $i \in \mathbb{N}$. The perverse cohomology ${}^p\mathcal{H}^{-i}(K)$ is dual to ${}^p\mathcal{H}^i(K)$ and the duality is compatible with the natural decomposition in i).

iii) The (perverse) spectral sequence with respect to the perverse filtration ${}^p\tau$ *on* K *(Equation 2.2.0.1) degenerates at the term* E_1*. We have a natural decomposition into a direct sum of polarized HS of weight* ${}^pw + i + j$

$$Gr_i^{{}^p\tau}\mathbb{H}^{i+j}(X, j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^j(V, {}^p\mathcal{H}^i(K)) \xrightarrow{\sim} \oplus_{S\in\mathcal{S}}\, \mathbb{H}^j(V, i_{S!*}{}^p\mathbf{L}_S^i). \tag{1.9}$$

Moreover, there exists an isomorphism in the derived category of constructible sheaves $D_c^b(V, \mathbb{Q})$

$$K \simeq \oplus_{i\in\mathbb{Z}}{}^p\mathcal{H}^i(K)[-i]. \tag{1.10}$$

Remark 1.1. i) Let V_0 be be the union of the zero dimensional strata of V and let $v \in V_0$, then by (Equation 1.8)
$H^j(i_v^!\, {}^p\mathcal{H}^i(K)) \simeq \oplus_{S\in\mathcal{S}} H^j(i_v^!(i_S)_{!*}{}^p\mathbf{L}_S^i) = 0$ if $j < 0$ and $H^0(i_v^!\, {}^p\mathcal{H}^i(K)) \simeq {}^p\mathbf{L}_v^i$
$H^j(i_v^*\, {}^p\mathcal{H}^i(K)) \simeq \oplus_{S\in\mathcal{S}} H^j(i_v^*(i_S)_{!*}{}^p\mathbf{L}_S^i) = 0$ if $j > 0$ and $H^0(i_v^*\, {}^p\mathcal{H}^i(K)) \simeq {}^p\mathbf{L}_v^i$
hence ${}^p\mathbf{L}_v^i \simeq \operatorname{Im} H^0(i_v^!\, {}^p\mathcal{H}^i(K)) \to H^0(i_v^!\, {}^p\mathcal{H}^i(K))$ and by Equation 1.10, we have:

$${}^p\mathbf{L}_v^i \simeq (\operatorname{Im} H^i(i_v^! K) \to H^i(i_v^! K)).$$

At a general point $s \in S$, we apply this argument to the intersection with a normal section to S at s, which explains the interest in the intersection morphisms (Equation 1.7).

ii) We have: $H^0({}^p\mathcal{H}^i(K)) \simeq \oplus_{v\in V_0}{}^p\mathbf{L}_v^i$.

iii) The local systems ${}^p\mathbf{L}_S^i$ are shifted polarized VHS on S, ${}^p\mathbf{L}_S^{-i}$ is dual to ${}^p\mathbf{L}_S^i$, and η^i is an isomorphism ${}^p\mathbf{L}_S^{-i} \xrightarrow{\sim} {}^p\mathbf{L}_S^i$ for each i and S.

Let $V_l := \cup_{S\in\mathcal{S},\dim S\leq l} S$ denote the union of the strata of V of dimension $\leq l$ and $k_j : (V \setminus V_j) \to V$. We have:

$$\operatorname{Im}\left({}^p\mathcal{H}^i(k_{j!}k_j^* K) \to {}^p\mathcal{H}^i(K)\right) = k_{j!*}k_j^*\, {}^p\mathcal{H}^i(K) \tag{1.11}$$

iv) The decomposition in (Equation 1.10) is not canonical (see [De 93] for a preferred choice of decomposition).

v) The proof is local on V. Hence, along the proof we may suppose X and V projective varieties by reduction to a Zariski affine open subset od V and then take its projective closure. Then, the proof is by induction and reduction to ample hyperplane sections.

1.2.1.2 Scheme of the proof

We give here a guide to follow the full proof in chapter 6, based on the necessary technical constructions in chapters $3-5$. The new method of proof introduced in the text occurs in the abelian category of infinitesimal mixed Hodge structures (IMHS) defined by M. Kashiwara (ch. 2, §2.3.2) and avoid the axiomatic definition of the category of Hodge $D-$modules in Saito's proof.

The elements necessary to read the statements such as perverse sheaf, intermediate extension, perverse truncation of a constructible complex of sheaves and perverse cohomology are recalled in chapter 2.

Induced filtration. First, to define the Hodge structure on perverse cohomology, we suppose the perverse truncation ${}^p\tau_i$ of $K := f_*j_{!*}{}^p\mathbf{L}$ defined by a sub-complex of K (see the remark by Deligne that any morphism of complexes is homotopic to an embedding of complexes in chapter 2 §2.3.2),

Thus, the filtration f_*F is induced on ${}^p\tau_i$ and then projected on ${}^p\tau_i/{}^p\tau_{i-1}$. In this way, we are entitled to apply the theory developed by Deligne in [De 72] on filtered and bi-filtered derived functor. Remark that the sub-complexes defined by the filtration F are not not constructible and their perverse filtration is not defined or used in the text. Another option is to use the concept of spectral objects by Verdier ([Ve 77], [De 93]).

Induction on $\dim X$. If X is singular, we consider a desingularization $\pi : X' \to X$, and then it suffices to prove the theorem for π and $\pi \circ f$ ([De 68], Proposition 2.16) (see ch. 6, 6.2.2.2). The case $\dim X = 0$ is trivial while the case $\dim X = 1$ follows essentially from [Zu 79].

We suppose V projective, X smooth and projective. Let $\mathcal{S}$ denotes a Thom-Whitney stratification of f (2.1.3.1) (including one Whitney stratification of V and one of X). The inverse image of a general section L of V transversal to the strata of V is a non singular subvariety L' of X. The perverse cohomology of f_*K has good reduction to L.

Hence by induction on $\dim X$, we can deduce the theorem on a neighborhood of L from the case $f' : L' \to L$ where $\dim L' = \dim X - 1$.

However, such argument apply over the open subset $V^* := V \setminus V_0$ complement of the isolated strata but does not apply for L through the strata of dimension zero in V_0 since then L is not general. In this way the proof is reduced to a local problem concerning a neighborhood at each point $v \in V_0$.

Proof via nearby and vanishing cycles. The first proof of purity in [DeG 81], is based on the nearby cycles complex defined by a general section $L := g^{-1}(0)$ through a point $v \in V_0$, that is transversal to the strata outside v. In this case the vanishing cycles complex is supported by v.

Let $h : U \to D$ be a proper analytic morphism defined on an open analytic subset of X onto a disc D of $\mathbb{C} \supset 0$ such that $U_0 := h^{-1}(0)$ is a NCD in U. We use Hodge theory in the context of normal crossing divisors (NCD) to construct a complex $\Psi^* I\mathcal{L} \simeq \psi_h j_{!*}{}^p\mathbf{L}$ in chapter 4 such that a Hodge filtration F and the weight monodromy filtration W are well defined on $\Psi^* I\mathcal{L}$. Moreover $\mathrm{Gr}_k^W \Psi^* I\mathcal{L}$ decomposes into a direct sum of intermediate extensions on the smooth intersections of components (see chapter 5).

Weight spectral sequence. Returning to $f : X \to V$ and using a Lefschetz pencil intersecting the projective variety V, the proof is reduced to the case where there exist $g : V \to \mathbb{P}^1_{\mathbb{C}}$, $v \in L := g^{-1}0$, L is transverse to the strata of $V \setminus v$ and moreover $f^{-1}L$ is a NCD in X consisting of the union of the fiber

$X_v := f^{-1}v$ and the strict transform L' of L (we may need many blowings up above the fiber $f^{-1}v$ to reach this situation).

In this situation we construct the complex $\Psi^* I\mathcal{L} \simeq \psi^u_{g\circ f} j_{!*}{}^p\mathbf{L}$ such that a Hodge filtration F and the weight monodromy filtration W are well defined on $\Psi^* I\mathcal{L}$ (see chapters 3 and 4).

In this situation, the weight spectral sequence defined by the monodromy filtration W on the nearby cycle complex $\psi^u_{g\circ f} j_{!*}{}^p\mathbf{L}$ splits into a spectral sequence degenerating at rank 1 and a spectral sequence of MHS degenerating at rank 2 (see chapter 5). This is an important simplification of the proof.

Precisely, the terms E_1 of the weight spectral sequence involves

1. the hypercohomology of intermediate extensions on the intersections of components of the NCD X_v
2. the derived image of intermediate extensions on the "horizontal" smooth component L' above L

The terms E_1 of the spectral sequence above v form a spectral sequence of Hodge structures $E_1^{p,q}$ of weight q degenerating at rank 2.

The other terms involving the decomposition of $L' \to L$ form a spectral sequence which degenerates at rank 1 ($E_2^{p,q} = E_1^{p,q}$).

Proof of the decomposition. The proof is reduced to the decomposition of the vanishing cycle complex $\varphi_g j_{!*}{}^p\mathbf{L}$ by application of Verdier's classification of extensions of a perverse sheaf along a divisor [V 85-1] (see §2.2.3).

Singular X. In the case where X is singular, let $j : U \hookrightarrow X$ be a non singular Zariski open subset, $(\mathbf{L}, F)$ a polarized VHS on U, $\pi : X' \to X$ a desingularization of X and $j' : U \hookrightarrow X'$. The Hodge filtration on the intermediate extension $j_{!*}{}^p\mathbf{L}$ on X is defined as a component of $\pi_* j'_{!*}{}^p\mathbf{L}$ by the decomposition theorem applied to π.

Relation to the proof based on the theory of Hodge $\mathcal{D}-$modules ([Sa 88] and [Sa 90]). M. Saito covers the weight spectral sequence of any $\psi_g K$; in this case the weight spectral sequence does not split, but degenerates at rank 2 in the category of Hodge modules still to define. Effectively this category is defined axiomatically and then the proof is carried coherently with the definition.

In our exposition, the category corresponding to Hodge $\mathcal{D}-$modules is not needed in the proof of decomposition since we use the degeneration at rank 2 only in the category of MHS. Moreover, since φ is concentrated on one point, the decomposition occurs in the abelian category of IMHS ([Ka 86], 4.3, see Definition 2.14). After the proof of the decomposition, the definition of the category of perverse VHS corresponding to Hodge $\mathcal{D}-$modules is developed easily in chapter 7.

The relation between $\mathcal{D}_X-$modules and perverse complexes is given by Kashiwara's proof that the de Rham complex of a holonomic $\mathcal{D}_X-$module (see [PS 08] Equation (XIII-24)) is a perverse complex [Ka 74].

This basic result has been completed by Mebkhout [Me 80] and [Me 89] into a Riemann-Hilbert correspondence between the derived category of

bounded complexes of coherent $\mathcal{D}-$modules with regular holonomic cohomology sheaves and $D^b_c(X,\mathbb{C})$ (see [PS 08], Theorem 13.65).

The intermediate extension $j_{!*}{}^p\mathbf{L}$ in terms of $\mathcal{D}-$modules figure in ([KaK 87], Lemma 3.4.1 and [KaK 86]). The correspondence with the theory in terms of $\mathcal{D}-$modules is not given here. For more information on $\mathcal{D}-$modules (see [Sch 20], [Shi 93] and [PS 08], ch.13 and 14).

1.2.1.3 Perverse extension of a polarized VHS

As an application of the decomposition theorem we give the definition of the extension of a polarized VHS $(\mathbf{L},F)$ defined on a stratum S of a stratified complex variety V (S is non singular but V may be singular along S).

Let $j:S\to V$ denote the embedding of the locally closed subset S (equivalently j is the embedding into the closure $\bar{S}$ of S). Let ${}^p\mathbf{L} := \mathbf{L}[\dim S]$ denote the associated perverse local system (by convention we shift the degree of $\mathbf{L}$ into a complex with cohomology $\mathbf{L}$ in degree $-\dim S$). There exists a unique intermediate extension $j_{!*}\,{}^p\mathbf{L}$ of ${}^p\mathbf{L}$ by a complex in the category of constructible complexes $D^b_c(V,\mathbb{C})$.

The problem is to construct the correct filtration F on $j_{!*}\,{}^p\mathbf{L}$, then the abelian category of VHS on singular varieties.

The category of VMHS is defined by the fact that the graded object with respect to the weight W is a polarized VHS.

We give a definition of the extension of the filtration F by desingularization in terms of algebraic topology constructions as follows.

Theorem 1.3. *Let $(\mathbf{L},F)$ be a polarized VHS on a non singular open algebraic subset U of a variety V. There exists a canonical filtration F on the intermediate extension $j_{!*}{}^p\mathbf{L}$ on V defined in the filtered derived category $D^bF(V,\mathbb{C})$.*

Let $j:U\to V$ (resp. $j':U\to V'$) be the inclusion in V (resp. in a desingularization V' of V with complement of U a NCD). We define the extension $(j'_*\,{}^p\mathbf{L},F)$ on V' by an explicit definition of a sub-complex $IC^*\,{}^p\mathcal{L}'$ of the derived image $\Omega^*\,{}^p\mathcal{L}'\simeq j'_*\,{}^p\mathbf{L}$ with the induced degenerated filtration F in the case of a NCD. Let $\pi:V'\to V$ be the canonical morphism. We push forward F into a filtration π_*F of $\pi_*IC^*\,{}^p\mathcal{L}'$.

Since the perverse filtration ${}^p\tau$ is naturally defined on $\pi_*IC^*\,{}^p\mathcal{L}'$, the two filtrations ${}^p\tau$ and π_*F define a bifiltered complex in the category $D^bF_2(V,\mathbb{C})$ of bifiltered complexes, then a natural induced filtration F is defined on the perverse cohomology ${}^p\mathcal{H}^i(\pi_*j'_{!*}\,{}^p\mathbf{L})$. We use the decomposition theorem to embed $j_{!*}{}^p\mathbf{L}$ in ${}^p\mathcal{H}^i(\pi_*j'_{!*}\,{}^p\mathbf{L})$ and to induce the filtration F on $j_{!*}{}^p\mathbf{L}$.

1.2.1.4 Local and global invariant cycle theorems

Basic results such as the local and global invariant cycle theorems with constant coefficients ([De 80] §3.6, [NaG 90], resp. [De 72] Corollaire 3.2.18, [De 80] §6.2.11, [Vo 2007] Théorème II.4, 24), are generalized to coefficients in an intermediate extension of polarized VHS (see ch. 7, Theorems 4.1 and 4.2), which allow us to adapt the proof of local purity [DeG 81] to complex varieties in chapter 8. The text give a faithful account of the geometrical constructions in [DeG 81].

However, in chapter 10, we see that the decomposition theorem leads to a simplification of the proof of the local purity theorem.

1.2.1.5 Perverse extension of polarized VMHS

The most general definition of polarized VMHS is in terms of $D-$modules by M. Saito [Sa 90]. We mention in (ch. 7, §7.1.1) similar results in terms of PVMHS under the conditions of admissibility and the existence of the relative monodromy filtration. Such conditions should coincide in general with the conditions stated in the theory of mixed Hodge modules.

1.2.1.6 Direct construction of the weight filtration and Relative local purity

In chapter 9, we give a direct construction of the weight filtration W on the logarithmic complex $\Omega^*\mathcal{L} \simeq j_*\mathbf{L}$ in terms of the local monodromy along the components of the NCD. The weight filtration W is induced by the monodromy filtration on the nearby cycles complex (see §5.2.1.4).

In chapter 10 a new combined direct geometric proof of the local purity (ch. 1, Theorem 1.1) and the decomposition theorem (ch. 1, Theorem 1.2) is given in terms of the relative version of local purity.

In particular, the proof of local purity differs from the proof in [DeG 81] and an interpretation in terms of Intersection morphisms is given at the end.

Chapter 2
Preliminaries

We start by a summary on derived categories [Ve 77] in order to fix the notations and conventions and the nature of the isomorphisms of complexes in the text (quasi-isomorphisms).

We give a guide to basic results on perverse sheaves introduced in [BBD 83], without proofs. Namely, we recall the definitions of perverse sheaves and perverse cohomology (see §2.2).

In the next chapter, perverse sheaves with singularities along NCD are described. Even in such case one should not under estimate the theoretical study. Indeed, the theory has not been developed until the Intersection cohomology sheaf has been discovered by M. Goresky, R. MacPherson, based on the search for an auto-dual cohomology of singular spaces [GMacP 83]. An exposition on the level of graduate texts is given in [Max 19].

It is remarkable that this abelian category has been defined first in the context of derived category. Its relation to the category of $\mathcal{D}_X-$modules has been established later by J.L. Brylinski and M. Kashiwara [KaS 90], which lead to the theory of Hodge modules by M. Saito ([Sa 88]).

In view of the application to Hodge theory, we recall also results on the local structure of a degenerating polarized variation of Hodge structure (VHS) (resp. Variation of mixed Hodge structure (VMHS)) (see §2.3), which is needed to define the weight and Hodge filtrations on the de Rham complex $\Omega^*\mathcal{L}$ with coefficients in a graded polarized admissible variation of MHS on a local system.

2.1 Derived categories

Let $\mathcal{A}$ denote an abelian category and $C(\mathcal{A})$ the abelian category of complexes of $\mathcal{A}$. Two morphisms of complexes $\psi, \varphi : X \to Y$ are homotopic if there exits a family $h_i : X^i \to Y^{i-1}$ such that $d^{i-1} \circ h_i + h^{i+1} \circ d^i = \psi^i - \varphi^i : X^i \to Y^i$. Let $Ht(X, Y)$ denote the subgroup of morphisms $X \to Y$ homotopic to 0.

The category $K(\mathcal{A})$ is defined with the same objects of $C(\mathcal{A})$ but the morphisms of complexes are equivalence classes up to homotopy:

$$Hom_{K(\mathcal{A})}(X,Y) := Hom_{\mathcal{A}}(X,Y)/Ht(X,Y).$$

Homotopical morphisms φ in a class induce on cohomology a unique family of morphisms $\varphi^i : H^i(X) \to H^i(Y)$ for $i \in \mathbb{Z}$.

Mapping Cone. The mapping cone of a morphism $\varphi : X \to Y$ is the complex $C(\varphi)$ defined by $d_C^n : C^n := X^{n+1} \oplus Y^n \to C^{n+1} := X^{n+2} \oplus Y^{n+1}$

$$d_C^n = ((-1)d_X^{n+1} + \varphi^{n+1}, d_Y^n)$$

The translation functor T (denoted also by $[1] : C \to C$) is defined by $(X[1])^n := X^{n+1}$ and $d_{(X[1])^n} := -d_X^{n+1}$.

There exist natural morphisms $i : Y \to C(\varphi)$ and $p : C(\varphi) \to X[1]$ satisfying the following sequence of morphisms

$$X \xrightarrow{\varphi} Y \xrightarrow{i} C(\varphi) \xrightarrow{p} X[1]$$

called a distinguished triangle. We remark that there exists a morphism in the category $K(\mathcal{A})$: $\phi : X[1] \to C(i)$ such that we have an isomorphism of sequences

$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d, "id"] & C(\varphi) \arrow[r, "p"] \arrow[d, "id"] & X[1] \arrow[r, "-\varphi[1]"] \arrow[d, "\phi"] & Y[1] \arrow[d, "id"] \\
Y \arrow[r, "i"] & C(\varphi) \arrow[r, "i'"] & C(i) \arrow[r, "p'"] & Y[1]
\end{tikzcd}$$

We refer to ([KaS 90] Lemma 1.4.2) for a proof. The morphism ϕ is not unique and it is an isomorphism in $K(\mathcal{A})$ (up to homotopy), but not in $C(\mathcal{A})$. As a consequence, $X[1]$ may be constructed as the cone $C(i)$. A distinguished triangle is embedded in a turning sequence of distinguished triangles.

Remark 2.1. i) Let φ and ψ be two morphisms: $X \to Y$. Given an homotopy h from φ to ψ, we can deduce an isomorphism $C(h) : C(\varphi) \to C(\psi)$ depending on h, hence the construction of the cone in $K(\mathcal{A})$ is unique up to an isomorphism but this isomorphism is not unique.

ii) Since the morphism φ is defined up to homotopy, the choice of $C(\varphi)$ is not natural which is a problem in the context of mixed Hodge complexes (MHC) [De 72]. In this case, there exists a problem of compatibility between the rational and complex level of construction.
To obtain the naturality of the mixed Hodge structure, Deligne used the concept of simplicial varieties to avoid compatibility problems.

iii) *Second homotopy.* If h and h' are two homotopies from φ to ψ, an homotopy from h to h' leads to an isomorphism of diagrams between the corresponding isomorphisms $C(h)$ and $C(h')$.

iv) The category $K(\mathcal{A})$ is not abelian ([Max 19] Example 4.5.3).

The derived category $D(\mathcal{A})$.
The classical cohomology theories of a topological space X may be defined as derived functors of the global section functor $F := \Gamma(X, *)$ on the category of sheaves on X [G 58].

A functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories extends to a functor $F^\bullet : C(\mathcal{A}) \to C(\mathcal{B})$ defined by $F^\bullet(X^\bullet) := Y^\bullet$ where $Y^i := F(X^i)$. A morphism of complexes $\varphi : X^\bullet \to Y^\bullet$ is called a *quasi-isomorphism* if it induces isomorphisms on cohomology in each degree. We remark that the image of a quasi-isomorphism is not a quasi-isomorphism unless F is exact.

When $\mathcal{A}$ has enough injectives (every object of $\mathcal{A}$ may be embedded in an injective object of $\mathcal{A}$) and F is left exact, for any bounded below complex $X^\bullet$ there exists an injective resolution (a quasi-isomorphism $X^\bullet \to I^\bullet$ and $F^\bullet$ transforms a quasi-isomorphism of complexes of injective objects $\mathbf{I}^\bullet \xrightarrow{\sim} J^\bullet$ into a quas-isomorphism.

To construction of the $i-th$ satellite functor R^iF (we should write $R^iF^\bullet$), we fix an injective resolution of $X^\bullet$, that is a quasi-isomorphism $X^\bullet \xrightarrow{\sim} \mathcal{I}^\bullet$ and define $R^iF(X^\bullet) := H^i(F(\mathcal{I}^\bullet)$ (we should write $R^iF^\bullet(X^\bullet)$).

To construct in general the derived category $D(\mathcal{A})$, we remark that a quasi-isomorphism is not necessarily an isomorphism in $K(\mathcal{A})$.

The purpose of derived categories is to construct a category of complexes where the quasi-isomorphisms are isomorphisms. The right derived functor RF of a left exact functor F is defined as a complex $RF(X) := F(\mathcal{I}^\bullet)$ and not merely its cohomology objects: the satellites R^iF. If F is right exact we consider projective resolutions $\mathcal{P}^\bullet \xrightarrow{\sim} X$ and define the left derived functor LF by $F(\mathcal{P}^\bullet)$.
Later in the text we often denote also by F the derived functor.

The class of quasi-isomorphisms $X \to Y$ form a multiplicative system S in the sense of Verdier [Ve 77]. By a construction, similar to the construction of $\mathbb{Q}$ out of $\mathbb{Z}$, it is possible to construct a derived category $D(\mathcal{A})$ with the same objects as $K(\mathcal{A})$ but with group of morphisms obtained by inverting quasi-isomorphisms: there exists a functor $Q : K(\mathcal{A}) \to D(\mathcal{A})$ such that the image of a quasi-isomorphism $X \to Y$ is an isomorphism in $Hom_{D(\mathcal{A})}(X, Y)$.

An important consequence is that any short exact sequence in $K(\mathcal{A})$ is isomorphic to a distinguished triangle in $D(\mathcal{A})$ ([KaS 90] proposition 1.7.5).

2.1.0.1 The complex of Homomorphisms $Hom^\bullet(*, *)$

We mention the following bifunctor on an abelian category $\mathcal{A}$ to the category of abelian groups: $Hom(*, *) : \mathcal{A}^\circ \times \mathcal{A} \to Ab : (X, Y) \mapsto Hom(X, Y)$.

If $X^\bullet$ and $Y^\bullet$ are complexes, we define the complex $Hom^\bullet(X^\bullet, Y^\bullet)$ by

$$Hom^n(X^\bullet, Y^\bullet) = \prod_{p \in \mathbb{Z}} Hom(X^p, Y^{p+n}) \tag{2.1}$$

$d^n = \prod_{p\in\mathbb{Z}} d^n_p$, $d^n_p : Hom(X^p, Y^{p+n}) \to Hom(X^p, Y^{p+n+1}) : d^n_p(\varphi_p) = d^{n+p}_{Y^\bullet} \circ \varphi_p + (-1)^{n+1}\varphi_{p+1} \circ d^p_{X^\bullet}$.

The group $H^i(Hom^\bullet(X^\bullet, Y^\bullet)) := H^0(Hom^\bullet(X^\bullet, Y^\bullet)[i])$ may be identified with the group of classes of morphisms of complexes of degree i modulo the sub-group of morphisms homotopic to zero.

Given a complex $X^\bullet := (X^p, d_p)_{p\in\mathbb{Z}}$, the complex $(X^\circ)^\bullet \in C(\mathcal{A})^\circ$ is defined by $(X^\circ)^r := X^{-r}$ and the complex $Hom^\bullet$ is defined by the double complex $S^{p,q} := Hom(X^{-p}, Y^q)$. We deduce a functor defined modulo homotopy of complexes:

$$Hom^\bullet : K(\mathcal{A})^\bullet \times K(\mathcal{A}) \to K(Ab).$$

If Y^i is injective for all i, then $Hom^\bullet(*, Y^\bullet)$ transforms quasi-isomorphisms into quasi-isomorphisms. Hence, if $\mathcal{A}$ admits enough injective objects, we deduce a bifunctor $Hom^\bullet : D(\mathcal{A})^\circ \times D^+(\mathcal{A}) \to D(Ab)$ (this bifunctor may be defined also by projective resolutions on the left). We deduce a right derived functor $RHom^\bullet(*,*) : (D(\mathcal{A})^\circ \times (D^+\mathcal{A}) \to D(Ab)$ such that:
$Hom_{D(\mathcal{A})}(X^\bullet, Y^\bullet) = H^0(RHom^\bullet(X^\bullet, Y^\bullet)) := Ext^0_{\mathbb{Z}}(X^\bullet, Y^\bullet)$ and
$Ext^i_{\mathbb{Z}}(X^\bullet, Y^\bullet) = H^0(RHom^\bullet(X^\bullet, Y^\bullet)[i])$ for all $i \in \mathbb{Z}$.

2.1.0.2 Triangulated categories

We list now some properties of the derived category $D(\mathcal{A})$. It is an advantage to abstract these properties into a set of axioms and define directly the concept of triangulated categories.

Let $\mathcal{C}$ be an additive category with a translation functor [1]. A triangle of $\mathcal{C}$ is defined as a sequence of morphisms $X \to Y \to Z \to X[1]$.

Definition 2.1. i)A triangulated category consists of an additive category $\mathcal{C}$ with a translation functor and a family of triangles called distinguished triangles satisfying four axioms TR1-4 stated in [Ve 77] such that any triangle isomorphic to a distinguished triangle is in the family.

ii) An additive functor $F : \mathcal{C} \to \mathcal{A}$ to an abelian category is called a cohomological functor if for any distinguished triangle $X \to Y \to Z \to X[1]$, the sequence $F(X) \to F(Y) \to F(Z)$ is exact.

Set $F^k := F \circ T^k$, a long exact sequence follows from the axioms:

$\cdots F^{-1}(Z) \to F(X) \to F(Y) \to F(Z) \to F^1(X) \to F^1(Y) \to F^1(Z) \cdots$

More about triangulated functors and the long exact sequence of cohomology groups defined by a triangle figure in ([KaS 90] §1.7).

2.1.0.3 Derived category of sheaves $D^+(X, A)$ on a locally compact space X

Let A a commutative Noetherian ring (for some results we need to add that A has finite homological dimension but we may suppose $A = \mathbb{Z}$ or $\mathbb{Q}$). We

mention classical functors on the category of sheaves $Sh(X, A)$ ($Sh(X)$ for short) and refer to a basic reference such as [G 58] or to [KaS 90] for a full study.

1) Tensor product $* \otimes *$ in $\mathcal{A} := Sh(X, A)$:

Let $\otimes_{\mathbb{Z}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A} : (B_1, B) \mapsto B_1 \otimes_A B$ denote the tensor product in $\mathcal{A}$. The complex tensor product of two complexes $B_1^\bullet$ and $B^\bullet$ in $K^-\mathcal{A}$ is

$$B_1^\bullet \otimes_A B^\bullet \in K^-\mathcal{A} \text{ defined by } (B_1^\bullet \otimes_A B^\bullet)^n = \textstyle\sum_{p+q=n} A^p \otimes_A B^q$$

with differential $d := d_{B_1} + (-1)^n d_B$. This bifunctor is well defined up to homotopy and it is right exact.

If F is a flat sheaf, the functor $F \otimes *$ is exact on $\mathcal{A}$. Since there exists flat resolutions in $D^-\mathcal{A}$, we deduce the left derived functor

$$\otimes_{\mathbb{Z}}^L : D^-\mathcal{A} \times D^-\mathcal{A} \to D^-\mathcal{A}.$$

2) *f_* and f^*.* Let $f : X \to V$, the direct image of a sheaf F on X is a sheaf denoted f_*F on V. The functor f_* is left exact with a derived functor Rf_* often denoted also by f_*.

The inverse image of a sheaf G on V is a sheaf denoted f^*G on X. The functor f^* is exact and its derived functor Rf^* is often denoted by f^* (we use the notation f^* instead of f^{-1} on the level of the category $Sh(V, A)$.

We have an isomorphism of groups $Hom(G, f_*F) \simeq Hom(f^*G, F)$. It is an example of the notion of adjoint functors: we say that f^* is left adjoint to f_* and f_* is right adjoint to f^*. On the level of morphisms of sheaves, we have: $\mathcal{H}om(G, f_*F) \simeq f_*\mathcal{H}om(f^*G, F)$.

In the case of a a subset Z of X, let $j : Z \hookrightarrow X$. We write

$$F|_Z := j^*F \text{ and } \Gamma(Z, F) := \Gamma(Z, j^*F).$$

3) *Direct image with proper support $f_!$.* The support of a section s of a sheaf F on U is the complementary S in U of the open subsets $V \subset U$ such that $s_{|V} = 0$, hence $S := \mathrm{supp}_U s$ is closed in U:

$$\mathrm{supp}_U s := \overline{\{x \in U : s_x \neq 0\}}$$

The support of a sheaf F on X, is the subset $S \subset X$ complementary of open sets $U \subset X$ such that $F_{|U} = 0$, hence $S := \mathrm{supp}\, F$ is closed in X.

Let $f : X \to V$ be a continuous map. We say that f is proper if the image of a closed set of X is closed in V, the fibers of f are compact and two distinct points in a fiber have two disjoint neighborhoods in X. If X and V are locally compact, f is proper if the inverse of any compact set in V is compact in X. The sub-sheaf $f_!F \subset f_*F$ is defined by setting for open sets $U \subset V$:

$$\Gamma(U, f_!F) := \{s \in F(f^{-1}U) \text{: } f|_{\mathrm{supp}\, s} : \mathrm{supp}\, s \to U \text{ is proper } \}$$

This sheaf is called the direct image with proper supports.
In the case where V is reduced to a unique point, we write

$$\Gamma_c(X, F) := \{s \in \Gamma(X, F) : \mathrm{supp}\, s \text{ is compact and Hausdorff } \}.$$

If X and V are locally compact then: $(f_!F)_x := \Gamma_c(f^{-1}(x), F)$.

4) *Extension by* 0. Let Z be locally closed in X, $j : Z \to X$ and F a sheaf on Z, then the sheaf $j_!F$ on X is the unique sheaf on X satisfying:

$$(j_!F)|_Z = F,\ (j_!F)|_{X\setminus Z} = 0 \tag{2.2}$$

If j is a closed immersion $j_!F = j_*F$.

Remark 2.2. Let $F^\bullet$ and $G^\bullet$ be two bounded complexes of sheaves on X such that $H^i(F^\bullet) = 0$ for $i > n$ and $H^i(G^\bullet) = 0$ for $i \leq n$, then the following natural map is an isomorphism ([GMacP 83] Proposition 1.15)

$$Hom_{D^b(X)}(F^\bullet, G^\bullet) \xrightarrow{\sim} Hom_{Sh(X)}(H^n(F^\bullet), H^n(G^\bullet))$$

2.1.1 Poincaré - Verdier duality

Sheafified version of the duality. Let F and G be two sheaves on a topological space X. We define the presheaf $\mathcal{H}om_A(F,G)$ (which is already a sheaf) by the correspondance : $U \mapsto Hom_A(F_{|U}, G_{|U})$.

The bifunctor $\mathcal{H}om_{Sh(X,A)}(*,*)$ is defined by

$$(F,G) \to \mathcal{H}om_A(F,G) \in Sh(X,A).$$

Let $C^+(X,A)$ denote the category of sheaves bounded below complexes, $F^\bullet$ and $G^\bullet$ be in $C^+(X,A)$, we deduce from (Equation 2.1) the definition of the complex of sheaves $\mathcal{H}om^\bullet(F^\bullet, G^\bullet) \in C^+(X,A)$.

The derived bifunctor of $\mathcal{H}om_{Sh(X,A)}(*,*)$ is defined by taking injective resolutions of G on the right side or projective resolutions on the left side. We deduce

$$\begin{aligned}(D^+(X,A))^\circ \times D^+(X,A) \ni (F,G) &\to R\mathcal{H}om^\bullet(F,G) \in D^+(X,A)\\ R\mathbb{H}om^\bullet(F,G) &\simeq R\Gamma(X, R\mathcal{H}om^\bullet(F,G)\\ Hom_{D^+(X,A)}(F,G[n]) &= \mathbb{H}^n(X, R\mathcal{H}om^\bullet(F,G))\end{aligned} \tag{2.3}$$

For F the constant sheaf defined by A, $Hom_{D^+(X,A)}(A_X, G[n]) = \mathbb{H}^n(X,G)$.

Theorem 2.1 (Duality [Ve 65]). *Let $f : X \to V$ be a continuous map of locally compact spaces. Assume $f_!$ has finite cohomological dimension. There exists a functor $f^! : D^+(V,A) \to D^+(X,A)$ right adjoint to the functor $Rf_! : D^+(X,A) \to D^+(V,A)$ satisfying for $F \in Ob(D^+(X,A))$ and $G \in Ob(D^+(V,A))$:*

$$R\mathcal{H}om^\bullet(Rf_!F, G) \simeq Rf_*\mathcal{H}om^\bullet(F, f^!G)\ in\, D^+(V,A) \tag{2.4}$$

$$\mathbb{H}om^\bullet_V(Rf_!F, G) \simeq \mathbb{H}om^\bullet_X(F, f^!G) \tag{2.5}$$

$$Hom_{D^+(V,A)}(Rf_!F, G) \simeq Hom_{D^+(X,A)}(F, f^!G) \tag{2.6}$$

To summarize the statement $f^!$ is called right adjoint to $Rf_!$.
In particular, we recover Equations 2.5 and 2.4 by taking $R\Gamma(V,*)$ (resp. $R^0\Gamma(V,*)$) in Equation 2.4. We refer to [Ve 65] and [KaS 90] for the construction of $f^!$ and the proof of the duality isomorphism.

The equation 2.6 helps to define $f^!$ directly on the level of complexes. For any open subset $U \subset X, j : U \subset X$, set $F = j_!\mathbb{Z}_U$; we have:

$$R\Gamma(U, f^!G) := RHom^\bullet_X(j_!\mathbb{Z}_U, f^!G) \simeq RHom^\bullet_V(Rf_!j_!\mathbb{Z}_U, G).$$

Definition 2.2 (dualizing complex). i) Let $f : X \to \{e\}$ be the map to a point. The complex $f^!A \in D^+(X, A)$ is denoted by $\mathbb{D}_{X,A}$ or simply $\mathbb{D}_X$ and called the dualizing complex on X.

ii) The (Verdier) dual of a complex of sheaves $F \in ObD^+(X, A)$ is:

$$\mathcal{D}_X F := R\mathcal{H}om^\bullet(F, \mathbb{D}_X) = R\mathcal{H}om^\bullet(F, f^!A)$$

$\mathcal{D}_X : D^+(X, A) \to D^+(X, A) : F \mapsto \mathcal{D}_X F$ is called the dualizing functor.

Properties of the dualizing functor. Let $f : X \to V$, it follows from duality and the adjoint properties: for all $F \in Ob^+(X, A), G \in Ob^+(V, A)$

$$\mathcal{D}_X f^*G \simeq f^!(\mathcal{D}_V G),\ \mathcal{D}_V(f_*F) \simeq f_!(\mathcal{D}_X F) \tag{2.7}$$

Example 2.1. i) In the case of an oriented manifold X of (real-)dimension n, $A = \mathbb{Q}$ and V is reduced to a point

$$\mathbb{D}_X := f^!\mathbb{Q} = \mathbb{Q}_X[n],\ \mathcal{D}_X F := R\mathcal{H}om^\bullet(F, \mathbb{Q}_X[n])$$

and for any open subset $U \subset X$, by duality:

$$Hom^\bullet_{\mathbb{Q}}(R\Gamma_c(U, F), \mathbb{Q})) \simeq R\Gamma(U, \mathcal{D}_X F),\ \mathbb{H}^{-q}(U, \mathcal{D}_X F) \simeq Hom_{\mathbb{Q}}(\mathbb{H}^q_c(U, F), \mathbb{Q}).$$

ii) Let $A = \mathbb{Q}$ and $F := \mathcal{L}$ a local system, then $R\mathcal{H}om^\bullet(\mathcal{L}, \mathbb{Q}_X)$ is reduced to a unique sheaf $\check{\mathcal{L}} := \mathcal{H}om(\mathcal{L}, \mathbb{Q}_X)$. Taking the $(-i)-$th group of cohomology, we have Poincaré duality:

$$Hom_{\mathbb{Q}}(H^i_c(X, \mathcal{L}), \mathbb{Q}) \simeq H^{n-i}(X, \check{\mathcal{L}})$$

Proposition 2.1. *([Ve 76b] §1.2) Let ϕ be a paracompact family of support and U an open subset of a complex algebraic variety, there exist isomorphisms:*

$$H^\phi_p(U, \mathbb{Z}) \xrightarrow{\sim} \mathbb{H}^{BM}_{p,\phi}(U, \mathbb{Z}) \xrightarrow{\sim} \mathbb{H}^{-p}_\phi(U, \mathbb{D}_{X,\mathbb{Z}})$$

of the homology groups (resp. Borel Moore homology) with support in ϕ and the cohomology of $D_{X,\mathbb{Z}}$ with support in ϕ.

2.1.2 Grothendieck - Serre duality of coherent sheaves

The development of duality in the frame of derived category started earlier with Grothendieck's extension of Serre duality [Gro 57]. Let V be a projective complex variety and $D^+(V, \mathcal{O})$ the category of sheaves of $\mathcal{O}-$modules.

Theorem 2.2 (Grothendieck). *([Gro 57]Proposition 5) Let $Y \subset V$ be an algebraic subset of dimension $n-p$ in a non singular complex algebraic variety V of dimension n and F a coherent module on Y, then:*

i) The sheaves $\mathcal{E}xt^{p+i}_{\mathcal{O}_V}(F, \Omega^n_V)$ *vanish for* $i < 0$*, and are independent of the choice of* V *and the embedding of* Y *for* $i \geq 0$*.*

ii) $$\mathcal{E}xt^p_{\mathcal{O}_V}(F, \Omega^n_V) \simeq \mathcal{H}om_{\mathcal{O}_Y}(F, \mathcal{E}xt^p_{\mathcal{O}_V}(\mathcal{O}_Y, \Omega^n_V).$$

iii) If Y *is non singular of dimension* $n-p$ *in* V *of dimension* n*:*

$$\mathcal{E}xt^{p+i}_{\mathcal{O}_V}(\mathcal{O}_Y, \Omega^n_V) = 0 \text{ if } i \neq 0 \text{ and } \mathcal{E}xt^p_{\mathcal{O}_V}(\mathcal{O}_Y, \Omega^n_V) \simeq \Omega^{n-p}_Y.$$

Moreover: $\mathbb{H}^n(V, \Omega^n_V)$*, of rank* 1*, is generated by the cohomology class of* V*.*

iv) Duality for coherent sheaves: Let F *be coherent on* V*, then:*

$$Ext^{n-p}_{\mathcal{O}_V}(F, \Omega^n_V) \simeq (Hom_{\mathbb{C}}(\mathbb{H}^p(V, \Omega^n_V), \mathbb{C})$$

With the notation of the theorem above, Grothendieck introduced the concept of dual with value in a "dualizing complex $\mathbf{K}_Y := R\mathcal{H}om^\bullet_X(\mathcal{O}_Y, \Omega^n_V[n])$ which is a basic improvement of the statements i) - iii) involving extension groups $\mathcal{E}xt^*(\mathcal{O}_Y, \mathbf{K}_X)$ ([Gro 58], [Gro 66]), [Ha 66] chapters III, VI).

For this purpose, it was necessary to construct the derived category ([Ve 77]), in order to state duality theorems in general.

In the case of a proper morphism $f : X \to Y$ of locally Noetherian preschemes and a complex of modules $F^\bullet$ on X with quasi-coherent cohomology, the duality theorem in the category $D_{qc}(X)$ is stated as follows [Ha 66] chapter VII, §3):

$$\mathcal{D}_X F^\bullet := R\mathcal{H}om^\bullet_X(F^\bullet, \mathbf{K}_X) \simeq (R\mathcal{H}om^\bullet_Y(Rf_* F^\bullet, \mathbf{K}_Y) := \mathcal{D}_Y Rf_* F^\bullet$$

2.1.3 Constructible sheaves

2.1.3.1 Thom-Whitney stratification $\mathcal{S}$ of f

Let V be a complex algebraic variety, and $\mathcal{S} = (S_\alpha)$ an algebraic Whitney stratification by a family of locally closed algebraic non singular sub-varieties called strata S_α of V ([LeT 83], [GMacP 88], [Ve 76]). The subspaces $V_l = \cup_{\dim S_\alpha \leq l} S_\alpha$ form an increasing family of closed algebraic subsets of V of dimension $\leq l$, with index $l \leq n$, where n is the dimension of V,

$$V_{-1} = \emptyset \subset V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V \tag{2.8}$$

($V_i \setminus V_{i-1}$ is a union of strata of $\mathcal{S}$ of dimension i but V may be singular along S_α). In addition, whenever $S_\alpha \subseteq \bar{S}_\beta$ Whitney conditions are satisfied in order to guarantee that the variety V is topologically equisingular along each stratum.

Let $f : X \to V$ be an algebraic map. If $Z \subset V$ is a sub-space of V, let $X_Z := f^{-1}(Z)$, in particular for each strata S of $\mathcal{S}$ let $X_S = f^{-1}(S)$.

Definition 2.3. A Thom-Whitney stratification of the algebraic map $f : X \to V$ consists of two Whitney stratifications one for V and one for X such that the inverse X_S of each stratum S of V is a union of connected strata of

X, the restriction $f_| : X_S \to S$ of f to each stratum of X_S has maximal rank equal to $\dim S$ and $f_| : X_S$ is a locally trivial topological fibration over S.

Using Thom-Mather first isotopy theorem, one can prove the following property (equisingularity along any stratum):

- (W) The link at any point of a stratum is a locally constant topological invariant of the stratum [Mat 12], [LeT 83].

We may also consider locally finite stratifications. There exist analytic Whitney stratifications of an analytic (resp. algebraic) variety and any stratification can be refined into a Whitney stratification. There exist algebraic Thom-Whitney stratifications for any algebraic map f [Ve 76] while such stratifications exist only for proper analytic morphisms.

2.1.3.2 Constructible sheaves

Let V be a complex algebraic variety and G an abelian group. Recall that the sheaf $\mathcal{G}$ associated to the presheaf $U \mapsto G$ is called a constant sheaf.

Definition 2.4 (Locally constant sheaves). A sheaf F is locally constant on V with respect to the transcendental topology (equivalently a local system), if there exists an open covering $V = \cup_i U_i$ such that the restriction $F_{|U_i}$ is constant on U_i.

For simply connected open subsets $U_2 \subset U_1$, the restriction $\Gamma(U_1, F) \to \Gamma(U_2, F)$ is an isomorphism (the locally constant sheaf is constant on simply connected open subsets).

For example, on $\mathbb{C}^*$ with real coordinates x, y, we consider the covering reduced to $\mathbb{C}^* \setminus \mathbb{R}^+$ and $\mathbb{C}^* \setminus \mathbb{R}^-$. On the intersection defined by the two subsets $y > 0$ and $y < 0$, a locally constant sheaf is defined by a $\mathbb{Z}-$module V and an invertible linear transformation T called monodromy.

We refer also to the relation with the fundamental group $\pi_1(V)$ ([De 70] I, Proposition 1.7).

Definition 2.5 (Constructible complex of sheaves). Let A be a commutative Noetherian ring

i) A sheaf F of $A-$modules is constructible on V if there exists a stratification V_i (Equation 2.8) such that $F_{|V_i \setminus V_{i-1}}$ is a sheaf of $A-$modules of finite type locally constant.

ii) A complex of sheaves is constructible if it is cohomologically bounded and constructible (finite non zero constructible cohomology sheaves).

By definition, a constructible sheaf restricts to a locally constant sheaf on a Zariski open dense subset of V. The properties of a constructible complex of sheaves F are listed in ([Ve 65], CC1-CC9).

Let $\in V$, a ball $B_\epsilon(x) \subset V$ is the trace on V of a general small ball with center x defined by a local embedding of V at x in $\mathbb{C}^N$ with interior $\mathring{B}_\epsilon(x)$.

Lemma 2.1 (Properties of a constructible complex). *Let $F \in D^b_c(V, \mathbb{Q})$ and $B_\epsilon(x)$ a small ball with center x*

$$\begin{aligned} \mathbb{H}^q(B_\epsilon(x) \setminus x, F) &\simeq \mathbb{H}^q(\partial B_\epsilon(x), F) \\ \mathcal{H}^q(F)_x \simeq H^q(i_x^* F) &\simeq \mathbb{H}^q(\mathring{B}_\epsilon(x), F) \simeq \mathbb{H}^q(B_\epsilon(x), F) \\ \mathbb{H}^q(i_x^! F) = \mathbb{H}^q_x(X, F) = H^q_c(\mathring{B}_\epsilon(x), F) &\simeq Hom_{\mathbb{Q}}(H^{-q}(i_x^* \mathcal{D}_X F) \end{aligned} \tag{2.9}$$

The boundary $\partial B_\epsilon(x)$ of a general ball intersect transversally the strata. The first isomorphism follows from the local topological invariance due to constructibility and a deformation of $B_\epsilon(x) \setminus x$ onto $\partial B_\epsilon(x)$ ([Max 19] §7.2).

Dévissage. Let $j_i : V_i \setminus V_{i-1} \to V$, then the sheaf $j_{i,!} F_{|V_i \setminus V_{i-1}}$ (extension by zero) is constructible and there exists on F a filtration by constructible sub-sheaves with successive quotients of the type $j_{i!} F_{|V_i \setminus V_{i-1}}$.

We refer to [Ve 76b] for more properties of constructible sheaves defined on analytic varieties.

Proposition 2.2. *i) The extension $\mathcal{E}xt^i_{A_X}(F, G) := R^i \mathcal{H}om^\bullet_{A_X}(F, G)$ of two constructible sheaves is constructible.*

ii) The dualizing complex D_X is constructible.

iii) Let $f : X \to V$ be a submersive morphism of analytic varieties, then: for $G \in ObD^+(V, A)$, $f^! G \simeq f^* G \otimes^L f^! A_V$.

iv) Let $f : X \to V$ be an algebraic map, and K a constructible complex of sheaves on X, then $f_* K$ is constructible on V ($R^q f_* K$ are constructible sheaves). In particular $R f_* \mathbb{Q}_X$ is constructible on V.

In general proofs on algebraic varieties are easier. The proof of i) is by dévissage. Since locally X can be embedded in a non singular variety Z, we deduce ii) from i) and the lemma

Lemma 2.2. *If $X \subset Z$ is closed in Z, then $\mathbb{D}_X \simeq R\mathcal{H}om^\bullet(\mathbb{Z}_X, \mathbb{D}_Z)$.*

iii) If $f = pr_V : Z \times V \to V$, pr_Z is the projection on Z and $g : Z \to \{e\}$ is the projection to a point $f^! A_V \simeq pr_Z^* g^! A_V$ ([Ve 65] §5).

We deduce iv) from the existence of Thom-Whitney stratifications.

2.1.3.3 Intermediate extensions

To motivate the definition of perverse sheaves later, we state here a definition of the intermediate extension of a local system given on a Zariski open subset. The topological equisingularity along a stratum is the basic property used in Intersection cohomology [GMacP 83]. By the *local normal triviality* property of the stratification in (Equation 2.8) at $x \in V_i \setminus V_{i-1}$, there exists a neighbourhood $U_x \subset V$ and a compact $2n - 2i - 1$ dimensional topologically stratified space

$$L = L_{2n-2i-1} \supset \ldots \supset L_3 \supset L_1 \supset L_{-1} = \emptyset \tag{2.10}$$

with open cone $\mathring{c}L := L \times [0,1)/L \times \{0\}$ and a homeomorphism

$$\varphi : U_x \xrightarrow{\sim} \mathbb{R}^{2i} \times \mathring{c}L \tag{2.11}$$

since $V_i \setminus V_{i-1}$ is a topological manifold of dimension $2i$.
Deligne's construction of the intermediate extension. Let $i \geq 1$ and

$$U_n = V \setminus V_{n-1}, \quad U_0 = V, \quad U_i = V \setminus V_{i-1}, \quad j_i : U_i \hookrightarrow U_{i-1},$$

Definition 2.6 (intermediate extension). Let $\mathbf{L}$ be a local system on U_n and set ${}^p\mathbf{L} := \mathbf{L}[n]$. We define the intermediate extension $j_{!*}{}^p\mathbf{L}$ on V inductively (here $j_{i*} := Rj_{i*}$)

$$j_{!*}{}^p\mathbf{L} := \tau_{\leq -1} j_{1*} \circ \cdots \circ \tau_{\leq -i} j_{i*} \cdots \circ \tau_{\leq -n} j_{n*}\, {}^p\mathbf{L} \tag{2.12}$$

Let $S \subset V_i \setminus V_{i-1}$ be a non singular stratum and $x \in S \subset U_i \setminus U_{i+1}$. There exists a local embedding $U_x \subset \mathbb{C}^N$ and a complex linear subspace $H_x \subset C^N$ normal to S at x, intersecting the strata of V_j near x transversally. We can suppose U_x as in equation 2.11 and $\mathring{c}L = U_x \cap H_x$. We have a diagram

$$B_x \cap S \overset{\sigma_i}{\hookrightarrow} B_x \cap U_i \overset{j_{i+1}}{\hookleftarrow} (B_x \cap U_{i+1}) \setminus S \quad \text{where } j_{i+1} : U_{i+1} \cap U_x \to U_i \cap U_x$$

Lemma 2.3. *i)* $(\mathcal{H}^k(j_{!*}{}^p\mathbf{L}))|V_i = 0$ *if* $k \geq -i, \forall i \geq 0$.
Precisely, if $x \in S \subset V_i \setminus V_{i-1}$*:*

$$(\mathcal{H}^k(j_{!*}{}^p\mathbf{L}))_x = \mathbb{H}^k(L_{2n-2i-1}, j_{i+1!*}\, {}^p\mathbf{L}) \text{ if } k \leq -i-1,\ 0 \text{ if } k \geq -i.$$

ii) Duality: $\mathcal{D} j_{!*}{}^p\mathbf{L} \simeq j_{!*}\mathcal{D}\, {}^p\mathbf{L}$. *If* ${}^p\mathbf{L}$ *is auto-dual, the intermediate extension is auto-dual, which is the case of a polarized local system. In particular:* $\mathcal{H}^k i^!_{V_i}(j_{!*}{}^p\mathbf{L})) = 0$ *if* $k \leq -i, \forall i \geq 0$.

i) We have $U_{i+1} \cap U_x = U_x \setminus S$. Let $j'_{i+2} : U_x \setminus (V_{n-1} \cap U_x) \to U_{i+1} \cap U_x$ and $j'_{i+1} : U_x \setminus (V_{n-1} \cap U_x) \to U_x$. We have by induction:

$$(j_{!*}\, {}^p\mathbf{L})|U_x = j'_{i+1!*}\, {}^p\mathbf{L} = \tau_{\leq -i-1} j_{i+1*}(j'_{i+2!*}({}^p\mathbf{L}|(U_x \setminus (V_{n-1} \cap U_x)))) =$$
$$\tau_{\leq -i-1} j_{i+1*} \circ \cdots \circ \tau_{\leq -n} j_{n*}({}^p\mathbf{L}|(U_x \setminus (V_{n-1} \cap U_x)).$$

By definition $(j_{i+1*} j'_{i+2!*}({}^p\mathbf{L}|U_x))_x \simeq R\Gamma(U_x \setminus S \cap H_x, j'_{i+2!*}({}^p\mathbf{L}|U_x))$ which, by constructibility and retraction of $U_x \setminus S$ onto $\mathring{c}L_{2n-2i-1} \setminus x$ and finally onto $L_{2n-2i-1}$, reduces to the hypercohomology of the link:

$$R\Gamma(\mathring{c}L_{2n-2i-1} \setminus x, j_{i+1!*}({}^p\mathbf{L}|U_x) \simeq R\Gamma(L_{2n-2i-1}, j_{!*}\, {}^p\mathbf{L}).$$

The lemma i) follows after truncation.

ii) The properties here give a characterization of the intermediate extension and apply to the dual complex ([GMacP 83] §6, [Max 19] 6.2.7 and 6.4.1).

Example 2.2. Let $\mathbf{L}$ be defined on $(\mathbb{C}^*)^n \overset{j}{\hookrightarrow} \mathbb{C}^n$ by a vector space L and n nilpotent endomorphisms, then the fiber $i_x^* j_{!*}\mathbf{L}$ at $x = 0$ is defined by the complex

$$L \to \cdots \to N_{k_i} L \cdots \oplus_{k_1 < \ldots < k_i} N_{k_1} \cdots N_{k_i} L \to \cdots \oplus_{k_1 < \ldots < k_n} N_{k_1} \cdots N_{k_n} L$$

with differenttials defined by N_r for $r \notin \{k_1, \ldots, k_i\}$ (see Definition 3.4). The dual complex is defined by various embedding $N_r^* N_{k_1}^* \cdots N_{k_i}^* L^* \hookrightarrow N_{k_1}^* \cdots N_{k_i}^* L^*$. Recall that by duality $Hom_{\mathbb{Q}}(H^{-i}(i_x^* j_{!*}{}^p\mathbf{L}), \mathbb{Q}) = H^i(i_x^! j_{!*}{}^p\check{\mathbf{L}})$.

The above constructions including the direct image j_{i*} and the truncation $\tau_{\leq -i}$ along the strata and the properties stated in the lemma give a motivation to the next definition of perverse sheaves.

2.2 Perverse sheaves

From now on, we write F instead of RF to denote the derived functors. We present a guideline to the abelian category of perverse sheaves for the middle perversity, including truncation functors and perverse cohomology. We are interested in the definition in terms of stratifications, but we start with an abstract definition of $t-$structures on abelian categories ([BBD 83], chapter 1.3, théorème 1.4.10) (see also [KaS 90] chapter X).

Definition 2.7 ($t-$structure). Two full subcategories $D^{\leq 0}$ and $D^{\geq 0}$ of a triangulated category D, define a $t-$structure on D if the following conditions are satisfied. Set $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$:

- $D^{\leq -1} \subset D^{\leq 0}$ and $D^{\geq 1} \subset D^{\geq 0}$
- $Hom_D(X, Y) = 0$ for $X \in D^{\leq 0}$ and $Y \in D^{\geq 1}$
- For any $X \in Ob\ D$ there exists a distinguished triangle $X_0 \to X \to X_1 \xrightarrow{+1}$ with $X_0 \in D^{\leq 0}$ and $X_1 \in D^{\geq 1}$

We have long exact sequences for all $Y \in ObD$:
$Hom_D(Y, X_1[-1]) \to Hom_D(Y, X_0) \to Hom(Y, X) \to Hom_D(Y, X_1)$ and
$Hom_D(X_0[1], Y) \to Hom_D(X_1, Y) \to Hom_D(X, Y) \to Hom_D(X_0, Y)$.

Example 2.3. The pair of the two strict full subcategories ${}^pD^{\geq 0}(V)$ and ${}^pD^{\leq 0}(V)$ of the category of constructible sheaves $D_c^b(V, \mathbb{Q})$ defined in 2.2.1.1 satisfy the above definition as proved in ([BBD 83], Proposition 2.1.3 and Corollary 2.1.4 applied to the middle perversity).

2.2.0.1 Truncation functor $\tau_{\leq n}$ and $\tau_{\geq n}$

We deduce the existence of truncation functors ([KaS 90] Proposition 10.1.4): $\tau_{\leq n} : D \to D^{\leq n}$ right adjoint to the inclusion $D^{\leq n} \to D$ and $\tau_{\geq n} : D \to D^{\geq n}$ left adjoint to the inclusion $D^{\geq n} \to D$ satisfying:
$\forall X \in ObD^{\leq n}, Y \in ObD : Hom_{D^{\leq n}}(X, \tau_{\leq n} Y) = Hom_D(X, Y)$ and
$\forall Y \in D^{\geq n}, X \in ObD : Hom_{D^{\geq n}}(\tau_{\geq n} X, Y) = Hom_D(X, Y)$
For any $X \in ObD$ there exists an exact triangle

$$\tau_{\leq 0}X \to X \to \tau_{>0}X \xrightarrow{+1}$$

where $\tau_{\leq 0}X \in D^{\leq 0}$ and $\tau_{>0}X \in D^{\geq 1}$ depend functorially on X. Set

$$\tau_{\leq n}X = (\tau_{\leq 0}(X[n]))[-n],\ \tau_{>n}X = (\tau_{>0}(X[n]))[-n].$$

The two full sub-categories with the three properties above, and the truncation functors $\tau_{\leq n}$, $\tau_{\geq n}$ form the t-structure. We have for all $n, m \in \mathbb{Z}$:

$$\tau_{\leq n}(X[m]) = (\tau_{\leq n+m}(X))[m],\ \tau_{\geq n}(X[m]) = (\tau_{\geq n+m}(X[n]))[m].$$

2.2.0.2 The (perverse) $t-$cohomology

The full subcategory P of D defined by $Ob\ P := Ob\ D^{\leq 0} \cap Ob\ D^{\geq 0}$ is called the heart of D or the category of perverse objects of D.

Proposition 2.3. *i) The heart $P := D^{\leq 0} \cap D^{\geq 0}$ is an abelian category.*

ii) If $X \to Y \to Z \xrightarrow{+1}$ is a distinguished triangle with X and Z in P, then Y is in P.

iii) If $0 \to X \to Y \to Z \to 0$ is an exact sequence in P, then there exists a unique morphism $\partial : Z \to X[1]$ such that $X \to Y \to Z \xrightarrow{\partial} X[1]$ is a distinguished triangle in D.

The definition of perverse cohomology $\mathcal{H}^*$ (or ${}^t\mathcal{H}^*$) is by truncation

$$\mathcal{H}^0(X) := \tau_{\geq 0}\tau_{\leq 0}(X) \in P,\ \mathcal{H}^n(X) := \mathcal{H}^0(X[n]) = \tau_{\geq 0}(\tau_{\leq 0}X[n])$$

The functor $\mathcal{H}^0 : D \to P$ defined on D to the abelian category of perverse sheaves on V is cohomological ([BBD 83] Th'eorème 1.3.6):

A distinguished triangle $\to A \to B \to C \xrightarrow{[1]}$ in D gives rise to the long exact sequence of perverse cohomology sheaves, where $\mathcal{H}^i := \mathcal{H}^0 \circ [i]$:

$$\cdots \to \mathcal{H}^i(A) \to \mathcal{H}^i(B) \to \mathcal{H}^i(C) \to \mathcal{H}^{i+1}(A) \to \cdots$$

In particular this is a sequence of complexes with morphisms in the abelian category P. The kernel and cokernel are computed as objects in the abelian category P ([BBD 83], Proposition 1.2.2).

2.2.0.3 Left and right t-exact functors

Let D_i be two triangulated categories for $i = 1$ and $i = 2$, with $t-$ structures $(D_i^{\leq 0}, D_i^{\geq 0})$ and perverse heart $P_i := D_i^{\leq 0} \cap D_i^{\geq 0}$, $\varphi : P_i \to D_i$.

Definition 2.8. i) A functor $F : D_1 \to D_2$ is left (resp. right) $t-$exact if $F(D_1^{\geq 0}) \subset D_2^{\geq 0}$ (resp. $F(D_1^{\leq 0}) \subset D_2^{\leq 0}$) is exact if it is left and right $t-$exact.

ii) Set $\varphi_1 : P_1 \to D_1$ and define ${}^pF := {}^p\mathcal{H}^0 \circ F \circ \varphi_1 : P_1 \to P_2$.

Lemma 2.4. *If F is left (resp. right) $t-$exact, then ${}^pF : P_1 \to P_2$ is left (resp. right) exact on the perverse heart abelian sub-categories.*

2.2.1 Perverse sheaves defined by the middle perversity

The concept of perversity introduced in [GMacP 83] is topological. However, on a complex algebraic variety V, the special case called middle perversity is adapted to Hodge theory.

A complex of sheaves of $\mathbb{Q}-$vector spaces $K^\bullet \in D^b(V,\mathbb{Q})$ is constructible with respect to a given stratification $\mathcal{S}$ of V, if the restriction of its cohomology sheaves $i_S^* H^i K$ to the strata S of a stratification $\mathcal{S}$ are locally constant sheaves of finite dimension. Let $D_c^b(V,\mathbb{Q})$ denote the derived category of bounded complexes with constructible cohomology on V. We give the definition of middle perversity in terms of a stratification.

Definition 2.9 (Perverse sheaves with respect to a stratification $\mathcal{S}$). Let $\mathcal{S}$ be a complex stratification of a complex variety V. A complex $K^\bullet$, constructible with respect to $\mathcal{S}$, is called perverse (with respect to the middle perversity) if the following conditions are satisfied:

$$\forall\, S \in \mathcal{S},\ H^i i_S^* K^\bullet = 0 \text{ for } i > -\dim S, \text{ and } H^i i_S^! K^\bullet = 0 \text{ for } i < -\dim S$$

where i_S^* is the restriction to S and $Ri_S^!$ (denoted $i_S^!$) is the derived functor of the functor of sections with support in S. Verdier duality transforms $i_S^!$ into i_S^* in $D_c^b(V,\mathbb{Q})$.

Due to Thom-Mather first isotopy theorem (§2.1.3.1) and the fact that the complex is constructible, the groups $H^k(i_S^* K^\bullet)$ and $H^k(i_S^! K^\bullet)$ are locally constant. Let $N_{S,x}$ be a normal section of S at x. The fibers at $x \in S$ are:

$$H^k(i_S^* K^\bullet)_x = H^k(i_x^*(i_{N_{S,x}}^* K^\bullet)) \text{ and } H^k(i_S^! K^\bullet)_x = H^k(i_x^!(i_{N_{S,x}}^* K^\bullet)).$$

Let $x \in S$, since for any locally constant sheaf $\mathbf{L}$ on S : $H_x^i(S,\mathbf{L}) = 0$ for $i < 2\dim S$, we deduce: $H_x^i(X,K^\bullet) := H^i(i_x^! K^\bullet) = 0$ for $i < \dim S$.

This definition is useful in the text where most of the proofs are by induction on the dimension of the strata. Perverse sheaves on a singular variety are independent of the stratification.

2.2.1.1 The $t-$structure defined by the middle perversity

Let $\mathcal{A} := C^+(V,\mathbb{Q})$ denote the abelian category of complexes of sheaves bounded at $-\infty$). Let $K\mathcal{A}$ denote the category with the same objects but with group of morphisms defined by homotopy classes of morphisms of complexes. The derived category $D^+(V,\mathbb{Q})$ i defined with the same objects but with the group of morphisms constructed by inverting the multiplicative system of quasi-isomorphisms (§2.1).

The derived category of bounded complexes with constructible cohomology sheaves $D_c^b(V,\mathbb{Q}) \subset D^+(V,\mathbb{Q})$ is the full sub-category defined by complexes with bounded constructible cohomology sheaves (vanishing cohomology in degrees $i > \infty$ and $i < \infty$).

The definition of perverse sheaves is based on the introduction of the t-structure with two full sub-categories $({}^pD_V^{\leq 0},\ {}^pD_V^{\geq 0})$ of $D_c^b(V,\mathbb{Q})$ defined by the following conditions:

$$Ob({}^pD_V^{\leq 0}) := \{K^\bullet \in D_c^b(V,\mathbb{Q}) : \forall j,\ \dim_{\mathbb{C}} \operatorname{supp} H^j(K^\bullet) \leq -j\}$$

where H^j is the local cohomology sheaf of degree j. Respectively, let D denotes the Verdier dual, then

$$Ob({}^pD_V^{\geq 0}) := \{K^\bullet \in D_c^b(V,\mathbb{Q}) : \forall j,\ \dim_{\mathbb{C}} \operatorname{supp} H^j(DK^\bullet) \leq -j\}.$$

Remark 2.3. i) The duality functor D interchanges ${}^pD_V^{\geq 0}$ and ${}^pD_V^{\leq 0}$, since for all $x \in V$: $D_V \circ i_x^* \simeq Ri_x^! \circ D_V$ or $i_x^*(D_V F) \simeq Hom^\bullet(Ri_x^! F, \mathbb{Q})$ and $Ri_x^!(D_V F) \simeq Hom^\bullet(i_x^* F, \mathbb{Q})$.

ii) The proof of the equivalence with the definition 2.9 is based on Thom-Mather first isotopy theorem §2.1.3.1 and the fact that the complex is constructible. In particular, the support of $H^j(K^\bullet)$ is a closed analytic subset (resp. algebraic), hence this definition is equivalent to the definition 2.1.2 in [BBD 83].

Definition 2.10. i) The category $P_V := {}^pD_V^{\leq 0} \cap {}^pD_V^{\geq 0}$ of perverse sheaves is the full sub-category of $D_c^b(V,\mathbb{Q})$ whose objects are in $Ob\,{}^pD_V^{\leq 0} \cap Ob\,{}^pD_V^{\geq 0}$.

ii) the perverse cohomology of a complex $K \in ObD_V$ is denoted by

$${}^p\mathcal{H}^0(K) := {}^p\tau_{\geq 0}\,{}^p\tau_{\leq 0}(K) \in P_V,\ {}^p\mathcal{H}^n(K) := {}^p\mathcal{H}^0(K[n]) = {}^p\tau_{\geq 0}(\tau_{\leq 0}K[n])$$

Remark 2.4. i) The perverse cohomology is a *cohomological functor* ${}^p\mathcal{H}^0 : D_c^b(V,\mathbb{Q}) \to P_V$ with value in the abelian category of perverse sheaves on V, where p refers to the specific perversity (the auto-dual perversity in this case).

ii) The family of functors ${}^p\mathcal{H}^i$ is *conservative*: a morphism $f : A \to B \in D_c^b(V,\mathbb{Q})$ is an isomorphism if and only if ${}^p\mathcal{H}^i(f) : {}^p\mathcal{H}^i(A) \simeq {}^p\mathcal{H}^i(B)$ for all $i \in \mathbb{Z}$ ([BBD 83] Proposition 1.3.7).

Moreover $K \in Ob({}^pD_V^{\leq 0})$ (resp. $Ob({}^pD_V^{\geq 0})$) if and only if ${}^p\mathcal{H}^i(K) = 0$ for $i > 0$ (resp. $i < 0$).

iii) On a singular variety V the usual cohomology sheaf $H^i(K)$ of a complex $K \in ObD_c^b(V,\mathbb{Q})$ is a constructible sheaf different than the perverse cohomology sheaf ${}^p\mathcal{H}^i(K)$ which is a complex of sheaves. The objects in P_V are called perverse sheaves, despite the fact that they are complexes of sheaves.

Example 2.4. Let $\mathbf{L}$ be a local system on the big stratum $U \subset V$ of equal dimension n and $j : U \to V$, then ${}^p\mathbf{L} := \mathbf{L}[n]$ is perverse on U. If $V \setminus U$ is a local principal divisor, $j_!{}^p\mathbf{L}$ and $j_*{}^p\mathbf{L}$ are perverse on X. The intermediate extension $j_{!*}{}^p\mathbf{L}$ is perverse .

2.2.1.2 Gluing t-structures along a stratification

In a typical proof by induction over a strata, we have a set of inclusions $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} F$ of an open subset U and its complement a closed subset F. Whenever F is a principal divisor, j is affine.
Usual functors are defined between the triangulated categories $D^b_c(U)$, $D^b_c(X)$ and $D^b_c(F)$: the direct image j_*, i_*, the inverse image j^*, i^*, the extension $j_!$ by 0, and $i^!$ the support in F ([BBD 83], 1.4.1).

Theorem 2.3 ([BBD 83] Theorem 1.4.10). *Let $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} F$ be defined by an open subset U and its closed complement F in an algebraic variety X. If a t-structure $({}^pD^{\leq 0}_U, {}^pD^{\geq 0}_U)$ (resp. $({}^pD^{\leq 0}_F, {}^pD^{\geq 0}_F)$) is defined on U (resp. on F), there exists a unique t-structure on X defined by*

$$ {}^pD^{\leq 0}_X := \{K^\bullet \in D^b_c(X) : j^*K^\bullet \in {}^pD^{\leq 0}_U \text{ and } i^*K^\bullet \in {}^pD^{\leq 0}_F\} $$
$$ {}^pD^{\geq 0}_X := \{K^\bullet \in D^b_c(X) : j^*K^\bullet \in {}^pD^{\geq 0}_U \text{ and } i^!K^\bullet \in {}^pD^{\geq 0}_F\} $$

Middle perversity $t-$structure. Let V be a complex algebraic variety and let $\mathcal{S}$ be a Whitney-stratification on V and V_i the union of strata of complex dimension $\leq i$. We have a family of locally closed sub-varieties: $V := V_n \supset \cdots V_m \supset V_{m-1} \supset \cdots \supset V_0 \supset V_{-1} = \emptyset$.

The t-structure on $D^b_c(V_m, \mathbb{Q})$ is defined by increasing induction on m.

If $m = 0$ we consider the standard t-structure, where a perverse sheaf is just a locally constant sheaf in degree $-\dim V_0$ on the non singular variety V_0. For $m > 0$, we consider the diagram

$$ U = (V_m \setminus V_{m-1}) \hookrightarrow V_m \hookleftarrow V_{m-1} = F $$

By induction, we suppose the perverse t-structure defined on $D^b_c(F, \mathbb{Q})$. On $D^b_c(U, \mathbb{Q})$ we put the standard t-structure shifted by m :

$$ {}^pD^{\leq 0}_U := D^{\leq -m}_U, \quad {}^pD^{\geq 0}_U := D^{\geq -m}_U. $$

The perverse t-structure on $D^b_c(V_m, \mathbb{Q})$ is defined by gluing the two t-structures on $D^b_c(F, \mathbb{Q})$ and $D^b_c(U, \mathbb{Q})$.

Remark 2.5. We mention the following properties:

1. Let $\mathcal{F} \in Ob\, {}^pD^{\leq 0}$ and $\mathcal{G} \in Ob\, {}^pD^{\geq 0}$ on V, then $H^i(R\mathcal{H}om(\mathcal{F}, \mathcal{G})) = 0$ for $i < 0$. In particular $U \mapsto Hom_{D(U)}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$ is a sheaf ([KaS 90] Proposition 10.2.7).
2. If $f : X \to V$ is smooth of relative dimension d, $f^* \circ [d]$ is exact, moreover if the fibers are non empty, then $f^* \circ [d] : P_V \to P_X$ is fully faithful ([BBD 83] Proposition 4.2.5).
3. *Gluing perverse sheaves.* Given an open covering $V = \cup_{i \in I} U_i$ a family of perverse sheaves $\mathcal{F}_i$ on U_i and isomorphisms $f_{ij} : \mathcal{F}_{j|U_{ij}} \simeq \mathcal{F}_{i|U_{ij}}$ on the double intersections of U_i satisfying the cocycle condition $(f_{ij|U_{ijk}}) \circ$

$(f_{jk|U_{ijk}}) = (f_{ik|U_{ijk}})$ on the triple intersections of U_i, there exists a perverse sheaf $\mathcal{F}$ on V and $f_i : \mathcal{F}_{|U_i} \simeq \mathcal{F}_i$ such that $f_{ij} \circ f_{j|U_{ij}} = f_{i|U_{ij}}$. Moreover the family $(\mathcal{F}, f_i)$ is unique up to isomorphisms ([KaS 90] Proposition 10.2.9).
If I is finite, the proof is by induction on $|I|$. In case $I = \{1, 2\}$, we define F by the distinguished triangle $i_{U_{12!}}\mathcal{F}_1|_{U_{12}} \xrightarrow{(a,b)} i_{U_{1!}}\mathcal{F}_1 \oplus i_{U_{2!}}\mathcal{F}_2 \to \mathcal{F} \xrightarrow{1}$ where a is natural, b is deduced from f_{21}. If $I = \mathbb{N}$, we can suppose $U_n \subset U_{n+1}$ and take the inductive limit of the family F_n constructed by induction.

4. **Lefschetz-Artin's vanishing theorem**: The behavior of perverse sheaves and affine morphisms is basic in the text, we use freely the results ([BBD 83] §4.1). In particular, if X is an affine variety (resp. Stein) ([KaS 90], Theorem 10.3.8, [Max 19], Corollary 8.6.6):
$$\mathbb{H}^j(X, \mathcal{F}) = 0 \text{ for } j > 0 \text{ if } \mathcal{F} \in Ob\, {}^p\mathcal{D}^{\leq 0}$$
$$\mathbb{H}^j_c(X, \mathcal{F}) = 0 \text{ for } j < 0 \text{ if } \mathcal{F} \in Ob\, {}^p\mathcal{D}^{\geq 0}$$
Moreover, let $f : X \to Y$ be an affine morphism, then f_* is right $t-$exact and $f_!$ is left $t-$exact ([BBD 83] §4.1).
5. Let $j : U \to X$ be a Zariski open subset and $\mathcal{F} \in P_X$, we have distinguished triangles in $D^b_c(X)$:
$$j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \overset{+1}{\to} \quad \text{and} \quad i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \overset{+1}{\to}$$
In the case where $U \subset X$ is the complement of a principal divisor D, $j : U \to X$ is an affine embedding, hence $j_!\mathcal{F}, j_*\mathcal{F}$ are perverse. Let $i : D \to X$, we deduce the following exact sequences in the category of perverse sheaves P_X ([BBD 83] Corollaries 4.1.10, 4.1.12)
$$0 \to i_*\, {}^p\mathcal{H}^{-1}(i^*\mathcal{F}) \to j_!j^*F \to \mathcal{F} \to i_*\, {}^p\mathcal{H}^0(i^*\mathcal{F}) \to 0$$
$$0 \to i_*\, {}^p\mathcal{H}^0(i^!\mathcal{F}) \to \mathcal{F} \to j_*j^*\mathcal{F} \to i_*\, {}^p\mathcal{H}^1(i^!\mathcal{F}) \to 0$$
$$i^*j_{!*}\mathcal{F}[-1] = {}^p\mathcal{H}^{-1}(i^*j_*\mathcal{F}) = {}^p\mathcal{H}^0(i^!j_!\mathcal{F})$$
$$i^!j_{!*}\mathcal{F}[1] = {}^p\mathcal{H}^0(i^*j_*\mathcal{F}) = {}^p\mathcal{H}^1(i^!j_!\mathcal{F})$$

2.2.1.3 Intermediate extensions

Let $U \overset{j}{\hookrightarrow} X$ be an algebraic open subset with complement $F \overset{i}{\hookrightarrow} X$. A perverse sheaf $\mathcal{F} \in P_X$ extends a perverse sheaf $\mathcal{G} \in P_U$ if $j^*\mathcal{F} \simeq \mathcal{G}$. There may be many extensions. For example both ${}^p\mathcal{H}^0(j_!\mathcal{G})$ and ${}^p\mathcal{H}^0(j_*\mathcal{G})$ extend $\mathcal{G}$.

Definition 2.11. There exists a unique perverse sheaf $\mathcal{F}$ on X extending a perverse sheaf $\mathcal{G}$ on U satisfying the following equivalent properties:

1. $i^*\mathcal{F} \in {}^pD^{\leq -1}_F$ and $i^!\mathcal{F} \in {}^pD^{\geq 1}_F$.
2. $\mathcal{F}$ is the image in P_X of the morphism ${}^p\mathcal{H}^0(j_!\mathcal{G}) \to {}^p\mathcal{H}^0(j_*\mathcal{G})$.

$\mathcal{F}$ is denoted by $j_{!*}\mathcal{G}$ and called the Intermediate extension of $\mathcal{G}$.

Remark 2.6. i) The following exact sequences associated to $\mathcal{F}$ in P_X and $F \subset X$

$$0 \to i_* \, {}^p\mathcal{H}^0(i^!\mathcal{F}) \to \mathcal{F}, \quad \mathcal{F} \to i_* \, {}^p\mathcal{H}^0(i^*\mathcal{F}) \to 0,$$

define the biggest sub-object (resp. the biggest quotient of $\mathcal{F}$) with support in F. Hence:

$$j_{!*}\mathcal{G} = {}^p\mathcal{H}^0(j_!\mathcal{G})/i_* \, {}^p\mathcal{H}^0(i^!j_!\mathcal{G}) = \mathrm{Ker}\,({}^p\mathcal{H}^0(j_*\mathcal{G}) \to i_* \, {}^p\mathcal{H}^0(i^*j_*\mathcal{G}))$$

where $i_* \, {}^p\mathcal{H}^0(i^!j_!\mathcal{G})$ is the biggest sub-object of ${}^p\mathcal{H}^0(j_!\mathcal{G})$ in P_F and $i_* \, {}^p\mathcal{H}^0(i^*j_*\mathcal{G})$ is the biggest quotient of ${}^p\mathcal{H}^0(j_*\mathcal{G})$ in P_F .

1) ${}^pj_*\mathcal{G}$ has no non trivial sub-object with support in F.

2) ${}^pj_!\mathcal{G}$ has no non trivial quotient-object with support in F.

3) $j_{!*}\mathcal{G}$ has no non trivial sub-object or quotient-object with support in F.

ii) Let F be a locally principal divisor in X and $\mathcal{G} \in P_U$. The derived extension $j_!\mathcal{G}$ of $\mathcal{G}$ by 0 and the derived direct image $j_*\mathcal{G}$ are perverse on X ([BBD 83] Corollaire 4.1.10) and $j_{!*}\mathcal{G} := \mathrm{Im}\,(j_!\mathcal{G} \to j_*\mathcal{G})$ ([BBD 83] Définition 1.4.22).

iii) Let $U \overset{j}{\hookrightarrow} Z \overset{i}{\hookrightarrow} X$ denote a a closed subvariety Z and U a smooth open Zariski subset of Z. Let $\mathcal{G} \in P_U$ and $k = i \circ j : U \to X$, then we write $k_{!*} \, {}^p\mathcal{G}$ for $i_*j_{!*} \, {}^p\mathcal{G}$ on X.

If $\mathcal{G}$ be a simple object in P_U, then $k_{!*} \, {}^p\mathcal{G}$ is a simple object of P_X.

iv) Let $\mathcal{G} := \mathbf{L}$ be a local system, we have: $D_X(j_{!*}{}^p\mathbf{L}) \simeq j_{!*}{}^p\check{\mathbf{L}}$ where $\check{\mathbf{L}}$ is the linear dual of $\mathbf{L}$.

Theorem 2.4. *i) On a complex algebraic variety X, the category of perverse sheaves $P_X \subset D^b_c(X, \mathbb{Q})$ is abelian (although $D^b_c(V, \mathbb{Q})$ is not abelian), stable under Verdier duality, Noetherian and Artinian ([BBD 83] Théorème 1.3.6).*

ii) The simple perverse sheaves are intermediate extensions of irreducible local systems defined on non singular open subsets of irreducible closed algebraic sub-varieties of X and any perverse complex sheaf has finite length.

2.2.2 $\psi_f j_{!*}\mathbf{L}$ and extensions of perverse sheaves

Let X be a complex analytic variety, $f : X \to D$ a non constant analytic morphism to a disc in $\mathbb{C}$ such that the analytic restriction $f_| : X_{D^*} \to D^*$ is a topological fibration over the punctured disc D^* with small radius. This is the case if f is proper or f is is the restriction of an algebraic morphism of complex algebraic varieties.

Let K be an object of the derived category $D^+_c(X^*, \mathbb{Q})$, i.e K is a complex of sheaves of $\mathbb{Q}$-vector spaces, bounded below, with locally constant cohomology sheaves on the strata of a Thom - Whitney stratification of X.

The nearby cycle complex $\psi_f K$ on $X_0 := f^{-1}(0)$ with monodromy action T, is defined by Deligne's construction ([DeK 73] II, Exposé XIV).

We introduce the universal covering $p : \widetilde{D}^* \to D^*$ of the punctured disc $D^* := D \setminus 0$. It is convenient (but not necessary) to consider the topological space $\widetilde{D} := \widetilde{D}^* \cup \{0\}$ where $\widetilde{D}^*$ is open and a fundamental set of neighborhoods of 0 is defined by $p^{-1}(U)$ where U is a fundamental set of a neighborhoods of 0 in D, then p extends to $\bar{p} : \widetilde{D} \to D$ by $p(0) = 0$.

Set $X^* = X \setminus X_0, \bar{X} := X \times_D \widetilde{D}^*, \widetilde{X}^* := X \setminus X_0$ and consider the following left diagram over the right diagram:

$$\begin{tikzcd} X_0 \arrow[r, "\bar{i}"] \arrow[d, "="] & \bar{X} \arrow[d, "\bar{p}"] & \widetilde{X}^* \arrow[l, "\bar{j}"'] \arrow[d, "p"] \arrow[dl, "\widetilde{p}"] \\ X_0 \arrow[r, "i"] & X & X^* \arrow[l, "j"] \end{tikzcd} \qquad \begin{tikzcd} \{0\} \arrow[r, "\tilde{i}"] \arrow[d, "="] & \widetilde{D} \arrow[d, "\bar{p}"] & \widetilde{D}^* \arrow[l, "\tilde{j}"'] \arrow[d, "p"] \\ \{0\} \arrow[r, "i"] & \mathbb{C} & \mathbb{C}^* \arrow[l, "j"] \end{tikzcd} \tag{2.13}$$

The space $\widetilde{X}^*$ is homotopy equivalent to any fiber X_t, as a topological space of a fiber bundle over $\widetilde{D}^*$ which is contractible.

This construction provides a complex analytic space $\widetilde{X}^*$ homotopic to the general fiber. The nearby cycle complex of sheaves $\psi_f K$ of K is defined in $D_c^+(X_0, \mathbb{Q})$ by:

$$\psi_f K := \bar{i}^* R\bar{j}_*(p^* K) = i^* R j_* R p_*(p^* K) = i^* R j_* R\mathcal{H}om^\bullet(p_! \mathbb{Q}_{\widetilde{X}^*}, K) \tag{2.14}$$

The first equality is defined by the natural morphism:

$$i^* R j_* R p_* p^* K \simeq i^* R\bar{p}_* R\bar{j}_* p^* K \to \bar{i}^* R\bar{j}_* p^* K := \psi_f K$$

which is an isomorphism by base change: $\bar{p}^* R j_* K \xrightarrow{\sim} R\bar{j}_* p^* K$. Remark that p_* and $p_!$ are exact functors and $p^! = p^*$. The second equality follows by Verdier duality: $R\mathcal{H}om^\bullet(p_! \mathbb{Q}_{\widetilde{X}^*}, K) \simeq p_* R\mathcal{H}om^\bullet(\mathbb{Q}_{\widetilde{X}^*}, p^* K) = p^* K$.

Under the above conditions the complex $\psi_f K$ is constructible on X_0. The functor ψ_f reflects the cohomological properties of the general fiber near the special fiber X_0. The cohomology of ψ_f at a point $x \in X_0$ is isomorphic to the hypercohomology of the Milnor fiber $X_{f,x}$ in a neighborhood of x small enough:

$$H^i((\psi_f K)_x) \simeq \mathbb{H}^i(X_{f,x}, K).$$

Vanishing cycle complex $\varphi_f K$. We suppose now $K \in D_c^+(X, \mathbb{Q})$ defined on X and we still denote by $\psi_f K$ the complex $\psi_f j^* K$. The monodromy action on $\widetilde{D}^*$ over D^* associated to a generator of $\pi_1(D^*)$ defines a monodromy automorphism T acting on $\psi_f K$.

There exists a natural morphism $sp : i^* K \to \psi_f K$. The vanishing cycle complex of sheaves $\varphi_f K$ is defined as the cone of sp; hence we have a triangle with a canonical morphism *can* fitting in a distinguished triangle:

$$i^*K \xrightarrow{sp} \psi_f K \xrightarrow{can} \varphi_f K \to i^*K[1] \tag{2.15}$$

Since the natural morphism $sp : i^*K \to \psi_f K$ is $T-$equivariant. The action of T extends to the cone and defines an action on $\varphi_f K$ denoted also by T.

Then, the action $Id - T : \psi_f K \to \psi_f K$ is well defined on the cone $\varphi_f K$ and induces a (topological) variation denoted $var : \varphi_f K \to \psi_f K$ such that $var \circ can = Id - T$.

Proposition 2.4 (Invariance under monodromy). *There exists a natural distinguished triangle:*

$$i^*Rj_*K \xrightarrow{sp} \psi_f K \xrightarrow{Id-T} \psi_f K \xrightarrow{[1]} i^*Rj_*K[1] \tag{2.16}$$

The monodromy T acts on Rp_*p^*K. The derived functor of the invariant sub-complex under the action of T is by definition the derived functor $R\mathcal{H}om^{\bullet}_{\mathbb{Z}[T]}(\mathbb{Z}, Rp_*p^*K)$ where $\mathbb{Z}[T]$ acts trivially on $\mathbb{Z}$ (it is denoted also by $R\Gamma(\mathbb{Z}, Rp_*p^*K)$). Such derived functor is computed after a result of Hochschild-Serre, which yields the following triangle in $D^+_c(X^*, \mathbb{Q})$ (see [Br 86] Théorème 1.2)

$$K \to Rp_*p^*K \xrightarrow{Id-T} Rp_*p^*K \xrightarrow{1} \tag{2.17}$$

We apply i^*Rj_* to the triangle 2.17 and deduce the proposition from $\psi_f K = i^*Rj_*Rp_*p^*K$ (Equation 2.14):

Corollary 2.1. *There is an exact sequence of perverse sheaves on X_0*
$0 \to {}^p\mathcal{H}^{-1}(i^*Rj_*j^*K) \to \psi^u_f K[-1] \xrightarrow{Id-T} \psi^u_f K[-1] \to {}^p\mathcal{H}^0(i^*Rj_*j^*K) \to 0$

The shift by $[-1]$ is explained by the fact that $\psi^u_f K[-1]$ is perverse. The corollary follows from the long exact sequence associated to the triangle in equation 2.16.

Proposition 2.5 (Duality). *There is a non canonical isomorphism in the derived category on X_0: $D_{X_0}(\psi_f K[-1]) \simeq \psi_f(D_X(K))[-1]$.*

Proof. See ([Br 86] Lemme 1.4). The proof by after Beilinson and Bernstein is based on a concept similar to the sections of finite determination introduced by Deligne. For a general complex F of objects of an abelian category $\mathcal{A}$ on which an automorphism T acts, and for each non zero element $P \in \mathbb{Q}[T, T^{-1}]$.

Let $F(P)$ denotes the simple complex associated to the double complex $P : F \to F$.

If P divides Q, there exists a morphism of double complexes $F(P) \to F(Q)$ defined by the morphism of double complexes: $(F \xrightarrow{P} F) \xrightarrow{(Id,\, Q/P)} (F \xrightarrow{Q} F)$. We suppose that the inductive limit exists in $\mathcal{A}$ and define:

$$F_{fin\,det} : \varinjlim F(P) \text{ directed by divisibility of polynomials.}$$

There is a natural morphism $F_{fin\,det} \to F$. We have

Lemma 2.5. *Let F be a bounded complex with constructible cohomology in the category of sheaves of $\mathbb{Q}-$vector spaces on X_0. Then:*
$F_{fin\,det} \xrightarrow{\sim} F$ *is an isomorphism*

Proof. The functor: $F \to F_{fin\,det}$ being exact, the proof may be reduced by "dévissage" to the case where F is a constructible sheaf in degree 0. Then, we check the lemma on the fibres of F at any point x: $(F_{fin\,det})_x \simeq (F_x)_{fin\,det}$ as $\mathbb{Q}[T,T^{-1}]-$ modules of *finite length* (by constructibility over $\mathbb{Q}$).

Let $B := \mathbb{Q}[T,T^{-1}]$, and M a $B-$ module of finite length with attached complex $M_{fin\,det}$, then

$$H^0(M_{fin\,det}) \simeq \varinjlim_P Hom_B(B/(P),M) = M,$$
$$H^1(M_{fin\,det}) \simeq \varinjlim_P Ext^1_B(B/(P),M) = 0$$

since every element of M is annihilated by some polynomial in B.

Lemma 2.6. *i)* $\psi_f K := i^*Rj_*p_*(p^*K) \simeq i^*Rj_*p_*(p^*K)_{fin\,det}$
ii) $(p_*p^*K)_{fin\,det} \simeq (p_!p^*j^*K)_{fin\,det}[1]$

In the diagram (Equation 2.13), $p_!\mathbb{Q}_{\widetilde{X}^*}$ is a locally constant sheaf of free $\mathbb{Q}[T,T^{-1}]-$modules of rank 1. In this case the previous lemma does not apply since the module M *is a free* $\mathbb{Q}[T,T^{-1}]-$*module* and not of finite length.

The complex M_P is quasi-isomorphic to $(M/P.M)[-1]$ (non zero cohomology in degree 1). We have:

$$(p_!\mathbb{Q}_{\widetilde{X}^*})_{fin\,det}[1] \simeq \varinjlim_P \mathbb{Q}[T,T^{-1}]/(P).$$

ii) While $p_*\mathbb{Q}_{\widetilde{X}^*} \simeq \mathbb{Q}[[T,T^{-1}]] \otimes_{\mathbb{Q}[T,T^{-1}]} p_!\mathbb{Q}_{\widetilde{X}^*}$ is a locally constant sheaf of $\mathbb{Q}[T,T^{-1}]-$modules locally isomorphic to $\mathbb{Q}[[T,T^{-1}]]$

The complex $\mathbb{Q}[[T,T^{-1}]]_{(P)}$ is non-canonically isomorphic to $\mathbb{Q}[T,T^{-1}]/(P)$ (the elements annihilated by P form a non zero sub-module just the opposite of the previous case of a free $\mathbb{Q}[T,T^{-1}]-$module and since any non zero polynomial P is invertible, the cohomology in degree 1 vanish). Moreover, we can define a commutative diagram of quasi-isomorphisms

$$\begin{array}{ccc}
\mathbb{Q}[T,T^{-1}]/(P) & \xrightarrow{Q/P} & \mathbb{Q}[T,T^{-1}]/(Q) \\
\downarrow{\scriptstyle q.iso.} & & \downarrow{\scriptstyle q.iso.} \\
\mathbb{Q}[[T,T^{-1}]]_{(P)} & \longrightarrow & \mathbb{Q}[[T,T^{-1}]]_{(Q)}
\end{array}$$

hence $(p_*\mathbb{Q}_{\widetilde{X}^*})_{fin\,det} \simeq \varinjlim_P \mathbb{Q}[T,T^{-1}]/(P) \simeq (p_!\mathbb{Q}_{\widetilde{X}^*})_{fin\,det}[1]$.
We deduce lemma ii) since $p_*p^*K \simeq K \otimes (p_*p^*\mathbb{Q}_{X^*})$.

Proof of of the proposition. Representing K by a complex of injective sheaves over $\mathbb{Q}[T,T^{-1}]$ we deduce from the lemma

$\psi_f K \simeq i^*Rj_*(p_*p^*K)_{fin\,det} \simeq i^*Rj_*(p_!p^*K)_{fin\,det}[1] = i^*Rj_*(p_!p^*K)[1]$
hence: $(\psi_f(DK))[-1] = (i^*Rj_*p_*p^*(DK))[-1] = i^*Rj_*p_!p^*(DK)_{fin\,det} =$

$D(i^! Rj_! p_* p^* K)_{fin\,det} = D(i^* Rj_* p_* p^* K[-1])_{fin\,det} = D(\psi_f K[-1])$
since $i^* j_* p_* p^* K[-1] \simeq i^! j_! p_* p^* K$.

The perversity follows from Artin's theorem and duality ([Br 86] Théorème 1.2, see also [Max 19] Theorem 10.3.13) and the references there:

Proposition 2.6 (Perversity). *i) Let K be a perverse sheaf on X, then $\psi_f K[-1]$ is perverse on X_0.*
ii) $\varphi_f K[-1]$ is perverse on X_0.
iii) $\psi_f K[-1], \varphi_f K[-1] : P_X \to P_{X_0}$ are exact functors of perverse sheaves.

i) The proof is local on X_0. Let U_x be a small ball with center $x \in X_0$. Since the Milnor fiber $X_{f,x}$ is a principal hypersurface in U_x, $K[-1]$ restricts to a perverse sheaf on the Stein subvariety $X_{f,x}$. We deduce by Artin's theorem (Remark 2.5) that the ordinary cohomology $\mathbb{H}^i(X_{f,x}, \psi_f K[-1]) \simeq (\psi_f K[-1])_x$ vanish for $i > 0$.

Since the cohomology $\mathcal{H}^k(P)$ sheaves of any perverse sheaf P on a variety Y vanish for $k > 0$, we deduce that the perverse cohomology ${}^p\mathcal{H}^i(\psi_f K[-1])$ vanish for $i > 0$. Since this inequality apply to the dual sheaf $D_X K$, we deduce by duality that ${}^p\mathcal{H}^i(\psi_f K[-1])$ vanish for $i < 0$; hence $\psi_f K[-1]$ is perverse.

ii) $\varphi_f K[-1]$ *is perverse.* We deduce from the perverse exact sequence defined by the triangle in (Equation 2.15), the exact sequence

$0 \to {}^p\mathcal{H}^{-2}(\varphi_f K) \to {}^p\mathcal{H}^{-1}(i^* K) \xrightarrow{can_{-1}} {}^p\mathcal{H}^{-1}(\psi_f K) \to {}^p\mathcal{H}^{-1}(\varphi_f K) \to$
${}^p\mathcal{H}^0(i^* K) \to 0$

where ${}^p\mathcal{H}^{-1}(\psi_f K) = \psi_f K[-1]$ and ${}^p\mathcal{H}^i(\psi_f K) = 0$ if $i \neq -1$.

We prove can_{-1} is injective. By the corollary 2.1, it is enough to prove that ${}^p\mathcal{H}^{-1}(i^* K) \xrightarrow{h_{-1}} {}^p\mathcal{H}^{-1}(i^* Rj_* j^* K)$ is injective. Since ${}^p\mathcal{H}^i(i^* K) = 0$ for $i \leq -2$ ([BBD 83], Corollary 4.1.10), the functor $K \to {}^p\mathcal{H}^{-1}(i^* K)$ from perverse sheaves on X to perverse sheaves on X_0 is left exact.

Let N denotes the perverse kernel of $K \to Rj_* j^* K$. Since the sequence of perverse sheaves $0 \to N \to K \to Rj_* j^* K$ is exact we deduce:
$0 \to {}^p\mathcal{H}^{-1}(i^* N) \to {}^p\mathcal{H}^{-1}(i^* K) \to {}^p\mathcal{H}^{-1}(i^* Rj_* j^* K)$ is exact.
Since N is supported by X_0, $i_* i^* N = N$ is perverse and ${}^p\mathcal{H}^{-1}(i^* N) = 0$, hence h_{-1} is injective.

iii) The assertion is admitted.

2.2.3 Verdier's classification of the extensions

Extensions of perverse sheaves along a locally principal divisor $D \subset X$ occur in the work of Deligne, Mac Pherson and Verdier in various ways. The description provided by Verdier [V 85-1] is useful to construct perverse sheaves;

in particular the case of singularities along a NCD. It is conveniently applied to prove the decomposition properties in Hodge theory.

We recall the result only in the case of an extension along an effective principal divisor defined by an equation f. Let $\psi_f^u K[-1]$ (resp. $\varphi_f^u K[-1]$) denotes the sub-sheaf on which the action of T is unipotent.

The result of Verdier covers the case of locally principal divisor $Y \subset X$ (and even a closed subvariety Y in general), useful in the construction of the weight filtration in the open case $X \setminus Y$ out of the monodromy filtration along the cone $C_Y X$ of Y in X ([V 85-1], [Sa 90], [ELM 10]).

The extension is expressed in terms of the nearby cycle functor $\psi_f^u[-1]$ and the vanishing cycle $\varphi_f^u[-1]$ (Equation 2.15).

Let $\mathrm{Perv}(X)$ denote the abelian category of perverse sheaves on an algebraic variety X and let $Z \subset X$ be a principal divisor defined by an equation $f = 0$, $U := X \setminus Z, i : Z \to$ and $j : U \to X$.

We associate to Z the category $V(Z)$ consisting of:

1. A perverse sheaf $G \in Perv(U)$ on U
2. A diagram in the category $Perv(Z)$

$$\psi_f^u(G)[-1] \xrightarrow{a} \Phi \xrightarrow{b} \psi_f^u(G)[-1] : b \circ a = Id - T \tag{2.18}$$

   where T is the monodromy of $\psi_f^u(G)[-1]$.
3. Morphisms are couple of morphisms of diagrams and over U.

The triple (a, Φ, b) is the variable in $V(Z)$ which determines the extension.

Lemma 2.7. *The functor* $V : \mathrm{Perv}(X) \to V(Z)$

$$V : F \mapsto (j^*F, \{\psi_f^u F[-1] \xrightarrow{can} \varphi_f^u F[-1] \xrightarrow{var} \psi_f^u F[-1]\})$$

is an equivalence of categories.

Proof. We adapt the proof ([V 85-1], §2 Proposition , §4 Corollary 1) to our case. By definition $\psi_f^u F = \psi_f^u j^* F$. Since F is perverse, the complexes $\psi_f^u F[-1]$ and $\varphi_f^u F[-1]$ are perverse sheaves on Z. We recall the triangles where *var* is defined by $Id - T$:

$$\to i^! F \to \varphi_f^u F[-1] \xrightarrow{var} \psi_f^u F[-1] \xrightarrow{[1]}, \ \to i^! F \to F \to j_* j^* F \xrightarrow{[1]} \tag{2.19}$$

where the first triangle is dual to (Equation 2.15).

V is faithful: If $m : F \to G$ satisfies $V(m) = 0$, then by definition $j_* j^* m = 0$ and $\psi_f^u(m) = 0$, $\varphi_f^u(m) = 0$. Hence by the above first triangle $i^! m = 0$ and by the second $m = 0$.

V is fully faithful: Let F, G be perverse on X and let $m' : V(F) \to V(G)$ be a morphism in $V(Z)$, hence $m' = (u, \alpha, \beta, \alpha)$ where $u : j^*F \to j^*G$, $\alpha = \psi_f^u(u)$ and $\beta = \varphi_f^u(u)$. We construct $m : F \to G$ with image $V(m) = m'$ as follows.

We deduce from $\beta : \varphi_f^u F[-1] \to \varphi_f^u G[-1]$ and $\alpha : \psi_f^u F[-1] \to \psi_f^u G[-1]$ a morphism $\gamma : i^! F \to i^! G$ compatible with the first triangle and then a morphism $m : F \to G$ compatible with γ and $j_* j^* u$ in the second triangle.
V is essentially surjective: Given a diagram $\psi_f^u(G)[-1] \xrightarrow{a} \Phi \xrightarrow{b} \psi_f^u(G)[-1]$ and $G \in Per(U)$, we construct $F \in Per(X)$ such that $\varphi_f^u F[-1] \simeq P$.

First we construct a triangle $\to F_1 \to \Phi \xrightarrow{b} \psi_f^u F[-1] \xrightarrow{[1]}$ with support in Y (where the complex F_1 is expected to be $i^! F$) with associated perverse exact sequence $0 \to {}^p\mathcal{H}^0(F_1) \to \Phi \xrightarrow{b} \psi_f^u G[-1] \to {}^p\mathcal{H}^1(F_1) \to 0$. Hence, we have $\operatorname{Ker} b \simeq {}^p\mathcal{H}^0(F_1)$, ${}^p\mathcal{H}^1(F_1) \simeq \operatorname{Coker} b$.

Then we deduce F_2 satisfying the second triangle $F_1 \to F_2 \to j_* G \xrightarrow{[1]}$ such that we have a long exact sequence of perverse sheaves:

$$0 \to {}^p\mathcal{H}^0(F_1) \to {}^p\mathcal{H}^0(F_2) \to {}^p\mathcal{H}^0(j_* G) \to {}^p\mathcal{H}^1(F_1) \to {}^p\mathcal{H}^1(F_2) \to 0$$

where ${}^p\mathcal{H}^{-1}(j_* G) = 0 = {}^p\mathcal{H}^1(j_* G)$ and ${}^p\mathcal{H}^0(j_* G) = j_* G$ since j is affine.
We deduce the exact sequence

$$0 \to \operatorname{Ker} b \to {}^p\mathcal{H}^0(F_2) \xrightarrow{v} j_* G \to \operatorname{Coker} b \to {}^p\mathcal{H}^1(F_2) \to 0$$

In particular, the restriction of v to U is an isomorphism and ${}^p\mathcal{H}^0(F_2)$ is a perverse extension of $j_* G$ since . Moreover, $F_1 \simeq i^! \, {}^p\mathcal{H}^0(F_2)$ since $\operatorname{Ker} b \simeq i^! \, {}^p\mathcal{H}^0(F_2)$, $\operatorname{Coker} b \simeq i^! \, {}^p\mathcal{H}^1(F_2)$ and $\Phi \simeq \varphi_f^u F[-1]$.

Remark 2.7. In the text, we will use a variation morphism *var* instead of the topological variation *var* defined by $N' := \pm(T_j^u - I)$ acting on ψ_f^u and $var : \varphi_f^u \to \psi_f^u$ satisfying: $var \circ can = N'$.

In chapters 4 and 5 we introduce the action ν on $\Psi^* I\mathcal{L} \simeq \psi_f^u \mathbf{L}$ (§4.21) with Hodge filtration. By (Equation 3.1), $-2i\pi\nu$ induces $N = LogT$ on $\psi_f^u \mathbf{L}$.

The action of N on $\psi_f^u K$ induces a morphism of mixed Hodge structure up to a shift . Since *sp* is equivariant for N, the action of N extends to $\varphi_f^u K$ (instead of T in topology) and we introduce a geometrical $var : (\varphi_f^u, W, F) \to (\psi_f^u, W, F)(-1)$ such that: $var \circ can = N$, instead of the *var* defined by N'.

However, there exists a polynomial A such that $N = A(N')$ (see Equation 3.2) and N' is nilpotent. Since A is invertible as a series and since N is nilpotent, we find a polynomial B such that $N' = B(N)$.

We deduce that $\ker N = \ker N'$, and N' is also compatible with MHS. Moreover, the monodromy filtrations $W(N), W(N')$ and $W(\nu)$ defined by the nilpotent endomorphisms N, N' or ν (§2.3.0.1) coincide and are all defined over $\mathbb{Q}$.

For each $k > 1$, $N^k : \operatorname{Gr}_{-k}^{W(N)} L^u \xrightarrow{\sim} \operatorname{Gr}_k^{W(N)} L^u$ is equal to N'^k or ν^k up to constants, which explains that there is no danger to cite references with one of these conventions.

2.2.3.1 Applications

Since the applications in view are in Hodge theory, we use henceforth the *var* defined by N instead of N'.

Due to the equivalence of the categories $\mathrm{Perv}(X)$ and $V(Z)$, the following correspondence will be useful in various proofs in the text (*var* denotes two morphisms, one on ψ_f and one on φ_f).

1) $j_!G$: The case where $a : \psi_f(G)[-1] \xrightarrow{\sim} \Phi$ is an isomorphism and $b = var \circ a^{-1}$: $\psi_f(G)[-1] \xrightarrow{a} \Phi \xrightarrow{b} \psi_f(G)[-1]$ corresponds to the extension $j_!G$ of G by zero across Z ($\ker a = \ker N = i_Z^* G[-1] = 0$) such that we have the diagram where *can* is an isomorphism

$$(\psi_f G[-1] \overset{a=can\simeq}{\longrightarrow} \varphi_f G[-1] \overset{b=var}{\longrightarrow} \psi_f G[-1]) \tag{2.20}$$

2) $j_{!*}G$: The case where a is an epimorphism and b a monomorphism such that $\Phi = \mathrm{Im}(\psi_f(G)[-1] \xrightarrow{N} \psi_f(G)[-1])$. Then, the diagram

$$(\psi_f G[-1] \overset{a=\mathrm{can}}{\longrightarrow} \varphi_f G[-1] \overset{b=\mathrm{var}}{\hookrightarrow} \psi_f G[-1]) \tag{2.21}$$

corresponds to the intermediate extension $j_{!*}G$.

3) The case where $G_{|U} = 0$ corresponds to the perverse sheaf Φ concentrated on Z.

4) A decomposition of the diagram in (Equation 2.18) corresponds to a decomposition of the extension. In particular, we the following Verdier's decomposition criteria:

$$\Phi \simeq \mathrm{Im}\, can \oplus \mathrm{Ker}\, var \iff j_{!*}G_{|U} \oplus \mathrm{Ker}\, var \tag{2.22}$$

is used in the text with $\mathrm{Ker}\, var$ concentrated on Z.

5) The extension $j_*G_{|U}$ corresponds to the case where b is an isomorphism and and $a = b^{-1} \circ var$ ($b \circ a = N$) (Equation 9.43) and the diagram where *var* is an isomorphism

$$(\psi_f G[-1] \overset{a=\mathrm{can}}{\longrightarrow} \varphi_f G[-1] \overset{b=\mathrm{var}\simeq}{\longrightarrow} \psi_f G[-1]) \tag{2.23}$$

6) In the case with an increasing filtration W^0 on G such that the relative monodromy $M := M(N, W^0)$ exists on $\psi_f G[-1]$, the filtration of weight

$$(N * W^0)_k := NW^0_{k+1} + (M_k \cap W^0_{k+1})\psi_f^u G[-1]$$

lifts to $\varphi_f^u G[-1]$ and corresponds to a filtration W on $j_*G_{|U}$ such that (W_k extends W^0_k (§9.7.1.1).

2.3 Degeneration of VMHS

The theory of nilpotent orbit (resp. admissible variation of MHS) is needed in the next chapters to define the limit Hodge filtration and the weight filtration. The theory, inspired by Deligne, appears in [Sc 73, GrSc 73, Gr 74, CaKSc 86, CaKSc 87, CEGL 14] (resp. [De 80], I.8.15, [E 83], [StZ 85]) but we adopt the notations and terminology of [Ka 86] summarized below.

The admissibility of a VMHS on $X^* := X \setminus Y$ refers to asymptotic properties of $(\mathbf{L}, W, F)$ along the NCD. Such asymptotic properties are expressed on Deligne's extension $\mathbf{L}_X$ of $\mathbf{L} \otimes_{\mathbb{Q}} \mathcal{O}_{X^*}$ defined along Y in terms of the 'multivalued' horizontal sections of ∇ on X^*. The extension $\mathcal{L}_X$ is a locally free analytic sheaf of modules, hence algebraic if X is projective [De 70].

By construction, the connection ∇ on $\mathcal{L}_{X^*}$ extends to $\mathcal{L}_X$ with logarithmic singularities along Y. The residues of the logarithmic singularities are endomorphisms on the restrictions $\mathcal{L}_{Y_i}$ along the components Y_i of Y. The connection is uniquely characterized by the choice of the logarithm of the eigenvalues of the monodromy (see §3.1).

The fibre $L := \mathcal{L}(x) := \mathcal{L}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$ of the vector bundle $\mathcal{L}_X$ is viewed as the space of the 'multivalued' horizontal sections of $\mathbf{L}$ at x (sections of a universal covering of the complementary of Y in a ball B_x at x).

The extension over Y of the Hodge filtration of a polarizable variation of HS in the case of a punctured disc constructed by Schmid [Sc 73, GrSc 73], ([Gr 74] Chapter IV) is a fundamental asymptotic property, that is required by assumption in the case of a graded polarizable variation of MHS, as a condition of admissibility. The results on the degeneration of a VHS on the complement of a NCD are based on [Sc 73], [CaKSc 86] and [CaKSc 87], see also ([Gr 74] Chapter V, [CEGL 14] Chapters 7, 8).

A filtration W of $\mathbf{L}$ over a punctured disc extends to a filtration by sub-bundles $\mathcal{W} \subset \mathcal{L}_X$. The local monodromy T_i around a component Y_i of Y, defines a nilpotent endomorphism $N_i := Log\, T_i^u$, logarithm of the unipotent part of T which acts on the restrictions to Y_i of the sub-bundles $\mathcal{W}$. The definition of admissibility is based on the introduction by Deligne of the relative monodromy filtration $M(\sum_i N_i, W))$ on the limit vector space L in ([De 80], I.8.15).

The required properties are proved in the case of geometric variation of MHS over a punctured disc in [E 83] and studied axiomatically as conditions of admissibility in [StZ 85]. In [Ka 86], the definition of admissibility along a NCD is reduced to the case over a punctured disc.

To state the degeneration of a variation of MHS locally, Kashiwara introduced the abelian category of infinitesimal MHS (IMHS) called also infinitesimal mixed Hodge modules in reference to D_X-modules. *A proof of the decomposition theorem is given in terms of distinguished pairs of IMHS.*

2.3.0.1 The monodromy filtration

On the cohomology group of a complex projective non singular variety X of dimension n: $L := \oplus_{k\in\mathbb{Z}} H^k(X, \mathbb{Q}[n])$ centered at 0, an increasing filtration W on L is defined by $W_i = \oplus_{j\geq -i} H^j(X, \mathbb{Q}[n])$ ($W_n = L$ and $W_{-n} = H^n(X, \mathbb{Q}[n])$). The Lefschetz structure, defined by the cup product with the class of an hyperplane section is a linear morphism $\eta : W^i \to W^{i-2}$ satisfying for each i an isomorphism $\eta^i : \mathrm{Gr}_i^W L \xrightarrow{\sim} \mathrm{Gr}_{-i}^W L$.

To state conjecturally the properties of the degeneration of a VHS, Deligne defines in general on an object L of an abelian category and a nilpotent endomorphism N on L, a unique increasing filtration W satisfying the following conditions ([De 80] 1.6)

(i) $\forall j \in \mathbb{N} : NW_j \subset W_{j-2}$.
(ii) The induced morphism $N^i : Gr_{-i}^{W(N)} L \xrightarrow{\sim} Gr_i^{W(N)} L$ is an isomorphism for all $i \in \mathbb{N}$.

Such filtration W is denoted by $W(N)$ and called the monodromy filtration. In fact $W_{i-1} = (N^i)^{-1} W_{-i-1}$ and $W_{-i} = N^i W_i$ for $i \geq 1$.
If $N^{k+1} = 0$ and $N^k \neq 0$ set $W_i = L$ for $i \geq k$ and $W_i = 0$ for $i < -k$, then defines W_i inductively for $i \in [-k, k-1]$, to start with $W_{-k} = N^k L, W_{k-1} = \mathrm{Ker}\, N^k$. It is surprising that the defining properties of W, will set various decomposition properties through the whole chapter.

For $i \geq 0$, let $P_i L := \mathrm{Ker}\, N^{i+1} : \mathrm{Gr}_i^W L \to \mathrm{Gr}_{-i-2}^W L$ denote the subspace called the primitive subspace, then $\mathrm{Gr}_i^W L = P_i L \oplus N\mathrm{Gr}_{i+2}^W L = \oplus_{j\geq 0} P_{i+2j} L$.

2.3.0.2 The relative monodromy filtration

Let $N : (L, W) \to (L, W)$ be a nilpotent endomorphism compatible with W ($NW_i \subset W_i$) in an abelian category $\mathfrak{A}$ where W is an increasing filtration of an object L of $\mathfrak{A}$.

There exists at most a unique filtration M satisfying ([De 80], I.8.15)

$$NM_j \subset M_{j-2}, \text{ and } \mathrm{Gr}_{i+k}^M \mathrm{Gr}_i^W L \xrightarrow{N^k} \mathrm{Gr}_{i-k}^M \mathrm{Gr}_i^W L$$

If M exists, it is called the relative monodromy filtration. In particular, if N_i is the endomorphism induced by N on $\mathrm{Gr}_i^W L$, then M induces on $\mathrm{Gr}_i^W L$ the monodromy filtration $W(N_i)[i]$ shifted by i ($(W(N_i)[i])_{i+j} := W(N_i)_j$). Properties of the filtration are given in [StZ 85] and [E 83].

2.3.1 Nilpotent orbit

Let $(L, F, S, N_1, \ldots, N_n, w')$ be defined by a finite dimensional complex vector space L with a $\mathbb{Q}$-structure, a decreasing filtration F by complex vector subspaces, an integer w', a non-degenerate rational bilinear form S satisfying

$$S(x, y) = (-1)^{w'} S(y, x) \text{ for } x, y \in L \text{ and } S(F^p, F^q) = 0 \text{ for } p + q > w',$$

and pairwise commuting nilpotent endomorphisms $N_1, \ldots, N_n$ defined over $\mathbb{Q}$, such that $S(N_i x, y) + S(x, N_i y) = 0$ and $N_i F^p \subset F^{p-1}$.

Definition 2.12 (Nilpotent orbit). ([Sc 73], [CaKSc 86], [Gr 74], [Ka 86], 4.1) The above data is called a (polarized) nilpotent orbit of weight w' if :

(i) The monodromy filtration $W(N)$ of the endomorphism $N = \sum_{j \in [1,n]} t_j N_j$ is independent of t_j if $t_j > 0$ for all j.
(ii) Let $M := W(N)[w']$ denote the filtration shifted to right ($M_{w'+k} := W(N)_k$). The data (L, M, F) is a MHS on L.
(iii) The bilinear form S_k such that $S_k(x, y) = S(x, N^k y)$ polarizes the primitive subspace $P_k = \mathrm{Ker}\,(N^{k+1} : \mathrm{Gr}^M_{w'+k} \to \mathrm{Gr}^M_{w'-k-2})$ with its induced HS of weight $w' + k$.

The various definitions in this chapter are based on the following result applied throughout this volume to Deligne's extension $\mathcal{L}_X$ (Equation 3.9):

Theorem (Cattani, Kaplan and Schmid [CaKSc 86]) Let $\mathbf{L}$ be polarized VHS on the complement of a NCD $Y \subset X$. At each point $x \in Y$, the extension of the Hodge filtration $\mathcal{F}$ on $\mathcal{L}_{X \setminus Y}$ defines a nilpotent orbit with local monodromy actions $(N_i, i \in I := [1, n])$ on the limit fiber $L := \mathcal{L}_X(x)$.

Let $N_K := W(\sum_{i \in K \subset I} N_i)$, for each partition $K \cup J = I$, $W(N_I)$ coincides with the relative monodromy filtration $M(N_K, W(N_J))$.

Remark 2.8. i) The results mentioned above, along a period starting around 1966, are difficult to prove but well covered by the references.

i) The proof of the theorem is based on on a generalization of the nilpotent and SL_2 orbit approximations of the period matrix to higher dimension [Sc 73, GrSc 73, CaKSc 86].

In general, for each $J \subset [1, n]$ and fixed $t_j > 0$, $N_J := \sum_{j \in J} t_j N_j$, is a nilpotent endomorphism of L such that the increasing monodromy filtration $W(N_J)$ is independent of the positive real integers t_j and we have a variation of limit MHS at points of Y^*_J depending on J.

ii) The concept of splitting of a MHS is necessary to the proof: there exists a real number c such that $(L, exp(i \sum_j t_j N_j) F)$ is a HS of weight w', polarized by S whenever $t_j > c$.

iii) The filtration $W(N)$ in the definition above is also said to define a MHS on L of weight w'.

iv) In the text, we refer to the summary of the results in ([KaK 87] §1.2).

2.3.2 Infinitesimal mixed Hodge structure (IMHS)

We consider a data $(L, W, F, N_1, \ldots, N_n)$, where L is a finite dimensional complex vector space with a $\mathbb{Q}$-structure, W is an increasing $\mathbb{Q}$-filtration, F is a decreasing $\mathbb{C}$-filtration and $N_1, \ldots, N_n$ are commuting nilpotent endomorphisms defined over $\mathbb{Q}$ such that: $N_j F^p \subset F^{p-1}$ and $N_j W_k \subset W_k$.

Definition 2.13 (Mixed nilpotent orbit). The above data is called a (graded polarizable) mixed nilpotent orbit if for each integer i the data with restricted structures

$$(\mathrm{Gr}_i^W L, F_|, (N_1)_|, \ldots, (N_n)_|)$$

is a nilpotent orbit of weight i for some polarization S_i.

We consider only graded polarized mixed nilpotent orbits (called pre-infinitesimal mixed Hodge module in [Ka 86], 4.2).

Definition 2.14 (IMHS). ([Ka 86], 4.3)

A mixed nilpotent orbit $(L, W, F, N_1, \ldots, N_n)$ is called an infinitesimal mixed Hodge structure (IMHS) if the following conditions are satisfied:

(i) For each $J \subset I = \{1, \ldots, n\}$, the monodromy filtration $M(J)$ of $\sum_{j \in J} N_j$ relative to W exists and satisfies $N_j M(J)_i \subset M(J)_{i-2}$ for all $j \in J$ and $i \in \mathbb{Z}$.

(ii) The filtrations $M(I)$ and F define a graded polarizable MHS on L such that the filtrations W and $M(J)$ are compatible with this MHS (filtrations by sub-MHS and the N_i are morphisms of type $(-1, -1)$.

We refer to the filtration $M(\sum_{i \in I} N_i, W)$ as the limit MHS.

Remark 2.9. (i) The filtration $M(J)$ is the weight filtration relative to W of any $N \in C(J) := \{\sum_{j \in J} t_j N_j ; t_j > 0\}$.

(ii) In the case of an admissible VMHS $(\mathbf{L}, W, F)$ on the polydisc $(D^*)^n = D^n \setminus Y$ where $Y \subset D^n$ is the natural NCD, an IMHS is attached to $\mathbf{L}$, at each $x \in Y_J^*, J \subset [1, n]$, on the fiber of Deligne's extension $\mathcal{L}(x)$. Since Y_J is defined by z_i for $i \in J = \{i_1, \cdots, i_j\}$, we introduce the local system on $\psi_J \mathbf{L}$ as the restriction to Y_J^* of $\psi_{i_j} \circ \cdots \circ \psi_{i_1} \mathbf{L}$ (independent of the order in J up to isomorphism; we write ψ_j for ψ_{z_j}) to define the VMHS $(\psi_J \mathbf{L}, \mathcal{L}_{Y_J^*}, M(J), F(J))$ on Y_J^* where $M(J)$ and $F(J)$ form the limit IMHS at varying $x \in Y_J^*$. Moreover, a combination of ψ_{z_i} and φ_{z_j} is necessary to characterize locally a perverse sheaf.

(iii) At $y \in Y_J \setminus Y_J^*$ there exist two IMHS on $\mathcal{L}(y)$, one defined directly by $(\mathbf{L}, W, F)$ and one defined by $(\psi_J \mathbf{L}, \mathcal{L}_{Y_J^*}, M(J), F(J))$. They coincide since $M(N_1 + \cdots + N_i)$ is the relative weight filtration of N_i with respect to $M(N_1 + \cdots + N_{i-1})$ ([Ka 86, Prop. 5.2.5]). The IMHS $(M(J), F(J))$ can be viewed as successive IMHS along the various Y_J^*.

Example 2.5. In the case of two variables, a remarkable result states that the limit $(W(N_1+N_2),F)_0$ is relative to the limit at 0 of the limit $(W(N_i),F)_x$ at $x_j, j \neq i$

$$\begin{tikzcd}
x_2 & & (\mathbf{L},F) \arrow[ll, "(W(N_1),F)_{x_2}"'] \arrow[dll, "(W(N_1+N_2),F)_0"'] \arrow[d, "(W(N_2),F)_{x_1}"] \\
0 \arrow[u] \arrow[rr] & & x_1
\end{tikzcd}$$

A morphism of two IMHS $f:(L,W,F,N_1,\dots,N_n)\to(L',W,F,N_1',\dots,N_n')$ is a linear morphism $L\to L'$ respecting W,F, commuting with N_i and N_i' inducing a morphism of a graded nilpotent orbits with respect to Gr_*^W.

A graded polarized MHS is an IMHS when identified with the object with $N_i=0$ for $i\in[1,n]$. If (L,N,F) is a nilpotent orbit, then $(L,W(N),F)$ is an IMHS with $N_i=0$ and $N:(L,W(N),F)\to(L,W(N)[2],F(-1))$ is a morphism of IMHS with $N_i=0$ on both sides.

Proposition 2.7. *([Ka 86] Prop. 5.2.6) The category of IMHS is abelian. The filtrations W,F, and the limit MHS $M(J)$ for any set J of indices, define exact functors on this category.*

A morphism $f:L\to L'$ of two IMHS is necessarily strict for W, since by the nilpotent orbit theorem W is the weight filtration and $exp(i\sum_j t_jN_jF)$ the Hodge filtration of a MHS on L (resp. L') for t_i large enough. Similarly $M(N_1,W)$ is the weight of a MHS on L (resp. L').

To prove that the restriction of $M(N_1,W)$ to $L''\coloneqq\operatorname{Ker}f$ is the monodromy filtration relative to $W\cap L''$, let W'' and M'' denote the restrictions to L'', then we deduce from the exact sequence $0\to\mathrm{Gr}_{k+i}^{M''}\mathrm{Gr}_k^{W''}\to\mathrm{Gr}_{k+i}^{M}\mathrm{Gr}_k^{W}\to\mathrm{Gr}_{k+}^{M'}\mathrm{Gr}_k^{W'}$ and the corresponding exact sequence for Gr_{k-i} that $N_1^i:\mathrm{Gr}_{k+i}^{M''}\mathrm{Gr}_k^{W''}\to\mathrm{Gr}_{k-i}^{M''}\mathrm{Gr}_k^{W''}$ is an induced isomorphism.

Definition 2.15 (Preadmissible VMHS). A variation of mixed Hodge structure $(\mathbf{L},W,F)$ graded polarized over the punctured unit disc D^* with local monodromy T and associated filtration $\mathcal{W}$ by subbundles of Deligne's extension $\mathcal{L}_{\mathcal{O}_D}$, is called *preadmissible* if the following conditions are satisfied:

i) The Hodge filtration $\mathcal{F}\subset\mathcal{L}_{\mathcal{O}_{D^*}}$ extends to a filtration $\mathcal{F}$ on $\mathcal{L}_{\mathcal{O}_D}$ by subbundles inducing for each k on the graded sheaf with respect to the filtration $\mathcal{W}$ defined by W $Gr_k^{\mathcal{W}}\mathcal{L}_{\mathcal{O}_D}$, Schmid's extension of the Hodge filtration.

ii) Let $W^0\coloneqq\mathcal{W}(0)$, $F_0\coloneqq\mathcal{F}(0)$ denote the filtrations on the fiber $L\coloneqq\mathcal{L}_{\mathcal{O}_D}(0)$ at $0\in D$ and T the local monodromy at 0, $N=\log T^u$; then the following two conditions must be satisfied:

1) $NF_0^p\subset F_0^{p-1}$ for all $p\in\mathbb{Z}$.

2) The weight filtration $M(N,W^0)$ relative to W^0 exists.

Notice that the extension of the filtration $\mathcal{F}$ to $\mathcal{L}_{\mathcal{O}_D}$ cannot be deduced in general from the various Schmid extensions to $Gr_k^{\mathcal{W}}\mathcal{L}_{\mathcal{O}_D}$. We remark that the filtrations $M := M(N, W^0)$ and F_0 at the origin define a MHS.

Definition 2.16 (Admissible VMHS). ([Ka 86], §1.8) Let X be a complex analytic space and U a nonsingular open subset, complement of a closed analytic subset of X. A graded polarized variation of mixed Hodge structure $(\mathbf{L}, W, F)$ on U is called *admissible* if, for every analytic morphism $f : D \to X$ on a unit disc which maps D^* to U, the inverse $(f_{|D^*})^*(\mathbf{L}, W, F)$ is a preadmissible variation on D^*.

In the case of locally unipotent VMHS, Kashiwara remarked that preadmissible VMHS on the unit disc are necessarily admissible [Ka 86].

Remark 2.10 (IMHS defined by an admissible VMHS). i) Let $Y \subset X$ be a NCD and $x \in Y_M^*$ for $M \subset Y$. In the text we introduce Deligne's extension $\mathcal{L}_X$ associated to a local system $\mathcal{L}$ on $X \setminus Y$ and the vector space $L := \mathcal{L}_X(x)$ (Definition 3.1). An admissible VMHS $(\mathbf{L}, W^0, F)$ on $X \setminus Y$ defines at x an

$$(\mathcal{L}, \mathcal{W}^0, \mathcal{F}) \Rightarrow \text{IMHS: } (L, W^0, F, P, N_i, i \in M) \text{ where } L := \mathcal{L}_X(x)$$

where the filtration W^0 of L, preserved by N_i, is the limit of the filtration $\mathcal{W}^0$ of $\mathcal{L}$. In this case, the relative monodromy filtration $M(\sum_{i\in M} N_i, W^0)$ defines the limit MHS on L.

ii) Let $Z := X \setminus U$ and let $\pi : X' \to X$ be a desingularization of X such that $Z' := \pi^{-1}Z$ is a NCD, then $(\pi^*\mathbf{L}, W, F)$ is admissible along Z'. In particular $(\pi^*\mathbf{L}, W, F)|_{D^*}$ defines an IMHS at any point in Z'.

In the text, we define an induced Hodge filtration F on the intermediate extension of a local system on $X \setminus Y$ (see §2.2.1.3) which reflects the behaviour of the limit filtration F on $\mathcal{L}$ at 'infinity' along the NCD Y.

2.4 Induced filtrations on the perverse cohomology

The category of bifiltered complexes is defined in ([De 72] and [De 75]). Let $(K', W, F) \in D^bF_2(X, \mathbb{C})$ be a bifiltered complex of abelian sheaves on the variety X. A morphism $\phi : (K_1, W, F) \to (K_2, W, F)$ in the category $D^bF_2(X, \mathbb{C})$ is an isomorphism if the morphism $Gr_*^W Gr_F^* K_1 \to Gr_*^W Gr_F^* K_2$ is a quasi-isomorphism. If W is the weight filtration defined with $\mathbb{Q}-$coefficients, a precise diagram of compatibility with the $\mathbb{Q}-$structure is satisfied.

Let $f : X \to V$. The derived direct image $(K'', W, F) := (f_*K', f_*W, f_*F)$ is defined in $D^bF_2(V, \mathbb{C})$ by taking f_*-acyclic bi-filtered resolutions of the complex (K', W, F) ([De 72] 1.4).

The perverse filtration ${}^p\tau_i$ on a complex K on a variety V is defined by a family of morphisms $\varphi_i : {}^p\tau_i \to K$. Given a filtration F of K, the problem is to define the induced filtration F on the perverse cohomology.

It is possible to define the resticrion of F to ${}^p\tau_i$ by $\varphi_i^{-1}(F)$, however by a remark of Deligne, it is possible to suppose ${}^p\tau$ a filtration by sub-complexes of K which has the advantage to apply directly the results of ([De 72]).

2.4.0.1 Embedding of the perverse filtration

For any morphism of complexes $f : L \to K$, let L' denote the acyclic cone $L' := cone(id : L \to L)$. We consider after ([BBD 83] 1.1.2) the following embedding σ_f defined by:

$$\sigma_f : L \to K' := K \oplus L' : \sigma_f^i : L^i \to K^i \oplus L^{i+1} \oplus L^i : v \mapsto (f^i(v) + 0 + v)$$

The injection $u : K \to K \oplus L'$ is a quasi-isomorphism such that $u \circ f$ is homotopic to σ_f, while the projection $p : K \oplus L' \to K$ defines a quasi-isomorphism $K' \xrightarrow{\sim} K$ such that $p \circ \sigma_f = f$.

Given a filtration F by sub-complexes of K, we define the filtration $F' := F \oplus L' \subset K'$ such that the restriction of F' to $\sigma_f(L)$ is isomorphic to the inverse image $f^{-1}F$ on L. We deduce:

Lemma 2.8. *Let K be a bounded constructible complex on a variety V, F_1 and F_2 two filtrations of K by sub-complexes (not necessarily constructibles) and let ${}^p\tau$ denote the perverse filtration of K.*

There exists a bifiltered complex (K', F_1', F_2') where K' is bounded and constructible with filtrations by sub-complexes of K' and a bifiltered quasi-isomorphism $\pi : (K, F_1, F_2) \to (K', F_1', F_2')$ such that the perverse filtration of K' is defined by sub-complexes of K'.

In the case of a finite sequence of complexes $g_i : {}^p\tau_i \to K$ we repeat the above process by a descending induction on the indices i.

Corollary 2.2. *i) A well defined filtration F on the perverse cohomology ${}^p\mathcal{H}^i(K)$ is obtained by inducing the filtration F on ${}^p\tau_i K$ followed by the projection on ${}^p\mathcal{H}^i(K)[-i] := {}^p\tau_i K / {}^p\tau_{i-1} K$.*

In the text, we suppose the filtrations such as ${}^p\tau, W, F$ defined by sub-complexes of a complex K on a variety V.

Remark 2.11. i) There is no perverse filtration defined by a sub-complex $F^i \subset K$ if F is not constructible.

The bifiltered complex $(K, {}^p\tau, F) \in D^b F_2(V, \mathbb{C})$ is defined in the bifiltered category of sheaves of complex vector spaces (forgetting constructibility).

ii) Given a functor T, we consider acyclic $T-$resolutions of $Gr_p^{{}^p\tau} Gr_F^q K$. In the case of the global section functor Γ, we have the canonical acyclic bi-filtered Godement resolution $\mathcal{C}^*(K, {}^p\tau, F)$ of $(K, {}^p\tau, F)$ to define $R(\Gamma, *)$. Since the resolution $\mathcal{C}^*$ is canonical, the complex $Gr_F^i Gr_j^{{}^p\tau} \mathcal{C}^*$ is also an acyclic resolution of $Gr_F^i Gr_j^{{}^p\tau} K$. In particular, we deduce in this way the correct Hodge filtration on $\mathbb{H}^k(V, {}^p\mathcal{H}^i(K))$.

iii) In presence of Hodge filtration F or weight filtration W we deduce induced filtrations on the perverse cohomology. The construction is similar to Deligne's definition of induced filtrations filtration such that we can apply [De 72]. Since the filtration W is perverse, we consider the perverse weight spectral sequence and the induced filtration F on ${}^p\mathcal{H}^{p+q}(Gr^W_{-p}K)$.

The image filtration $\mathrm{Im} : {}^p\mathcal{H}^{p+q}(W_{-p}K) \to {}^p\mathcal{H}^{p+q}(K)$ in the abelian category of perverse sheaves defines a filtration W on ${}^p\mathcal{H}^{p+q}(K)$ with induced filtration F on the limit of the spectral. By abuse of language, we say the filtered perverse spectral sequence or perverse filtration to refer to such induced filtrations.

2.4.0.2 Abbreviations and conventions

1. Polarized VHS (resp. VMHS) stands for polarized variation of Hodge structure (resp. mixed Hodge structure). A polarized VHS $(\mathbf{L}, F)$ of weight w' is given by a local system $\mathbf{L}$ on a smooth algebraic open subset $U \subset V$ and a decreasing filtration F of $\mathcal{L} = \mathbf{L} \otimes \mathcal{O}_U$. The complex ${}^p\mathbf{L} := \mathbf{L}[\dim V]$ shifted by the dimension of V, is a perverse sheaf and $({}^p\mathbf{L}, F)$ is a complex of weight ${}^pw = w' + \dim V$. We say that the perverse sheaf ${}^p\mathbf{L}$ is a shifted polarized VHS.
2. Given a functor $T : \mathfrak{A} \to \mathfrak{B}$ of abelian categories, its derived functor is also denoted by $T : D^+\mathfrak{A} \to D^+\mathfrak{B}$ after chapter 3 instead of RT in chapter 2 [Ve 77] (with exceptions if confusion may arise such $R\Gamma$ for example). However, we still denote the $i-$th cohomology of the derived image by T by $R^iT(K) := H^i(T(K)$.
3. On a complex variety V, the topological middle perversity truncations ${}^p\tau_{\leq i}$ on K ([BBD 83] section 2 and Proposition 2.1.17) define an increasing filtration denoted ${}^p\tau$ on the hypercohomology:
$$ {}^p\tau_i\mathbb{H}^k(V, K) := \mathrm{Im}\left\{\mathbb{H}^k(V, {}^p\tau_i K) \to \mathbb{H}^k(V, K)\right\}. $$
4. An intermediate extension is denoted by $j_{!*}{}^p\mathbf{L}$, while ${}^p\mathcal{H}^i(K)$ denotes the $i-$th perverse cohomology of a complex K.

Chapter 3
Perverse sub-sheaves of the logarithmic complex $\Omega^{*p}\mathcal{L}$

In [De 72] Deligne introduced the concept of Hodge complex (HC) of sheaves on a compact complex variety X as a filtered complex (K, F) such that its hypercohomlogy $(R\Gamma(X, K), F)$ is a Hodge structure (HS). The de Rham complex of a non singular complex variety with its Hodge filtration is a HC. Originally Deligne distinguished between the cohomological HC of sheaves and the HC of its hypercohomology. By abuse of language, only the terminology of HC survived.

If X is not compact, a bifiltered mixed Hodge complex (MHC) (K, W, F) is constructed on a compactification $\bar{X}$ such that $(R\Gamma(\bar{X}, K), W, F)$ is a mixed Hodge structure (MHS).

The introduction of the intersection complex by Goresky and MacPherson [GMacP 88] lead to a new type of HC.

Let $\mathbf{L}$ be a local system on the complement of a NCD $Y \subset X$ in a complex non singular variety X and let $j := (X \setminus Y) \to X$. We describe in this chapter the well known logarithmic complex $\Omega^*\mathcal{L} := \Omega^*_X(LogY) \otimes \mathcal{L}_X \simeq j_*\mathbf{L}$ witlh coefficients in Deligne's extension $\mathcal{L}_X$ [De 70] in order to define the Intersection complex $IC^*\mathcal{L} \subset \Omega^*\mathcal{L}$ embedded as a sub-complex with its Hodge filtration (see §3.1).

Let ${}^p\mathbf{L} := \mathbf{L}[\dim X]$,the complex $\Omega^*\,{}^p\mathcal{L} := \Omega^*\mathcal{L}[\dim X]$ is the associated perverse sheaf with the conventional shifted degree. It is common to say perverse logarithmic complex with or without shift.

Let $i_Z : Z \to X$be the embedding of a NCD sub-divisor of $Y \subset X$ and $j_Z := (X \setminus Z) \to X$. We construct an embedded logarithmic complex isomorphic to $j_{Z*}j_Z^*(j_{!*}\mathbf{L})$ denoted by $IC^*\,{}^p\mathcal{L}(LogZ) \subset \Omega^*\,{}^p\mathcal{L}$ (cf §3.2).

To motivate t The constructions are motivated by the application of Kashiwara's criteria for a decomposition of a perverse sheaf into a direct sum of intermediate extensions (cf §3.2.6).

We recall the behavior of perverse sheaves along locally principal divisor (see §3.2.4), to construct later more bifiltered logarithmic complexes, such as

$$j_{Z!}j_Z^*(j_{!*}{}^p\mathbf{L}),\ i_Z^*j_{Z*}j_Z^*(j_{!*}{}^p\mathbf{L}),\ i_Z^!j_{!*}{}^p\mathbf{L} \text{ and } i_Z^*j_{!*}{}^p\mathbf{L}$$

We refer to such complexes as bifiltered logarithmic complexes.

3.1 Perverse logarithmic sub-complexes of $\Omega^{*p}\mathcal{L} \simeq j_*{}^p\mathbf{L}$

Let $\mathbf{L}$ be a locally constant sheaf (finite dimensional local system of vector spaces) on the complement U of a NCD Y in a smooth complex algebraic variety X and $\mathcal{L}_U := \mathbf{L}\otimes\mathcal{O}_U$ the analytic locally free sheaf of modules defined by $\mathbf{L}$ on U.

There is a correspondence between local systems $\mathbf{L}$ and connections $(\mathcal{L}_U, \nabla)$ with vanishing curvature and horizontal sheaf of sections of ∇.

Locally at each point $x \in Y$, a ball B_x with center x and $B_x^* := B_x \cap U$ are Stein. The restriction of $\mathcal{L}$ to B_x^* is free. Each trivialization gives rise to a free extension of $\mathcal{L}_{|B_x^*}$.

Moreover, since the local system $\mathbf{L}$ is defined on U ($\mathcal{L}_U$ is flat), there exists a locally free global extension of $\mathcal{L}_U$ with a regular singular connection (or with logarithmic singularities), that is a fibre bundle $\mathcal{L}_X$ extending $\mathcal{L}_U$ and a connection $\nabla : \mathcal{L}_X \to \Omega^1_X(LogY) \otimes \mathcal{L}_X$ ([De 70], [Ma 87] definition 3.1). The residue of ∇ is defined along a component Y_i of the NCD Y as an endomorphism of the restriction $\mathrm{Res}_{Y_i}\nabla : \mathcal{L}_{Y_i} \to \mathcal{L}_{Y_i}$, of $\mathcal{L}_X$ to Y_i.

The eigenvalues of the residue are constant along a connected component of Y_j and related to the local monodromy T_j of $\mathbf{L}$ at a general point of Y_j by the formula ([De 70], theorem 1.17, proposition 3.11):

$$Log\, T_j = -2i\pi \mathrm{Res}_{Y_j}\nabla \tag{3.1}$$

Deligne's idea is to fix the choice of the residues of the connection by the condition that the eigenvalues of the residue belong to the image of a section of the projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$, determined by fixing the real part of $z : m \leq \mathcal{R}(z) < m+1$, hence fixing the determination of the logarithm $Log : \mathbb{C}^* \to \mathbb{C}$, which forces the uniqueness of the local extensions. Hence the local constructions glue into a global bundle on X. The extension in the case $m = 0$ is called Deligne's extension.

The connection ∇ extends naturally into a complex, called the logarithmic complex $\Omega^*_X(LogY) \otimes \mathcal{L}_X$ and denoted by $\Omega^*\mathcal{L}$.

3.1.1 Local definitions along a NCD $Y \subset X$

Let $Y = \cup_{i\in I} Y_i$ denote the decomposition of Y into smooth irreducible components with finite index I.
For $M \subset I$, set $Y_M := \cap_{i\in M} Y_i$, $Y_M^* := Y_M \setminus \cup_{i\notin M}(Y_i \cap Y_M)$ ($Y_\emptyset^* := X \setminus Y$).
We denote uniformly the various embeddings by $j : Y_M^* \to X$.

To describe $\mathcal{L}_X$ in local coordinates, let $n := |M|$, $x \in Y_M^*$ and B_x a small polydisc in X with center x, isomorphic to $D^{n+\ell}$ with coordinates y_i. It is not restrictive to suppose $\ell = 0$, such that $Y \cap B_x$ is defined by $y_1 \cdots y_n = 0$

and $B_x^* := B_x \setminus (Y \cap B_x)$ is isomorphic to $(D^*)^n$. We identify the ball B_x with $D^n \subset \mathbb{C}^n$ the product of complex discs.

The restriction of the VHS $\mathbf{L}_{|B_x \setminus (B_x \cap Y)}$ corresponds to a VHS $\mathbf{L}$ on $B^* = (D^*)^n$ defined by a set $(L, T_i, i \in [1,n])$ including a rational vector space L isomorphic to the multivalued horizontal sections of $\mathbf{L}$ and $n-$commuting linear automorphisms called the monodromy transformations of L.

Let $\pi : \widetilde{B} \to B^*$ denote a universal covering and $z = (z_1, \cdots, z_n)$ a set of coordinates on $\widetilde{B} \subset \mathbb{C}^n$ such that $\pi(z) = y$ has components $y_j := \exp 2\pi i z_j \in (D^*)^n$. For each j, the monodromy automorphism T_j on L along a closed path around a component Y_j of Y, is defined over $\mathbb{Q}$ and decomposes into a semi-simple T_j^s and a unipotent T_j^u automorphisms: $T_j = T_j^s T_j^u$. On the complex vector space L, all automorphisms T_i^s are simultaneously diagonalizable and conjugate to the diagonal matrix of their eigenvalues.

We define the logarithm of T_j^u by:

$$N_j := \log T_j^u := \sum_{k \geq 1} (-1)^{k+1} (T_j^u - I)^k / k \text{ for } j = 1, \ldots n. \tag{3.2}$$

Along any closed path in B^* the logarithm of the monodromy is an integral linear combination of the nilpotent endomorphisms N_j.

Let $(a.) = (a_1, \cdots, a_n)$, where a_j is an eigenvalue of T_j, then $L = \oplus_{(a.)} L^{a.}$ where $L^{a.} = \cap_{i \in [1,n]} \ker(T_i^s - a_i I)$. Let $\alpha_j \in \mathbb{C}$ with real part in $[0,1[$ such that $\mathbf{e}(\alpha_j) := \exp(-2i\pi\alpha_j) = a_j$. For $v \in L^{a.}$:

$$T_j^s(v) = a_i v,\ T_j^u(v) = (\exp N_j)v \text{ and } T_j(v) = \exp(-2\pi i \alpha_j + N_j)v.$$

Set $L_\infty := \Gamma(\widetilde{B}, \pi^{-1}\mathbf{L}) \simeq L$, T_j acts on L_∞ by translation $z_j \mapsto z_j + 1$:

$$\forall v \in L_\infty := \Gamma(\widetilde{B}, \pi^{-1}\mathbf{L}), v(z_1, \ldots, z_j + 1, \ldots, z_n) = T_j v(z_1, \ldots, z_j, \ldots, z_n).$$

Let $\mathcal{L}_{B^*} := \mathbf{L} \otimes_{\mathbb{Q}} \mathcal{O}_{B^*}$ denote the locally free analytic bundle on B^*. It is free since B^* is Stein. For each $v \in L^{a.}$, the analytic section $\tilde{v} \in (\widetilde{B}, \pi^* \mathcal{L}_{B^*})$

$$\tilde{v} = \exp(\sum_{j \in [1,n]} z_j (2i\pi\alpha_j - N_j))v, \quad (a_j := \exp -2i\pi\alpha_j,\ N_j := \log T_j) \tag{3.3}$$

([De 70], 5.2.1-3) is the inverse image of a (single valued) analytic section of $\mathcal{L}_{B^*}$ on B^*, since $\tilde{v}$ is invariant by each T_j:

$$\tilde{v}(z_1, \ldots, z_j + 1, \ldots, z_n) = \tilde{v}((T_j^s)^{-1}(T_j^u)^{-1} T_j v) = \tilde{v}(z_1, \ldots, z_j, \ldots, z_n).$$

Since $L = \oplus_{(a.)} L^{a.}$, the above tilde correspondence extends to L:

Definition 3.1 (Tilde correspondence). There exists a canonical tilde embedding

$$\sim : \Gamma(\widetilde{B}, \pi^{-1}\mathbf{L}) \xrightarrow{\sim} \widetilde{L} \subset \mathcal{L}_{B,0} \subset \Gamma(B^*, \mathcal{L}), \quad \sim : v \mapsto \tilde{v}.$$

The fibre $\mathcal{L}_{B,0}$ is the $\mathcal{O}_{B,0}$-module generated by the image $\widetilde{L}$ of the tilde embedding. It is the fiber of a free analytic sheaf $\mathcal{L}_B$ on B.

The vector space $\mathcal{L}_B(0) \simeq L_\infty$ will be denoted also by L. A basis v_a of L lifts into a basis $\tilde{v}_a$ of $\mathcal{L}_{B,0}$.

The connection ∇ extends to a differential of the logarithmic de Rham complex $\Omega^*_X \log Y) \otimes_{\mathcal{O}_X} \mathcal{L}_X$ ([De 70] Définition I.2.11, (3.12, b), [Ma 87] §4). The endomorphisms $\alpha_i I$ and N_i correspond to endomorphisms on the image sections $\widetilde{v}$ in $\mathcal{L}_{X,x}$ denoted by the same symbols $\alpha_i I$ and N_i.

They define the connection ([De 70], Théorème 1.17)

$$\nabla \tilde{v} = \Sigma_{j\in J} \left(\widetilde{\alpha_j v} - \frac{1}{2i\pi} \widetilde{N_j v} \right) \otimes \frac{dy_j}{y_j}.$$

Since $2\pi i z_j := \log y_j$, the tilde correspondence is often expressed in terms of the local coordinates y_j as follows

$$\tilde{v}((y_1, \ldots, y_n) = \Pi_{j\in J} {y_j}^{\alpha_j} \exp(\Sigma_{j\in J} - 1/2i\pi(\log y_j)N_j).v \qquad (3.4)$$

Recall, that there exists a correspondence between representation of $\pi_1(M)$, local systems and integrable connections ([De 70], [Vo 2007]).

3.1.2 The complex $\Omega^* L$ and the tilde embedding

The proofs are carried for the transcendental topology although we omit the analytic notation X^{an}.

In view of the local description of perverse sheaves on a product of discs (see 3.2.6.1 and 3.2.6), many constructions on the logarithmic complex, such as the weight filtration, are described first locally on a complex of vector spaces $\Omega^* L$, defined at a point $x \in Y$, then carried to the logarithmic complex by an extension of the tilde correspondence $L \hookrightarrow \mathcal{L}_x$ to a tilde embedding into the de Rham complex: $\Omega^* L \to (\Omega^*_X(\log Y) \otimes \mathcal{O}_X \mathcal{L})_x$ (see Lemma 3.1).

First, we give a general definition of the complex $\boldsymbol{s}(L_J, f_i)$ determined by:

1. a vector space L, a nonnegative integer n and a linear subspace L_J of L for each $J \subset [1, n]$;
2. a collection of commuting endomorphisms $f_i, i = 1, \ldots, n$ of L.

This data should satisfy the condition that $f_i L_{J\setminus\{i\}} \subset L_J$ for each $i \in J$.

The subsets J of length $k := |J|$ correspond to strictly increasing sequences $i_1 < \ldots < i_k \subset [1, n]$). Let S_k denote the set of such strictly increasing sequences. For $J \in S_k$ and $j \in J$, let $\varepsilon_j = (-1)^r$ when j is the r-th element of J in the increasing order.

Definition 3.2 ($\boldsymbol{s}(L_J, f_i)$). The complex $\boldsymbol{s}(L_J, f_i)$ is defined by

$$\boldsymbol{s}(L_J, f_i)^k = \bigoplus_{J\in S_k} L_J \text{ for } k = 0, \ldots, n$$

with differentials $d_{k-1} : \boldsymbol{s}(L_J, f_i)_{k-1} \to \boldsymbol{s}(L_J, f_i)_k$ for $k = 1, \ldots, 1$ given by

$$(d_{k-1}v)_J = \textstyle\sum_{j\in J} \varepsilon_j f_j v_{J\setminus\{j\}}.$$

Returning to the case where L is a complex vector space with commuting endomorphisms $\alpha_j Id - (1/2i\pi)N_j$ for $j \in [1,n]$ where $\alpha_j \in \mathbb{C}$ and N_j is nilpotent, let $L^u \subset L$ be the subspace on which the action of each monodromy T_i is unipotent. We define $\Omega^* L^u$ and $\Omega^* L$ as follows:

Definition 3.3 ($\Omega^* L^u \subset \Omega^* L$). i) The complex $\Omega^* L$ is defined by the special case where $L_J = L$ for each J and $f_j = \alpha_j Id - (1/2i\pi)N_j$ for $j \in [1,n]$.

ii) The sub-complex $\Omega^* L^u \subset \Omega^* L$ is defined by the special case where $L_J = L^u$ for each J and $f_j = N_j$ for $j \in [1,n]$.

$$\Omega^* L^u := 0 \to L^u \to \cdots \to \oplus_{\{i_1<\ldots<i_{k-1}\}} L^u \to \oplus_{\{i_1<\ldots<i_k\}} L^u \cdots \to 0$$

With the same notations, for each $J \in S_k$ let $f_J := \Pi_{j\in J}(\alpha_j Id - \frac{1}{2i\pi}N_j)$ denote the composition morphism for $j \in J$.

Definition 3.4 ($IC^* L$). The complex $\mathrm{IC}^* L$ is the subcomplex of $\Omega^* L$ defined by $L_J = f_J L := \mathrm{Im}\,(f_J : L \to L)$ for each $J \subset [1,n]$ and $f_j = \alpha_j Id - \frac{1}{2i\pi}N_j$ for $j \in [1,n]$.

Remark 3.1. For each sequence $\alpha. = (\alpha_1, \cdots, \alpha_n)$, set $\mathbf{e}(\alpha_j) := \exp -2i\pi\alpha_j$, such that $\mathbf{e}(\alpha.) = \mathbf{e}(\alpha_1, \cdots, \alpha_n)$ is a sequence of eigenvalues of $T. = (T_1, \cdots, T_n)$. Set $IC^* L^{\mathbf{e}(\alpha.)}$ the subcomplex of $\Omega^* L$ where $L_J = f_J L^{\mathbf{e}(\alpha.)} := \mathrm{Im}\,(f_J : L^{\mathbf{e}(\alpha.)} \to L^{\mathbf{e}(\alpha.)})$ for each $J \subset [1,n]$ and $f_j = \alpha_j Id - \frac{1}{2i\pi}N_j$ for $j \in [1,n]$. We have $IC^* L := \oplus_{\alpha.} IC^* L^{\mathbf{e}(\alpha.)}$.

We introduce for each sequence $\alpha.$ the subset $M(\alpha.) \subset [1,n]$ such that $j \in M(\alpha.)$ if and only if $\alpha_j = 0$. Let $N_{J\cap M(\alpha.)} = \Pi_{j\in J\cap M(\alpha.)} N_j$ (it is the identity if $J \cap M(\alpha.) = \emptyset$). For each $J \subset [1,n]$, we have the equality of the image subspaces: $f_J L^{\mathbf{e}(\alpha.)} = N_{J\cap M(\alpha.)} L^{\mathbf{e}(\alpha.)}$ since the endomorphism $(\alpha_j Id - \frac{1}{2i\pi}N_j)$ is an isomorphism on $L^{\mathbf{e}(\alpha.)}$ whenever $\alpha_j \neq 0$, hence

$$IC^* L \simeq \oplus_{\alpha.} s(N_{J\cap M(\alpha.)} L^{\mathbf{e}(\alpha.)})_{J\subset M} \tag{3.5}$$

Lemma 3.1 (The tilde embedding). *Let $x \in Y_M^*$ be a general smooth point on $\cap_{i\in M} Y_i$. The above correspondence $v \mapsto \tilde{v}$ from L to $\mathcal{L}_{X,x}$ (Equation and Definition), extends to an tilde embedding:*

$$\sim : \Omega^* L \to (\Omega_X^*(\log Y) \otimes \mathcal{L}_X)_x \simeq (j_*\mathbf{L})_x$$

$$v_J \mapsto \tilde{v}\,\frac{dy_{i_1}}{y_{i_1}} \wedge \cdots \wedge \frac{dy_{i_j}}{y_{i_k}} \text{ for each } J = (i_1 < \cdots < i_k) \in S_k, v_J = v \in L_J$$

inducing the quasi-isomorphism

$$\Omega^* L \cong (\Omega_X^*(\log Y) \otimes \mathcal{L}_X)(x). \tag{3.6}$$

The complex $\Omega^* L$ and its sub-complex $IC^* L \subset \Omega^* L$ are acyclic if there exists an index j such that $\alpha_j \neq 0$:

Definition 3.5 (L unipotent along Y). The local system $\mathbf{L}$ on the complement of a NCD $Y = \cup_{i\in I} Y_i$ is *unipotent along* Y if the local monodromies T_i at $x \in Y_i$ are unipotent for $i \in I$.

Example 3.1. Let $M = \{1, 2\}$. Then $\mathrm{IC}^* L$ is isomorphic to the complex

$$L \xrightarrow{N_1 \oplus N_2} N_1 L \oplus N_2 L \xrightarrow{N_2 \oplus -N_1} N_1 N_2 L.$$

3.1.3 The complex $\Omega^*\mathcal{L} \simeq j_*\mathbf{L} \in D^b_c(X, \mathbb{C})$

The following de Rham complex with coefficient represent the complex $j_*\mathbf{L}$ in the derived category of constructible sheaves of vector spaces $D^b_c(X, \mathbb{C})$.

Proposition 3.1 ($\Omega^*\mathcal{L}$). *Let $j : X \setminus Y \to X$. The derived direct image $j_*\mathbf{L} \in D^b_c(X, \mathbb{C})$ of $\mathbf{L}$ is quasi-isomorphic to the complex of analytic modules ([De 70, Corollaire 3.14]):*

$$j_*\mathbf{L} \xrightarrow{\sim} \Omega^*\mathcal{L} := \Omega^*_X(\log Y) \otimes_{\mathcal{O}_X} \mathcal{L}_X. \tag{3.7}$$

In particular, the hyper-cohomology of the complex is isomorphic to the cohomology groups: $H^i(R\Gamma(X, \Omega^*\mathcal{L})) \simeq \mathbb{H}^i(X \setminus Y, \mathcal{L})$.

The quasi-isomorphism $j_*\mathcal{L} \xrightarrow{\sim} \Omega^*\mathcal{L}$ follows from Deligne's generalization to local systems ([De 70], definition 3.1) of Grothendieck's comparison theorem between the topological cohomology and algebraic de Rham cohomology on a compact complex algebraic variety [Gro 58], [Gro 66]. We recall also that if X is projective the complex $\Omega^*\mathcal{L}$ is necessarily algebraic by Serre's comparison theorem.

Lemma 3.2. *Let $M \subset I$, $x \in Y^*_M$ a general non singular point on $\cap_{i\in M} Y_i$, $L := \mathcal{L}_X(x)$ with monodromy action $T_i, i \in M$ and $\Omega^* L^u \subset \Omega^* L$ (Definition 3.3). The tilde correspondence induces a quasi-isomorphism $\Omega^* L^u \simeq (\Omega^*\mathcal{L})_x$.*

We prove that for each power of the maximal ideal $\mathfrak{m}^r \subset \mathcal{O}_{Xx}$ at x, the subcomplex $\mathfrak{m}^r(\Omega^*\mathcal{L})_x$ is acyclic for $r > 0$, and for $r = 0 : \Omega^* L^u \simeq \Omega^* L$ are quasi-isomorphic with their image by the tilde correspondence.
For each monomial $s := y^{m_1} \cdots y^{m_n}$, $\nabla(s\widetilde{v}) = \sum_i (s(m_i + \alpha_i)\widetilde{v} - \frac{1}{2i\pi} s\widetilde{(N_i v)}) \otimes \frac{dy_i}{y_i}$, hence the image by the tilde correspondence $\widetilde{L} :=\sim L$ multiplied by s generates in $\mathfrak{m}^r(\Omega^*\mathcal{L})_x$, the sub-complex of $\mathbb{C}-$vector spaces

$$\Omega^* sL := (0 \to sL \to \cdots \to \oplus_{\{i_1 < \ldots < i_k\}} sL \frac{dy_{i_1}}{y_{i_1}} \wedge \cdots \wedge \frac{dy_{i_k}}{y_{i_k}} \cdots \to 0) \tag{3.8}$$

which is quasi-isomorphic to the complex

$$\Omega^* L := \left(0 \to L \to \oplus_{i\in[1,n]} L \to \cdots \to \oplus_{\{i_1 < \ldots < i_k\}} L \cdots \to 0\right)$$

with differential in degree $k-1$:

$\forall v_\bullet,\ d_{k-1}(v_\bullet)_{\{i_1<\ldots<i_k\}} = \sum_j (-1)^j((m_{i_j}+\alpha_{i_j})Id - \frac{1}{2i\pi}N_{i_j})v_{\{i_1<\ldots\hat{i}_j\ldots<i_k\}}$.

It is acyclic if at least one $m_j+\alpha_j \neq 0$ in which case $(m_j+\alpha_j)Id - \frac{1}{2i\pi}N_j$ is an isomorphism of L as N_j is nilpotent; indeed, the complex may be written as a cone over such morphism. We deduce the proposition as $\alpha_j \notin \mathbb{Z}$ unless $\alpha_j = 0$, then $m_j+\alpha_j = 0$ iff $m_j = 0$ and $\alpha_j = 0$.

Remark 3.2. i) Let $\mathcal{I}_Y$ denote the product of the ideals of the components Y_i. The product $\mathcal{I}_Y\Omega^*\mathcal{L}$ is an acyclic sub-complex by the lemma 3.2.

ii) The above description of $(j_*\mathbf{L})_x$ is the model for the description of various perverse sheaves in the rest of the chapter.

For example, to describe the global sub-complex of analytic sheaves $IC^*\mathcal{L} \subset \Omega^*_X(\log Y)\otimes\mathcal{L}_X$ in the next definition 3.6, we lift locally the subspaces N_JL for various J into subspaces $\widetilde{N_JL} \subset \mathcal{L}_{X,x}$ by the tilde embedding (Lemma 3.1).

iii) The complex $\Omega^*\mathcal{L}$ is defined over $\mathbb{Q}$.

3.1.4 The intermediate extension $j_{!*}{}^p\mathbf{L} \simeq IC^{*p}\mathcal{L} \subset \Omega^{*p}\mathcal{L}$

For convenience, we still say intermediate extension for $j_{!*}\mathbf{L}$ or shifted $j_{!*}{}^p\mathbf{L}$. The intermediate extension $j_{!*}{}^p\mathbf{L}$ on X of ${}^p\mathbf{L}$ on U is defined in the category of constructible sheaves $D^b_c(X,Q$ ([GMacP 83], [BBD 83], see 2.2.1.3). We describe below a sub-complex $IC^*\mathcal{L} \subset \Omega^*\mathcal{L}$ representing $j_{!*}\mathbf{L}$ in $D^b_c(X,\mathbb{C})$. Let $\mathcal{I}_Y$ denote the ideal product of the ideals of the components Y_i. The product $\mathcal{I}_Y\Omega^*\mathcal{L}$ is an acyclic sub-complex (Remark 3.2 i). The complex $IC^*\mathcal{L} \supset \mathcal{I}_Y\Omega^*\mathcal{L}$ is characterized by the quotient complex supported by Y.

Definition 3.6 ($IC^*\mathcal{L} \subset \Omega^*\mathcal{L}$). The complex of sheaves $IC^*\mathcal{L}$ is the sub-complex of $\Omega^*\mathcal{L}$ whose stalk at a point $x \in Y^*_M$ is defined as the $\mathcal{O}_{X,x}$ sub-module generated by the sections $\widetilde{v_J} \wedge \Omega^*_{X,x} \in \Omega^*\mathcal{L}$ for all $v_J \in f_JL$ (Lemma 3.1) and $J \subset M$ and by $\mathcal{I}_Y\Omega^*\mathcal{L}$.

To check that the complex of sheaves is well defined at a point $x \in Y_M$ of the intersection of Y_i for $i \in M$. Let $z_j = fy_j$ be a change of coordinate equations for Y_j at x for one $j \in M$ at a time with f invertible at x. Since $\frac{dz_j}{z_j} = \frac{dy_j}{y_j} + \frac{df}{f}$, where $\frac{df}{f}$ is regular at x, the germ of $IC^*\mathcal{L}$ at x is independent of the choice of coordinates.

Moreover, if $x \in Y_M \cap Y_K$ for $K \cap M = \emptyset$, let $\omega = \wedge_{j\in M} dz_j/z_j$ and $\omega' = \wedge_{j\in K} dz_j/z_j$, then $\omega\wedge\omega'$ restricts to a section of $IC^*\mathcal{L}$ on $Y_M \setminus Y_M \cap (\cup_{j\in K} Y_j)$ since the restriction of ω' becomes regular. We say for short that the complex $IC^*\mathcal{L}$ is generated locally by the tilde embedding of IC^*L.

Lemma 3.3. *The complex $j_{!*}\mathbf{L} \in D^b_c(X,\mathbb{C})$ is quasi-isomorphic to the sub-complex of analytic modules of $IC^*\mathcal{L} \subset \Omega^*\mathcal{L}$:*

$$j_{!*}\mathbf{L} \xrightarrow{\sim} IC^*\mathcal{L}. \tag{3.9}$$

Let $\mathcal{S}$ denote the stratification defined by the NCD $Y \subset X$ with strata Y_J^* for $J \subset I$. Let $n := \dim X$ (the complex dimension of X), $V_k := \cup_{J,|J|\leq k} Y_J^*$ ($V_1 = Y, V_{n+1} = \emptyset$) for $k \leq n$, $U_k := X \setminus V_k$ ($U_1 = U, U_{n+1} = X$) and $j_k : U_k \to U_{k+1}$ for $k \leq n$, then $j_{!*}\mathbf{L}$ is defined by induction on k: $j_{!*}\mathbf{L} = \mathbf{L}$ on U_1 and $j_{!*}\mathbf{L}_{|U_{k+1}} := \tau_{n-k} j_{k*} j_{!*}\mathbf{L}_{|U_k}$.

The construction of $IC^*\mathcal{L} \subset \Omega^*\mathcal{L}$ is local. If $IC^*\mathcal{L}_{|U_k}$ is defined by coherent analytic sub-complex, the simple direct image $j_{k*}IC^*\mathcal{L}_{|U_k}$ coincides with the derived direct image since j_k is Stein. At a point $x \in Y_J^* \subset (V_k \backslash V_{k+1})$ the local definition of $(IC^*\mathcal{L}_{|U_k})_x$ corresponds to the construction of $IC^*L \subset \Omega^*L$.

Remark 3.3. Let L^u denote the subspace of L such that all local monodromies are unipotent, then we have quasi-isomorphisms $IC^*L^u \simeq IC^*L \simeq (IC^*\mathcal{L})(x)$.

3.2 The logarithmic complex $j_{Z*}j_Z^*j_{!*}\mathbf{L} \simeq IC^*\mathcal{L}(\log Z)$

With the notations of Proposition 3.1, let $Z \subset Y$ be a NCD which is a sub-divisor of Y and $j_Z : X \setminus Z \to X$ the open embedding.

We define a subcomplex $\mathrm{IC}^*\mathcal{L}(\log Z) \subset \Omega^*\mathcal{L}$ quasi-isomorphic to the derived direct image $j_{Z*}j_Z^*(j_{!*}\mathcal{L}) \in D_c^b(X, \mathbb{C})$.

Recall that $Y = \cup_{i\in I} Y_i$ is a NCD containing the singularities of $\mathcal{L}$. For any NCD Z such that $Z \cup Y$ is still a NCD, we may always suppose that $\mathcal{L}$ is defined on $X - (Y \cup Z)$ (by enlarging Y) and consider Z equal to a union of components of $Y \cup Z$.

Let $Z = \cup_{i\in I_Z} Y_i \subset Y$ with indices in a subset I_Z of I. Fix a set of coordinates y_i at $x \in Y_M^*$ such that $Y_M = V(y_i, i \in M)$ near x. The local definition of $\mathrm{IC}^*\mathcal{L}(\log Z)$ at x in terms of these coordinates is as follows.
For each subset J of M let:

$$J_Z := J \cap I_Z,\ J'_Z = J \setminus J_Z \text{ so } J = J_Z \cup J'_Z,\ M'_Z := M \setminus M_Z. \tag{3.10}$$

Let $M = [1, n]$. With the notations of Definition 3.4, for each index $J \subset M$, the subspaces $f_{J'_Z}L \subset L$ form a sub-family of L with induced morphisms f_j for $j \in [1, n]$. Recall that $\Omega^*L = \boldsymbol{s}(L_J, f_j)$ with $L_J = L$ for all $J \subset M$. Replacing L_J by $f_{J'_Z}L$ in this complex we obtain

Definition 3.7. Let $f_{J'_Z}L$ denote the image of the composition morphism $f_{J'_Z}$ in L. We define $\mathrm{IC}^*L(\log Z) := \boldsymbol{s}(f_{J'_Z}L, f_j) \subset \Omega^*L$.

Example 3.2. On the 3-dimensional disc $D^3 \subset \mathbb{C}^3$, let $Y = Y_1 \cup Y_2 \cup Y_3$ be the NCD defined by the coordinate equations $y_1y_2y_3 = 0$ and $\mathbf{L}$ the unipotent local system defined by a vector space L with the action of 3 nilpotent endomorphisms N_i.

Let $Z = Y_1 \cup Y_2$ be defined by $y_1y_2 = 0$, $M_Z = \{1,2\}$ and $M'_Z = \{3\}$, then $IC^*(Log\, Z)$ is defined by the diagram with differentials defined by $\pm N_i$:

$$\begin{array}{ccccc} L & \xrightarrow{N_1,N_2} & L \oplus L & \xrightarrow{N_2,-N_1} & L \\ \downarrow N_3 & & N_3 \downarrow \oplus N_3 & & \downarrow -N_3 \\ N_3L & \xrightarrow{-N_2,-N_1} & N_3L \oplus N_3L & \xrightarrow{N_2,-N_1} & N_3L \end{array} \tag{3.11}$$

We have a quasi-isomorphism $R\Gamma(D^3 \setminus (D^3 \cap Z), j_{!*}\mathbf{L}) \simeq \mathrm{IC}^*L(\log Z)$.

Remark 3.4. We extend (Definition 3.2) in the locally unipotent case, to families of complexes instead of vector spaces, in order to write the complex $\mathrm{IC}^*L(\log Z)$ as the simple complex associated to a double complex in two ways as in the diagram above:

(i) For each fixed subset $J' \subset M'_Z$, we define the complex $\Omega^*(N_{J'}L, N_i)_{i\in M_Z} := \boldsymbol{s}(N_{J'}L, N_i, i \in M_Z)_{J'\subset J\subset M_Z}$ by summing over $J \subset M_Z$ (the complex on each line in Example).
Let $N_j : \Omega^*(N_{J'-j}L, N_i)_{i\in M_Z} \to \Omega^*(N_{J'}L, N_i)_{i\in M_Z}$ for $j \in J' \subset M'_Z$.
We sum over $J' \subset M'_Z$ the family of complexes endowed with morphisms N_j for $j \in M'_Z$:

$$\mathrm{IC}^*(\Omega^*(N_{J'}L, N_i)_{i\in M_Z}, N_i)_{i\in M'_Z} := \boldsymbol{s}(\Omega^*(N_{J'}L, N_i)_{i\in M_Z}, N_i, i \in M'_Z)_{J'\subset M'_Z}$$

$$\mathrm{IC}^*L(\log Z) = \mathrm{IC}^*(\Omega^*(N_{J'}L, N_i)_{i\in M_Z}, N_i)_{i\in M'_Z} \tag{3.12}$$

(ii) Let $\mathrm{IC}^*(L, N_i)_{i\in M'_Z} := \boldsymbol{s}(N_{J'}L, N_i, i \in M'_Z)_{J'\subset M'_Z}$ be the sum over $J' \subset M'_Z$ (the complex on each column in Example 3.2).
As each N_i for $i \in M_Z$ acts on the complex $\mathrm{IC}^*(L, N_i)_{i\in M'_Z}$, we define:

$$\Omega^*(IC^*(L, N_i)_{i\in M'_Z}, N_i)_{i\in M_Z} := \boldsymbol{s}(IC^*(L, N_i)_{i\in M'_Z}, N_i, i \in M_Z)_{J\subset M_Z}$$

as the sum over $J \subset M_Z$ of the complex independant of J, then:

$$\mathrm{IC}^*L(\log Z) = \Omega^*(\mathrm{IC}^*(L, N_i)_{i\in M'_Z}, N_i)_{i\in M_Z} \tag{3.13}$$

If $Z \neq Y$, the cohomology $H^i(\mathrm{IC}^*L(\log\, Z))$ vanishes in degree $i \geq |M| - 1$.

Definition 3.8. [$\mathrm{IC}^*\mathcal{L}(\log\, Z) \subset \Omega^*\mathcal{L}$] Let $I_Z \subset I$ denote the set of indices of the components of Z and for $J \subset M$, let $J_Z = J \cap I_Z, J'_Z = J \setminus J_Z$. The subcomplex $\mathrm{IC}^*\mathcal{L}(\log\, Z) \subset \Omega^*\mathcal{L}$ is generated at $x \subset Y_M^*$ as an $\mathcal{O}_{X,x}$-submodule, by the sections $\tilde{v} \wedge_{j\in J} \frac{dy_j}{y_j} \wedge \Omega^*_{X,x} \in \Omega^*\mathcal{L}$ for $J \subset M$ for all $v \in N_{J'_Z}L$, where $J'_Z = J \setminus J_Z$ and by $\mathcal{I}_Y\Omega^*\mathcal{L}(\log Y)$.

Proposition 3.2. *Let $L := \mathcal{L}_X(x)$. There exist quasi-isomorphisms*

$$j_{Z*}j_Z^*j_{!*}\mathbf{L} \xrightarrow{\sim} \mathrm{IC}^*\mathcal{L}(\log\, Z), \quad (j_{Z*}j_Z^*j_{!*}\mathbf{L})_x \xrightarrow{\sim} \mathrm{IC}^*L(\log\, Z). \tag{3.14}$$

Proof. Let $j' : X \setminus Y \to X \setminus Z$ be the inclusion map such that $j = j_Z \circ j' : X \setminus Y \to X$. There exists a convergent spectral sequence with respect to the truncation filtration on $j'_{!*}\mathcal{L}$:

$E_2^{p,q}(\mathbf{L}) = (R^p j_{Z*} H^q(j'_{!*}\mathbf{L})) \to (R^{p+q} j_{Z*} j'_{!*}\mathbf{L}) := H^{p+q}(j_{Z*} j_Z^* j_{!*}\mathbf{L})$.

We write the complex $IC^*\mathcal{L}(Log\, Z)$ as a double complex, depending on the logarithmic terms, locally as in (Remark 3.4) and then we deduce globally by the tilde correspondence a similar spectral sequence.

Let $I'_Z = I \setminus I_Z$. We introduce the sub-NCD $Z' = \cup_{i \in I'_Z} Y_i$ complement to Z, such that $Y = Z \cup Z'$. Then Z' is *transversal to* Z in the sense that intersections of components of Z', intersects transversally Z.

Let $Z'_k = \cup_{J \subset I'_Z, |J| \geq k} Y_J$ ($Z'_0 = X$), $Z'_k(*) = (Z'_k \setminus Z'_{k+1})$, $j_k : Z'_k(*) \to Z'_k$ and $i_k : Z'_k \to X$. Remark that Z'_k is decreasing and $Z \cap Z'_k(*)$ is a NCD in $Z'_k(*)$.

At a point $x \in Z \cap Z'_s(*)$, let $L := \mathcal{L}_X(x)$. By (Definition 3.8) the tilde correspondence induces a quasi-isomorphism $IC^*L(Log\, Z) \simeq i_x^* \mathrm{IC}^*\mathcal{L}(\log\, Z)$.

To prove the quasi-isomorphism $i_x^* j_{Z*} j_Z^* j_{!*}\mathbf{L} \simeq IC^*L(Log\, Z)$, we use the structure of double complex on $IC^*L(Log\, Z) = \Omega^*(IC^*(L, N_i)_{i \in M'_Z}, N_i)_{i \in M_Z}$ (Equation 3.13). The corresponding spectral sequence with respect to the truncation filtration on $IC^*(L, N_i)_{i \in M'_Z}$, is:

$$E_2^{p,q}(L) = H^q(\Omega^*(H^p(IC^*(L, N_i)_{i \in M'_Z}), N_i)_{i \in M_Z})) \to H^{p+q}(IC^*L(Log\, Z))$$

We check the isomorphism $i_x^* E_2^{p,q}(\mathbf{L}) \simeq E_2^{p,q}(L)$.

Let $U := D^r \times D^s$ be a product of discs isomorphic to a neighborhood with center x in X with local coordinates $(z., t.) := (z_1, \cdots, z_r, t_1, \cdots, t_s)$, such that $\mathcal{L}$ induces a locally unipotent local system on $U^* := (D^*)^r \times (D^*)^s$.

Let $p : D^r \times D^s \to D^s$ denote the first projection and W the NCD in D^r defined by the product $z_1 \cdots z_r = 0$ such that $Z = p^{-1}(W)$.

Let $U' := U \setminus Z$, $j' : U^* \to U'$, $j_Z : U' \to U$ such that $j = j_Z \circ j'$.

$$\begin{array}{ccccc} U^* \xrightarrow{j'} U' = (D^r \setminus W) \times D^s & \xrightarrow{j_Z} & D^r \times D^s & \xrightarrow{p} & D^r \\ & & \cup & & \cup \\ & & Z = p^{-1}(W) = W \times D^s & \xrightarrow{p_|} & W \end{array}$$

The cohomology sheaves $\mathbf{L}^i := R^i(p_{|U'})_* j'_{!*}\mathbf{L}$ are locally constant on $(D^*)^r$ with fiber at a point $z = (z.) \in (D^*)^r$ isomorphic to the cohomology $H^i((j'_{!*}\mathbf{L})_z)$, which is in turn isomorphic to the Intersection cohomology of a local system on the sphere in $(D^*)^r$ at the center 0.

Let $j_W : (D^r \setminus W) \to D^r$, since the restriction of p to the family of spheres in the fibers of p is proper, the functor Rp_* commute with Rj_{W*} and Rj_{Z*}:

$$(Rj_{W*}\mathbf{L})_0 \simeq \Omega^*(H^p\{IC^*(L, N_i)_{i \in M'_Z}\}, N_i)_{i \in M_Z},$$
$$E_2^{p,q}(L) \simeq (R^q j_{W*}\mathbf{L})_0 \simeq (R^p j_{Z*} H^q(j'_{!*}\mathbf{L}))_x \simeq E_2^{p,q}(\mathbf{L})_x.$$

Lemma 3.4 (Transversality). *Let $j'_Z : X \setminus Y \to X \setminus Z', j_{Z'} : X \setminus Z' \to X$. We have a commutation property: $j_{Z*}j'_{!*}\mathbf{L} \simeq j_{Z'!*}j'_{Z*}\mathbf{L}$.*

The proof is local at each point of the intersection $Z \cap Z'$ and it is reduced, by induction on the dimension, to the case of a point in the intersection of n components of Y and $Z \neq Y$.

The computation is carried with respect to the stratification defined by Y. In terms of (Remark 3.4) where $|M| = n$ and a point $0 \in D^n \subset \mathbb{C}^n$, we check that $(j_{Z'!*}j'_{Z*}\mathbf{L})_0$ is obtained by truncation in degree $\leq n-1$ of $(j_{Z*}j'_{!*}\mathbf{L})_0$. By the remark the complex $(j_{Z*}j'_{!*}\mathbf{L})_0$ has no cohomology in degrees $\geq n$.

3.2.1 Thom isomorphism and Gysin morphism

It is well-known that Thom isomorphism extends to stratified varieties under a normally nonsingular embedding [GMacP 88], ([Max 19] Definition 3.4.5). The following result is cited in ([CaMi 5], Lemma 3.5.4)

Lemma 3.5. *Let $i : H \to V$ be a normally nonsingular inclusion of complex codimension d of complex varieties transversal to every stratum of a stratification $\mathcal{S}$ of V and K a constructible complex with respect to $\mathcal{S}$, then $i^!K \simeq i^*K[-2d]$.*

With the previous notations $Z \subset Y \subset X$ and ${}^p\mathbf{L}$ defined on $X \setminus Y$, let $i_W : W \to X$ be a closed embedding of a non singular variety of codimension d transverse to Z, Y and to the strata of an adequate stratification $\mathcal{S}$ of X, then we have Thom isomorphism

$$i_W^! j_{!*}{}^p\mathbf{L}[2d] \simeq i_W^* j_{!*}{}^p\mathbf{L}$$

Let $W^* := W \setminus (Z \cap W)$, then:

$$\mathbb{H}^i(W^*, j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^{i+2d}_{W^*}(X \setminus Z, j_{!*}{}^p\mathbf{L}).$$

Let $W = H$ of codimension 1, the *Thom isomorphism*:

$$\mathbb{H}^i(H \setminus (Z \cap H), j_{!*}\mathbf{L}) \xrightarrow{\sim} \mathbb{H}^{i+2}_{H\setminus(Z\cap H)}(X \setminus Z, j_{!*}\mathbf{L})(1) \tag{3.15}$$

is defined on logarithmic complexes as follows. We have isomorphisms

$$i_H^! IC^*\mathcal{L}(\log Z)[1] \xleftarrow{\sim c} IC^*\mathcal{L}(\log Z \cup H)/IC^*\mathcal{L}(\log Z)) \xrightarrow{\sim r} i_{H*}IC^*i_H^*\mathcal{L}(\log Z \cap H)[-1]$$

where on the right r is induced by the residue map

$$R_H : IC^*\mathcal{L}(\log Z \cup H) \to i_{H*}IC^*i_H^*\mathcal{L}(\log Z \cap H)[-1]$$

vanishing on $IC^*\mathcal{L}(\log Z)$ and on the left c is the connecting isomorphism with the cohomology with support. Later, we deduce from this description that the *Gysin morphism* is compatible with MHS.

3.2.2 Polarized Variation of HS (VHS)

The complex $\Omega^*\mathcal{L}$ is specifically interesting in the case of a variation of HS (VHS) defined by a Hodge filtration F by sub-bundles of $\mathbf{L} \otimes \mathcal{O}_U$.

Let $x \in Y_M^*$ and a punctured disc $\Delta^* \subset B^* \subset D^n$ with center 0. The monodromy T acting on the restriction $\mathbf{L}_{|\Delta^*}$ generates a group Γ acting on the classifying space $\mathbb{D}$ of polarized VHS ([Gr 74] chapter I). The period map $\phi : \Delta^* \to \mathbb{D}/\Gamma$ lifts to the covering space $\widetilde{\phi} : \widetilde{\Delta}^* \to \mathbb{D}$. Let $N = \log T^u$ (in general we suppose the VHS unipotent along Y by reduction to a finite local covering [Sc 73]). The map $\Psi : \Delta^* \to \check{\mathbb{D}} : z \to \exp(-zN\widetilde{\phi}(z)$ where $\check{\mathbb{D}}$ is the dual space of D, extends at 0 and define a filtration F_0 on $\mathcal{L}_B(0)$ ([Gr 74] chapter IV, Theorem 2 and Proposition 10).

Globally, since the Hodge filtration F extends to Hodge sub-bundles $\mathcal{F}$ of $\mathcal{L}_X$ satisfying Griffith's transversality $\nabla : \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1$, we deduce the Hodge filtration by sub-complexes of $\Omega^*\mathcal{L}$

$$F^p = \left[0 \to \mathcal{F}^p\mathcal{L}_X \cdots \to \Omega_X^i(\log Y) \otimes \mathcal{F}^{p-i}\mathcal{L}_X \to \cdots\right] \tag{3.16}$$

It should be noted that the existence of a Hodge filtration on $\mathbf{L}$ induces strong properties used in the definition of the weight filtration on $\Omega^*\mathcal{L}$.

3.2.3 Duality and Polarization

The topological properties of duality of the intersection cohomology (see [GMacP 83], [Max 19]) must be reinforced by the assumption of polarization of the VHS $\mathbf{L}$ inducing the auto-duality property, in order to prove the decomposition theorem. The polarization of the intermediate extension of a polarized VHS with singularities along a NCD is defined in [KaK 87].

As we reduce the proof to the case of IMHS (Definition 2.14), we use essentially the local definitions of graded polarized limit MHS at infinity in chapter 2.3. As the intermediate extension is canonically defined by the local system on a Zariski open non singular subset, *the polarization of the intermediate extension is also canonically defined by the polarization of the VHS on the open subset.* Hence, to refer to the polarization of the intermediate extension, *it is sufficient to check only the polarization of the VHS* on the open subset U and in turn to check the polarization at each point $x \in U$ (§2.3).

Poincaré -Verdier duality. Let $f : X \to V$ a proper morphism of complex algebraic varieties, $K \in D^b(X, \mathbb{Q})$ a complex of sheaves, $\mathbb{D}_X$ the dualizing complex and D_X the dualizing functor $D_X K := \mathcal{H}om^\bullet(K, \mathbb{D}_X)$ (derived $\mathcal{H}om^\bullet$), then by (Equation 2.4):

$$\mathbb{H}^{-i}(X,\, D_X K) \simeq D(\mathbb{H}^i(X, K)) := Hom(\mathbb{H}^i(X, K), \mathbb{Q})$$

If $\mathbf{L}$ is a VHS, to apply this statement of duality with weights, we recall some conventions on duality of filtrations and polarization in Hodge theory.

i) *Translation of filtered complexes with weights.* The couple of fields $(\mathbb{Q}, \mathbb{C})$ form a HS of weight 0. If H is a HS of weight w', we write $H(r)$ for the HS of weight $w' - 2r$.

A Hodge complex of vector spaces $(K, K_{\otimes_{\mathbb{Q}}}\mathbb{C}, F)$ defined over $\mathbb{Q}$, is of weight w if its cohomology $H^i(K)$ is a HS of weight $w + i$.
We set $\mathbb{Q}[r]$ for the complex with $\mathbb{Q}$ in degree $-r$ and zero otherwise. It is a HC of weight r since $H^{-r}(\mathbb{Q}[r]) = \mathbb{Q}$ is a HS of weight 0.

Let $(\mathbf{L}, F)$ be a VHS of weight w'. The filtration F of the shifted VHS ${}^p\mathbf{L} := \mathbf{L}[n]$ is defined by: $F_i({}^p\mathbf{L}) := F_i(\mathbf{L})[n]$. Then the weight of $({}^p\mathbf{L}, F)$ as a complex of sheaves, is $n + w'$ since the weight of its cohomology satisfy:

$$w(H^{-n}({}^p\mathbf{L}, F)) = w(H^0(\mathbf{L}, F)) = w' + n - n = w'.$$

Let $(\mathbf{L}, W, F)$ be a VMHS. The filtration W and F of the shifted VHMS ${}^p\mathbf{L} := \mathbf{L}[n]$ are defined by: $W^i({}^p\mathbf{L}) := W^{i-n}(\mathbf{L})[n]$ and $F_i({}^p\mathbf{L}) := F_i(\mathbf{L})[n]$, then $({}^p\mathbf{L}, W, F))$ is called a perverse VMHS.

ii) *Dual filtrations.* The dual of a bifiltered complex (K, W, F) in the bifiltered derived category of an abelian category with a dualizing functor D, is denoted by (DK, W, F) with dual filtrations defined by:

$$W_{-i}DK := D(K/W_{i-1}), F^{-i}DK := D(K/F^{i+1})$$

such that: $DGr_i^W K \xrightarrow{\sim} Gr_{-i}^W DK$ and $DGr_F^i K \xrightarrow{\sim} Gr_F^{-i} DK$.

Below, we are interested in various dualities with value in $\mathbb{Q}$ or with value in a dualizing complex $\mathbb{D}_X$ with weights or without.

iii) *Dual filtrations with weights.* The dual of a variation of MHS $(\mathbf{L}, W, F)$ with value in $\mathbb{Q}$ on a non singular complex algebraic variety X is a variation of MHS $(\mathbf{L}^*, W, F)$ $(\mathbf{L}^* := Hom_{\mathbb{Q}}(\mathbf{L}, \mathbb{Q}))$ with filtrations dual to W and F on $\mathbf{L}$, such that $Gr_{-i}^W(\mathbf{L}^*) \simeq (Gr_i^W \mathbf{L})^*$ and $Gr_F^{-i}(\mathbf{L}^*) \simeq (Gr_F^i \mathbf{L})^*$.

iv) *Verdier dual filtrations with weights* (the notations extend to the singular case). In the category of VHS $(\mathbf{L}, F)$ of weight w' on a non singular complex algebraic variety X of dimension n, we introduce a non trivial Hodge filtration on the dualizing complex: $\mathbb{D}_X^{w'} = \mathbb{D}_X(n - w')$ (that is $\mathbb{D}_X \simeq \mathbb{Q}_X[2n]$ tensor a HS of weight $2w' - 2n$).
Then, Verdier-Poincaré duality states the following (derived $Hom^\bullet$):

$$D_X^{w'}(\mathbf{L}, F) : Hom^\bullet((\mathbf{L}, F), \mathbb{Q}_X[2n](n - w')) = Hom^\bullet((\mathbf{L}, F), \mathbb{Q}_X)[2n](n - w')$$
$$\mathbb{H}^{-i}(X,\, D_X^{w'}(\mathbf{L}, F)) \simeq D^{w'}(\mathbb{H}^i(X, (\mathbf{L}, F)) := Hom(\mathbb{H}^i(X, (\mathbf{L}, F)),\, \mathbb{Q}(-w'))$$

Hence the weight of the left term of the second line, is $w' - i$ since $w' = -w' + 2n - 2n + 2w'$; while the weight of $D^{w'}\mathbb{H}^i(X, (\mathbf{L}, F))$ is also $-w' - i + 2w' = w' - i$ (the same weight as: $w(H^{2n-i}(X, \mathbf{L}^*) = w' - i)$. Moreover:

$$D_X^{w'}(\mathbf{L}, F) \text{ is a VHS of weight } w',\ D_X^{w'}\mathrm{Gr}_F^i(\mathbf{L}, F) \simeq \mathrm{Gr}_F^{w'-i} D_X^{w'}(\mathbf{L}, F) \tag{3.17}$$

v)*Polarization of HS.*

Definition 3.9. i) A polarization of a HS of weight w' is defined by a non degenerate $w'-$ symmetric morphism $S : L \otimes L \to \mathbb{Q}$ such that its complex extension $S : L \otimes L \otimes \mathbb{C} \to \mathbb{C}(-w')$ satisfy :

1. $\forall x \in L^{p,q}, x \neq 0, S(i^{p-q}x, \bar{x}) > 0$
2. $\forall x \in L^{p,q}, \forall y \in L^{p' \neq q, q' \neq p} : S(x, y) = 0$
3. $\forall p, q : p + q > w', S(F^p, F^q) = 0.$

Since $\overline{S(x, y)} = S(\bar{x}, \bar{y})$, the last condition is equivalent to $S(F^p, F^q) = 0$ for $p + q < w'$, hence the conditions (2) and (3) are equivalent.

Corollary 3.1. *A polarized HS is auto-dual.*

Proof. By definition the isomorphism $S^* : (L, F) \xrightarrow{\sim} Hom((L, F), \mathbb{C}(-w')) : a \mapsto (b \mapsto S(a, b)$ defined by S, in the filtered category of HS of weight w', sends $F^p \mapsto F^p$ as $S(F^p, F^{w'-p+1}) = 0$. We deduce from (Equation 3.17) that the dual filtration coincides with the filtration F by transport by S. We say that the polarized HS is self dual.

Polarization of a VHS. A polarization S of a VHS $(\mathbf{L}, F)$ of weight w' on a non singular complex algebraic variety X induces an isomorphism of HS of equal weight w':

$$(\mathbf{L}, F) \xrightarrow{\sim} (\mathbf{L}, F)^*(-w').$$

The dual of $(\mathbf{L}, F)$ with value in $\mathbb{Q}(-w')$ of weight $2w'$ is a VHS $(\mathbf{L}, F)^*$ of weight $w((\mathbf{L}, F)^*) = -w'$. Then S induces a quasi-isomorphism in the filtered category of VHS

$$(\mathbf{L}, F) \xrightarrow{\sim} \mathcal{H}om((\mathbf{L}, F), \mathbb{C}(-w')).$$

3.2.3.1 Polarization of an intermediate extension and purity

We introduce the concept of non degenerate bilinear pairing in the category of intermediate extensions of VHS of weight w'.

Let $(\mathbf{L}, F)$ denote a polarized VHS on an open non singular Zariski open subset U of an algebraic variety X of dimension n. In the case where U is the complement of a NCD, the complex $IC^*\mathcal{L} \subset \Omega^*\mathcal{L}$ (Definition 3.6) with its Hodge filtration induced by F on $\Omega^*\mathcal{L}$ (Equation 3.16), defines a Hodge filtration on the intermediate extension: $(j_{!*}\mathbf{L}, F) := (IC^*\mathcal{L}, F)$.

The complex $\Omega^*\mathcal{L}_2$ of square summable $\mathcal{C}^\infty-$ differential forms with its Hodge filtration F is defined respectively in ([Zu 79], [KaK 87], [CaKSc 87])

and the embedding $(IC^*\mathcal{L}, F) \simeq (\Omega^*\mathcal{L}_2, F)$ defines a filtered acyclic resolution of $(j_{!*}\mathbf{L}, F)$. Moreover, the natural pairing by wedge product and the $(-1)^{w'}$−symmetric product S on $\mathcal{L}$ extends to the complex

$$S : (\Omega^*\mathcal{L}_2, F) \otimes (\Omega^*\mathcal{L}_2, F) \xrightarrow{S\circ\wedge} (\Omega_X^{2n}, F) \tag{3.18}$$

and it is compatible with F ([KaK 87], Theorem 6.4.2).

Let σ denotes the symmetry on the tensor product $\sigma(\omega^p \otimes \omega^q) := (-1)^{pq}\omega^q \otimes \omega^p$, then $S \circ \sigma = (-1)^{w'}S$ (compatible with the convention of signs in [Sa 88] §5.2). We deduce the definitions below.

Definition 3.10. i) A bilinear $\pm$ symmetric pairing in the category $D_c^b(X, \mathbb{Q})$ is defined by a morphism $S : \mathcal{F} \otimes \mathcal{F} \to \mathbb{D}_X$ satisfying $S \circ \sigma = \pm S$.
S is non degenerate if it induces a quasi-isomorphism: $\mathcal{F} \xrightarrow{\sim} \mathcal{H}om^\bullet((\mathcal{F}, \mathbb{D}_X)$.

ii) A polarization of the intermediate extension of a polarized VHS $(j_{!*}{}^p\mathbf{L}, F) := (IC^*\mathcal{L}, F)$ of weight ${}^pw$ on a complex algebraic variety X of dimension n, is defined by a ${}^pw-$ symmetric morphism

$$S : (j_{!*}{}^p\mathbf{L}, F) \otimes (j_{!*}{}^p\mathbf{L}, F) \to \mathbb{D}_X(n - {}^pw). \tag{3.19}$$

Equivalently $S^* : (j_{!*}{}^p\mathbf{L}, F) \xrightarrow{\sim} \mathcal{H}om^\bullet((j_{!*}{}^p\mathbf{L}, F), \mathbb{D}_X(n - {}^pw))$ induced by S is a filtered quasi-isomorphism.

iii) A graded polarized variation of MHS is defined with a family of polarizations S_i on $(Gr_i^W{}^p\mathbf{L}, F)$ inducing isomorphisms:

$$(Gr_i^W j_{!*}{}^p\mathbf{L}, F) \xrightarrow{\sim} D_X^i(Gr_i^W j_{!*}{}^p\mathbf{L}, F).$$

iv) If the weight is fixed, we still use the symbol of duality D_X instead of $D_X^{{}^pw}$. We write $D_X(j_{!*}{}^p\mathbf{L}) \simeq j_{!*}(D_X{}^p\mathbf{L})$ as a consequence of the polarization.

Now we can state the following purity result after ([KaK 87], Theorem 6.4.2):

Theorem 3.1 (Purity). *i) Hard Lefschetz: Let X be a complex non singular projective variety. The cup product with the class of an hyperplane section induces isomorphisms of Hodge structures*

$$\eta^i : \mathbb{H}^{-i}(X, (j_{!*}{}^p\mathbf{L}, F)) \xrightarrow{\sim} \mathbb{H}^i(X, (j_{!*}{}^p\mathbf{L}, F))(i).$$

ii) The following pairing induced by ${}^pS$ (Equations 3.18, 3.19) on hypercohomology is denoted by $(,)$ ($D_X^{{}^pw}$ denoted D_X):

$$\mathbb{H}^{-i}(X, j_{!*}{}^p\mathbf{L}) \otimes \mathbb{H}^i(X, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^0(X, j_{!*}{}^p\mathbf{L} \otimes j_{!*}{}^p\mathbf{L}) \xrightarrow{\mathbb{H}^0({}^pS)} \mathbb{H}^0(X, D_X) \xrightarrow{trace} \mathbb{C}$$

Set $P^{-i} := \ker \eta^{i+1} : \mathbb{H}^{-i}(X, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^{i+2}(X, j_{!*}{}^p\mathbf{L})$, then the pairing defined by $(-1)^{i(i+1)/2}(a, \eta^i b)$ induces a polarization of the HS of weight ${}^pw-i$ on P^{-i}. Hence, the filtration F induces on $\mathbb{H}^i(X, j_{!*}{}^p\mathbf{L})$ a polarized HS of weight ${}^pw + i$.

The purity theorem is also stated as follows:

Let $({}^p\mathbf{L}, F)$ be a shifted polarized variation of HS of weight ${}^pw$ on a dense open subset of a compact variety X. We have a duality isomorphism with equal weights

$$\mathbb{H}^i(X, j_{!*}{}^p\mathbf{L})^* \xrightarrow{\sim} \mathbb{H}^{-i}(X, j_{!*}{}^p\mathbf{L})({}^pw) \tag{3.20}$$

where $w(\mathbb{H}^i(X, j_{!*}{}^p\mathbf{L})^*) = -({}^pw+i)$ and $w(\mathbb{H}^{-i}(X, j_{!*}\mathbf{L})({}^pw)) = {}^pw-i-2\,{}^pw$.

The statement is deduced from the purity theorem, the polarization and Verdier duality with weights $j_{!*}{}^p\mathbf{L} \xrightarrow{\sim} D(j_{!*}{}^p\mathbf{L})$.

Definition 3.11. i) If the weight is fixed, we still use the symbol of duality D_X instead of $D_X^{{}^pw}$. We write $D_X(j_{!*}{}^p\mathbf{L}) \simeq j_{!*}(D_X{}^p\mathbf{L})$ as a consequence of the polarization.

ii) A graded polarized variation of MHS $({}^p\mathbf{L}, W, F)$ on a Zariski open subset is defined by a family of polarizations S_i on $(Gr_i^W{}^p\mathbf{L}, F)$. It induces isomorphisms:

$$(Gr_i^W j_{!*}{}^p\mathbf{L}, F) \xrightarrow{\sim} D^i(Gr_i^W j_{!*}{}^p\mathbf{L}, F).$$

Example 3.3. Duality of a shifted VHS. As a complex of sheaves, $({}^p\mathbf{L}, F) := (\mathbf{L}[n], F)$ has weight ${}^pw = w' + n$. Its dual in the filtered derived category

$$D^{{}^pw}({}^p\mathbf{L}) := \mathcal{H}om^\bullet({}^p\mathbf{L}, \mathbb{Q}[2n](n - {}^pw)) \xrightarrow{\sim} \mathbf{L}^*(n - {}^pw)[n]$$

has weight $-w' + n - 2n = {}^pw$.

Polarization of a shifted VHS. A pairing $S : \mathbf{L} \otimes \mathbf{L} \to \mathbb{C}$ translates into a pairing ${}^pS : {}^p\mathbf{L} \otimes {}^p\mathbf{L} \to \mathbb{C}[2n] : a \otimes b \mapsto S(a, b) \in \mathbb{C}[2n]$. If S is $(-1)^{w'}$–symmetric, the pairing ${}^pS$ is $(-1)^{{}^pw}$–symmetric:

$\sigma : {}^p\mathbf{L} \otimes {}^p\mathbf{L} \to {}^p\mathbf{L} \otimes {}^p\mathbf{L} : a \otimes b \mapsto (-1)^{n^2} b \otimes a$, then ${}^pS \circ \sigma = (-1)^{{}^pw}\,{}^pS$ since: ${}^pS \circ \sigma(a, b) = (-1)^{n^2} S(b, a) = (-1)^{(n+w')} S(a, b) = (-1)^{{}^pw}\,{}^pS(a, b)$.

If S is a polarization of a VHS of weight w' then ${}^pS$ is a polarization of ${}^p\mathbf{L}$ in the sense that $\mathcal{H}^{-n}({}^p\mathbf{L}) \simeq \mathbf{L}$ is polarized and equivalently ${}^pS$ induces an isomorphism: ${}^p\mathbf{L} \xrightarrow{\sim} D^{{}^pw}({}^p\mathbf{L})$ (${}^p\mathbf{L}$ is auto-dual).

Graded polarized VMHS. As a complex of sheaves, a shifted VMHS $({}^p\mathbf{L} := \mathbf{L}[n], W, F)$ has its weight increased by n, such that $W^i({}^p\mathbf{L}) := W^{i-n}(\mathbf{L})[n]$, and $F_i({}^p\mathbf{L}) := F_i(\mathbf{L})[n]$. The filtration on its cohomology $H^{-n}({}^p\mathbf{L})$ satisfy: $(H^{-n}({}^p\mathbf{L}), W) = (\mathbf{L}, W)$.

Remark 3.5. i) The pairing and the polarization are transported to the direct image and its decomposition by duality applied to the proper morphism $f : X \to V$ (Equation 2.5)

$$D_X^{w'}(\mathbf{L}, F) := \mathcal{H}om^\bullet((\mathbf{L}, F), D_X^{w'}) \simeq \mathcal{H}om^\bullet(f_*(\mathbf{L}, F), D_V^{w'}) := D_V^{w'} f_*(\mathbf{L}, F). \tag{3.21}$$

ii) *Mixed cone.* The mixed cone over a morphism $f : (K, W, F) \to (K', W', F')$ is the cone $(C(f), W, F)$ with the filtrations on $C(f) := K[1] \oplus K'$ defined by the convention $W_{j+1}(K[1]) := (W_jK)[1]$ and $F_j(K[1]) := (F_jK)[1]$.

The weight filtration on the mixed cone is not always naturally defined (see the second homotopy problem in §2.1), however we will frequently use this concept whenever it is naturally defined.

For example, the diagonal filtration defined by a simplicial covering in [De 75] is a repeated use of mixed cone. Simplicial coverings define natural weights on cohomology of algebraic varieties (see also [E 83]).

3.2.4 Perverse sheaves along a NCD

Let $Y \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$ denote the embeddings of a normal crossing divisor (NCD) Y and of $U := X \setminus Y$. Let $\mathbf{L}$ be a local system on U, then ${}^p\mathbf{L} := \mathbf{L}[\dim X]$ is perverse on U.
Let $Z \subset Y \subset X$ be a NCD sub-divisor of Y, $i_Z : Z \to X$ and $j_Z : X \setminus Z \to X$. The previous constructions on the perverse sheaves $j_*{}^p\mathbf{L}$ and $j_{!*}{}^p\mathbf{L}$ extend to the following complexes:

$$i_Z^! j_{!*}{}^p\mathbf{L},\ i_Z^* j_{!*}{}^p\mathbf{L},\ i_Z^! j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L},\ i_Z^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L},\ j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L},\ j_{Z!} j_Z^* j_{!*}{}^p\mathbf{L}.$$

Let $\mathbf{L}$ be VHS of weight w', then ${}^p\mathbf{L} := \mathbf{L}[\dim X]$ is perverse of weight ${}^pw = w' + \dim X$.

Let D denote Verdier duality. By definition of auto-duality of ${}^p\mathbf{L}$, we have an isomorphism $D{}^p\mathbf{L} \simeq {}^p\mathbf{L}({}^pw)$ where $({}^pw)$ is a twist of the weight.
Let $Z \subset Y$ be a NCD sub-divisor, we deduce the weight filtration on $i_Z^! j_{!*}{}^p\mathbf{L}$ from $(IC^*\mathcal{L}(Log\, Z), W, F)$ and the weight filtration on $i_Z^* j_{!*}{}^p\mathbf{L}$ by duality with $i_Z^! j_{!*}{}^p\mathbf{L}$:

$$D(j_{!*}{}^p\mathbf{L} \simeq j_{!*}{}^p\mathbf{L}({}^pw),\ Di_Z^! j_{!*}{}^p\mathbf{L} \simeq i_Z^* D(j_{!*}{}^p\mathbf{L}),\ Dj_{Z*} j_Z^* j_{!*}{}^p\mathbf{L} \simeq j_{Z!} j_Z^* D(j_{!*}{}^p\mathbf{L})$$

3.2.5 $i_{Z*}i_Z^*j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L}$

The complex $i_{Z*}i_Z^*j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L}$ occurs in the following triangle:

$$j_{Z!}j_Z^*j_{!*}{}^p\mathbf{L} \xrightarrow{\text{can}} j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L} \to i_{Z*}i_Z^*j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L} \xrightarrow{[1]} \tag{3.22}$$

The complexes $j_{Z!}j_Z^*j_{!*}{}^p\mathbf{L}$ and $j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L}$ are perverse since j_Z is affine (Stein in the analytic case). Let $\mathcal{K} := j_{Z*}j_{!*}{}^p\mathbf{L}$.

The associated perverse long exact sequence is reduced to

$$0 \to i_{Z*}{}^p\mathcal{H}^{-1}i_Z^*j_{Z*}\mathcal{K} \to j_{Z!}\mathcal{K} \xrightarrow{\text{can}} j_{Z*}\mathcal{K} \to i_{Z*}{}^p\mathcal{H}^0 i_Z^*j_{Z*}\mathcal{K} \to 0 \tag{3.23}$$

as $i_Z^*j_{Z*}\mathcal{K}$ has only two non trivial perverse cohomology sheaves ([BBD 83] Corollaire 4.1.10 ii). Since $j_{!*}{}^p\mathbf{L} = j_{Z!*}\mathcal{K} := \mathrm{Im} : j_{Z!}\mathcal{K} \xrightarrow{\text{can}} j_{Z*}\mathcal{K}$, we deduce (([BBD 83], Corollaire 4.1.12) and [Br 82], Proposition 2.3.8):

$$\begin{aligned} i_Z^* j_{!*}{}^p\mathbf{L}[-1] &\simeq {}^p\mathcal{H}^{-1}(i_Z^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}) \simeq \operatorname{Ker}\operatorname{can}, \\ i_Z^! j_{!*}{}^p\mathbf{L}[1] &\simeq {}^p\mathcal{H}^{0}(i_Z^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}) \simeq \operatorname{Coker}\operatorname{can}. \end{aligned} \tag{3.24}$$

where $\operatorname{Coker}\operatorname{can} \simeq j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}/j_{!*}{}^p\mathbf{L}$.

In the text we describe the filtration W on $IC^{*\,p}\mathcal{L}(LogZ) \simeq j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}$. Then, we deduce the filtrations W and F on

$$i_Z^* i_Z^! j_{!*}{}^p\mathbf{L}[1] \simeq (j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L})/j_{!*}{}^p\mathbf{L} = IC^{*\,p}\mathcal{L}(LogZ)/IC^{*\,p}\mathcal{L} \tag{3.25}$$

and by duality the filtration W on $i_Z^* j_{!*}{}^p\mathbf{L}[-1]$.

3.2.6 Combinatorial description

To deal with perverse sheaves including perverse sub-sheaves of the complex $\Omega^*\mathcal{L}$, for $\mathcal{L}$ locally constant on $U := X \setminus Y$, we recall the description of perverse sheaves locally constant on the non singular strata Y_J^* defined the NCD $Y := V(z_1 \cdots z_n) := \{z_\bullet : z_1 \cdots z_n = 0\}$.

In [V 83], J. L. Verdier describes the extensions of a general perverse sheaf $\mathcal{P}$ along a divisor by reduction to a local divisor defined by an equation f in terms of the nearby cycles Ψ_f and the vanishing cycles φ_f. Since $\Psi_f\mathcal{P}$ and $\varphi_f\mathcal{P}$, are shifted perverse, Verdier's description can be repeated inductively in the case of NCD to obtain the combinatorial description below.

3.2.6.1 Combinatorial description of perverse sheaves along a NCD $Y \subset \mathbb{C}^n$ as a de Rham family

The NCD Y has a natural stratification. For each $M \subset [1,n]$,let $Y_M := \{z \in \mathbb{C}^n : z_j = 0 \text{ for } j \in M\}$ and $Y_M^* := \{z \in Y_M : z_j \neq 0 \text{ for } j \notin M\}$ (the non singular locus of Y_M). The perverse sheaves P such that for each $M \subset [1,n]$, the cohomology of $P_{|Y_M^*}$ is locally constant, are described by the following combinatorial construction ([Ka 86] §2.1, see also [GGM 85] following a description by Deligne).

The category $\mathcal{P}$ of perverse sheaves P on $\mathbb{C}^n$, with respect to the NCD stratification Y_M^* of Y is equivalent to the abelian category defined by :

(i) A family of vector spaces L_A for $A \subset [1,n]$,
(ii) A family of morphisms
$f_{AB}\colon L_B \to L_A$ and $h_{BA}\colon L_A \to L_B$ for $B \subset A \subset [1,n]$ such that :
$f_{AB} \circ f_{BC} = f_{AC}$, $h_{CB} \circ h_{BA} = h_{CA}$ for $C \subset B \subset A$
$f_{AA} = h_{AA} = id$, $h_{A,A\cup B} \circ f_{A\cup B,B} = f_{A,A\cap B} \circ h_{A\cap B,B}$ for all A, B
and if $A \supset B, \mid A \mid = \mid B \mid +1$, then $1 - h_{BA} f_{AB}$ is invertible.

Intermediate extensions. For each for $A \subset [1,n]$, the description of the category IE_A of intermediate extensions of a locally constant sheaf $\mathbf{L}$ on $Y_A^\star$ in terms of the family of vector spaces L_B for $B \subset [1,n]$ is as follows:

(i) $L_B = 0$ for $A \subset B, A \neq B$
(ii) f_{BA} is surjective and g_{AB} is injective for $A \subset B$.

We denote by IE the objects isomorphic to a direct sum of objects in $\cup_A IE_A$.

3.2.6.2 Distinguished pair and Kashiwara's criteria of decomposition into direct sum of intermediate extensions

Let $P \in \mathcal{P}$ and for all $B \subset A \subset [1,n]$, $L_B \xrightarrow{f_{AB}} L_A \xrightarrow{h_{BA}} L_B$, then by

([Ka 86] Proposition 2.3.1): *The perverse sheaf $P \in \mathcal{P}$ is a direct sum of intermediate extensions if and only if the following distinguished pair criteria is satisfied*

$$\forall\, B \subset A \subset [1,n], A, B \neq [1,n], \quad L_A \simeq \operatorname{Im} f_{AB} \oplus \operatorname{Ker} h_{BA} \tag{3.26}$$

Moreover, it is enough to consider $|A| = |B| + 1$.
Let $P_B(P) = \cap_{C \subset B, C \neq B} \operatorname{Ker} h_{CB}$. The above condition is equivalent to the isomorphism:

$$\oplus_{B \subset A} f_{AB}(P_B(P)) \quad \xrightarrow{\sim} \quad L_A \tag{3.27}$$

The induced morphism $h_{BA} : f_{AB}(P_B(P) \to P_B(P)$ is injective for $B \subset A$.

Remark 3.6. Originally, the local description of IC^*L started with Deligne's local purity conjecture (see [CaKSc 87]). It took sometime to set the notations and reach the above statement. The reader would need as much time to meditate on the various ways to apply this combinatorial description.

i) The first version of the description in [GGM 85] is based on the cohomology with supports in the strata of the stratification defined by the NCD. Since then, the description by Verdier's extension prevail.

ii) In the text, we extract from this data the concept of de Rham family which determines the fiber $i_0^*\Omega^*\mathcal{L} := \Omega^*L$ (resp. $L_A := N_A L$ in the case of the complex $IC^*L = i_0^*IC^*\mathcal{L}$ (Definition 3.2)).

By the tilde embedding (Lemma 3.1) L_A above for $A = \{i_1, \ldots, i_a\}$, corresponds to $Ldz_{i_1} \wedge \cdots \wedge dz_{i_a}$ (see Definition3.7 for the general case $IC^*L(LogZ)$).

iii) In the case of $\psi_f\mathbf{L}$ where $f = z_1 \cdots z_n$, set $L_A := \Psi_A L$. In this case $L_\emptyset = 0$ since $\psi_f\mathbf{L}$ is supported by Y. The de Rham family is applied to describe the fiber of the nearby cycle complex Ψ^*L (see ch. 4 and 5).

On the (Example 5.1) in the case of Ψ^*L, we could as well represents the local system by (L, N_1, N_2) on the generic stratum. At a point $z_2 \in Y_1^*$, $(L_\emptyset = L, N_1, L_{\{1\}} = L)$ represents the complex $(j_{1*}\mathbf{L})_{z_2}$, if we add the action of N_2 it represents $i_{Y_1^*}^* j_{1*}\mathbf{L}$ as a complex of two local system on Y_1^*.

iv) In the proof of the decomposition theorem of the direct image by a morphism $f : X \to V$, it is an advantage to introduce a map g locally on V to $\mathbb{C}$ since ψ_g is exact perverse and satisfy $\psi_g f_* j_{!*}{}^p\mathbf{L} \simeq \psi_{g\circ f} j_{!*}{}^p\mathbf{L}$ (similarly for and φ_g), then if we add the canonical morphism *can*, we can deduce $i^*_{g^{-1}(0)} f_* j_{!*}{}^p\mathbf{L}$.

v) For a general perverse sheaf, Saito in ([Sa 90], §3) express the terms of a combinatorial description by repeated application of the functors ψ_{z_i} and φ_{z_j} with the canonical and variation morphisms which determine all the terms of a perverse sheaf on a small ball including the local monodromy. We can recover the perverse sheaf completely by such data (see [Sa 90], 3.1).

Example 3.4. To describe $j_{!*}{}^p\mathbf{L}$ on $\mathbb{C}^n$ with singularities along Y, we find $L = {}^p\psi_{z_n} \circ \cdots {}^p\psi_{z_1} j_{!*}{}^p\mathbf{L}$ where ${}^p\psi_{z_i} = \psi_{z_i}[-1]$.

If $n = 1$, $i_0^* j_{!*}{}^p\mathbf{L} = (L \xrightarrow{N} NL)[1], \psi_{z_1} j_{!*}{}^p\mathbf{L} = L[1]$, ${}^p\psi_{z_1} j_{!*}{}^p\mathbf{L} = L$.

By definition $\varphi_{z_1} j_{!*}{}^p\mathbf{L}$ is the cone of the morphism: $i_0^* j_{!*}{}^p\mathbf{L} \to \psi_{z_1} j_{!*}{}^p\mathbf{L}$, hence $\varphi_{z_1} j_{!*}{}^p\mathbf{L} = NL[1]$ and ${}^p\varphi_{z_1} j_{!*}{}^p\mathbf{L} = NL$.

vi) In the case where $\mathbf{L}$ is a VHS the Hodge filtration F is induced on the elements of the de Rham family (recall that F is not a filtration by perverse sub-sheaves). Then on Y_1^* we add the limit structures on Y_1^* and Y_2^* which are different than the limit structure at 0.

vii) We will consider a decomposition on the graded level of the graded polarized nilpotent orbit defined by the VHS. Although the decomposition above (Equation 3.27) is topological, the decomposition will be compatible with Hodge filtration which leads to a direct decomposition of the Hodge filtration by the following fact:

Let H be a MHS, A and B two sub-MHS, if H is a direct sum of the sub-spaces A and B, then it is a direct sum of MHS by a simple count of dimensions of the filtrations:

Let $\mathrm{Gr}_i^W H = \oplus_{p,q\in\mathbb{Z}:p+q=i} H^{p,q}$, and $h^{p,q} := \dim H^{p,q}$, then $h^{p,q} \geq 0$ and $\sum_i h^{p,q} = \dim H$.

By application to A and B, since $H^{p,q}(A) \oplus H^{p,q}(B) \subset H^{p,q}$, we find $h^{p,q}(A) + h^{p,q}(B) = h^{p,q}(H)$ and $H^{p,q}(A) \oplus H^{p,q}(B) = H^{p,q}$.

Chapter 4
The Nearby cycles complex $\Psi^* I\mathcal{L}$

Fouad El Zein, Joseph Steenbrink

In this new section consisting of the next chapters $4-8$, we develop the proof of the local purity and the decomposition theorems announced in §1.2.1.2.

A natural structure of bifiltered mixed Hodge complex on the nearby and vanishing cycles complex with coefficients in an intermediate extension $j_{!*}{}^p\mathbf{L}$ of a (shifted) polarized variation of Hodge structure $(\mathbf{L}, F)$ is defined in the first two chapters 4 and 5.

The structure of mixed Hodge complex is proved inductively with the decomposition theorem in chapter 6. The main feature is the reduction of the proof to the case of a vanishing cycles complex supported by the isolated strata on V. Hence the decomposition occurs in the category of IMHS.

By using a general hyperplane section, a remarkable fact in this case is the splitting of the weight spectral sequence which is the main simplification of the proof compared with [Sa 88].

4.1 Introduction to chapters $4-8$ (section 2)

In view of Verdier's decomposition criteria (Equation 2.22), the decomposition theorem (see ch. 1, Theorem 1.2) can be reduced to a decomposition of the vanishing cycle complex.

First in the case where X is a non singular complex variety, let $f : X \to D$ be a projective morphism to a complex disc with a NCD as central fiber and $j_{!*}\mathbf{L}$ on X. We construct in this chapter a (unipotent) limit nearby cycles complex $\Psi^* I\mathcal{L} \simeq \psi_f^u j_{!*}\mathbf{L}$ with coefficients in an intermediate extension $j_{!*}\mathbf{L}$ of a polarized variation of Hodge structure (VHS) $\mathbf{L}$ (resp. shifted ${}^p\mathbf{L}$) with a limit Hodge filtration F (see ch. 2, §2.2.2) and (§4.2.1).

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Since the definition of the complex $\psi_f j_{!*}\mathbf{L}$ is topological (§2.2.2) and the polarized VHS $\mathbf{L}$ is on $X \setminus Y$, the definition of the Hodge filtration on the limit complex is subtle (the filtration $\psi_f F^p j_{!*}\mathbf{L}$ is not interesting and $F^p\mathcal{L}$ is not constructible).

The extension of F appears as Schmid's limit Hodge filtration [Sc 73] in the one variable case. On Steenbrink's limit complex [St 76] the filtration F is deduced from the limit filtration on the logarithmic complex with constant coefficients.

The structure of mixed Hodge complex on $\Psi^* I\mathcal{L}$ is defined by induction in chapter 5 in order to carry the proof mentioned in §1.2.1.2.

The decomposition theorem is proved by reduction to isolated strata and general hyperplane sections in chapter 6.

When the vanishing cycles complex is supported by the isolated strata on V, the decomposition occurs in the category of IMHS. Since the hyperplane section is general, the weight special sequence splits into a direct sum of a spectral sequence concentrated on the isolated strata which degenerates at rank 2 in the category of mixed Hodge structures (MHS), and a spectral sequence which degenerates at rank 1 (see ch.6, Corollary 6.3).

The structure of mixed Hodge complex on $\psi_f^u j_{!*}\mathbf{L}$ in the case of a singular variety is defined by application of the decomposition theorem to the desingularization $\pi : \widetilde{V} \to V$ in chapter 6 as follows. Let $j : U \to V$ denote the open embedding of a non singular Zariski open subset U of a complex projective variety V on which $\mathbf{L}$ is defined, and $j' : U \to \widetilde{V}$ the embedding into the desingularization $\pi : \widetilde{V} \to V$.

By the derived push forward functor, we have a Hodge filtration on $\pi_*(j'_{!*}\,{}^p\mathbf{L}, F)$ and by the decomposition theorem $j_{!*}\,{}^p\mathbf{L}$ is a component of ${}^p\mathcal{H}^0(\pi_* j'_{!*}\,{}^p\mathbf{L})$ with natural induced Hodge filtration F. This construction is independent of the desingularization up to isomorphism (see ch. 7, §7.1).

The category of intermediate extension of a polarized VHS on algebraic varieties is abelian (see ch. 7, §7.1).

In chapter 8, we fulfill our aim to adapt faithfully to complex varieties the original local purity theorem in positive characteristic, although the proof may be simplified by combination with the decomposition (see ch. 10).

For convenience, we do not use the letter R in general for a derived functor (see ch.1, §2.4.0.2 for conventions and terminology) and we still say (shifted) intermediate extension for $j_{!*}\mathbf{L}$.

4.1.0.1 Weigt and Hodge filtrations on $\psi_f j_{!*}{}^p\mathbf{L}$

Let $f : X \to D$ be the restriction over a disc D of an algebraic morphism on a non singular complex variety to $\mathbb{C}$ and $\mathbf{L}$ a polarized VHS on an open subset U complement of a NCD Y of X containing the central fiber $Z \subset Y$.

The Hodge filtration F on $\mathbf{L}$ extends to Deligne's canonical extension $\mathcal{L}_X$ of $\mathbf{L} \otimes_{\mathbb{Z}} \mathcal{O}_U$ and to the logarithmic complex $\Omega^*\mathcal{L} := \Omega^*_X(\log Y) \otimes_{\mathcal{O}_X} \mathcal{L}_X$

(see ch.3, §3). We suppose the fiber of f at $0 \in D$ a NCD $Z := f^{-1}(0)$ contained as a sub-divisor of Y. Let $j : U \to X$ and $j_Z : (X \setminus Z) \to X$, then $j_*\mathbf{L} \simeq \Omega^*\mathcal{L}$ and there exists a sub-complex $IC^*\mathcal{L}(\log Z)$ of $\Omega^*\mathcal{L}$, with induced Hodge filtration F, such that $j_{Z*}j_Z^*(j_{!*}{}^p\mathbf{L}) \simeq IC^{*p}\mathcal{L}(\log Z)$ (see ch.3, Definition 3.8).

To define the Hodge filtration F on the unipotent sub-sheaf of nearby cycles complex $\psi_f^u j_{!*}{}^p\mathbf{L}$ (see ch. 2, §2.2.2), first in the case of a VHS $\mathbf{L}$ locally unipotent along a NCD Y, we extend the construction of Steenbrink's limit complex defined in the case of constant sheaf [St 76].

We define in §4.2.1 the limit complex $(\Psi^* I\mathcal{L}, F)$ using the logarithmic complex $(IC^*\mathcal{L}(LogZ), F)$. The Hodge filtration F on $\psi_f^u j_{!*}{}^p\mathbf{L}$ is defined by the isomorphism with the complex $\Psi^* I\,{}^p\mathcal{L}$ (see Proposition 4.1).

The weight filtration W is deduced in chapter 5, from the monodromy weight filtration $W(N)$ shifted by the weight ${}^pw$ of ${}^p\mathbf{L}$.

The filtrations W and F on $\Psi^* I\mathcal{L}$ are used to define the structure of a mixed Hodge complex on the nearby and vanishing cycles complexes ψ_f^u and ϕ_f^u with coefficients in a polarized VHS on a Zariski open subset complement of a NCD in a complex variety (see chapter 5).

The decomposition of the graded monodromy weight filtration into direct sums of intermediate extensions (see ch.5, Theorem 5.3) is based on combinatorial constructions of perverse sheaves along a NCD [Ka 86] (see ch.3, §3.2.6 and §3.2.6.1).

The definition of the filtrations W and F on $\psi_f j_{!*}{}^p\mathbf{L}$ for $\mathbf{L}$ not necessarily locally unipotent along Y is deduced directly from the locally unipotent case see (ch.5, §5.2).

4.1.0.2 Splitting of the perverse weight spectral sequence

Let $f : X \to V$ be a projective morphism of complex algebraic varieties. A proof of the decomposition theorem (ch.1, Theorem 1.2) is given in chapter 6 by reduction to the case of an isolated stratum $v \in V$ and a general hyperplane section H through v.

Since the proof may be reduced to a desingularization of X with a NCD as inverse image of v, we suppose already X non singular and v an isolated stratum in V. By considering a general pencil of hyperplane sections of V containing H and blowing up the axis of the pencil, the proof is reduced to the case where there exists a morphism $g : V \to \mathbb{P}$ to the projective complex line such that $g^{-1}(g(v)) = H$ is general and $X_H := f^{-1}(H)$ is a NCD in X.

Let $h := g \circ f$. The perverse weight spectral sequence of the nearby cycles $(f_*\psi_h j_{!*}{}^p\mathbf{L}, W)$ *splits in this case as H is general*, into a direct sum of a spectral sequence concentrated on v which degenerates at rank 2 in the category of mixed Hodge structures (MHS), and a spectral sequence which degenerates at rank 1 (see ch.6, Corollary 6.3).

Moreover, the complex of vanishing cycles $\phi_h f_* j_{!*}{}^p\mathbf{L}$ is concentrated on v which means that we are reduced to the case of a decomposition in the category of IMHS (see ch. 2, §2.3.2). Recall that the induced Hodge filtration on perverse cohomology is defined by restriction (see ch.2, §2.4).

4.1.0.3 Decomposition theorem on singular varieties

After the proof of the decomposition theorem we extend Hodge theory to singular varieties in (§6.2.2.2) and chapter 7. The definition of the abelian category of intermediate extensions of VHS (ch.7, Definition 7.1) on a singular variety V and the definition of perverse variation of MHS (ch.7, Definition 7.2) are deduced from the definition on a desingularization of V by application of the decomposition theorem which is already proved.

This is a significant simplification with respect to the proof in the theory of Hodge $D-$modules [Sa 88, Sa 90].

4.1.1 Invariant local and global cycles theorems

Mixed Hodge structure defined along divisors. As a consequence of the decomposition theorem, we construct a MHS on the hypercohomology of *open or closed algebraic subsets of a nonsingular variety* X with coefficients in an intermediate extension in chapter 7.

To state the local purity theorem, we construct a MHS on the hypercohomology of a small analytic tubular neighborhood $B^*_{X_v}$ of a fiber X_v of $f : X \to V$ with the central fiber X_v deleted.

Invariant theorems. Let V be a complex algebraic variety, $(\mathbf{L}, F)$ a polarized VHS on a nonsingular Zariski open subset $U \subset V$, $j : U \to V$ and $j_{!*}{}^p\mathbf{L}$ its intermediate extension. Let $g : U' \to D$ be a projective morphism defined on an open subset $U' \subset V$ onto a disc with center 0 and central fiber $H = g^{-1}(0)$. The unipotent nearby cycles complex $\psi^u_g j_{!*}{}^p\mathbf{L}[-1]$ is perverse (Proposition 2.6) and we have:

Theorem 4.1 (Local invariant cycle theorem). *There exists an exact sequence*

$$\mathbb{H}^q(H, j_{!*}{}^p\mathbf{L}) \xrightarrow{sp} \mathbb{H}^q(H, \psi^u_g j_{!*}{}^p\mathbf{L}) \xrightarrow{N} \mathbb{H}^q(H, \psi^u_g j_{!*}{}^p\mathbf{L})(-1) \tag{4.1}$$

where N is the logarithm of the unipotent monodromy.

The proof is based on the existence of mixed Hodge structures on the various hypercohomology groups of the sequence (ch.7, Proposition 7.5).
Let $B_0 \subset D$ be a small disc, the specialization morphism sp is precisely the composition map (where ρ is surjective):
$\mathbb{H}^q(H, j_{!*}{}^p\mathbf{L}) \xrightarrow{\sim} \mathbb{H}^q(g^{-1}B_0, j_{!*}{}^p\mathbf{L}) \xrightarrow{\rho} \mathbb{H}^0(B_0, R^q g_* j_{!*}{}^p\mathbf{L}) \xrightarrow{\sim} \mathbb{H}^q(H, \psi^u_g j_{!*}{}^p\mathbf{L})^T$.

Theorem 4.2 (Global invariant cycle theorem). *Let $g : V \to T$ be a morphism of projective varieties where V is pure dimensional, $\mathbf{L}$ a polarized VHS on a nonsingular Zariski open subset $U \subset V$ and $j_{!*}{}^p\mathbf{L}$ the intermediate extension of ${}^p\mathbf{L}$.*

There exists a Zariski open dense subset $T' \subset T$ of parameters t over which $R^q g_* j_{!*}{}^p\mathbf{L}$ is a polarized VHS, and for each fiber $V_t := g^{-1}(t)$ at $t \in T'$, the image of the restriction map

$$\mathbb{H}^q(V, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^q(V_t, j_{!*}{}^p\mathbf{L}) \tag{4.2}$$

is the invariant subset $\mathbb{H}^q(V_t, j_{!*}{}^p\mathbf{L})^{\pi_1(T',t)}$ of fixed elements under the monodromy action $\pi_1(T', t) \to Aut\, H^q(V_t, j_{!*}{}^p\mathbf{L})$.

Precisely, we have the following factorization where u is surjective:
$\mathbb{H}^q(V, j_{!*}{}^p\mathbf{L}) \xrightarrow{u} \mathbb{H}^0(T', R^q g_* j_{!*}{}^p\mathbf{L}) \xrightarrow{\sim} \mathbb{H}^q(V_t, j_{!*}{}^p\mathbf{L})^{\pi_1(T',t)}$.

4.1.2 Purity and local purity theorems

The global purity result (Theorem 3.1) after [CaKSc 87] and [KaK 87], is deduced in [Sa 88] and [Sa 90] from the decomposition of the derived direct image $f_* j_{!*}\mathbf{L}$ by a projective morphism onto a curve and Zucker's purity result in the curve case. The same method apply here to prove the purity first on a non singular variety and then deduce the purity on singular complex varieties. The HS obtained by various methods are forced to be isomorphic by their own duality properties [KaK 86].

Theorem 4.3 (Purity Theorem). *Let $\mathbf{L}$ be a polarized VHS of weight ${}^pw$ defined on the complement $X \setminus Y$ of a NCD Y in a projective variety X. The limit Hodge filtration F on the intermediate extension $j_{!*}\mathbf{L}$ defines a HS of weight ${}^pw + i$ on $\mathbb{H}^i(X, j_{!*}{}^p\mathbf{L})$.*
See (Corollary 6.2 ii) if X is singular and not projective.

Local purity (ch. 1, Theorem 1.1). We give a faithful interpretation on complex varieties of the proof in positive characteristic ([DeG 81] using the local and global invariant cycle theorems for the perverse cohomology (ch.7, Proposition 7.5, Lemma 7.2).

A new simple proof of the local purity theorem related to the decomposition theorem is given in the last section (chapter 10).

4.2 The limit Hodge filtration on $\Psi^* I\mathcal{L} \simeq \psi_f^u j_{!*}\mathbf{L}$

Let X be a complex algebraic variety, $Y \subset X$ a NCD, $\mathbf{L}$ a local system on $X \setminus Y$ (for the analytic topology) and $j : (X \setminus Y) \to X$.

Let $f : X \to \mathbb{C}$ be an algebraic map. The analytic restriction of f over the punctured disc D^* with small radius is a topological fibration. We suppose the central fiber $Z = f^{-1}(0) \subset Y \subset X$ a NCD sub-divisor of Y.

A **filtered logarithmic limit complex** $(\Psi^* I\mathcal{L}, F)$ is defined in §4.2.1 with the following isomorphism (see Proposition 4.1):

$$\Psi^* I\mathcal{L} \xrightarrow{\sim} \psi_f^u j_{!*}\mathbf{L}.$$

The Hodge filtration F on $\psi_f^u j_{!*}\mathbf{L}$ is deduced by this isomorphism.

Since we go back to the original work by Deligne ([DeK 73], Exposé XIV) various intermediary complexes are mentioned such as $\tilde{\Psi}^* IL$ (Definition 4.1) and $\bar{\Psi}^* I\mathcal{L}$ (Equation 4.31). The construction extends Steenbrink's limit complex with constant coefficients [St 76] to the case of coefficients in an intermediate extension of a polarized VHS $j_{!*}\mathbf{L}$. We refer to [NaG 90] for a clear exposition in the case of constant coefficients.

Since the definition of the complex $\psi_f j_{!*}\mathbf{L}$ is topological, the problem is to define correctly the Hodge filtration on the limit complex. The main contribution to the definition of $F^p\psi_f j_{!*}\mathbf{L}$ started with Schmid's extension of F to Deligne's canonical $\mathcal{L}_X$ of $\mathbf{L}\otimes_{\mathbb{Z}}\mathcal{O}_U$ [De 70], [Ma 87] and the extension of F to the logarithmic complex $\Omega^*\mathcal{L} := \Omega_X^*(\log Y) \otimes_{\mathcal{O}_X} \mathcal{L}_X$. Already, j_*F on $j_*\mathbf{L}$ is different from F on $\Omega^*\mathcal{L}$.

Let $j_Z : (X \setminus Z) \to X$, the sub-complex $IC^*\mathcal{L}(\log Z)$ of $\Omega^*\mathcal{L}$ with the induced Hodge filtration F represents $j_{Z*}(j_{!*}\mathbf{L})_{|X\setminus Z}$ (ch. 3, Definition 3.8).

4.2.0.1 Definition of $\psi_f j_{!*}\mathbf{L}$

We resume the definition of the sheaf of nearby cycles (§2.2.2, [DeK 73]) and the following diagram:

$$\widetilde{X}^* \xrightarrow{\widetilde{p}} X^* \xrightarrow{j_Z} X \xleftarrow{i_Z} Z \tag{4.3}$$

where $Z := f^{-1}(0), X^* = X \setminus Z, \widetilde{X}^* := X \times_D \widetilde{D}^*$ the fiber product with a universal covering $p : \widetilde{D}^* \to D^*$ of the punctured disc D^*.

Let $\mathcal{K} \in D_c^+(X^*, \mathbb{Q})$ be a complex of sheaves of $\mathbb{Q}$-vector spaces, bounded below, with cohomology sheaves locally constant on the strata of a Whitney stratification of X, then

$$\psi_f\mathcal{K} := i_Z^*\widetilde{j}_{Z*}\widetilde{p}^*\mathcal{K} \in D^b(Z, \mathbb{C}) \quad \text{where} \quad \widetilde{j}_Z = j_Z \circ \widetilde{p} \tag{4.4}$$

(we write $\widetilde{j}_{Z*}$ for $R\widetilde{j}_{Z*}$). The universal covering $\widetilde{D}^*$ of D may be represented by $\{u \in \mathbb{C}; Re\, u < 0\}$ with the projection $z = p(u) = e^u$ (equivalently $p(u) = e^{2i\pi u}$ for u in the Poincaré upper half plane). The action of the counter clockwise generator of the fundamental group of D^* on the hypercohomology of the Milnor fiber is the mondromy automorphism T of $\psi_f\mathcal{K}$.

If the complex $\mathcal{K}$ is perverse with respect to the middle perversity structure (§2.2.1 and [BBD 83], §4.0), the shifted complex $\psi_f\mathcal{K}[-1]$ as well $\phi_f\mathcal{K}[-1]$ are also perverse on Z (Equation 2.15, Proposition 2.6, [Br 82], [Br 86] Théorème 1.2, [GMacP 88] §6.13). We denote by $\psi_f^u\mathcal{K}$ the perverse sub-sheaf unipotent under the action of T.

In general, if $T = T^sT^u$ is the decomposition of T into its semisimple and unipotent part, the nilpotent endomorphism $N := \log T^u$ determines the monodromy weight increasing filtration $W_*(N)$ of $\psi_f(\mathcal{K})$ (§2.3.0.1). Explicitly, the filtration is given by $W_j(N) = \sum_i \operatorname{Im} N^i \cap \operatorname{Ker} N^{i+j+1}$.

Let us suppose that the central fiber $Z = f^{-1}(0)$ of the morphism f is a NCD in a nonsingular complex algebraic variety X. We suppose that Z is contained as a sub-divisor in a NCD $Y \subset X$ and that a local system $\mathbf{L}$ is given on the open subset $U := X \setminus Y$. In the next paragraph 4.2.1 we construct the limit complex $\Psi^* I\mathcal{L}$ in terms of the logarithmic complex $IC^*\mathcal{L}(\log Z) \subset \Omega^*\mathcal{L}$ (chapter 3, Definition 3.8).

Since $\mathbf{L}$ *underlies a polarized VHS, there exists a limit filtration* F [Sc 73], [Gr 74] and (§2.3.1) (Equation 3.16) on the restriction $i_Y^*\mathcal{L}_X$ of Deligne's extension $\mathcal{L}_X$ [De 70].

4.2.1 The limit complex $\Psi^* I\mathcal{L}$

Let T denote the monodromy action defined by a generator of $\pi_1(D^*)$ on the sheaf $\psi_f(j_{!*}\mathbf{L})$. We denote by $\psi_f^u(j_{!*}\mathbf{L})$ the unipotent sub-sheaf and $\psi_f^\alpha(j_{!*}\mathbf{L})$ the unipotent sub-sheaf of the action $T - \alpha Id$ (which makes sense in the abelian category of perverse sheaves after an adequate shift of the complex). We construct a complex $\Psi^* I\mathcal{L}$ on Z which is a representative of the perverse sheaf $\psi_f^u(j_{!*}\mathbf{L})$ by a complex of analytic $\mathcal{O}_X-$modules with support on Z and bounded cohomology $\Psi^* I\mathcal{L} \simeq \psi_f^u(j_{!*}\mathbf{L})$ (Proposition 4.1), convenient for the representation of the perverse sheaf $\psi_f^u(j_{!*}\mathbf{L})[-1]$ by the associated de Rham family §4.2.2.2.

To define the limit complex, we introduce the logarithmic complex with coefficients in Deligne's extension, since the Hodge filtration F is not defined by the filtration $\widetilde{j}_{Z*}\widetilde{j}_Z^*F$, but it is the expression of a limiting process.

Technically, since the Hodge filtration F is not defined by the filtration $\widetilde{j}_{Z*}\widetilde{j}_Z^*F$, but it is the expression of a limiting process, we use $(IC^*\mathcal{L}(\log Z), F)$ to construct various intermediary complexes:

1. the complex $\tilde{\Psi}^* I\mathcal{L} \simeq \Psi^* I\mathcal{L}$ (4.2.1.4)
2. the complex $\bar{\Psi}^* I\mathcal{L}$ (4.2.2.2)
3. local constructions (4.2.2.2, 4.2.1.2, 4.2.1.3)

The main result is: $\psi_f^u(j_{!*}\mathbf{L}) \simeq \Psi^* I\mathcal{L}$ (§4.21, Proposition 4.1). Since $\psi_f^u(j_{!*}\mathbf{L})$ is perverse, the proof and constructions are based on the local version $\Psi^* IL$

(§4.2.1.3, Definition 4.2) of $\Psi^* I\mathcal{L}$ and its associated de Rham family (Equation 4.11).

4.2.1.1 $\Psi^* IL$ and its associated de Rham family

Let L be a finite dimensional $\mathbb{C}$-vector space endowed with a set of commuting nilpotent endomorphisms N_i for $i \in M \simeq [1,n]$, $\mathbb{C}[N]$ the polynomial ring in one variable N, and $\mathbb{C}[N,N^{-1}]$ the algebra of Laurent polynomials. We denote by N the endomorphism on $L[N,N^{-1}] := L \otimes \mathbb{C}[N,N^{-1}]$ defined by multiplication by N. It maps the subspaces $L[N]$ and $NL[N]$ to themselves and acts on $L[N^{-1}] = L \otimes \mathbb{C}[N^{-1}]$ when we consider this module as $L[N,N^{-1}]/NL[N]$ (so $N=0$ on LN^0).
The action of the commuting nilpotent endomorphisms N_i extends as $N_i \otimes Id$, still denoted by N_i, on $L \otimes \mathbb{C}[N,N^{-1}]$ (resp. on $L[N]$ and $L[N^{-1}]$). Given a subset $M_Z \subset M$ and a sequence of positive integers m_i for $i \in M_Z$, set

$$A_i = m_i N - N_i \text{ if } i \in M_Z, \text{ and } A_i = -N_i \text{ if } i \notin M_Z. \tag{4.5}$$

It is a family of commuting endomorphisms acting on $L[N,N^{-1}], L[N]$ and on $L[N^{-1}]$.

Remark 4.1. The sign $-N_i$ in A_i is due to the definition $N_i := \log T_i$ such that $Res_{Y_i}\nabla = -\frac{1}{2i\pi}N_i$. In general there is a problem of signs in the equations which goes back to the fact that monodromy is taken in opposite directions in the two schools in the field by Deligne and Griffith.

We associate to $L[N^{-1}]$ endowed with A_i the *de Rham family* of modules with index $J \subset M$:

$$(L[N^{-1}]_J = L[N^{-1}] \text{ for } J \subset M,\ A_i : L[N^{-1}]_J \to L[N^{-1}]_{J\cup i} \text{ for } J \cup i \subset M, i \notin J).$$

4.2.1.2 Local differential complex $\tilde{\Psi}^* IL$ defined by de Rham family

We give first a local description, of a global complex $\tilde{\Psi}^* I\mathcal{L}$, at a point $x \in Y^*_M$ where $M \subset [1,n]$ is a set of indices of equations of Y^*_M at x.

Let $L := \mathcal{L}_X(x), M_Z \subset M \subset [1,n]$. For each $J \subset M$ set:

$$J_Z := J \cap M_Z \text{ and } J'_Z = J \setminus J_Z \text{ such that } J = J_Z \cup J'_Z \tag{4.6}$$

With the notations of (ch. 3, §3.1.2), we consider two de Rham families:

$$\begin{aligned} L[N]_J :=& L[N], A_i : L[N]_{J-i} \to L[N]_J, J \subset M, i \in M \\ L[N]_J :=& N_{J'_Z} L[N] \subset L[N], (A_i)_|, J \subset M, i \in M \end{aligned} \tag{4.7}$$

where $N_\emptyset = Id$ and with induced filtration by F since $\mathbf{L}$ is a VHS.

Definition 4.1 ($\tilde{\Psi}^* L$ and $\tilde{\Psi}^* IL$). The filtered complexes

$$\tilde{\Psi}^* L := \boldsymbol{s}(L[N]_J = L[N], A_i)_{J\subset M, i\in M},\ F^p\tilde{\Psi}^* L := \boldsymbol{s}((F^{p-|J|}L)[N], A_i)_{J\subset M, i\in M}$$

$$\tilde{\Psi}^* IL := \boldsymbol{s}(N_{J'_Z} L[N], A_i)_{J\subset M, i\in M} \subset \tilde{\Psi}^* L$$

are associated to the de Rham families in equation 4.7, with filtration F on $\tilde{\Psi}^* IL$ induced by F on $\tilde{\Psi}^* L$.

Example 4.1. On the 3-dimensional disc $D^3 \subset \mathbb{C}^3$, let $W = W_1 \cup W_2 \cup W_3$ be the NCD defined by the coordinates $z_1 z_2 z_3 = 0$. A unipotent local system $\mathcal{L}$ is defined by a vector space L with the action of 3 nilpotent endomorphisms $(L, N_i, i \in [1,3])$.

Let $M_Z = \{1,2\}$ and $Z = Z_1 \cup Z_2$ be defined by $z_1 z_2 = 0$, then $\tilde{\Psi}^* IL$ is the simple complex associated to the double complex with differentials defined by A_i and N_i up to sign ($A_3 = -N_3$).

$$\begin{array}{ccccc} L[N] & \xrightarrow{A_1+A_2} & L[N]\oplus L[N] & \xrightarrow{A_2-A_1} & L[N] \\ \downarrow N_3 & & \downarrow (N_3\oplus N_3) & & \downarrow N_3 \\ N_3 L[N] & \xrightarrow{A_1+A_2} & N_3 L[N]\oplus N_3 L[N] & \xrightarrow{A_2-A_1} & N_3 L[N] \end{array} \tag{4.8}$$

4.2.1.3 The complex $\Psi^* IL$

The construction of a global bounded logarithmic limit complex $\Psi^* I\mathcal{L}$ where $\mathcal{L}$ is locally unipotent, is easily reduced to the construction of the finite complex $\Psi^* IL$ in the local case below, fit for the computation of the monodromy filtration.

For each subset $J \subset M$, let $A_J = A_{i_1} \circ \cdots \circ A_{i_j}$ for $J = \{i_1, \ldots, i_j\}$ and let $\operatorname{Im} A_J \subset L[N]$ denote the image of the morphism

$$A_J : L[N] \to L[N], \quad \operatorname{Im} A_J \subset N_{J'_Z} L[N] \subset L[N]. \tag{4.9}$$

To check that $\operatorname{Im} A_J \subset N_{J'_Z} L[N]$, note that $A_J = A_{J_Z} \circ A_{J'_Z} = \pm A_{J_Z} \circ N_{J'_Z}$.

We remark that the family $\operatorname{Im} A_J$ defines a sub-complex

$$\operatorname{Im}^* A := \boldsymbol{s}(\operatorname{Im} A_J, A_i)_{J\subset M, i\in M} \subset \widetilde{\Psi}^* IL$$

Definition 4.2 ($\Psi^* IL$). With the notations of (Equation 4.6), we define the finite dimensional quotient complex

$$\Psi^* IL := \widetilde{\Psi}^* IL / \operatorname{Im}^* A. \tag{4.10}$$

The de Rham family associated to $\Psi^* IL$ is defined by the finite quotient vector spaces

$$\Psi_J(IL) := (N_{J'_Z}L)[N]/\mathrm{Im}\, A_J, A_i : \Psi_J(IL) \to \Psi_{J\cup i}(IL) \text{ for } i \notin J \tag{4.11}$$

We write $\Psi^* L$ when $M_Z = M$.

To satisfy the conditions in (§3.2.6.1), set for $J \subset K$: $f_{JK} := A_{K\setminus J}$: $\Psi_J(IL) \to \Psi_K(IL)$ which is well defined since $A_{K\setminus J} : N_{J'_Z}L[N] \to N_{K'_Z}L[N]$ and $A_{K\setminus J}(\mathrm{Im}\, A_J) \subset (\mathrm{Im}\, A_K)$. Set $p_{KJ} := \Psi_K(IL) \to \Psi_J(IL)$ be the natural projection since $N_{K'_Z}L[N] \subset N_{J'_Z}L[N]$ and $\mathrm{Im}\, A_K \subset \mathrm{Im}\, A_J$.

Lemma 4.1. *There exists a natural filtered quasi-isomorphism*

$$(\widetilde{\Psi}^* IL, F) \xrightarrow{\sim} (\Psi^* IL, F). \tag{4.12}$$

Proof. We prove that the complex $\mathrm{Im}^* A$ is acyclic. We write the complex $\mathrm{Im}^* A$ as the cone over the morphism A_i. Let $i \in M_Z \neq \emptyset$ such that $m_i > 0$, and $Z' := Z \setminus \{i\}$. We write the complex $\mathrm{Im}^* A$ as a cone complex over the morphism

$$A_i : \mathrm{Im}^* A(\hat{i}) \to \mathrm{Im}^* A(\hat{i})$$

where $\mathrm{Im}^* A(\hat{i})$ is defined by deleting the index i. We prove that A_i is an isomorphism of complexes. Since N_i is nilpotent, the inverse of A_i on $L[N, N^{-1}]$ is defined and equal to the finite sum

$$A_i^{-1} = \Sigma_{j\geq 0} m_i^{-j-1} N_i^j N^{-j-1}. \tag{4.13}$$

In particular, the endomorphism A_i is injective on $L[N]$. Since $A_i : \mathrm{Im}\, A_J \to \mathrm{Im}\, A_{J\cup\{i\}}$ is clearly surjective for $i \notin J$, we deduce that the morphism A_i is an isomorphism of complexes and its cone $\mathrm{Im}^* A$ is acyclic.

Example 4.2. In Diagram 4.8, the complex $\mathrm{Im}^* A$

$$\begin{array}{ccccc} L[N] & \xrightarrow{A_1+A_2} & A_1L[N] \oplus A_2L[N] & \xrightarrow{A_2-A_1} & A_2A_1L[N] \\ \downarrow N_3 & & \downarrow (N_3 \oplus N_3) & & \downarrow N_3 \\ N_3L[N] & \xrightarrow{A_1+A_2} & N_3A_1L[N] \oplus N_3A_2L[N] & \xrightarrow{A_2-A_1} & N_3A_2A_1L[N] \end{array} \tag{4.14}$$

is the cone over the isomorphism defined in terms of A_2:

$$\begin{array}{ccccc} L[N] & \xrightarrow{A_1+N_3} & A_1L[N] \oplus N_3L[N] & \xrightarrow{N_3-A_1} & A_1N_3L[N] \\ \simeq\downarrow A_2 & & \simeq\downarrow (A_2 \oplus -A_2) & & \simeq\downarrow A_2 \\ A_2L[N] & \xrightarrow{-A_1+N_3} & A_2A_1L[N] \oplus A_2N_3L[N] & \xrightarrow{-N_3-A_1} & A_2A_1N_3L[N] \end{array}$$

Lemma 4.2. *Let $M_Z \subset M$, $J \subset M$, $J_Z = J \cap M_Z$, $l = |J|$ and $l_z = |J_Z|$, then*

$$\Psi_J(IL) := (N_{J'_Z}L)[N]/\mathrm{Im}\, A_J \simeq \Psi_{J_Z}(N_{J'_Z}L) \simeq \oplus_{j\leq l_z-1} N_{J'_Z}L \otimes N^j \tag{4.15}$$

In particular, for $M_Z = M$, we have $A_J L \simeq L[N]/\mathrm{Im}\, A_J \simeq \oplus_{j\leq l-1} L \otimes N^j$.

Let $A_{J_Z} := A_{l_z} \cdots A_1$, by definition:

$\Psi_{J_Z}(N_{J'_Z}L) := (N_{J'_Z}L)[N]/\mathrm{Im}\,(A_{J_Z} : (N_{J'_Z}L)[N] \to (N_{J'_Z}L)[N])$
Since $A_i = -N_i$ for $i \in J'_Z$, $A_{J_Z}(N_{J'_Z}L) \subset \mathrm{Im}\, A_J$, the isomorphism follows.

Since A_{J_Z} acts as a polynomial in the variable N on $\Psi_{J_Z}(N_{J'_Z}L)$ and vanish, we develop A_{J_Z} in terms of the elementary symmetric functions σ_j, of $\frac{1}{m_i}N_i$ for $i \in J_Z$ to deduce the relation on $\Psi_{J_Z}(N_{J'_Z}L)$:

$$N^{l_z} = \Sigma_{1\le j\le l_z}(-1)^{j+1}\sigma_j(\frac{1}{m_i}N_i, i \in J_Z) \otimes N^{l_z-j}.$$

The action of the endomorphism N is the multiplication by N on $N_{J'_Z}L \otimes N^j$ for $j < l_z - 1$ and acts on $a \in N_{J'_Z}L \otimes N^{l_z-1}$ by the relation:

$$N(a \otimes N^{l_z-1}) = \Sigma_{1\le j\le l_z}(-1)^{j+1}\sigma_j(\frac{1}{m_i}N_i, i \in J_Z)(a) \otimes N^{l_z-j}. \tag{4.16}$$

Example 4.3. On the 2-dimensional disc $D^2 \subset \mathbb{C}^2$, let $W = W_1 \cup W_2$ be the NCD defined by the equation $z_1z_2 = 0$. A unipotent local system $\mathbf{L}$ is defined by a vector space L with the action of 2 nilpotent endomorphisms $(L, N_i, i \in [1, 2])$.

Let $M_Z = \{1, 2\}$ and $Z = W$, then $\Omega^* L[N](\log Z)$ is defined by the diagram

$$L[N] \xrightarrow{A_1\oplus A_2} L[N] \oplus L[N] \xrightarrow{A_2-A_1} L[N]$$

with differentials defined by A_i and N_i up to sign. Taking the quotient in $\Psi_J L$, we obtain the complex $\Psi^* IL$ in degree 1 and 2

$$\begin{array}{ccccc} \Psi^1 IL := \Psi_1 \oplus \Psi_2 = L & & \oplus & & L \\ \downarrow d \quad \downarrow d & & & \downarrow d & \\ \Psi^2 IL := \quad \Psi_{12} & = & L & \oplus & LN \end{array} \tag{4.17}$$

where for $(a, b) \in \Psi_1 \oplus \Psi_2 : d(a, b) = -N_1b - N_2a \oplus (m_2a + m_1b)N \in \Psi_{12}$.
The action of N on Ψ is a matrix with entries defined by N_1 and N_2 as follows:

1. $N|\Psi_1 \oplus \Psi_2 :\ Na = N_1a,\ Nb = N_2b,$
2. $N|\Psi_{12} : N(a + bN) = -\frac{1}{m_1m_2}N_1N_2b + ((\frac{N_1}{m_1} + \frac{N_2}{m_2})b + a)N$

Remark 4.2 (The complex $\bar{\Psi}^* L$). We introduce the following complex:

(i) $\bar{\Psi}^* L$ is the exterior differential algebra (or Koszul complex) defined as the simple complex associated to the de Rham family

$$(L[N^{-1}]_J = L[N^{-1}], A_i, i \in M)_{J\subset M};$$

(ii) $\bar{\Psi}^* IL \subset \bar{\Psi}^* L$ is the simple complex associated to the de Rham subfamily $(N_{J'_Z}L[N^{-1}] \subset L[N^{-1}], A_i, i \in M)_{J\subset M}$ where $N_\emptyset L = L$

$$\bar{\Psi}^* IL := \boldsymbol{s}((N_{J'_Z}L[N^{-1}] \subset L[N^{-1}], A_i)_{J\subset M,\, i\in M}. \tag{4.18}$$

(iii) The exterior complex $\Omega^*(IL[N,N^{-1}],A_i)_{i\in M_Z}$ is acyclic since A_i is invertible for $i\in M_Z$ (Equation 4.13). Equivalently, the simple complex associated to the de Rham sub-family $(N_{J'_Z}L[N,N^{-1}]$ is acyclic and we have a quasi-isomorphism: $\bar{\Psi}^*IL\simeq\widetilde{\Psi}^*IL$.

4.2.1.4 The logarithmic limit complex $\Psi^*\,I\mathcal{L}\simeq\psi^{\boldsymbol{u}}_{\boldsymbol{f}}(j_{!*}\mathbf{L})$

We resume the global hypothesis with a morphism $f:X\to D$ and a NCD $Z:=\sum_{i\in I_Z}m_iY_i$ as central fiber and let $\mathbf{L}$ be a local system defined on a NCD $Y\supset Z$.

The global limit logarithmic complex $\Psi^*I\mathcal{L}$ with indices in the right half plane is constructed as follows. Consider the double complex with gradings $p\geq 0$ and q:

$$\tilde{\Psi}^{p,q}I\mathcal{L}:=\mathbb{C}u^{-(p+1)}\otimes_{\mathbb{C}}i_Z^*\mathrm{IC}^{p+q+1}\mathcal{L}(\log Z) \tag{4.19}$$

with differentials

$$\begin{aligned} d':\tilde{\Psi}I\mathcal{L}^{p,q}\to&\tilde{\Psi}I\mathcal{L}^{p+1,q},\quad d'(u^{-(p+1)}\otimes\varphi)=-(p+1)u^{-(p+2)}\otimes\frac{df}{f}\wedge\varphi,\\ d'':\tilde{\Psi}I\mathcal{L}^{p,q}\to&\tilde{\Psi}I\mathcal{L}^{p,q+1},\quad d''(u^{-(p+1)}\otimes\varphi)=u^{-(p+1)}\otimes\nabla\varphi,\end{aligned}$$

where d'' is deduced from the differential ∇ on $i_Z^*IC^*\mathcal{L}(\log Z)$, hence the total differential is: $d(u\otimes\varphi)=u\otimes\frac{df}{f}\wedge\varphi\pm 1\otimes\nabla\varphi$. The associated simple complex is denoted by $\tilde{\Psi}^*I\mathcal{L}$.

Definition 4.3 ($\Psi^*I\mathcal{L}$). There exists a unique complex $\Psi^*I\mathcal{L}:=\tilde{\Psi}^*I\mathcal{L}/\mathcal{I}m^*A$, defined as a quotient complex by a sub-complex $\mathcal{I}m^*A\subset\tilde{\Psi}^*I\mathcal{L}$ such that at a point $x\in Y_M^*$, the tilde embedding (chapter 3, Lemma 3.1) is a quasi-isomorphism:

$$\Psi^*IL[1]\xrightarrow{\sim}\Psi^*I\mathcal{L}_x,\ L:=\mathcal{L}(x) \tag{4.20}$$

(We remark that the embedding of Ω^*LN^j is equal to the embedding of Ω^*L in (ch. 2, Lemma 2.8) with a coefficient $(-2i\pi)^j$ times).

The complex $\Psi^*I\mathcal{L}$ coincides in the case $Y=Z$ and $\mathcal{L}=\mathbb{C}$, with the complex A^* introduced in Steenbrink [St 76].

4.2.1.5 The morphism ν of bidegree $(1,-1)$ on $\Psi^*I\mathcal{L}$

The morphism ν of bidegree $(1,-1)$ on $\Psi^*I\mathcal{L}$ is induced by

$$\tilde{\nu}:\tilde{\Psi}I\mathcal{L}^{p,q}\to\tilde{\Psi}I\mathcal{L}^{p+1,q-1} \tag{4.21}$$

defined by $\tilde{\nu}(\varphi):=u^{-p}\otimes\varphi\to u^{-p+1}\otimes\varphi$ for $\varphi\in i_Z^*\Omega^{p+q+1}\mathcal{L}(\log Z))$.

Proposition 4.1. *The unipotent perverse sheaf $\psi_f^u(j_{!*}\mathbf{L})$ is isomorphic to the $\Psi^* I\mathcal{L}$ in the derived category $D^b(Z,\mathbb{C})$, precisely*

$$\psi_f^u(j_{!*}\mathbf{L}) \simeq \tilde{\Psi}^* I\mathcal{L} \simeq \Psi^* I\mathcal{L} \tag{4.22}$$

The proof may be reduced to the original Lemma in ([DeK 73], Exposé XIV, Lemma 4.18.5). The main point is the comparison theorem between the topological definition of $\psi_f^u(j_{!*}\mathbf{L})$ and the analytic definition of $\Psi^* I\mathcal{L}$ which allows the later use of Hodge filtration on logarithmic differential forms. By construction, we need to construct an adequate representative of the complex $\psi_f j_{!*}\mathbf{L} := i_Z^* \widetilde{j}_{Z*}\widetilde{p}^* j_{!*}\mathbf{L} \in D^b(Z,\mathbb{C})$ (see Equation 2.14) in order to define later the correct filtration F on it. For this purpose we introduce another complex.

4.2.2 Proof of proposition 4.1

4.2.2.1 The complex $\bar{\Psi}^* I\mathcal{L}$

We suppose the central fiber of the morphism $f : X \to D$ a NCD $Z \subset Y$ included but eventually distinct from the singular set Y of $\mathcal{L}$. The wedge product

$$\smile \eta : i_Z^* \mathrm{IC}^* \mathcal{L}(\log Z) \to i_Z^* \mathrm{IC}^* \mathcal{L}(\log Z)[1] \colon \varphi \mapsto \varphi \wedge \frac{1}{2\pi i}\frac{df}{f}$$

defines a morphism of degree 1. Recall that $t = \pi(u) = \exp 2\pi i u$. Then $c := [\frac{1}{2\pi i}\frac{dt}{t}] \in H^1(D^*,\mathbb{Q})$ and $I \smile \eta$ corresponds to the cup product with the inverse image $\eta = f^* c \in H^1(X^*,\mathbf{Q})$:

$$\smile \eta : H^1(X^*,\mathbf{Q}) \otimes H^i(X^*, j_{!*}\mathbf{L}) \to H^{i+1}(X^*, j_{!*}\mathbf{L})$$

The associated simple complex with indices in the left half plane

$$\bar{\Psi}^* I\mathcal{L} := \mathbb{C}[u] \otimes_{\mathbb{C}} i_Z^* \mathrm{IC}^* \mathcal{L}(\log Z) = \Sigma_{p\le 0}\mathbb{C}u^{-p} \otimes_{\mathbb{C}} i_Z^* \mathrm{IC}^* \mathcal{L}(\log Z)[p] \tag{4.23}$$

where $du = \frac{df}{f}$, is the simple complex associated to the double complex with indices (p,q), $p \le 0$, $p+q \ge -\dim X$

$$\begin{aligned} \bar{\Psi} I\mathcal{L}^{p,q} &= \mathbb{C}u^{-p} \otimes_{\mathbb{C}} i_Z^* IC^{q+p}\mathcal{L}(\log Z), \quad p \le 0, \\ d' : \bar{\Psi} I\mathcal{L}^{p,q} \to & \bar{\Psi} I\mathcal{L}^{p+1,q} : \quad d'(u^{-p} \otimes \varphi) = -pu^{-p-1} \otimes \frac{df}{f} \wedge \varphi, \\ d'' : \bar{\Psi} I\mathcal{L}^{p,q} \to & \bar{\Psi} I\mathcal{L}^{p,q+1} : \quad d''(u^{-p} \otimes \varphi) = u^{-p} \otimes \nabla\varphi, \end{aligned} \tag{4.24}$$

Definition 4.4. The complex $\bar{\Psi}^* I\mathcal{L}$ is the simple complex associated to this double complex:

$$\bar{\Psi}^* I\mathcal{L} := \boldsymbol{s}(\bar{\Psi} I\mathcal{L}^{p,q}, d', d'')_{p\leq 0, q\in\mathbb{Z}} = \boldsymbol{s}(i_Z^* \mathrm{IC}^* \mathcal{L}(\log Z)[p], \wedge \frac{1}{2\pi i}\frac{f}{df})_{p\leq 0}.$$

In the case $Z = Y$, the complex $\bar{\Psi}^* I\mathcal{L} = \bar{\Psi}^*\mathcal{L}$ has been introduced in [St 76], [NaG 90]. It embeds into $i_Z^* \widetilde{j}_{Z*} \widetilde{p}^* j_{!*}\mathbf{L}$. This embedding extends easily to the complex $\bar{\Psi}^* I\mathcal{L}$ in the case $Z \neq Y$.

4.2.2.2 The quasi-isomorphism $\bar{\Psi}^* I\mathcal{L} \simeq \tilde{\Psi}^* I\mathcal{L}$

The morphism $\mu : \bar{\Psi}^* I\mathcal{L} \to \tilde{\Psi}^* I\mathcal{L}$ defined by: $\mu(u^p \otimes \omega_p) = 0$ if $p \neq 0$ and $\mu(u^0 \otimes \omega_0) = (-1)^{|\omega_0|} u^{-1} \otimes \frac{1}{2\pi i}\frac{df}{f} \wedge \omega_0$ is a quasi-isomorphism (to compare with [NaG 90] Lemma 2.5).

The morphisms $\nu \circ \mu$ and $\mu \circ \nu$ are homotopic. The homotopy $h : \bar{\Psi}^* I\mathcal{L} \to \tilde{\Psi}^* I\mathcal{L}$ of bidegree $(0, 1)$ is defined by

$h(u^p \otimes \omega_p) = 0$ if $p \neq 0$ and $h(u^p \otimes \omega_p) = (-1)^{|\omega_0|} u^{-1} \otimes \frac{df}{f} \wedge \omega_0$ if $p = 0$.

Proposition 4.2. *Let* $\mathbf{L}$ *be a polarized VHS on* $X \setminus Y$*. Then there exists a natural quasi-isomorphism* $\bar{\Psi}^* I\mathcal{L} \xrightarrow{\sim} \Psi_f^u j_{!*}\mathbf{L}$*.*

The local proof below (Lemma 4.4) is based on an extension of the following comparison result by Deligne [De 70] between analytic and topological complexes:

Lemma 4.3. *Let* $Y = Z$*. There exists a quasi-isomorphism* $\bar{\Psi}^*\mathcal{L} \xrightarrow{\sim} \Psi_f^u \mathbf{L}$*.*

The proof is based on a comparison with the *meromorphic complex* $\Psi_f^m(\Omega^*_{X^*}(\mathcal{L}))$ on a complex manifold $X^* := X \setminus Z$. We remark that the embedding j is Stein, hence j_* is acyclic on coherent sheaves on $X \setminus Z$. The following isomorphism is proved in ([DeK 73], Exposé XIV, Proposition 4.15)

$$\Psi_f^m(\Omega^*_{X^*}(\mathcal{L})) \xrightarrow{\sim} \Psi_f^u \mathbf{L}. \tag{4.25}$$

The complex $\bar{\Psi}^*\mathcal{L}$ is embedded as a sub-algebra in $\widetilde{j}_{Z*}\widetilde{p}^*\Omega^*\mathcal{L}$ by the relation

$$I(u^p \otimes \varphi) := \frac{1}{p!} u^p \otimes \varphi \text{ for } p \geq 0 \text{ where } du = \frac{df}{2\pi i f} \text{ since } 2\pi i u = \log f.$$

This embedding defines the quasi-isomorphism

$$\bar{\Psi}^*\mathcal{L} \xrightarrow{\sim} \Psi_f^m(\Omega^*_{X^*}(\mathbf{L})) \xrightarrow{\sim} \Psi_f^u \mathbf{L} \tag{4.26}$$

as we check by the following local computation due to Deligne.

Let $Y = Z = \cup_{i\in I} Z_i$ be the decomposition into smooth irreducible components Z_i with indices $i \in I$ and let $\mathbf{L}$ denote a local system on $X \setminus Z$, B_y

a polydisc in X with center $y \in Z$, $B_y^* = B_y \setminus Z$ and $\widetilde{B}_y^* := \widetilde{D}^* \times_{D^*} B_z$. The unipotent cohomology $H^i(\Psi_f^u \mathcal{L})_y \simeq H^i(\widetilde{B}_y^*, \mathbf{L})^u$ is related to $H^i(B_y^*, \mathbf{L})$ as follows.

Let U be a product of discs D^e with center 0 embedded as a polydisc $B_y \subset X$ such that $f : U \to D$ is defined by $\prod_{i\in[0,d]} z_i^{m_i}$ on U, $U^* = U \cap f^{-1}D^*$ and $\widetilde{U}^* := \widetilde{D}^* \times_D U \simeq \widetilde{B}_y^* \subset \widetilde{X}$.

A generator c of $H^1(D^*, \mathbb{Q})$ defines an element $\eta := f^*(c) \in H^1(U^*, \mathcal{L})$ and the cup-product map $\smile \eta : H^{i-1}(U^*, \mathcal{L}) \to H^i(U^*, \mathcal{L})$.

On the one hand, the cohomology of the fiber of $\Psi_f^u \mathcal{L}$ at y is isomorphic to

$$H^i(\Psi_f^u \mathbf{L})_y = H^i(\widetilde{U}^*, \mathbf{L})^u \simeq H^i(U^*, \mathbf{L})/\eta H^{i-1}(U^*, \mathbf{L}) \tag{4.27}$$

(see [DeK 73], Exposé XIV, Lemma 4.18.5).

On the other hand, since $Y = Z$, the morphism $\smile \eta$ is induced by $\wedge \frac{df}{2\pi\imath f}$, and a spectral sequence argument proves the isomorphism

$$H^i(\bar{\Psi}^* \mathcal{L}) \simeq H^i(U^*, \mathbf{L})/\eta H^{i-1}(U^*, \mathbf{L}).$$

4.2.2.3 The general case: $Y \neq Z$

Let $Y' := \cup_{i\in I-I_Z} Y_i$ denote the subdivisor of Y complement to Z such that $Y = Z \cup Y'$. Let $X' := X \setminus Y'$ and $f' : X' \to D$ the restriction of f to X'.

The natural quasi- isomorphism in Proposition 4.2 is an extension of the quasi- isomorphism $\bar{\Psi}^* \mathcal{L} \xrightarrow{\sim} \Psi_{f'}^u \mathbf{L}$ in Lemma 4.3. It remains to prove the quasi-isomorphism locally, at a point $x \in Y_M^* \subset Z_{M_Z}^*$.
With the notations of §4.2.1.2, and (Definition 4.1), we introduce the complex:

$$\bar{\Psi}^* IL := \boldsymbol{s}((N_{J'_Z} L[N^{-1}] \subset L[N^{-1}], A_i)_{J\subset M,\, i\in M}. \tag{4.28}$$

such that there exists an isomorphism: $\bar{\Psi}^* IL \simeq (\bar{\Psi}^* I\mathcal{L})_x$ defined by the tilde embedding (see chapter 1, Lemma 3.1). The proof of the proposition is reduced to the following

Lemma 4.4. $\bar{\Psi}^* IL \xrightarrow{\sim} (\Psi_f^u j_{!*}\mathbf{L})_x$

The lemma follows from the isomorphism: $(j_{Z*} j_Z^* j_{!*}\mathbf{L})_x \simeq \mathrm{IC}^* L(\log Z)$ (chapter 3, Proposition 3.2). The proof is similar.

Let $U := D^p \times D^q$ with local coordinates $(z., t.) := (z_1, \cdots, z_p, t_1, \cdots, t_q)$ centered at x, such that $\mathbf{L}$ induces on $U^* := (D^*)^p \times (D^*)^q$ a local system unipotent along $Y \cap U$.

Let $g : D^p \to D$ denote the morphism $g(z.) = z_1^{m_1} \cdots z_p^{m_p}$ and $p_1 : D^p \times D^q \to D^p$ the first projection such that $f = g \circ p_1$. The central fiber $W \subset D^p$ of g is defined by $g(z.) = 0$ and its pre-image $Z \cap U = p_1^{-1}(W)$ is the central fiber of f in U.

We reduce the proof of the lemma to the case $\psi_g^u p_{1*}(j_{!*}\mathbf{L}_{|U})$, where the derived functor $p_{1*}(j_{!*}\mathbf{L}_{|U})$ has locally constant cohomology on $D^p - W$. We

consider the diagram

$$\begin{array}{ccccc} U' := U \setminus Z = (D^p \setminus W) \times D^q & \xrightarrow{j_Z} & D^p \times D^q & \xrightarrow{p_1} & D^p & \xrightarrow{g} & D \\ & & \cup & & \cup & & \cup \\ U \cap Z = p_1^{-1}(W) = U \cap f^{-1}(0) = & & W \times D^q & \xrightarrow{p_{1|}} & W = g^{-1}(0) & \xrightarrow{g_|} & 0 \end{array}$$

The cohomology sheaves $\mathbf{L}^i := R^i(p_{1|U'})_*(j_{!*}\mathbf{L}_{|U'})$ are locally constant on $(D^*)^p$. Their fiber at a point $z. \in (D^*)^p$ is isomorphic to the intersection cohomology of a local system on $(D^*)^q$ at the center $0 \in D^q$.

Let $j_W : (D^p \setminus W) \to D^p$ be the inclusion. Since the restriction of p_1 to the family of spheres in the fibers of p_1 is proper, the derived functor p_{1*} commutes with j_{W*} and j_{Z*} and similarly, the functors ψ_f on U and ψ_g on D^p commute with p_{1*} ([DeK 73] §2.1.7):

$$p_{1|*}\psi_f^u j_{!*}\mathbf{L} \xrightarrow{\sim} \psi_g^u (p_{1|U'})_*(j_{!*}\mathbf{L}_{|U'}).$$

By constructibility, we deduce for U small enough, the following isomorphisms

$$\begin{aligned}(\psi_f^u j_{!*}\mathbf{L})_0 \simeq R\Gamma(U, \psi_f^u j_{!*}\mathbf{L}) = R\Gamma(D^p, R(p_{1|_*}\psi_f^u j_{!*}\mathbf{L}) \\ \xrightarrow{\sim}(Rp_{1|*}\psi_f^u(j_{!*}\mathbf{L}_{|U'}))_0 \xrightarrow{\sim} (\psi_g^u R(p_{1|U'})_*(j_{!*}\mathbf{L}_{|U'}))_0\end{aligned}$$

By a spectral sequence argument, Deligne's comparison applies to the complex $R(p_{1|U'})_* j'_{!*}\mathbf{L}$ since it has locally constant cohomology.

We check now that the induced morphism $\bar{\Psi}^* IL \to (\psi_g^u R(p_{1|U'})_*(j_{!*}\mathbf{L}_{|U'}))_0$ is an isomorphism. We write $\bar{\Psi}^* IL$ as the associated single complex of a double complex as in (ch. 1, Remark 3.4). Let $M'_Z := M \setminus M_Z, J'_Z := J \setminus J_Z$. For each fixed subset $J_Z \subset M_Z$ we define the following Koszul complex :

$$\mathrm{IC}^*(L[N^{-1}], N_i)_{i \in M'_Z} := \boldsymbol{s}(N_{J'_Z} L[N^{-1}], N_i)_{J'_Z \subset M'_Z,\, i \in M'_Z}$$

by summing over $J'_Z \subset M'_Z$, on which A_j acts for $j \in M_Z$. We define

$$\Omega^*(\mathrm{IC}^*(L[N^{-1}], N_i)_{i \in M'_Z}, A_j)_{j \in M_Z}$$

as the sum over $J_Z \subset M_Z$ of these constant complexes

$$\bar{\Psi}^* IL := \boldsymbol{s}\big(\boldsymbol{s}(N_{J'_Z} L[N^{-1}], N_i)_{J'_Z \subset M'_Z, i \in M'_Z}, A_j\big)_{J_Z \subset M_Z, j \in M_Z} \tag{4.29}$$

We use this structure of double complex to define filtrations on both terms of the lemma with isomorphic spectral sequences.

Let H^j denote the fiber of the local system $\mathcal{H}^j := R^j(p_{|U'})_*(j_{!*}\mathcal{L}_{|U'})$, with an action of A_i for $i \in M_Z$ and let $\bar{\Psi}^* H^j$ be defined by H^j as in (Remark 4.2, i).

Sub-lemma.

(i) We have $H^j[N^{-1}][-j] := \mathrm{Gr}_j^\tau \boldsymbol{s}(N_{J'_Z} L[N^{-1}], N_i)_{J'_Z \subset M'_Z, i \in M'_Z}$, *where* τ *denotes the truncation filtration of the complex.*

(ii) The filtration τ *extends to a filtration* W' *on* $\bar{\Psi}^* IL$ *satisfying*

$$\mathrm{Gr}_j^{W'} \bar{\Psi}^* IL := \boldsymbol{s}(H^j[N^{-1}][-j], A_i)_{J_Z \subset M_Z, i \in M_Z} \simeq \bar{\Psi}^* H^j. \tag{4.30}$$

Proof of the lemma. Let $\tau'_j \Psi_g^u(R(p_{|U'})_*(j_{!*}\mathbf{L}_{|U'})) := \Psi_g^u \tau_j R(p_{1|U'})_*(j_{!*}\mathbf{L}_{|U'}))$, then

$$(\mathrm{Gr}_j^{\tau'} \Psi_g^u(R(p_{|U'})_*(j_{!*}\mathbf{L}_{|U'}))_0 = (\Psi_g^u \mathcal{H}^j[-j])_0.$$

Since $\mathcal{H}^j$ is a local system we apply Proposition 4.3 to deduce

$$\bar{\Psi}^* \mathrm{Gr}_j^\tau(\boldsymbol{s}(N_{J'_Z} L, N_i)_{J'_Z \subset M'_Z, i \in M'_Z}) = \bar{\Psi}^* H^j \simeq (\Psi_g^u \mathcal{H}^j)_0$$

from which we deduce an isomorphism of spectral sequences defined by the filtrations $(\bar{\Psi}^* IL, W')$ and $(\Psi_g^u(R(p_{|U'})_*(j_{!*}\mathbf{L}_{|U'}), \tau')_0$, hence an isomorphism of complexes. This proves Proposition 4.2 and completes the proof of Proposition 4.1 as we established the following isomorphisms

$$\psi_f^u j_{!*}\mathbf{L} \xleftarrow{\sim} \bar{\Psi}^* I\mathcal{L} \xrightarrow{\sim} \tilde{\Psi}^* I\mathcal{L} \xrightarrow{\sim} \Psi^* I\mathcal{L} \tag{4.31}$$

Chapter 5
Limit mixed Hodge structure on $\Psi^* I\mathcal{L}$

*Hypothesis.*Let $f : X \to D$ be a projective morphism on a non singular complex variety X to a complex disc and $\mathbf{L}$ a local system defined on an open subset $U := X \setminus Y$ complement of a NCD $Y = \bigcup_{i\in I} Y_i \subset X$. We suppose the central fiber equal to a NCD $Z = f^{-1}(0)$ contained in Y as a sub-divisor. The divisor Z is a union of components of Y indexed by a subset $I_Z \subset I$. For $i \in I_Z$ let $m_i > 0$ denote the multiplicity of Y_i in the divisor of f, so $(f) = \sum_{i\in I_Z} m_i Y_i$.
*Results.*Let $\mathbf{L}$ be a polarized VHS on U, the filtration F is transported onto the nearby cycle complex by the isomorphism $\psi_f^u j_{!*}\mathbf{L} \simeq \Psi^* I\mathcal{L}$ (see chapter 4, Proposition 4.1). The monodromy filtration $W(N)$ shifted by the weight w' of $\mathbf{L}$ define the weight filtration $W := W(N)[w']$ on $\psi_f^u j_{!*}\mathbf{L}$ such that F and W form a structure of mixed Hodge complex (MHC) on $\psi_f^u j_{!*}\mathbf{L}$.

The finite increasing filtration $W(\nu)$ defined by the nilpotent endomorphism ν on $\Psi^* I\mathcal{L}$ corresponds to the monodromy filtration $W(N)$ on $\psi_f^u j_{!*}\mathbf{L}$.

We define also the vanishing cycles complex $\varphi_f(j_{!*}\mathbf{L})$ as a mixed cone over the specialization sp : $i_Z^* j_{!*}\mathbf{L} \to \psi_f(j_{!*}\mathbf{L})$ (§5.1.3.2) with filtrations W and F, on which the action of the nilpotent endomorphism N extends.

The main result of this chapter is the decomposition of $Gr_*^{W(N)}\Psi_f^u j_{!*}\mathbf{L}$ and $Gr_*^{W(N)}\varphi_f^u j_{!*}\mathbf{L}$ into a direct sum of intermediate extensions on the smooth and projective components of the NCD Z.

Remark 5.1 (Conventions and terminology). With the notations of (§2.4.0.2), we denote by $({}^p\mathbf{L}, F) := (\mathbf{L}[\dim X], F)$ the perverse shifted polarized VHS.
If $L := \mathcal{L}_X(x)$ is the fiber of Deligne's extension, then ${}^pL := {}^p\mathcal{L}_X(x)$. If ν acts on a perverse sheaf $\mathcal{P}$ as a nilpotent endomorphism, the filtration $W(\nu)$ is well defined in the abelian category of perverse sheaves.
A shifted perverse sheaf $\mathcal{P}$ on an algebraic variety X is a complex such that $\mathcal{P}[k]$ is perverse when shifted by an integer k.
Consequently, the notation $\Psi^* I^p\mathcal{L} = \Psi^* I\mathcal{L}[\dim X]$ means a corresponding shift in the degrees. It is not a perverse sheaf on Z but $\Psi^* I^p\mathcal{L}[-1]$ is perverse, hence we introduce the notations:

$$ {}^p\psi_f^u j_{!*}{}^p\mathcal{L} := \psi_f^u j_{!*}{}^p\mathcal{L}[-1], \quad {}^p\varphi_f^u j_{!*}{}^p\mathcal{L} := \varphi_f^u j_{!*}{}^p\mathcal{L}[-1]. \tag{5.1} $$

ii) *Conventions on the weight filtration W.* In the text, we define a weight filtration on $\psi_f^u j_{!*}\mathbf{L}$ correspondind to a weight W on ${}^p\psi_f^u j_{!*}{}^p\mathbf{L}$. We apply the following conventions:

1) Let W be an increasing filtration on an object of an abelian category, the filtration $W[k]$ is defined in [De 72] and [De 75] by: $W(k)_n := W_{n-k}$. For example, this convention apply to $\psi_f^u j_{!*}\mathbf{L}$ where the weight W is related to the monodromy filtration by $W = W(N)[w']$.

2) In the case of a HS (H, F), the weight of the twisted HS $(H, F)(k)$ drops by $-2k$.

3) The weight W on a complex K is shifted with the complex: $W_{i+1}(K[1]) = (W_i K)[1]$ but not F $(F^i(K[1]) = (F^i K)[1])$.

4) A bifiltered complex of sheaves (K, W, F) on a topological space X was called for a while: cohomological MHC if its hypercohomology on X is a mixed Hodge complex. This terminology did not survive, so by extension we call this bifiltered complex a MHC.

After the introduction of perverse sheaves, we define in the text the more rich concept of perverse variation of MHS, which yields also a MHC, so for a while we use both terminology.

Definition 5.1. The morphism $N := Log\, T$ acts on $\psi_f K$, and extends to the cone $C(sp)$ to define a morphism N on $\varphi_f^u K$. The morphism N induces a morphism $var : \varphi_f^u K \to \psi_f^u K$ such that $can \circ \mathrm{var} = N$ and $var \circ \mathrm{can} = N$.

With the above conventions, we have:

$$ \begin{aligned} &\mathrm{Gr}^W_{a+{}^p w-1}\, {}^p\psi_f^u j_{!*}{}^p\mathbf{L} = (\mathrm{Gr}^{W(N)}_a \psi_f^u j_{!*}\mathbf{L})[n-1] = \mathrm{Gr}^W_{a+{}^p w}\psi_f^u j_{!*}{}^p\mathbf{L}[-1] \\ &\mathrm{Gr}^W_{a+{}^p w}\psi_f^u j_{!*}{}^p\mathbf{L} = (\mathrm{Gr}^W_{a+w'}\psi_f^u j_{!*}\mathbf{L})[n] \end{aligned} \tag{5.2} $$

In the case of the vanishing cycle, the lemma 5.1 apply and adds 1 to the weight of the structure of nilpotent orbit

$$ \begin{aligned} &\mathrm{Gr}^W_{a+{}^p w}\varphi_f^u j_{!*}{}^p\mathbf{L} = (\mathrm{Gr}^W_{a+{}^p w+1}\varphi_f^u j_{!*}{}^p\mathbf{L})[-1] = (\mathrm{Gr}^W_{a+w'+1}\varphi_f^u j_{!*}\mathbf{L})[n-1] \\ &\mathrm{Gr}^W_{a+w'+1}\varphi_f^u j_{!*}\mathbf{L} = \mathrm{Gr}^{W(N)}_a \varphi_f^u j_{!*}\mathbf{L} \end{aligned} \tag{5.3} $$

Theorem 5.1 (Decomposition of ${}^p\psi_f^u j_{!*}{}^p\mathbf{L}$ and ${}^p\varphi_f^u j_{!*}{}^p\mathbf{L}$). *The weight filtration W and the Hodge filtration F on ${}^p\psi_f^u j_{!*}{}^p\mathbf{L}$ (resp. ${}^p\varphi_f^u j_{!*}{}^p\mathbf{L}$) satisfy the following decompositions:*

$$ \begin{aligned} Gr_i^W\, {}^p\psi_f^u j_{!*}{}^p\mathbf{L} &\simeq \oplus_{Z_J\subset Z}({}^p\mathcal{P}_i^J, F) \\ Gr_i^W\, {}^p\varphi_f^u j_{!*}{}^p\mathbf{L} &\simeq \oplus_{Z_J\subset Z}({}^p\mathcal{Q}_i^J, F) \end{aligned} \tag{5.4} $$

into intermediate extensions of shifted polarized VHS on Y_Z^*: $({}^p\mathcal{P}_i^J, F)$ (resp. $({}^p\mathcal{Q}_i^J, F)$) (see Equation 5.31). Moreover:

$Gr^W_{a+{}_pw-1}\,{}^p\psi^u_f j_{!*}{}^p\mathbf{L} \simeq Gr^{W(N)}_a\,{}^p\psi^u_f j_{!*}{}^p\mathbf{L}$, $Gr^W_{a+{}_pw}\,{}^p\varphi^u_f j_{!*}{}^p\mathbf{L} \simeq Gr^{W(N)}_a\,{}^p\varphi^u_f j_{!*}{}^p\mathbf{L}$ *and there exists a diagram of morphisms :*

$$({}^p\psi^u_f j_{!*}{}^p\mathbf{L}, W, F) \overset{can}{\to} ({}^p\varphi^u_f j_{!*}{}^p\mathbf{L}, W, F) \overset{var}{\to} ({}^p\psi^u_f j_{!*}{}^p\mathbf{L}, W, F)(-1)$$

such that: $var \circ can = N$ and $can \circ var = N$.

The statement here is a decomposition in terms of *intermediate extensions* in opposition with classical statements on a decomposition of cohomology in terms of HS (before the introduction of perverse sheaves).
As a consequence of this decomposition and the existence of a pure HS on the hypercohomology of intermediate extensions of polarized VHS on the smooth projective components of Z (proved by induction on $\dim Z$), we deduce the existence of a MHS on $\mathbb{H}^*(Z, \psi^u_f j_{!*}\mathbf{L}, W, F)$ and $\mathbb{H}^*(Z, \varphi^u_f j_{!*}\mathbf{L}, W, F)$.

Organization of the chapter. After a local study (§5.1), the theorem follows from (Proposition 5.3). The proof of the local version of the theorem (see Equation 5.23) relies on the local description of perverse sheaves by de Rham families (§3.2.6.1, Remark 3.6) and a criterion by Kashiwara on the decomposition into a direct sum of intermediate extensions, based on the property of distinguished pairs (§3.2.6.2).

For the global version (see §5.1.3.1) and for the results on vanishing cycles $(\varphi_f j_{!*}\mathbf{L}, F)$ (see §5.1.3.2).

5.1 Local proof of theorem 5.1

The proof is local at a point $x \in Z$ (see §5.1.2 and Corollary 5.2). It is necessary to use the local description of a perverse sheaf by the associated *canonical de Rham families (Remark 3.6, Equation 4.11), on which natural structures of IMHS are defined below (§5.1.1).*

Since the decomposition occurs locally in the abelian category of MHS, it is necessarily accompanied by the decomposition of the filtration F.

The fiber of Deligne's extension $L := \mathcal{L}_X(x)$ at a point $x \in Z$ is endowed with a structure of nilpotent orbit and there exists a local decomposition of the complex $\mathrm{Gr}^{W(N)}_i \Psi^* IL$ into a direct sum of shifted complexes $IC^* P^J_a L$ defined by nilpotent orbits $(P^J_a L, W, F, N_i, i \in M \setminus J)$ of weight $a+w'+1-|J|$ with induced filtrations W and F defined by the equation (5.5) below.

We refer to the nilpotent orbit structure on $\Psi_J(L)$ (see Equation 5.7) proved by Kashiwara ([Sa 90], Appendix), and to the decomposition for distinguished pairs following the proof by Saito ([Sa 88], Lemma 5.2.15) (see Equation 5.17 below).

When x varies on Z^*_J, the $P^J_a L$ form a polarized VHS $\mathcal{P}^J_a$ (§5.1.2). We deduce from the purity theorem on the components of Z that the complex $\mathrm{Gr}^{W(\nu)}_a \Psi^* I\mathcal{L}$ is a Hodge complex (see §5.1.3.1).

The purity theorem on the hypercohomology of the complex $\mathrm{Gr}^{W(N)}_i \Psi^* I\mathcal{L}$ may be deduced from (Theorem 3.1) or by induction on $\dim X$, since $\dim Z <$

$\dim X$. The purity theorem is proved in the same time with the decomposition theorem. The main difference with the proof in [Sa 88] and [Sa 90], is the reduction to the use of the category of IMHS at isolated strata on V.

The main property of the limit MHS is the functoriality of the monodromy filtration with respect to the derived direct image f_* (§7.1.1.1). This is proved by induction on $\dim X$.

The theory of nilpotent orbit (resp. admissible variation of MHS) appears in [Sc 73, GrSc 73, CaKSc 86] (resp. [StZ 85]) but we adopt the notations and terminology of [Ka 86] as summarized in §2.3 of chapter 2.

5.1.1 Infinitesimal mixed Hodge structure on $\Psi_J(L)$

We consider again a polarized VHS $\mathbf{L}$ of weight w', $x \in Z_M^* \subset Y$ and the nilpotent orbit $(L, F, P, N_i, i \in M)$ of weight w' on $L := \mathcal{L}_X(x)$ with monodromy $N_i, i \in M$ and polarization P. The filtrations F and $W = W(\sum_{i\in M} N_i)[w']$ form a limit MHS on L. These filtrations are used to construct a MHS on $\Psi_J(L)$ (see ch. 4, Definition 4.2).

First, two filtrations W and F are defined on $L[N]$ as follows:

$$W_k(L[N]) = \sum_i W_{k+2i}L \otimes N^i \,,\; F^p(L[N]) = \sum_i F^{p+i}L \otimes N^i \tag{5.5}$$

such that

$$(\mathrm{Gr}_k^W L[N], F) = \oplus_i (\mathrm{Gr}_{k+2i}^W L, F[i]) \otimes N^i.$$

where $F[i]^p := F^{p+i}$ defines a HS of weight k for each index i, since $(\mathrm{Gr}_{k+2i}^W L, F)$ is of weight $k+2i$. The morphisms N and N_i induce morphisms:

$$W_k(L[N]) \to W_{k-2}(L[N]) \text{ and } F^p(L[N]) \to F^{p-1}(L[N])$$

(see §4.2.1.3, Equation 5.5).

Since for each $j \in M$, the endomorphisms $A_j = m_j N - N_j$ shift W by -2 and F_J by -1, $\mathrm{Im}A_J$ is compatible with F and W on $L[N]$, hence the filtrations W and F induce a MHS on the quotient (see Definition 4.2)

$$\Psi_J L = (L[N]/\mathrm{Im}A_J, W^J, F_J). \tag{5.6}$$

The filtration W^J in Equation (5.6) is in fact related to the monodromy filtration $W(N + \sum_{i\in M} N_i)$ ([Sa 90], Appendix):

Theorem (Kashiwara) *Let $(L, F, P, N_i, i \in M)$ be a polarized nilpotent orbit of weight w' and $m_i > 0$ for $i \in M$ a set of integers, then there exists on*

$$\Psi_J(L) = (L[N]/ImA_J, N, N_1, \ldots, N_n; F, P_J) \tag{5.7}$$

a structure of polarized nilpotent orbit of weight $\delta = w' + 1 - |J|$ defined by the nilpotent endomorphism $N_J := (N + \sum_{i\in M} N_i)|_{\Psi_J(L)}$ and the polarization

P_J (Equation 5.10). The monodromy filtration shifted by δ and the filtration F_J define a MHS on $\Psi_J(L)$:

$$W^J = W(N_J)[\delta] : W^J_{k+\delta} = W(N_J)_k \tag{5.8}$$

Precisely: $\forall k \geq 0,\ N^k_J : \mathrm{Gr}^{W^J}_{\delta+k}\Psi_J(L) \xrightarrow{\sim} \mathrm{Gr}^{W^J}_{\delta-k}\Psi_J(L)$.
We recall that polarized means that $\mathrm{Gr}_i^{W(N_J)}\Psi_J(L)$ is a direct sum of primitive polarized structure of weight $\delta + i$.

Let $\ell = |J|$ We recall the construction by Kashiwara of the polarization on $\Psi_J L$. Since the endomorphisms A_j shift W by -2 and F by -1, the isomorphism in Equation (5.51) is an isomorphism of MHS

$$(\Psi_J L, W^J, F_J) \simeq (\oplus_{j\leq \ell-1} L \otimes N^j, W, F) \tag{5.9}$$

The action of N on the left hand side is defined by Equation 4.16:

$$N(a \otimes N^{l-1}) = \sum_{j=1}^{\ell} (-1)^{j+1}\sigma_j((N_i/m_i), i \in J)(a) \otimes N^{\ell-j}.$$

In order to define a polarization we introduce a product P_J on $\Psi_J L$ as follows

$$P_J(a \otimes N^i, b \otimes N^j) = P(a, (-1)^i \operatorname{Res}(A_J^{-1}(b \otimes N^{i+j}))) \tag{5.10}$$

where A_J^{-1} is defined on $L[N, N^{-1}]$ and the map $\operatorname{Res} : L[N, N^{-1}] \to L$ is defined by $\operatorname{Res}(\sum_i u_i N^i) := u_{-1}$. This relation shows directly that the product is well defined on $\operatorname{Coker} A_J$: note that $P_J(a \otimes N^i, A_J(c \otimes N^j)) = P(a, (-1)^i \operatorname{Res}(c \otimes N^{i+j}) = 0$ as $\operatorname{Res}(c \otimes N^{i+j}) = 0$.

Using the explicit expression of A_i^{-1} (see ch. 4, Equation 4.13), we find

$$\forall a, b \in L :\ P_J(a, b \otimes N^j) = P(a, b) \text{ if } j = \ell-1, \text{ and } 0 \text{ if } 0 \leq j < \ell-1. \tag{5.11}$$

Example 5.1. Let $\mathbf{L}$ be a local system on $\mathbb{C}^2 \setminus Y$ where $Y = Y_1 \cup Y_2$ is defined by $z_1 z_2 = 0$. Then, $\Psi_i L$ at $x \in Y_i^*$ defines a VMHS for a varying x on Y_i^* and $W(N) = W(N_i)$, while $\Psi_{12} L$ is a nilpotent orbit for $L = \mathcal{L}(0)$ with limit MHS defined by $W^{\{12\}}$ a shifted weight filtration $W(N + N_1 + N_2)$.

$$\begin{tikzcd}
(\Psi_1 L, W^{\{1\}}, F)_{x\in Y_1^*} \arrow[d] & (\mathbf{L}, F) \arrow[l] \arrow[d] \arrow[dl] \\
(\Psi_{12} L, W^{\{12\}}, F)_{x=0\in Y_{12}^*} & (\Psi_2 L, W^{\{2\}}, F)_{x\in Y_2^*} \arrow[l]
\end{tikzcd}$$

where W^J is defined by (Equation 5.8) for $J = \{1\}, \{2\}, \{12\}$.

Remark that the decomposition (Equation 5.15) depends on the point x. For example, $(\Psi_1 L, W^{\{1\}}, F)_{x\in Y_{12^*}} \neq (\Psi_1 L, W^{\{1\}}, F)_{x\in Y_{1^*}}$ in the decomposition of $(\mathrm{Gr}^{W(N)}\Psi_{12}L, W^{\{12\}}, F)_0$ into a direct sum:

$(N_2\mathrm{Gr}^{W(N)}\Psi_1, W^{\{1\}}L, F)_0 \oplus (N_1\mathrm{Gr}^{W(N)}\Psi_2 L, W^{\{2\}}, F)_0 \oplus HS.$

5.1.1.1 Distinguished pairs of $\mathrm{Gr}_*^{W(N)}\Psi^* IL$

A diagram of modules $A \xrightarrow{\rho} B \xrightarrow{G} C$ satisfy the *decomposition property* if $B = \mathrm{Im}\,\rho \oplus \mathrm{Ker}\,G$. In this case, we say that the diagram forms a *distinguished pair* and G induces an isomorphism $\mathrm{Im}\,\rho \simeq \mathrm{Im}\,G \circ \rho$. If $C = A$, we say that the diagram forms a *distinguished pair* ([Ka 86, §2.2]). Such decompositions appear in Deligne's proof of the Hard Lefschetz Theorem.

The decomposition of $\mathrm{Gr}_a^{W(N)}\Psi^* L$ is based on the concept of distinguished pairs. The following proposition is proved by an argument similar to ([Sa 88] Lemma 5.2.15):

Proposition 5.1. *Let $(L, F, P, N_i, i \in M)$ be a polarized nilpotent orbit of weight w', $m_i > 0$ a family of integers for $i \in M$ and $J \subset M$ $(Y = Z)$. Let $W(N)$ denote the filtration defined on $\Psi_J L$ by the nilpotent endomorphism N (§4.2.1.3, Lemma 4.2 in the case $M_Z = M$). For each $J \subset M$, $a \in \mathbb{Z}$ and $i \in J$, the diagram*

$$\mathrm{Gr}_a^{W(N)}\Psi_{J\setminus i}L \xrightarrow{A_i} \mathrm{Gr}_a^{W(N)}\Psi_J L(-1) \xrightarrow{p_i} \mathrm{Gr}_a^{W(N)}\Psi_{J\setminus i}L(-1) \tag{5.12}$$

where $A_i = m_i N - N_i$ and p_i is the canonical projection, is a distinguished pair:

$$\mathrm{Gr}_a^{W(N)}\Psi_J L(-1) \quad \simeq \quad \mathrm{Im}\,A_i \oplus \mathrm{Ker}\,p_i. \tag{5.13}$$

and p_i induces an isomorphism of $\mathrm{Im}\,A_i \subset Gr_a^{W(N)}\Psi_J L$ with $\mathrm{Im}\,(N_i : Gr_a^{W(N)}\Psi_{J\setminus i}L \to Gr_a^{W(N)}\Psi_{J\setminus i}L)$.

Proof. Here $L := \mathcal{L}_X(x)$ at $x \in Y_M^*$ is endowed with the limit MHS W of an IMHS defined in (§5.1.1) at x. Let $i \in M$, the induced morphisms N, N_i and A_i, shift the limit MHS W (Equation 5.5) by -2 and F by -1.

Let and $N_0 := \sum_{k\in M\setminus\{i\}} N_k$, since $W(N + N_0 + N_i)$ is the relative weight filtration with respect to $W(N + N_0)$ on $\Psi_J(L)$ (resp. $W(N + N_0)$ is relative to $W(N)$) ([CaKSc 86], [Ka 86] Theorem 3.2.9, [StZ 85] Definition 2.5), we have a HS of weight $a + j + r + \delta$ where $\delta = w' + 1 - |J|$ on

$$\mathrm{Gr}_{j+r+a}^{W(N+N_0+N_i)}\mathrm{Gr}_{r+a}^{W(N_0+N)}\mathrm{Gr}_a^{W(N)}\Psi_J L \simeq \mathrm{Gr}_j^{W(N_i)}\mathrm{Gr}_r^{W(N_0)}\mathrm{Gr}_a^{W(N)}\Psi_J L$$

For all $j, r \in \mathbb{N}$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}_j^{W(N_i)}\mathrm{Gr}_r^{W(N_0)}\mathrm{Gr}_a^{W(N)}\Psi_{J\setminus i}L \arrow[r, "-A_i"] \arrow[d, "N_i"] & \mathrm{Gr}_{j-1}^{W(N_i)}\mathrm{Gr}_r^{W(N_0)}\mathrm{Gr}_a^{W(N)}\Psi_J L(-1) \arrow[d, "N_i"] \arrow[dl, "p_i"] \\
\mathrm{Gr}_{j-2}^{W(N_i)}\mathrm{Gr}_r^{W(N_0)}\mathrm{Gr}_a^{W(N)}\Psi_{J\setminus i}L(-1) \arrow[r, "-A_i"] & \mathrm{Gr}_{j-3}^{W(N_i)}\mathrm{Gr}_r^{W(N_0)}\mathrm{Gr}_a^{W(N)}\Psi_J L(-2)
\end{tikzcd} \tag{5.14}$$

Since N shifts $W(N)$ by -2, we deduce that $p_i \circ N = 0$, hence $p_i \circ A_i = -N_i$. The weight of the HS on the first line of (Equation 5.14) are the same, $a+j+r+(\delta+1)$ on the left and $a+(j-1)+r+\delta+2$ (after the twist by -1) on the right since it is defined by the limit MHS of the IMHS at $J\setminus i$ resp. J.

Moreover the maps N_i satisfy Lefschetz isomorphisms:
$(N_i)^j : \mathrm{Gr}_j^{W(N_i)} \simeq \mathrm{Gr}_{-j}^{W(N_i)}$. We also have:

$$P_J(A_i u, v) \;=\; P_{J-i}(u, p_i v) \text{ for all } u \in \Psi_{J-i}L \text{ and } v \in \Psi_J L.$$

In this situation, a result of M. Saito ([Sa 88] 5.2.15) shows the decomposition in Equation 5.13 (see also Kashiwara [Ka 86] Proposition 3.3.1, Lemma 5.6.5, Equation 5.5.16 with correct index $k-1$ on the right term). When j and r vary, we deduce the decomposition over all $\mathrm{Gr}_a^{W(N)}\Psi_J L$.

-Remark that we could consider only $W(N+N_i)$, since by (Remark 2.9) or (Proposition 2.7), for i fixed, there exists a filtration F_i and a polarization P_i such that (L, F_i, P_i, N, N_i) is a nilpotent orbit.
The proof proceeds without N_0, to deduce the decomposition of the vector space in (Equation 5.13), but it is necessarily in the category of IMHS.

5.1.1.2 The case of $\Psi_J(IL)$.

We extend the above result to the case where $Z \neq Y$. Let $M_Z \subset M$, $J \subset M$ such that $m_i = 0$ for $i \notin M_Z$, $J_Z = J \cap M_Z$, $J' = J \setminus J_Z$, $l = |J|$ and $l_z = |J_Z|$, then the proposition above extends to $(\Psi_J(IL), W^J, F_J)$.

Corollary 5.1. *For each $J \subset M$ the data $(\Psi_J IL, W^J, F_J, \{A_j, j \in M\})$ is an IMHS on $\Psi_J IL$. For each $J \subset M$, $a \in \mathbb{Z}$ and $i \in J$ we have the following distinguished pair:*

$$\mathrm{Gr}_a^{W(N)}\Psi_{J\setminus i}IL \xrightarrow{A_i} \mathrm{Gr}_a^{W(N)}\Psi_J IL(-1) \xrightarrow{p_i} \mathrm{Gr}_a^{W(N)}\Psi_{J\setminus i}IL(-1)$$

where $A_i = m_i N - N_i$ and p_i is the natural projection. The decomposition:

$$\mathrm{Gr}_a^{W(N)}\Psi_J IL \quad \simeq \quad \mathrm{Im}\, A_i \oplus \mathrm{Ker}\, p_i \tag{5.15}$$

is compatible with the primitive decomposition of $\mathrm{Gr}_a^{W(N)}\Psi_J IL$.

In particular, p_i induces an isomorphism of $\mathrm{Im}\, A_i$ in $\mathrm{Gr}_a^{W(N)}\Psi_J IL$ onto $\mathrm{Im}\,(p_i \circ A_i)$ in $\mathrm{Gr}_a^{W(N)}\Psi_{J\setminus i}IL$. Moreover, $\mathrm{Ker}\, p_i = 0$ if $i \notin M_Z$.

Proof. By (§4.2.1.3, Lemma 4.2): $\Psi_J(IL) \simeq \Psi_{J_Z}(N_{J'_Z}L)$. Set $\Psi_{J_Z}(N_{J'_Z}L) := \Psi_{J_Z,J'}$. First let $i \in J_Z$ and consider the sequence

$$\mathrm{Gr}_a^{W(N)}\Psi_{J_Z\setminus i,J'} \xrightarrow{A_i} \mathrm{Gr}_a^{W(N)}\Psi_{J_Z,J'}(-1) \xrightarrow{p_i} \mathrm{Gr}_a^{W(N)}\Psi_{J_Z\setminus i,J'}(-1)$$

where $A_i = m_i N - N_i$ and p_i is the natural projection.

By repeated application of the lemma (5.1) below, $N_{J'}L$ is a nilpotent orbit. Hence the above proposition applies to $N_{J'_Z}L$ and we have the decomposition:

$$\mathrm{Gr}_a^{W(N)}\Psi_{J_Z,J'} \quad \simeq \quad \mathrm{Im}\, A_i \oplus \mathrm{Ker}\, p_i \tag{5.16}$$

Second, for $i \in M'_Z$, hence $A_i = -N_i$, we check that the following sequence

$$\mathrm{Gr}_a^{W(N)}\Psi_{J_Z,J'\setminus i}L(1) \xrightarrow{N_i} \mathrm{Gr}_a^{W(N)}\Psi_{J_Z,J'}L \xrightarrow{p_i} \mathrm{Gr}_a^{W(N)}\Psi_{J_Z,J'\setminus i}L(-1)$$

satisfies the decomposition property

$$\mathrm{Gr}_a^{W(N)}\Psi_{J_Z,J'}L \quad \simeq \quad \mathrm{Im}\, N_i \oplus (\mathrm{Ker}\, p_i = 0). \tag{5.17}$$

which is clear via the isomorphism (§4.2.1.3, Lemma 4.2):

$$(\Psi_{J_Z,J'}L, W, F) \simeq (\oplus_{j\le l_z-1} N_{J'}L \otimes N^j, W, F)$$

compatible with the filtrations. The surjectivity of N_i (resp. the injectivity of p_i) follows from the surjection $N_{J'\setminus i}L \xrightarrow{N_i} N_{J'}L$ (resp. the injection $N_{J'}L \hookrightarrow N_{J'\setminus i}L$). The proof is a consequence of:

Lemma 5.1 (Vanishing cycle). *([KaK 87] Theorem. 2.1.5, [CaKSc 87] Descent Lemma 1.16). Let $(L, F, P, N_1, ..., N_n)$ be a nilpotent orbit of weight w' polarized by P. Set $F^{k+1}(N_1L) := N_1F^{k+1}L = N_1L \cap F^kL$, then*

$$(N_1L, N_1F, P_1, (N_i)_{|N_1L}), \text{ where } P_1(N_1a, N_1b) = P(a, N_1b), \forall a, b \in L$$

is a polarized nilpotent orbit of weight $w' + 1$.

i) Set $N_0 := \sum_{i\in[1,n]} N_i$. The monodromy filtrations centered at 0 satisfy:

$$W_k(N_0|_{N_1L}) = N_1W_{k+1}(N_0) = N_1L \cap W_{k-1}(N_0)$$

and $W_{k+w'+1}N_1L = W(N_0|_{N_1L})_k = N_1W_{k+w'+1}L = N_1W_{k+1}(N_0)$.

ii) The morphism N_1 is written as the composition of morphisms of MHS:

$$N_1 : (L, W, F) \xrightarrow{can} (N_1L, F, W) \overset{N'_1}{\hookrightarrow} (L, W, F)(-1)$$

where can is surjective and called canonical such that $F^{k+1}(N_1L) := N_1F^{k+1}L = N_1L \cap F^kL$ and $W_{k+1}(N_1L) := N_1W_{k+1}L = N_1L \cap W_{k-1}L$.

The difficulty in the lemma is to prove the structure of nilpotent orbit on N_1L as in the following diagram. Let $\widetilde{N}_0 := (N_0)|_{N_1L}$ the restriction of N_0, then for all $k \ge 0$, the induced morphism by N_0^k on the first line, is an isomorphism:

$$\begin{array}{ccc} N_1\mathrm{Gr}_{k+1}^{W(N_0)}L & \xrightarrow{N_0^k\sim} & N_1\mathrm{Gr}_{-k+1}^{W(N_0)}L \\ \downarrow = & & \downarrow = \\ \mathrm{Gr}_k^{W(\widetilde{N}_0)}N_1L & \xrightarrow{\widetilde{N}_0^k\sim} & \mathrm{Gr}_{-k}^{W(\widetilde{N}_0)}N_1L \end{array}$$

Remark 5.2. i) Later, the lemma is used to deduce the weight filtration on the perverse vanishing cycle sheaf ${}^p\varphi_f^u j_{!*}{}^p\mathbf{L}$ from the weight on the perverse nearby cycle ψ_f^u which explains the name of the result.

ii) In [CaKSc 87], the convention in the Descent lemma is: $F^kN_1L = N_1F^{k+1}L$, which explains the weight $w'-1$ there, hence we have the factorization: $N_1 : (L,W,F) \to (N_1L,F,W)(-1) \hookrightarrow (L,W,F)(-1)$.

Example 5.2. We illustrate the vanishing cycle lemma as follows. Given (L,N), let $L' := NL$, denote by $N' : L' \to L'$ the restriction of N and by $\widetilde{N} : L \to L'$ the morphism induced by N. We have:

$$\begin{array}{ccccc} & & P_2 = \mathrm{Ker}\,N^3 \subset \mathrm{Gr}_2^W L & \xrightarrow{\widetilde{N}\sim} & \widetilde{N}P_2 = \mathrm{Ker}\,N'^2 \subset \mathrm{Gr}_1^{W'}L' \\ & & \downarrow N & & \downarrow N'\sim \\ P_0 = \mathrm{Ker}\,\widetilde{N} & \longrightarrow & P_0 \oplus N\mathrm{Gr}_2^W L \subset \mathrm{Gr}_0^W L & \xrightarrow{\widetilde{N}} & \mathrm{Gr}_{-1}^{W'}L' \simeq N^2\mathrm{Gr}_2^W L \\ & & \downarrow N & & \downarrow N' \\ N^2P_2 = \mathrm{Ker}\,\widetilde{N} & \longrightarrow & N^2\mathrm{Gr}_2^W L \simeq \mathrm{Gr}_{-2}^W L & \xrightarrow{\widetilde{N}} & \mathrm{Gr}_{-3}^{W'}L' \simeq N\mathrm{Gr}_{-2}^W L \end{array}$$

$$\begin{array}{ccc} P_1 \subset \mathrm{Gr}_1^W L & \xrightarrow{\widetilde{N}\sim} & \widetilde{N}P_1 = \mathrm{Ker}\,N' \subset \mathrm{Gr}_0^{W'}L' \\ \downarrow N & & \downarrow N' \\ NP_1 = \mathrm{Ker}\,\widetilde{N} \subset \mathrm{Gr}_{-1}^W L & \xrightarrow{\widetilde{N}} & \mathrm{Gr}_{-2}^{W'}L' \end{array}$$

5.1.2 Local decomposition of $\mathrm{Gr}_*^{W(N)}\Psi^*IL$

The proof of Theorem 5.1 relies on the local description of perverse sheaves by de Rham families and a criterion by Kashiwara on the decomposition into a direct sum of intermediate extensions, based on the property of distinguished pairs (§3.2.6.2).

The restriction of the shifted perverse sheaf $\Psi^*I\mathcal{L}$ to a small polydisc with center $x \in Y_M^*$ isomorphic to D^{m+l} in the product $Y_M \times N_{Y_M}$ of Y_M with the normal section to Y_M at x, is described by a de Rham family attached to $L := \mathcal{L}(x)$ ([Ka 86], §2):

$$DR(\Psi IL) := \{\Psi_J IL, p_J^K : \Psi_J IL \to \Psi_K IL, A_{J\setminus K} : \Psi_K IL \to \Psi_J IL\}_{K\subset J\subset M}$$
$$p_J^K := \prod_{i\in J\setminus K} p_i : \Psi_J IL \to \Psi_K IL,\ A_{J\setminus K} := \prod_{i\in J\setminus K} A_i : \Psi_K IL \to \Psi_J IL \tag{5.18}$$

where the morphisms p_J^K and $A_{J\setminus K}$ satisfy compatibility relations ([Ka 86], (2.1.1) to (2.1.4)). The morphism N acts on the de Rham family $DR(\Psi IL)$ and we consider the graded de Rham family

$$\mathrm{Gr}_a^{W(N)} DR(\Psi IL) := \{\mathrm{Gr}_a^{W(N)}\Psi_J IL, p_J^K, N_{J\setminus K}\}_{K\subset J\subset M} \tag{5.19}$$

where we write p_J^K for $\mathrm{Gr}_a^{W(N)} p_J^K$. Moreover, $\mathrm{Gr}_a^{W(N)} A_{J\setminus K}$ coincides with $N_{J\setminus K}$ up to a sign, since $N : \mathrm{Gr}_a^{W(N)}\Psi_{J\setminus i} IL \to \mathrm{Gr}_a^{W(N)}\Psi_J IL$ vanishes for all $i \in J$. So the distinguished pairs property is satisfied by Corollary 5.1 for all $K \subset J \subset M$:

$$\mathrm{Gr}_a^{W(N)}\Psi_J IL \simeq \mathrm{Im}\, N_{J\setminus K} \oplus \mathrm{Ker}\, p_J^K. \tag{5.20}$$

Proposition 5.2. *Let* $\mathbf{L}$ *be defined on* $X \setminus Y$*. For each* $J \subset M$*,* $x \in Z_J$*, let* $L := \mathcal{L}_X(x)$*. We define* $P_a^J(IL) \subset \mathrm{Gr}_a^{W(N)}\Psi_J IL$ *by*

$$P_a^J(IL) := \cap_{K\subset J, K\neq J}\mathrm{Ker}\,(p_J^K : \mathrm{Gr}_a^{W(N)}\Psi_J IL \to \mathrm{Gr}_a^{W(N)}\Psi_K IL) \tag{5.21}$$

then $P_a^J(IL) \neq 0$ *if and only* $J \subset J_Z$*.*

i) If $x \in Y_M^*$*, then* $x \in Y_J$ *for all* $J \subset M, J \neq M$ *and we have the decomposition as a direct sum of MHS (hence including the decomposition of the Hodge filtration* F*)*

$$\mathrm{Gr}_a^{W(N)}\Psi_M IL \xrightarrow{\sim} \oplus_{J\subset M_Z} N_{M\setminus J} P_a^J(L) \tag{5.22}$$

Let $\mathrm{Im}\, N_{M\setminus J} : P_a^J(L) \to P_a^J(L)$ *denote the image of* $N_{M\setminus J}$ *acting on* $P_a^J(L)$*. Then* p_M^J *induces an isomorphism:*

$$\mathrm{Gr}_a^{W^M}\Psi_M IL \supset N_{M\setminus J} P_a^J(L) \overset{p_M^J}{\simeq} \mathrm{Im}\, N_{M\setminus J} \subset P_a^J(L).$$

ii) Let $J = M_Z$ *and* $x \in Y_J^*$*, then* $P_a^J(L)$ *is a HS of weight* $a + w' + 1 - |J|$*. For varying* $x \in Y_J^*$*,* $P_J\mathbf{L}$ *defines a polarize VHS on* Y_J^**. iii) Let* $x \in Y_J \setminus Y_J^*$ *and* $x \in Y_M^*$*, then* $(P_a^J(L), F, N_i, i \in M \setminus J)$ *with induced filtration* F *and nilpotent endomorphisms* N_i*, is a nilpotent orbit of weight* $a + w' + 1 - |J|$*.*

Corollary 5.2. *When* $x \in Y_M^*$ *the decomposition in Equation 5.22 is interpreted as the following decomposition*

$$(\mathrm{Gr}_a^{W(N)}\Psi^* IL, F) \xrightarrow{\sim} \oplus_{K\subset M_Z}(IC^* P_a^K L, F)[-|K|]. \tag{5.23}$$

The corollary follows by (Equation 5.22) since the filtered morphism

$$\oplus_{J\subset M_Z}(N_{M\setminus J} P_a^J L, W, F) \xrightarrow{\sim} (\mathrm{Gr}_a^{W(N)}\Psi_M IL, W, F) \tag{5.24}$$

is necessarily an isomorphism in the category of MHS by (Remark 3.6vii) .

The local decomposition by (Equation 5.23) leads later to the global decomposition (Equation 5.31).

Example 5.3. The decomposition of Diagram 4.17 in chapter 4, where $Z = Y$, is as follows

$$\begin{array}{lllll}
\mathrm{Gr}_a^{W(N)}\Psi^1 L := & \mathrm{Gr}_a^{W(N)}\Psi_1 \oplus & \mathrm{Gr}_a^{W(N)}\Psi_2 = & \mathrm{Gr}_a^{W(N_1)}L \oplus & \mathrm{Gr}_a^{W(N_2)}L \\
\downarrow & \downarrow & & \downarrow N_2 & \downarrow N_1 \\
\mathrm{Gr}_a^{W(N)}\Psi^2 L := & \mathrm{Gr}_a^{W(N)}\Psi_{12} & = & N_2 P_a^1(L) \oplus & N_1 P_a^2(L) \ \oplus P_a^{12}(L)
\end{array}$$

where the term $P_a^1(L) = \mathrm{Gr}_a^{W(N_1)}\Psi^1 L$, (resp. $P_a^2 L = \mathrm{Gr}_a^{W(N_2)}\Psi^2 L$) is a direct sum of polarized HS for $L := \mathcal{L}_X(x)$, $x \in Y_1^*$ (resp. $x \in Y_2^*$). For $x \in Y_{12}^*$ the space $N_2 P_a^1 L \oplus N_1 P_a^2 L$ is endowed with the induced MHS $W(N_1 + N_2)$ on $\mathrm{Gr}_a^{W(N)}\Psi^2 L$. Hence $W(N_2)$ on $N_2 P_a^1 L$ is induced by the nilpotent orbit $(P_a^1 L, N_2)$, resp. $W(N_1)$ on $N_1 P_a^2 L$.
Thus, the two columns are fibres of intersection complexes on Y_i.
The term $P_a^{12} L$ is endowed with a sub-polarized HS defined by $W(N_1 + N_2)$.
We have a decomposition as a direct sum of fibres of complexes at 0:

$$IC^* P_a^1 L[-1] \oplus IC^* P_a^2 L[-1] \oplus P_a^{12} L[-2]$$

including the decomposition of Hodge filtrations. In addition there exists natural morphisms: $p_{12}^1 : \mathrm{Gr}_a^{W(N)}\Psi_{12} \to \mathrm{Gr}_a^{W(N)}\Psi_1$ vanishing on $N_1 P_a^1 L \oplus P_a^{12} L$ and injective on $N_2 P_a^2 L$ (resp. $p_{12}^2 : \mathrm{Gr}_a^{W(N)}\Psi_{12} \to \mathrm{Gr}_a^{W(N)}\Psi_2$ vanishing on $N_2 P_a^2 L \oplus P_a^{12} L$ and injective on $N_1 P_a^2 L$).

Proof of Proposition 5.2. By (Equation 5.17), $\ker p_i = 0$ if $i \notin M_Z$.

i) The decomposition follows functorially from the property of distinguished pairs in the graded family 5.19 and properties of the morphisms p_J^K and $N_{J\setminus K}$ ([Ka 86], Proposition 2.3.1, Equations (2.1.1) to 2.1.4)).

We suggest a direct proof by induction on the length of $M : n = |M|$ of the general formula

$$\mathrm{Gr}_a^{W(N)}\Psi_M IL \xrightarrow{\sim} \oplus_{J\subset M} N_{M\setminus J} P_a^J(IL).$$

For $n = 1$ the decomposition is reduced to the property of graded distinguished pair. Let $M' := \{1, \cdots, n-1\}$. We consider two sub-families:

$$(\mathrm{Gr}_a^{W(N)}\Psi_J IL, p_J^K, N_{J\setminus K})_{K\subset J\subset M'};$$

$$(\mathrm{Gr}_a^{W(N)}\Psi_{J\cup n} IL, p_{J\cup n}^{K\cup n}, N_{J\setminus K})_{K\subset J\subset M'}.$$

By induction, we suppose the decomposition for the two families with indices $J \subset M'$, valid on the two rows of the following diagram

$$\begin{array}{ccc}
\mathrm{Gr}_a^{W(N)}\Psi_{M'} IL & \xrightarrow{\sim} & \oplus_{J\subset M'} N_{M'\setminus J} P_a^J(IL) \\
p_M^{M'} \uparrow\downarrow N_n & & p_{J\cup n}^J \uparrow\downarrow N_n \\
\mathrm{Gr}_a^{W(N)}\Psi_M IL & \xrightarrow{\sim} & \oplus_{J\subset M'} N_{M'\setminus J} P_a^{J\cup n, M'}(IL)
\end{array} \tag{5.25}$$

where $P_a^{J\cup n,M'}(IL)$ is defined with respect to the family $K\cup n$ for $K\subset M'$, while $P_a^{J\cup n}(IL)$ (for the family with index in M) is defined with respect to the extended family with index $K\subset J\cup n$, i.e. $K\subset J\subset M'$ and $K\cup n\subset J\cup n$. The columns are written in terms of the property of distinguished pairs, for each $J\subset M'$:

$$\begin{aligned} &P_a^{J\cup n,M'}(IL)\simeq\ N_nP_a^J(IL)\oplus \operatorname{Ker} p^J_{J\cup n}\\ &P_a^{J\cup n}(IL)=\operatorname{Ker}\left(p^J_{J\cup n}:P_a^{J\cup n,M'}(IL)\to P_a^J(IL)\right)\end{aligned}\tag{5.26}$$

We deduce from the second line of the diagram the sum over all subsets of M, that is $J\subset M'$ and $J\cup n, n\notin J$:

$$\begin{aligned} &\operatorname{Gr}_a^{W(N)}\Psi_J(IL)\simeq \oplus_{J\subset M'}(N_nN_{M'\setminus J}P_a^J(IL)\oplus N_{M'\setminus J}P_a^{J\cup n}(IL))\\ &=\oplus_{J\subset M}N_{M\setminus J}P_a^J(IL)\end{aligned}\tag{5.27}$$

ii) For $i\in J\subset M_Z$ and $K=J\setminus i$, $N_i:\operatorname{Gr}_a^{W(N)}\Psi_JL\to\operatorname{Gr}_a^{W(N)}\Psi_JL$ is equal to the composition morphism $\operatorname{Gr}_a^{W(N)}\Psi_JL\xrightarrow{p_J^K}\operatorname{Gr}_a^{W(N)}\Psi_KL\xrightarrow{N_i}\operatorname{Gr}_a^{W(N)}\Psi_JL$. Hence for all $i\in J$, N_i vanishes on $P_a^JL\subset\operatorname{Ker}p_J^K$.

iii) For $x\in Y_J^*$, since $N_0:=\sum_{i\in J}N_i$ vanishes on P_a^JL and P_a^JL is a direct summand of $\operatorname{Gr}_a^{W(N)}\Psi_JL$, we deduce that $P_a^JL=\operatorname{Gr}_0^{W(N_0)}P_a^JL$ is a pure direct summand of weight $a+w'+1-|J|$ of $\operatorname{Gr}_0^{W(N_0)}\operatorname{Gr}_a^{W(N)}\Psi_JL$ with respect to the MHS induced on $\operatorname{Gr}_a^{W(N)}\Psi_JL$ by W^J.

iii) When x varies on Z_J^*, $P_a^J(L)$ defines a global polarized VHS, hence for $x\in Z_J$, $P_a^J(L)$ is a nilpotent orbit of weight $a+w'+1-|J|$.

Remark 5.3. Let $\mathbf{L}$ be a shifted local system on the open subset U union of the big strata of a stratification $\mathcal{S}$ of X, $j:U\to X$ and $f:X\to D$ a morphism to the disc $D\in\mathbb{C}$ with central fiber Z a NCD in X intersecting transversally the strata of $\mathcal{S}$ and let $j':Z\cap U\to Z$, then $\operatorname{Gr}_*^{W(N}\psi_f^uj_{!*}\mathbf{L}\simeq j'_{!*}\operatorname{Gr}_*^{W(N}\psi^u_{f_{|U}}\mathbf{L}$.

Remark 5.4. [Relation with the open embedding] Let $Z\subset X$ be the NCD defined by f and $j:X\setminus Z\to X$. The filtration $W(N)$ on ψ_f^u defines the weight filtration on j_* in the general case of $VMHS$ (see §9.7.1.1).

5.1.3 Perverse VMHS structure on $\psi_f^uj_{!*}{}^p\mathbf{L}$ and $\varphi_f^uj_{!*}{}^p\mathbf{L}$

With the notations of §4.2.1.4 in chapter 4, we give below the global definition of the weight and Hodge filtrations on $\Psi^*I\mathcal{L}$ when $\mathbf{L}$ is a polarized VHS on the complement of Y. The action of the logarithm of the monodromy is defined by ν on $\Psi^*I\mathcal{L}$ (Equation 4.21); the weight filtration is defined by $W(\nu)$.

By the preceding local results, the graded weight complex decomposes into a direct sum of intermediate extensions. In particular, when the NCD Z

is projective, the derived global sections functor is a mixed Hodge complex (MHC) inducing MHS on its cohomology. The same results apply to the vanishing cycle complex $\Phi^* I\mathcal{L}$ (see §5.1.3.2 below).

5.1.3.1 Weight and Hodge filtrations on $\Psi^* I\mathcal{L}$

The *Hodge filtration F on the quotient complex* $\Psi^* I\mathcal{L}$ is deduced from the Hodge filtration F on $\tilde{\Psi}^* I\mathcal{L}$:

$$F^p\tilde{\Psi}^* I\mathcal{L} := \sum_{q\geq 0} \mathbb{C}u^{-(q+1)} \otimes_{\mathbb{C}} (i_Z^* IC^*\mathcal{L}(\log Z)[q+1], F^{q+p+1}) \tag{5.28}$$

The weight and the monodromy filtration $W(\nu)$ on $\Psi^* I\mathcal{L}$.
The action of the nilpotent endomorphism ν defines a unique monodromy filtration $W(\nu)$ on $\Psi^* I\mathcal{L}$. This filtration $W(\nu)$ on $\Psi^* I\mathcal{L}$ corresponds locally at a point $x \in Y_M^* \subset Z$, to the filtration $W(N)$ on $\Psi^* IL[1] \simeq (\Psi^* I\mathcal{L})_x$ (Equation 4.20).

In terms of a set of coordinate equations y_i for $i \in M$ at x of Y_M, the subspace $W(\nu)_k$ is generated as an $\Omega^*_{X,x}$ sub-module by the sections $\tilde{v} \wedge_{j\in J} \frac{dy_j}{y_j} \in \Psi^* I\mathcal{L}_x$ for all $v \in W(N)_k \Psi_J IL[1]$ and $J \subset M$ such that the quasi-isomorphism

$$(\Psi^* I\mathcal{L}_x, W(\nu), F) \simeq (\Psi^* IL[1], W(N), F) \tag{5.29}$$

(see Equation 4.20) is compatible with both filtrations. The theorem (5.1) follows in the unipotent case from the following remark that the components $P_a^K L$ in Equations 5.22 and 5.23 are local version of a global VHS: .

Definition 5.2 ($IC^*\mathcal{P}_a^K\mathcal{L} \subset \mathrm{Gr}_a^{W(\nu)}\Psi^* I\mathcal{L}$). There exists a unique complex $IC^*\mathcal{P}_a^K\mathcal{L}$, defined as a sub-complex of $\mathrm{Gr}_a^{W(\nu)}\Psi^* I\mathcal{L}[|K|-1])$ such that at a point $x \in Y_M^*$, the tilde embedding by (Equation 4.20) induces a quasi-isomorphism:

$$IC^*(P_a^K L) \xrightarrow{\sim} (IC^*\mathcal{P}_a^K\mathcal{L})_x \subset (\mathrm{Gr}_a^{W(\nu)}\Psi^* I\mathcal{L}[|K|-1])_x,\ L := \mathcal{L}(x) \tag{5.30}$$

The corollary below follows from the local decomposition in (Equation 5.23).

Corollary 5.3. *There exists a polarized VHS: $P_a^J\mathbf{L}$ of weight $a+w'+1-|J|$ on Z_J^*, such that $(IC^*\mathcal{P}_a^J\mathcal{L}, F)$ is a resolution of the intermediate extension of $P_a^J\mathbf{L}$ by $j_Z : Z_J^* \to X$:$(IC^*\mathcal{P}_a^J\mathcal{L}, F) \simeq (j_{Z!*}P_a^J\mathbf{L}, F)$.*

Proposition 5.3 (Decomposition of $\mathrm{Gr}_*^{W(N)}\psi_f^u j_{!*}\mathbf{L}$). *Let $\mathbf{L}$ be of weight w' on $X\setminus Y$ locally unipotent along Y and let $W(\nu)$ be the monodromy filtration on $\Psi^* I\mathcal{L}$. For each $a \in \mathbb{Z}$, there exists a decomposition into a direct sum over all $J \subset I_Z$ of intermediate extensions by $j_Z : Z_J^* \to X$ of weight $a+w'$:*

$$(\mathrm{Gr}_a^{W(\nu)}\Psi^* I\mathcal{L}, F) \simeq \oplus_{J\subset I_Z, J\neq\emptyset}(IC^*\mathcal{P}_a^J\mathcal{L}, F)[1-|J|](1-|J|) \tag{5.31}$$

Proof. By definition of $IC^*\mathcal{P}_a^J\mathcal{L}$, the decomposition in (Equation 5.31) corresponds to the local version (Equation 5.23) and (Equation 4.20):

$$(\mathrm{Gr}_a^{W(\nu)}\Psi^* I\mathcal{L}_x, F) \xrightarrow{\sim} \oplus_{K\subset M_Z}(IC^* P_a^K L, F)[1-|K|](1-|K|) \qquad (5.32)$$

Remark the twist of the weight by $-2(1-|J|)$ and the shift by $[1-|J|]$, since $P_a^J L$ is a nilpotent orbit of weight $a+w'+1-|J|$, hence:
$IC^*(P_a^J L)[1-|J|](1-|J|)$ is of weight $a+w'+2(1-|J|)-2(1-|J|) = a+w'$.

Remark 5.5. i) The decomposition is stated in the category of filtered derived complex and follows from the local quasi-isomorphism with the mixed Hodge complex (Equation 5.23) and the isomorphism of (Equation 5.24).

ii) The notations are compatible with [St 76] in the case of a constant system $\mathbf{L}$ of weight w' on the complement of a NCD as central fiber, where the hypercohomology $\mathbb{H}^i(Z, \psi_f\mathbf{L})$ is of weight $w'+i$.

5.1.3.2 Unipotent vanishing cycles: $\varphi_f^u(j_{!*}\mathbf{L})$

The complex of sheaves of vanishing cycles $\varphi_f(j_{!*}\mathbf{L})$ (resp. $\varphi_f^u(j_{!*}\mathbf{L})$) is defined as the cone over the specialization morphism $\mathrm{sp} : i_Z^* j_{!*}\mathbf{L} \to \psi_f(j_{!*}\mathbf{L})$ (resp. $\mathrm{sp} : i_Z^* j_{!*}\mathbf{L} \to \psi_f^u(j_{!*}\mathbf{L})$) such that we have a triangle:

$$\begin{aligned} i_Z^* j_{!*}\mathbf{L} &\xrightarrow{\mathrm{sp}} \psi_f(j_{!*}\mathbf{L}) \xrightarrow{\mathrm{can}} \varphi_f(j_{!*}\mathbf{L}) \xrightarrow{+1} \\ i_Z^* j_{!*}\mathbf{L} &\xrightarrow{\mathrm{sp}} \psi_f^u(j_{!*}\mathbf{L}) \xrightarrow{\mathrm{can}} \varphi_f^u(j_{!*}\mathbf{L}) \xrightarrow{+1} \end{aligned} \qquad (5.33)$$

The monodromy T extends to $\varphi_f(j_{!*}\mathbf{L})$ since T on $\psi_f(j_{!*}\mathbf{L})$ satisfy $T\circ\mathrm{sp} = \mathrm{sp}$. There exists a variation morphism $var : \varphi_f^u j_{!*}\mathbf{L} \to \psi_f^u j_{!*}\mathbf{L}(-1)$ such that $var \circ can = N : \psi_f^u j_{!*}\mathbf{L} \to \psi_f^u j_{!*}\mathbf{L}(-1)$ (§2.2.3, Remark 2.7).

5.1.3.3 Weight and Hodge filtrations on ${}^p\psi_f^u j_{!*}{}^p\mathbf{L}$ and ${}^p\varphi_f^u j_{!*}{}^p\mathbf{L}$

The following relations in terms of Ker and Im are well defined in the abelian category of perverse sheaves (see Equation 5.49 and 5.50):

$$i_Z^* j_{!*}{}^p\mathbf{L}[-1] \simeq \mathrm{Ker}(N : {}^p\psi_f^u j_{!*}{}^p\mathbf{L} \to {}^p\psi_f^u j_{!*}{}^p\mathbf{L}(-1)),\ {}^p\varphi_f^u j_{!*}{}^p\mathbf{L} \simeq \mathrm{Im}\, N \quad (5.34)$$

We also write by abuse of notations: $i_Z^* j_{!*}\mathbf{L} \simeq \mathrm{Ker}(N : \psi_f^u j_{!*}\mathbf{L} \to \psi_f^u j_{!*}\mathbf{L})$ and $\varphi_f^u j_{!*}\mathbf{L} \simeq \mathrm{Im}\, N$.

Definitions of F and W. The filtration W_k on $i_Z^* j_{!*}{}^p\mathbf{L}[-1]$ (resp. ${}^p\varphi_f^u j_{!*}{}^p\mathbf{L}$) is defined as $\ker N|_{W_k\, {}^p\psi_f^u j_{!*}{}^p\mathbf{L}}$ (resp. $\mathrm{Im}\, N$).

We may suppose $\ker N \xrightarrow{\sigma} {}^p\psi_f^u j_{!*}{}^p\mathbf{L}$ embedded as a sub complex of ${}^p\psi_f^u j_{!*}{}^p\mathbf{L}$, then the filtration F is induced on $\ker N$ (resp. $\mathrm{Im}\, N$ as quotient

complex). The filtrations W and F on ${}^p\varphi^u_f j_{!*}{}^p\mathbf{L}$ may be defined also on the mixed cone $C_M(\sigma)$.

This is easily checked on the de Rham family associated to $\psi^u_f j_{!*}\mathbf{L}$. We suppose $\mathbf{L}$ locally unipotent. At each point $x \in Z$, set $L := \mathcal{L}(x)$ and with the notations of the de Rham family of nilpotent orbits:

$$K_J IL := \ker(N : \Psi_J IL \to \Psi_J IL(-1), \quad \phi_J IL := \mathrm{Im}(N : \Psi_J IL \to \Psi_J IL(-1)$$

The morphisms of MHS A_i on $\Psi_{J-i}IL \to \Psi_J IL$ for each $i \in J$, induce corresponding morphisms on $K_{J-i}IL$ (resp. $\phi_{J-i}IL$) The total complex:

$$\boldsymbol{s}(K_J IL, A_i)_{J\subset M_Z, i\in M_Z}, \text{ (resp. } s(\phi_J IL, A_i))$$

is isomorphic to the complex $K^*IL := \ker(N : \Psi^*IL \to \Psi^*IL) \simeq \ker N$ (resp. $\Phi^*IL := \mathrm{Im}\,(N : \Psi^*IL \to \Psi^*IL) \simeq \mathrm{Im}\, N$ in the category of perverse sheaves. Moreover the induced filtrations define a structure of MHC (if Z is projective) and more precisely a structure of perverse $VMHS$ to be made precise later (§, 9.7.1.3).

Remark 5.6. We write the endomorphism ν as the composition of a canonical morphism *can* and a variation morphism *var*:

$$\nu : \Psi^* I^p\mathcal{L}[-1] \xrightarrow{can} \Phi^* I^p\mathcal{L}[-1] \xrightarrow{var} \Psi^* I^p\mathcal{L}[-1](-1).$$

with short exact sequences of perverse sheaves of MHC

$$0 \to i_Z^* j_{!*}{}^p\mathbf{L}[-1] \xrightarrow{sp} \Psi^* I^p\mathcal{L}[-1] \xrightarrow{can} \Phi^* I^p\mathcal{L}[-1] \to 0$$

and we have an isomorphism with the mixed cone: $C_M(sp) \simeq (\Phi^* I^p\mathcal{L}[-1], W, F))$.

Let N' be the nilpotent endomorphism on Φ^*IL induced by N. The vanishing cycle lemma 5.1 relate the weight filtration on ${}^p\varphi^u_f j_{!*}{}^p\mathbf{L}$ to the monodromy filtration defined by the nilpotent endomorphism N'.

By construction, we have morphisms

$$({}^p\psi^u_f j_{!*}{}^p\mathbf{L}, W, F) \xrightarrow{can} ({}^p\varphi^u_f j_{!*}{}^p\mathbf{L}, W, F) \xrightarrow{var} ({}^p\psi^u_f j_{!*}{}^p\mathbf{L}, W, F)(-1).$$

where in this case *can* is an epimorphism and *var* is a monomorphism.

The graded complex $\mathrm{Gr}_*^{W(N')}\Phi^*IL$. As in the case Ψ^*IL (5.1.2), the graded complex $\mathrm{Gr}_*^{W(N')}\Phi^*IL$ decomposes. We deduce a decomposition similar to (Equation 5.22)

$$\mathrm{Gr}_a^{W(N)}\Phi_M IL \xrightarrow{\sim} \oplus_{J\subset M} N_{M\setminus J} Q_a^J(IL) \tag{5.35}$$

Definition 5.3. The sub-complex $\Phi^*I\mathcal{L}$ of $\Psi^*I\mathcal{L}$ is generated locally at $x \in Z$ by the image of the tilde embedding of the complex Φ^*IL for $L := \mathcal{L}_X(x)$.

We denote by ν' the nilpotent endomorphism on $\Phi^*I\mathcal{L}$ induced by ν.

The endomorphism ν' corresponds locally to the nilpotent endomorphism N' (up to a constant) on Φ^*IL induced by N.

The shifted perverse filtration $W(\nu')$ is defined locally by $\mathrm{Gr}_i^{W(N')}\Phi^*IL$.

Proposition 5.4. *i) The complex $\Phi^* I\mathcal{L}$ coincides with the image of ν:*

$$\Phi^* I\mathcal{L} \xrightarrow{\sim} \mathrm{Im}\,(\nu : \Psi^* I\mathcal{L} \to \Psi^* I\mathcal{L}).$$

It is endowed with filtrations W and F.

ii) Let $W(\nu')$ denote the monodromy weight filtration on $\Phi^* I\mathcal{L}$ and $W := W(\nu')[w'+1]$, then $(\mathrm{Gr}^W_{a+w'+1}\Phi^* I\mathcal{L}, F) := (Gr_a^{W(\nu')}\Phi^* I\mathcal{L}, F)$ decomposes into the direct sum of shifted and twisted intermediate extensions of polarized VHS of weight $a+w'+1$ on the various components of the NCD.

The shift of $w'+1$ on $W(\nu')$ is compatible with the vanishing cycle Lemma. The proof is similar to (Proposition 5.3) or may be deduced by the same lemma.

Corollary 5.4. *There exists a weight filtration W and a Hodge filtration F on ${}^p\psi^u_f j_{!*}{}^p\mathbf{L}$ (resp. ${}^p\varphi^u_f j_{!*}{}^p\mathbf{L}$) such that:*

$$\begin{aligned} Gr_a^W\, {}^p\psi^u_f j_{!*}{}^p\mathbf{L} &\simeq \oplus_{Z_J\subset Z}({}^p\mathcal{P}^J_a, F) \\ Gr_a^W\, {}^p\varphi^u_f j_{!*}{}^p\mathbf{L} &\simeq \oplus_{Z_J\subset Z}({}^p\mathcal{Q}^J_a, F) \end{aligned} \tag{5.36}$$

where $({}^p\mathcal{P}^J_a, F)$ (resp. $({}^p\mathcal{Q}^J_a, F)$) is an intermediate extension of weight a of a shifted polarized VHS on Y^*_Z (see Equation 5.31) such that:
$Gr^W_{a+{}^pw-1}\, {}^p\psi^u_f j_{!*}{}^p\mathbf{L} \simeq Gr_a^{W(N)}\, {}^p\psi^u_f j_{!*}{}^p\mathbf{L}$, $Gr^W_{a+{}^pw}\, {}^p\varphi^u_f j_{!*}{}^p\mathbf{L} \simeq Gr_a^{W(N)}\, {}^p\varphi^u_f j_{!*}{}^p\mathbf{L}$).

This result is used later to put a structure of mixed Hodge complex on $\psi^u_f(j_{!*}\mathbf{L})$ and $\varphi^u_f(j_{!*}\mathbf{L})$ by induction on dim X as long as the purity of the hypercohomolgy of intermediate extension of VHS is proved.

5.2 Perverse VMHS on ${}^p\psi_f j_{!*}{}^p\mathbf{L}$ and ${}^p\varphi_f j_{!*}{}^p\mathbf{L}$

We extend Hodge theory to the perverse sheaves $\psi_f(j_{!*}{}^p\mathbf{L})$ and $\varphi_f(j_{!*}{}^p\mathbf{L})$ by reduction to the case where $\mathbf{L}$ is not necessarily locally unipotent along Y.

5.2.0.1 Reduction to VHS unipotent along Y

In the case of a VHS ${}^p\mathbf{L}$ quasi-unipotent along Y, we introduce a ramified covering locally at a point $x \in Y$, to reduce the proof to the locally-unipotent case. Let $U \simeq D^n$ be a neighborhood of x such that Y is defined by a product of $q \le n$ local coordinates $t_1 \cdots t_q$. We can suppose $q = n$ such that $U \setminus Y \cap U \simeq (D^*)^n$.

Since the local monodromy T_i on the limit vector space $L := \mathcal{L}(x)$ is quasi-unipotent, there exists ν_i such that $T_i^{\nu_i} - Id$ is nilpotent.

Let $Z \subset X$ be the central fiber of $f : X \to D$. We suppose $Z \subset Y$ be a sub-NCD with irreducible components indexed by $I_Z \subset I$. Let g denote the local ramified covering of a product of complex balls

$$(Z \subset Y \subset D^n, t_.) \xleftarrow[g]{} (Z' \subset Y' \subset D_1^n, u_.) \tag{5.37}$$

defined by $g : D_1^n \to D^n : (u_1, \ldots, u_n) \to (t_1 = u_1^{\nu_1}, \ldots, t_n = u_n^{\nu_n})$.

The inverse image ${}^p\mathbf{L}' := g^*({}^p\mathbf{L}_{|D^n})$ is a polarized VHS on $D_1^n \setminus Y'$ with locally unipotent monodromy $T_i' := T_i^{\nu_i}$ around the component $Y_i' \subset Y'$.

Let $j' : D_1^n \setminus Y' \to D_1^n$. The finite group of covering transformations G_i generated by $\xi_i := exp(2i\pi/\nu_i), i \in [1, n]$ acts on D_1^n and on ${}^p\mathbf{L}'$ by T_i' as well on its intermediate extension $j'_{!*}\,{}^p\mathbf{L}'$.

Lemma 5.2. *The perverse sheaf $j_*{}^p\mathbf{L}$ (resp. $j_{!*}{}^p\mathbf{L}$) is the invariant subspace of $g_*j'_*\,{}^p\mathbf{L}'$ (resp. $g_*j'_{!*}\,{}^p\mathbf{L}'$) by the action of the group $G = G_1 \times \cdots \times G_n$.*

Proof. Since the subgroups G_i act independently and commutatively, the proof may be reduced to the case $n = 1$.

The case $n = 1$. Let $\nu_i := \nu$ be fixed. Since the monodromy T is quasi-unipotent, let $L_b \subset L$ denote the subspace on which T^s acts with eigenvalue $a_b = exp(-2i\pi\alpha_b)$ where $\alpha_b = b/\nu$ for $b \in [0, \nu - 1[$ and let $N_b := Log\, T^u$ on L_b. Let $z = (1/2i\pi)log\, t$ denote the coordinate on $\widetilde{D}$.
Recall the holomorphic section on D

$$\tilde{v}(t) = \exp(log\, t(b/\nu - (1/2i\pi)N_b))v(z) = t^{b/\nu} \exp\left(-(1/2i\pi)(log\, t)N_b\right) v(z)$$

satisfying $\widetilde{v}(te^{2i\pi}) = \widetilde{v}(t)T^{-1}.Tv(z) = \widetilde{v}(t)$, where $v(z) \in L_b$ is a multiform section.

If $g : (D_1, u) \to (D, t)$ is of order ν, $T' = T^\nu$ acts as $exp(\nu N_b)$ on L_b. The action of the transformation group G over D, generated by $\xi = e^{2i\pi/\nu}$, extends to the logarithmic complex $\Omega^*\mathcal{L}'_{D_1}$ as follows.

Let $z = (1/2i\pi)log\, u$ and $v(z) \in L'_b$, then we have a section of ${}^p\mathcal{L}'_{D_1}$:

$$\widetilde{v}^1 := \exp\left(-(1/2i\pi)(log\, u)\nu N_b\right) v(z)$$

satisfying $\widetilde{v}^1(ue^{2i\pi}) = \widetilde{v}^1(u)(T')^{-1}T'v(z) = \widetilde{v}^1(z)$.

The generator $\xi = e^{2i\pi/\nu}$ acts on the inverse image g^*f of a holomorphic section $f(t) \in \Gamma(D, \mathcal{O}_D)$. We have: $g^*f(\xi u) = g^*f(u)$ since $f(\xi^\nu u^\nu) = f(u^\nu)$.

Reciprocally, let $h \in \Gamma(D_1, \mathcal{O}_{D_1})$ be a section on D_1. Set $(\xi.h)(u) := h(\xi u)$. If $\xi.h = h$, then $h(\xi u) = h(u)$ and h is the inverse image of a holomorphic section $f(t)$ on D. This action extends to sections of $g_*\,{}^p\mathcal{L}'_{D_1}$ and $g_*\Omega^*\,{}^p\mathcal{L}'_{D_1}$.

Sub-lemma. *Let G acts on the linear combination of sections: $h(u)\widetilde{v}^1(u)$ of Deligne's extension ${}^p\mathcal{L}'_{D_1}$ as follows:*

$$\xi.(h(u)\widetilde{v}^1(u)) := h(\xi u).\widetilde{v}^1(\xi u) \tag{5.38}$$

The invariant sections ω of $g_*\,{}^p\mathcal{L}'_{D_1}$ satisfying $\xi.\omega = \omega$ correspond to sections of ${}^p\mathcal{L}_D$ on D. This action extends to $g_*\Omega^*\,{}^p\mathcal{L}'_{D_1}$ since $g^*dt/t = \nu du/u$. We have:

$$\Omega^*\,{}^p\mathcal{L}_D \simeq (g_*\Omega^*\,{}^p\mathcal{L}'_{D_1})^{\xi}$$

The inverse image of a section $f.\tilde{v}$ of Deligne's extension ${}^p\mathcal{L}_D$ is a section of the extension ${}^p\mathcal{L}'_{D_1}$, since $\widetilde{v}(u^\nu) = u^b\widetilde{v}^1(u)$, hence:

$$g^*(f.\widetilde{v}(t)) = f(u^\nu)\widetilde{v}(u^\nu) = f(u^\nu)u^b\widetilde{v}^1(u)$$

Let $h(u)\widetilde{v}^1(u) = g^*(f.\widetilde{v}(t))$, we deduce:

$h(u) = f(u^\nu)u^b$ and $h(\xi u) = f(u^\nu)\xi^b u^b = h(u)\xi^b$.

If we develop $h(u) = \sum_i a_i u^i$: $h(\xi u) = \sum_n a_n u^n = \sum_n a_n u^n \xi^b \Longrightarrow a_n\xi^n = a_n\xi^b$; hence $a_n = 0$ or $\xi^{n-b} = 1, n-b = k\nu$ and finally $n = k\nu + b$.

Reciprocally, sections $h(u)\widetilde{v}^1(u)$ where $h(u) = \sum_k a_{k\nu+b}u^{k\nu+b}$ are written as inverse image of sections on D.

Corollary 5.5. *The filtration F on $(\Omega^*\,{}^p\mathcal{L})_D$ is defined by the invariant sections*

$$F^q(\Omega^*\,{}^p\mathcal{L})_D \simeq (F^q\Omega^*\,{}^p\mathcal{L}')^G_{D_1}$$

and extends to a global filtration F on $(\Omega^*\,{}^p\mathcal{L})_X$. Hence (Theorem 5.1 and Equation 5.36) apply to polarized VHS non necessarily unipotent along Y.

The case of $j_{!*}{}^p\mathbf{L}$. The restriction to D of the perverse direct image: $(j_*\,{}^p\mathbf{L})_{|D}$ is the invariant perverse subspace of $g_*((j'_*\,{}^p\mathbf{L}')_{|D_1})$. Hence, the cohomology $\mathbb{H}^*(D^*, {}^p\mathbf{L})$ is isomorphic to the invariant subspace of $\mathbb{H}^*(D_1^*, {}^p\mathbf{L}')$. Since $\mathbb{H}^*(D, j_{!*}{}^p\mathbf{L})$ is defined by truncation, we deduce $j_{!*}{}^p\mathbf{L}$ by the invariance construction as well.

Let $Z \subset Y$ be a sub-divisor (resp. $Z' \subset Y'$ above Z), $j_Z : D^n \setminus (Z \cap D^n) \to D^n$ and $j'_{Z'} : D_1^n \setminus (Z' \cap D_1^n) \to D_1^n$, then $\mathbb{H}^*(D^n, j_{Z*}j_Z^*j_{!*}\mathbf{L})$ is isomorphic to the invariant subspace of $\mathbb{H}^*(D_1^n, j'_{Z'_*}j'_{Z'}{}^*j'_{!*}\,{}^p\mathbf{L}')$.

The case of $\psi_f^u j_{!*}{}^p\mathbf{L}$. The properties of $\psi_f^u j_{!*}{}^p\mathbf{L}$ may be recovered again from the definition of the filtration F on $(\Omega^*\,{}^p\mathcal{L})_X$. Hence, the case of $\psi_f^u j_{!*}{}^p\mathbf{L}$ may be deduced from the corollary 5.5.

The reduction to the locally unipotent case apply. The ramified covering g fits in the following diagram

$$\begin{tikzcd}
(\widetilde{D}^*)^n \arrow[d, "\pi"] \arrow[dr, "\pi_1"] & & D^n \times_D \widetilde{D}^* \arrow[dl, "p_1"'] \arrow[d, "p'_1"] & D_1{}^n \times_D \widetilde{D}^* \arrow[l, "g\times id"'] \arrow[dl] \\
(Z \subset Y \subset D^n, t_.) \arrow[d, "f"] & (Z' \subset Y' \subset D_1^n, u_.) \arrow[l, "g"] \arrow[dl, "f_1"] & & \\
D & & D^* \arrow[ll] &
\end{tikzcd}$$

where $f : D^n \to D$ is defined by $f = t_1^{m_1} \cdots t_p^{m_p}$ and $Z = f^{-1}(0) \subset Y$ are NCD, $f_1 = f \circ g, Y' = f^{-1}(Y), Z' = f^{-1}(Z)$ and $Y \cap f^{-1}(D^*), Y' \cap f_1^{-1}(D^*)$ are relative NCD over D^*.

We deduce $\psi_f^u j_{!*}{}^p\mathbf{L}$ as the invariant subspace by the action of G on $g_*\psi_{f_1} j'_{!*}{}^p\mathbf{L}'$. Let $N_i := Log\, T_i^u$, $N'_i := Log\, T'_i, N' := N'_1 + \cdots + N'_n$ and $N := N_1 + \cdots + N_n$. The filtration $W(N)$ defined by N on $\psi_f^u j_{!*}{}^p\mathbf{L}$ corresponds to the invariant subspace of the filtration $W(N')$ defined by N' on $\psi_{f_1}^u j'_{!*}{}^p\mathbf{L}'$. The decomposition of $Gr_*^{W(N)}\psi_f^u j_{!*}{}^p\mathbf{L}$ is deduced from the decomposition of $g_* Gr_*^{W(N')}\psi_{f_1}^u j'_{!*}{}^p\mathbf{L}'$ by the action of G.

5.2.0.2 Weight and Hodge filtrations on ${}^p\psi_f j_{!*}{}^p\mathbf{L}$ and ${}^p\varphi_f j_{!*}{}^p\mathbf{L}$

Let ${}^p\mathbf{L}$ be a quasi-unipotent shifted VHS. Let $\alpha \in [0,1[$ such that $exp(-2i\pi\alpha)$ is an eigenvalue of ${}^p\psi_f j_{!*}{}^p\mathcal{L} := \psi_f j_{!*}{}^p\mathcal{L}[-1]$. Set $\beta = exp(-2i\pi\alpha)$ and ${}^p\psi_f^\beta j_{!*}{}^p\mathbf{L}$ the sub-perverse sheaf invariant by $T^s - \beta Id$ where T^s is the semi simple action of T. There exists a natural decomposition into a direct sum of perverse sub-sheaves:

$$ {}^p\psi_f j_{!*}{}^p\mathcal{L} \simeq \oplus_{\alpha\in[0,1[}\, {}^p\psi_f^\beta j_{!*}{}^p\mathbf{L}, \quad {}^p\varphi_f j_{!*}{}^p\mathcal{L} \simeq \oplus_{\alpha\in[0,1[}\, {}^p\varphi_f^\beta j_{!*}{}^p\mathbf{L} $$

Proposition 5.5 (perverse VMHS on ${}^p\psi_f j_{!*}{}^p\mathbf{L}$ and ${}^p\varphi_f j_{!*}{}^p\mathbf{L}$). *With the notations of (Equation 5.36 and Theorem 5.1), let ${}^p\mathbf{L}$ be a shifted VHS quasi-unipotent along Y.*

There exists a natural structure of perverse VMHS on ${}^p\psi_f j_{!*}{}^p\mathcal{L}$ (resp. ${}^p\varphi_f j_{!*}{}^p\mathcal{L}$) with support on the NCD Z satisfying the following decomposition according to the action of the monodromy. Let $\beta = exp(-2i\pi\alpha)$, then:

$$ \begin{aligned} (\mathrm{Gr}_a^W\, {}^p\psi_f j_{!*}{}^p\mathbf{L}, F) &\simeq \oplus_{\alpha\in[0,1[}(\mathrm{Gr}_a^W\, {}^p\psi_f^\beta j_{!*}{}^p\mathbf{L}, F) \simeq \oplus_{\alpha\in[0,1[}\, \oplus_{Z_J\subset Z}\, ({}^p\mathcal{P}_a^{\beta,J}, F) \\ (\mathrm{Gr}_a^W\, {}^p\varphi_f j_{!*}{}^p\mathbf{L}, F) &\simeq \oplus_{\alpha\in[0,1[}(\mathrm{Gr}_a^W\, {}^p\varphi_f^\beta j_{!*}{}^p\mathbf{L}, F) \simeq \oplus_{\alpha\in[0,1[}\, \oplus_{Z_J\subset Z}\, ({}^p\mathcal{Q}_a^{\beta,J}, F) \end{aligned} \tag{5.39} $$

where ${}^p\psi_f^\beta j_{!*}{}^p\mathbf{L}$ (resp. ${}^p\varphi_f^\beta j_{!*}{}^p\mathbf{L}$) is the perverse sub-sheaf on which the semi-simple monodromy action T_s has eigenvalue β and ${}^p\mathcal{P}_a^{\beta,J}$ (resp. ${}^p\mathcal{Q}_a^{\beta,J}$) is an intermediate extension of a polarized VHS of weight a on Z_J.

Proof. In the case of quasi-unipotent ${}^p\mathbf{L}$, the decomposition into a direct sum over α is proved in ([DeK 73]: Proposition 4.17, Lemma 4.18.5 and Remark 4.18.6). By the same reference, we introduce on D^* the local system U_α of rank 1 with monodromy $e^{2i\pi\alpha}$ and trivial Hodge structure of type $(0,0)$. We have:

$$ {}^p\psi_f^u(j_{!*}{}^p\mathcal{L} \otimes f^*U_\alpha) \simeq {}^p\psi_f^\beta j_{!*}{}^p\mathcal{L} $$

where for each Milnor fiber at $z \in Z$: ${}^p\psi_f^u j_{!*}({}^p\mathcal{L} \otimes f^*U_\alpha)_z \simeq {}^p\psi_f^\beta j_{!*}({}^p\mathcal{L})_z$. Then

$$ {}^p\psi_f j_{!*}{}^p\mathcal{L} \simeq \oplus_{\alpha\in[0,1]}{}^p\psi^u_f(j_{!*}{}^p\mathcal{L}\otimes f^*U_\alpha) \tag{5.40} $$

Hence the proof is reduced to the unipotent case ${}^p\psi^u_f$. However, since $\mathcal{L}\otimes f^*U_\alpha$, is a VHS, quasi-unipotent along Z, we apply (§5.2.0.1) to reduce the decomposition of ${}^p\psi^u_f(j_{!*}{}^p\mathcal{L}\otimes f^*U_\alpha)$ over $Z_J\subset Z$ to the case of a locally unipotent $\mathbf{L}$, where we apply the equations (5.36) and (5.31) to write: $({}^p\mathcal{P}^J_a, F) = (j_{!*}\mathcal{P}^J_{a-\,{}^pw+1}[\dim X - |J|], F)(1-|J|)$.

By (§5.1.3.3, Corollary 5.1) the proof apply to give a similar decomposition for $(\mathrm{Gr}^W_a\,{}^p\varphi_f j_{!*}{}^p\mathbf{L}, F)$. For $\alpha\neq 0$, the canonical morphism induces an isomorphism ${}^p\psi^\beta_f j_{!*}{}^p\mathcal{L}\simeq {}^p\varphi^\beta_f j_{!*}{}^p\mathcal{L}$.

5.2.0.3 Perverse VMHS with support

The following structure of perverse VMHS with support on Z is satisfied by ${}^p\psi_f j_{!*}{}^p\mathbf{L}$ and ${}^p\varphi_f j_{!*}{}^p\mathbf{L}$.

Definition 5.4 (Perverse VMHS on X with support on a NCD). A perverse VMHS is a bifiltered complex (K', W, F) with support on a NCD $Z\subset X$ satisfying the following conditions:

1. W is an increasing perverse filtration on K' defined over $\mathbb{Q}$,
2. F is a decreasing filtration defined over $\mathbb{C}$ such that for all $r\in\mathbb{Z}$,

$$ (\mathrm{Gr}^W_r K', F) \simeq \oplus_{Z_J\subset Z}(j_{!*}{}^p\mathbf{L}^J_r, F) \tag{5.41} $$

is a direct sum of intermediate extension of polarized VHS ${}^p\mathbf{L}^J_r$ of weight r on non empty smooth open subsets Z^*_J, with embedding $j: Z^*_J\to Z_J$.

This concept is a generalization of MHC in [De 72] and by extension it is sometimes denoted MHC or PVMHS.

5.2.1 Functoriality of the monodromy filtration

We illustrate the proof of the decomposition theorem in §6 and its relation to the purity theorem, in the case of $\dim V = 1$.

We deduce the structure of PVMHS on $\psi^u_f j_{!*}{}^p\mathbf{L}$ by the isomorphism $\psi^u_f j_{!*}{}^p\mathbf{L}\simeq \Psi^* I\,{}^p\mathcal{L}$ and similarly the structure of PVMHS on the vanishing cycles $\varphi^u_f(j_{!*}{}^p\mathbf{L})$ by the isomorphism $\varphi^u_f(j_{!*}{}^p\mathbf{L})\simeq\varphi^* I\,{}^p\mathcal{L}$.

5.2.1.1 Bigraded polarized HS

We assume the purity theorem 3.1. In the text, this purity theorem is proved by induction on $\dim X$. The induction starts in dimension 0 or 1 with the result on a projective smooth curve [Zu 79].

As a consequence, if we assume the purity theorem in the case $\dim X = n-1$, we define for X of dimension n:

Definition 5.5. The limit MHS on $\mathbb{H}^*(Z, \Psi^* I\, {}^p\mathcal{L})$ is defined by $(\Psi^* I\, {}^p\mathcal{L}, W, F)$ and transported onto the hypercohomology $\mathbb{H}^*(Z, \psi_f^u j_{!*}{}^p\mathbf{L})$ (Equation 4.22) with monodromy endomorphism N induced by the endomorphism $-2i\pi\nu$ on $\Psi^* I\, {}^p\mathcal{L}$.

We deduce from the definition, the next statement on the functoriality of the monodromy filtration with the derived direct image f_* (this functoriality is used later to extend the induction on the purity theorem to dimension n):

Theorem 5.2 (Monodromy filtration). *The iterated morphism*

$$N^r : Gr^W_{{}^pw+q+r}\mathbb{H}^q(Z, \psi_f^u j_{!*}{}^p\mathbf{L}) \simeq Gr^W_{{}^pw+q-r}\mathbb{H}^q(Z, \psi_f^u j_{!*}{}^p\mathbf{L})(-r)$$

is an isomorphism of Hodge structures induced by $-2i\pi\nu$. Hence, the filtration induced by (Equation 5.5) is the monodromy filtration of N.

Proof. We recall the statement of (Equation 5.31) in terms of the monodromy filtration $W(\nu)$ on the perverse sheaf $\Psi^* I^p\mathcal{L}[-1]$, where the shift by $[-1]$ induces a shift on the weight filtration, hence $W := W(\nu)[{}^pw - 1]$:

$$(\mathrm{Gr}_a^W\, {}^p\psi_f^u j_{!*}{}^p\mathbf{L}, F) := (\mathrm{Gr}^{W(\nu)}_{a-{}^pw+1}\Psi^* I^p\mathcal{L}[-1], F) \simeq \oplus_{Z_J\subset Z}({}^p\mathcal{P}_a^J, F) \quad (5.42)$$

where: $({}^p\mathcal{P}_a^J, F) = (j_{Z!*}\mathcal{P}^J_{a-{}^pw+1}[\dim X - |J|], F)(1-|J|)$ (Equation 5.31).

The shift in the degree of the perverse sheaf allows symmetry in the equations and the use of the next result.

Bigraded polarized HS: The weight spectral sequence with respect to the weight filtration

$$E_1^{p,q} = \mathbb{H}^{p+q}(Z, Gr^W_{-p}\Psi^* I^p\mathcal{L}[-1]) \Longrightarrow Gr^W_q\mathbb{H}^{p+q}(Z, \Psi^* I^p\mathcal{L}[-1]) \quad (5.43)$$

degenerates at rank 2. The proof is based on Hodge theory. Let $n := \dim X$. Since $\dim Z = n-1$, the E_1 terms carry a bigraded polarized HS by the Induction hypothesis . For symmetry reasons we introduce the following family of spaces L_p^q ([Sa 88] Proposition 4.2.2, [St 76], [Shi 93] Lemma 4.2.2):

$$L_p^q := E_1^{-(p+{}^pw-1),p+{}^pw-1+q} = \mathbb{H}^q(Z, Gr_p^{W(\nu)}\Psi^* I^p\mathcal{L}[-1])$$

satisfying the following properties:

1. L_p^q is a HS of weight $p+q+{}^pw-1$, since we have the decomposition

$$L_p^q \simeq \oplus_{J\subset I_Z}\mathbb{H}^q(Z, j_{Z!*}\mathcal{P}^J_{p-|J|}[n-|J|])(1-|J|).$$

2. The cup product with the class of an hyperplane section of X induces a morphism $l : L_p^q \to L_p^{q+2}(1)$ such that $l^q : L_p^{-q} \xrightarrow{\sim} L_p^q(q)$ is an isomorphism for $q \geq 0$ by Hard Lefschetz on the components of Z.
3. The endomorphism ν induces the monodromy morphism $\nu : L_p^q \to L_{p-2}^q(-1)$ which commutes with l: $\nu \circ l = l \circ \nu$, such that $\nu^p : L_p^q \xrightarrow{\sim} L_{-p}^q(-p)$ is an isomorphism for $p \geq 0$ by the property of the monodromy filtration $W(\nu)$.
4. We deduce from the self-duality of $\mathbf{L}$ and of the monodromy filtration a scalar product polarizing the HS

$$S : L_{-p}^{-q} \times L_p^q =\to \mathbb{Q}(-{}^p w + 1)$$

5. Let ${}_0L_p^{-q} = \operatorname{Ker} N^{p+1} \cap \operatorname{Ker} l^{q+1}$, then: $S(x, N^p l^q y)$ induces a polarization of the HS on ${}_0L_p^{-q} \times {}_0L_p^{-q}$ up to sign.
6. The differential d_1 on the terms E_1 of the spectral sequence induces a morphism $d : L_p^q \to L_{p-1}^{q+1}$ satisfying $d^2 = 0$, $d \circ l = l \circ d$, $d \circ \nu = \nu \circ d$ and $S(dx, y) = \pm S(x, dy)$.

The above properties form a HL module ([Sa 88] Section 4.2, see also [St 76] §5) such that the following theorem apply:

Theorem ([Sa 88] Proposition 4.2.2) Let $L := \oplus_{p,q\in\mathbb{Z}} L^{p,q}$ denote a bigraded Hodge Lefschetz polarized differential module, then the cohomology

$$H = \operatorname{Ker} d/\operatorname{Im} d = \oplus_{p,q\in\mathbb{Z}} H^{p,q}$$

is also a bigraded Hodge Lefschetz polarized differential module with the induced polarization.

The proof by Saito is a correction to the proof in ([St 76] §5). The statement is also valid in the abelian category of Hodge modules.

The proof of the theorem follows .

5.2.1.2 Special case of the Decomposition and Purity theorems

Let $f : X \to \mathbb{P} := \mathbb{P}^1_{\mathbb{C}}$ be a morphism such that $X_0 := f^{-1}(0)$ is a NCD in X. Let $\Sigma \subset \mathbb{P}$ denote the set of critical values of f, so $0 \in \Sigma$ and the restriction $f_| : X \setminus f^{-1}(\Sigma) \to \mathbb{P}^* = \mathbb{P} \setminus \Sigma$ is smooth. We consider the diagram

$$\begin{array}{ccccc} X_0 & \overset{i}{\hookrightarrow} & X & \overset{k}{\hookleftarrow} & X \setminus f^{-1}(\Sigma) \\ \downarrow & & \downarrow f & & \downarrow \\ 0 & \overset{i'}{\hookrightarrow} & \mathbb{P} & \overset{k'}{\hookleftarrow} & \mathbb{P}^* \end{array} \tag{5.44}$$

Let $\mathbf{L}$ denote a polarized VHS on the complement of a NCD Y containing X_0. We write $Y = X_0 \cup X'$ where X' is the closure of $Y \setminus X_0$ in X. We

suppose X' horizontal on $\mathbb{P}^*$ in the sense that $X'_t \subset X_t$ is a NCD in the smooth fiber X_t for $t \notin \Sigma$ and $X'_{|\mathbb{P}^*}$ is topologically locally trivial over $\mathbb{P}^*$ such that $R^i f^* \mathbf{L}_{|\mathbb{P}^*}$ is locally constant over $\mathbb{P}^*$ with fiber $\mathbb{H}^i(X_t, j_{!*}\mathbf{L})$.

Let $K := f_* j_{!*}{}^p\mathbf{L}$. The restriction to $\mathbb{P}^*$ of ${}^p\mathcal{H}^i(K)$ is a shifted polarized VHS and we have ${}^p\mathcal{H}^i(K)_{|\mathbb{P}^*} \simeq \mathbf{L}^i_{\mathbb{P}^*}[1]$.

Proposition 5.6 (Decomposition). *Let $K := f_* j_{!*}{}^p\mathbf{L}$ be the derived image. There exists a canonical splitting*

$$({}^p\mathcal{H}^i(K), F)_0 = (L^i_0, F) \oplus (IC^*\mathcal{L}^i_{\mathbb{P}^*}[1], F)_0$$

into the polarized HS (L^i_0, F) and $IC^*\mathcal{L}^i_{\mathbb{P}^*}[1] = k'_{!*}\mathbf{L}^i_{\mathbb{P}^*}[1]$ with its limit filtration F.

We recall that by the uniqueness of the decomposition of ${}^p\mathcal{H}^i(K)$ we have the isomorphism $L^i_0 \simeq \mathrm{Im}(\mathbb{H}^i_{X_0}(X, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^i(X_0, j_{!*}{}^p\mathbf{L})$.

The proof is local at 0, hence we consider the preimage $X_D := f^{-1}(D) \subset X$ of a disc D with center 0 and coordinate z. Since

$${}^p\psi^u_z = \psi^u_z[-1],\ {}^p\psi^u_{z\circ f} = \psi^u_{z\circ f}[-1] \text{ and } {}^p\varphi^u_z := \varphi^u_z[-1],\ {}^p\varphi^u_{z\circ f} = \varphi^u_{z\circ f}[-1]$$

transform perverse sheaves into perverse sheaves, we have:

$$\begin{aligned}({}^p\psi^u_z\, {}^p\mathcal{H}^i(K)) &= {}^p\mathcal{H}^i(f_{|X_0*}{}^p\psi^u_{z\circ f} j_{!*}{}^p\mathcal{L}) = H^{i-1}(X_0, \psi_{z\circ f} j_{!*}{}^p\mathcal{L}) =: H_\psi\\ ({}^p\varphi^u_z\, {}^p\mathcal{H}^i(K)) &= {}^p\mathcal{H}^i(f_{|X_0*}{}^p\varphi^u_{z\circ f} j_{!*}{}^p\mathcal{L}) = H^{i-1}(X_0, \varphi_{z\circ f} j_{!*}{}^p\mathcal{L}) =: H_\varphi.\end{aligned}$$

Moreover, since $\dim X_0 = n-1$, the conditions of Theorem 5.2 are satisfied by induction. Hence, the monodromy filtration on H_ψ is deduced from the monodromy filtration on $\psi_{z\circ f} j_{!*}{}^p\mathbf{L}$. Let ${}^pw$ be the weight of ${}^p\mathbf{L}$. Then H_ψ (resp. H_φ) is a nilpotent orbit of weight ${}^pw + i - 1$ (resp. ${}^pw + i$). Hence $H^m_\psi := \mathrm{Gr}^{W(N)}_m H_\psi$ is a polarized HS of weight ${}^pw + i - 1 + m$ and $H^{m-1}_\varphi := \mathrm{Gr}^{W(N)}_{m-1} H_\varphi$ is a PHS of weight ${}^pw + i + m - 1$.

For all $m \in \mathbb{Z}$ the diagrams $H^m_\psi \xrightarrow{C} H^{m-1}_\varphi \xrightarrow{V} H^{m-2}_\psi(-1)$ of polarized HS, induced by the canonical and the variation morphisms (§5.1.3.2), satisfy the conditions of ([Sa 88], Lemma 5.2.15) and (Remark 2.7), hence for all m we have a decomposition

$$H^{m-1}_\varphi = \mathrm{Im}\, C \quad \oplus \quad \mathrm{Ker}\, V \tag{5.45}$$

Since the morphisms C and V are morphisms of MHS, they are strict. Hence we deduce the following decomposition

$${}^p\varphi^u_z\, {}^p\mathcal{H}^i \simeq \mathrm{Im}\,(C : {}^p\psi^u_z\, {}^p\mathcal{H}^i \to {}^p\varphi^u_z\, {}^p\mathcal{H}^i) \oplus \mathrm{Ker}\,(V : {}^p\varphi^u_z\, {}^p\mathcal{H}^i \to {}^p\psi^u_z\, {}^p\mathcal{H}^i(-1)) \tag{5.46}$$

where ${}^p\mathcal{H}^i := {}^p\mathcal{H}^i(K)$. Let $L^i := {}^p\psi^u_z\, {}^p\mathcal{H}^i(K) = {}^p\psi^u_z\, {}^p\mathcal{L}^i_{\mathbb{P}^*} = \psi^u_z \mathcal{L}^i_{\mathbb{P}^*}$, then $\mathrm{Im}\, C \simeq \mathrm{Im}\,(N : L^i \to L^i)$. Set: $L^i_0 := \mathrm{Ker}\,(V : \mathbb{H}^i(X_0, {}^p\varphi^u_{z\circ f} j_{!*}{}^p\mathbf{L}) \to L^i(-1))$. We deduce:

$$ {}^p\varphi_z^u\ {}^p\mathcal{H}^i(K) = \mathbb{H}^i(X_0, {}^p\varphi^u_{z\circ f} j_{!*}{}^p\mathbf{L}) \simeq L_0^i \oplus (\operatorname{Im} N : L^i \to L^i) \tag{5.47} $$

The term L^i with the monodromy filtration is a nilpotent orbit (Theorem 5.2), hence the term $\operatorname{Im} N : L^i \to L^i$ is endowed with the induced MHS as in the definition of the intermediate extension of $k'_{!*}(\mathcal{L}^i_{P^*}[1]$ on $\mathbb{P}$.

Remark 5.7. The induced filtration F on $\mathbb{H}^j(\mathbb{P}, k'_{!*}(\mathcal{L}^i_{P^*}[1])$ defines a HS ([Zu 79]), hence $\mathbb{H}^j(\mathbb{P}, K)$ is a direct sum of two HS.

Corollary 5.6. *Let $K := f_* j_{!*}{}^p\mathbf{L}$. We have the following splitting*

$$ K \simeq \oplus_{i\in\mathbb{Z}}({}^p\mathcal{H}^i(K)[-i] \in D^b_c(\mathbb{P}, \mathbb{Q}) $$

The iterated cup-product with the class c_1 of an hyperplane section of X induces Lefschetz isomorphisms

$$ \eta^i : {}^p\mathcal{H}^{-i}(K) \simeq {}^p\mathcal{H}^i(K)(i) $$

on the components of the decomposition of ${}^p\mathcal{H}^{-i}(K)$ including L_0^i by the bigraded polarized setting on ${}^p\psi_z^u$ and ${}^p\varphi_z^u$ (§5.2.1.1).

Then, the existence of a splitting of the complex K follows from Deligne's general argument.

5.2.1.3 Perverse cohomology along a locally principal divisor

Let $j_{!*}{}^p\mathbf{L}$ be an intermediate extension on the complex algebraic variety X. Let Z be a locally principal divisor, $i : Z \to X$, $U = X \setminus Z$ and $j_Z : U \to X$. The complex $i_* i^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}$ occurs in two triangles:

$$ \begin{gathered} j_{Z!} j_Z^* j_{!*}{}^p\mathbf{L} \xrightarrow{\text{can}} j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L} \to i_* i^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L} \xrightarrow{[1]} \\ i^! j_{!*}{}^p\mathbf{L} \xrightarrow{I} i^* j_{!*}{}^p\mathbf{L} \to i^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L} \xrightarrow{[1]} \end{gathered} \tag{5.48} $$

The complexes $j_{Z!} j_Z^* j_{!*}{}^p\mathbf{L}$ and $j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}$ are perverse since j_Z is affine (resp. Stein).

Let Z be a principal divisor and set $\mathcal{K} := (j_{!*}{}^p\mathbf{L})_{|U}$. The perverse long exact sequences defined by the corresponding triangles, including (Equation 2.16) are reduced to

$$ \begin{gathered} 0 \to {}^p\mathcal{H}^{-1} i^* j_{Z*}\mathcal{K} \to \psi_f^u\mathcal{K}[-1] \xrightarrow{N} \psi_f^u\mathcal{K}[-1] \to {}^p\mathcal{H}^0 i^* j_{Z*}\mathcal{K} \to 0 \\ 0 \to {}^p\mathcal{H}^{-1} i^* j_{Z*}\mathcal{K} \to \psi_f^u\mathcal{K}[-1] \xrightarrow{can} \varphi_f^u\mathcal{K}[-1] \to {}^p\mathcal{H}^0 i^* j_{Z*}\mathcal{K} \to 0 \\ 0 \to i_*{}^p\mathcal{H}^{-1} i^* j_{Z*}\mathcal{K} \to j_{Z!}\mathcal{K} \xrightarrow{can'} j_{Z*}\mathcal{K} \to i_*{}^p\mathcal{H}^0 i^* j_{Z*}\mathcal{K} \to 0 \end{gathered} \tag{5.49} $$

as $i^* j_{Z*}\mathcal{K}$ has only two non trivial perverse cohomology sheaves ([BBD 83] Corollaire 4.1.10 ii). Recall that $j_{!*}{}^p\mathbf{L} = j_{Z!*}\mathcal{K} := \operatorname{Im}(can : j_{Z!}\mathcal{K} \to j_{Z*}\mathcal{K})$.

If Z is defined by an equation f and $N := Log\, T^u$, we have (([BBD 83], Corollaire 4.1.12) and [Br 82] Proposition 2.3.8, [Br 86]):

$$\begin{aligned} i^* j_{!*}{}^p\mathbf{L}[-1] &\simeq {}^p\mathcal{H}^{-1} i^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L} \simeq \operatorname{Ker}\operatorname{can} \simeq \operatorname{Ker} N, \\ i^! j_{!*}{}^p\mathbf{L}[1] &\simeq {}^p\mathcal{H}^0 i^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L} \simeq \operatorname{Coker}\operatorname{can} \simeq \operatorname{Coker} N \\ j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}/j_{!*}{}^p\mathbf{L} &\simeq i_*\operatorname{Coker}(N : {}^p\psi_f^u j_{!*}{}^p\mathbf{L} \to {}^p\psi_f^u j_{!*}{}^p\mathbf{L}(-1)) \end{aligned} \tag{5.50}$$

where $j_{!*}{}^p\mathbf{L}$ is viewed as a sub-complex.

5.2.1.4 Hodge theory along a NCD

Let Y be a NCD and $\mathbf{L}$ a local system on $X \setminus Y$ with intermediate extension $j_{!*}{}^p\mathbf{L}$ on the complex algebraic variety X. Let Z be a NCD sub-divisor of Y, $i : Z \to X$, $U = X \setminus Z$, $j : X \setminus Y \hookrightarrow X$ and $j_Z : U \to X$.

We construct below various logarithmic complexes along a NCD, based on the properties of perverse cohomology sheaves (ch. 3, §3.2.4). We extend the definition of the weight and Hodge filtrations W and F to the logarithmic complexes $i_Z^! j_{!*}{}^p\mathbf{L}$, $i_Z^* j_{!*}{}^p\mathbf{L}$ and $i_Z^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}$ using the relations with ${}^p\psi_f^u(j_{!*}{}^p\mathbf{L})$.

Let Z be a principal divisor defined by an equation f, the filtration $W(N)$ (resp. $\operatorname{Ker} N, \operatorname{Coker} N$) corresponds to the filtration $W(\nu)$ (resp. $\operatorname{Ker}\nu, \operatorname{Coker}\nu$) by the isomorphism $\psi_f^u(j_{!*}{}^p\mathbf{L}) \simeq \Psi^*(I^p\mathcal{L})$.

The weight filtration W is deduced from $W(N)$ as follows:

$$\begin{aligned} &\operatorname{Gr}^W_{l+{}^pw-1}\, {}^p\psi_f^u j_{!*}{}^p\mathbf{L} = (\operatorname{Gr}^W_{l+{}^pw}\psi_f^u j_{!*}{}^p\mathbf{L})[-1] = \\ &(\operatorname{Gr}^W_{l+w'}\psi_f^u j_{!*}\mathbf{L})[n-1] = (\operatorname{Gr}^{W(N)}_l \psi_f^u j_{!*}\mathbf{L})[n-1] \end{aligned} \tag{5.51}$$

from which we deduce the weight filtration on $\operatorname{Coker} N|\, {}^p\psi_f^u(j_{!*}{}^p\mathbf{L})$ (resp. $\operatorname{Ker} N|\, {}^p\psi_f^u(j_{!*}{}^p\mathbf{L})$) as follows:

$$W_{l+{}^pw-1}(\operatorname{Coker} N) := W(N)_l(\operatorname{Coker} N) = \operatorname{Im}(\operatorname{Ker} N^{l+1} \to \operatorname{Coker} N) \text{ if } l \geq 0$$

$$\begin{aligned} &W_{l+{}^pw-1}(\operatorname{Ker} N) := W(N)_l(\operatorname{Ker} N) = \operatorname{Ker} N \cap \operatorname{Im} N^{-l} \text{ if } l \leq 0, \\ &W(N)_l(\operatorname{Coker} N) = 0 \text{ if } l < 0, \quad W(N)_l(\operatorname{Ker} N) = \operatorname{Ker} N \text{ if } l > 0 \\ &N^l : \operatorname{Gr}^W_{l+{}^pw-1}(\operatorname{Coker} N) \simeq \operatorname{Gr}^W_{-l+{}^pw-1}(\ker\ N)(-l) \text{ if } l \geq 0, \end{aligned} \tag{5.52}$$

Weight filtration on $j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}$.

If Z is a locally principal divisor, we can suppose Z defined locally by an equation f. The monodromy weight filtration $W(N)$ on ${}^p\psi_f^u(j_{!*}{}^p\mathbf{L})$ induces a filtration on $\operatorname{Coker} N|\, {}^p\psi_f^u(j_{!*}{}^p\mathbf{L})$ transported on $i^! j_{!*}{}^p\mathbf{L}[1]$ (resp. on $\operatorname{Ker} N|\, {}^p\psi_f^u(j_{!*}{}^p\mathbf{L})$ transported on $i^* j_{!*}{}^p\mathbf{L}[-1]$) is independent of the local equation f ([Sa 88], [Sa 90], [ELM 10]). If Z is principal and we consider the filtrations W and F in (Equation 5.49), we have:

$$0 \to {}^p\mathcal{H}^{-1} i^* j_{Z*}\mathcal{K} \to {}^p\psi_f^u \mathcal{K} \xrightarrow{N} {}^p\psi_f^u \mathcal{K}(-1) \to {}^p\mathcal{H}^0 i^* j_{Z*}\mathcal{K} \to 0 \tag{5.53}$$

since N drops the weight by 2, hence we write

$$(\mathrm{Coker}(N|\,{}^p\psi_f^u j_{!*}{}^p\mathbf{L}, W, F)(-1) \simeq (j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L})/j_{!*}{}^p\mathbf{L}, W, F).$$

As a consequence, the weight W on $j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L}$ is defined as follows.

Let $q : j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L} \to j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L}/j_{!*}{}^p\mathbf{L} \simeq \mathrm{Coker}\, N$. We deduce from (Equation 5.52), that there exists unique filtrations W and F on $j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L}$, independent of the local equation f of Z, satisfying (Equations 5.53, 5.52 and 5.2):

$$\begin{aligned} &W_l(j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L}) = 0 \text{ if } l < {}^pw, \quad W_l j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L} = j_{!*}{}^p\mathbf{L} \text{ if } l = {}^pw; \\ &\quad W_{l+{}^pw+1} j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L} = q^{-1}(W(N)_l \mathrm{Coker}\, N|\,{}^p\psi_f^u j_{!*}{}^p\mathbf{L}) \text{ for } l \geq 0; \\ &\mathrm{Gr}_l^W j_{Z*}j_Z^*(j_{!*}{}^p\mathbf{L}) = \mathrm{Gr}_{l-2}^W \mathrm{Coker}(N|\,{}^p\psi_f^u j_{!*}{}^p\mathbf{L})(-1) \text{ if } l > {}^pw. \end{aligned} \tag{5.54}$$

Example: $\mathrm{Gr}_{{}^pw+1}^W j_{Z*}j_Z^*(j_{!*}{}^p\mathbf{L}) \simeq (\mathrm{Gr}_0^{W(N)} \psi_f^u j_{!*}\mathbf{L})[n-1](-1)$(check for $n = 1$)

Proposition 5.7 (W and F). *Let Z be a NCD sub-divisor of Y in X, and ${}^p\mathbf{L}$ a polarized VHS of weight ${}^pw$ on $X \setminus Y$.*

1. *There exist unique global filtrations W and F on $i^* j_{!*}{}^p\mathbf{L}[-1]$ (resp. $i^! j_{!*}{}^p\mathbf{L}[1]$), locally compatible with the filtrations deduced from W and F in Equation 5.52 by the isomorphisms :*

$$\begin{aligned} &(i^! j_{!*}{}^p\mathbf{L}[1], W, F)) \simeq (\mathrm{Coker}\, N|\,{}^p\psi_f^u j_{!*}{}^p\mathbf{L}, W, F)(-1) \\ &i^* j_{!*}{}^p\mathbf{L}[-1], W, F) \simeq (\mathrm{Ker}\, N|\,{}^p\psi_f^u j_{!*}{}^p\mathbf{L}, W, F) \end{aligned} \tag{5.55}$$

2. *There exist unique global filtrations W and F on $j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L}$ such that $W_{{}^pw} j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L} = j_{!*}{}^p\mathbf{L}$ and locally for $l \geq 0$:*

$$i^* Gr_{l+{}^pw+1}^W j_{Z*}j_Z^*(j_{!*}{}^p\mathbf{L}) \simeq Gr_l^{W(N)} \mathrm{Coker}\, N|\,{}^p\psi_f^u j_{!*}{}^p\mathbf{L}$$

3. *Dually, $i^* j_{!*}{}^p\mathbf{L}[-1] \simeq \mathrm{Ker}\, N|\,{}^p\psi_f^u j_{!*}{}^p\mathbf{L}$ embeds into $j_{Z!}j_Z^* j_{!*}{}^p\mathbf{L}$ and the weight filtration W on $j_{Z!}j_Z^* j_{!*}{}^p\mathbf{L}$ is deduced from $W(N)$ on $\mathrm{Ker}\, N$: $\mathrm{Gr}_{{}^pw}^W j_{Z!}j_Z^* j_{!*}{}^p\mathbf{L} = j_{!*}{}^p\mathbf{L}$ and locally for $l \leq 0$:*

$$i^* \mathrm{Gr}_{l+{}^pw-1}^W j_{Z!}j_Z^* j_{!*}{}^p\mathbf{L} \simeq \mathrm{Gr}_l^{W(N)} \mathrm{Ker}\ N|\,{}^p\psi_f^u j_{!*}{}^p\mathbf{L}$$

4. *If X is projective, the filtrations W and F induced on the hypercohomology of X by the filtrations on*

$$i^* j_{!*}{}^p\mathbf{L}[-1], \quad i^! j_{!*}{}^p\mathbf{L}[1], \quad j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L}, \quad j_{Z!}j_Z^* j_{!*}{}^p\mathbf{L}$$

define respectively a structure of mixed Hodge complex.

Proof of the proposition 5.7. The definition of the filtrations W and F follows from (Equation 5.53).

The proof relies on the gluing of perverse sheaves as they behave like sheaves (Remark 2.5): morphisms of perverse sheaves on a covering $\mathcal{U}$ of X which coincide on the intersection of two open subsets in $\mathcal{U}$ glue into a global morphism on X ([BBD 83], Corollaires 2.1.21, 2.1.22 and 2.1.23).

Hence, the local definitions of the perverse weight filtration W define a global filtration on X. The Hodge filtrations have been defined globally on $\Omega^*\mathcal{L}$ and on $\Psi^* I\mathcal{L}$ (Equation 5.28).

An elegant alternative proof is based on the deformation to the normal cone and the specialization functor introduced by Verdier [V 83], which reduces the local construction of the ψ functor in the case of a divisor to the case of a principal divisor and gives a global filtration of W [ELM 10].

Example 5.4. 1) The polarization on the primitive spaces is defined by the bilinear product, composition of the duality with the isomorphisms for $l \geq 0$:

$$\begin{array}{ccc}
\mathrm{Gr}^W_{l+{}^pw+1} i^! j_{!*}{}^p\mathbf{L}[1] \xrightarrow{\equiv} \mathrm{Gr}^{W(N)}_l \mathrm{Coker}(N|\,{}^p\psi^u_f j_{!*}{}^p\mathbf{L}) & & \\
\Big\downarrow N^l \simeq & & \\
\mathrm{Gr}^{W(N)}_{-l} \mathrm{Ker}(N|\,{}^p\psi^u_f j_{!*}{}^p\mathbf{L}) & \xrightarrow{=} & \mathrm{Gr}^W_{-l+{}^pw-1} i^* j_{!*}{}^p\mathbf{L}[-1]
\end{array}$$

2) Let $\mathbf{L}$ be a unipotent VHS of weight w' on a punctured disc with center 0 in $\mathbb{C}$, then $(\psi_z\mathbf{L})_0 \simeq L$ is a nilpotent orbit of weight w'.

Moreover, $j_*\mathbf{L}/j_{!*}\mathbf{L} \simeq L/NL[-1] \simeq L/NL[-1]$ is of weight $\geq w'-1$ and $\mathrm{Gr}^W_{w'+1} j_*\mathbf{L} \simeq \mathrm{PrimGr}^W_{w'-1}(L)[-1]$.

3) Let $\mathbb{P}^* := \mathbb{P}^1_{\mathbb{C}} - \Sigma$ with Σ finite and $0 \in \Sigma$, ${}^p\mathbf{L}$ be defined on $\mathbb{P}^*$ and $j : \mathbb{P}^* \to \mathbb{P}^1_{\mathbb{C}}$. The weight filtration W on $j_*{}^p\mathbf{L}$ is defined by $W_{{}^pw} := j_{!*}{}^p\mathbf{L}$. The quotient $j_*{}^p\mathbf{L}/j_{!*}{}^p\mathbf{L}$ is isomorphic to $L/NL := \mathrm{Coker}\, N : L \to L$ in degree 0, where $L := \psi_z(\mathbf{L})$ is endowed with the limit MHS.

Hence for $i > 0$: $W_{{}^pw+i} j_*{}^p\mathbf{L}$ corresponds to the filtration induced on L/NL by $W_{w'+i-1}(L/NL)(-1)$ (Equation 5.2).

4) Let $i_0 : 0 \to \mathbb{P}^1_{\mathbb{C}}$. The weight filtration W on $i_0^* j_*{}^p\mathbf{L} \simeq (L \xrightarrow{N} L)[1]$ is defined by $W_{i+{}^pw-1} = Gr^W_i(N)(\mathrm{Ker}\, N : L \to L)[1]$ (in degree -1) if $i \leq 0$.

Remark 5.8. The following graded perverse sheaves decompose into direct sum of intermediate extensions of polarized perverse sheaves on smooth components of Z.

$$\begin{aligned}
\mathrm{Gr}^W_l i^* j_{!*}{}^p\mathbf{L}[-1] &\simeq \mathrm{PrimGr}^{W(\nu)}_{l-{}^pw+1} \Psi^* I^p\mathcal{L}[-1] \quad \text{if} \quad l < {}^pw \\
\mathrm{Gr}^W_l j_{Z!} j_Z^* j_{!*}{}^p\mathbf{L} &\simeq i_* \mathrm{PrimGr}^{W(\nu)}_{l-{}^pw+1} \Psi^* I^p\mathcal{L}[-1] \quad \text{if} \quad l < {}^pw \\
\mathrm{Gr}^W_l i^! j_{!*}{}^p\mathbf{L}[+1] &\simeq \mathrm{PrimGr}^{W(\nu)}_{l-{}^pw-1} \Psi^* I^p\mathcal{L}[-1](-1) \quad \text{if} \quad l > {}^pw \\
\mathrm{Gr}^W_l j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L} &\simeq i_* \mathrm{PrimGr}^{W(\nu)}_{l-{}^pw-1} \Psi^* I^p\mathcal{L}[-1](-1) \quad \text{if} \quad l > {}^pw
\end{aligned}$$

where by convention: $W(\nu)_{i-1}(\Psi^* I^p\mathcal{L}[-1]) := (W(\nu)_i \Psi^* I^p\mathcal{L})[-1]$,
$\mathrm{Gr}^{W(\nu)}_{l-{}^p w+1}\mathrm{Ker}\,\nu|\Psi^* I^p\mathcal{L}[-1] \simeq \mathrm{PrimGr}^{W(\nu)}_{l-{}^p w+1}\Psi^* I^p\mathcal{L}[-1]$ if $l < {}^p w$ and
$\mathrm{Gr}^{W(\nu)}_{l-{}^p w+1}\mathrm{Coker}\,\nu|\Psi^* I^p\mathcal{L}[-1] \simeq \mathrm{PrimGr}^{W(\nu)}_{l-{}^p w+1}\Psi^* I^p\mathcal{L}[-1]$ if $l > {}^p w$.

Corollary 5.7. *Let $Z \subset$ be NCD in a projective variety X and ${}^pL$ a shifted polarized VHS of weight ${}^pw$ on $X \setminus Y$:*

1. *The induced MHS on $\mathbb{H}^q(X \setminus Z, j_{!*}{}^p\mathbf{L})$ has weights $w \geq {}^pw + q$. Moreover, the natural map $\mathbb{H}^i(X, j_{!*}{}^p\mathbf{L}) \to W_{{}^pw+i}\mathbb{H}^i(X \setminus Z, j_{!*}{}^p\mathbf{L})$ is surjective.*
2. *The induced MHS on $\mathbb{H}^q(Z, j_{!*}{}^p\mathbf{L})$ has weights $w \leq {}^pw + q$.*
3. *Dually: $w \leq {}^pw + q$ on $\mathbb{H}^q_c(X \setminus Z, j_{!*}{}^p\mathbf{L})$ and $w \geq {}^pw + q$ on $\mathbb{H}^q_W(X, j_{!*}{}^p\mathbf{L})$.*

The corollary follows from the purity theorem 4.3. There exists a complex endowed with filtrations $(j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L}, W, F)$, isomorphic to $(IC^{*p}\mathcal{L}(LogZ), F)$ in the derived filtered category $D^+F(X, \mathbb{C})$, such that: $W_{{}^pw} j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L} = j_{!*}{}^p\mathbf{L}$ and locally, if $l > {}^pw$

$\mathrm{Gr}^W_l j_{Z*}j_Z^*(j_{!*}{}^p\mathbf{L}) = i_*\mathrm{Gr}^W_l(i^! j_{!*}{}^p\mathbf{L}[1]) = i_*\mathrm{Gr}^{W(\nu)}_{l-{}^pw-1}\mathrm{Coker}\,\nu_{|\Psi^* I^p\mathcal{L}[-1]}(-1)$.

The complex $(R\Gamma(X, j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L}, W, F)$ is a MHC and the weight spectral sequence

$$E_1^{p,q} := \mathbb{H}^{p+q}(X, Gr^W_{-p} j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L}) \Rightarrow Gr^W_q \mathbb{H}^{p+q}(X \setminus Z, j_{!*}{}^p\mathbf{L})$$

degenerates at rank 2. Since $W_{-p} j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L} = 0$ if $-p < {}^pw$:

- $E_1^{p,q} = \mathbb{H}^{p+q}(X, \mathrm{Gr}^W_{-p} j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L}) = 0$ if $-p < {}^pw$;
- $E_1^{-{}^pw,q} = \mathbb{H}^{q-{}^pw}(X, \mathrm{Gr}^W_{{}^pw} j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L})$;
- $E_2^{p,q} = \mathrm{Gr}^W_{-p}\mathbb{H}^{p+q}(X \setminus Z, j_{!*}{}^p\mathbf{L}) = 0$ if $q < p + q + {}^pw$.

The term $E_2^{p,q}$ is a HS of weight q.

Since $E_1^{1-{}^pw,q} = \mathbb{H}^{q+1-{}^pw}(X, \mathrm{Gr}^W_{{}^pw-1} j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L}) = 0$, the projection

$$E_1^{-{}^pw,q} = \mathbb{H}^{q-{}^pw}(X, \mathrm{Gr}^W_{{}^pw} j_{Z*}j_Z^* j_{!*}{}^p\mathbf{L}) \to E_2^{-{}^pw,q} = \mathrm{Gr}^W_{{}^pw}\mathbb{H}^{q-{}^pw}(X \setminus Z, j_{!*}{}^p\mathbf{L})$$

is surjective, and $E_2^{-{}^pw,q} := \mathrm{Gr}^W_{{}^pw}\mathbb{H}^{q-{}^pw}(X \setminus Z, j_{!*}{}^p\mathbf{L})$ is a HS of weight q (the weight on the hypercohomology in degree $q - {}^pw$ is obtained after a shift on the index of W by the degree $q - {}^pw$).

The statements in ii) and iii) follow by duality or standard exact sequences.

Remark 5.9 (Relation between the weight filtration on $\Psi^* L$ (Proposition 5.1) and the weight filtration on $\Omega^* L$). i) The isomorphism:

$$i_* i^! j_{!*}{}^p\mathbf{L}[1] \simeq \mathrm{Coker}\,\mathrm{can}/j_*{}^p\mathbf{L} \simeq \mathrm{Coker}\,N/\Psi^*{}^p\mathbf{L}[-1] \text{ (Equation 5.50)}$$

is equivalent to a de Rham family of canonical isomorphisms for each $J \subset M$:

$$\mathrm{Coker}\,(N : \Psi_J L \to \Psi_J L) \simeq Q'_J := \mathrm{Coker}\,(\mathrm{can} : N_J L \to L).$$

The induced filtration W' on Q'_J by $W(N)$ satisfy:

$$Gr_k^{W(N)}(\Psi_J L/NL) \simeq Prim_k(Gr_k^{W(N)}\Psi_J L) \simeq Gr_k^{W'} Q'_J$$

with induced MHS by M defined by $M(N_0)$ on $L_J := L$ shifted by $1-|J|$ (§5.1.1, Equation 5.5).

ii) The filtration $W(N)$ on $\Psi^* L$ corresponds to a filtration W' on $\Omega^* L$ (see §9.2, Equation 9.10). Then, the local decomposition in terms of $P_k^J L$ (Equations 5.22 and 9.21) corresponds to a local decomposition of $\Omega^* L$ in terms of the spaces $C_k^J L := \mathrm{Coker}(N/P_k^J L)$ (Equation 9.23).

The direct definition of W' in chapter 9, is based on a new filtration $N * W$ on L (Remark 2.21) whose construction from (L, W) by Kashiwara is motivated by the results in (Lemma **??**). The filtration on $\Omega^* L$ is studied directly in (ch. 9, Equation 9.6, 9.23, see also (Remark 5.4)).
When $L = \mathcal{L}(x)$ for $x \in Z_j^*$, since $N_0 := \sum_{i\in J} N_i$ vanishes on $C_k^J L$, we deduce, that $C_k^J L = \mathrm{Gr}_0^{W(N_0)} C_k^J L$ is a pure polarized direct summand of weight $k+\delta$ of $\mathrm{Gr}_0^{W(N_0)}\mathrm{Gr}_k^{W'} Q'_J$ with respect to the MHS defined by $W(N_0)$ on the primitive part of $\mathrm{Gr}_k^{W(N)}\Psi_J L$ shifted by $k+\delta$.

5.2.2 Hodge theory on $i^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}$

Let W' denote the weight filtration on $i^* j_{!*}{}^p\mathbf{L}[-1]$ (resp. $i^! j_{!*}{}^p\mathbf{L}[1]$). We deduce t the weight W defined by

$$(i^* j_{!*}{}^p\mathbf{L}, W) := ((i^* j_{!*}{}^p\mathbf{L}[-1])[1], W'[1])$$
$$(i^! j_{!*}{}^p\mathbf{L}, W) := ((i^! j_{!*}{}^p\mathbf{L}[1])[-1], W'[-1])$$

such that $(i^* j_{!*}{}^p\mathbf{L}, W)$ and $(i^! j_{!*}{}^p\mathbf{L}, W)$ keep the structure of MHC.
Moreover: $W_{{}^pw-k} i^! j_{!*}{}^p\mathbf{L} = 0$ if $k \geq 1$ and $W_{{}^pw+k} i^* j_{!*}{}^p\mathbf{L} = i^* j_{!*}{}^p\mathbf{L}$ if $k \geq 0$ and we have filtered morphism:

$$\mathrm{can} : j_{Z!} j_Z^* j_{!*}{}^p\mathbf{L} \to j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}, \quad I : i^! j_{!*}{}^p\mathbf{L} \to i^* j_{!*}{}^p\mathbf{L}. \tag{5.56}$$

where $I : (i^! j_{!*}{}^p\mathbf{L}, W) \to (j_{!*}{}^p\mathbf{L}, W) \to (i^* j_{!*}{}^p\mathbf{L}, W)$ is the composition morphism and $j_{!*}{}^p\mathbf{L}$ is pure of weight ${}^pw$.

We deduce from the triangles in (Equation 5.48) that $i^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}$ is quasi-isomorphic to the cone over each of the morphisms can and I.

Definition 5.6. i) The structure of MHC on $K' := i^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}$ is defined as a mixed cone C_M over I. K' is not perverse and:

$$\mathrm{Gr}_r^W C_M(I) = (\mathrm{Gr}_{r-1}^W i^! j_{!*}{}^p\mathbf{L})[1] \oplus Gr_r^W i^* j_{!*}{}^p\mathbf{L}.$$

Remark 5.10. If Z is defined by an equation f, we may define $i^* j_{Z*} j_Z^* {}^p\mathbf{L}$ as a mixed cone C_M over $N : {}^p\psi_f^u j_{!*}{}^p\mathbf{L} \to {}^p\psi_f^u j_{!*}{}^p\mathbf{L}$.

Lemma 5.3. *i) We have a graded long exact sequence:*

$\mathrm{Gr}_k^W \mathbb{H}^{j+1}(X, i^! j_{!*}{}^p\mathbf{L}) \to \mathrm{Gr}_k^W \mathbb{H}^j(X, i^* j_{!*}{}^p\mathbf{L}) \to \mathrm{Gr}_k^W \mathbb{H}^j(X, i^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}) \xrightarrow{+1} \mathrm{Gr}_k^W \mathbb{H}^{j+1}(X, i^! j_{!*}{}^p\mathbf{L})$.

ii) We suppose $i^* j_{!*}{}^p\mathbf{L} \subset K'$, then $K'/i^* j_{!*}{}^p\mathbf{L} \xrightarrow{\simeq} i^! j_{!*}{}^p\mathbf{L}[1]$.

Let $q : K' \to i^! j_{!*} {}^p\mathbf{L}[1]$. The weight filtration on K' is defined as follows:
$W_l K' := q^{-1}(W_l i^! j_{!*} {}^p\mathbf{L}[1])$ *if* $l > {}^p w$ *and* $\mathrm{Gr}_l^W K' = \mathrm{Gr}_l^W i^! j_{!*} {}^p\mathbf{L}[1]$
$W_l K' = W_l i^* j_{!*} {}^p\mathbf{L} = (W_{l-1} i^* j_{!*} {}^p\mathbf{L}[-1])[1]$ *if* $l \leq {}^p w$ *(*$W_{{}^p w} = i^* j_{!*} {}^p\mathbf{L}$*).*
$\mathrm{Gr}_l^W K' = \mathrm{Gr}_l^W i^* j_{!*} {}^p\mathbf{L} = (\mathrm{Gr}_l^W i^* j_{!*} {}^p\mathbf{L}[-1])[1]$.

The weight spectral sequence degenerates at rank 2 as follows:
$E_1^{p,q} = \mathbb{H}^{p+q}(X, \mathrm{Gr}_{-p}^W i^! j_{!*} {}^p\mathbf{L}[1])$ if $-p > {}^p w$
$E_1^{p,q} = \mathbb{H}^{p+q}(X, \mathrm{Gr}_{-p-1}^W i^* j_{!*} {}^p\mathbf{L}[-1])[1]$ if $-p \leq {}^p w$.
For $p = -{}^p w - 1$ and $q = {}^p w$:
$E_1^{-{}^p w-1, {}^p w} = \mathbb{H}^{-1}(X, \mathrm{Gr}_{{}^p w+1}^W i^! j_{!*} {}^p\mathbf{L}[1]) \Longrightarrow \mathrm{Gr}_{{}^p w}^W \mathbb{H}^{-1}(X, i^* j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L})$
$E_1^{-{}^p w, {}^p w} = \mathbb{H}^0(X, \mathrm{Gr}_{{}^p w}^W i^* j_{!*} {}^p\mathbf{L}) \Longrightarrow \mathrm{Gr}_{{}^p w}^W \mathbb{H}^1(X, i^* j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L}[-1])$
The terms are dual, since in both cases the cohomology is with value in $\mathrm{Gr}_0^{W(N)} \mathrm{Coker}\, N = \mathrm{Gr}_0^{W(N)} \mathrm{Ker}\, N$ (see Equation 5.55).

Remark 5.11. The spectral sequence of the mixed cone $K' := C_M(\mathrm{can})$ with a structure of MHC defined by
$W_l K' = W_l i^* j_{!*} {}^p\mathbf{L} = (W_{l-1} i^* j_{!*} {}^p\mathbf{L}[-1])[1] \oplus W_l j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L}$
coincide at rank 2 with the above spectral sequence (see §9.5.1 and the octahedron diagrams (§7.2.0.2 and Equation 10.42)).

5.2.2.1 Thom isomorphism and Gysin morphism

With the previous notations ($Z \subset Y \subset X$ and ${}^p\mathbf{L}$ defined on $X \setminus Y$), let $i_W : W \to X$ be a closed embedding of codimension d transverse to Y and the strata of $\mathcal{S}$ on X, then we have Thom isomorphism

$$i_W^! j_{!*} {}^p\mathbf{L}[2d] \simeq i_W^* j_{!*} {}^p\mathbf{L}$$

(see [GMacP 88] on stratified varieties and [CaMi 5], Lemma 3.5.4).
Let $W^* := W \setminus (Z \cap W)$, then:

$$\mathbb{H}^i(W^*, j_{!*} {}^p\mathbf{L}) \simeq \mathbb{H}^{i+2d}_{W^*}(X \setminus Z, j_{!*} {}^p\mathbf{L}).$$

In particular *Gysin morphism* is compatible with MHS.
Let $W = H$ of codimension 1, the *Thom isomorphism*:

$$\mathbb{H}^i(H \setminus (Z \cap H), j_{!*}\mathbf{L}) \xrightarrow{\sim} \mathbb{H}^{i+2}_{H \setminus (Z \cap H)}(X \setminus Z, j_{!*}\mathbf{L})(1) \tag{5.57}$$

is given by the residue map:

$$IC^* i_H^* \mathcal{L}(\log Z \cap H) \xrightarrow{\sim} i_H^! IC^* \mathcal{L}(\log Z)[2](1).$$

On the one hand we have an isomorphism of the quotient complex with the cohomology with support:

$$IC^*\mathcal{L}(\log Z \cup H)/IC^*\mathcal{L}(\log Z)) \xrightarrow{\sim} i_H^! IC^*\mathcal{L}(\log Z)[1]$$

defined as a connecting isomorphism. On the other hand, the residue map

$$R_H : IC^*\mathcal{L}(\log Z \cup H)[1] \to i_{H*} IC^* i_H^* \mathcal{L}(\log Z \cap H)$$

vanishes on $IC^*\mathcal{L}(\log Z)$ and induces an isomorphism on the quotient.

Chapter 6
Proof of the Decomposition Theorem

The proof of the decomposition theorem (ch.1, Theorem 1.2) and the purity theorem (ch.4, 4.3) is by induction on the dimension of X. We can suppose f surjective, so $\dim V \leq \dim X$. We introduce the following assertion:

$D(n)$: *The decomposition theorem applies to projective morphisms $f : X \to V$ on irreducible varieties X of dimension $\leq n$.*

In particular, in the case $\dim V = 0$, the purity theorem holds for projective varieties X of dimension $\leq n$ by the same induction.

The assertion $D(0)$ is obvious. Let X be non singular of dimension n. The proof is by reduction to a general hyperplane section of V through an isolated stratum v with a NCD as inverse image $f^{-1}(v) := X_v \subset X$. In particular $D(n-1)$ apply by induction on the components of X_v of dimension $n-1$.

Using a Lefschetz pencil of hyperplanes on V, the proof is reduced to the case where there exists a morphism g on V to the projective line such that $g(v) = 0$. Let $j_{!*}{}^p\mathbf{L}$ be on X, the decomposition of $f_* j_{!*}{}^p\mathbf{L}$ is reduced by Verdier's criteria, to the decomposition of $\varphi_g\, {}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L})$ locally at v.

In the case where $g^{-1}g(v)$ is a general hyperplane through v, the weight spectral sequence of $\psi_g\, {}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L})$ (Equation 6.12 below) splits into a direct sum of a spectral sequence of MHS concentrated on v degenerating at rank 2 and a spectral sequence degenerating at rank one (Corollary 6.3).

As a consequence of the splitting of the spectral sequence, we only need the category of IMHS (Definition 2.14) to prove the decomposition.

The decomposition theorem follows from the decomposition of ${}^p\varphi_g\, {}^p\mathcal{H}^i(K)$ (§6.2.0.4) and Verdier's criteria of extension of a perverse sheaf defined on $V \setminus V(g)$ (§2.2.3, Equation 2.18 and §2.2.3.1). Verdier's extension apply in general along a locally principal divisor [V 85-1] but we need only the principal divisor case.

If we consider all possible local projections g, the spectral sequence degenerates at rank 2. Hence, the existence of the category of Hodge $\mathcal{D}-$modules is needed to prove the degeneration at rank 2 (see [Sa 88, Sa 90]), while for a general projection the proof is reduced to the category of IMHS.

The construction of the category of Hodge perverse sheaves is carried, after the proof of the decomposition theorem in the next chapter, which simplify considerably the proof and the exposition.

In chapters 10 and 11, we give another proof of the decomposition, combined with a direct proof of local purity, based on the weight on the logarithmic complex.

6.1 Proof of $D(n-1) \Rightarrow D(n)$ for non-singular X

The proof is divided in two cases according to the dimension of V.

6.1.0.1 The case $0 < \dim V \leq n$

Let $f : X \to V$ be defined on a non singular complex variety X of dimension n, $\mathcal{S}$ a Thom-Whitney stratification of f adapted to $j_{!*}{}^p\mathbf{L}$, V_0 the union of zero-dimensional strata of $\mathcal{S}$ on V, $i_{V_0} : V_0 \to V$, $V^* := V \setminus V_0$.

Assuming $D(n-1, V)$ and $\dim V > 1$, we introduce an *intermediate statement* $D(n, V^*)$, to deduce the decomposition theorem over V^* by considering hyperplane sections of V transversal to the strata $S \in \mathcal{S}$ (avoiding V_0).

Lemma 6.1 ($D(n, V^*)$)**.** *i) Let $K := f_* j_{!*}{}^p\mathbf{L}$. The restriction of the $i-$perverse cohomology of K to V^* decomposes into a direct sum of intermediate extensions of shifted polarized VHS ${}^p\mathbf{L}^i_S$ of weight ${}^pw + i$ over all strata S of dimension > 0.*
Let $k : V^* \to V$, with the notations of (ch. 1, Equations 1.7 and 1.8):

$$ {}^p\mathcal{H}^i(k^*K) \xrightarrow{\sim} \bigoplus_{S \in \mathcal{S},\, S \subset V^*} k^* i_{S!*}{}^p\mathbf{L}^i_S \tag{6.1} $$

ii) The iterated cup-product η with the class of an hyperplane section of X, induces an isomorphism on the restrictions to V^*

$$ \eta^i : {}^p\mathcal{H}^{-i}(f_* j_{!*}{}^p\mathbf{L})_{|V^*} \xrightarrow{\sim} {}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L})_{|V^*} \tag{6.2} $$

Proof. We remark that on a stratum S of dimension $s > 0$, $\mathbf{L}^i_S$ is uniquely defined in the decomposition as $\mathbf{L}^i_S = i^*_S H^{-s}({}^p\mathcal{H}^i(K))$.
The decomposition (6.1) is checked by induction on a general normal section N through a point $s \in S$, of dimension $< \dim V$ transversal to all strata, since perverse cohomology commutes with shifted restrictions to transversal sections.

Remark that $R^{i-s} f_{S*}(i^!_{X_S} j_{!*}{}^p\mathbf{L})$ (resp. $R^{i-s} f_{S*}(i^*_{X_S} j_{!*}{}^p\mathbf{L})$) is a variation of MHS of weights $w \geq {}^pw + i - s$ (resp. $w \leq {}^pw + i - s$) and $\mathbf{L}^i_S$ is a local system

on S, hence the image of I_S is a polarized VHS of weight $w = {}^p w + i - s$ (see also ch. 9, Lemma 9.7).

The isomorphisms in (6.2) follow by induction since the inverse image X_N of N, is non singular of $\dim X_N < \dim X$ as N is general.

Example 6.1. Over the big stratum U in V, the restriction of f to $X_U := f^{-1}(U)$ is smooth and the restrictions of the cohomology sheaves $\mathcal{H}^i(K)$ are locally constant on U, hence ${}^p\mathcal{H}^i(K)_{|U} = \mathcal{H}^{i-\dim V}(K)_{|U}[\dim V]$ is a shifted polarized VHS on U.

By the induction hypothesis $D(n-1)$, the fiber $(R^i f_* j_{!*}\mathbf{L})_v \simeq \mathbb{H}^i(X_v, j_{!*}\mathbf{L})$, at a point $v \in U$, is isomorphic to the intersection cohomology of the restriction of $\mathbf{L}$ to the fiber X_v and Hard Lefschetz applies on X_v.

6.1.1 The crucial case at isolated strata

The main problem is to extend the decomposition over isolated strata in V_0.

Proposition 6.1. *Assume $D(n, V^*)$ and the fiber X_v at $v \in V_0$ is a NCD.*

(i) The decomposition (6.1) of ${}^p\mathcal{H}^i(k^*K)$ over V^* extends to a decomposition over V, including the terms $\mathbf{L}_v^i$ (ch. 1, Equation 1.7)

$$ {}^p\mathcal{H}^i(K) \simeq \oplus_{v \in V_0} \mathbf{L}_v^i \oplus k_{!*} k^* \, {}^p\mathcal{H}^i(K) \tag{6.3} $$

(ii) The Lefschetz morphisms η^i are isomorphisms:

$$ \eta^i : {}^p\mathcal{H}^{-i}(K) \xrightarrow{\sim} {}^p\mathcal{H}^i(K) \tag{6.4} $$

The statement ii) implies the existence of a splitting of K in the derived category $D_c^b(V, \mathbb{Q})$ by Deligne's general argument ([De 68] Théorème 1.5, [De 93]).

We recall that the splitting of K means essentially the degeneration of the spectral sequence with respect to the perverse filtration at the E_1-term. Since the proof is local at v, we suppose V_0 reduced to a single stratum v. Moreover, we suppose first the fiber X_v a NCD in X.

The proof will occupy the rest of the chapter.

6.2 Nearby and vanishing cycles at an isolated stratum

The decomposition at isolated stratum $v \in V_0 \subset V$, is based on the properties of perverse nearby and vanishing cycles along a general hyperplane section H through v.

We suppose first, there exists a morphism $g : V \to \mathbb{P}$ to the complex projective line $\mathbb{P} := \mathbb{P}^1_{\mathbb{C}}$ such that $g(v) = 0$ and $g^{-1}(0) := H$ is transverse to all strata $S \subset V^* = V \setminus v$ as in the diagram

$$\begin{array}{ccccccc} X_v & \hookrightarrow & Z & \overset{i'}{\hookrightarrow} & X & \hookleftarrow & X \setminus Z \\ \downarrow & & \downarrow & & \downarrow f & & \downarrow \\ v & \hookrightarrow & H & \overset{i}{\hookrightarrow} & V & \overset{k'}{\hookleftarrow} & V \setminus H \\ \downarrow & & \downarrow & & \downarrow g & & \downarrow \\ 0 & \to & 0 & \overset{i''}{\hookrightarrow} & \mathbb{P} & \hookleftarrow & \mathbb{P} \setminus 0 \end{array} \tag{6.5}$$

where $Z := (g \circ f)^{-1}(0) = f^{-1}(H) = X_v \cup Z_s$ is a NCD in X, $X_v := f^{-1}(v)$, and Z_s is the strict transform of H defined as the closure of $f^{-1}(H \setminus v)$. For a general section H, Z_s is smooth and $X_v \cap Z_s$ is a NCD in Z_s.

Let I_Z (resp. $I_v \subset I_Z$) denote the set of indices of the components $Z_j \subset Z$ (resp. $Z_j \subset X_v$) and let $s \in I_Z$ corresponds to the component Z_s, hence $I_Z = I_v \cup s$. For each $J \subset I_Z$, let $Z_J = \cap_{j \in J} Z_j$, Z_J^* the regular locus of Z_J and set uniformly $j : Z_J^* \hookrightarrow Z_J$.
We suppose the stratification $\mathcal{S}$ compatible with f, H, Z and the singularities Y of $j_{!*}{}^p\mathbf{L}$. Let $i_S : S \to \bar{S}$ denote the embedding of any stratum S in its closure $\bar{S}$.

6.2.0.1 Decomposition hypothesis

Let (K', W, F) be a perverse VMHS with support on Z (ch. 5, Definition 5.4). By definition

$$(\mathrm{Gr}_r^W K', F) \simeq \oplus_{J \in I_Z} (j_{!*}{}^p\mathbf{L}_r^J, F) \tag{6.6}$$

is a direct sum of intermediate extension of VHS ${}^p\mathbf{L}_r^J$ of weight r on Z_J^*.

The weight spectral sequence: $E_{W,r}^{p,q} \Rightarrow \mathbb{H}^{p+q}(X, K', F)$ defined by W on $R\Gamma(X, K')$ degenerates at rank 2, since the terms $E_{W,1}^{p,q} := \mathbb{H}^{p+q}(X,\ Gr_{-p}^W K', F)$ with their filtrations induced by F are HS of weight q by the assumption $D(n-1)$ applied to intersections of components of Z,

We still denote $(f_{|Z})_*$ by f_*. The derived direct image $(f_* K', W, F)$, where W and F are derived direct image of W and F on K', has support in H.
By definition, (K', W, F) satisfy the *decomposition hypothesis if:*
for each component $j_{!*}{}^p\mathbf{L}_r^J$ of $(\mathrm{Gr}_r^W K', F)$ in equation 6.6, the following non canonical decomposition property is satisfied:

$$f_* j_{!*}{}^p\mathbf{L}_r^J \simeq \oplus_{i \in \mathbb{Z}, S \in \mathcal{S}, S \subset H}\ i_{S!*}({}^p\mathbf{L}_r^J)_S^i[-i] \tag{6.7}$$

where $({}^p\mathbf{L}_r^J)_S^i$ is the component of the decomposition of $f_* j_{!*}{}^p\mathbf{L}_r^J$ according to the notations of (ch 1, Equation 1.7).

6.2.0.2 The cases ${}^p\psi_{g\circ f}j_{!*}{}^p\mathbf{L}$ and ${}^p\varphi_{g\circ f}j_{!*}{}^p\mathbf{L}$

The complex $K' := {}^p\psi_{g\circ f}j_{!*}{}^p\mathbf{L}$ (resp. $K' := {}^p\varphi_{g\circ f}j_{!*}{}^p\mathbf{L}$) is a perverse VMHS by (ch. 5, §5.1.3.3, Proposition 5.5, and Equation 5.42) and we have

$$(\mathrm{Gr}_a^W K', F) \simeq \oplus_{Z_J\subset X_v}((\,{}^p\mathcal{P}_a^J, F) \oplus_{Z_J\cap Z_s} (\,{}^p\mathcal{P}_a^{J\cup s}, F)) \oplus_{Z_s} (\,{}^p\mathcal{P}_a^s, F) \quad (6.8)$$

where ${}^p\mathcal{P}_a^J$ (resp. ${}^p\mathcal{P}_a^{J\cup s}$ and ${}^p\mathcal{P}_a^s$) are intermediate extensions of VHS on Z_J for $J \subset I_v$ (resp. $Z_{J\cup s}$, Z_s) of weight a. Respectively, we have similar decomposition for $K' := {}^p\varphi_{g\circ f}j_{!*}{}^p\mathbf{L}$ (Theorem 5.1).

Since Z_s is transverse to X_v and to the NCD Y singular set of ${}^p\mathbf{L}$, the component ${}^p\mathcal{P}_a^s$ vanishes for $a \neq {}^pw - 1$:

$$(\,{}^p\mathcal{P}_a^s, F) = ((j_{!*}{}^p\mathbf{L})|_{Z_s}[-1], F) \text{ if } a = {}^pw - 1 \text{ and } {}^p\mathcal{P}_a^s = 0 \text{ if } a \neq {}^pw - 1 \quad (6.9)$$

Moreover, since $\dim H < \dim V$, K' satisfy the decomposition hypothesis with respect to the restriction of f to Z_J for all non empty $J \subset I_Z$.

Since $f_*\,{}^p\psi_{g\circ f}j_{!*}{}^p\mathbf{L} \simeq \psi_g f_* j_{!*}{}^p\mathbf{L}$ and $f_*\,{}^p\varphi_{g\circ f}j_{!*}{}^p\mathbf{L} \simeq \varphi_g f_* j_{!*}{}^p\mathbf{L}$ and since ${}^p\psi_g$ (resp. ${}^p\varphi_g$) are exact functors, we define the filtrations on ${}^p\psi_g K$ (resp. ${}^p\varphi_g K$), where $K = f_* j_{!*}{}^p\mathbf{L}$, as follows:

Definition 6.1. The filtrations W and F on ${}^p\psi_g K$ (resp. ${}^p\varphi_g K$) are induced by W and F on f_*K':

$$K' := {}^p\psi_{g\circ f}j_{!*}{}^p\mathbf{L},\ {}^p\psi_g K = f_*K', W_r{}^p\psi_g K := f_*W_rK',\ F^r{}^p\psi_g K := f_*F^rK'$$
$$K' := {}^p\varphi_{g\circ f}j_{!*}{}^p\mathbf{L},\ {}^p\varphi_g K = f_*K', W_r{}^p\varphi_g K := f_*W_rK',\ F^r{}^p\varphi_g K := f_*F^rK'$$

The filtrations W and F on ${}^p\psi_g\,{}^p\mathcal{H}^i(K)$ (resp. ${}^p\varphi_g\,{}^p\mathcal{H}^i(K)$) are induced by W and F on f_*K':

$${}^p\psi_g\,{}^p\mathcal{H}^i(K) \simeq {}^p\mathcal{H}^i(f_*K'), \quad \text{resp. } {}^p\varphi_g\,{}^p\mathcal{H}^i(K) \simeq {}^p\mathcal{H}^i(f_*K')$$

In the case of the vanishing cycles $K' := {}^p\varphi_{g\circ f}j_{!*}{}^p\mathbf{L}$, all the terms of the decomposition are supported by X_v since H is general through v, hence K' is a perverse VMHS supported by X_v and ${}^p\varphi_g K = f_*K'$ is supported by v.

Proposition 6.2. *Let $K := f_*j_{!*}{}^p\mathbf{L}$. The isomorphism*

$${}^p\mathcal{H}^i(\,{}^p\psi_g K) \simeq {}^p\mathcal{H}^i(f_*\,{}^p\psi_{g\circ f}j_{!*}{}^p\mathbf{L}) \ \textit{(resp. } {}^p\mathcal{H}^i(f_*K') \simeq {}^p\varphi_g\,{}^p\mathcal{H}^i(K)\textit{)}$$

transports the action of ν on K' to the action of the monodromy logarithm N on the right hand term. Let η denote the cup-product by the class of the hyperplane section of X, we have:

$$\eta^j : \mathrm{Gr}_a^W\,{}^p\mathcal{H}^{-j}(\,{}^p\psi_g K) \xrightarrow{\sim} \mathrm{Gr}_a^W\,{}^p\mathcal{H}^j(\,{}^p\psi_g K)(j) \ \textit{(resp. in the case of } {}^p\varphi_g K\textit{)}$$
$$N^i : \mathrm{Gr}^W_{{}^pw-1+j+i}\,{}^p\mathcal{H}^j(\,{}^p\psi_g K) \xrightarrow{\sim} \mathrm{Gr}^W_{{}^pw-1+j-i}\,{}^p\mathcal{H}^j(\,{}^p\psi_g K)(-i)$$
$$\textit{resp. } N^i : \mathrm{Gr}^W_{{}^pw+j+i}\,{}^p\mathcal{H}^j(\,{}^p\varphi_g K) \xrightarrow{\sim} \mathrm{Gr}^W_{{}^pw+j-i}\,{}^p\mathcal{H}^j(\,{}^p\varphi_g K)(-i)$$

We prove first the lemma:

Lemma 6.2. *i) Let* $K' := {}^p\psi_{g\circ f} j_{!*}{}^p\mathbf{L}$ *and* $w_0 := {}^pw - 1$. *If* $a \neq w_0$:

$$f_*\mathrm{Gr}_a^W K' = \oplus_{i\in\mathbb{Z}}\, {}^p\mathcal{H}^i(f_*\ Gr_a^W K')[-i] = \oplus_{i\in\mathbb{Z}} A_a^i[-i]$$

where A_a^i *are PHS of weight* $a+i$ *supported by* v.

Let $a = w_0$, Z_s *the strict transform of* H *and* $u : (H \setminus v) \to V$:

$$\begin{aligned} &f_*\mathrm{Gr}_{w_0}^W K' = \oplus_{i\in\mathbb{Z}}\, {}^p\mathcal{H}^i(f_*\mathrm{Gr}_{w_0}^W K')[-i] = \\ &\oplus_{i\in\mathbb{Z}}\, (A_{w_0}^i[-i] \oplus B_{w_0}^i[-i] \oplus u_{!*}u^*\, {}^p\mathcal{H}^i((f|_{Z_s})_* j_{!*}{}^p\mathbf{L}|_{Z_s}[-1])[-i]) \end{aligned} \tag{6.10}$$

where $B_{w_0}^i$ *is the polarized HS of weight* w_0+i, *component of* ${}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L}|_{Z_s}[-1])$ *with support* v, $A_{w_0}^i$ *is also a polarized HS of weight* w_0+i *with support* v *and* $u_{!*}u^*\, {}^p\mathcal{H}^i(f_*(j_{!*}{}^p\mathbf{L}|_{Z_s}[-1]) \simeq u_{!*}u^*\, {}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L})|_H[-1]$ *is a direct sum of intermediate extensions of polarized VHS of weight* w_0+i *supported by* H.

i′) Respectively, let $K' := \varphi_{g\circ f} j_{!*}{}^p\mathbf{L}$,

$$f_*\mathrm{Gr}_a^W K' = \oplus_{i\in\mathbb{Z}}\, {}^p\mathcal{H}^i(f_*\mathrm{Gr}_a^W K')[-i] = \oplus_{i\in\mathbb{Z}} C_a^i[-i]$$

where C_a^i *are polarized HS of weight* $a+i$ *supported by* v.

Remark 6.1. To follow the proof, we explain the indices in the lemma and the degeneration at rank 1 on the restriction to $V \setminus v$. By transversality, the restriction $i^*\, {}^p\mathcal{H}^j(f_* j_{!*}{}^p\mathbf{L})[-1]$ to $H \setminus v$ is perverse, direct sum of shifted VHS of pure weight $j + {}^pw - 1 = j + w^0$ over the strata.

By induction, the derived direct image $f_* i_{Z_s}^* j_{!*}{}^p\mathbf{L}[-1]$ is of weight w_0 and decomposes into a direct sum of shifted ${}^p\mathcal{H}^j(f_* i_{Z_s}^* j_{!*}{}^p\mathbf{L}[-1])$ of pure weight $j+w_0$ whose restriction to $H\setminus v$ coincide with $u^*\, {}^p\mathcal{H}^j(f_* j_{!*}{}^p\mathbf{L})[-1]$ by transversality, which is consistent with the degeneration at rank 1.

By the decomposition by induction of $Z_s \to H$, we know more: the weight spectral sequence on H degenerates to $u_{!*}u^*\, {}^p\mathcal{H}^j(f_{|Z_s *} j_{!*}{}^p\mathbf{L}_{|Z_s}[-1]) \oplus B_{w_0}^j$. The spaces $B_{w_0}^j$ are pure HS of weight $w_0 + j$ with support on v.

Proof. The Hodge filtration F is induced on ${}^p\mathcal{H}^i(f_*\mathrm{Gr}_a^W K')$ as explained in (ch.1, §2.4). The lemma follows from (Equations 6.8, 6.9) as the terms A_a^i are the image of terms with support in X_v, except the derived image $f_*(j_{!*}{}^p\mathbf{L}|_{Z_s}[-1])$ which has support on H. In the case $a = w_0$ we have, by induction, the decomposition on H (see Equations 6.7 and 6.9) :

$${}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L}|_{Z_s}[-1]) \simeq u_{!*}u^*\, {}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L}|_{Z_s}[-1]) \oplus ({}^p\mathbf{L}|_{Z_s}[-1])_v^i.$$

Recall that $u^*\, {}^p\mathcal{H}^i(f_*(j_{!*}{}^p\mathbf{L}|_{Z_s}[-1]) \simeq u^*\, {}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L})|_H[-1]$ by transversality of H except at v. With the notations of (Equations 6.7, 1.7), set: $B_{w_0}^i := ({}^p\mathbf{L}|_{Z_s}[-1])_v^i$.

Lemma 6.3 (Splitting of the spectral sequence). *i) The perverse weight spectral sequence of* (f_*K', W, F)

$$E_1^{p,q} := {}^p\mathcal{H}^{p+q}(\mathrm{Gr}_{-p}^W f_*K') \Rightarrow \mathrm{Gr}_q^W\, {}^p\mathcal{H}^{p+q}(f_*K') \tag{6.11}$$

splits into the direct sum of two spectral sequences:

$$E_1^{p,q} = E_1^{p,q}(v) \oplus E_1^{p,q}(H) \tag{6.12}$$

where $E(v)$ is a spectral sequence of HS supported by v which degenerates at the E_2-term while $E(H)$ is supported by H and degenerates at the E_1-term.

i′) Respectively, if $K' := {}^p\varphi_{g\circ f} j_{!*}{}^p\mathbf{L}$, the perverse weight spectral sequence $E_1^{p,q}$ is supported by v and degenerates at rank 2.

ii) $$\mathrm{Gr}_q^W\, {}^p\mathcal{H}^{q-w_0}(f_*K') \simeq E_2^{-w_0,q}(v) \oplus u_{!*}u^*\, {}^p\mathcal{H}^{q-w_0}(f_{|Z_s*}j_{!*}{}^p\mathbf{L}_{|Z_s}[-1]) \quad ,$$
$$\mathrm{Gr}_q^W\, {}^p\mathcal{H}^{p+q}(f_*K') \simeq E_2^{p,q}(v) \quad \textit{if } p \neq -w_0.$$

$E_2^{p,q}(v)$ is a HS supported by v and $u_{!*}u^*\, {}^p\mathcal{H}^{q-w_0}(f_*j_{!*}{}^p\mathbf{L}_{|Z_s}[-1])$ is an intermediate extension of a VHS.

ii′) Similarly, $K' := {}^p\varphi_{g\circ f} j_{!*}{}^p\mathbf{L}$ is supported by v:

$$\mathrm{Gr}_q^W\, {}^p\mathcal{H}^{p+q}(f_*K') \simeq E_2^{p,q} \quad \textit{for all } p, q.$$

Proof. i) The weight filtration W_i on ${}^p\mathcal{H}^j(f_*K')$ is defined by the perverse image of ${}^p\mathcal{H}^j(f_*W_i) \to {}^p\mathcal{H}^j(f_*K')$, while the Hodge filtration is defined by restriction as explained in (ch. 1, §2.4).

$$E_1^{p,q} = {}^p\mathcal{H}^{p+q}(Gr_{-p}^W f_*K') \xrightarrow{d_1} E_1^{p+1,q} = {}^p\mathcal{H}^{p+q+1}(Gr_{-p-1}^W f_*K')$$
$$E_1^{p,q} \Longrightarrow Gr_q^W\, {}^p\mathcal{H}^{p+q}(f_*K')$$

The differentials on the terms $E_1^{p,q}$ are written as follows:

$$\begin{aligned} E_1^{-w_0-1,q} &\xrightarrow{d_1} E_1^{-w_0,q} \xrightarrow{d_1} E_1^{-w_0+1,q} \\ E_1^{-w_0,q} &= E_1^{-w_0,q}(v) \oplus u_{!*}u^*\, {}^p\mathcal{H}^{q-w_0}(f_*j_{!*}{}^p\mathbf{L}|_{Z_s}[-1]) \\ E_1^{-w_0-1,q} &= E_1^{-w_0-1,q}(v), \quad E_1^{-w_0+1,q} = E_1^{-w_0+1,q}(v) \end{aligned} \tag{6.13}$$

Set

$$E_1^{p,q}(v) := A_{-p}^{p+q} \quad \text{if } p \neq -w_0,\ E_1^{-w_0,q}(v) := A_{w_0}^{q-w_0} \oplus B_{w_0}^{q-w_0} \tag{6.14}$$

then $E_1^{p,q}(v)$ consists of HS in degree 0 supported by v, while

$$E_1^{-w_0,q}(H) := u_{!*}u^*\, {}^p\mathcal{H}^{q-w_0}(f_*j_{!*}{}^p\mathbf{L}|_{Z_s}[-1]),\ E_1^{p,q}(H) = 0 \quad \text{if } p \neq -w_0. \tag{6.15}$$

Since $H^0(i_v^* u_{!*}u^*\, {}^p\mathcal{H}^{q-w_0}(f_*j_{!*}{}^p\mathbf{L}_{|Z_s}[-1]) = 0$, the induced differentials by d_1 on $E_1^{p,q}(H)$ vanish.

The spectral sequence of HS $E_1^{p,q}(v)$ degenerates at rank 2 ([De 72]) while the spectral sequence $E_1^{p,q}(H)$ degenerates at rank 1 since its terms vanish if $p \neq -w_0$. The statement ii') is clear.

Proof of the proposition 6.2. Concerning the terms $E_1^{p,q}(H)$, the statement is part of the decomposition theorem by induction over H and the degeneration at rank one of $E_1^{p,q}(H)$:

1) N induces 0 on ${}^p\mathcal{H}^i(f_*(j_{!*}{}^p\mathbf{L}|_{Z_s}[-1])$

2) $\eta^j : {}^p\mathcal{H}^{-j}(f_*j_{!*}{}^p\mathbf{L}|_{Z_s}[-1]) \simeq {}^p\mathcal{H}^j(f_*j_{!*}{}^p\mathbf{L}|_{Z_s}[-1])(j)$ are isomorphisms, hence including the terms $B^i_{w_0} := ({}^p\mathbf{L}|_{Z_s}[-1])^i_v$.

Concerning the statement of the terms $E_2^{p,q}(v)$, the following properties of the terms $E_1^{p,q}(v)$ are satisfied also for $a = w_0$:

1) $N^a : A^i_a \simeq A^i_{-a}(-a)$

2) $\eta^j : A^{-j}_a \simeq A^j_a(j)$

We refer to ([Sa 88] Proposition 4.2.2) to deduce the corollary as follows. For symmetry reasons, set

$$L^q_p := E_1^{-p,p+q}(v) \tag{6.16}$$

The following *properties of bigraded polarized HS* are satisfied (see §5.2.1.1):

1) L^q_p is a HS of weight $p+q$,

2) The cup product with the class of an hyperplane section of X induces a morphism $l : L^q_p \to L^{q+2}_p(1)$ such that $l^q : L^{-q}_p \xrightarrow{\sim} L^q_p(q)$ is an isomorphism for $q \geq 0$ by Hard Lefschetz on the components of Z.

3) The endomorphism ν induces the monodromy morphism $\nu : L^q_p \to L^q_{p-2}(-1)$ which commutes with l: $\nu \circ l = l \circ \nu$, such that $\nu^p : L^q_p \xrightarrow{\sim} L^q_{-p}(-p)$ is an isomorphism for $p \geq 0$ by the property of the monodromy filtration $W(\nu)$.

4) We deduce from the autoduality of $\mathbf{L}$ and the auto-duality of the monodromy filtration, a scalar product polarizing the HS

$$S : L^{-q}_{-p} \times L^q_p \to \mathbb{Q}(-w-n+1)$$

5) The differential d_1 on the terms E_1 of the spectral sequence induces a morphism $d : L^q_p \to L^{q+1}_{p-1}$ satisfying $d^2 = 0$, $d \circ l = l \circ d$, $d \circ \nu = \nu \circ d$ and $S(dx, y) = \pm S(x, dy)$.

6.2.0.3 Reduction to a fibration by g

The following reduction is a method introduced by Grothendiek's school ([De 80] §4.2) based on Lefschetz pencils and will be used also later in chapter 8. Let $(H_t, t \in \mathbb{P})$ be a general pencil of hyperplane sections of V with axis A of codimension 2 away from v. The only section H_0 through v is transverse to the strata of a Whitney stratification $\mathcal{S}$ of $V \setminus v$.

We set $H := H_0$. The sections H_t, for $t \neq 0$ varying in a small disc $D \subset \mathbb{P}$ with center 0, intersect the strata of a Whitney stratification $\mathcal{S}$ of V transversally (H_t is normally embedded in $V \setminus v$).

The blow-up $\widetilde{V}$ of V along A is the variety $\widetilde{V}$ of disjoint union of strict transforms $\widetilde{H}_t \simeq H_t$ of H_t with natural projections on V and $\mathbb{P}$

$$\widetilde{V} = \{(x,t) : x \in X, t \in \mathbb{P}, x \in H_t\},\ p_1 : \widetilde{V} \to V,\ p_2 : \widetilde{V} \to \mathbb{P}, p_1 : \widetilde{H}_t \simeq H_t.$$

The projection p_2 defines a fibration on $\widetilde{V}$ onto $\mathbb{P}^1$, while the projection p_1 induces an isomorphism $\widetilde{V} \setminus p_1^{-1}(A) \simeq V \setminus A$. Hence there exists a point $\widetilde{v} \in \widetilde{V}$ over v such that p_1 projects a neighborhood $B_{\widetilde{v}}$ of $\widetilde{v}$ isomorphically onto a neighborhood B_v of v.
We consider the fiber product morphism $\widetilde{f} : \widetilde{X} \to \widetilde{V}$ on $\widetilde{X} := X \times_V \widetilde{V}$. Since the morphism remains unchanged over a neighborhood of v, the decomposition of $\widetilde{f}$ at v is equivalent to the decomposition of f.

6.2.0.4 Decomposition of ${}^p\varphi_g\, {}^p\mathcal{H}^i(K)$

Let $K := f_* j_{!*} {}^p\mathbf{L}$. The decomposition of ${}^p\varphi_g\, {}^p\mathcal{H}^i(K)$ below (Equation 6.17) follows from (Corollary 6.2) by a method due to M. Saito ([Sa 88], Lemma 5.2.15, see also [Ka 86] Proposition 3.3.1 for example).

Lemma 6.4. *In the diagram of morphisms of perverse sheaves*

$$ {}^p\psi_g\, {}^p\mathcal{H}^i(K) \overset{\text{can}}{\to} {}^p\varphi_g\, {}^p\mathcal{H}^i(K) \overset{\text{var}}{\to} {}^p\psi_g\, {}^p\mathcal{H}^i(K) $$

${}^p\varphi_g\, {}^p\mathcal{H}^i(K)$*, supported by* v *in degree* 0*, is a MHS direct sum of two sub-MHS:*

$$ {}^p\varphi_g\, {}^p\mathcal{H}^i(K) \simeq \operatorname{Im}\text{can} \oplus \operatorname{Ker}\text{var} \tag{6.17} $$

Since H is general, ${}^p\varphi_g^u\, {}^p\mathcal{H}^i(K)$ is a MHS supported by v in degree 0 (Proposition 6.2 and Corollary 6.3). The decomposition is in the category of IMHS as it occurs on degree 0:

$$ H^0({}^p\psi_g\, {}^p\mathcal{H}^i(K)) \overset{\text{can}}{\to} H^0({}^p\varphi_g\, {}^p\mathcal{H}^i(K)) \overset{\text{var}}{\to} H^0({}^p\psi_g\, {}^p\mathcal{H}^i(K)) $$

The proof is carried on $\operatorname{Gr}_*^W {}^p\varphi_g^u({}^p\mathcal{H}^i(K)$. The morphism *can* vanishes on the summand of $\operatorname{Gr}_*^W {}^p\psi_g^u({}^p\mathcal{H}^i(K)$ with support on H with cohomology 0 in degree 0 as intermediate extensions.

For each $m \in \mathbb{Z}$, let $H_\psi^m(v)$ denote $\operatorname{Gr}_m^{W(N)} H^0({}^p\psi_g^u\, {}^p\mathcal{H}^i(K))$ with support on v. Then $H_\psi^m(v)$ is a polarized HS of weight ${}^pw - 1 + i + m$. Similarly, let $H_\phi^{m-1} := \operatorname{Gr}_{m-1}^{W(N)} {}^p\varphi_g^u\, {}^p\mathcal{H}^i(K)$. Then H_ϕ^{m-1} is of weight ${}^pw + i + m - 1$.

For all $m \in \mathbb{Z}$ we have diagrams $H_\psi^m \xrightarrow{C} H_\phi^{m-1} \xrightarrow{V} H_\psi^{m-2}(-1)$ induced by the canonical and the variation morphisms. Such diagrams of polarized HS satisfy the conditions of ([Sa 88], Lemma 5.2.15) by (Corollary 6.2). Hence for all m we have a decomposition: $H_\phi^{m-1} = \operatorname{Im} C \quad \oplus \quad \operatorname{Ker} V$ from which we deduce the decomposition of ${}^p\phi_g^u$, since Imcan and $\ker$var are sub-MHS.

6.2.1 Decomposition of ${}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L})$

The decomposition is deduced by Verdier's criteria on the extension of perverse sheaves in terms of the diagram defined by the canonical and variation morphisms (Equation 2.22). We stress the splitting of the filtration F.

By the reduction to a general hyperplane section, the proof of the decomposition theorem, following the decomposition of the spectral sequence, is limited to the use of the abelian category of IMHS.

Lemma 6.5. *Let* $K := f_*j_{!*}{}^p\mathbf{L}, i : H \to V, k : V \setminus v \to V$.
i) There exists a canonical splitting

$$({}^p\mathcal{H}^i(K), F) = (L_v^i, F) \oplus (k_{!*}k^*\,{}^p\mathcal{H}^i(K), F) \tag{6.18}$$

where: (L_v^i, F) *is a pure HS supported by* v *and* $k_{!*}k^*\,{}^p\mathcal{H}^i(K)$ *is an intermediate extension of a polarized VHS on* $V \setminus v$ *of weight* ${}^pw + i$.
Moreover we have induced Hodge Lefschetz isomorphisms
$\eta^i : i^*({}^p\mathcal{H}^{-i}(K), F) \to i^*({}^p\mathcal{H}^i(K), F)$.

Proof. Let $k' : V \setminus H \to V$. The proof is based on Verdier's description of the extensions of perverse sheaves by the equivalence of the categories $\mathrm{Perv}(X)$ and $V(H)$ (§2.2.3, Equation 2.18). The decomposition $\varphi \simeq \mathrm{Im}\,can \oplus \mathrm{Ker}\,var$ (Equation 6.17) corresponds to the extension over H

$$k'_{!*}k'^*\,{}^p\mathcal{H}^i(K) \oplus \mathrm{Ker}\,var.$$

Let $k'' : V \setminus H \to V \setminus v$, hence $k' = k \circ k''$. By construction:

$$k'_{!*}k'^*\,{}^p\mathcal{H}^i(K) = k_{!*}(k''_{!*}k'^*\,{}^p\mathcal{H}^i(K))$$

(first we extend to $V \setminus v$, then to V according to the stratification defined by H and v). Since by hypothesis H is transversal to the stratification away from v: $k''_{!*}k'^*\,{}^p\mathcal{H}^i(K) = k^*\,{}^p\mathcal{H}^i(K)$, hence:

$$k'_{!*}k'^*\,{}^p\mathcal{H}^i(K) = k_{!*}k''_{!*}k'^*\,{}^p\mathcal{H}^i(K) = k_{!*}k^*\,{}^p\mathcal{H}^i(K).$$

With the notations of (ch. 1, Equation 1.7) and by the uniqueness of the decomposition $L_v^i := \mathrm{Ker}\,\mathrm{var}$ is a pure HS and by induction $k^*\,{}^p\mathcal{H}^i(K)$ is a direct sum of intermediate extensions, which proves (Equation 6.18).

The morphisms $\eta^i : i^*({}^p\mathcal{H}^{-i}(K), F) \to i^*({}^p\mathcal{H}^i(K), F)$ are isomorphisms since they are already isomorphisms on ${}^p\psi_g\,{}^p\mathcal{H}^{-i}(K)$ and on ${}^p\varphi_g\,{}^p\mathcal{H}^{-i}(K)$, which ends the proof of the lemma.

Remark 6.2. In view of the decomposition 6.17 the proof is direct as follows:

Lemma 6.6. *i)* $\mathrm{Ker}\,var \simeq \mathcal{H}^0(i_v^*\,{}^p\mathcal{H}^i(K))$
ii) $\mathcal{H}^k(i_v^*\,{}^p\mathcal{H}^i(K)) = \mathcal{H}^{k+1}(i_v^{*p}\psi_g^u\,{}^p\mathcal{H}^i(K))$ *if* $k < -1$
iii) $0 \to \mathcal{H}^{-1}(i_v^*\,{}^p\mathcal{H}^i(K)) \to \mathcal{H}^0(i_v^{*p}\psi_g^u\,{}^p\mathcal{H}^i(K)) \to \mathrm{Im}\,can \to 0$

where the MHS $\mathrm{Im}\,can$ *is determined exactly by the terms* $E_2^{p,q}(v)$ *supported by* v *(Lemma 6.3, ii) while the split vector space* L_v^i *is* $\mathrm{Ker}\,var$.

Since ${}^p\varphi_g^u\,{}^p\mathcal{H}^i(K)$ is the cone of $sp : i^*\,{}^p\mathcal{H}^i(K)[-1] \to {}^p\psi_g^u\,{}^p\mathcal{H}^i(K)$, we have a distinguished triangle:

$$ {}^p\psi_g^u\,{}^p\mathcal{H}^i(K) \overset{can}{\to} {}^p\varphi_g^u\,{}^p\mathcal{H}^i(K)) \overset{[+1]}{\to} i^*\,{}^p\mathcal{H}^i(K) \to \tag{6.19} $$

and its long perverse exact sequence

$$ 0 \to {}^p\mathcal{H}^{-1}(i^*\,{}^p\mathcal{H}^i(K)) \to {}^p\psi_g^u\,{}^p\mathcal{H}^i(K) \overset{can}{\to} {}^p\varphi_g^u\,{}^p\mathcal{H}^i(K) \to {}^p\mathcal{H}^0(i^*\,{}^p\mathcal{H}^i(K)) \to 0 $$

Since ${}^p\varphi_g^u\,{}^p\mathcal{H}^i(K) \simeq \operatorname{Im} can \oplus \operatorname{Ker} var$ is a vector space supported by v, we deduce the following isomorphism and short exact sequences:

$$ \operatorname{Ker} var \simeq {}^p\mathcal{H}^0(i^*\,{}^p\mathcal{H}^i(K)) $$

$$ 0 \to {}^p\mathcal{H}^{-1}(i^*\,{}^p\mathcal{H}^i(K)) \to {}^p\psi_g^u\,{}^p\mathcal{H}^i(K) \to \operatorname{Im} can \to 0 $$

We remark also, since var is a monomorphism on $\operatorname{Im} can$ and $var \circ can = N$:

$$ \operatorname{Im} can \xrightarrow{var\sim} \operatorname{Im}(N : {}^p\psi_g^u\,{}^p\mathcal{H}^i(K) \to {}^p\psi_g^u\,{}^p\mathcal{H}^i(K)) $$

$$ {}^p\mathcal{H}^{-1}(i^*\,{}^p\mathcal{H}^i(K)) \simeq \operatorname{Ker}(N : {}^p\psi_g^u\,{}^p\mathcal{H}^i(K) \to {}^p\psi_g^u\,{}^p\mathcal{H}^i(K)) $$

Then, by (Equation 5.49) and ([BBD 83] corollary 4.1.12):

$$ {}^p\mathcal{H}^{-1}(i^*\,{}^p\mathcal{H}^i(K)) \simeq i^*(k'_{!*}k'^*\,{}^p\mathcal{H}^i(K))[-1] = \operatorname{Ker}(N|{}^p\psi_g^u\,{}^p\mathcal{H}^i(K). $$

Since ${}^p\mathcal{H}^0(i^*\,{}^p\mathcal{H}^i(K)) \simeq \operatorname{Ker} var = \mathcal{H}^0(i_v^*\,{}^p\mathcal{H}^i(K))$ is supported by v, we deduce ${}^p\tau_{\leq -1} i^*\,{}^p\mathcal{H}^i(K) = \tau_{\leq -1} i^*\,{}^p\mathcal{H}^i(K) = \operatorname{Im} can$ and

$$ i_v^*\,{}^p\mathcal{H}^i(K) \simeq \tau_{\leq -1} i^*\,{}^p\mathcal{H}^i(K) \oplus \mathcal{H}^0(i_v^*\,{}^p\mathcal{H}^i(K). $$

Moreover, we extract from the long cohomology exact sequence, associated to the triangle (Equation 6.19) restricted to v, the exact sequence (iii) of the lemma.

Lemma 6.7 (Splitting of the filtration F). *The decomposition of $i^*\,{}^p\mathcal{H}^i(K)$ splits the induced Hodge filtration.*

At this point, we did prove the decomposition into the direct sum of the perverse cohomologies. The splitting of the induced filtration F into the induced filtrations on each term is a consequence of the existence of MHS on $i_v^*\,{}^p\mathcal{H}^i(K)$.

By (Lemma 6.3) each term ψ_g^u and φ_g^u has an induced MHS. The MHS on $i_v^*\,{}^p\mathcal{H}^i(K) = i_v^* i_H^*\,{}^p\mathcal{H}^i(K)$ is deduced from the mixed cone of the morphism $can : \psi_g^u \to \varphi_g^u$. The decomposition in (Equation 6.17) is compatible with MHS. Since $L_v^i = \operatorname{Ker} var$ has a pure HS and its embedding is compatible with the MHS, we deduce the splitting of the MHS and consequently the splitting of the filtration F at v.

Remark 6.3. The MHS on $i_v^*\,{}^p\mathcal{H}^i(K) = i_v^* i_H^*\,{}^p\mathcal{H}^i(K)$ may be defined by desingularization and compatibility of MHS with the perverse filtration.

Such compatibility of Hodge and perverse filtrations is a general property on various cohomology groups on subspaces of V and will be repeatedly useful in the text.

6.2.2 Hodge structure on $\mathbb{H}^*(X, j_{!*}{}^p\mathbf{L})$ ($\dim V = 0$)

We refer to ([Zu 79]) in the case where $\dim V = 0$ and $\dim X = 1$.

If $\dim X > 1$, we use a Lefschetz fibration by a pencil of hyperplane section to reduce the proof to the case where f admits a factorization by a morphism to the projective complex line and the constant map:

$$X \xrightarrow{p_2} \mathbb{P} \xrightarrow{c_{\mathbb{P}}} \operatorname{spec} \mathbb{C}, \quad \mathrm{f} = \mathrm{c}_{\mathbb{P}} \circ \mathrm{p}_2$$

Let $(X_t, t \in \mathbb{P})$ be a pencil of hyperplane sections of X with axis A of codimension 2. The disjoint union of the hyperplane sections X_t forms an algebraic variety known as the blow-up $\widetilde{X}$ of X along A with projections p_1 and p_2

$$\widetilde{X} = \{(x,t) : x \in X_t, t \in \mathbb{P}\}, p_1 : \widetilde{X} \to X, p_2 : \widetilde{X} \to \mathbb{P}, p_1 : \widetilde{X}_t \simeq X_t$$

([De 80], 4.3.1). Let $j' : U \to \widetilde{X}$ denote the embedding of an open subset of definition of the polarized VHS $\mathbf{L}$. If the pencil is general with axis A transverse to a stratification of $(X, j_{!*}{}^p\mathbf{L})$, then $j'_{!*}{}^p\mathbf{L} \simeq p_1^* j_{!*}{}^p\mathbf{L}$ and $p_{1*} j'_{!*}{}^p\mathbf{L} \simeq p_{1*} p_1^* j_{!*}{}^p\mathbf{L}$ since p_1 is a fibration by simply connected projective lines $\mathbb{P}$: The decomposition follows a from §6.1.0.1 since $\dim \widetilde{X} = \dim X = n$, hence we have in the derived category:

$$p_{1*} j'_{!*}{}^p\mathbf{L} \simeq j_{!*}{}^p\mathbf{L} \oplus {}^p\mathcal{H}^2(p_{1*} j'_{!*}{}^p\mathbf{L})[-2] \simeq j_{!*}{}^p\mathbf{L} \oplus i_A^* j_{!*}{}^p\mathbf{L}[-2] \tag{6.20}$$

We have: $\mathrm{Gr}_2^{{}^p\tau} \mathbb{H}^k(\widetilde{X}, j'_{!*}{}^p\mathbf{L}, F) \simeq \mathbb{H}^{k-2}(A, j_{!*}{}^p\mathbf{L}, F)$ and ${}^p\tau_0 \mathbb{H}^k(\widetilde{X}, j'_{!*}{}^p\mathbf{L}, F) \simeq \mathbb{H}^k(X, j_{!*}{}^p\mathbf{L}, F)$.

First we deduce a HS on $(\mathbb{H}^k(\widetilde{X}, j'_{!*}{}^p\mathbf{L}), F)$ since the decomposition theorem apply to p_2 and [Zu 79] applies to all intermediate extensions of polarized VHS of the perverse cohomologies of $(p_{2,*} j'_{!*}{}^p\mathbf{L}, F)$ on $\mathbb{P}$. Hence

1. $({}^p\tau_2 \mathbb{H}^k(\widetilde{X}, j'_{!*}{}^p\mathbf{L}), F) = (\mathbb{H}^k(\widetilde{X}, j'_{!*}{}^p\mathbf{L}), F)$ is a HS.
2. ${}^p\tau_1 \mathbb{H}^k(\widetilde{X}, j'_{!*}{}^p\mathbf{L}), F) = (\mathbb{H}^k(X, j_{!*}{}^p\mathbf{L}), F)$

Moreover, and $\mathrm{Gr}_2^{{}^p\tau} \mathbb{H}^k(\widetilde{X}, j'_{!*}{}^p\mathbf{L}), F) \simeq (\mathbb{H}^{k-2}(A, j_{!*}{}^p\mathbf{L}), F)$ is a HS by induction since $\dim A < \dim X$. Hence $(\mathbb{H}^k(X, j_{!*}{}^p\mathbf{L}), F)$ is a HS as the kernel of a morphism of HS. Recall, that the decomposition is natural for constant coefficients (see [Vo 2007], Theorem 7.31).

Corollary 6.1. *An hyperplane section defines a polarization on* $\mathbb{H}^k(X, j_{!*}{}^p\mathbf{L})$. *Let* η *denote the cup-product with the class of a hyperplane section of* X, *then* η^i *defines an isomorphism:* $\mathbb{H}^{-i}(X, j_{!*}{}^p\mathbf{L}) \xrightarrow{\sim} \mathbb{H}^i(X, j_{!*}{}^p\mathbf{L})$.

Remark 6.4. The HS on $\mathbb{H}^k(X, j_{!*}{}^p\mathbf{L})$ coincides with the HS defined in [KaK 87], [CaKSc 87] and [De 71]. The proof follows from [KaK 86] where the HS are shown to coincide due to the property of auto-duality.

This ends the induction process on $\dim X$ *for non singular* X

6.2.2.1 Purity on a singular variety X

We complete the proof of the purity theorem in the case of a singular variety X (ch. 4, §4.1.2, Theorem 4.3) by reduction to a complete non-singular algebraic variety X'. We consider the finite covering of the variety X by the disjoint union of its irreducible components to reduce the proof to the case where X is irreducible and of dimension n and $\mathbf{L}$ a polarized VHS of weight w' on a non singular Zariski-open subset $U \subset X$. The weight of ${}^p\mathbf{L} := \mathbf{L}[\dim X]$ is ${}^pw := w' + n$.

Let $Y = X \setminus U$, $\pi : X' \to X$ a non singular modification of X such that $Y' := \pi^{-1}(Y)$ is a NCD. Let $j' : U \hookrightarrow X'$, then $j'_{!*}{}^p\mathbf{L}$ is endowed with a Hodge filtration F induced by the isomorphism with $IC^{*p}\mathcal{L}$ on X'.

There exists on ${}^p\mathcal{H}^0(\pi_* j'_{!*}{}^p\mathbf{L})$ a Hodge filtration F induced by the push-forward filtration $\pi_* F$. The Hodge filtration F on $j_{!*}{}^p\mathbf{L}$ is induced by the decomposition of ${}^p\mathcal{H}^0(\pi_* j'_{!*}{}^p\mathbf{L})$ compatible with F. This Hodge filtration in the derived filtered category $D^bF(X, \mathbb{C})$ is independent of the choice of the designgularization X' up to isomorphism.

Lemma 6.8. *The perverse filtration ${}^p\tau$ on $\mathbb{H}^k(X', j'_{!*}{}^p\mathbf{L})$ with respect to the desingularization map $\pi : X' \to X$ is a filtration by sub-HS.*

The definition of the filtration F is obtained by the decomposition theorem applied to the morphism $\pi : X' \to X$.
As in the previous case, ${}^p\tau_0\mathbb{H}^k(X', j'_{!*}{}^p\mathbf{L})$ is a Hodge sub-structure since the quotient $\mathbb{H}^k(X', j'_{!*}{}^p\mathbf{L})/{}^p\tau_0\mathbb{H}^k(X', j'_{!*}{}^p\mathbf{L})$ is a HS by induction on dimension, as the components of the decomposition over strata of dimension strictly lower than $\dim X$ appears in the quotient.
By a similar argument ${}^p\tau_{-1}\mathbb{H}^k(X', j'_{!*}{}^p\mathbf{L})$ is a sub-HS, hence $Gr_0^{{}^p\tau}\mathbb{H}^k(X', j'_{!*}{}^p\mathbf{L})$ is a HS which contains $\mathbb{H}^k(X, j_{!*}{}^p\mathbf{L})$ as a natural sub-HS (see also a general method to prove that ${}^p\tau$ is a filtration by sub-HS (Equation 10.7)).

Corollary 6.2 (Purity Theorem on a singular variety X). *Let $\mathbf{L}$ be defined on a Zariski open subset of an algebraic complex and compact variety. Then, the Hodge filtration F induces a HS pf weight ${}^pw + k$ on $\mathbb{H}^k(X, j_{!*}{}^p\mathbf{L})$.*

For any complex algebraic variety, there exists a birational morphism $g : X' \to X$ where X' is projective and non-singular over $\mathbb{C}$ [De 68], 5.3.4). The decomposition theorem extends to g since it is local on X. By the lemma, the HS on $\mathbb{H}^k(X, j_{!*}{}^p\mathbf{L})$ is a direct summand of $Gr_0^{{}^p\tau}\mathbb{H}^k(X', j'_{!*}{}^p\mathbf{L})$.

The independence of the choice of the designgularization X' up to isomorphism is proved by comparison of two non singular modifications X'_1 and X'_2 with a non singular modification X' of the fiber product $X'_1 \times_X X'_2$ such that $X' \to X'_1$ (resp. $X' \to X'_2$). Then the HS on $\mathbb{H}^i(X, j_{!*}{}^p\mathbf{L})$ deduced from X'_1 or X'_2 coincides with the HS deduced from X'.

6.2.2.2 Decomposition theorem in the case of a singular variety X

The proof is by reduction to the non singular case. Let $f : X \to V$ and $j_{!*}{}^p\mathbf{L}$ on the singular variety X. The induced Hodge filtration F on $j_{!*}{}^p\mathbf{L}$ on X is deduced from a non singular modification X' of X (see Corollary 6.2). We have a diagram

$$X' \xrightarrow{\pi} X \xrightarrow{f} V, \quad h := f \circ \pi \tag{6.21}$$

There exist Thom-Whitney stratifications $\mathcal{S}$ of f, π and $h := f \circ \pi$, compatible with $Y := X \setminus U$ and the NCD $Y' := \pi^{-1}(Y)$, and Whitney stratifications of X, X' and V underlying the Thom-Whitney stratifications $\mathcal{S}$.

Let V_0 (resp. X_0) denote the set of zero dimensional strata in V (resp. X) and V_m (resp. X_m) the maximal strata.We suppose the following conditions:

1) The inverse image X'_{V_0} (resp. X'_{X_0}) is a NCD
2) The inverse image Y'_{V_m} (resp. Y'_{X_m}) is empty or a relative NCD.

where relative means that the fiber Y'_v at each point $v \in V_m$ (resp. $v \in X_m$) is a NCD in the non singular fiber X'_v.
These conditions may be easily satisfied over isolated strata in V_0. It is even possible to ask that the inverse image of each stratum S of V is a NCD in X' and a fibration by NCD (chapter 12 §11.2, [EL 14]).

The decomposition theorem for f is deduced from the case of π and $h = f \circ \pi$ after ([De 68], Proposition 2.16) as follows.
Recall that $j' : U \to X'$ denote the embedding and $j'_{!*}{}^p\mathbf{L}$ is endowed with a Hodge filtration F. Let $K' := \pi_* j'_{!*}{}^p\mathbf{L}$. By the decomposition theorem applied to π:

a) $$K' \simeq \oplus_i {}^p\mathcal{H}^i(K')[-i] \in D^b_c(X)$$

Let $K := f_*K' = (f \circ \pi)_* j'_{!*}{}^p\mathbf{L}$. By the decomposition theorem applied to $f \circ \pi$:

b) $$K \simeq \oplus_i {}^p\mathcal{H}^i(K)[-i] \in D^b_c(V)$$

From a) we deduce $f_*K' \simeq \oplus_i f_* {}^p\mathcal{H}^i(K')[-i]$. Then by ([De 68], Corollary 1.12) for all k we have

$$f_* {}^p\mathcal{H}^k(K') \simeq \oplus_i {}^p\mathcal{H}^i(f_* {}^p\mathcal{H}^k(K'))[-i] \in D^b_c(V) \tag{6.22}$$

Corollary 6.3 (Decomposition). *Let $f : X \to V$ and $j_{!*}{}^p\mathbf{L}$ on X (Equation 6.21). There exists a decomposition in the derived category*

$$f_* j_{!*}{}^p\mathbf{L} \simeq \oplus_i {}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L})[-i] \tag{6.23}$$

We apply (Equation 6.22) for $k = 0$, since $j_{!*}{}^p\mathbf{L}$ is a summand of ${}^p\mathcal{H}^0(K')$ where $K' := \pi_* j'_{!*}{}^p\mathbf{L}$.

Chapter 7
Perverse VMHS

Let V be a complex projective variety possibly singular. We define the abelian category $I\mathcal{H}$ of intermediate extension of polarized VHS with support on irreducible sub-varieties of V (Lemma 7.1) and the category of perverse variation of mixed Hodge structures PVMHS (Definition 7.2).

The weight spectral sequence of PVMHS degenerates in general at rank 2 (Corollary 7.2) with E_1 terms in $I\mathcal{H}$ (§7.1).

For each closed subvariety $S \subset V$, there exists a MHS on $\mathbb{H}^i(S, j_{!*}{}^p\mathbf{L})$ of weights $w \leq {}^pw+i$, respectively a MHS on $\mathbb{H}^i_S(V, j_{!*}{}^p\mathbf{L})$ of weights $w \geq {}^pw+i$.

At a closed point v, with $i_v : v \to V$ and $k : V \setminus v \to V$, there exists a MHS on $H^i(i_v^* k_* k^* j_{!*}{}^p\mathbf{L})$ (§7.1.1).

Let $g : U' \to \mathbb{C}$ be a projective morphism defined on an open subset $U' \subset V$. We define a PVMHS on ${}^p\psi_g j_{!*}{}^p\mathbf{L}$ and ${}^p\varphi_g j_{!*}{}^p\mathbf{L}$ by desingularization. Then, we can state the local and global Invariant cycle theorems (§7.2).

At the end of the chapter (§9.7) we recover results of M. Saito ([Sa 90]) on mixed Hodge modules in terms of PVMHS (see Corollary 7.2).

7.1 Perverse variation of mixed Hodge structures

We define the abelian category $I\mathcal{H}$ of intermediate extensions of polarized VHS. Then, we deduce the definition of perverse variations of MHS as extensions of objects of $I\mathcal{H}$.

7.1.0.1 Hodge filtration by desingularization

Let U be a non singular open dense subset of an irreducible projective variety V and ${}^p\mathbf{L}$ a polarized VHS on U of weight ${}^pw$. Let $\pi : \widetilde{V} \to V$ be a desingularization of V with exceptional divisor a NCD Y inverse image of $V \setminus U$. Let $j : U \to V$ and $j' : U \to \widetilde{V}$ denote the open embeddings into V and V'.

A Hodge filtration F is defined on $j'_{!*}\,{}^p\mathbf{L}$, from which we deduce by push-forward a filtration π_*F on $\pi_*j'_{!*}\,{}^p\mathbf{L}$.

We recall that the Hodge filtration F on ${}^p\mathcal{H}^0(\pi_*j'_{!*}\,{}^p\mathbf{L})$ is induced by the filtration π_*F on V. Since $j_{!*}{}^p\mathbf{L}$ is a summand of $({}^p\mathcal{H}^0(\pi_*j'_{!*}\,{}^p\mathbf{L}), F)$ by the decomposition theorem (ch.2, §2.4), we deduce again the Hodge filtration by restriction of F.

The construction is performed in the bifiltered derived category $D^b_{{}^p\mathcal{T},F}(V,\mathbb{C})$. The filtration F induced on $j_{!*}{}^p\mathbf{L}$ does not depend in $D^b_F(V,\mathbb{C})$ on the choice of the desingularization of V. A functor T is derived using bigraded bifiltered $T-$acyclic resolutions of $(\pi_*j'_{!*}\,{}^p\mathbf{L},\ {}^p\mathcal{T}, F)$.

7.1.0.2 The category $I\mathcal{H}$ of Hodge intermediate extensions

Let $j : U \to V$ denote the embedding of a smooth open subset of V. A morphism of polarized VHS (PVHS) $g : ({}^p\mathbf{L}, F) \to ({}^p\mathbf{L}', F)$ on U extends uniquely to the intermediate extensions $j_{!*}g : (j_{!*}{}^p\mathbf{L}, F) \to (j_{!*}{}^p\mathbf{L}', F)$ on V.

Lemma 7.1. *The category $I\mathcal{H}_V$ of filtered intermediate extensions of polarized VHS on smooth open subsets of V of the form $(j_{!*}\,{}^p\mathbf{L}, F)$ with support V and morphisms of intermediate extensions is abelian.*

Proof. We prove for example the existence of the kernel and image of a morphism. Let $g : ({}^p\mathbf{L}, F) \to ({}^p\mathbf{L}', F)$ be a morphism of polarized VHS on the non singular open subset $U \subset V$ and $j : U \to V$. Let $j' : U \to \widetilde{V}$ denote the embedding of U in the desingularization of V and $Y = \cup_{i\in I} Y_i$ the NCD complement of $U \subset \widetilde{V}$.
Let $u : {}^p\mathbf{K} \to {}^p\mathbf{L}$ denote the kernel of g (resp. $v : {}^p\mathbf{L} \to {}^p\mathbf{I}$ the image).

Sub-lemma. *The morphism g (resp. u, v) extends into a morphism of of intermediate extensions of polarized VHS with induced filtration F by push-forward from $j'_{!*}g : (j'_{!*}{}^p\mathbf{L}, F) \to (j'_{!*}{}^p\mathbf{L}', F)$ (resp. $j'_{!*}u : j'_{!*}\,{}^p\mathbf{K} \to j'_{!*}{}^p\mathbf{L}$ and $j'_{!*}v : j'_{!*}\,{}^p\mathbf{L} \to j'_{!*}\,{}^p\mathbf{I}$) on $\widetilde{V}$.*
Moreover $j'_{!*}u$ and $j'_{!*}v$ are kernel and image of $j'_{!*}g$.

The proof is local. Let $J \subset I$, locally at a point of Y_J, we have a sequence of nilpotent orbits : $0 \to K \to L \to I \to 0$ endowed with monodromy weight filtration. With the notations of (ch.3, Definition 3.4), by induction on the length of J and the vanishing cycle lemma (see ch. 5, Lemma 5.1), the sequences:

$$0 \to N_JK \to N_JL \to N_JI \to 0,\ 0 \to \operatorname{Ker} N_JK \to \operatorname{Ker} N_JL \to \operatorname{Ker} N_JI \to 0 \tag{7.1}$$

are exact. The graded exact sequence $\mathrm{Gr}_*^{W(N_J)}$ defined by the weight filtration, split into primitive exact sequences, from which we deduce the exactness in (Equation 7.1). We deduce a distinguished triangle of the intermediate extensions

$$j'_{!*}\,{}^p\mathbf{K} \xrightarrow{j'_{!*}u} j'_{!*}{}^p\mathbf{L} \xrightarrow{j'_{!*}v} j'_{!*}\,{}^p\mathbf{I} \xrightarrow{[1]}$$

The associated long exact sequence of perverse cohomology on V splits, due to the weights, into short exact sequences

$$0 \to {}^p\mathcal{H}^0(\pi_* j'_{!*}\, {}^p\mathbf{K}) \xrightarrow{u_0} {}^p\mathcal{H}^0(\pi_* j'_{!*}\, {}^p\mathbf{L}) \xrightarrow{v_0} {}^p\mathcal{H}^0(\pi_* j'_{!*}\, {}^p\mathbf{I}) \to 0$$

In particular, we have induced sub-exact sequence of perverse sheaves on V:

$$0 \to j_{!*}\, {}^p\mathbf{K} \xrightarrow{j_{!*}u} j_{!*}{}^p\mathbf{L} \xrightarrow{j_{!*}v} j_{!*}\, {}^p\mathbf{I} \to 0$$

Hence $(j_{!*}\, {}^p\mathbf{K}, j_{!*}u)$ is a kernel, and $j_{!*}\, {}^p\mathbf{I}$ is the image of $j_{!*}g : (j_{!*}{}^p\mathbf{L}, F) \to (j_{!*}\mathbf{L}'p, F)$. The proof of the existence of co-image and co-kernel is similar.

Remark 7.1. The morphism $j_{!*}g : j_{!*}({}^p\mathbf{L}, F) \to (j_{!*}{}^p\mathbf{L}', F)$ is a canonical extension of g and all morphisms are obtained in this way in the category.

For each irreducible closed subset $S \subset V$, we have a natural functor $I\mathcal{H}_S \to \mathcal{P}(V)$ to the category $\mathcal{P}(V)$ of perverse sheaves on V.

Definition 7.1 ($I\mathcal{H}$). The category of Hodge intermediate extensions $I\mathcal{H}$ attached to V is the direct sum for all closed irreducible subsets S of V: $I\mathcal{H} := \oplus_{S\subset V} I\mathcal{H}_S$.

Since a morphism on intermediate extensions of polarized VHS with support S is determined by its restriction to the big stratum S^*, a morphism $i_{S!*}{}^p\mathbf{L} \to i_{S'!*}{}^p\mathbf{L}'$ between two of intermediate extensions of polarized VHS with support S and S' is necessarily 0 unless $S = S'$.

Proposition 7.1. *Let $f : X \to V$ be a projective morphism and $j_{!*}{}^p\mathbf{L}$ a polarized VHS on X. For all integers i, the perverse cohomology sheaves $({}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L}), F)$ are in $I\mathcal{H}$ on V.*

We prove that any component $i_{S!*}{}^p\mathbf{L}^i_S$ of ${}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L})$ with support on the closure $\bar{S}$ of a stratum S of V is in $I\mathcal{H}$ on V.
Let $\pi : S' \to \bar{S}$ be a desingularization of $\bar{S}$ and $k : S \hookrightarrow S'$.

We have to prove that $(i_{S!*}{}^p\mathbf{L}^i_S, F)$ is a component of $({}^p\mathcal{H}^0(\pi_* k_{!*}{}^p\mathbf{L}^i_S), F)$, We construct a variety $\widetilde{V}$ in the diagram

$$\begin{tikzcd}
 & X \arrow[d, "f"] & \widetilde{X} \arrow[l, "\pi_1"'] \arrow[d, "f'"] \arrow[dl, "h"'] & \\
S \arrow[r] & V & \widetilde{V} \arrow[l, "\pi"] & \widetilde{S} \arrow[l]
\end{tikzcd}$$

by blowings up above the singularities S_s of $\bar{S}$ (including the singularities of ${}^p\mathbf{L}^i_S$) such that there exists a desingularization $\widetilde{S} \subset \widetilde{V}$ of S, strict transform of S in $\widetilde{V}$ and an embedding $i_{\widetilde{S}} : S \hookrightarrow \widetilde{S}$ such that $V \setminus S_s \simeq \widetilde{V} \setminus \pi^{-1}S_s$ (without modification of V outside the singularities $S_s \subset S$).

Set $\widetilde{X} := X \times_V \widetilde{V}, X_s := f^{-1}(S_s)$ such that $X \setminus X_s \simeq \widetilde{X} \setminus \pi_1^{-1} X_s$ ($\widetilde{X}$ is a modification of X). Let $j'_{!*}\, {}^p\mathbf{L}'$ denote the intermediate extension of ${}^p\mathbf{L}$ on $\widetilde{X}$ ($\widetilde{X}$ may be singular, we recall that $j_{!*}{}^p\mathbf{L}$ on X and $j'_{!*}\, {}^p\mathbf{L}'$ on $\widetilde{X}$ are constructed out of a desingularization of $\widetilde{X}$).

The component $i_{S!*}{}^p\mathbf{L}^i$ may be constructed out of the decomposition of h. In the decomposition of ${}^p\mathcal{H}^i(f'_* j'_{!*}\, {}^p\mathbf{L}')$ the component $i_{\widetilde{S}!*}{}^p\mathbf{L}^i$ is exactly the extension of ${}^p\mathbf{L}^i$ on the desingularization $\widetilde{S}$ of S and by the decomposition theorem

$$(i_{S!*}{}^p\mathbf{L}^i_S, F) \subset ({}^p\mathcal{H}^0(\pi_* i_{\widetilde{S}!*}{}^p\mathbf{L}^i_S), F) \subset ({}^p\mathcal{H}^i(h_* j'_{!*}\, {}^p\mathbf{L}'), F).$$

Corollary 7.1. *Let V be an algebraic variety. The category of Hodge intermediate extensions $I\mathcal{H}$ is well defined on V.*

We can suppose V irreducible, since the category is the direct sum over irreducible components. In the previous definitions, the fact that V is projective was convenient for the proof but not essential.

By uniqueness of the extension we can remove the condition V is projective. Let ${}^p\mathbf{L}$ be defined on an open subset U. For any affine open subset $A \subset V$, the extension of the restriction ${}^p\mathbf{L}|_{A\cap U}$ to a projective closure is well defined, hence it is well defined on A. All extensions over an affine covering of V glue into a unique extension on V.

Definition 7.2 (Perverse variation of MHS (PVMHS)). A bifiltered complex (K, W, F) is a perverse variation of mixed Hodge structures (PVMHS) on a singular X if:

1. K is a $\mathbb{Q}-$perverse sheaf on X,
2. W is an increasing filtration by $\mathbb{Q}-$perverse sub-sheaves,
3. F is a decreasing filtration by $\mathbb{C}-$sub-sheaves of $K \otimes_{\mathbb{Q}} \mathbb{C}$ such that: $\forall r \in \mathbb{Z}, (\mathrm{Gr}^W_r K, F) \in I\mathcal{H}$ is a direct sum of of intermediate extensions of polarized VHS on V.

In the text all PVMHS are graded polarized. The following result may be deduced as in ([Sa 90], 2.d):

Corollary 7.2 (M. Saito). *Let $f : X \to V$ be a projective morphism and (K, W, F) a PVMHS on X. The perverse cohomology $({}^p\mathcal{H}^i(f_*K), W, F)$ with induced filtrations W and F is a PVMHS on V and we have:*

$$ {}^pE_1^{p,q} := {}^p\mathcal{H}^{p+q}(f_* Gr^W_{-p} K) \Longrightarrow Gr^W_q\, {}^p\mathcal{H}^{p+q}(f_*K) \tag{7.2}$$

degenerating at rank 2: ${}^pE_2^{p,q}(f_*K, W) \simeq Gr^W_q\, {}^p\mathcal{H}^{p+q}(f_*K)$.

By the decomposition theorem, already proved (see ch.6, Corollary 6.3) the terms of the perverse weight spectral sequence ${}^pE_r^{p,q}(f_*K', W)$ are in the abelian category $I\mathcal{H}$ on V and the spectral sequence degenerates at rank 2 by an argument similar to Deligne's results on spectral sequences with HS ([De 72] revisited in [De 75]).

The main interest in the abelian category $I\mathcal{H}$ is to construct the terms ${}^pE_2^{p,q}$ as cohomology of the complex of terms ${}^pE_1^{p,q}$.

7.1.1 Hodge theory along locally principal divisors

Let $j_{!*}{}^p\mathbf{L}$ be an intermediate extension of a VHS ${}^p\mathbf{L}$ of weight ${}^pw$ on a non-singular dense algebraic open subset U of a variety V and $j : U \to V$.

Let $H \subset V$ be a principal divisor defined by an equation g. There exists a perverse variation of mixed Hodge structure (PVMHS) on ${}^p\psi_g j_{!*}{}^p\mathbf{L}$ (resp. ${}^p\varphi j_{!*}{}^p\mathbf{L}$) (see Proposition 7.3 below).

Let $i : H \to V$. There exists a PVMHS on $i^* j_{!*}{}^p\mathbf{L}[-1]$ (resp. $i^! j_{!*}{}^p\mathbf{L}[1]$)

Let $v \in V$ be a closed point, B_v a small ball with center v and $B_v^* := B_v \backslash v$. There exists a MHS on $\mathbb{H}^j(B_v^*, j_{!*}{}^p\mathbf{L})$ (Proposition 7.4 below).

7.1.1.1 Functoriality of the monodromy filtration

Let $f : X \to V$ be a projective morphism of algebraic complex varieties, $U' \subset V$ an open subset (algebraic or analytic) and $g : U' \to \mathbb{C}$ a morphism. Let $H := g^{-1}(0)$. We suppose *X non singular and $f^{-1}H \subset X$ a NCD.*

Let $j_{!*}{}^p\mathbf{L}$ be the intermediate extension of a shifted polarized VHS on X, $h := g \circ f$ and $(K, F) := f_*(j_{!*}{}^p\mathbf{L}, F)$, then:

$$ {}^p\psi_g\, {}^p\mathcal{H}^i(f_*K) = {}^p\mathcal{H}^i(f_*\, {}^p\psi_h j_{!*}{}^p\mathbf{L}), \quad {}^p\varphi_g\, {}^p\mathcal{H}^i(f_*K) = {}^p\mathcal{H}^i(f_*\, {}^p\varphi_h j_{!*}{}^p\mathbf{L}) \tag{7.3} $$

The weight filtration W is related to the monodromy filtration $W(N_h)$ on ${}^p\psi_h j_{!*}{}^p\mathbf{L}$ (resp. on $\varphi_h j_{!*}{}^p\mathbf{L}$) (see Theorem 5.1). The image of the perverse cohomology ${}^p\mathcal{H}^i(f_*W)$ defines a filtration on ${}^p\psi_g\, {}^p\mathcal{H}^i(f_*K)$ (resp. ${}^p\varphi_g\, {}^p\mathcal{H}^i(f_*K)$). We have a perverse spectral sequence

$$ \begin{aligned} {}^pE_1^{p,q} &:= {}^p\mathcal{H}^{p+q}(f_* Gr^W_{-p}\, {}^p\psi_h j_{!*}{}^p\mathbf{L}) \Longrightarrow Gr^W_q\, {}^p\psi_g\, {}^p\mathcal{H}^{p+q}(f_* j_{!*}{}^p\mathbf{L}) \\ {}^pE_1^{p,q} &:= {}^p\mathcal{H}^{p+q}(f_* Gr^W_{-p}\, {}^p\varphi_h j_{!*}{}^p\mathbf{L}) \Longrightarrow Gr^W_q\, {}^p\varphi_g\, {}^p\mathcal{H}^{p+q}(f_* j_{!*}{}^p\mathbf{L}) \end{aligned} \tag{7.4} $$

Proposition 7.2 (Saito [Sa 88] Proposition 5.3.5). *i) The perverse weight spectral sequences (7.4) degenerate at rank 2 in the abelian category $I\mathcal{H}$ on H (Definition 7.1).*

ii) The filtration induced by $\mathrm{Im}\, {}^p\mathcal{H}^i(f_*W(N_h)) \to {}^p\psi_g\, {}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L})$ *(resp. ${}^p\varphi_g\, {}^p\mathcal{H}^{p+q}(f_* j_{!*}{}^p\mathbf{L})$) coincides with the monodromy filtration $W(N_g)$.*

iii) The terms ${}^pE_2^{p,q}$ are in $I\mathcal{H}$ and we have a structure of $PVMHS$ on ${}^p\psi_g\, {}^p\mathcal{H}^{p+q}(f_*K)$ (resp. ${}^p\varphi_g\, {}^p\mathcal{H}^{p+q}(f_*K)$).

The proof is based on the bigraded objects of intermediate extensions of polarized VHS on H in the abelian category $I\mathcal{H}$ by comparison with the bigraded polarized HS in chapter 5, §5.2.1.1.

We use the perverse Hard Lefshetz theorem (resp. the morphism N) on the terms ${}^pE_1^{p,q}$.

7.1.2 Nearby and vanishing cycles on a variety V

Let $j_{!*}{}^p\mathbf{L}$ be the intermediate extension of a polarized VHS on a non singular Zariski open subset U dense in a projective variety V possibly singular and $g : U \to \mathbb{C}$ a projective analytic morphism.

Proposition 7.3. *Let $j_{!*}{}^p\mathbf{L}$ on V and $N := \log T^u$. There exists on the perverse nearby cycle complex $\psi_g j_{!*}{}^p\mathbf{L}[-1]$ (resp. $\varphi_g j_{!*}{}^p\mathbf{L}[-1]$) a filtration F inducing with the filtration $W := W(N)[{}^pw - 1]$ (resp. $W := W(N)[{}^pw]$) a structure of PVMHS where the weight W is a shifted monodromy weight filtration $W(N)$.*

Let W be an increasing filtration, recall: $W[n]_{i+n} = W_i$. The proof is by reduction to the non singular case. Let $H := g^{-1}(0)$ and Y the singularities of ${}^p\mathbf{L}$ in V including the singularities of V. Let $\pi : \widetilde{V} \to V$ be a desingularization such that $Y' := \pi^{-1}(Y)$ and $H' := \pi^{-1}(H)$ are NCD. We suppose that H' is a sub-divisor of Y' as we can enlarge Y if necessary. Let $j' : U \to \widetilde{V}$ ($U \simeq (\widetilde{V} \setminus Y') \simeq V \setminus Y$. We apply the proposition 7.2 to $j'_{!*}{}^p\mathbf{L}$ on $\widetilde{V}$. The proof follows, since $j_{!*}{}^p\mathbf{L}$ is a component of ${}^p\mathcal{H}^0(\pi_* j'_{!*}{}^p\mathbf{L})$.

The uniqueness follows by considering a diagram $\widetilde{V}_1 \leftarrow \widetilde{V} \to \widetilde{V}_2$ of modifications of V in the case of two choices $\widetilde{V}_1$ and $\widetilde{V}_2$.

Definition 7.3. We define $\psi_g j_{!*}{}^p\mathbf{L}$ as a component of $\psi_g{}^p\mathcal{H}^0(\pi_* j_{!*}{}^p\mathbf{L}) := {}^p\mathcal{H}^0(\pi_*\psi_{g\circ\pi}(j'_{!*}{}^p\mathbf{L}))$. A similar definition of $\varphi_g j_{!*}{}^p\mathbf{L}$ apply.

Corollary 7.3. *Let V be projective and $i : H \hookrightarrow V$ the embedding of a locally principal divisor.*

i) There exists on $i^* j_{!*}{}^p\mathbf{L}[-1]$ (resp. on $i^! j_{!*}{}^p\mathbf{L}[1]$) a natural PVMH inducing on $\mathbb{H}^j(H, j_{!*}{}^p\mathbf{L})$ (resp. on $\mathbb{H}^j_H(V, j_{!*}{}^p\mathbf{L}[1])$) a filtration by MHS of weights $w \le {}^pw + j$ (resp. $w > {}^pw + j$).

ii) Let $k : V \setminus H \hookrightarrow V$. There exists on $k_* k^* j_{!*}{}^p\mathbf{L}$ a natural PVMH of weights $w \ge {}^pw$ and a MHS on $\mathbb{H}^j(V \setminus H, j_{!*}{}^p\mathbf{L})$ of weights $w \ge {}^pw + j$.

i) Let g be a local equation of H. The corollary follows since $i^* j_{!*}{}^p\mathbf{L}[-1]$ may be constructed locally as kernel of the morphism $N : \psi_g^u \psi_g j_{!*}{}^p\mathbf{L}[-1] \to \psi_g^u \psi_g j_{!*}{}^p\mathbf{L}[-1]$ (see ch.5, Equation 5.50) with induced weight filtration. The local constructions glue globally on H (the case $i^! j_{!*}{}^p\mathbf{L}[1]$ is dual).

It is equivalent to construct the filtrations W and F by push-forward from a desingularization of V.

ii) The quotient $k_* k^* j_{!*}{}^p\mathbf{L}/j_{!*}{}^p\mathbf{L}$ is isomorphic to $i^! j_{!*}{}^p\mathbf{L}[1]$.

Proposition 7.4 (MHS on $\mathbb{H}^j(B_v^*, j_{!*}{}^p\mathbf{L})$). *Let $v \in V, k_v : (V \setminus v) \to V$. There exists a natural MHS on $H^j(v, k_{v*} k_v^* j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^j(B_v^*, j_{!*}{}^p\mathbf{L})$.*

Proof. Since the proof is local at v, we change V into a small ball B_v with center $v \in V$. By blowing up $v \in B_v$, we have a diagram

$$\begin{array}{ccccc} \widetilde{Z} & \overset{\widetilde{i}}{\hookrightarrow} & \widetilde{B}_v & \overset{\widetilde{k}}{\hookleftarrow} & \widetilde{B}_v^* \\ \pi_v \downarrow & & \downarrow \pi & & \simeq\downarrow \pi' \\ v & \overset{i_v}{\hookrightarrow} & B_v & \overset{k}{\hookleftarrow} & B_v^* \end{array} \tag{7.5}$$

where $B_v^* := B_v \setminus \{v\}$ and $\pi : \widetilde{B}_v \to B_v$ is the blow up of $v \in B_v$.

Let P be a complex on $\widetilde{B}_v^*$ such that $\pi'_* P = (j_{!*}{}^p\mathbf{L})_{B_v^*}$. Since π' is an isomorphism, we have:

$$i_v^* k_* \pi'_* P \simeq \pi_{v*} \widetilde{i}^* \widetilde{k}_* P$$

where $\widetilde{i}^* \widetilde{k}_* P$ has two non vanishing perverse cohomology on the locally principal divisor $\widetilde{Z}$:

$$^p\mathcal{H}^{-1}(\widetilde{i}^* \widetilde{k}_* P) = \widetilde{i}^* \widetilde{k}_{!*} P[-1],\ {}^p\mathcal{H}^0(\widetilde{i}^* \widetilde{k}_* P) = \widetilde{i}^! \widetilde{k}_{!*} P[1] \tag{7.6}$$

The complex $\widetilde{i}^* \widetilde{k}_* P$ is isomorphic to a mixed cone over the intersection complex $I : \widetilde{i}^! \widetilde{k}_{!*} P \to \widetilde{i}^* \widetilde{k}_{!*} P$ with filtrations W and F such that

$$(H^i(\pi_{v*} \widetilde{i}^* \widetilde{k}_* P), W, F) \simeq \mathbb{H}^i(\widetilde{Z}, \widetilde{i}^* \widetilde{k}_* P, W, F) \tag{7.7}$$

$H^i(v, k_{v*} k_v^* j_{!*}{}^p\mathbf{L}) \simeq H^i(\pi_{v*} \widetilde{i}^* \widetilde{k}_* P)$ has a natural MHS.

Remark 7.2. 1) Let $i : H \hookrightarrow V$ is a closed subvariety of V, then the results of Corollary 7.3) can be generalized to this case:

i) There exists on $i^* j_{!*}{}^p\mathbf{L}[-1]$ (resp. on $i^! j_{!*}{}^p\mathbf{L}[1]$) a natural PVMH inducing on $\mathbb{H}^j(H, j_{!*}{}^p\mathbf{L})$ (resp. on $\mathbb{H}^j_H(V, j_{!*}{}^p\mathbf{L}[1])$) a filtration by MHS of weights $w \leq {}^pw + j$ (resp. $w > {}^pw + j$).

ii) Let $k : V \setminus H \hookrightarrow V$. There exists on $k_* k^* j_{!*}{}^p\mathbf{L}$ a natural PVMH of weights $w \geq {}^pw$ and a MHS on $\mathbb{H}^j(V \setminus H, j_{!*}{}^p\mathbf{L})$ of weights $w \geq {}^pw + j$.

Proof. We blow-up H to construct $H' \subset V'$ and $f : V' \to V$ such that $f^{-1}(H) = H'$, then apply (Corollary 7.3) to the locally locally principal divisor H' to define a PVMHS, then apply (Proposition 7.2).

2) Then Proposition 7.4 follows since $i_v^* k_{v*} k_v^* j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^j(B_v^*, j_{!*}{}^p\mathbf{L})$ is a mixed cone over the intersection morphism $I : i_v^! \to i_v^*$ (Equation 2.1.3.1, §5.2.2, Lemma 5.3).

Remark 7.3 (Compatibility with the perverse filtration). (See a general method to prove the compatibility with the perverse filtration by (Equation 10.7)).

i) Let $f : X \to V$ be a projective morphism (X may be singular), $H \subset V$ a locally principal divisor on V and $(j_{!*}{}^p\mathbf{L}, W, F)$ on X. Set $H' := f^{-1}(H) \subset X$, $i_H : H \to V$, $i_{H'} : H' \to X$.

Let $(K', W, F) := (i_{H'}^* j_{!*}\, {}^p\mathbf{L}, W, F)$ (resp. $(K', W, F) := (i_{H'}^! j_{!*}\, {}^p\mathbf{L}, W, F)$) and denote again $f_{|H*}W$ by W, $f_{|H*}F$ by F on $f_* K'$.

We deduce from the perverse filtration ${}^p\tau$ on $f_* j_{!*}\, {}^p\mathbf{L}$ a filtration $i_H^*\, {}^p\tau$ on $i_H^* f_* j_{!*}\, {}^p\mathbf{L}$ (resp. $i_H^!\, {}^p\tau$ on $i_H^! f_* j'_{!*}\, {}^p\mathbf{L}$) and by transport of structure a filtration

$i_H^*\,{}^p\tau$ on $f_{|H*}K'$ by the isomorphism $f_{|H*}K' \simeq i_H^* f_* j_{!*}\,{}^p\mathbf{L}$ (resp. $i_H^!\,{}^p\tau$ on $i_H^! f_* j_{!*}\,{}^p\mathbf{L}$). Thus we have three filtrations

$$(f_{|H*}K', W, F, i_H^*\,{}^p\tau) \text{ (resp. } (f_{|H*}K', W, F, i_H^!\,{}^p\tau)).$$

We deduce from (Corollary 7.3 and Proposition 7.4) an isomorphism of MHS of weights $w \leq {}^pw + j$ (resp. of weights $w \geq {}^pw + j$).

$$\begin{aligned}(\mathbb{H}^j(H, {}^p\mathcal{H}^i(f_* j_{!*}\,{}^p\mathbf{L})), W, F) &\simeq (Gr_i^{i_H^*\,{}^p\tau}\mathbb{H}^{i+j}(X_H, j_{!*}\,{}^p\mathbf{L}), W, F)\\ (\mathbb{H}^j_H(V, {}^p\mathcal{H}^i(f_* j_{!*}\,{}^p\mathbf{L})), W, F) &\simeq (Gr_i^{i_H^!\,{}^p\tau}\mathbb{H}^{i+j}_{X_H}(X, j_{!*}\,{}^p\mathbf{L}), W, F)\end{aligned} \tag{7.8}$$

ii) We may even consider the induced filtration in the category of PVMHS on the desingularization to deduce PVMHS on a singular variety.

iii) *MHS on* $H^j(B_v^*, {}^p\mathcal{H}^i(f_* j_{!*}\,{}^p\mathbf{L}))$: Let $v \in V$ be a closed point such that $Z = X_v := f^{-1}(v)$ is a locally principal divisor in X. Set $k : V \setminus \{v\} \to V, i_v : v \to V$, $i : Z \to X$ and $j_Z : X \setminus Z \to X$ and $j_{!*}\,{}^p\mathbf{L}$ on X.

We deduce from the bifiltered complex $(K', W, F) := (i^* j_{Z*}(j_{!*}\,{}^p\mathbf{L})_{|(X\setminus Z)}, W, F)$ a structure of MHC on the derived global sections complex $\Gamma(Z, K', W, F)$. We denote again by W and F the filtrations $(f_{|Z})_*W, (f_{|Z})_*F$ on $(f_{|Z})_*K'$.

We use the isomorphism $(f_{|Z})_*K' \simeq i_{v*}k_*((f_* j_{!*}\,{}^p\mathbf{L})_{|(V\setminus v)})$ to deduce a filtration ${}^p\tau^*$ on $(f_{|Z})_*K'$ from the perverse filtration ${}^p\tau$ on $(f_* j_{!*}\,{}^p\mathbf{L})_{|(V\setminus v)}$

$${}^p\tau^*(f_{|Z})_*K' \simeq {}^p\tau^*(i_{v*}k_*(f_* j_{!*}\,{}^p\mathbf{L})_{|(V\setminus v)}) := i_{v*}k_*({}^p\tau(f_* j_{!*}\,{}^p\mathbf{L})_{|(V\setminus v)})$$

Thus we have three filtrations $W, F, {}^p\tau^*$ on the complex $(f_{|Z})_*K'$ inducing filtrations W, F and ${}^p\tau^*$ on the hypercohomology space $\mathbb{H}^j(Z, j_{Z*}j_Z^* j_{!*}\,{}^p\mathbf{L})$:

1) The induced filtration ${}^p\tau^*$ on $\mathbb{H}^j(X_v, K')$ is a filtration by MHS.

2) There exists a natural MHS on $H^j(v, k_*k^*\,{}^p\mathcal{H}^i(f_* j_{!*}\,{}^p\mathbf{L})$ compatible with the isomorphism of MHS:

$$H^j(v, k_*k^*\,{}^p\mathcal{H}^i(f_* j_{!*}\,{}^p\mathbf{L}) \simeq H^j(v, \mathrm{Gr}_i^{{}^p\tau^*}(f_{|X_v})_*K', W, F) \tag{7.9}$$

If Z is a NCD in a non singular X, the bifiltered complex $K' := i^* j_{Z*}j_Z^* j_{!*}\,{}^p\mathbf{L}$ with induced W and F is defined by (chapter 5, §5.2.2, Definition 5.6).

Corollary 7.4 (Perverse filtration by sub-HS). *With the notations of (Corollary 6.3), the filtration ${}^p\tau$ on $f_* j_{!*}\,{}^p\mathbf{L}$ induces a filtration by Hodge substructures of $\mathbb{H}^{i+j}(X, j_{!*}\,{}^p\mathbf{L})$ such that*

$$(Gr_i^{{}^p\tau}\mathbb{H}^{i+j}(X, j_{!*}\,{}^p\mathbf{L}), F) \simeq (\mathbb{H}^j(V, {}^p\mathcal{H}^i(f_* j_{!*}\,{}^p\mathbf{L})), F).$$

We have a decomposition into a direct sum of polarized HS of weight ${}^pw+i+j$:

$$\mathbb{H}^j(V, {}^p\mathcal{H}^i(f_* j_{!*}\,{}^p\mathbf{L})) \xrightarrow{\sim} \oplus_{S\in\S}\ \mathbb{H}^j(X, i_{S!*}(j_{!*}\,{}^p\mathbf{L})^i_S)$$

7.2 Local and global invariant cycle theorems

We prove the local and global invariant cycle theorems on eventually singular varieties as announced in the introduction (ch. 4, Theorems 4.1 and 4.2). The results on non singular varieties extend to singular varieties with coefficients in intermediate extensions of VHS by the previous remark 7.3.
Also, Deligne's proof on non singular varieties extends to singular varieties.

The octahedron diagram Diagram (7.13) and Equation (7.15) are applications of the decomposition of φ_g (see ch.6, Equation 6.17).

7.2.0.1 Local invariant cycle theorem on singular varieties

Let V be an algebraic variety, ${}^p\mathbf{L}$ a shifted polarized VHS of weight ${}^pw$ on a non singular Zariski open subset $U \subset V$ and $j_{!*}{}^p\mathbf{L}$ the intermediate extension of ${}^p\mathbf{L}$ endowed with a Hodge filtration F (see ch.6, Definition 6.2).

Let $g : V_D \to D$ be a projective morphism from a complex open subset V_D to the complex disc $D \in \mathbb{C}$ with central fiber $H := V_0$.

Proposition 7.5 (Local invariant cycle). *The sequence*

$$\mathbb{H}^q(H, j_{!*}{}^p\mathbf{L}[-1]) \xrightarrow{sp} \mathbb{H}^q(H, \psi_g^u j_{!*}{}^p\mathbf{L}[-1]) \xrightarrow{N} \mathbb{H}^q(H, \psi_g^u j_{!*}{}^p\mathbf{L}[-1])(-1) \tag{7.10}$$

is an exact sequence of MHS.

The proof is similar to Deligne's proof for constant coefficients [De 80]. For D small enough, V_D is a retract by deformation of H as g is proper. Hence the restriction map $H^i(V_D, j_{!*}{}^p\mathbf{L}) \to H^i(H, j_{!*}{}^p\mathbf{L})$ is an isomorphism. Let ${}^pw$ be the weight of ${}^p\mathbf{L}$. We consider the diagram

$$\begin{tikzcd}
H_H^{i+1}(V_D, j_{!*}{}^p\mathbf{L}) & & \\
H^i(V_D \setminus H, j_{!*}{}^p\mathbf{L}) \arrow[u] \arrow[r, "\beta"] & \mathbb{H}^i(H, \psi_g^u j_{!*}{}^p\mathbf{L})^N \arrow[r] & 0 \\
H^i(V_D, j_{!*}{}^p\mathbf{L}) \arrow[u, "\alpha"] \arrow[ur, "sp"] & &
\end{tikzcd}$$

Since the topological Wang exact sequence

$$\mathbb{H}^r(V_D \setminus H, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^r(H, \psi_g^u j_{!*}{}^p\mathbf{L}) \xrightarrow{N} \mathbb{H}^r(H, \psi_g^u j_{!*}{}^p\mathbf{L}) \to \tag{7.11}$$

is compatible with MHS (see Remark 7.3 ii), the morphism β is surjective on the invariant subspace by N and compatible with MHS.

Let $j_H : V \backslash H \to V$. The whole diagram can be constructed in the category of MHS, since the following hypercohomology spaces carry MHS:

1) $\mathbb{H}^i(V_D \setminus Z, j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^i(H, i_H^* j_{H*} j_H^* j_{!*}{}^p\mathbf{L})$ (V_D retracts on H),

2) $H^i(H, \psi_f^u j_{!*}{}^p\mathbf{L})$, (Proposition 7.3)

3) $\mathbb{H}^i(V_D, j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^i(H, j_{!*}{}^p\mathbf{L})$ of weight $\leq {}^pw + i$ (Corollary 7.3) and

4) $\mathbb{H}_Z^{i+1}(V_D, j_{!*}{}^p\mathbf{L})$ of weight $\geq {}^pw + i + 1$ (Corollary 7.3).

Since $H_H^{i+1}(V_D, j_{!*}{}^p\mathbf{L})$ has weights $\geq {}^pw + i + 1$, and the weight of $\operatorname{Im}\alpha$ is $\leq {}^pw + i$, we have: $W_{{}^pw+i}\mathbb{H}^i(V_D \setminus H, j_{!*}{}^p\mathbf{L}) \subset \operatorname{Im}\alpha$.

Since β is surjective by Wang sequence, we deduce that $sp := \beta \circ \alpha$ is surjective as the weight of $\mathbb{H}^i(H, \psi_f^u j_{!*}{}^p\mathbf{L})^N$ is $\leq {}^pw + i$.

7.2.0.2 The octahedron diagram

Let $g : V \to \mathbb{P}$ be a morphism to the complex projective line $\mathbb{P} := \mathbb{P}^1_{\mathbb{C}}$, $H := g^{-1}(0)$ and $f : X \to V$ such that $Z := (g \circ f)^{-1}(0) = f^{-1}(H)$ is a NCD in X

$$\begin{array}{ccccc} Z & \overset{i'}{\hookrightarrow} & X & \hookleftarrow & X \setminus Z \\ \downarrow & & \downarrow f & & \downarrow \\ H & \overset{i}{\hookrightarrow} & V & \overset{k'}{\hookleftarrow} & V \setminus H \\ \downarrow & & \downarrow g & & \downarrow \\ 0 & \overset{i''}{\hookrightarrow} & \mathbb{P} & \hookleftarrow & \mathbb{P} \setminus 0 \end{array} \tag{7.12}$$

Let $K := f_* j_{!*}{}^p\mathbf{L}$. We attach to the Diagram 7.12, the following upper and lower 'caps' of the octahedron diagram ([BBD 83], 1.1.6):

$$\begin{tikzcd} i^!K \arrow[dr, "d"] \arrow[dd, "I"'] & & {}^p\psi_g^u K \arrow[ll, "(+1)"'] \\ & {}^p\varphi_g^u K \arrow[ur, "var"] \arrow[dl, "d"] & \\ i^*K \arrow[rr, "+1"] & & {}^p\psi_g^u K \arrow[ul, "can"'] \arrow[uu, "N"'] \end{tikzcd} \qquad \begin{tikzcd} i^*K \arrow[dr, "c'"] \arrow[dd, "+1"'] & & i^!K \arrow[ll, "I"'] \\ & C(N) \arrow[ur, "d\ (+1)"] \arrow[dl, "(+1)"] & \\ {}^p\psi_g^u K \arrow[rr, "N"] & & {}^p\psi_g^u K \arrow[ul, "d"'] \arrow[uu, "+1"'] \end{tikzcd} \tag{7.13}$$

where the cones $C(N) \simeq C(I)$ are isomorphic. The middle term of the sequence:

$$({}^p\mathcal{H}^i({}^p\psi_g^u K) \overset{can}{\to} {}^p\mathcal{H}^i({}^p\varphi_g^u K)) \overset{var}{\to} {}^p\mathcal{H}^i({}^p\psi_g^u K))$$

where ${}^p\mathcal{H}^i({}^p\psi_g K) = {}^p\psi_g\, {}^p\mathcal{H}^i(K)$ and ${}^p\mathcal{H}^i({}^p\varphi_g K) = {}^p\varphi_g\, {}^p\mathcal{H}^i(K)$, splits into a direct sum by (Lemma 6.4)

$${}^p\mathcal{H}^i({}^p\varphi_g^u K) \overset{\sim}{\to} \operatorname{Im} can \oplus \operatorname{Ker} var \tag{7.14}$$

Typically for an octahedron diagram, this decomposition is equivalent to the long exact sequence with four terms defined by the common boundary of the diagrams:

$$ {}^p\mathcal{H}^{i-1}(i^*K) \to {}^p\mathcal{H}^i({}^p\psi_g^u K) \xrightarrow{N} {}^p\mathcal{H}^i({}^p\psi_g^u K)(-1) \to {}^p\mathcal{H}^{i+1}(i^!K) \xrightarrow{I} {}^p\mathcal{H}^{i+1}(i^*K) \to $$

or the exact sequence

$$ i^*\,{}^p\mathcal{H}^i(K)[-1] \to {}^p\psi_g^u\,{}^p\mathcal{H}^i(K) \xrightarrow{N} {}^p\psi_g^u\,{}^p\mathcal{H}^i(K)(-1) \to i^!\,{}^p\mathcal{H}^i(K)[1] \to i^*\,{}^p\mathcal{H}^{i+2}(K)[-1] \to $$

since ${}^p\mathcal{H}^{i-1}(i^*K) = i^*\,{}^p\mathcal{H}^i(K)[-1]$, ${}^p\mathcal{H}^{i+1}(i^!K) = i^!\,{}^p\mathcal{H}^i(K)[1]$.

Since H is locally principal and ${}^p\mathcal{H}^i(K)$ decomposes into a sum of intermediate extensions, this exact sequence splits into various perverse exact sequences

$$ 0 \to i^*\,{}^p\mathcal{H}^i(K)[-1] \to {}^p\psi_g^u\,{}^p\mathcal{H}^i(K) \xrightarrow{N} {}^p\psi_g^u\,{}^p\mathcal{H}^i(K)(-1) \to i^!\,{}^p\mathcal{H}^i(K)[1] \to 0 \tag{7.15} $$

This exact sequence appears in ([Sa 88] 5.2.2.2), see also [BBD 83]. The local invariant theorem follows again:

Corollary 7.5. *i) We have an exact sequence of MHS:*

$$ \mathbb{H}^q(H, {}^p\mathcal{H}^0(K)[-1]) \xrightarrow{sp} \mathbb{H}^q(H, \psi_g^u\,{}^p\mathcal{H}^0(K)[-1]) \xrightarrow{N} \mathbb{H}^q(H, \psi_g^u\,{}^p\mathcal{H}^0(K)[-1])(-1) $$

ii) Let ${}^p\mathbf{L}$ be on V. We have an exact sequence of MHS:

$$ \mathbb{H}^{q-1}(H, j_{!*}{}^p\mathbf{L}) \xrightarrow{sp} \mathbb{H}^{q-1}(H, \psi_g^u j_{!*}{}^p\mathbf{L}) \xrightarrow{N} \mathbb{H}^{q-1}(H, \psi_g^u j_{!*}{}^p\mathbf{L})(-1) $$

i) The weights $w(\mathbb{H}^q(H, {}^p\mathcal{H}^0(K)[-1]) \leq {}^pw+q-1$ and $w(\operatorname{Ker} N) \leq {}^pw+q-1$. The exact sequence (Equation 7.15) is reduced to two short exact sequences:

$$ 0 \to i^*\,{}^p\mathcal{H}^0(K)[-1] \to {}^p\mathcal{H}^0({}^p\psi_g^u K) \to \operatorname{Im} N \to 0, $$
$$ 0 \to \operatorname{Im} N \to {}^p\mathcal{H}^0({}^p\psi_g^u K)(-1) \to i^!\,{}^p\mathcal{H}^0(K)[1] \to 0 $$

We deduce the associated long exact sequences

$$ \mathbb{H}^q(H, {}^p\mathcal{H}^0(K)[-1]) \xrightarrow{sp} \mathbb{H}^q(H, \psi_g^u\,{}^p\mathcal{H}^0(K)[-1]) \xrightarrow{\alpha} \mathbb{H}^q(H, \operatorname{Im} N) \to $$
$$ \mathbb{H}^{q-1}(H, i^!\,{}^p\mathcal{H}^0(K)[1]) \xrightarrow{\delta} \mathbb{H}^q(H, \operatorname{Im} N) \xrightarrow{\beta} \mathbb{H}^q(H, \psi_g^u\,{}^p\mathcal{H}^0(K)[-1])(-1) \to $$

where $N = \beta \circ \alpha$. Since the weights w of $\mathbb{H}^q(H, {}^p\mathcal{H}^0(i^*K)[-1])$ are $\leq {}^pw+q-1$, we prove that any element of $a \in \operatorname{Ker} N \subset \mathbb{H}^q(H, \psi_g^u j_{!*}{}^p\mathbf{L}[-1])$ of weight $w(a) \leq {}^pw + q - 1$ satisfy necessarily $\alpha(a) = 0$, hence a is in the image of sp.

We have $Na = \beta \circ \alpha(a) = 0$ and $w(\alpha(a)) = w(a) \leq {}^pw + q - 1$. Since the weights w of $\mathbb{H}^{q-1}(H, i^!\,{}^p\mathcal{H}^0(K)[1])$ satisfy $w \geq {}^pw + q$ we have: $w(\operatorname{Im}, \delta) = w(\operatorname{Ker} \beta) \geq {}^pw + q$ and $\alpha(a) \subset \operatorname{Ker} \beta$, hence $\alpha(a) = 0$.

ii) We apply i) to the diagram defined by a desingularization $\widetilde{V}$ of V and to the intermediate extension of ${}^p\mathbf{L}$ on $\widetilde{V}$. Since $j_{!*}{}^p\mathbf{L} \subset {}^p\mathcal{H}^0(K)$ in this case, we deduce the statement ii) from i).

Remark 7.4. i) We deduce from the exact sequence of the boundary, the decomposition in the lower cap of the octahedron on its turn:

$${}^p\mathcal{H}^i({}^p\varphi_g^u K) \xrightarrow{\sim} (\mathrm{Im}: {}^p\mathcal{H}^i(i^! K) \to {}^p\mathcal{H}^i({}^p\varphi_g^u K)) \oplus (\mathrm{Ker}: {}^p\mathcal{H}^i({}^p\varphi_g^u K) \to {}^p\mathcal{H}^i(i^* K))$$

ii) The exact sequence in the corollary may be extended by the same method, to a long exact sequence of four terms known after Clemens-Schmid in the case of constant coefficients:

$$\begin{aligned} &\to \mathbb{H}^{q-1}(H, j_{!*}{}^p\mathbf{L}) \xrightarrow{sp} \mathbb{H}^{q-1}(H, \psi_g^u j_{!*}{}^p\mathbf{L}) \xrightarrow{N} \mathbb{H}^{q-1}(H, \psi_g^u j_{!*}{}^p\mathbf{L})(-1) \\ &\to \mathbb{H}_H^{q+1}(V, j_{!*}{}^p\mathbf{L}) \xrightarrow{I} \mathbb{H}^{q+1}(H, j_{!*}{}^p\mathbf{L}) \end{aligned}$$

Remark 7.5. i) Let V_D be a tubular analytic neighbourhood of H defined over a small disc $D \subset \mathbb{C}$. We have a long exact Wang sequence

$$\mathbb{H}^r(V_D \setminus H, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^r(H, \psi_g^u j_{!*}{}^p\mathbf{L}) \xrightarrow{N} \mathbb{H}^r(H, \psi_g^u j_{!*}{}^p\mathbf{L})(-1) \to$$

where the weight is deduced from the weight on $\Psi_g^u j_{!*}{}^p\mathbf{L}$. From this we extract the short exact sequence of MHS

$$0 \to \mathrm{Coker}\, N | \mathbb{H}^{r-1}(H, \psi_g^u j_{!*}{}^p\mathbf{L})(-1) \to \mathbb{H}^r(V_D \backslash H, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^r(H, \psi_g^u j_{!*}{}^p\mathbf{L})^N \to 0$$

ii) In Corollary 7.5 ii), we may change $\mathbb{H}^q(H, j_{!*}{}^p\mathbf{L})$ into $\mathbb{H}^q(V_D, j_{!*}{}^p\mathbf{L})$ since V_D is a tubular neighborhood of Z.

iii) Let H be transverse to the strata of V except at a stratum $v \in V$ of dimension 0. A weaker version of the theorem where invariant cycles are lifted on $V \setminus \{v\}$ is used in [DeG 81].

7.2.1 Global invariant cycle theorem

Let $h : X \to T$ be a morphism of complex projective varieties, ${}^p\mathbf{L}$ a shifted polarized VHS of weight ${}^pw$ on a nonsingular Zariski open subset $U \subset X$ and $j_{!*}{}^p\mathbf{L}$ the intermediate extension of the perverse sheaf ${}^p\mathbf{L}$.

Lemma 7.2 (Global invariant cycle). *Let $T' \subset T$ denote the Zariski open dense subset of parameters $t \in T$ over which $R^q h_* j_{!*}{}^p\mathbf{L}$ is a polarized VHS for all q. Let $t_0 \in T'$(note that $T' \neq \emptyset$). The image of the restriction map*

$$\mathbb{H}^q(X, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^q(X_{t_0}, j_{!*}{}^p\mathbf{L}) \tag{7.16}$$

is the invariant subset $\mathbb{H}^q(X_{t_0}, j_{!*}\mathbf{L})^\pi$ of fixed elements under the monodromy action $\pi := \pi_1(T', t_0) \to Aut\, H^q(X_{t_0}, j_{!*}\mathbf{L})$.

Proof. (Similar to the non singular case ([De 80] Corollary 6.2.12)). By the invariant theory of local systems, the restriction morphism:
$\mathbb{H}^0(T', R^q h_* j_{!*} {}^p\mathbf{L}) \overset{c}{\hookrightarrow} (R^q h_* j_{!*} {}^p\mathbf{L})_{t_0} = \mathbb{H}^q(X_{t_0}, j_{!*} {}^p\mathbf{L})$ is injective with image the invariant subspace $(R^q h_* j_{!*} {}^p\mathbf{L})^{\pi}_{t_0}$. In the diagram

$$\mathbb{H}^q(X, j_{!*} {}^p\mathbf{L}) \xrightarrow{a} \mathbb{H}^q(X_{T'}, j_{!*} {}^p\mathbf{L}) \xrightarrow{b} \mathbb{H}^0(T', R^q h_* j_{!*} {}^p\mathbf{L}) \overset{c}{\hookrightarrow} \mathbb{H}^q(X_t, j_{!*} {}^p\mathbf{L})$$

the morphism b is surjective by the Decomposition Theorem. The restriction morphisms $c \circ b$ and $c \circ b \circ a$ have the same image since $\mathbb{H}^q(X, j_{!*} {}^p\mathbf{L})$ and $\mathbb{H}^q(X_{t_0}, j_{!*} {}^p\mathbf{L})$ are pure of weight ${}^p w + q$ while $\mathbb{H}^q(X_{T'}, j_{!*} {}^p\mathbf{L})$ is mixed of weights $w \geq {}^p w + q$.

Moreover, the morphism a is surjective on $W_{{}^p w+q}\, \mathbb{H}^q(X_{T'}, j_{!*} {}^p\mathbf{L})$, since we may apply an extension of (ch. 5, Corollary 5.7 i) to this case. The lemma follows by strictness of morphisms of MHS (we may suppose $W := X \setminus X_{T'}$ a NCD by blowing up subvarieties above W without modification of $X_{T'}$).

Chapter 8
Deligne Gabber local purity

We illustrate, on complex algebraic varieties, the proof of the local purity theorem in positive characteristic which preceded the decomposition theorem (see [DeG 81] and Theorem 1.1) .

In some sense, we have already expanded the proof in the previous chapters, including the local and global invariant cycle theorems in chapter 7. Hence, we adapt to complex algebraic varieties the original dense text in [DeG 81] without further comments.

In chapter 6 the method has been applied to give a simple proof of the decomposition once we have at our disposal the preliminaries on perverse sheaves in chapter 2.

A simplified proof combined with the decomposition theorem is given in chapter 10.

8.0.0.1 Induction

Let V be a complex algebraic variety. Since the statement of the local purity theorem is local at a point $v \in V$, we can suppose V projective by considering an affine neighborhood that can be embedded into a projective space. Let $j_{!*}{}^p\mathbf{L}$ be the intermediate extension of a shifted polarized VHS on V of weight ${}^pw$ and $B_v \subset V$ a small ball with center v.

The inequalities on the the weights w on $\mathbb{H}^i(B_v \setminus \{v\}, j_{!*}{}^p\mathbf{L})$, except the crucial case $i = 0, w = {}^pw$ are easily deduced by induction from the inequalities on the intersection with a general hyperplane section H of V through v by results on the Stein inclusion $B_v \setminus (H \cap B_v) \hookrightarrow B_v$ similar to Artin-Lefschetz theorems on affine morphisms ([BBD 83] 4.1). The proof will concentrate essentially on the crucial case $i = 0, w = {}^pw$ in §8.1,

Lemma 8.1 (Induction). *If we assume the local purity theorem (Theorem 1.1) in the case of dimension* $\dim V = m - 1$*, the inequalities on the weights* w *of the MHS on* $\mathbb{H}^i(B_v \setminus \{v\}, j_{!*}{}^p\mathbf{L})$ *(Proposition 7.4) are satisfied in the*

case $\dim V = m$*:*

$$w \leq {}^p w + i \text{ if } i \leq -1 \text{ and } w > {}^p w + i \text{ if } i \geq 0.$$

except: $Gr^W_{{}^p w}\mathbb{H}^0(B_v \setminus \{v\}, j_{!*}{}^p\mathbf{L}) = 0$ *and* $Gr^W_{{}^p w}\mathbb{H}^{-1}(B_v \setminus \{v\}, j_{!*}{}^p\mathbf{L}) = 0$.

The intersection cohomology $\mathbb{H}^*(\partial B_v, j_{!*}{}^p\mathbf{L})$ satisfy Poincaré's selfduality. The boundary ∂B_v intersects transversely a stratification of V when B_v is small enough. Then B_v^* retracts by deformation to its boundary and the duality by transport of structure transforms one set of inequalities on w into the other set. Hence, the proof is reduced to either one of the set of inequalities.

If $\dim V = 0$, the assertion is trivially satisfied. If $\dim V = 1$ and V is nonsingular at v, then B_v is isomorphic to a disc in $\mathbb{C}$. Let $L := \psi_z({}^p\mathbf{L})(v)$ be a nilpotent orbit of weight ${}^p w - 1$; as in (Example 5.4, 2) in §5.2.1.4, the weights w satisfy:

$$\mathrm{Gr}^W_w \mathbb{H}^{-1}(B_v \setminus \{v\}, j_{!*}{}^p\mathbf{L}) = \mathrm{Gr}^W_w(\mathrm{Ker}\, N : L \to L) \text{ for } w \leq {}^p w - 1$$
$$\mathrm{Gr}^W_w \mathbb{H}^0(B_v \setminus \{v\}, j_{!*}{}^p\mathbf{L}) = \mathrm{Gr}^W_{w-1}(\mathrm{Coker}\, N : L \to L) = \mathrm{Gr}^W_{w-1}\mathrm{Prim}\, L \text{ for } w > {}^p w.$$

If V is singular at v, the proof is reduced to the previous case by a desingularization $\pi : \widetilde{V} \to V$, since at each point $\widetilde{v} \in \pi^{-1}(v)$, the map π induces an isomorphism from $B^*_{\widetilde{v}}$ to a connected component of B_v^*.

If $\dim V > 1$, let ${}^p\mathbf{L}$ be defined on the complement of a divisor $Y \subset V$, $\mathcal{S}$ a stratification of V compatible with v and Y and H a general hyperplane section of V through v, transverse to all strata $S \subset V \setminus v$.

Proof (of the lemma by induction). Let $K = j_{!*}{}^p\mathbf{L}$, $V^* := V \setminus v, H^* := H \setminus v, j : X \setminus Y \to Y$ and $i_H : H \to V$. Let $i \neq 0, B_v^* := B_v \setminus v, H_v^* := H \cap B_v^*$ and $K_H = i_{H*}i_H^* K[-1]$. By transversality, Thom isomorphisms apply on V^* with a twist of MHS:

$$\mathbb{H}^{i-1}(H_v^*, K_H) = \mathbb{H}^{i-2}(H_v^*, K) \simeq \mathbb{H}^i_{H_v^*}(X_v^*, K)(1)$$

Hence, the exact sequence $\mathbb{H}^i_{H_v^*}(B_v^*, K) \to \mathbb{H}^i(B_v^*, K) \to \mathbb{H}^i(B_v^* \setminus H_v^*, K) \xrightarrow{1}$ may be rewritten as

$$\mathbb{H}^{i-1}(B_v^* \setminus H_v^*, K) \to \mathbb{H}^{i-1}(H_v^*, K_H) \to \mathbb{H}^i(B_v^*, K) \to$$

Since $B_v^* \setminus H_v^*$ is Stein, a version of the Artin-Lefschetz vanishing theorem applies in this case to prove that $\mathbb{H}^i(B_v^* \setminus H_v^*, K) = 0$ for $i > 0$. Hence we have: $\mathbb{H}^{i-1}(H_v^*, K_H)(-1) \simeq \mathbb{H}^i(B_v^*, K)$ if $i > 1$ and $H^0(H_v^*, K_H)(-1) \to \mathbb{H}^1(B_v^*, K) \to 0$ is surjective.

Let $j' : H \setminus (H \cap Y) \to H$. Since, by transversality, the shifted restriction $K_H \simeq j'_{!*}{}^p\mathbf{L}_{|H})_{|V^*}[-1]$ is perverse of weight ${}^p w - 1$, the inequalities $w > {}^p w - 1 + i - 1 = {}^p w + i - 2$ apply to $\mathbb{H}^{i-1}(H_v^*, K_H)$ by induction and necessarily $w > {}^p w + i$ holds for $\mathbb{H}^i(B_v^*, K)$ for $i \geq 1$.

We prove that $\mathbb{H}^0(B_v^*, K)$ has weights $w \geq {}^p w$ (weaker than $w > {}^p w$): Let $H_1 \not\ni v$ be a general hyperplane section not containing v and let $k : V^* \to V$. From the distinguished triangle $k_! k^* K \to k_* k^* K \to i_{v*} i_v^* k_* k^* K$,

we deduce the exact sequence of hypercohomology on the affine open subset $U_1 := V \setminus H_1$

$$\mathbb{H}^0(U_1, k_*k^*K) \xrightarrow{\gamma_0} H^0(i_v^*k_*k^*K) \to \mathbb{H}^1(U_1, k_!k^*K) = 0$$

where the last term vanishes as $k_!k^*K \in {}^pD_c^{\leq 0}$ ([BBD 83] Théorème 4.1.1) .

This proves the lemma since the weights w of $\mathbb{H}^0(U_1, k_*k^*K)$ satisfy $w \geq {}^pw$ on the open set $U_1 \setminus v$, and γ_0 is surjective.

8.0.1 The crucial case: $i = 0, -1, w = {}^pw$

The proof by induction on $\dim V$ is based on a fibration of V defined by a Lefschetz pencil of hyperplane sections. This will occupy the rest of this section.

Proposition 8.1. *Let $j_{!*}{}^p\mathbf{L}$ be the intermediate extension of a shifted polarized VHS on a complex algebraic variety V and $v \in V$, then :*

$$\mathrm{Gr}^W_{{}^pw}\mathbb{H}^{-1}(B_v^*, j_{!*}{}^p\mathbf{L}) = 0 \textit{ and dually } \mathrm{Gr}^W_{{}^pw}\mathbb{H}^0(B_v^*, j_{!*}{}^p\mathbf{L}) = 0.$$

8.0.1.1 Extension to hyperplane sections

Let H be a general hyperplane section of V through v. By induction, we suppose that the local Purity theorem holds on H.

Recall the notations: $V^* := V \setminus \{v\}$, $H^* := H \setminus \{v\}, k : V^* \to V, h : H^* \to H$, $i_v : \{v\} \to H$, $i_H : H \to V, H_v := B_v \cap H$, $H_v^* := H_v \setminus \{v\}$ and $B_v^* := B_v \setminus \{v\}$.

Two distinguished triangles are attached to any complex $L \in D_c^b(H)$:

$$h_!h^*L \to L \to i_{v*}i_v^*L \xrightarrow{+1} \text{ and } h_!h^*L \to h_*h^*L \to i_{v*}i_{h*}h^*L \xrightarrow{+1} \tag{8.1}$$

from which two exact sequences are deduced:

$$\begin{array}{ccccccc} \mathbb{H}^{-1}(H, L) \to & \mathbb{H}^{-1}(H_v, L) & \xrightarrow{\partial} & \mathbb{H}_c^0(H^*, L) & \xrightarrow{\gamma} & \mathbb{H}^0(H, L) \\ \downarrow R & \downarrow R_v & & \downarrow\simeq & & \downarrow \\ \mathbb{H}^{-1}(H^*, L) & \xrightarrow{\rho} \mathbb{H}^{-1}(H_v^*, L) & \xrightarrow{\partial} & \mathbb{H}_c^0(H^*, L) & \to & \mathbb{H}^0(H^*, L) \end{array} \tag{8.2}$$

Let $L := i_H^*j_{!*}{}^p\mathbf{L}$. Since $B_v - H_v$ is Stein and the fact that the cohomology of $i_v^*j_{!*}{}^p\mathbf{L}$ is concentrated in degree < 0 we have the first isomorphism:

$$\mathbb{H}^{-1}(H_v, L) \simeq \mathbb{H}^{-1}(B_v, j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^{-1}(B_v^*, j_{!*}{}^p\mathbf{L})$$

Sine $H^r(i_v^!j_{!*}{}^p\mathbf{L}) = 0$ if $r \leq 0$, we have the second isomorphism.

An element $\alpha \in \mathbb{H}^{-1}(B_v, j_{!*}{}^p\mathbf{L})$ will be identified with its restrictions to H_v and B_v^* by the above isomorphisms. Let $R_v(\alpha) := \alpha_{|H_v^*}$ be the restriction to H_v^*, then we identify $\partial(\alpha_{|H_v^*}) = \partial(\alpha) \in \mathbb{H}_c^0(H^*, L)$.
The space $\mathbb{H}^{-1}(B_v^*, j_{!*}{}^p\mathbf{L})$ being endowed with a MHS (Proposition 7.4), all these spaces are endowed with MHS. In particular the MHS on $\mathbb{H}^*(H, L)$ is defined by (Corollary 7.3).

Lemma 8.2 (Extension). *i) Given an element $\alpha \in \mathrm{Gr}_{{}^pw}^W \mathbb{H}^*(B_v^*, j_{!*}{}^p\mathbf{L})$, its restriction $\alpha_{|H_v^*}$ to H_v^* extends to an element*

$$\alpha_{H^*} \in \mathrm{Gr}_{{}^pw}^W \mathbb{H}^{-1}(H^*, j_{!*}{}^p\mathbf{L}) \text{ with } \rho(\alpha_{H^*}) = \alpha_{|H_v^*}.$$

ii) Equivalently, $\alpha \in \mathrm{Gr}_{{}^pw}^W \mathbb{H}^{-1}(H_v, j_{!*}{}^p\mathbf{L})$ extends to a section

$$\alpha_H \in \mathrm{Gr}_{{}^pw}^W \mathbb{H}^{-1}(H, j_{!*}{}^p\mathbf{L}).$$

The obstruction to extend $\alpha_{|H_v^*}$ is the element $\partial(\alpha_{|H_v^*}) = \partial(\alpha)$ in $\mathbb{H}_c^0(H^*, L)$ (Equation 8.2). We prove:

$\partial(\alpha) = 0 \in \mathbb{H}_c^0(H^*, L)$, which is equivalent to $\partial(\alpha_{|H_v^*}) = 0$.
We remark that the image of the obstruction $\gamma(\partial\alpha) = 0 \in \mathbb{H}^0(H, L)$.

Let $L' = h_{!*}(j_{!*}{}^p\mathbf{L})_{|H^*} := \tau_{\leq -2} h_*(j_{!*}{}^p\mathbf{L})_{|H^*} = [1] \circ h_{!*}((j_{!*}{}^p\mathbf{L})_{|H^*}[-1])$.
It differs from L only at v since $L_{|H^*} \simeq L'_{|H^*}$. As $L'[-1]$ is perverse, $H^r(i_v^! L') = 0$ if $r \leq -1$ and $H^r(i_v^* L') = H^{r+1}(i_v^* L'[-1]) = 0$ if $r \geq -1$. Hence

- If $i \leq 0$: $\mathbb{H}^i(H, L') \simeq \mathbb{H}^i(H^*, L')$ is pure of weight $a + i$, and $\mathbb{H}^i(H_v, L') \simeq \mathbb{H}^i(H_v^*, L')$ satisfy local purity inequalities by induction on H.
- If $i \geq 0$: $\mathbb{H}_c^i(H^*, L') \simeq \mathbb{H}^i(H, L')$ is pure of weight ${}^pw + i$.

Let H' be a general hyperplane section of V. Since $H \setminus H'$ is affine and $L_{|H^*}[-1]$ perverse, $\mathbb{H}^0(H \setminus H', h_!(L_{|H^*})) = 0$,

We apply the global section functor $\Gamma(H \setminus H', *)$ to the second triangle on the left of (8.1), to deduce the exact sequence of hypercohomology:

$$\mathbb{H}^{-1}(H \setminus H', L) \to \mathbb{H}^{-1}(H_v, L)) \xrightarrow{\partial} \mathbb{H}^0(H \setminus H', h_!(L_{|H^*})) = 0.$$

It follows that the obstruction to extend α to $\beta \in \mathrm{Gr}_{{}^pw}^W \mathbb{H}^{-1}(H \setminus H', L)$ is zero. Let $u : H \setminus H' \to H$. We apply $\Gamma(H, *)$ to the distinguished triangle:

$$i_{H \cap H' *} i_{H \cap H'}^! L \to L \to u_* u^* L \to$$

to deduce the exact sequence:

$$\mathbb{H}^{-1}(H, L) \to \mathbb{H}^{-1}(H \setminus (H \cap H'), L) \xrightarrow{\partial} \mathbb{H}_{H \cap H'}^0(H, L).$$

The obstruction to extend β to an element in $\mathrm{Gr}_{{}^pw}^W \mathbb{H}^{-1}(H, L)$ is the element

$$\xi = \partial\beta \in \mathrm{Gr}_{{}^pw}^W \mathbb{H}_{H \cap H'}^0(H, L) = \mathbb{H}_{H \cap H'}^0(H, L') \text{ as } v \notin H \cap H'.$$

Remark 8.1. The image ξ' of ξ by the morphism $\mathbb{H}^0_{H\cap H'}(H, L') \to \mathbb{H}^0(H, L')$ coincides with the image $(\partial\alpha)'$ of $\partial(\alpha)$ by the isomorphism:
$\mathbb{H}^0(H, h_! h^* L) \simeq \mathbb{H}^0(H, L')$.

To apply the global invariant cycle theorem on H (Lemma 7.2), we fix H and let H' vary in a Lefschetz pencil P of hyperplane sections of V. The hypercohomology spaces $\mathbb{H}^*(H \cap H', L')$ form a local system on a Zariski open subset U of P.

We choose an embedding of $H \hookrightarrow V \hookrightarrow \mathbb{P}$ into a projective space $\mathbb{P}$ and construct the diagram

$$\begin{array}{ccccc} H & \overset{q}{\leftarrow} & \widetilde{H}_{\check{\mathbb{P}}} & \overset{i}{\hookleftarrow} & \widetilde{H} \\ & & g\downarrow & & \downarrow g \\ & & \check{\mathbb{P}} & \hookleftarrow & P \end{array}$$

where $\widetilde{H}_{\check{\mathbb{P}}}$ is the disjoint union of all hyperplane sections of H, and $\check{\mathbb{P}}$ the paramater space of all hyperplane sections. Moreover P is a general line in the the dual projective space $\check{\mathbb{P}}$ and $\widetilde{H}$ is the restriction of $\widetilde{H}_{\check{\mathbb{P}}}$ over P. The morphism $q \circ i$ is defined by the blow-up of the axis of the pencil.

By the transversality of a general hyperplane section, $R^i g_* q^* L'$ is a local system over a dense Zariski open subset W of $\check{\mathbb{P}}$.

Let $U := W \cap P$ and $t \in U$. The fundamental group $\pi := \pi_1(U, t)$ acts as a subgroup of $\mathrm{GL}(\mathbb{H}^*(\widetilde{H}_t, L'))$ (In this case the action of $\pi_1(W, t)$ as a subgroup of $\mathrm{GL}(\mathbb{H}^*(\widetilde{H}_t, L'))$ coincides with the action of π). We remark that $\widetilde{H}_t := H \cap H'$ where H' is the hyperplane section in V.

We apply the global invariant cycle theorem to the morphism g over the Zariski-dense open subset U of P. For $t \in U$, the fundamental group $\pi_1(U, t)$ acts linearly on $\mathbb{H}^*(\widetilde{H}_t, L')$.

For a general H' with $v \notin H'$, we can view $\xi \in \mathbb{H}^0_{H\cap H'}(H, L')$ as an element in $\mathbb{H}^{-2}(H \cap H', L')$ via the Thom isomorphism.

The image of $\mathbb{H}^{-2}(H, L')$ in $\mathbb{H}^{-2}(H \cap H', L')$ under the restriction to $H \cap H'$ coincides with the image of $\mathbb{H}^0(P, R^{-2} g_*(i^* q^* L'))$ and is equal to the subspace of fixed elements $\mathbb{H}^{-2}(H \cap H', L')^\pi$ under the monodromy action of the fundamental group π; this subspace contains ξ ([De 80] 6.2.12, Corollary 4.1 ii).

By duality, the Gysin morphism

$$G : \mathbb{H}^{-2}(H, L')^\pi(-1) \hookrightarrow \mathbb{H}^0(H, L')$$

is injective. With the notation of Remark 8.1, if $\partial(\alpha) \neq 0$, we have $\xi \neq 0$.

Moreover, by transversality at a general point $t \in U$, the restriction $L'_{|H\cap H'}$ is a shifted intermediate extension on $H \cap H'$.

We introduce the following diagram of Gysin morphisms with coefficients in the derived category of sheaves of vector spaces:

$$\begin{array}{ccc} \mathrm{Gr}^W_{p_w-4}\mathbb{H}^{-2}(H\cap H', j_{!*}{}^p\mathbf{L}_{|V^*}) & \stackrel{G_1}{\to} & \mathrm{Gr}^W_{p_w-2}\mathbb{H}^0(H', j_{!*}{}^p\mathbf{L}_{|V^*}) \\ \downarrow G_2 & & \downarrow G_3 \\ \mathrm{Gr}^W_{p_w-2}\mathbb{H}^0(H, k_{!*}(j_{!*}{}^p\mathbf{L}_{|V^*})) & \stackrel{G_4}{\to} & \mathrm{Gr}^W_{p_w}\mathbb{H}^2(V, k_{!*}(j_{!*}{}^p\mathbf{L}_{|V^*})) \end{array} \tag{8.3}$$

where $j_{!*}{}^p\mathbf{L} := k_{!*}(j_{!*}{}^p\mathbf{L}_{|V^*})$. Recall that for any subvariety S' of codimension s in V which intersects transversely the strata of $\mathcal{S}$ (i.e. S' is *normally embedded* in the sense of [GMacP 88] [Max 19] Definition 3.4.5), one has the definition of an extension element $\eta_V^{2s} \in Ext^{2s}(\mathbb{Q}_{S'}, \mathbb{Q}_V)$ called the fundamental class [Ve 76].

It is based on the Thom isomorphism $i^!_{S'}\mathbb{Q} \simeq i^*_{S'}\mathbb{Q}[-2s]$ and extends by tensor product to coefficients in $j_{!*}{}^p\mathbf{L}$ since we have: $i^!_{S'}j_{!*}{}^p\mathbf{L} \simeq i^*_{S'}j_{!*}{}^p\mathbf{L}[-2s]$. This construction applies to G_1 and G_2 (resp. G_3) in the case of $H\cap H'$ (resp. H'). Special attention is needed in the case of the embedding $i_H : H \hookrightarrow V$, since H contains the stratum v. In this case we have a morphism in the derived category $\eta^2_{V^*} : \mathbb{Q}_{H^*}[-2] \to \mathbb{Q}_{V^*}$ defined by the fundamental class.

To construct the morphism G_4 after $\eta^2_{V^*}$, let L'' denote $j_{!*}{}^p\mathbf{L} = k_{!*}(j_{!*}{}^p\mathbf{L}_{|V^*})$ and introduce the long exact sequences

$$\begin{array}{ccccccc} H^1(i_v^*L'') & \stackrel{\delta_1}{\to} & \mathbb{H}^2_c(V^*, L'') & \stackrel{\gamma\simeq}{\to} & \mathbb{H}^2(V, L'') & \to & H^2(i_v^*L'') \\ \uparrow \eta^2_{V^*} & & \uparrow \eta^2_{cV^*} & & & & \\ H^{-1}(i_v^*L'') & \stackrel{\delta_{-1}}{\to} & \mathbb{H}^0_c(H^*, L'') & \stackrel{\gamma'}{\to} & \mathbb{H}^0(H, L'') & \to & H^0(i_v^*L'') \end{array}$$

defined by the distinguished triangle $k_!k^*L'' \to L'' \to i_{v*}i_v^*L'' \stackrel{[1]}{\to}$.
Here γ is an isomorphism and γ' is surjective, since $H^i(i_v^*L'') = 0$ for $i = 0, 1$.

We have $\delta_1 \circ \eta^2_{V^*} = \eta^2_{cV^*} \circ \delta_{-1}$, hence $\gamma \circ \eta^2_{cV^*} \circ \delta_{-1} = 0$.
Indeed, $\eta^2_{V^*}$ induces two morphisms $\eta^2_{B_v^*} : \mathbb{H}^{-1}(B_v \cap H^*, L'') \to \mathbb{H}^1(B_v^*, L'')$ (equivalently $H^{-1}(i_v^*L'') \to H^1(i_v^*L'')$) and $\eta^2_{cV^*} : \mathbb{H}^0_c(H^*, L'') \to \mathbb{H}^2_c(V^*, L'')$.

We deduce the morphism $G_4 : \mathbb{H}^0(H, L'') \to \mathbb{H}^2(V, L'')$.
Moreover, it is clear by this construction that G_4 is compatible with MHS.

We have $G_2(\xi) = \gamma(\partial\alpha) = 0$, hence $G_3 \circ G_1(\xi) = 0$. The morphism G_3 is injective since $\mathbb{H}^1(X - H', k_{!*}({}^p\mathcal{H}^i)) = 0$, hence $G_1(\xi) = 0$.

Finally, we fix H' and let H vary in a Lefschetz pencil of hyperplane sections H through v and we apply the global invariant cycle theorem again to prove that the Gysin morphism G_1 is injective, hence $\xi = 0$.

Remark 8.2. The original proof is based on a diagram defined by cohomology classes with support

$$\begin{array}{ccc} \mathbb{Q}_{H\cap H'}(-1)[-2] & \to & \mathbb{Q}'_H \\ \downarrow & & \downarrow \\ \mathbb{Q}_H & \to & \mathbb{Q}_V(1)[2] \end{array}$$

which generates by tensor product with coefficients in $L'' := j_{!*}{}^p\mathbf{L}$ the commutative diagram 8.3 of topological morphisms. However, the reference [Ve 76] applies only for nonsingular V; moreover H is not normally embedded since

it contains the stratum v. We modified the proof to apply directly to the Diagram 8.3.

8.0.1.2 Reduction to a fibration

To reduce the crucial case to the case where a fibration $g : V \to \mathbb{P}^1$ exists, we use a general Lefschetz pencil of hyperplanes as already mentioned in chapter 6, §6.2.0.3. Then, the local and global invariant cycle theorems for perverse cohomology sheaves (Proposition 7.5, Corollary 7.5) and the extension property (Lemma 8.2) apply.

Let $g(v) = 0$, D a complex disc with center 0 in $\mathbb{P}^1$ and $V_D := g^{-1}(D)$. We have the diagram

$$\begin{array}{ccccc} H \setminus \{v\} & \xrightarrow{u} & V_D \setminus \{v\} & \xhookleftarrow{\phi} & V_D \setminus H \\ \downarrow & & \downarrow g & & \downarrow \\ 0 & \xrightarrow{u'} & D & \xhookleftarrow{\phi'} & D^* \end{array} \tag{8.4}$$

Lemma 8.3. *Let H be a general hyperplane section and P a perverse sheaf on $V_D \setminus \{v\}$. With the notations of Diagram 8.4, there exists a decomposition in the derived category of complexes of sheaves of vector spaces on $H \setminus \{v\}$:*

$$u^*P \oplus u^*P[-1] \simeq u^*\phi_*\phi^*P.$$

The splitting of complexes of $\mathbb{Q}$-vector spaces at $0 \in D$

$$(u')^*\phi'_*\mathbb{Q} \simeq \mathbb{Q} \oplus \mathbb{Q}[-1](-1)$$

corresponds to the isomorphism $H^*(D^*, \mathbb{Q}) \simeq H^0(D^*, \mathbb{Q}) \oplus H^1(D^*, \mathbb{Q})$. In terms of the de Rham resolution $\mathcal{E}^*_{D^*}$ of $\mathbb{C}$, the generators are $1 \in \mathbb{Q}$ and $\frac{dz}{z}$ respectively.

In terms of a Whitney stratification $\mathcal{S}$ of V compatible with the perverse sheaf P, the restriction of the cohomology sheaves of P to a stratum S near a point $a \in H \setminus v$ is constant (there is no monodromy near a), hence we deduce that the natural morphism $P \otimes g^*\phi'_*\mathcal{E}^*_{D^*} \to \phi_*\phi^*P$ induces an isomorphism in the derived category of complexes of sheaves of vector spaces

$$u^*P \oplus u^*P[-1] \xrightarrow{\gamma} u^*\phi_*\phi^*P.$$

In view of this description we denote the decomposition as:

$$\forall i \in \mathbb{Z}, \quad \mathbb{H}^i(H \setminus v, u^*\phi_*\phi^*P) \simeq \mathbb{H}^i(H \setminus v, P) \oplus \eta\, \mathbb{H}^{i-1}(H \setminus v, P) \tag{8.5}$$

8.0.1.3 Lifting of α_H

Let $P := j_{!*}{}^p\mathbf{L}$ on V, $\mathrm{Coker}^{-2} N := \mathrm{Coker}\,(N : \mathbb{H}^{-2}(H, \psi_g^u P) \to \mathbb{H}^{-2}(H, \psi_g^u P)$.

By (Remark 7.5), we have an exact sequence:

$$0 \to \mathrm{Coker}^{-2} N(-1) \to \mathbb{H}^{-1}(V_D \setminus H, P) \to \mathbb{H}^{-1}(H, \psi_g^u P)^N \to 0$$

.

Since V_D retracts by deformation onto H, the element $\alpha_H \in \mathrm{Gr}^W_{{}^pw}\,\mathbb{H}^{-1}(H, P)$ in the Extension Lemma 8.2 corresponds to an element $\alpha_D \in Gr^W_{{}^pw}\mathbb{H}^{-1}(V_D, P)$, with restriction

$$\alpha_D^* \in Gr^W_{{}^pw}\mathbb{H}^{-1}(V_D \setminus H, P) \text{ on } V_D \setminus H.$$

The specialization (Equation 7.10)

$$\mathrm{sp}(\alpha_H) := \tilde{\alpha}_D^* \in \mathrm{Gr}^W_{{}^pw}\mathbb{H}^{-1}(H, \psi_g^u P)^N = 0$$

must vanish since the space $\mathbb{H}^{-1}(H, \psi_g^u P)^N$ is of weight $\leq {}^pw - 1$, while $w(\alpha_H) = {}^pw$.

Hence the section α_D^* lifts to a section

$$\tilde{\beta} \in \mathrm{Gr}^W_{{}^pw}(\mathbb{H}^{-2}(H, \psi_g^u P)(-1)) = \mathrm{Gr}^W_{{}^pw-2}(\mathbb{H}^{-2}(H, \psi_g^u P) : (\tilde{\beta} \mapsto \alpha_D^*)$$

$\tilde{\beta}$ is unique in $\mathrm{Coker}\, N := \mathrm{Coker}\,(N : \mathrm{Gr}^W_{{}^pw}\mathbb{H}^{-2}(H, \psi_g^u P) \to \mathrm{Gr}^W_{{}^pw-2}\mathbb{H}^{-2}(H, \psi_g^u P)$.

Since $\mathrm{Ker}\, N := \mathrm{Ker}\,(N : Gr^W_{{}^pw-2}\mathbb{H}^{-2}(H, \psi_g^u P) \to Gr^W_{{}^pw-4}\mathbb{H}^{-2}(H, \psi_g^u P)$ projects isomorphically onto $\mathrm{Coker}\, N$ as both are isomorphic to the space $Prim\, \mathrm{Gr}^W_{{}^pw-2}\mathbb{H}^{-2}(H, \psi_g^u P)$, we can choose the element $\tilde{\beta} \in Gr^W_{{}^pw-2}\mathbb{H}^{-2}(H, \psi_g^u P)^N$.

By the local invariant cycle theorem (Corollary 7.5), we lift this canonical choice of $\tilde{\beta}$ to an element $\beta_D \in Gr^W_{{}^pw-2}\mathbb{H}^{-2}(V_D, P)$.

Let $\beta_D^* \in \mathrm{Gr}^W_{{}^pw-2}\mathbb{H}^{-2}(V_D \setminus H, P)$ denote the restriction of β_D; we have

$$\eta \wedge \beta_D^* = \alpha_D^* \in \mathrm{Gr}^W_{{}^pw}\mathbb{H}^{-1}(V_D \setminus H, P) \tag{8.6}$$

8.0.1.4 First reduction to prove $\alpha = 0$

Let $B_v^* = B_v \setminus \{v\}, V_D^* = V_D \setminus \{v\}, H^* = H \setminus \{v\}, H_v = H \cap B_v, H_v^* = H_v \setminus \{v\}$, and consider the diagram

$$\begin{array}{ccc}
\alpha_*^* \in \mathrm{Gr}^W_{{}^pw}\mathbb{H}^{-1}(B_v \setminus H_v, j_{!*}{}^p\mathbf{L}) & \leftarrow & \alpha \in \mathrm{Gr}^W_{{}^pw}\mathbb{H}^{-1}(B_v^*, j_{!*}{}^p\mathbf{L}) \\
\uparrow & & \uparrow \\
\alpha_D^* \in \mathrm{Gr}^W_{{}^pw}\mathbb{H}^{-1}(V_D \setminus H, j_{!*}{}^p\mathbf{L}) & \leftarrow & \alpha_D \in \mathrm{Gr}^W_{{}^pw}\mathbb{H}^{-1}(V_D^*, j_{!*}{}^p\mathbf{L}) \\
\sigma \downarrow & & \downarrow \\
A & \xleftarrow{\sigma_1} & \alpha_H \in \mathrm{Gr}^W_{{}^pw}\mathbb{H}^{-1}(H^*, j_{!*}{}^p\mathbf{L})
\end{array}$$

where $\quad A := \mathrm{Gr}^W_{{}^pw}(\mathbb{H}^{-1}(H^*, P) \oplus \eta\, \mathrm{Gr}^W_{{}^pw-2}\mathbb{H}^{-2}(H^*, P)$ (by Equation 8.5)

The morphism $\sigma := (\sigma_1, \sigma_2)$ is defined by Equation 8.5. The element α_D restricts to $\alpha_D^* \in Gr^W_{{}^pw}\mathbb{H}^{-1}(V_D \setminus H, P)$ and to $\alpha_H \in Gr_a^W\mathbb{H}^{-1}(H^*, P)$ satisfying

the relation:

$$\sigma(\alpha_D^*) = \sigma_1(\alpha_H) \in Gr_{{}^p w}^W(\mathbb{H}^{-1}(H^*, P) \tag{8.7}$$

The diagram above is commutative and by Lemma 8.2 the restriction of α_D coincides with $\alpha \in \mathrm{Gr}_{{}^p w}^W \mathbb{H}^{-1}(B_v^*, P)$.

Lemma 8.4. *An element $\alpha \in \mathrm{Gr}_{{}^p w}^W \mathbb{H}^{-1}(B_v^*, j_{!*}{}^p\mathbf{L})$ vanishes if and only if its restriction α_*^* to $B_v \setminus H_v$ is zero.*

Proof. Since we have the exact sequence

$$\mathrm{Gr}_{{}^p w-2}^W \mathbb{H}^{-3}(H_v^*, j_{!*}{}^p\mathbf{L}) \xrightarrow{G} \mathrm{Gr}_{{}^p w}^W \mathbb{H}^{-1}(B_v^*, j_{!*}{}^p\mathbf{L}) \to \mathrm{Gr}_{{}^p w}^W \mathbb{H}^{-1}(B_v \setminus H_v, j_{!*}{}^p\mathbf{L})$$

it is enough to prove $\mathrm{Gr}_{{}^p w-2}^W \mathbb{H}^{-3}(H_v^*, j_{!*}{}^p\mathbf{L}) = 0$.

This follows by the inductive hypothesis on the small neighborhood H_v of v in H since $\mathbb{H}^{-3}(H_v^*, j_{!*}{}^p\mathbf{L}) = \mathbb{H}^{-2}(H_v^*, j_{!*}{}^p\mathbf{L}[-1])$ satisfy $w \leq {}^p w - 3$ by induction.

8.0.1.5 Proof of $\alpha_*^* = 0$ in $\mathrm{Gr}_{{}^p w}^W \mathbb{H}^{-1}(B_v \setminus H_v, j_{!*}{}^p\mathbf{L})$

We consider the diagram

$$\begin{array}{ccc}
\beta^* \in \mathrm{Gr}_{{}^p w-2}^W \mathbb{H}^{-2}(B_v^*, P) & \xrightarrow{\eta\circ\rho} & \alpha_*^* \in \mathrm{Gr}_{{}^p w}^W \mathbb{H}^{-1}(B_v \setminus H_v, P) \\
\uparrow & & \uparrow \\
\beta_D \in \mathrm{Gr}_{{}^p w-2}^W \mathbb{H}^{-2}(V_D^*, P) & \xrightarrow{\eta\circ\rho} & \alpha_D^* \in \mathrm{Gr}_{{}^p w}^W \mathbb{H}^{-1}(V_D \setminus H, P) \\
\downarrow & & \downarrow \sigma \\
\beta_H \in \mathrm{Gr}_{{}^p w-2}^W \mathbb{H}^{-2}(H^*, P) & \xrightarrow{\eta\cup} & A
\end{array} \tag{8.8}$$

where ρ denotes the restriction of β^* to $B_v \setminus H_v$, resp. β_D to $V_D \setminus H$.

Lemma 8.5. *The restriction $\beta_H \in \mathrm{Gr}_{{}^p w-2}^W \mathbb{H}^{-2}(H^*, P)$ of β_D vanishes.*

Proof. The element $\beta_D \in \mathrm{Gr}_{{}^p w-2}^W \mathbb{H}^{-2}(V_D^*, j_{!*}{}^p\mathbf{L})$ restricts to an element $\rho(\beta_D)$ on $V_D \setminus H$ such that $\eta \cup \rho(\beta_D) = \alpha_D^* \in \mathrm{Gr}_{{}^p w}^W \mathbb{H}^{-1}(V_D \setminus H, j_{!*}{}^p\mathbf{L})$ by Equation (8.6). By restricting to H^* we deduce:

$$\eta \cup \beta_H = \sigma(\eta \cup \rho(\beta_D)) = \sigma(\alpha_D^*) = \sigma_2(\alpha_D^*) \in \eta\, \mathrm{Gr}_{{}^p w-2}^W(\mathbb{H}^{-2}(H^*, j_{!*}{}^p\mathbf{L})) \subset A$$

By (8.7) we have the equality $\sigma(\alpha_D^*) = \sigma_1(\alpha_H)$, $\sigma_2(\alpha_D^*) = \sigma(\alpha_D^*) = \sigma_1(\alpha_H) = 0$ by the decomposition property of A; then $\beta_H = 0$ since the cup product $\eta\cup$ is injective on $\mathrm{Gr}_{a-2}^W \mathbb{H}^{-2}(H^*, j_{!*}{}^p\mathbf{L})$.

We denote by β^* the restriction of β_D to B_v^* in Diagram (8.8), and by $\beta_{H_v^*}^*$ the restriction of β_H to H_v^*.

Lemma 8.6. *i) The restriction $\mathrm{Gr}_{{}^p w-2}^W \mathbb{H}^{-2}(B_v^*, j_{!*}{}^p\mathbf{L}) \to \mathrm{Gr}_{{}^p w-2}^W \mathbb{H}^{-2}(H_v^*, j_{!*}{}^p\mathbf{L}) : \beta^* \mapsto \beta_{H_v^*}^*$ is injective.*

ii) We have: $\beta^* = \beta_{H_v^*}^* = 0$ and $\alpha_*^* = \eta \cup \rho(\beta^*) = 0$.

Proof. i) Let $j_H : B_v \setminus H_v \to B_v$ denote the open embedding, since the open subset $B_v \setminus H_v$ is Stein, we have
$\mathbb{H}^{-2}(B_v \setminus H_v, j_{H!} j_H^* P) = 0 = \mathbb{H}^{-1}(B_v \setminus H_v, j_{H!} j_H^* P)$.

ii) We have $\beta^*_{H_v^*} = 0$ as the restriction of $\beta_H = 0$, hence $\beta^* = 0$ and
$\alpha_*^* = \eta \cup \rho(\beta^*) = 0 \in \mathrm{Gr}^W_{p_w - 2} \mathbb{H}^{-2}(B_v \setminus H_v, j_{!*} {}^p\mathbf{L})$.

The local invariant cycle theorem is needed here as the lemma applies exactly to β_D and not to α_D. The final conclusion $\alpha = 0 \in \mathrm{Gr}^W_{p_w} \mathbb{H}^{-1}(B_v^*, j_{!*} {}^p\mathbf{L})$ follows by (Lemma 8.4).

Remark 8.3. Let $f : X \to V$ be a projective morphism of complex algebraic varieties and $v \in V$, with fiber $X_v := f^{-1}(v) \subset X$. The local purity theorem may be interpreted as a relative local purity (ch. 9, Theorem 9.2) and in terms of the intersection morphism I: $\mathrm{Gr}_j^{{}^p\tau} \mathbb{H}^i_{X_v}(X, j_{!*} {}^p\mathbf{L}) \xrightarrow{{}^p\tau I_i^j} \mathrm{Gr}_j^{{}^p\tau} \mathbb{H}^i(X_v, j_{!*} {}^p\mathbf{L})$ (§10.4).

Chapter 9
Weight filtration, decomposition and local purity

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We start a new section of this volume consisting of chapters $9-12$, to give a new combined direct geometric proof of the local purity (ch. 1, Theorem 1.1) and the decomposition theorem (ch. 1, Theorem 1.2).

In this section, we do not use the nearby or vanishing cycles complex to construct the weight filtration on the logarithmic complex $\Omega^*\mathcal{L} \simeq j_*\mathbf{L}$.

The proof of local purity differs from the proof in [DeG 81]. An interpretation in terms of Intersection morphisms is given at the end.

We refer to the necessary preliminaries in chapters 2 and 3, namely §2.3 on the degeneration of VMHS, §2.3.2 on infinitesimal mixed Hodge structure (IMHS), Remark 2.10 on IMHS defined by an admissible VMHS, chapter 2.4.0.1 on the embedding of the perverse filtration, §3 on perverse sub-sheaves of the logarithmic complex $\Omega^{*p}\mathcal{L} \simeq j_*{}^p\mathbf{L}$ and Remark 3.6 vii on the decomposition of the Hodge filtration following the decomposition of spaces with underlying MHS.

9.1 Introduction to chapters $9-11$ (section 3)

Given a graded polarized admissible variation of MHS $(\mathbf{L}, W^0, F)$ on the complement of a NCD $Y \subset X$, we develop in §9.1.0.2 a direct definition of the weight filtration W on the logarithmic complex $\Omega^*\mathcal{L}$ (§3.1) by perverse sub-sheaves, extending naturally the weight filtration W^0 on $\mathbf{L}$, as an application of the local study by Kashiwara [Ka 86] (see ch. 2, 2.3 and ch. 3, §3.2.6).

The weight filtration W is defined in this chapter in terms of the local monodromy defined by the coordinates (§3.1.1) at points of the NCD Y.

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Since Y is defined locally by an equation $f = z_{i_1} \cdots z_{i_n}$ product of local coordinates, the weight filtration is related to the monodromy weight filtration on the nearby cycle complex $\psi_f \mathbf{L}$ by (Equation 5.54, §5.1.2 Remark 5.4).

Since $Gr_*^W \Omega^* \mathcal{L}$ satisfy a decompositon property (§9.2.2) into a direct sum of (shifted) intermediate extensions with Hodge filtration induced by $(\Omega^* \mathcal{L}, F)$, we deduce from the purity theorem on intermediate extensions with induced Hodge filtration (Theorem 3.1), the existence of a structure of MHC on $(\Omega^* \mathcal{L}, F, W)$ as well on the sub-complex $IC^* \mathcal{L}(LogZ)$.

We introduce a relative version of the local purity theorem to reduce the proof to the non singular case.

The decomposition theorem is proved by induction on the dimension of X, first by a reduction to general hyperplane sections and then to isolated strata. The proof in chapter 10 is combined with the proof of the relative version of the local purity theorem.

Originally, the work on this section preceded the previous sections (see [EL 14] and [EY 14]).

9.1.0.1 Weight filtration on $\Omega^* \mathcal{L}$

Let $Y \subset X$ be a NCD and $(\mathbf{L}, W^0, F)$ a graded admissible polarized variation of MHS on $X^* := X \setminus Y$. We define the weight W on $\Omega^* \mathcal{L}$, extending W^0 over $X^* := X \setminus Y$, in terms of various local weight monodromy filtrations at points $x \in Y$. We prove the following theorem:

Theorem 9.1. *Let $(\mathbf{L}, W^0, F)$ be a graded polarized admissible VMHS on $X \setminus Y$. There exists a weight filtration W on the logarithmic complex with coefficients in Deligne's extension $\mathcal{L}_X$ (§3.1) by complexes of analytic perverse sub-sheaves and a Hodge filtration F by complexes of analytic sub-sheaves such that the bi-filtered complex*

$$\Omega^* \mathcal{L} := (\Omega_X^*(Log\, Y) \otimes \mathcal{L}_X, W, F) \tag{9.1}$$

induces a MHS on the cohomology groups $\mathbb{H}^i(X \setminus Y, \mathbf{L})$.

The construction of W and the proof of the theorem will occupy the rest of the chapter.

9.1.0.2 Logarithmic complexes along a NCD

Various complexes with weight and Hodge filtrations related to a NCD subdivisor Z of $Y \subset X$ are defined by restriction to Z, with support in Z or Verdier's duality.

Let ${}^p\mathbf{L} := \mathbf{L}[\dim X]$ denote the associated perverse sheaf on $X \setminus Y$ with the conventional shifted degree. It is called a shifted VMHS.

Let $i_Z : Z \to X$, $j := (X \setminus Y) \to X$ and $j_Z := (X \setminus Z) \to X$. We construct the weight and Hodge filtrations on the sub-complex $IC^*\mathcal{L}(Log\, Z) \subset \Omega^*\mathcal{L}$ isomorphic to $j_{Z*}(j_{!*}\mathbf{L})_{|X\setminus Z}$, from which we deduce logarithmic complexes isomorphic to $j_{Z!}(j_{!*}{}^p\mathbf{L})_{|X\setminus Z}, i_Z^* j_{Z*}(j_{!*}{}^p\mathbf{L})_{|X\setminus Z}, i_Z^! j_{!*}{}^p\mathbf{L}$ and $i_Z^* j_{!*}{}^p\mathbf{L}$ (see ch.3, §3.2.4). *We refer to such complexes as bifiltered logarithmic complexes in the derived category of bifiltered sheaves of abelian groups.*

9.1.0.3 Relative local purity

We modify the statement of the local purity theorem (ch. 1, Theorem 1.1), to explore its relation with the decomposition theorem.

Let $K \in D_c^b(V, \mathbb{Q})$ be a complex in the derived category of complexes with constructible cohomology on an analytic space V. The topological middle perversity truncations of K define an increasing *perverse filtration* ${}^p\tau$ of K on V ([BBD 83] section 2 and proposition 2.1.17).

We deduce for each subset S of V, an increasing filtration ${}^p\tau$ on the hypercohomology of the restriction of K to S:

$$ {}^p\tau_i\mathbb{H}^k(S, K) := \operatorname{Im}\left\{\mathbb{H}^k(S, {}^p\tau_i K) \to \mathbb{H}^k(S, K)\right\}. \tag{9.2} $$

Let $f : X \to V$ be a projective morphism and let $j_{!*}{}^p\mathbf{L}$ be the intermediate extension on X of a (shifted) polarized variation of Hodge structures (VH) of weight ${}^pw$ on a smooth Zariski open subset of X.

Let $B_{X_v} := f^{-1}(B_v)$ be the inverse image of a small ball B_v with center $v \in V$ and $X_v := f^{-1}(v) \subset B_{X_v}$. A *perverse filtration* ${}^p\tau$ on $\mathbb{H}^k(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L})$ is defined by the natural equality $\mathbb{H}^k(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L}) = \mathbb{H}^k(B_v \setminus v, f_* j_{!*}{}^p\mathbf{L})$:

$$ {}^p\tau_i\mathbb{H}^k(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L}) := {}^p\tau_i\mathbb{H}^k(B_v \setminus v, f_* j_{!*}{}^p\mathbf{L}) \tag{9.3} $$

(f_* is the derived functor, in general we do not use henceforth the letter R for a derived functor (see chapter 1, §2.4 and 2.4.0.2 for more conventions).

The hypercohomology $\mathbb{H}^k(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L})$ is independent of the choice of B_v with small radius and carry a natural MHS such that the perverse filtration is a filtration by sub-MHS (ch.11.1 Proposition 10.2 and ch. 10 Proposition 10.1). We prove in chapter 10:

Theorem 9.2 (Relative local purity). *Let $f : X \to V$ be a projective morphism of complex algebraic varieties, $v \in V$, $X_v := f^{-1}(v) \subset X$ and $B_{X_v} := f^{-1}(B_v)$ the inverse image of a small ball $B_v \subset V$.*

Let $j_{!*}{}^p\mathbf{L}$ be a (shifted) polarized variation of HS on X of weight ${}^pw$, the weight w of the MHS on the graded-cohomology spaces with respect to the middle perversity filtration ${}^p\tau$ satisfy the inequalities: $wGr_i^{{}^p\tau}\mathbb{H}^r(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L})$:

$$ w \leq {}^pw + r \text{ if } r - i \leq -1; \qquad w > {}^pw + r \text{ if } r - i \geq 0. \tag{9.4} $$

Remark 9.1. i) We say that $j_{!*}{}^p\mathbf{L}$ is locally pure at v relative to f.

ii) Let $K := f_* j_{!*}{}^p\mathbf{L}$. If the decomposition theorem apply to K on $V \setminus v$, then $j_{!*}{}^p\mathbf{L}$ is locally pure at v relative to f if and only if ${}^p\mathcal{H}^i(K)$ is locally pure at v for each i, since $Gr_i^{{}^p\tau}\mathbb{H}^r(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^{r-i}(B_v \setminus v, {}^p\mathcal{H}^i(K))$. In this case, we say that K is locally pure at v.

iii) In the case of a finite morphism and a non singular variety X, $\dim X = \dim V = n$, X_v is a finite number of points and $B_{X_v} \setminus X_v$ is a finite number of punctured discs of dimension n. Since $f_* j_{!*}{}^p\mathbf{L}$ is perverse, the ${}^p\tau$-filtration is reduced to one term ${}^p\mathcal{H}^0(f_* j_{!*}{}^p\mathbf{L}) = f_* j_{!*}{}^p\mathbf{L}$.

The statement is reduced to a condition on the weight w of $\mathbb{H}^r(D^n \setminus 0, j_{!*}{}^p\mathbf{L})$. In this special case, the theorem is proved in ([CaKSc 87], Corollary 1.13, [KaK 87], Theorem 4.0.1) since $\mathbb{H}^r(D^n \setminus 0, j_{!*}{}^p\mathbf{L}) = \mathbb{H}^r(D^n, j_{!*}{}^p\mathbf{L})$ for $r \leq -1$.

9.1.0.4 The decomposition theorem

The combined new proof of the decomposition theorem (ch.1, Theorem 1.2) and the local purity (ch.1, Theorem 1.1), is by induction on the dimension of X and reduction to isolated strata (ch.10 and ch.10.3), in terms of a Thom-Whitney stratification.

9.1.0.5 Local purity and Intersection isomorphisms

The relative local purity theorem with coefficients in a polarized variation of HS is related to the Intersection morphism in the case of a divisor on a variety X contracting to $v \in V$ (ch. 10.3, §10.4).

In the case of constant coefficients, the intersection morphism is studied in ([CaMi 5] section 6.3) where a generalized Grauert contractibility criterion is proved. This statement is in fact equivalent to the purity theorem (see §10.4.1) and it is related to the proof of the decomposition theorem in [CaMi 5].

Since MHS on cohomology can be assumed in the constant coefficients case, the proofs in [CaMi 5] are different from the general case where the construction of mixed Hodge theory is simultaneous and included in the proof. The reader will find in [CaMi 5] numerous interesting examples.

9.1.0.6 Fibration by NCD over the strata

The proof of the decomposition theorem is always reduced to the case of a fiber by NCD above an isolated stratum. This reduction is local. By induction, this amounts to the case where the inverse of a stratum is locally a fibration by NCD.

We remark at the end of chapter 11 (§11.2) that it is possible to reduce all proofs to the case of a family of projective morphisms fibered by NCD above the strata of a global fibration of V.

9.1.0.7 Admissible perverse VMHS on $X \setminus Z$

We added in this chapter complements on admissible perverse VMHS on $X \setminus Z$ ([Sa 90] and §9.7) to answer a question by the referee. We discuss the subject with hints to the proof as it is a motivation to the introduction of the filtration $N * W$ and shows its relation to Verdier's classification.

9.2 The weight filtration W

Let $\mathbf{L}$ be a VHS. The relation between the weight filtration on $\Psi^* L$ (Proposition 5.1) and the weight filtration W on $\Omega^* L$ is defined by (Equation 5.54):

$$\mathrm{Gr}_l^W j_{Z*} j_Z^*(j_{!*}{}^p\mathbf{L}) = \mathrm{Gr}_{l-1}^W(\mathrm{Coker}\, N) \text{ if } l > {}^p w$$

where N acts on the nearby cycle complex (Equation 9.43).

This chapter, is a definition, in the case of a VMHS $\mathbf{L}$, of the weight filtration W on $\Omega^* L$ described by repeated star transformation $N * W$ of a filtration W of L defined by Kashiwara ([Ka 86], 3.4, §9.2.1).

The main results of [Ka 86] is a test of admissibility in codimension one and an important property of the relative monodromy useful to extend results from the pure case to the mixed case:

Proposition 9.1 (Kashiwara ([Ka 86], Theorem 3.2.9)). *Let L be an object of an abelian category with an increasing filtration W and $N : (L, W) \to (L, W)$ a nilpotent endomorphism compatible with W. The relative monodromy filtration M satisfy the following natural decomposition*

$$Gr_l^M L \simeq \oplus_k Gr_k^W Gr_l^M L \tag{9.5}$$

In particular, there exist natural sections $\mathrm{Gr}_r^M \mathrm{Gr}_k^W L \to \mathrm{Gr}_r^M L$.

We give at the end of the paragraph (9.7.1.3 below) informations on the proof and the consequences.

However, we find in [Ka 86] various local statements needed to define the weight filtration W on $\Omega^* \mathcal{L}$ which was probably the projected result but do not figure there, neither the motivation of the definition $N * W$.

In this chapter the properties of weight filtration W on $\Omega^* \mathcal{L}$ are discussed and the decomposition of $Gr_*^W \Omega^* \mathcal{L}$ is proved, which leads to a new proof of the decomposition theorem (Theorem 10.1, see also Theorem 11.1).

We consider a graded polarized admissible variation of MHS on the underlying local system $\mathbf{L}$ on $X^* := X \setminus Y$, satisfying asymptotic properties at points of Y, summarized in the condition of admissibility (ch. 2, §2.3).

The Hodge filtration F denotes a filtration $\mathcal{F}_{X^*}$ by sub-bundles of $\mathcal{L}_{X^*} := \mathbf{L} \otimes \mathcal{O}_{X^*}$. By definition, it extends into a filtration of Deligne's extension $\mathcal{L}_X$ by analytic sub-bundles $\mathcal{F}_X$.

The filtration by sub-local systems W^0 of $\mathbf{L}$ extends to a filtration by perverse analytic sub- sheaves $\mathcal{W}^0 \subset \mathcal{L}_X$ which restricts to a filtration by sub-bundles on each stratum of Y.

By the admissibility of $\mathbf{L}$, the monodromy filtration relative to the extension $\mathcal{W}^0$ along Y exists. Both filtrations $\mathcal{W}^0$ and $\mathcal{F}_X$ are combined to define the global weight filtration W on $\Omega^*\mathcal{L}$ (Definition 9.2), although the construction below of W is local. Such results are used in chapters 10 and 10.3 to prove the theorem 11.1.

Remark 9.2. The weight W is a filtration by $\mathbb{Q}$-perverse sub-sheaves although the definition is by analytic sub-sheaves of $\Omega^i\mathcal{L} := \Omega^i_X(Log\, Y) \otimes \mathcal{L}_X$ in each degree. The reader should be aware that the filtration W on $\Omega^*\mathcal{L}$ is called weight since it induces the weight of the MHS on the hypercohomology.

In the case of intermediate extension the weight is trivial. However, we will see that the local cohomology carry also weight filtrations as local orbits or IMHS (Definition 2.14) and the local purity property is satisfied (it is the original conjecture by Deligne to prove purity).

The weight of a nilpotent orbit of dimension n is defined by $N_1 + \cdots + N_n$, otherwise we write $M(A, W)$ for the relative weight filtration defined by an endomorphism A compatible with W.

9.2.1 Local weight filtration on Ω^*L

In view of the local description of perverse sheaves on a product of discs ([Ka 86] §2 and ch.3, §3.2.6) many constructions on logarithmic complexes, such as the weight filtrations, will appear first on the complex of vector spaces Ω^*L (ch.3, §3.1.2) where $L := \mathcal{L}_X(x)$ at a point $x \in Y$, and will be carried later on the logarithmic complex by the tilde embedding (Lemma 3.1).

The properties of IMHS (Definition 2.14) are used in the description of the decomposition of $Gr^W_* \Omega^*L$.

9.2.1.1 The filtration $N * W$

Let (L, W, N) denote an increasing filtration W on a vector space L with a nilpotent endomorphism compatible with W ($NW_i \subset W_i$) such that the relative monodromy filtration $M(N, W)$ exists.

A new filtration $N * W$ of L is defined by Kashiwara ([Ka 86], 3.4):

$$(N*W)_k := NW_{k+1} + M_k(N, W) \cap W_k = NW_{k+1} + M_k(N, W) \cap W_{k+1} \quad (9.6)$$

where the last equality follows from ([Ka 86], Prop 3.4.1). The endomorphism $N : L \to L$ as well the identity $I : L \to L$ shift the degrees:

$$N : W_k \to (N * W)_{k-1}, \quad I : (N * W)_{k-1} \to W_k \tag{9.7}$$

9.2.1.2 Properties of $N * W$

1) *The couple* $((L, W) \xrightarrow{N} (L, N * W))$ *form a (graded) distinguished pair* (proved in ([Ka 86], Lemma 3.4.2 and admitted here): we have the decomposition property of $N * W$

$$\begin{aligned}
&Gr_k^{N*W} L \simeq \mathrm{Im}\ (N : Gr_{k+1}^W L \to Gr_k^{N*W} L) \oplus \ker\ (I : Gr_k^{N*W} L \to Gr_{k+1}^W L) \\
&Im\,(N : Gr_{k+1}^W L \to Gr_k^{N*W} L) \xrightarrow{\sim} Im\,(N : Gr_{k+1}^W L \to Gr_{k+1}^W L) \\
&\ker\ (I : Gr_k^{N*W} L \to Gr_{k+1}^W L) \simeq \mathrm{Coker}(N : W_k Gr_{k+2}^M \to W_k Gr_k^M)
\end{aligned} \tag{9.8}$$

where $M := M(N, W)$ (see also [Ka 86], Corollary 3.4.3)

2) We have the following properties of M and the induced filtration W on $Gr_k^{N*W} L$ by ([Ka 86], Lemma 3.4.2):

$$\begin{aligned}
M(N, W) =& M(N, N * W) \\
W_{k+1} Gr_k^{N*W} L =& Gr_k^{N*W} L, \\
Gr_{k+1}^W Gr_k^{N*W} L \simeq& \mathrm{Im}(N : Gr_{k+1}^W L \to Gr_{k+1}^W L), \\
W_k Gr_k^{N*W} L =& \ker\ (I : Gr_k^{N*W} L \to Gr_{k+1}^W L) \\
\forall p \le k :\ Gr_p^W (Gr_k^{N*W} L) \simeq& \mathrm{Coker}\,(N : Gr_{k+2}^M Gr_p^W L \to Gr_k^M Gr_p^W L)
\end{aligned} \tag{9.9}$$

Since N is strict with respect to W (as an IMHS), we have ([Ka 86], Corollary 3.4.3):

$$Gr_k^{N*W} L \simeq \mathrm{Im}(N : Gr_{k+1}^W L \to Gr_{k+1}^W L) \oplus \mathrm{Coker}\,(N : W_k Gr_{k+2}^M \to W_k Gr_k^M) \tag{9.10}$$

Lemma 9.1. *([Ka 86], Proposition 5.3.1, Corollary 5.5.4).*

Let $(L, W, F, N_1, \ldots, N_n)$ *be an IMHS, then* $(L, N_1 * W, F, N_1, \ldots, N_n)$ *and* $(L, M(N_1, W), F, N_i, i \neq 1)$ *are IMHS.*

Hint to the proof. By (§2.3.2, Remark 2.9), there exists on (L, N_1, W) an adequate filtration F such that (L, N_1, W, F) is an IMHS, hence $N_1 * W$ is well defined.

Let $M := M(N_1, W)$. By the nilpotent orbit structure on $Gr_k^W L$, the HS defined $F(t) := exp(i \sum_j t_j N_J)$ for $t_j \gg 0$ is polarized, hence $Gr_l^M Gr_k^W L$ is polarized and the polarization is independent of t. Then by (Equation 9.1), $Gr_l^M L$ is polarized. From which we deduce that $(L, M(N_1, W), F, N_i, i \neq 1)$ is a mixed nilpotent orbit. By (Equation 9.10) applied to N_1, $\mathrm{Coker}\, N_1$ is a

nilpotent orbit, as well $N_1 Gr^W_{k+1} L$ by the descent lemma (ch.5, lemma 5.1), hence the direct sum $Gr_k^{N_1 * W} L$ is also a nilpotent orbit.

Corollary 9.1. *The decomposition of $Gr_k^{N_i * W} L$ (Equation 9.8) is in the abelian category of IMHS.*

The commutativity of the star transformation ([Ka 86], Proposition 5.5.5):

$$N_1 * N_2 * W = N_2 * N_1 * W \tag{9.11}$$

is deduced by Kashiwara from to the relation ([Ka 86], Theorem 5.5.1)

$$N_2 * M(N_1, W) = M(N_1, N_2 * W)$$

9.2.1.3 The filtration W^J associated to an IMHS

Let $(L, W^0, F, N_i, i \in M)$ denote an IMHS and let $J \subset M$.
By convention, we set for $J = \emptyset \in M$, $W^\emptyset := W^0$ on L.

Since by induction, $(L, W^J, F, N_i, i \in M)$ is an IMHS and by the commutativity of the star transformation, we define by induction

$$W^J := N_{i_1} * (\ldots (N_{i_j} * W) \ldots) \text{ for } J = \{i_1, \ldots, i_j\} \tag{9.12}$$

(W^J is denoted by $\Psi_J W$ in [Ka 86], 5.8.2).

The increasing filtration W^J on L does not depend on the order of composition of the respective transformations $N_{i_k} *$ since by (Equation 9.11) for all $i_p, i_q \in J$: $N_{i_p} * (N_{i_q} * W) = N_{i_q} * (N_{i_p} * W)$.
The relative weight filtration with respect to W is denoted by

$$M(J, W) := M(N, W) \text{ for } N \in C(J) := \{\Sigma_{j \in J} t_j N_j, t_j > 0\}, \tag{9.13}$$

Let $i \in J \subset M$. We have: $M(N_i, W^{J \setminus i}) = M(N_i, W^J)$ by ([Ka 86], Lemma 3.4.2) and in general, the following equalities:

$$\forall J, K \subset M : M(J \cup K, W) = M(J, M(K, W)) \tag{9.14}$$

are proved in ([Ka 86], Proposition 5.2.5). The star transformation satisfy the relations:

$$\begin{aligned} &\forall J \subset K \subset M : M(K, W^J) = M(K, W) \\ &\forall J_1, J_2 \subset M : M(J_1, W^{J_2}) = M(J_1, W)^{J_2} \end{aligned} \tag{9.15}$$

([Ka 86], Equations 5.8.6 and 5.8.5). In particular, let $N_J = \sum_{i \in J} N_i$

$$\forall i \in J, \quad M(N_J, W^J) = M(N_J, W) = M(N_i, M(N_{J \setminus i}, W) \tag{9.16}$$

9.2.1.4 The weigt W on $\Omega^* L$

The filtrations W^J fit together to define the weight filtration W on $\Omega^* L$:

Definition 9.1 (Weigt W on $\Omega^* L$). i) Let $(L, W, F, N_i, i \in M)$ be an IMHS. On the de Rham complex $\Omega^* L$ (ch.3, Definition 3.3), the filtrations W and F are defined by the families $W^J_{k-|J|}L$ and $F^{k-|J|}L$ for $J \subset M$:

$$W_k \Omega^* L := \boldsymbol{s}(W^J_{k-|J|}L, N_i, i \in M)_{J \subset M},\ F^p \Omega^* L := \boldsymbol{s}(F^{k-|J|}L, N_i, i \in M)_{J \subset M}$$

ii) The weight and Hodge filtrations on $IC^* L$ (ch.3, Definition 3.4) are induced by the embedding $IC^* L \hookrightarrow \Omega^* L$

where $W^\emptyset = W^0$. For example, in dimension 2:
$W_k \Omega^* L := (W^0_k L \xrightarrow{N_1, N_2} W^1_{k-1} L \oplus W^2_{k-1} L \xrightarrow{N_1, -N_2} W^{1,2}_{k-2} L.$

Remark 9.3. The induced morphism $N_i : W^J_k \to (N_i * W^J)_{k-1}$ drops the degree of the filtration W^J. For further use, it is important to add to the above data, the canonical inclusion $I : (N_i * W^J)_{k-1} \to W^J_k$.

Beware that the weight filtration W^0 on L underlying the variation of MHS, extends as a constant filtration W^0 on $\Omega^* L$ but it is different from the weight W. Later we shall see that the local definition of W on $\Omega^* L$ extends to a global weight filtration on the de Rham complex.

9.2.2 Decomposition of $Gr^W_* \Omega^* L$

The following proposition is the basic local result in the construction of the structure of MHC on the logarithmic de Rham complex

Proposition 9.2 (local decomposition). *The graded complex with respect to the filtration W on $\Omega^* L$ (Definition 9.1) satisfy the decomposition property*

$$(\mathrm{Gr}^W_k \Omega^* L, F) \xrightarrow{\sim} \oplus_{K \subset M} (IC^* P^K_{k-|K|} L, F)[-|K|](-|K|) \tag{9.17}$$

into a direct sum of intersection complexes (ch.3, Definition 3.4) of nilpotent orbit $(P^K_{k-|K|}L, F, N_i, i \in M \setminus K)$ of weight $k - |K|$ for all $K \subset M$.
The component $(IC^* \mathrm{Gr}^{W^0}_k L, F)$ figures in the direct sum for $K = \emptyset$.

Remark that $P^K_{k-|K|}L, F)[-|K|](-|K|)$ is of weight k. The decomposition follows from the combinatorial description ([Ka 86] §2 and ch.3, §3.2.6) of perverse sheaves satisfying the relation of distinguished pairs (Equation 9.8). Such decomposition is described by the following de Rham family attached to the IMHS L

$$DR(L) := \{L_J, W^J, F_J, I^K_J : L_J \to L_K, N_{J \setminus K} : L_K \to L_J\}_{K \subset J \subset M} \tag{9.18}$$

where: $L_J = L$, $F_J = F$ and W^J is the filtration defined by (Equation 9.12).

The morphisms $I_J^K := Id : L \to L$, $N_{J\setminus K} := \prod_{i\in J\setminus K} N_i : L \to L$. satisfy compatibility relations ([Ka 86], (5.6.1) to (5.6.7)) and induce the graded morphisms:

$$GrN_{J\setminus K} : Gr_a^{W^K} L_K \to Gr_{a-|J\setminus K|}^{W^J} L_J, \text{ and } GrI_J^K : Gr_{a-|J\setminus K|}^{W^J} L_J \to Gr_a^{W^K} L_K$$

For each $J \subset M$: $(L_J, W^J, F_J, N_j, j \in M)$ is an IMHS structure on L. Hence we have a decomposition into a graded distinguished pair for each set of subsets $K \subset J \subset M$:

$$Gr_{a-|J\setminus K|}^{W^J} L_J \simeq \operatorname{Im} GrN_{J\setminus K} \oplus \operatorname{Ker} GrI_J^K. \tag{9.19}$$

which follows from ([Ka 86], Corollary 5.5.4) and the decomposition in the case $|J \setminus K| = 1$ (Equation 9.8).

Hence, the distinguished pairs criteria apply to the graded de Rham family

$$\mathrm{Gr}_a^W DR(L) := \{\mathrm{Gr}_{a-|J|}^W L_J, F_J, GrI_J^K, GrN_{J\setminus K}\}_{K\subset J\subset M} \tag{9.20}$$

Lemma 9.2 ([Ka 86], Lemma 5.6.2). *Let* $\mathbf{L}$ *and* $\mathcal{L}_X$ *be as in theorem 11.1. For each* $J \subset M$, $x \in Y_J$, *and* $L := \mathcal{L}_X(x)$, *we define:*

$$P_k^J L := \cap_{K\subset J, K\neq J} Ker(GrI_J^K : Gr_k^{W^J} L \to Gr_{k+|J\setminus K|}^{W^K} L) \subset Gr_k^{W^J} L \tag{9.21}$$

i) For $x \in Y_J^*$ *and* $N_J = \sum_{j\in J} N_j$: $P_k^J L$ *is a polarized HS of weight* k, *direct summand of* $Gr_k^{M(N_J,W^J)} Gr_k^{W^J} L$.
ii) For $x \in Y_M^*$ *and* $L := \mathcal{L}_X(x)$ *(hence* $x \in Y_J$ *for all* $J \subset M, J \neq M$ *):*

$$Gr_{k-|M|}^{W^M} L \xrightarrow{\sim} \oplus_{J\subset M} N_{M\setminus J} P_{k-|J|}^J(L) \tag{9.22}$$

where $P_k^\emptyset L = \mathrm{Gr}_k^{W^0} L$ *and the direct sum contains the component* $N_M \mathrm{Gr}_k^{W^0} L$.
iii) There exist a filtered isomorphism

$$(IC_k^* L, F) \simeq \oplus_{J\subset M} (IC^* P_{k-|J|}^J L, F)[-|J|] \tag{9.23}$$

Proof. i) For $i \in J$ and $K = J \setminus i$, the endomorphism $N_i : Gr_k^{W^J} L \to Gr_k^{W^J} L$ is equal to the composition morphism $Gr_k^{W^J} L \xrightarrow{I_{J,K}} Gr_{k+1}^{W^K} L \xrightarrow{N_i} Gr_k^{W^J} L$ (Equation 9.7), hence N_i vanish on $P_k^J L \subset \operatorname{Ker} I_{J,K}$ for all $i \in J$.

Since by the property of distinguished pair $P_k^J L$ is a direct summand of $Gr_k^{W^J} L$ ([Ka 86] Lemma 5.6.2 and Proposition 3.3.1), $Gr_0^{W(N_J)} P_k^J L$ is a direct summand of $Gr_k^{M(N_J,W^J)} Gr_k^{W^J} L = Gr_0^{W(N_J)} Gr_k^{W^J} L$.

Since N_J vanish on $P_k^J L$, $P_k^J = Gr_0^{W(N_J)} P_k^J$ is a direct summand of $Gr_k^{M(N_J)} Gr_k^{W^J} L$ of weight k.

ii) The statement is proved by induction on length $|M| = n$ of M. For $n = 1$ the decomposition is reduced to the property of graded distinguished pair. Let $M' := \{1, \cdots, n-1\}$ and $M = M' \cup \{n\}$.

Let $W^n = N_n * W$, we decompose the data $DR(L)$ into two data where $(W^n)^J = W^{J\cup n}$ for all $J \subset M'$:

$DR(L, W, M') = (L_J, W^J, F_J, I_J^K : L_J \to L_K, N_{J\setminus K} : L_K \to L_J)$ and

$DR(L, W^n, M') = (L_J, W^{J\cup n}, F^{J\cup n}, I_{J\cup n}^{K\cup n}, N_{J\setminus K})$ for $K \subset J \subset M'$.

By induction, the decomposition is valid for M' on the below

$$\begin{array}{ccc} Gr_{k-|M'|}^{W^{M'}} L & \xrightarrow{\sim} & \oplus_{J\subset M'} N_{M'\setminus J} P_{k-|J|}^J(L, W) \\ I_M^{M'} \uparrow\downarrow N_n & & I_{J\cup n}^J \uparrow\downarrow N_n \\ Gr_{k-|M|}^{W^M} L & \xrightarrow{\sim} & \oplus_{J\subset M'} N_{M'\setminus J} P_{k-1-|J|}^J(L, W^n) \end{array} \tag{9.24}$$

Moreover $Gr_{k-|M|}^{W^M}(L, W) = Gr_{k-1-|M'|}^{(W^n)^{M'}}(L, W^n)$.

The last column satisfy the property of distinguished pairs (Corollary 9.1) for each $J \subset M'$:

$$\begin{aligned} P_{k-1-|J|}^J(L, W^n) &\simeq N_n P_{k-|J|}^J(L, W) \oplus \operatorname{Ker} I_{J\cup n}^J \\ P_{k-|J\cup n|}^{J\cup n}(L, W) &= \operatorname{Ker}\,(I_{J\cup n}^J : P_{k-1-|J|}^J(L, W^n) \to P_{k-|J|}^J(L, W)) \end{aligned} \tag{9.25}$$

We deduce from the second line of the diagram the sum over all subsets of M, that is $J \subset M'$ and $J \cup n, n \notin J$:

$$\begin{aligned} &Gr_{k-|M|}^{W^M} L \simeq \oplus_{J\subset M'}(N_n N_{M'\setminus J} P_{k-|J|}^J(L, W) \oplus N_{M'\setminus J} P_{k-|J\cup n|}^{J\cup n}(L, W)) \\ &= \oplus_{J\subset M} N_{M\setminus J} P_{k-|J|}^J(L, W) \end{aligned} \tag{9.26}$$

The proposition 9.2 follows from (Lemma 9.2 *ii*) and the definition of IC^*L by identification of the terms of the complexes in the equation 9.17.

iii) The isomorphism is an interpretation of (Equation 9.22). Moreover it is compatible with F since we have an isomorphism in the category of MHS

$$\oplus_{J\subset M}(N_{M\setminus J} P_{k-|J|}^J L, W, F) \xrightarrow{\sim} (Gr_{k-|M|}^{W^M} L, W, F) \tag{9.27}$$

The decomposition of the filtrations W and F follows (Equation 9.22) by (Remark 3.6vii).

Remark 9.4. i) The decomposition is related to the decomposition in (§5.1.2, Remark 5.9) of $\mathrm{Gr}_*^{W(N)} \Psi^* IL$.

ii) For each $i \in J$, let $N_{0,i} := \sum_{j\in J, j\neq i} N_j$, then:
$M_k(N_0) = M_k(N_i, M(N_{0,i}))$ and $\operatorname{Coker}\,(N_i : W_k Gr_{k+2}^M L \to W_k Gr_k^M L)$ in (Equation 9.8) may be written as $W_k^{J\setminus i} Prim_k^{M(N_i, M(N_{0,i}))} \subset \mathrm{Gr}_k^{M(N_0)} L$, then:
$P_k^J := \cap_{\mathrm{Gr}_k^{M(N_0)} L} W_k^{J\setminus i} Prim_k^{M(N_i, M(N_{0,i}))} \subset \mathrm{Gr}_k^{M(N_0)} L$.

See also (chapter 11, Equation 11.22) for a different computation of P_k^J.

iii) The equation 9.25 leads to the inductive definition:

$$P_i^{J\cup n}(L,W) = \mathrm{Ker}\,(I^J_{J\cup n} : P_i^J(L,W^n) \to P_{i+1}^J(L,W)).$$

9.2.3 Weight filtration on IC^*L

We denote by W and F the induced filtrations on $IC^*L \subset \Omega^*L$.

Proposition 9.3. *Let $(L, W^0, F, N_i, i \in M)$ be an IMHS, then:*

$$Gr_k^W(IC^*L, F) \xrightarrow{\sim} (IC^*(Gr_k^{W^0}L, F). \tag{9.28}$$

Proof. We prove the isomorphism: $(Gr^{W^J}_{k-|J|}N_JL, F) \simeq (N_JGr_k^{W^0}L, F)$ for all $J \subset M$. Since $(L, W^J, F, N_i, i \in M)$ is an IMHS, by induction on the length of J, it is enough to consider the case $J = \{j\}$ for $j \in M$.

By the remark (**??**), up to shift in indices, we may suppose the morphism: $N_j : (L, W, F, N_i, i \in M) \to (L, W^j, N_i, i \in M)$ a morphism of IMHS, then by (Proposition 2.7), the morphism N_j is strict with respect to W and W^j.

9.3 Global definition and properties of the weight W

Let $\mathbf{L}$ be a *graded - polarized* admissible VMHS locally unipotent along Y.

The complex of sheaves $\Omega^*\mathcal{L}$ is perverse with constructible cohomology sheaves with respect to the stratification defined by the NCD Y.

Recall that $Y := \cup_{i\in I}Y_i$ is the union of smooth irreducible components with index I, $Y_J := \cap_{i\in J}Y_i$ for $J \subset I$ and $Y_J^* := Y_J - \cup_{i\notin J}(Y_i \cap Y_J)$ ($Y_\emptyset^* := X^* = X \setminus Y$). We denote uniformly the embeddings of the subsets Y_J^* in X by $j : Y_J^* \to X$. On the big stratum $U := X \setminus Y$, the complex $\Omega^*\mathcal{L}$ is a resolution of its cohomology $\mathbf{L}$.

The weight filtration W on $\Omega^*\mathcal{L}$ by analytic sub-complexes of $\Omega^*\mathcal{L}$, is defined locally in terms of the tilde embedding.

Let $M \subset I$, the filtration W is determined in a neighborhood of y in terms of the IMHS $(L = \mathcal{L}_X(y), W, F, N_i)$ at $y \in Y_M^*$, by a set of coordinates y_i at y including local equations y_i for $i \in M$ of Y at y and the filtrations W^JL of L for $J \subset M$ (Equation 9.12), as follows:

Definition 9.2 (Weight W on $\Omega^*\mathcal{L}$). The term $W_r(\Omega_X^*(Log\,Y) \otimes \mathcal{L}_X)_y$ of the filtration at $y \in Y_M^*$, is the $\mathcal{O}_{X,y}-$ sub-module generated by the germs of the sections $\widetilde{v} \otimes \wedge_{j\in J}\frac{dy_j}{y_j} \wedge \Omega^*_{X,y} \in \Omega^*\mathcal{L}$ where $\widetilde{v} \in \mathcal{L}_{X,y}$ is the tilde embedding of $v \in W^J_{r-|J|}L$ for varying $J \subset M$ (ch.3, Lemma 3.1).

In particular, for $M = \emptyset$, at a point $y \in U$, W_r is the $\Omega^*_{X,y}-$ sub-module generated by the germs of the sections $v \in W_r\mathcal{L}_y$.

The definition of W is independent of the choice of coordinates on a neighborhood $U(y)$, since if we change the coordinate of index i into $y'_i = fy_i$ where f is holomorphic and invertible at y, the difference $\frac{dy'_i}{y'_i} - \frac{dy_i}{y_i} = \frac{df}{f}$ is holomorphic at y. Let $\widetilde{\alpha} \in W^J_{r-|J|}\mathcal{L}_{X,y}$, the difference of the sections $\widetilde{\alpha} \otimes \wedge_{j\in J}\frac{dy'_j}{y'_j} - \widetilde{\alpha} \otimes \wedge_{j\in J}\frac{dy_j}{y_j}$ is still a section of the $\Omega^*_{X,y}$−sub-module generated by the germs of the sections $W^J_{r-|J|}\mathcal{L}_{X,y} \otimes \wedge_{j\in(J-i)}\frac{dy_j}{y_j}$. We remark that the sections defined at y restrict to sections defined on $(U(y) \setminus Y^*_M \cap U(y))$.

9.3.0.1 Purity of the intersection complex $IC^*\mathcal{L}$

Recall the definition of $IC^*\mathcal{L} \subset \Omega^*\mathcal{L}$ (ch.3, Definition 3.6) and set:
$(IC^{*\,p}\mathcal{L}, W, F) := (IC^*\mathcal{L}, W, F)[\dim X]$, $(\Omega^{*\,p}\mathcal{L}, W, F) := (\Omega^*\mathcal{L}, W, F)[\dim X]$ such that: $IC^{*\,p}\mathcal{L} \simeq j_{!*}{}^p\mathbf{L}$ with induced filtration W and $\Omega^{*\,p}\mathcal{L} \simeq j_*{}^p\mathbf{L}$.

In the case of a pure polarized VHS, we recall (Purity Theorem 3.1):

Proposition 9.4. *Let $({}^p\mathbf{L}, F)$ be a polarized VHS of weight ${}^pw$, then the subcomplex $(IC^{*\,p}\mathcal{L}, F)$ of the logarithmic complex with induced filtration F is a Hodge complex which defines a pure HS of weight $a+i$ on its hypercohomology $\mathbb{H}^*(X, IC^{*\,p}\mathcal{L})$, equal to the Intersection cohomology $IH^*(X, {}^p\mathbf{L})$.*

9.3.0.2 Decomposition into intersections complexes of $P^J_k\mathcal{L}$

The local study (§9.2.2) ended with the local decomposition of the graded weight filtration into a direct sum of Intersection complexes (Equation 9.17). We develop now the corresponding global result.

By definition of the weight W on $(\Omega^*\mathcal{L}, F)$, there exists at each $x \in Y$, an induced embedding of

$$(\mathrm{Gr}^W_k IC^*L, F) \hookrightarrow (\mathrm{Gr}^W_k \Omega^*\mathcal{L}, F)_x,\ L := \mathcal{L}(x). \tag{9.29}$$

Definition 9.3. $(IC^*_{Y_K}\mathcal{P}^K_{k-|K|}\mathcal{L} \subset \mathrm{Gr}^W_k \Omega^*\mathcal{L}[|K|])$. There exists a unique complex $IC^*_{Y_K}\mathcal{P}^K_{k-|K|}\mathcal{L}$, defined as a sub-complex of $\mathrm{Gr}^W_k \Omega^*\mathcal{L}[|K|]$ such that at $x \in Y^*_M$, $L := \mathcal{L}(x)$ and $\forall K \subset M$, the consecutive embeddings by (Equations 9.17 and 9.29):

$$(IC^*P^K_{k-|K|}L, F)[-|K|](-|K|) \subset (\mathrm{Gr}^W_k \Omega^*L, F) \subset (\mathrm{Gr}^W_k \Omega^*\mathcal{L}[|K|])_x$$

induce a quasi-isomorphism:

$$IC^*P^K_{k-|K|}L) \xrightarrow{\sim} (IC^*_{Y_K}\mathcal{P}^K_{k-|K|}\mathcal{L})_x \subset (\mathrm{Gr}^W_k \Omega^*\mathcal{L}[|K|])_x,\ L := \mathcal{L}(x) \tag{9.30}$$

The definition and the corollary below follow from the local decomposition by (Equation 9.17).

Corollary 9.2. *i) There exists a polarized VHS: $P_a^K\mathbf{L}$ of weight a on Y_K^* for $K \subset M$, such that $(IC^*_{Y_K}\mathcal{P}_a^K\mathcal{L}, F)$ is a resolution of the intermediate extension by $j_K : Y_K^* \to X$ of $P_a^K\mathbf{L}$:*

$$(IC^*_{Y_K}\mathcal{P}_a^K\mathcal{L}, F) \simeq (j_{K!*}P_a^K\mathbf{L}, F).$$

ii) For $x \in Y_K \setminus Y_K^*$ and $x \in Y_M^*$, $(P_k^K L, F, N_i, i \in M \setminus K)$ is a nilpotent orbit of weight k.

Proposition 9.5. *i) The graded perverse sheaves for the weight filtration, satisfy the decomposition property into intermediate extensions for all k*

$$(Gr_k^W \Omega^*\mathcal{L}, F) \xrightarrow{\sim} \oplus_{J\subset I}(IC^*_{Y_J}\mathcal{P}^J_{k-|J|}\mathcal{L}[-|J|], F)(-|J|). \tag{9.31}$$

where $IC^*_{Y_J}\mathcal{P}^J_{k-|J|}\mathcal{L} \simeq j_{J!*}P^J_{k-|J|}\mathbf{L}$.
The shifted intersection complex $j_{!*}\mathrm{Gr}_k^{W^0}\mathbf{L}$ figures in the sum for $J = \emptyset$.

ii) The weight W is a filtration by sub-complexes of $j_*{}^p\mathbf{L}$ consisting of perverse sheaves defined over $\mathbb{Q}$.

i) The complex $IC^*_{Y_J}\mathcal{P}^J_{k-|J|}(\mathcal{L})$ is a shifted intermediate extension on Y_J. We remark that $P^J_{k-|J|}(\mathbf{L})[-|J|]$ on Y_J^* is a VHS of weight $k - 2|J|$, its twist by $(-|J|)$ is a VHS of weight k and corresponds to $\wedge_{j\in J}\frac{dy_j}{y_j}$ in the definition of the weight on $\Omega^*\mathcal{L} := \Omega^*_X(Log\, Y) \otimes \mathcal{L}_X$.

The decomposition in the category of derived filtered complexes (Equation 9.31) is based on the decomposition of filtered vector spaces (Equation 9.27).

ii) The various graded complexes $Gr_k^W(\Omega^*_X(Log\, Y) \otimes \mathcal{L}_X)$ are direct sum of Intersections complexes defined over $\mathbb{Q}$. The filtration W_k on the de Rham complex is defined over $\mathbb{Q}$ and perverse with respect to the stratification defined by Y_J^*, since the spaces $W_k\Omega^*L$ and $P_k^J(L)$ are defined over $\mathbb{Q}$. When x vary on Y_J^*, $P_k^J L$ defines a VHS and $P_k^J\mathbf{L}$ is a nilpotent orbit at $x \in Y_J \setminus Y_J^*$.

9.3.0.3 Restriction to $IC^*\,{}^p\mathcal{L}$

The Intersection complex $(IC^*\,{}^p\mathcal{L}, W, F)$ of an admissible VMHS is a mixed Hodge complex, embedded as a sub-complex of $(\Omega^*\,{}^p\mathcal{L}, W, F)$ with induced filtrations.

Proposition 9.6. *Let $\mathcal{L}, W, F)$ be an admissible VMHS. The induced filtrations on the complex $(IC^*\,{}^p\mathcal{L}, W, F) \subset (\Omega^*\,{}^p\mathcal{L}, W, F)$ define a structure of a mixed Hodge complex such that for all k:*

$$(IC^*(Gr_k^{W^0}\,{}^p\mathcal{L}), F) \simeq (Gr_k^W IC^*\,{}^p\mathcal{L}, F)$$

The proposition follows from Proposition 9.3 in the local case.

In general the Intersection complex of an extension of two local systems, is not the extension of the intersection complex of the local systems.

In the case of a pure VHS ${}^p\mathbf{L}$, the weights of $\Omega^*\, {}^p\mathcal{L}$ start with the subcomplex $IC^*\, {}^p\mathcal{L}$. The next result on weights satisfy a property similar to ([De 80], cor 3.3.5) in the case of varieties of positive characteristic.

Corollary 9.3. *The de Rham logarithmic mixed Hodge complex* $(\Omega^*\, {}^p\mathcal{L}, W, F)$ *of a variation of HS of weight* $\omega \geq {}^pw$ *induces on the hypercohomolgy* $\mathbb{H}^i(X \setminus Y, {}^p\mathbf{L})$ *a MHS of weight* $\omega \geq {}^pw + i$.

Indeed, $W_k = 0$ on the logarithmic complex for $k \leq {}^pw$.

9.4 The logarithmic complex $IC^*\mathcal{L}(Log\, Z)$

Let $Z = \cup_{i \in I_Z \subset I} Y_i \subset Y \subset X$ be a sub-divisor of the NCD Y union of components of Y with index in a subset I_Z of I, $j_Z := (X \setminus Z) \to X$ and $\mathbf{L}$ a VMHS on $X \setminus Y$. The induced filtrations W and F define a structure of mixed Hodge complexe (MHC) on the logarithmic sub-complex $IC^*\mathcal{L}(Log\, Z) \subset \Omega^*\mathcal{L}$ isomorphic to $j_{Z*}j_Z^*j_{!*}\mathbf{L} \in D_c^b(X, \mathbb{C})$ (ch.3, §3.2).

9.4.0.1 Local decomposition of $Gr_i^W(IC^*L(Log\, Z))$

Lemma 9.3. *Let* $P_k^J(L)$ *be defined by Equation 9.21 for all* $J \subset M_Z := M \cap I_Z$, *and* $IC^*(P_k^J L)$ *the Intersection complex defined by the nilpotent orbit* $(P_k^J(L), N_i, i \in M \setminus J)$, *then:*

$$(Gr_k^W IC^* L(Log\, Z), F) \xrightarrow{\sim} \oplus_{J \subset M_Z} IC^* P_{k-|J|}^J L, F, i \in M)[-|J|](-|J|) \tag{9.32}$$

Proof. The statement is similar to Proposition 9.2 with $J \subset M_Z$ instead of $J \subset M$. The proof is similar to the decomposition case of $\Omega^* L$ in (§9.2.2).

In terms of (ch. 3, §3.2, Remark 3.4), we introduce for $J' \subset M'_Z := M \setminus M_Z$ the IMHS $(N_{J'}L, W^{J'}, F, N_i, i \in M_Z)$ and the following data $DR(N_{J'}L)$:

$$\{(N_{J'}L)_J, W^{J' \cup J}, F^{J' \cup J}, I_J^K : N_{J'}L \to N_{J'}L, N_{J \setminus K} : N_{J'}L \to N_{J'}L\}_{K \subset J \subset M_Z}$$

with indices $J \subset M_Z$, constant term $(N_{J'}L)_J = N_{J'}L$, and induced filtrations $W^{J' \cup J}$, $F^{J' \cup J} = F$ on $N_{J'}L$.

For all $K \subset J \subset M_Z$, $N_{J \setminus K}$ is induced by $\prod_{i \in J \setminus K} N_i$ (resp. I_J^K by $Id : L \to L$). The property of distinguished pairs is satisfied, in particular for consecutive terms $N_i : N_K L \to N_J L$ for $J = K \cup i$. Set for each $J \subset M_Z$

$$P_k^J(N_{J'}L, W^{J'}) := \cap_{K \subset J, K \neq J} \mathrm{Ker}\,(Gr I_{J,K} : Gr_k^{W^{J \cup J'}} N_{J'}L \to Gr_{k+|J \setminus K|}^{W^{K \cup J'}} N_{J'}L).$$

As in (§9.2.2), we deduce the decomposition:

$$Gr^W_{k-|J'|}\Omega^*(N_{J'}L, N_i)_{i\in M_Z} = \oplus_{J\subset M_Z} IC^*(P^J_{k-|J\cup J'|}(N_{J'}L, W^{J'}), i \in M_Z)[-|J|]$$

where IC^* is limited to the variables $i \in M_Z$.
For each $J' \subset M'_Z$, we have a surjective induced morphism $N_{J'}$

$$P^J_{k-|J|}(L) \xrightarrow{N_{J'}} P^J_{k-|J\cup J'|}(N_{J'}L, W^{J'}) \simeq N_{J'}P^J_{k-|J|}(L)$$

and a surjective morphism $Gr^W_k\Omega^*(L, N_i)_{i\in M_Z} \xrightarrow{N_{J'}} Gr^W_{k-|J'|}\Omega^*(N_{J'}L, N_i)_{i\in M_Z}$ which splits into a direct sum of surjective morphisms
$\oplus_{J\subset M_Z}(IC^*(P^J_{k-|J|}L, i \in M_Z) \xrightarrow{N_{J'}} \oplus_{J\subset M_Z} IC^*(N_{J'}P^J_{k-|J|}L, i \in M_Z))[-|J|]$

By summing the double complex over $J \subset M_Z$ and $J' \subset M'_Z$, we find the statement of the lemma.

9.4.0.2 Structure of MHC on $IC^*\mathcal{L}(Log\, Z) \subset \Omega^*\mathcal{L}$

We remark that (Equation 9.32) is similar to (Equation 9.23) except that the sum is over $J \subset M_Z$ instead of $J \subset M$, hence (Definition 9.3) and (Corollary 9.2) apply. We deduce from (Lemma 9.3):

Theorem 9.3. *Let $Z := \cup_{i\in I_Z} Y_i$. With the notations of (Definition 9.3) and (Corollary 9.2), the bifiltered complex $(IC^*\mathcal{L}(Log\, Z), W, F) \subset (\Omega^*\mathcal{L}, W, F)$, is a MHC on X.*

The perverse weight graded sheaves satisfy the decomposition property into intermediate extensions for all k

$$\begin{aligned}(Gr^W_k IC^*\, {}^p\mathcal{L}(Log\, Z), F) &\xrightarrow{\sim} \oplus_{J\subset I_Z}(IC^*_{Z_J}\mathcal{P}^J_{k-|J|}\, {}^p\mathcal{L}[-|J|], F)(-|J|)\\ j_{Z!*}P^J_{k-|J|}({}^p\mathbf{L}) &\simeq IC^*_{Z_J}\mathcal{P}^J_{k-|J|}\, {}^p\mathcal{L}\end{aligned} \tag{9.33}$$

where $j_Z : Z^*_J \to Z_J$ is the inclusion for each $J \subset I_Z$. Moreover, $j_{!*}Gr^{W^0}_k\, {}^p\mathbf{L}$ for $Z = \emptyset$, otherwise the summands are supported by $Z_J \subset Z$ for $J \neq \emptyset$.

The theorem follows from (Lemma 9.3). We remark that $P^J_{k-|J|}({}^p\mathbf{L})[-|J|]$ for $J \subset I_Z$ are perverse, as shifted local systems on Z^*_J by $[\dim X - |J|] = \dim Z^*_J$.

Definition 9.4. The MHS on $\mathbb{H}^*(X \setminus Z, j_{!*}\, {}^p\mathcal{L})$ is defined by the bifiltered complex $(IC^*\, {}^p\mathcal{L}(Log\, Z), W, F)$.

Remark 9.5. i) For any NCD Z such that $Z \cup Y$ is still a NCD, we may always suppose that $\mathcal{L}$ is a variation of MHS on $X - (Y \cup Z)$ (by enlarging Y), then Z is a union of components of $Y \cup Z$.

ii) For any subset $Z \subset X$, the MHS on $\mathbb{H}^*(X \setminus Z, j_{!*}{}^p\mathbf{L})$ is defined by blowing up Z in a non singular X into a NCD without modifications above $X \setminus Z$.

9.4.0.3 Transversality

We denote by $Z' := \cup_{i \notin I_Z \subset I} Y_i \subset Y$ the sub-NCD of Y complement to Z, such that $Y = Z \cup Z'$. We say that Z' is *transversal to* Z. The equation 9.33 is interpreted as follows:

Lemma 9.4. *Let $X \setminus Y \xrightarrow{j''} X \setminus Z' \xrightarrow{j_{Z'}} X$ and $X \setminus Y \xrightarrow{j'} X \setminus Z \xrightarrow{j_Z} X$ such that $j_Z \circ j' = j = j_{Z'} \circ j''$. We have the commutation isomorphism:*

$$j_{Z*} {j'}_{!*}{}^p\mathbf{L} \simeq j_{Z'!*} {j''}_*{}^p\mathbf{L}.$$

The proof is local at each point of the intersection $Z \cap Z'$, by induction on the strata of the stratification defined by Z and it is reduced to the case of one point in the intersection.

In the case of the remark (ch. 3, Remark 3.4) where $|M| = n$ and a point $0 \in D^n \subset \mathbb{C}^n$, we check that $(j_{Z'!*} {j''}_*{}^p\mathbf{L})_0$ is obtained by truncation in degree ≤ -1 of $(j_{Z*} j'_{!*}{}^p\mathbf{L})_0$. However, by the same remark the complex $(j_{Z*} j'_{!*}{}^p\mathbf{L})_0$ has no cohomology in degrees ≥ 0 (after the shift in degrees on ${}^p\mathbf{L}$).

9.5 Hodge theory on perverse cohomology along a divisor

For basic results on perverse heaves along a divisor we refer to (ch. 3, §3.2.4), and to (ch. 2, §2.4) for induced filtrations on perverse hypercohomology. In the case of a NCD $Z \subset Y \subset X$, let $i : Z \to X$. We deduce from the bifiltered complex $(IC^*\mathcal{L}(Log\, Z), W, F)$ various complexes (called logarithmic) with weight and Hodge filtrations, realizing the following classes of complexes in the derived category:

$$i^! j_{!*}{}^p\mathbf{L},\ i^* j_{!*}{}^p\mathbf{L},\ j_{Z!} j_Z^* j_{!*}\mathbf{L} \text{ and } i^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}.$$

In particular we construct a MHS on the hypercohomology of the boundary of a tubular neighborhood of Z with its central fiber Z deleted (§3.2.5).

Thus, we recover most of the results of (Proposition 5.7) without introducing the monodromy action.

Theorem 9.4 (W and F). *Let $Z \subset Y$ be a locally principal divisor NCD in X, and ${}^p\mathbf{L}$ a PVHS of weight ${}^pw$ on $X \setminus Y$ and let D denote Verdier dual*

1. *There exist unique global filtrations W and F on $j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}$ such that*

$$\begin{aligned} &W_{{}^pw} j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L} = j_{!*}{}^p\mathbf{L}, \\ &W_l j_{Z*} j_Z^* (j_{!*}{}^p\mathbf{L}) = W_l IC^*\, {}^p\mathcal{L}(Log\, Z) \text{ if } l > {}^pw \text{ and } 0 \text{ if } l < {}^pw \end{aligned} \tag{9.34}$$

2. *There exist unique global filtrations W and F on $i^*j_{!*}{}^p\mathbf{L}[-1]$ (resp. $i^!j_{!*}{}^p\mathbf{L}[1]$) such that*

$$\begin{aligned} i_*Gr_l^W(i^!j_{!*}{}^p\mathbf{L}[1]) &\simeq Gr_l^W j_{Z*}j_Z^*(j_{!*}{}^p\mathbf{L}) \text{ if } l > {}^pw \text{ and } 0 \text{ if } l \leq {}^pw, \\ i^*j_{!*}{}^p\mathbf{L}[-1](^pw) &\simeq D(i^!j_{!*}{}^p\mathbf{L}[1]) : Gr_l^W(i^*j_{!*}{}^p\mathbf{L}[-1]) = 0 \text{ if } l \geq {}^pw, \\ Gr_l^W(i^*j_{!*}{}^p\mathbf{L}[-1]) &= DGr^W_{2\,{}^pw-l}(i^!j_{!*}{}^p\mathbf{L}[1]) \text{ if } l < {}^pw, \\ W_l(i^*j_{!*}{}^p\mathbf{L}) &:= (W_{l-1}(i^*j_{!*}{}^p\mathbf{L}[-1]))[1] \end{aligned} \tag{9.35}$$

3. *Dually, $i^*j_{!*}{}^p\mathbf{L}[-1]$ embeds into $j_{Z!}j_Z^*j_{!*}{}^p\mathbf{L}$ and the filtration W on $j_{Z!}j_Z^*j_{!*}{}^p\mathbf{L}$ satisfy $Gr^W_{{}^pw}j_{Z!}j_Z^*j_{!*}{}^p\mathbf{L} = j_{!*}{}^p\mathbf{L}$ and:*

$$Gr_l^W j_{Z!}j_Z^*j_{!*}{}^p\mathbf{L} \simeq i_*Gr_l^W(i^*j_{!*}{}^p\mathbf{L}[-1]) \text{ if } l < {}^pw.$$

4. *If X is projective, the filtrations W and F on*

$$i^*j_{!*}{}^p\mathbf{L}[-1], \quad i^!j_{!*}{}^p\mathbf{L}[1], \quad j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L}, \quad j_{Z!}j_Z^*j_{!*}{}^p\mathbf{L}$$

define respectively a structure of mixed Hodge complex.

Example 9.1. i) For a pure ${}^p\mathbf{L}$ of weight ${}^pw$, $i_*i^!j_{!*}{}^p\mathbf{L}$ is supported by Z, has weights $w \geq {}^pw$, and $i_*i^*j_{!*}{}^p\mathbf{L}$ has weights $w \leq {}^pw$ such that: $\forall \ell \geq {}^pw$:

$D(Gr_\ell^W i^!j_{!*}{}^p\mathbf{L}) \simeq Gr^W_{2\,{}^pw-\ell}i^*j_{!*}{}^p\mathbf{L} = Gr^W_{2\,{}^pw-\ell-1}i^*j_{!*}{}^p\mathbf{L}[-1]$.

The Intersection morphism I induces morphisms:

$Gr_i^W i_*i^!j_{!*}{}^p\mathbf{L} \to Gr_i^W j_{!*}{}^p\mathbf{L} \to Gr_i^W i_*i^*j_{!*}{}^p\mathbf{L}$ for $i = {}^pw$ and 0 for $i \neq {}^pw$.

ii) Let ${}^p\mathbf{L}$ be a polarized VHS on $\mathbb{C}^*$ with a nilpotent endomorphism N on L. The duality at 0 between $Gr_r^W i^*(IC^*\mathcal{L}(Log\,Z)/IC^*\mathcal{L})$ and $Gr^W_{-r+1}i^*(j_{!*}{}^p\mathbf{L}))[-1]$ corresponds to the duality between $Gr^W_{-r+1}\mathrm{Ker}\,N$ and $Gr^W_{r-1}L/NL$.

For ${}^p\mathbf{L}$ on $(\mathbb{C}^*)^n$, this duality may be interpreted as a duality of perverse sheaves for the action of N on the functor of nearby cycles or in terms of a strict simplicial coverings of a NCD.

9.5.0.1 Weights of the MHS

As a MHC, the weights of $IC^*\,{}^p\mathcal{L}(Log\,Z)$ are $\geq {}^pw$ by construction. The weights of $i^!j_{!*}{}^p\mathbf{L}$ are $\geq {}^pw$ since the weights of $IC^*\,{}^p\mathcal{L}(Log\,Z)/IC^*\,{}^p\mathcal{L}$ are $\geq {}^pw+1$.

Corollary 9.4. *The weights w satisfy the following inequalities:*

$$\begin{gathered} w\mathbb{H}^i(X\setminus Z, j_{!*}{}^p\mathbf{L}) \geq {}^pw+i, w\mathbb{H}^i_c(X\setminus Z, j_{!*}{}^p\mathbf{L}) \leq {}^pw+i \\ w\mathbb{H}^i_Z(X, j_{!*}{}^p\mathbf{L}) \geq {}^pw+i, w\mathbb{H}^i(Z, j_{!*}{}^p\mathbf{L}) \leq {}^pw+i \\ \mathbb{H}^i(Z, Gr_j^W i^*j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^{-i}(Z, (Gr^W_{-j}i^!j_{!*}{}^p\mathbf{L}(^pw))(-j))^* \end{gathered}$$

We have the following duality isomorphisms for the weights of $\mathbb{H}^i(Z, j_{!*}{}^p\mathbf{L})$:
$\mathbb{H}^i(Z, Gr_j^W i^* j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^i(Z, Gr_j^W Di^! j_{!*}{}^p\mathbf{L}({}^pw))$
$\simeq \mathbb{H}^i(Z, D(Gr_{-j}^W i^! j_{!*}{}^p\mathbf{L}({}^pw))(-j)) \simeq \mathbb{H}^{-i}(Z, (Gr_{-j}^W i^! j_{!*}{}^p\mathbf{L}({}^pw))(-j))^*$
where $w(Gr_{-j}^W i^! j_{!*}{}^p\mathbf{L}({}^pw))(-j)) = -{}^pw - j + 2j$, hence
$w\mathbb{H}^{-i}(Z, (Gr_{-j}^W i^! j_{!*}{}^p\mathbf{L}({}^pw))(-j))^* = {}^pw + i - j \leq {}^pw + i$ as it vanishes if $j < 0$.

Lemma 9.5. *Let ${}^p\mathbf{L}$ be a shifted polarized VHS of weight ${}^pw$ on a Zariski open subset $X \setminus Y$ and Z a closed subvariety of a projective variety X.*

i) There exists a long exact sequence of MHS

$$\to \mathbb{H}^i(X, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^i(X, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^i(X \setminus Z, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}_Z^{i+1}(X, j_{!*}{}^p\mathbf{L}) \to \cdots \tag{9.36}$$

with weights: $w(\mathbb{H}_Z^i(X, j_{!*}{}^p\mathbf{L})) \geq {}^pw + i$, $w(\mathbb{H}^i(X, j_{!*}{}^p\mathbf{L})) = {}^pw + i$, and $w(\mathbb{H}^i(X \setminus Z, j_{!*}{}^p\mathbf{L})) \geq {}^pw + i$.

ii) We have a dual exact sequence of MHS

$$\to \mathbb{H}_c^i(X \setminus Z, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^i(X, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^i(Z, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}_c^{i+1}(X \setminus Z, j_{!*}{}^p\mathbf{L}) \to \tag{9.37}$$

with weights: $w(\mathbb{H}_c^i(X \setminus Z, j_{!*}{}^p\mathbf{L})) \leq {}^pw + i, w(\mathbb{H}^i(X, j_{!*}{}^p\mathbf{L})) = {}^pw + i$, and $w(\mathbb{H}^i(Z, j_{!*}{}^p\mathbf{L})) \leq {}^pw + i$.

The lemma is proved first for X smooth projective, Y a NCD and $Z \subset Y$ a sub-NCD. The exact sequence is deduced from the distinguished triangle: $i_* i^! j_{!*}{}^p\mathbf{L} \to j_{!*}{}^p\mathbf{L} \to j_{Z*}({}^p\mathbf{L}_{|X\setminus Z})$ and the inequalities on the weights follow from the corollary above. In the general case, the lemma will be deduced from the case of NCD by the decomposition theorem.

9.5.0.2 Compatibility of Thom isomorphism with MHS

Thom isomorphism (ch. 3, §3.2.1) is compatible with weight and Hodge filtrations up to a twist.

Lemma 9.6. *Let $Z \subset X$ such that $Y \cup Z$ is a NCD and let a smooth hypersurface H intersects transversally $Y \cup Z$ such that $H \cup Y \cup Z$ is a NCD. There is an isomorphism compatible with the weight and Hodge filtrations up to a twist*

$$IC^*(i_H^*\mathcal{L})(Log\, Z \cap H) \xrightarrow{\sim} i_H^* IC^*(\mathcal{L})(Log\, Z \cap H) \xrightarrow{\sim} i_H^! IC^*\mathcal{L}(Log\, Z)[2](1) \tag{9.38}$$

By transversality, we have an isomorphism of the restriction $i_H^* IC^*\mathcal{L}(Log\, Z)$ with the logarithmic complex $IC^*(i_H^*\mathcal{L})(Log\, Z \cap H)$ constructed directly on H. The connecting isomorphism in the distinguished triangle

$$(i_H)_* i_H^! IC^*\mathcal{L}(Log\, Z) \to IC^*\mathcal{L}(Log\, Z) \to IC^*\mathcal{L}(Log\, Z \cup H) \xrightarrow{[1]}$$

defines an isomorphism of the quotient complex with the cohomology with support

$$(IC^*\mathcal{L}(Log\, Z \cup H)/IC^*\mathcal{L}(Log\, Z)) \xrightarrow{\sim} Ri^!_H IC^*\mathcal{L}(Log\, Z)[1]$$

The isomorphism $i^!_H IC^*\mathcal{L}(Log\, Z)[2](1) \xrightarrow{\sim} IC^* i^*_H \mathcal{L}(Log\, Z \cap H)$ is induced by the residue $Res_H : IC^*\mathcal{L}(Log\, Z \cup H)[1](1) \to i_{H*}IC^*(i^*_H\mathcal{L})(Log\, Z \cap H)$ compatible with the filtrations up to a shift in degrees since it vanishes on $IC^*\mathcal{L}(Log\, Z)$.

Corollary 9.5. *Under the hypothesis of the lemma, let $i : H \to X$ be a general embedding of complex varieties of codimension d, transversal to the strata of a stratification $\mathcal{S}$ of X including Z and Y.*

Thom isomorphism $\mathbb{H}^i(H \setminus Z \cap H, j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^{i+2d}_{H\setminus Z\cap H}(X \setminus Z, j_{!*}{}^p\mathbf{L})(d)$, *and*

Gysin morphism $\mathbb{H}^i(H \setminus Z \cap H, j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^{i+2d}(X \setminus Z, j_{!*}{}^p\mathbf{L})(d)$

are compatible with MHS.

In general, let $i_W : W \to X$ be a closed embedding of a non singular variety of codimension d transverse to Z, Y and to the strata of an adequate stratification $\mathcal{S}$ of X, then: $i^!_W j_{!*}{}^p\mathbf{L}[2d] \simeq i^*_W j_{!*}{}^p\mathbf{L}$.
The following isomorphism is compatible with MHS:

$$\mathbb{H}^i(W \setminus (Z \cap W), j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^{i+2d}_{W\setminus(Z\cap W)}(X \setminus Z, j_{!*}{}^p\mathbf{L}).$$

9.5.1 Hodge theory on $i^*j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L}$

We have a distinguished triangle

$$j_{Z!}j^*_Z j_{!*}{}^p\mathbf{L} \xrightarrow{\text{can}} j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L} \to i_*i^*j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L} \xrightarrow{[1]} \tag{9.39}$$

Definition 9.5. The structure of MHC on $K' := i^*j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L}$ is defined by the mixed cone $C_M(\text{can})$ with the following weight filtration:
$W_rC_M(\text{can}) = W_{r-1}j_{Z!}j^*_Z j_{!*}{}^p\mathbf{L}[1] \oplus W_r j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L}$

We recover the results of (Lemma 5.3). The terms of the weight spectral sequence are $E_1^{p,q} = \mathbb{H}^{p+q}(X, Gr^W_{-p}C_M(\text{can}))$ where
$\text{Gr}^W_r C_M(\text{can}) = \text{Gr}^W_{r-1}j_{Z!}j^*_Z j_{!*}{}^p\mathbf{L}[1] \oplus \text{Gr}^W_r j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L}$:

$E_1^{p,q} = \mathbb{H}^{p+q}(X, \text{Gr}^W_{-p}i^! j_{!*}{}^p\mathbf{L}[1])$ if $-p > {}^pw + 1$

$E_1^{p,q} = \mathbb{H}^{p+q}(X, j_{!*}{}^p\mathbf{L}[1] \oplus \text{Gr}^W_{{}^pw+1}i^! j_{!*}{}^p\mathbf{L}[1])$ if $-p = {}^pw + 1$

$E_1^{p,q} = \mathbb{H}^{p+q}(X, \text{Gr}^W_{{}^pw-1}i^* j_{!*}{}^p\mathbf{L}[-1])[1] \oplus j_{!*}{}^p\mathbf{L})$ if $-p = {}^pw$

$E_1^{p,q} = \mathbb{H}^{p+q}(X, \text{Gr}^W_{-p-1}i^* j_{!*}{}^p\mathbf{L}[-1])[1])$ if $-p < {}^pw$.

The terms with value in $j_{!*}{}^p\mathbf{L}[1]$ for $-p = {}^pw + 1$ and $j_{!*}{}^p\mathbf{L}$ for $-p = {}^pw$ vanish at rank 2. In particular, for $-p = {}^pw + 1$ and $q = {}^pw$, $p + q = -1$ the terms:

1) $E_1^{-\,p_{w-1},\,p_w} = j_{!*}{}^p\mathbf{L}[1] \oplus \mathrm{Gr}^W_{p_{w+1}} i^! j_{!*}{}^p\mathbf{L}[1]$ where
$\mathrm{Gr}^W_{p_{w+1}} i^! j_{!*}{}^p\mathbf{L}[1] = \mathrm{Gr}^W_{p_{w-1}} \mathrm{Coker}\, N|\psi_f j_{!*}{}^p\mathbf{L}$ gives the term:

$$E_2^{-\,p_{w-1},\,p_w} = \mathbb{H}^{-1}(X, \mathrm{Gr}^W_{p_{w-1}} \mathrm{Coker}\, N|\psi_f j_{!*}{}^p\mathbf{L}) \Longrightarrow \mathrm{Gr}^W_{p_w} \mathbb{H}^{-1}(X, i^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L})$$

2) $E_1^{-\,p_w,\,p_w} = \mathrm{Gr}^W_{p_w} j_{Z!} j_Z^* j_{!*}{}^p\mathbf{L}[1] \oplus j_{!*}{}^p\mathbf{L}$ where
$\mathrm{Gr}^W_{p_w} j_{Z!} j_Z^* j_{!*}{}^p\mathbf{L})[1] = (\mathrm{Gr}^W_{p_{w-1}} i^* j_{!*}{}^p\mathbf{L}[-1])[1] \oplus j_{!*}{}^p\mathbf{L}$ gives the term:

$$E_2^{-\,p_w,\,p_w} = \mathbb{H}^1(X, \mathrm{Gr}^W_{p_{w-1}} \mathrm{Ker}\, N|\psi_f j_{!*}{}^p\mathbf{L}) \Longrightarrow \mathrm{Gr}^W_{p_w} \mathbb{H}^0(X, i^* j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L})$$

The duality between the terms $E_2^{p,q}$ above induces the duality at the limit. The limits vanish according to the local purity theorem (crucial case).

The terms $E_2^{p,q}$ coincide with the terms of the weight spectral sequence of $C_M(I)$ (§5.2.2) since $j_{!*}{}^p\mathbf{L}$ in the terms $E_1^{p,q}$ disappear at rank 2.

Remark 9.6. For any deformation retract neighborhood B_Z of Z, let $B_Z^* := B_Z \setminus Z$ with Z deleted in B_Z, the MHS on the hypercohomology of B_Z^* is defined by

$$\mathbb{H}^i(B_Z^*, j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^i(Z, i^* j_{Z*}((j_{!*}{}^p\mathbf{L})_{|X\setminus Z}))$$

We have an exact sequence of MHS

$$\cdots \to \mathbb{H}^i(X \setminus Z, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^i(B_Z^*, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^{i+1}(X, j_!(j_{!*}{}^p\mathbf{L}_{|X\setminus Z})) \to \cdots$$

and a duality isomorphism

$$i^* j_{Z*}(j_{!*}{}^p\mathbf{L}_{|X\setminus Z})[-1] \xrightarrow{\sim} D(i^* j_{Z*}(j_{!*}{}^p\mathbf{L}_{|X\setminus Z})).$$

9.6 The bi-filtered relative complexes $IC_f^*\mathcal{L} \subset \Omega_f^*\mathcal{L}$

Let $f : X \to V$ be a smooth proper morphism of smooth complex algebraic varieties, and let Y be a NCD in X. The relative logarithmic complex $\Omega_f^*\mathcal{L} := \Omega^*_{X/V}(LogY) \otimes \mathcal{L}_X$ is defined in ([De 70], §2.22).

Definition 9.6. i) A normal crossing divisor Y in a non singular algebraic X is said "relative horizontal" if for each point v in a big stratum of V, the fiber $Y_v \subset X_v$ is a NCD with smooth components in the smooth fiber X_v and the restriction of f to Y is a topological fiber bundle over the big strata.

ii) A NCD inverse image of a strict subset $W \subset V$ is called vertical NCD.

iii) The fibration is adapted to a local system ${}^p\mathbf{L}$ defined on a Zariski open algebraic subset if the restriction of the cohomology groups of the intermediate extension $j_{!*}{}^p\mathbf{L}$, to the various strata of X, are locally constant.

The fibration is adapted to a NCD Y in X, if Y is a union of strata of the underlying whitney stratification $\mathcal{S}$ of X.

Let Y be relative horizontal over V. For each point $v \in V$, the fiber Y_v is a NCD in X_v and the various intersections $Y_{i_1,\cdots,i_p}$ of p- components are smooth over V and $Y \to V$ is a topological fiber bundle.

Let $U := X \setminus Y$ and $j : U \to X$. The sheaf of modules $i^*_{X_v}\mathcal{L}_X$ induced by the canonical extension $\mathcal{L}_X$ on each fiber X_v, is isomorphic to the canonical extension $(i^*_{U_v}\mathcal{L})_{X_v}$ of the induced local system $i^*_{U_v}\mathcal{L}$.

The cohomology spaces $H^i(U_v, \mathcal{L})$ (resp. $\mathbb{H}^i(X_v, j_{!*}\mathcal{L})$) for $v \in V$, form a variation of MHS. The logarithmic complex $\Omega^*_{X/V}(LogY) \otimes \mathcal{L}_X$ satisfy, in the case of horizontal NCD: $i^*_{X_v}\Omega^*_{X/V}(LogY) \otimes \mathcal{L}_X \simeq \Omega^*_{X_v}(LogY_v) \otimes (i^*_{U_v}\mathcal{L})_{X_v}$.

When $\mathbf{L}$ underlies an admissible graded polarized VMHS: $(\mathbf{L}, W, F)$, its restriction to the open subset U_v in X_v is also admissible. The image of the filtrations W and F, by the map $\Omega^*\mathcal{L} \to \Omega^*_f\mathcal{L}$

$$\begin{aligned} F :=& \mathrm{Im}(R^i f_* F_X \to R^i f_*(\Omega^*_{X/V}(LogY) \otimes \mathcal{L}_X)), \\ W :=& \mathrm{Im}(R^i f_* W_X \to R^i f_*(\Omega^*_{X/V}(LogY) \otimes \mathcal{L}_X)) \end{aligned}$$

define a variation of MHS on $R^i(f \circ j)_*\mathbf{L}$ inducing at each point $v \in V$ the corresponding weight W and Hodge F filtrations of the MHS on $(\mathbb{H}^i(U_v, \mathbf{L}), W, F)$.

Proposition 9.7. *Let $f : X \to V$ be a proper smooth morphism of non singular complex varieties.*
i) The direct image $R^i(f \circ j)_*\mathbf{L}$ is a local system on V and

$$R^i(f \circ j)_*\mathbf{L} \otimes \mathcal{O}_V \simeq R^i f_*(\Omega^*_{X/V}(LogY) \otimes \mathcal{L}_X)$$

ii) The connecting morphism in Katz-Oda's construction [KaOd 68] coincides with the connection on V defined by the local system $R^i(f \circ j)_*\mathbf{L}$

$$R^i f_*(\Omega^*_{X/V}(LogY) \otimes \mathcal{L}_X) \xrightarrow{\nabla_V} \Omega^1_V \otimes R^i f_*(\Omega^*_{X/V}(LogY) \otimes \mathcal{L}_X)$$

iii) The filtration F satisfy Griffith's transversality with respect to ∇_V, W is locally constant and $(R^i(f \circ j)_*\mathbf{L}, W, F)$ is a graded polarized variation of MHS on V.

Deligne's proof of ([De 70], proposition 2.28) extends in (i), as well the connecting morphism [KaOd 68] in (ii). Since Y is horizontal $R^i(f \circ j)_*\mathbf{L}$ is a local system and on each fiber W and F induce a MHS, hence W is locally constant and F is a filtration by analytic sub-bundles.

9.6.1 The relative Intersection complex $IC^*_f\mathcal{L}$

Let $IC^*_f\mathcal{L} := \mathrm{Im}(IC^*\mathcal{L} \to \Omega^*_f\mathcal{L})$ with image filtrations W and F:

$$W := Im(f_*W \to f_*(IC^*_f\mathcal{L})),\ F := Im(f_*F \to f_*(IC^*_f\mathcal{L})).$$

We deduce the filtrations on the i-th direct image $(R^i f_* j_{!*}\mathbf{L}, W, F)$:

$$F^p R^i f_* j_{!*}\mathbf{L} := (\mathrm{Im}\,(R^i f_* F^p IC^*\mathcal{L} \to R^i f_* IC^*\mathcal{L}),$$
$$W_q R^i f_* j_{!*}\mathbf{L} := Im\,(R^i f_* W_q \to R^i f_* IC^*\mathcal{L}))$$

where W_q is locally constant on V, and F^p is an analytic sub-bundle on V.

Proposition 9.8. *i) The direct image $R^i f_* j_{!*}\mathbf{L}$ of the intersection complex is a local system on V and*

$$R^i f_* j_{!*}\mathbf{L} \otimes \mathcal{O}_V \simeq R^i f_*(IC_f^*\mathcal{L})$$

ii) The filtrations W and F define a structure of variation of MHS on $R^i f_* j_{!*}\mathbf{L}$.

We remark that $R^i f_* IC_f^*\mathcal{L}$ is locally constant since f is a topological fibration and induces a topological fibration on $X\backslash Y$. Moreover, at each $v \in V$, the Hodge decomposition $Gr_q^W \mathbb{H}^i(X_v, j_{!*}\mathbf{L}) = \sum_{p+q=i} \mathbb{H}^{p,q}(X_v, Gr_q^W j_{!*}\mathbf{L})$ shows that the Hodge terms $R^{p,q} f_* IC_f^*\mathcal{L}$ are locally free $\mathcal{O}_V-$modules ([De 68], Théorème 5.5).

Lemma 9.7. *Let Y be a relative NCD over the maximal strata of V and S a stratum of V such that X_S is a relative NCD over S:*

i) $R^i f_* i^!_{X_S} j_{!*}{}^p\mathbf{L}$ (resp. $R^i f_* i^*_{X_S} j_{!*}{}^p\mathbf{L}$) are polarized VMHS on S of weight $w \geq {}^p w + i$ (resp. $w \leq {}^p w + i$).

ii) The image of the intersection morphism ${}^p\mathbf{L}^i_S := \mathrm{Im}\, I$ is a polarized VHS of weight ${}^p w + i$.

Proof. Set $Z := X_S$ and $IC_f^*\,{}^p\mathcal{L}(logZ) := \mathrm{Im}(IC^*\,{}^p\mathcal{L}(logZ) \to \Omega_f^*\,{}^p\mathcal{L})$. Let $v \in S$ and N_v a normally embedded section to S in V at v. Then $Z_v := X_v$ is a NCD in X_{N_v} and for each $v \in S$:

$$(f_* IC_f^*\,{}^p\mathcal{L}(logZ))_v \simeq IC^*_{f|X_{N_v}}\,{}^p\mathcal{L}_{|X_{N_v}}(logX_v) \subset \Omega^*_{f|X_{N_v}}\,{}^p\mathcal{L}_{|X_{N_v}}$$

$$(f_* i^!_{X_S}\,{}^p\mathbf{L})_v \simeq i^!_{X_v}({}^p\mathbf{L}_{|X_{N_v}}),\ (f_* i^*_{X_S}\,{}^p\mathbf{L})_v \simeq i^*_{X_v}({}^p\mathbf{L}_{|X_{N_v}})$$

By Thom-Whitney stratifications properties, $R^i f_* i^!_{X_S} j_{!*}{}^p\mathbf{L}$ and $R^i f_* i^*_{X_S} j_{!*}{}^p\mathbf{L}$ are local systems. The dimensions $\dim F^p \cap W_q$ of the filtrations defined by the MHS on $\mathbb{H}^i(i^!_{X_v}({}^p\mathbf{L}_{|X_{N_v}}), \mathbb{Q})$ are constant when $v \in S$ vary since no jump in the dimension of the fiber of the coherent analytic sheaf :
$(W_q \cap F^p) R^i f_* i^*_{X_S} j_{!*}{}^p\mathbf{L}$ occurs.

Remark 9.7. The above results in the relative version are applied generically over the strata (ch. 11, §11.2) since refinements of Thom-whitney stratifications may be adapted.

9.7 Admissible perverse VMHS on $X \setminus Z$

It has been suggested by the referee to give an exposition of the theory of perverse PVMHS (Chapter 7, Definition 7.2) in terms of Grothendieck's six functors (see M. Saito's theory of mixed Hodge modules [Sa 90] §2 and [Shi 93]).

Such treatment is not in the spirit of this text.

Instead, we add this paragraph to give an informal study of the definition of the direct image of an admissible PVMHS, from which we deduce various cases (Remark 9.10).

In the same time, this is a motivation of the definition of $N * W$ (§9.2.1.1). *Below a VMHS is always defined on a local system, a PVMHS is defined on a perverse sheaf, while in general we extend the notation of mixed Hodge complex (MHC) to a complex of sheaves (K, W, F) with the property that its global hypercohomology is a MHC, that is the hypercohomology on a complete complex algebraic is a MHS with the induced filtrations in each degree.*

9.7.1 PVMHS on $j_*{}^p\mathbf{L}$

*Hypothesis.*We resume the notations of chapter 7. Let $Y \subset X$ be a NCD, $Z \subset Y$ be a sub-NCD in Y and $x \in Z_M^* \subset Y$, $L := \mathcal{L}_X(x)$ (Definition 3.1). An admissible VMHS $(\mathbf{L}, W^0, F)$ (Definition 2.16) on $U := X \setminus Y$ defines at x an IMHS (Remark 2.10) where the filtration W^0 on the limit L (preserved by N_i) is the natural extension of the filtration $\mathcal{W}^0$ of $\mathcal{L}$. The relative monodromy filtration $M(\sum_{i \in M} N_i, W^0)$ exists by definition of admissibility and defines a limit MHS on L:

$(\mathcal{L}, \mathcal{W}^0, \mathcal{F})$ defines the IMHS: $(L, W^0, F, P, N_i, i \in M)$ where $L := \mathcal{L}_X(x)$.

Let Z be defined by an algebraic map $f : X \to \mathbb{C}$ on X, then $W^0{}^p\mathbf{L}$ defines a filtration: $W^0 = W^0({}^p\psi_f j_{!*}{}^p\mathbf{L}) := {}^p\psi_f j_{!*} W^0{}^p\mathbf{L}$. By admissibility, the relative filtration $M(N, W^0)$ exists on ${}^p\psi_f j_{!*}{}^p\mathbf{L}$ (resp. ${}^p\varphi_f j_{!*}{}^p\mathbf{L}$).

Let $W := M(N, W^0)[-1]$ on ${}^p\psi_f j_{!*}{}^p\mathbf{L}$ such that $W_r := M(N, W^0)_{r+1}$.

Proposition 9.9 (Perverse VMHS on ${}^p\psi_f j_{!*}{}^p\mathbf{L}$ and ${}^p\varphi_f j_{!*}{}^p\mathbf{L}$). *With the notations of (Equation 5.36), let ${}^p\mathbf{L}$ be a shifted admissible VMHS quasi-unipotent along Y.*

i) There exists a natural structure of perverse VMHS on ${}^p\psi_f j_{!*}{}^p\mathcal{L}$ (resp. ${}^p\varphi_f j_{!*}{}^p\mathcal{L}$) supported by the NCD Z such that the canonical morphism (denoted can) is a filtered morphism while the morphism: N and the variation (denoted var) are of type $(-1, -1)$.

ii) Let $W^0 = W^0({}^p\psi_f j_{!*}{}^p\mathbf{L}) := {}^p\psi_f j_{!*} W^0{}^p\mathbf{L}$ and $W := M(N, W^0)[-1]$, then $\mathrm{Gr}_i^W {}^p\psi_f j_{!*}{}^p\mathbf{L}$ (resp. $\mathrm{Gr}_i^{M(N,W^0)} {}^p\varphi_f j_{!*}{}^p\mathbf{L}$) decomposes into a direct sum of intermediate extension of polarized VHS of weight i on the various intersection of components of Z.

Proof. The proof is local. We suppose also $\mathbf{L}$ locally unipotent along Y. We consider the de Rham family attached to $L := \mathcal{L}(x)$ and defined by the IMHS $\Psi_J L$ for various $J \subset M_Z$ as in (Equation 5.18).
The relative monodromy filtration $M(N, W^0)$ is well defined on the de Rham complex $\Psi^* IL$ and satisfy the decomposition of the relative monodromy (Equation 9.5)

$$\mathrm{Gr}_i^{M(N,W^0)} \Psi^* IL \simeq \oplus_k \mathrm{Gr}_i^{M(N,W^0)} \mathrm{Gr}_k^{W^0} \Psi^* IL \tag{9.40}$$

As in (Equation 5.5), we define $L[N]$ with the filtrations F and the relative filtration $M(\sum_{i \in M^J} N_i, W_0)$. Then, for each $J \subset M_Z$, we define similarly a MHS on

$$(\Psi_J L = (L[N]/\mathrm{Im} A_J, W_J^0, M^J, F_J), W_J^0 = W_J^0(\Psi_J L) := \Psi_J W^0 L \tag{9.41}$$

Let $B^J := N + \sum_{i \in M^J} N_i$. Then, the relative monodromy filtration $M^J := M(B^J, W^0)$ exists on $\Psi_J L$.

Let B_k^J denote the nilpotent morphism induced on $\mathrm{Gr}_k^{W_J^0} \Psi_J L$. By definition M_{j+k}^J induces the monodromy filtration $W_j(B_k^J)$ on $\mathrm{Gr}_k^{W_J^0} \Psi_J L$. Hence $\mathrm{Gr}_{j+k}^{M^J} \mathrm{Gr}_k^{W_J^0} \Psi_J L$ is of weight $j + k - 1$. Let $W^J := M^J[-1]$ such that $W_r^J := M_{r+1}^J$, then $\mathrm{Gr}_{j+k}^{W^J} \mathrm{Gr}_k^{W_J^0} \Psi_J L$ is of weight $j + k$ on $\mathrm{Gr}_k^{W_J^0} \Psi_J L$.

Thus, (W^J, M^J, F_J) induces a MHS on each $\mathrm{Gr}_i^{W_J^0} \Psi_J L$ and defines a MHS on $\Psi_J L$ since $\mathrm{Gr}_r^{W^J} \Psi_J L \simeq \oplus_i \mathrm{Gr}_r^{W^J} \mathrm{Gr}_i^{W_J^0} \Psi_J L$.

The case of $\varphi_f^u j_{!*} \mathbf{L}$ is deduced by the vanishing cycle lemma (5.1).

Lemma 9.8 ($j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L}$ and $\varphi_f^u(j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L})$). *Let $i : Z \to X$ and*

$$i^* j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L} \simeq \ \mathrm{Cone}\ ({}^p\psi_f^u(j_{!*} {}^p\mathbf{L}) \xrightarrow{N} {}^p\psi_f^u(j_{!*} {}^p\mathbf{L})(-1)$$

be the mixed cone by (Equation 2.16, [Br 82], Proposition 2.3.6)
and let $\varphi_f j_{!*} \mathbf{L}$ be defined as the mixed cone over the specialization sp:
$\varphi_f^u(j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L}) := C_M(sp : i^* j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L} \to \psi_f^u(j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L})$, then:

$$var : \varphi_f^u(j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L}) \to \psi_f^u(j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L})(-1) \tag{9.42}$$

is an isomorphism.

We have the triangles where $\psi_f^u(j_{!*} {}^p\mathbf{L}) = \psi_f^u(j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L})$

$$\begin{array}{ccccccc}
i^* j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L} & \xrightarrow{sp} & \psi_f^u(j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L}) & \xrightarrow{can} & \varphi_f^u(j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L}) & \xrightarrow{[1]} & \\
\downarrow Id & & \downarrow Id & & \downarrow var & & \\
i^* j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L} & \xrightarrow{sp} & \psi_f^u(j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L}) & \xrightarrow{N} & \psi_f^u(j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L})(-1) & \xrightarrow{[1]} &
\end{array} \tag{9.43}$$

where N induces the vertical isomorphism var. Hence The morphism var is an isomorphism and this is a characterization of the extension $j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L}$ in Verdier's classification (§2.2.3.1 4)).
To define the limit Hodge filtrations on $j_{!*}\mathbf{L}$ as well on $\Psi^*I\mathcal{L} \simeq \psi_f^u j_{!*}\mathbf{L}$ (ch.4, 4.2) we used the complexes $IC^*\mathcal{L}, F) \subset (IC^*\mathcal{L}(\log Z), F) \subset (\Omega^*\mathcal{L}, F)$ (ch. 3, §3.8) isomorphic respectively to $j_{!*}\mathbf{L} \subset j_{Z*}j_Z^*j_{!*}\mathbf{L} \subset j_*j_{!*}\mathbf{L}$.

Example 9.2. i) Let $\mathbf{L}$ be on $\mathbb{C}^*, i : 0 \to \mathbb{C}, j : \mathbb{C}^* \to \mathbb{C}$, then:

1) $\psi_z^u\mathbf{L} = L$ at 0, $i^*j_*L \simeq (L \xrightarrow{N} L(-1))$. The action of N on i^*j_*L induces 0 on cohomology. Then $\varphi_z^u j_*\mathbf{L} :=$ cone $(i^*j_*\mathbf{L} \xrightarrow{sp} \psi_z^u\mathbf{L})$ is isomorphic to

$$\varphi_z^u j_*\mathbf{L} := (L \xrightarrow{d_{-1}:=(-N,id)} L(-1) \oplus L) \text{ in degrees } -1 \text{ and } 0$$

with unique cohomology in degree 0

$$H^0 := (L(-1) \oplus L)/\mathrm{Im}\, d_1 \xrightarrow{var\simeq} \psi_z^u\mathbf{L}(-1) = L(-1) : (b, c) \mapsto b + Nc.$$

2) The canonical map $can : \psi_z^u\mathbf{L} \to \varphi_z^u j_*\mathbf{L}$ is defined in degree 0 by
$Id : L \xrightarrow{(0,id)} L(-1) \oplus L : a \mapsto (0, a)$.
3) The variation map $var_\psi = var \circ can : L \to L(-1) : a \to Na$.
ii) In the case of $j_{!*}\mathbf{L}$: $\varphi_z^u j_{!*}\mathbf{L} \simeq (L \xrightarrow{d_{-1}:=(-N,id)} \mathrm{Im}\, N \subset L(-1) \oplus L)[1]$,
$var : \varphi_z^u j_{!*}\mathbf{L} \simeq \mathrm{Im}\, N \subset L(-1) : (b, c) \mapsto b + Nc \in \mathrm{Im}\, N \subset L(-1)$.
4) This example apply to perverse sheaves, as we can see for example on the associated de Rham family.

9.7.1.1 $(j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L}, W, F)$ and $N * W$

Proposition 9.10. *Let ${}^p\mathbf{L}$ be a polarized VMHS on the complement of a NCD Y of an algebraic variety X, ilmp its intermediate extension, $Z \subset X$ a NCD and $(j_Z^*j_{!*}{}^p\mathbf{L}, W^0, F)$ on X.*

i) There exists a weight filtration W on the complex $(j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L}, F) \simeq (IC^{*p}\mathcal{L}(\log Z), F)$ by perverse sub-sheaves on X such that $(j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L}, W, F)$ is a PVMHS and W_k is a perverse extension of $j_Z^*W_k^0 j_{!*}{}^p\mathbf{L}$.

ii) There exits a short exact sequence of perverse sheaves

$0 \to (j_{!*}{}^p\mathbf{L}, W, F) \to (j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L}, W, F) \to i_{Z*}i_Z^!j_{!*}{}^p\mathbf{L}[1] \to 0$

iii) Moreover: $(Gr_k^W j_{!*}{}^p\mathbf{L}, F) = (j_{!*}Gr_k^{W^0}{}^p\mathbf{L}, F)$

The proof of the proposition follows from (Corollary 9.9) below, as we need to describe the weight filtration W first.

Let Z be a locally principal NCD, the proof is reduced to the case where Z is principal defined by an equation f, in which case the weight and Hodge filtrations are defined on $\psi_f^u(j_{!*}{}^p\mathbf{L})$ and $\varphi_f^u(j_{!*}{}^p\mathbf{L})$. Such filtrations defined locally, will induce a unique filtration on $\mathrm{Coker}\, N$, hence on $j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L}/j_{Z!*}j_Z^*j_{!*}{}^p\mathbf{L}$ which is a characterization of W.

Interpretation of $N * W^0$ in terms of Verdier's classification (§2.2.3). Let Z be defined by an equation f. We deduce from the filtered perverse sheaf $({}^p\mathbf{L}, W^0)$, a perverse filtration $W^0 j_Z^* j_{!*}{}^p\mathbf{L} := j_Z^* j_{!*} W_0$. Let

$$W^0 := {}^p\psi_f^u(W^0 j_Z^* j_{!*}{}^p\mathbf{L}) \text{ on } {}^p\psi_f^u(j_Z^* j_{!*}{}^p\mathbf{L}).$$

To apply Verdier's classification, we need to define a sequence

$$W_k^0\, {}^p\psi_f^u(j_Z^* j_{!*}{}^p\mathbf{L}) \xrightarrow{a} W_k \xrightarrow{b} W_k^0\, {}^p\psi_f^u(j_Z^* j_{!*}{}^p\mathbf{L}) \tag{9.44}$$

with the correct weight and Hodge filtrations on W_k. Below, there will be weight filtrations on various spaces all denoted by W.

It is convenient to define the filtration $N * W^0$ by perverse sub-sheaves of ${}^p\psi_f^u(j_Z^* j_{!*}{}^p\mathbf{L})$ with a change of indices. Set:

$$*W_k^0 := NW_k^0 + M_{k-1}(N, W^0) \cap W_k^0 \subset {}^p\psi_f^u(W^0)$$

where $*W^0 = N * W^0[1]$ such that $*W_{k+1}^0 := (N * W^0)_k$.

Remark 9.8. $*W^0$ is defined here in the abelian category of perverse sheaves. Equivalently it si defined on the de Rham family of vector spaces associated to ${}^p\psi_f^u(W^0)$. It is related to the weight on $j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}$ by Verdier's correspondence and the isomorphism $j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L}/j_{!*}{}^p\mathbf{L} \simeq \operatorname{Coker} N$.

We have by (Equations 9.8 and 9.9):

$$M(N, *W^0) = M(N, N * W^0)[1] = M(N, W^0)[1]$$

since $M(N, N * W^0) = M(N, W^0)$. We have now: $W_k^0 \xrightarrow{N} *W_k^0 \xrightarrow{I} W_k^0$.

From now on, we write φ_f^u for ${}^p\varphi_f^u(j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L})$ and ${}^p\psi_f^u$ for ${}^p\psi_f^u(j_Z^* j_{!*}{}^p\mathbf{L})$. We have $\operatorname{Ker}(\varphi_f \xrightarrow{can} \psi_f) = \operatorname{Ker}(N : \psi_f \to \psi_f)$. Set
$W_k^0(\varphi_f^u) := var^{-1}(W_k^0 \psi_f^u)$ and $NW_k^0(\varphi_f^u) := var^{-1}(NW_k^0 \psi_f^u)$, then:
$NW_k^0(\varphi_f^u) = \operatorname{Im}(C : W_k^0 \to \varphi_f^u) \xrightarrow{var\simeq} NW_k^0(\psi_f)$.

The weights. As perverse sheaves, φ_f^u and ψ_f^u are isomorphic, but with different weights.
The sub-quotient sheaf $(\mathrm{Gr}_{k-1}^{M(N,W^0)}, F)$ of ${}^p\psi_f^u$ is pure Hodge of weight $k-2$.
Set $(W_{k-2}\, {}^p\psi_f^u(j_Z^* j_{!*}{}^p\mathbf{L}), F) := (M_{k-1}(N, W^0), F)$, then (W, F) is a PVMHS of ${}^p\psi_f^u(j_Z^* j_{!*}{}^p\mathbf{L})$, while $(W', F) := (W, F)(-1)$ is a PVMHS of ${}^p\varphi_f^u(j_{Z*} j_Z^* j_{!*}{}^p\mathbf{L})$ such that by (Equation 9.42):

$$({}^p\varphi_f^u, W', F) \xrightarrow{var\simeq} ({}^p\psi_f^u, W, F)(-1) \tag{9.45}$$

Remark 9.9. In order to apply the results of [Ka 86] and the classification of Verdier in their terminology, we keep the convention $W_k^0(\psi_f^u) := \psi_f^u(W_k^0)$ instead of $W_k^0(\psi_f^u) := \psi_f^u(W_{k+1}^0)$ in ([Shi 93] §3.1).

We introduce the subspace

$$\begin{aligned} &W_k\,{}^p\varphi^u_f := NW^0_k + W'_k \cap W^0_k : W_k\,{}^p\varphi^u_f \xrightarrow{var\simeq} (*W^0_k)\,{}^p\psi^u_f \\ &({}^p\psi^u_f, W^0) \xrightarrow{C} ({}^p\varphi^u_f, W) \xrightarrow{V} ({}^p\psi^u_f, W^0) \end{aligned} \tag{9.46}$$

satisfying $V \circ C = N$ and $C \circ V = N$. Set $W_k := W_k\,{}^p\varphi^u_f(j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L})$ in (Equations 9.44 and 9.46). The extension $W_k\,{}^p\varphi^u_f(j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L})$ above, corresponds to a term W_k of the weight filtration on $(j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L}, F)$. The filtrations W' and F are essential in the definition of W_k and in the proof below.

Lemma 9.9. *The sequence*

$$\psi^u_f(W^0_k j^*_Z j_{!*}{}^p\mathbf{L}) \xrightarrow{C_k} \phi_k := W_k\,{}^p\varphi^u_f j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L} \xrightarrow{V_k} \psi^u_f(W^0_k j^*_Z j_{!*}{}^p\mathbf{L})$$

defines by Verdier's classification, the perverse extension $W_k j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L}$ of $W^0_k j^*_Z j_{!*}{}^p\mathbf{L}$ such that (W, F) is a PVMHS.

Proof. We write $W_k\,{}^p\varphi^u_f$ for $W_k\,{}^p\varphi^u_f(j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L})$. In terms of the sequence in (Equation 9.31) and with the definition of W' (Equation 9.45) and by (Equations 9.8 and 9.9):

$$\begin{aligned} &Gr^W_k\,{}^p\varphi^u_f \simeq \\ &\mathrm{Im}\,(C_k : Gr^{W^0}_k\,{}^p\psi^u_f \to Gr^W_k\,{}^p\varphi^u_f) \oplus \ker\,(V_k : Gr^W_k\,{}^p\varphi^u_f \to Gr^{W^0}_k\,{}^p\psi^u_f) \end{aligned} \tag{9.47}$$

$$\begin{aligned} &\forall l < k, Gr^W_l Gr^{W^0}_k\,{}^p\varphi^u_f = 0 \\ &Gr^W_k Gr^{W^0}_k\,{}^p\varphi^u_f \simeq \mathrm{Im}\,(N : Gr^{W^0}_k\,{}^p\psi^u_f \to Gr^{W^0}_k\,{}^p\psi^u_f) \\ &\forall l \geq k+1, Gr^W_l Gr^{W^0}_k \simeq \mathrm{Coker}\,(N : Gr^{W'}_{l+2} Gr^{W^0}_k\,{}^p\varphi^u_f \to Gr^{W'}_l Gr^{W^0}_k\,{}^p\varphi^u_f). \end{aligned} \tag{9.48}$$

Hence the corresponding perverse filtration W_k on $j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L}$ satisfies the equation: $(Gr^W_k j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L}, F) =$
$(j_{Z!*}j^*_Z Gr^{W^0}_k j_{!*}{}^p\mathbf{L}, F) \oplus_{p\leq k} \mathrm{Coker}\,(N : Gr^{W'}_{k+2} Gr^{W^0}_p \varphi^u_f \to Gr^{W'}_k Gr^{W^0}_p \varphi^u_f)$

Moreover, the filtration W induces on $(Gr^{W^0}_k j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L}, F)$ the weight filtration on $(j_{Z*}Gr^{W^0}_k j^*_Z j_{!*}{}^p\mathbf{L}, F)$

Finally, by the splitting of $(Gr^{W'}_k\,{}^p\varphi^u_f, F) \simeq \oplus_p Gr^{W'}_k Gr^{W^0}_p\,{}^p\varphi^u_f, F)$ (Equation 9.5), we deduce a perverse HS of weight k on

$(Gr^W_k j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L}, F) \simeq (j_{Z!*}j^*_Z Gr^{W^0}_k j_{!*}{}^p\mathbf{L}, F) \oplus_{p<k} Gr^W_k j_{Z*} Gr^{W^0}_p (j^*_Z j_{!*}{}^p\mathbf{L}, F)$.

The above results extend to the singular case as follows:

Corollary 9.6. *Let $({}^p\mathbf{L}, W^0, F)$ be a graded polarized admissible VMHS on the non singular set V^* of an algebraic variety V, $j_{!*}{}^p\mathbf{L}$ its intermediate extension and $g : V \to \mathbb{C}$ an algebraic map with central fiber $Z := g^{-1}(0)$:*
i) the nearby (resp. vanishing) cycles complex ${}^p\psi_g j_{!*}{}^p\mathbf{L}$ (resp. ${}^p\varphi^u_g j_{!*}{}^p\mathbf{L}$) is a PVMHS on Z is a PVMHS on Z.
ii) let $U := V \setminus Z, j_Z : U \to V$, then ${}^p\varphi^u_g j^*_Z j_{!*}{}^p\mathbf{L}$ is a PVMHS on Z.
iii) the direct image $j_{Z*}j^*_Z j_{!*}{}^p\mathbf{L}$ is a PVMHS on X.

Proof. We consider a desingularization $\pi : V' \to V$ such that $Z' := \pi^{-1}(Z)$ is a NCD. Let $f := g \circ \pi$, $u : V^* \to V'$ and $u_Z : V' \setminus Z' \to V'$. We suppose $Y' := \pi^{-1}(V \setminus V^*)$ a NCD.

i) The PVMHS: ${}^p\psi_f u_Z^* u_{!*} {}^p\mathbf{L}$ and ${}^p\varphi_f u_{Z*} u_Z^* u_{!*} {}^p\mathbf{L}$ are well defined by assumption of admissibility. We apply the weight spectral sequence to the derived direct image π_* (Equation 7.4, Proposition 7.2):

$$ {}^pE_1^{p,q} := {}^p\mathcal{H}^{p+q}(\pi_* Gr_{-p}^W {}^p\psi_f u_{!*} {}^p\mathbf{L}) \Longrightarrow Gr_q^W {}^p\psi_g {}^p\mathcal{H}^{p+q}(\pi_* u_{!*} {}^p\mathbf{L}) $$

which degenerate at rank 2, to deduce the PVMHS. In the case $p + q = 0$, the limit is the weight filtration on ${}^p\psi_g {}^p\mathcal{H}^0(\pi_* u_{!*} {}^p\mathbf{L}) \supset {}^p\psi_g j_{!*} {}^p\mathbf{L}$.

The case of ${}^p\varphi_f u_{Z*} u_Z^* u_{!*} {}^p\mathbf{L}$ is similar.

ii) We have: $var : {}^p\varphi_g j_Z^* j_{!*} {}^p\mathbf{L} \simeq {}^p\psi_g j_{!*} {}^p\mathbf{L}(-1)$.

iii) In the case $j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L}$, we apply the weight spectral sequence to the derived direct image π_* (Equation 7.4):

$$ {}^pE_1^{p,q} := {}^p\mathcal{H}^{p+q}(\pi_* Gr_{-p}^W u_{Z*} u_Z^* u_{!*} {}^p\mathbf{L}) \Longrightarrow Gr_q^W {}^p\mathcal{H}^{p+q}(\pi_* u_{Z*} u_Z^* u_{!*} {}^p\mathbf{L}) $$

which degenerates at rank 2.

Since u_Z and j_Z are affine and π projective, we have:

$\pi_* u_{Z*} u_Z^* u_{!*} {}^p\mathbf{L} = j_{Z*} j_Z^* \pi_* u_{!*} {}^p\mathbf{L}$. We consider $\pi_* W$ induced on

$j_{Z*} j_Z^* \pi_* u_{!*} {}^p\mathbf{L} \supset {}^p\tau_{\leq 0} j_{Z*} j_Z^* \pi_* u_{!*} {}^p\mathbf{L}$ and then projected on:

$$ {}^p\mathcal{H}^0(j_{Z*} j_Z^* \pi_* u_{!*} {}^p\mathbf{L}) = j_{Z*} j_Z^* {}^p\mathcal{H}^0(\pi_* u_{!*} {}^p\mathbf{L}) \supset j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L}. $$

Corollary 9.7. *The direct definition of $*W^0$ on V corresponds by Verdier's classification (Equation 9.46) to the structure of PVMHS on $j_{Z*} j_Z^* j_{!*} {}^p\mathbf{L}$.*

Proof.(Equation 9.46) is defined directly on V by (Corollary 9.6 i) and ii)). In both constructions, the induced structure on $j_{Z*}(Gr_*^{W^0} j_Z^* j_{!*} {}^p\mathbf{L}, W)$ coincide.

9.7.1.2 $(j_{Z*} j_Z^* \mathbf{V}, W, F)$ for V a PVMHS and $N * W$

Hypothesis. We assume now that $(\mathbf{V}, W, F)$ is a PVMHS on a complex algebraic variety V, $Z \subset V$ a locally principal divisor, $U := V \setminus Z, j_Z : U \to V$ and $i_Z : Z \to V$.

To construct the PVMHS on $(j_{Z*} j_Z^* \mathbf{V}, W, F)$, we suppose Z principal defined by a morphism f defined on the algebraic complex variety V, $Z := f^{-1}(0)$. Indeed, the various constructions for different local equations f will glue together. The functor ψ_f (resp. ϕ_f) is well defined on $(\mathbf{V}, W)$.

Let W_i denote the filtration on $\psi_f \mathbf{V} : W_i \psi_f \mathbf{V} := \psi_f W_i \mathbf{V}$ and the filtration F is well defined on $\psi_f \mathrm{Gr}_i^W(\mathbf{V}, F)$ since $\mathrm{Gr}_i^W(\mathbf{V}, F)$ is perverse pure Hodge.

Definition 9.7. A PVMHS $(\mathbf{V}, W, F)$ on $V \setminus Z$ is admissible along Z if:

1) There exists a filtration F on $\psi_f(\mathbf{V}, W)$ inducing, for each index i, the filtration F on $\psi_f \mathrm{Gr}_i^W(\mathbf{V}, F)$

2) Let W on $\psi_f(\mathbf{V})$ be defined by $W_i \psi_f(\mathbf{V}) := \psi_f W_i \mathbf{V}$. The relative monodromy filtration $M := M(N, W)$ exists on $\psi_f(\mathbf{V})$.

Corollary 9.8. *Let $\widetilde{W} := M[-1]$ be the filtration defined by $\widetilde{W}_{k-1} = M_k$ and $j_Z : U \to V$, then $(\psi_f j_Z^* \mathbf{V}, \widetilde{W}, W, F)$ is a PVMHS on Z. Moreover: $(\varphi_f j_{Z*} j_Z^* \mathbf{V}, \widetilde{W}, W, F) \xrightarrow{var\simeq} (\psi_f j_{Z*} j_Z^* \mathbf{V}, \widetilde{W}, W, F)(-1)$ defines a structure of PVMHS on φ_f.*

Hint to the proof. By (Equation 9.5): $\mathrm{Gr}_r^M(\psi_f \mathbf{V}, F) \simeq \oplus_k(\mathrm{Gr}_r^M \mathrm{Gr}_k^W \psi_f K, F)$. Hence the proof is reduced to the case of a direct sum of intermediate extensions of perverse VHS (Definition of $I\mathcal{H}$ 7.1) then (Corollary 9.6) apply.

Corollary 9.9. *Let $Z \subset V$ be a locally principal divisor and j the embedding of $U := V \setminus Z$ in V. The derived direct image of an admissible PVMHS $(\mathbf{V}, W^0, F)$ on U, is a PVMHS $(j_* \mathbf{V}, W, W^0, F)$.*

Hint to the proof. We suppose Z locally defined by an equation f (as local constructions on a local covering of Z glue together). The relative monodromy filtration $M_k := M_k(N, \psi_f W^0)$ on $\psi_f j_{Z*} \mathbf{V}$ defines the weight filtration W such that $(W_{k-1}, F) := (M_k, F)$ from which we deduce by the isomorphism $var : \varphi_f \to \psi_f(-1)$ the definition of $(\varphi_f j_{Z*} j_Z^* \mathbf{V}, W', F)$

In this case (Equation 9.46) is well defined (see also Corollary 9.7). Verdier's extension (§2.2.3, equation 2.23 the case 5) applies and defines the weight filtration W on $(j_{Z*} j_Z^* \mathbf{V}, W, F)$ as extension of W^0.

Moreover, the filtration induced by W on $(j_{Z*} Gr_*^{W^0} j_Z^* \mathbf{V}, W, F)$ coincide with the filtration constructed by desingularization in the case of $(j_{Z*} Gr_*^{W^0} j_Z^* \mathbf{V}, W, F)$. By (Equations 9.47 and 9.48), the filtrations W is unique.

Corollary 9.10. *The induced filtration W on $(j_{Z!*} j_Z^* \mathbf{V}, F)$ satisfy:*

$$(\mathrm{Gr}_i^W j_{!*} \mathbf{V}, F) \simeq (j_{!*} \mathrm{Gr}_i^{W^0} \mathbf{V}, F)$$

Remark 9.10. As in (§5.2.1.4, Corollary 9.4 and Theorem 9.4), we deduce from the open embedding case the filtration W for various functors:

1) The filtrations (W, F) on $j_! \mathbf{V}$ may be defined by duality with $j_*(\mathbf{V}, W^0, F)$. A direct proof is based on the filtration $(N_! W)_k := W_{k-1} + M_k(N, W) \cap N^{-1} W_{k-1}$ dual to $N * W$ ([Ka 86] Equation 3.4.2).

2) The filtrations (W, F) on $(i_Z^! \mathbf{V}, W, F)[1]$ may be defined as a mixed cone of the morphism ρ in the natural sequence $i_Z^! \mathbf{V} \to \mathbf{V} \xrightarrow{\rho} j_{Z*} j_Z^* \mathbf{V}$.

3) The filtrations (W, F) on $i_Z^* \mathbf{V}$ may be defined by duality.

9.7.1.3 Proof of (Equation 9.5)

The result of Kashiwara involves sections $\mathrm{Gr}_r^M \mathrm{Gr}_k^W L \to \mathrm{Gr}_r^M L$ (Equation 9.5). The following description in ([Sa 90]) of the sections is clear.

Since the filtration induced by M on Gr_k^W coincides the shifted monodromy filtration $W(N)[k]$, we can define the following primitive object $P\mathrm{Gr}_{i+k}^M \mathrm{Gr}_k^W L$, denoted by $P_{i+k,k}$

$$P_{i+k,k} := \mathrm{Ker}(\mathrm{Gr}^M_{i+k}\mathrm{Gr}^W_k N^{i+1} : \mathrm{Gr}^M_{i+k}\mathrm{Gr}^W_k L \to \mathrm{Gr}^M_{-i-2+k}\mathrm{Gr}^W_k L)$$

The definition of the section is by induction and compatible with the action of N, such that the definition is reduced to the case of the primitive subspace $s_{i+k,k} : P_{i+k,k} \to \mathrm{Gr}^M_{i+k} W_k L$ as follows. If we suppose $s_{j+l,l}$ defined for $l \leq k-2$ for all j, we define the subspace $V_{i+k,k}$ by the sum:

$$V_{i+k,k} = \textstyle\sum_{j,m,l \in S_k} \mathrm{Im}\, N^m s_{j+l,l} \subset \mathrm{Gr}^M_{-i-2+k} W_k L \text{ where}$$

$S_k := \{j, m \geq 0, l \leq k-2 \text{ such that } j \geq m, j+l < i+k, j+l-2m = k-i-2\}$

and $P_{j+l,l} \xrightarrow{s_{j+l,l}} \mathrm{Gr}^M_{j+l} W_k L \xrightarrow{N^m} \mathrm{Gr}^M_{j+l-2m} W_k L$. Then,

$$s_{i+k,k} : P_{i+k,k} \xrightarrow{\sim} \mathrm{Ker}(N^{i+1} : \mathrm{Gr}^M_{i+k} W_k L \to \mathrm{Gr}^M_{-i-2+k} W_k L / V_{i+k,k}).$$

The section $s_{i+k,k}$ extends (by the primitive decomposition) to define the section on $\mathrm{Gr}^M_{i+k}\mathrm{Gr}^W_k L$ ([Sa 90] Proposition 1.5).

Chapter 10
Relative local purity and Decomposition

We prove the decomposition theorem (ch. 1,Theorem 1.2) and the local purity (ch. 4, Theorem 4.1) in terms of a Thom-Whitney stratification, by induction on $n := \dim X$ the dimension of X and reduction to isolated strata.

Let $\pi : \widetilde{X} \to X$ be a desingularization of X. The decomposition theorem for $f : X \to V$ follows directly from the case of π and $f \circ \pi$ ([De 68] Proposition 2.16), hence to start the proof of Theorem 1.2, we suppose X non singular and f surjective such that $\dim V \leq \dim X$.

The proof of the decomposition (§10.3, Theorem 10.1), is reduced, by induction (§10.0.0.1), to the case of isolated strata of $v \in V$, and it is related to the properties of the cohomology of the NCD above v stated in terms of the relative local purity (§10.6).

10.0.0.1 Induction statement $D(n)$

$D(n)$: *Theorem 9.2 on relative local purity and the decomposition theorem (ch.1, Theorem 1.2) apply to projective morphisms $f : X \to V$ on projective varieties X of dimension $\leq n$.*

The assertion $D(0)$ is obvious. To prove $D(n-1) \Rightarrow D(n)$, we assume first that X is nonsingular. Recall that $\mathbf{L}$ is a PVHS of weight w' on a Zariski-open subset $U \subset X$ such that $Y = X \setminus U$ is a NCD, ${}^p\mathbf{L} := \mathbf{L}[\dim X]$ is of weight ${}^pw := w' + \dim X$ and $j_{!*}{}^p\mathbf{L} \simeq IC^*\,{}^p\mathcal{L}$ (§3.1.4) is endowed with a Hodge filtration F.

10.0.1 The case $\dim V = 0$ *and* $\dim X = n$

We distinguish the case where $\dim V = 0$ which is admitted since the decomposition is reduced to the purity of the HS on $\mathbb{H}^*(X, j_{!*}\mathbf{L})$ proved in

[KaK 87] and [CaKSc 87] by comparison with L^2-cohomology (see Theorem 3.1). A proof by reduction to the case where X is over a curve is in [Sa 88].

Hard Lefschetz theorem is proved by the classical reduction to an hyperplane section ([KaK 87]. In particular, $\mathbb{H}^i(X, j_{!*}{}^p\mathbf{L})$ is a polarized HS of weight ${}^pw + i$ as a direct sum of primitive polarized HS. By the property of auto-duality [KaK 86], all HS coincide with the HS defined in this article by the subcomplex $(IC^*\,{}^p\mathcal{L}, F) \subset (\Omega^*\,{}^p\mathcal{L}, F)$.
The statement of relative local purity is trivial in this case since $B^*_{X_v} = \emptyset$.

10.0.2 Intermediate statement $D(n, V^*)$ in the case $0 < \dim V \leq n$

Hypothesis. Let $f : X \to V$ be defined on a non singular complex variety X of dimension n, $K := f_* j_{!*}{}^p\mathbf{L}$, $\mathcal{S}$ a Thom-Whitney stratification of f adapted to the intermediate extension of a shifted polarized VHS $j_{!*}{}^p\mathbf{L}$ on X.

We denote by $V_0 \subset V$ the union of zero-dimensional strata of $\mathcal{S}$, $i_{V_0} : V_0 \to V$, $V^* := V \setminus V_0$ and $k : V^* \to V$.

Assuming $D(n-1)$ and $\dim V > 1$, the first step is to prove the decomposition theorem over V^*. The assumption $D(n)$ over V^*, denoted by $D(n, V^*)$, is proved by reduction to hyperplane sections of V transversal to the strata of V.

Lemma 10.1 ($D(n, V^*)$). *Under the Hypothesis above, assuming $D(n-1)$, and $\dim V > 1$:*

i) The restriction of the perverse cohomology to $V \setminus V_0$ decomposes in each degree i into a direct sum of intermediate extensions of shifted polarized VHS ${}^p\mathbf{L}^i_S$ (ch. 1, Equation 1.7) of weight ${}^pw + i$ on the strata S of dimension $0 < \dim S \leq n$

$$ {}^p\mathcal{H}^i(k^*K) \xrightarrow{\sim} \oplus_{S\in\mathcal{S},\, S\subset V\setminus V_0}\, k^* i_{S!*}{}^p\mathbf{L}^i_S \tag{10.1} $$

ii) The iterated cup-product η with the class of an hyperplane section of X, induces an isomorphism on the restrictions to $V \setminus V_0$

$$ \eta^i : {}^p\mathcal{H}^{-i}(f_* j_{!*}\,{}^p\mathbf{L})_{|V\setminus V_0} \xrightarrow{\sim} {}^p\mathcal{H}^i(f_* j_{!*}\,{}^p\mathbf{L})_{|V\setminus V_0} \tag{10.2} $$

Proof. Let $f : X \to V$ be a surjective morphism, $K := f_* j_{!*}\mathbf{L}$ on V and let N denote a normally embedded subvariety of V of dimension $d < \dim V$ intersecting a stratum S of dimension $s > 0$ at a point u, such that X_N intersect transversally the strata of X. If X is non singular, then X_N is also non singular for a general N. The induction apply to $f \circ i_{X_N}$ since we have good reductions:

$$ i_N^* K \simeq (f \circ i_{X_N})_* j_{!*}\mathbf{L}, \quad i_N^*\,{}^p\mathcal{H}^i(K) \simeq {}^p\mathcal{H}^i((f \circ i_{X_N})_* j_{!*}\mathbf{L}). \tag{10.3} $$

Remark that ${}^p\mathbf{L}^i_S$ is uniquely defined by the decomposition as ${}^p\mathbf{L}^i_S = i^*_S H^{-s}({}^p\mathcal{H}^i(K))$.
Since $R^{i-s}f_{S*}(i^!_{X_S}j_{!*}{}^p\mathbf{L})$ and $R^{i-s}f_{S*}(i^*_{X_S}j_{!*}{}^p\mathbf{L}))$ are variation of MHS of weights $w \geq {}^pw+i-s$ (resp. $w \leq {}^pw+i-s$), the image ${}^p\mathbf{L}^i_S$ of I_S is a PVHS of weight $w = {}^pw+i-s$ (ch. 9, . Lemma 9.7) including a reduction to the case where X_S is a fibration by NCD).
The isomorphisms in (10.2) follow by induction since $\dim X_N < \dim X$.

Example 10.1. By definition of the big stratum U in V, the restriction $f_| : X_U \to U$ of the morphism f to $X_U := f^{-1}(U)$ is smooth and the restrictions of the cohomology sheaves $\mathcal{H}^i(K)$ are locally constant on U. Hence: ${}^p\mathcal{H}^i(K)_{|U} = \mathcal{H}^{i-\dim V}(K)_{|U}[\dim V]$ is a shifted polarized VHS on U. At a point $u \in U$: $(R^if_*j_{!*}\mathbf{L})_u \simeq \mathbb{H}^i(X_u, j_{!*}\mathbf{L})$ is isomorphic to the intersection cohomology of the restriction of $\mathbf{L}$ to the fiber X_u, hence Hard Lefschetz apply.

10.1 Proof of the relative local purity

In the next step, we deduce the relative local purity at points $v \in V_0$ from the decomposition over $V \setminus V_0$ (Lemma $D(n, V^*)$ 10.1). As the statement is local at points in V_0, we can suppose V_0 reduced to one point v.

10.1.0.1 Perverse filtration by sub-MHS on the Link

Since we assume X non singular, we may also assume that ${}^p\mathbf{L}$ is a VHS on the complement of a NCD $Y \subset X$ so to apply the results of chapters 3 and 9. Let $f : X \to V$, $v \in V$ a closed point. We also assume that $X_v := f^{-1}(v)$ and $X_v \cup Y$ are NCD in X.

Let $i : X_v \to X$, the complex $(K', W, F) := (i^*j_{X_v*}(j_{!*}{}^p\mathbf{L})_{|(X-X_v)}, W, F)$ (ch. 9, §9.5.1, Definition 9.35) defines a structure of MHC on the space of global sections $R\Gamma(X_v, K', W, F)$.

The topological interpretation of this intrinsic cohomology space in terms of balls with center v is more suggestive. A small ball $B_v \subset V$ with center v is defined as the trace of a ball $B \in \mathbb{C}^N$ with center 0. The link at a point $v \in V$ is a topological invariant represented by the boundary of a ball B_v of V. As B^*_v retracts on the boundary, we use the hypercohomology of B^*_v.

The inductive limit of the hypercohomology when the radius of B_v is small is an intrinsic group representing the hypercohomology of the link.

Let $X_v := f^{-1}(v)$, $X_{B_v} := f^{-1}(B_v)$ its tubular neighborhood and $X_{B^*_v} = X_{B_v} \setminus X_v$. There exists an isomorphism

$$\mathbb{H}^r(X_{B^*_v}, j_{!*}{}^p\mathbf{L}) \xrightarrow{\sim} \mathbb{H}^r((X_v, i^*(j_{X_v})_*j^*_{X_v}j_{!*}{}^p\mathbf{L}) \tag{10.4}$$

Set $k : V \setminus \{v\} \to V, j_{X_v} : X \setminus Z \to X$ and $i_v : v \to V$. We denote again by W and F the filtrations $(f_{|X_v})_* W, (f_{|X_v})_* F$ on $(f_{|X_v})_* K'$. The isomorphism $(f_{|X_v})_* K' \simeq i_{v*} k_*((f_* j_{!*} {}^p\mathbf{L})_{|(V-v)})$, defines by transport of structure, a filtration ${}^p\tau^*$ on $(f_{|X_v})_* K'$ deduced from the perverse filtration ${}^p\tau$ on $(f_* j_{!*} {}^p\mathbf{L})_{|(V\setminus v)}$

$$ {}^p\tau^*(f_{|X_v})_* K' \simeq {}^p\tau^*(i_{v*} k_*(f_* j_{!*} {}^p\mathbf{L})_{|(V\setminus v)}) := i_{v*} k_*({}^p\tau(f_* j_{!*} {}^p\mathbf{L})_{|(V\setminus v)}) \tag{10.5} $$

Thus we have three filtrations W, F, ${}^p\tau^*$ on the complex $(f_{|X_v})_* K'$ inducing filtrations W, F and ${}^p\tau^*$ on the cohomology space $\mathbb{H}^j(X_v, i^* j_{X_v *}(j_{!*} {}^p\mathbf{L})_{|(X\setminus X_v)})$.

Lemma 10.2 (Compatibility with MHS). *i) With the notations of (Definition 9.35), let $K' := i^* j_{X_v *} j^*_{X_v} j_{!*} {}^p\mathbf{L}$. Under the decomposition over V^* ($D(n, V^*)$) the filtration ${}^p\tau^*$, induced by the filtration ${}^p\tau$ on $f_* j_{!*} {}^p\mathbf{L}$, is compatible with the MHS on $\mathbb{H}^j(X_v, j_{!*} {}^p\mathbf{L})$:*
The induced filtrations W and F on $Gr_i^{{}^p\tau^*}(f_{|X_v})_* K'$ define a MHS and we have an isomorphism of MHS:

$$ H^j(v, \mathrm{Gr}_i^{{}^p\tau^*}(f_{|X_v})_* K', W, F) \simeq \mathrm{Gr}_i^{{}^p\tau^*} \mathbb{H}^{i+j}(X_v, K', W, F) \tag{10.6} $$

ii) Equivalently, we have an isomorphism of graded polarized MHS:

$$ \mathbb{H}^j(B^*_v, {}^p\mathcal{H}^i(K), W, F) \simeq Gr_i^{{}^p\tau} \mathbb{H}^{i+j}(B^*_{X_v}, j_{!*} {}^p\mathbf{L}, W, F). $$

Proof. The filtration ${}^p\tau^*$ in the lemma corresponds to the perverse filtration ${}^p\tau$ on the hypercohomology of $B^*_{X_v}$ (Equation 11.5).

Let $B^j := \mathbb{H}^j(B^*_{X_v}, j_{!*} {}^p\mathbf{L}, W, F)$ with the filtration ${}^p\tau$, $B := \oplus_j B^j$ and $K := f_* j_{!*} {}^p\mathbf{L}$. The morphism $\eta : K \longrightarrow K[2]$, defined by cup product with the Chern class c_1 of an ample line bundle, induces a morphism on the perverse cohomology sheaves ([BBD 83], Théorème 5.4.10).

We deduce a morphism of MHS $\eta : B \to B(1)$. Let A be the following increasing filtration on B:

$$ \begin{aligned} A_i &:= \oplus_{j<-i} {}^p\tau_{j-1} B^j \oplus {}^p\tau_{-i} B^{-i} \oplus_{j>-i} {}^p\tau_j B^j \\ A_{i-1} &:= \oplus_{j<-i} {}^p\tau_{j-1} B^j \oplus {}^p\tau_{-i-1} B^{-i} \oplus_{j>-i} {}^p\tau_j B^j \\ A_{i-2} &:= \oplus_{j+2<-i+2} {}^p\tau_{j+1} B^{j+2} \oplus {}^p\tau_{-i+2} B^{-i+2} \oplus_{j+2>-i+2} {}^p\tau_{j+2} B^{j+2} \\ \eta : A_i \to A_{i-2}, \eta : \mathrm{Gr}_i^A &= \mathrm{Gr}_{-i}^{{}^p\tau} B^{-i} \to \mathrm{Gr}_{i-2}^A = \mathrm{Gr}_{-i+2}^{{}^p\tau} B^{-i+2} \end{aligned} \tag{10.7} $$

We deduce: $\eta^i : \mathrm{Gr}_i^A = \mathrm{Gr}_{-i}^{{}^p\tau} B^{-i} \simeq \mathrm{Gr}_i^{{}^p\tau} B^i(i) = \mathrm{Gr}_{-i}^A$. Hence A_i is the unique increasing filtration defined by the nilpotent endomorphism η on B in the category of MHS. The lemma follows by the fact that A is necessarily a filtration by MHS.

Once the above MHS is defined, we can express the conditions of local purity and give a meaning to the purity theorem in [DeG 81].

We prove now (Theorem 9.2) in the case $\dim X = n$:

Proposition 10.1 (Local purity). *The weight w satisfy the inequalities :*

$$\begin{array}{lll} w > {}^p w + i + j & \text{on} & H^j(B_v \setminus v, {}^p\mathcal{H}^i(f_* j_{!*} {}^p\mathbf{L})) \text{ if } j \geq 0, \\ w \leq {}^p w + i + j & \text{on} & H^j(B_v \setminus v, {}^p\mathcal{H}^i(f_* j_{!*} {}^p\mathbf{L})) \text{ if } j \leq -1 \end{array} . \tag{10.8}$$

Remark 10.1. Equivalently, the inequalities may be written as:

$$\begin{array}{lll} w > {}^p w + r & \text{on} & {}^p\tau_{\leq r}\mathbb{H}^r(B_{X_v} \setminus X_v, j_{!*} {}^p\mathbf{L}), \\ w \leq {}^p w + r & \text{on} & \mathbb{H}^r(B_{X_v} \setminus X_v, j_{!*} {}^p\mathbf{L}) / {}^p\tau_{\leq r}\mathbb{H}^r(B_{X_v} \setminus X_v, j_{!*} {}^p\mathbf{L}). \end{array} \tag{10.9}$$

The proof in [DeG 81] is based on the local invariant cycle theorem. Our proof is based on duality and the polarization of the Intersection cohomology $\mathbb{H}^*(X, j_{!*} {}^p\mathbf{L})$.
In the case of a variety V with an isolated singularity and X is its desingularization, the polarization occurs also in the proof in the case of constant coefficients by ([Na 85], Prop. 5.1).

As in [DeG 81], the proof of the inequalities below is by induction on an hyperplane section of X, except the case: $Gr^W_{{}^p w} Gr_0^{{}^p\tau} \mathbb{H}^{-1}(B_{X_v} \setminus X_v, j_{!*} {}^p\mathbf{L}) = 0$ ($i = 0$ and $j = -1$, equivalently by duality $j = 0$ and $i = 0$). The reader may skip the induction and go directly to the crucial case (§10.2).

10.1.0.2 Duality

Lemma 10.3. *The dual of $Gr^W_{{}^p w+l} \mathbb{H}^{-1}(B_v \setminus v, {}^p\mathcal{H}^i(f_* j_{!*} {}^p\mathbf{L}))$ for all i and $l \leq i - 1$ is $Gr^W_{{}^p w-l} \mathbb{H}^0(B_v \setminus v, {}^p\mathcal{H}^{-i}(f_* j_{!*} {}^p\mathbf{L}))$ for all i and $-l \geq -i + 1$.*

Proof. Let $k : V \setminus \{v\} \to V, K := f_* j_{!*} {}^p\mathbf{L}$ and $K(*v) := i_v^* k_* k^* K$.
We have quasi-isomorphisms of derived complexes:

$$K(*v) \xrightarrow{\sim} \Gamma(B_v \setminus v, K) \xrightarrow{\sim} \Gamma(B_{X_v} \setminus X_v, j_{!*} {}^p\mathbf{L}).$$

The duality isomorphism $D(K(*v))[1] \xrightarrow{\sim} K(*v)$ where D stands for Verdier dual, is deduced from the auto-duality of of $j_{!*} {}^p\mathbf{L}$, Verdier's duality formula for the direct image by a proper morphism f and the duality between k_* and $k_!$ (resp. i_v^* and $i_v^!$).

The dual of the distinguished triangle $k_! k^* K \to k_* k^* K \to i_{v*} i_v^* k_* k^* K$ is the distinguished triangle: $i_{v*} i_v^! k_* k^* K \to k_! k^* K \to k_* k^* K$.

By identifying the terms of the triangles, we deduce the isomorphisms

$$D(K(*v))[1] \xrightarrow{\sim} i_{v*} i_v^! k_! k^* K[1] \xrightarrow{\sim} K(*v).$$

The duality isomorphism is compatible with the weight filtration:

$$D(Gr^W_{{}^p w-q} H^{-j}(K(*v))) \xrightarrow{\sim} Gr^W_{{}^p w+q} H^j(DK(*v)) \xrightarrow{\sim} Gr^W_{{}^p w+q} H^{j-1}(K(*v)).$$

and corresponds to the duality of the Intesection cohomology of the boundary: $\partial B_{X_v} = f^{-1}(\partial B_v)$ with coefficients in the restriction of $j_{!*} {}^p\mathbf{L}$.

Since the perverse filtration is compatible with duality and weight, we deduce an isomorphism: $D({}^p\mathcal{H}^{-i}(K)) \xrightarrow{\sim} {}^p\mathcal{H}^i(K)$ and

$$D(Gr^W_{{}^pw-q} Gr^{{}^p\tau}_{-i} H^{-j}(K(*v))) \xrightarrow{\sim} Gr^W_{{}^pw+q} Gr^{{}^p\tau}_{i} H^{j-1}(K(*v)).$$

Hence, the dual of $Gr^W_{{}^pw+l}\mathbb{H}^j(B_v \setminus v, {}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L}))$ for all i, $l \leq i-1$ and $j \leq -1$ is $Gr^W_{{}^pw-l}\mathbb{H}^{-j-1}(B_v \setminus v, {}^p\mathcal{H}^{-i}(f_*j_{!*}{}^p\mathbf{L}))$ for all i, $-l \geq -i+1$ and $-j-1 \geq 0$.

The proof of (Proposition 10.1) is reduced by the above duality, to one of the two cases in degree $j \geq 0$ or $j \leq -1$.

10.1.1 Preliminaries on hyperplane sections of X

Let H be a general hyperplane section of X transversal to all strata of a stratification $\mathcal{S}$ of X. We prove here by induction the inequalities of the equation 10.8, except the case $i = 0, j = 0, w = {}^pw$ (see §10.2).
Recall that ${}^p\mathbf{L}$ is defined on the complement $X \setminus Y$ of a NCD Y.

We suppose $H \cap Y$ a NCD. Let $j : (X \setminus Y) \to X$, $j' : H \setminus (H \cap Y) \to H$, and $i_H : H \to X$. By transversality, the restriction $i_H^* j_{!*}{}^p\mathbf{L}[-1] \simeq j'_{!*}({}^p\mathbf{L}_{|H}[-1])$ is perverse. Let $K_H = f_* i_{H*} i_H^* j_{!*}{}^p\mathbf{L}[-1]$.

By induction, the local purity theorem apply to the perverse cohomology sheaves of K_H since $\dim H < \dim X$. We use Artin-Lefschetz vanishing theorem to deduce the local purity for the perverse cohomology sheaves of K in degrees $i \neq 0$.

Lemma 10.4. *Let $j_H : (X \setminus H) \to X$ and $K_c = f_*(j_H)_! j_H^* j_{!*}{}^p\mathbf{L}$.*
i) The complex $K_c \in {}^pD_V^{\geq 0}$, equivalently: ${}^p\mathcal{H}^i(K_c) = 0$ for $i < 0$.
ii) Dually $K(*) := f_*(j_H)_* j_H^* j_{!*}{}^p\mathbf{L} \in {}^pD_V^{\leq 0}$ (${}^p\mathcal{H}^i(K(*)) = 0$ for $i > 0$).

Proof. As $f \circ j_H : (X \setminus H) \to V$ is affine, by a version of Artin-Lefschetz vanishing theorem, the functor $f_* \circ j_{H!}$ transforms ${}^pD_{X-H}^{\geq 0}$ into ${}^pD_V^{\geq 0}$ (the functor is left t-exact [BBD 83], corollary 4.1.2), hence $K_c \in {}^pD_V^{\geq 0}$.
The statement ${}^p\mathcal{H}^i(K(*)) = 0$ for $i > 0$ follows by duality.

Let $K = f_* j_{!*}{}^p\mathbf{L}$. The restriction morphism $\rho : K \to K_H[1]$, and dually the Gysin morphism $G : K_H[-1] \to K$, induce morphisms

$$\rho_i : {}^p\mathcal{H}^i(K) \to {}^p\mathcal{H}^{i+1}(K_H),\ G_i : {}^p\mathcal{H}^{i-1}(K_H) \to {}^p\mathcal{H}^i(K), \tag{10.10}$$

The cup product with the Chern class c_1 of an ample line bundle (or equivalently relative ample) defines a morphism $\eta : K \longrightarrow K[2]$ inducing morphisms of perverse sheaves ([BBD 83], Théorème 5.4.10)

$$\eta_i = G_{i+2} \circ \rho_i : {}^p\mathcal{H}^i(K) \longrightarrow {}^p\mathcal{H}^{i+2}(K)$$

Corollary 10.1. *i) The restriction* $\rho_i : {}^p\mathcal{H}^i(K) \to {}^p\mathcal{H}^{i+1}(K_H)$ *is an isomorphism for each integer* $i < -1$ *and a monomorphism for* $i = -1$.
Dually, the Gysin morphism $G_i : {}^p\mathcal{H}^{i-1}(K_H) \to {}^p\mathcal{H}^i(K)$ *is an isomorphism for* $i > 1$, *and an epimorphism for* $i = 1$.

ii) The filtrations W *and* F *define a MHS for* $i < 1$ *on:*

$$Gr_i^{{}^p\tau}\mathbb{H}^{i+j}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}, W, F) \simeq Gr_i^{{}^p\tau}\mathbb{H}^{i+j}(B_{H_v} \setminus H_v, i_H^* j_{!*}{}^p\mathbf{L}[-1]).$$

We have a dual statement for $i > 1$.

Proof. Here monomorphism (resp. epimorphism) stands for the perverse kernel is zero (resp. the perverse cockernel is zero). We deduce from the distinguished triangle $(j_H)_! j_H^* j_{!*}{}^p\mathbf{L} \to j_{!*}\mathbf{L} \to i_{H*} i_H^* j_{!*}{}^p\mathbf{L} \xrightarrow{(1)}$, and its derived direct image by f_*, an exact sequence of perverse cohomology ([BBD 83], Théorème 1.3.6)

$$\cdots \to {}^p\mathcal{H}^i(K_c) \xrightarrow{\mathrm{can}_i} {}^p\mathcal{H}^i(K) \xrightarrow{\rho_i} {}^p\mathcal{H}^{i+1}(K_H) \xrightarrow{\partial_i} {}^p\mathcal{H}^{i+1}(K_c) \to \cdots \quad (10.11)$$

The corollary follows from Lemma 10.4 as ${}^p\mathcal{H}^i(K_c) = 0$ for $i < 0$.

We consider the diagrams:

$$\begin{array}{l} {}^p\mathcal{H}^{-1}(K_H)(-1) \xrightarrow{G_0} {}^p\mathcal{H}^0(K) \xrightarrow{\rho_0} {}^p\mathcal{H}^1(K_H) \\ {}^p\mathcal{H}^{-1}(K)_{|V\setminus v} \xrightarrow{\rho_{-1}} {}^p\mathcal{H}^0(K_H)_{|V\setminus v} \xrightarrow{G_1} {}^p\mathcal{H}^1(K)_{|V\setminus v}(1) \end{array} \quad (10.12)$$

Lemma 10.5 (Induction). *The morphism* G_0 *is injective,* ρ_0 *is an epimorphism and by induction* $D(n-1)$ *(§10.0.0.1)*

$${}^p\mathcal{H}^0(K) \xrightarrow{\sim} \operatorname{Im} G_0 \oplus \operatorname{Ker} \rho_0. \quad (10.13)$$

Respectively, ρ_{-1} *is a monomorphism ,* G_1 *is an epimorphism and by induction (Lemma* $D(n, V^*)$ *10.1)*

$${}^p\mathcal{H}^0(K_H)_{|V\setminus v} \xrightarrow{\sim} \operatorname{Im} \rho_{-1} \oplus \operatorname{Ker} G_1 \quad (10.14)$$

Proof. Since $\rho_0 \circ G_0 = \eta_H$ in the equation 10.12 is an isomorphism on H by induction D(n-1), we deduce the decomposition in Equation 10.13.

Since $G_1 \circ \rho_{-1} = \eta$ in the equation 10.12 is an isomorphism on $V \setminus v$ by induction (Lemma $D(n, V^*)$ 10.1), we deduce the decomposition in the equation 10.14.

Still we have no information on $\operatorname{Ker} \rho_0$, for example if $f(H)$ is closed of dimension strictly less than $\dim V$. This will be the crucial case in the proof.

10.1.1.1 Reduction to an hyperplane section

Let $H_v := X_v \cap H$, $B_v^* := B_v \setminus v$, $B^*_{X_v} := B_{X_v} \setminus X_v$ and $B^*_{H_v} := B^*_{X_v} \cap H$.

$$\begin{array}{ccc}
\mathbb{H}^j(B_v^*, {}^p\mathcal{H}^i(K), W, F) & \xrightarrow{\rho_i} & \mathbb{H}^j(B_v^*, {}^p\mathcal{H}^i(K_H[1]), W, F) \\
\downarrow D_X \simeq & & \downarrow D_H \simeq \\
Gr_i^{{}^p\tau}\mathbb{H}^{i+j}(B_{X_v}^*, j_{!*}{}^p\mathbf{L}, W, F) & \xrightarrow{\rho_i} & Gr_i^{{}^p\tau}\mathbb{H}^{i+j}(B_{H_v}^*, i_H^* j_{!*}{}^p\mathbf{L}[-1], W, F)
\end{array} \tag{10.15}$$

where the isomorphisms (by the decomposition) D_X and D_H transport the filtration W and F by induction over V^* and ρ_i respects the filtrations W and F. This diagram will be used to check that W and F define MHS satisfying local purity by (including a new proof of the compatibility of ${}^p\tau$ with MHS) by induction on an hyperplane section H. We resume the notations of the proof of (Corollary 10.1).

Let $K := f_* j_{!*}{}^p\mathbf{L}$, $K_H = f_* i_{H*} i_H^* j_{!*}{}^p\mathbf{L}[-1], \rho : K \to K_H[1]$, and dually the Gysin morphism $G : K_H[-1] \to K$.

Corollary 10.2. *The local purity in proposition 10.1 applies to ${}^p\mathcal{H}^i(K)$ with the MHS defined by W and F, except eventually for the component* $\operatorname{Ker}\rho_0 \subset {}^p\mathcal{H}^0(K)$ *in Equation 10.13.*

Proof. The conditions on the weights $wH^j(B_v \setminus v, {}^p\mathcal{H}^{i+1}(K_H))$: $w > {}^pw - 1 + i + 1 + j$ if $j \geq 0$ are assumed by induction, since K_H is of weight ${}^pw - 1$.

Since for $i < -1$ the restriction morphisms ρ_i are isomorphisms, the filtrations W and F on $H^j(B_v \setminus v, {}^p\mathcal{H}^i(K))$ define a MHS satisfying the conditions on the weight by transport of structure.

In the case $i = -1$, ρ_{-1} induces an isomorphism onto a direct summand $\operatorname{Im}\rho_{-1}$ of ${}^p\mathcal{H}^{-1}(f_*(i_H)_* i_H^* j_{!*}{}^p\mathbf{L})_{|V-v}$ (Equation 10.14), hence the MHS is carried to $\mathbb{H}^j(B_v^*, {}^p\mathcal{H}^{-1}(K), W, F)$ and satisfy local purity. The case $i \geq 1$ follows by duality.

In the case $i = 0$, local purity holds for $\operatorname{Im} G_0$ since G_0 is an isomorphism by (Lemma 10.5).

Only the case of $\operatorname{Ker}\rho_0 \subset {}^p\mathcal{H}^0(K)$ is not deduced by induction from K_H.

Lemma 10.6 (The case $i = 0, j \neq 0, -1$). *Let $j_{!*}{}^p\mathbf{L}$ be of weight a. The MHS on*

$$Gr_0^{{}^p\tau} H^j(B_{X_v}^*, j_{!*}{}^p\mathbf{L}) \simeq H^j(B_v^*, {}^p\mathcal{H}^0(K))$$

is of weight $\omega > {}^pw + j$ for $j > 0$ and dually of weight $\omega \leq {}^pw + j$ for $j < -1$.

Proof. Let H be a general hyperplane section of V containing v, $H_v = B_v \cap H$ and $H_v^* = B_v^* \cap H$. Since H is general, it is normally embedded outside v so that the perverse cohomology commute with the restriction to $H \setminus v$.

We consider the exact sequence

$$\mathbb{H}^j_{H_v^*}(B_v^*, {}^p\mathcal{H}^0(K)) \xrightarrow{\alpha_j} \mathbb{H}^j(B_v^*, {}^p\mathcal{H}^0(K)) \to \mathbb{H}^j(B_v \setminus H_v, {}^p\mathcal{H}^0(K))$$

Since $\mathbb{H}^j_{H_v^*}(B_v^*, {}^p\mathcal{H}^0(K)) \simeq \mathbb{H}^{j-2}(H_v^*, {}^p\mathcal{H}^0(K))(-1)$ by Thom isomorphism (see Corollary 9.5), we deduce $w > {}^pw + j$ on $\mathbb{H}^j_{H_v^*}(B_v^*, {}^p\mathcal{H}^0(K))$ as $w > {}^pw + j - 2$ on $\mathbb{H}^{j-2}(H_v^*, {}^p\mathcal{H}^0(K))$ by induction.

By Artin-Lefshetz hyperplane section theorem, extended and applied to the Stein open subset $B_v \setminus H_v$ with coefficients in ${}^p\mathcal{H}^0(K) \in {}^pD_c^{\leq 0}(V, \mathbb{Q})$: $\mathbb{H}^j(B_v \setminus H_v, {}^p\mathcal{H}^0(K)) \simeq 0$ for $j > 0$.

Then, α_j is an isomorphism for $j > 1$ and surjective for $j = 1$, and $w > {}^pw + j$ on $\mathbb{H}^j(B_v^*, {}^p\mathcal{H}^0(K))$ by the surjectivity of α_j for $j > 0$.

10.1.2 The large inequality: $w \geq {}^pw$ for $j = 0$

We prove by induction:

Lemma 10.7. *i) $\mathbb{H}^0(B_v^*, {}^p\mathcal{H}^0(K))$ is of weight $w \geq {}^pw$.*
ii) Dually: $\mathbb{H}^{-1}(B_v^*, {}^p\mathcal{H}^0(K))$ is of weight $w \leq {}^pw$.

Proof. Let $H_1 \not\ni v$ be a general hyperplane section not containing v, $k : (V \setminus v) \to V$. We deduce from the distinguished triangle:

$$k_!k^*\,{}^p\mathcal{H}^0(K) \to k_*k^*\,{}^p\mathcal{H}^0(K) \to i_{v*}i_v^*k_*k^*\,{}^p\mathcal{H}^0(K),$$

the exact sequence of hypercohomology on $U_1 := V \setminus H_1$

$$\mathbb{H}^0(U_1, k_*k^*\,{}^p\mathcal{H}^0(K)) \xrightarrow{\gamma_0} H^0(i_v^*k_*k^*\,{}^p\mathcal{H}^0(K)) \to \mathbb{H}^1(U_1, k_!k^*\,{}^p\mathcal{H}^0(K)) = 0$$

where $\mathbb{H}^1(U_1, k_!k^*\,{}^p\mathcal{H}^0(K)) = 0$ since U_1 is affine ([BBD 83] Théorème 4.1.1) as $k_!k^*\,{}^p\mathcal{H}^0(K) \in {}^pD_c^{\leq 0}$.

We deduce the lemma since, the weight w of $\mathbb{H}^j(U_1, k_*k^*\,{}^p\mathcal{H}^0(K))$ satisfy $w \geq {}^pw$ on the open set $U_1 \setminus v$, and γ_0 is surjective.

10.2 The crucial case

It remains to prove:

$$Gr^W_{{}^pw} Gr_0^{{}^p\tau} \mathbb{H}^{-1}(B^*_{X_v}, j_{!*}\mathbf{L}) \xrightarrow{\sim} Gr^W_{{}^pw} \mathbb{H}^{-1}(B_v^*, {}^p\mathcal{H}^0(K)) \xrightarrow{\sim} 0. \tag{10.16}$$

and by duality $Gr^W_{{}^pw} Gr_0^{{}^p\tau} \mathbb{H}^0(B^*_{X_v}, j_{!*}\mathbf{L}) \xrightarrow{\sim} 0$.

Let $k : (V \setminus V_0) \to V$, $i_{V_0} : V_0 \to V$ and $K := Rf_*j_{!*}\mathbf{L}$. As the proof below is local, it applies to all isolated strata $v \in V_0$ of dimension 0 at the same time. We can suppose V_0 reduced to a unique stratum $i_v : v \to V$.

The proof is based on the idea that the cohomology of B_v^* fits in two exact sequences associated to distinguished two triangles involving $i_v^*k_*k^*K$

$$k_!k^*K \to k_*k^*K \to i_{v*}i_v^*k_*k^*K \text{ and } i_v^!K \to i_v^*K \to i_v^*k_*k^*K \tag{10.17}$$

from which we deduce the commutative diagram

$$\begin{tikzcd} \mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) \arrow[r, "\partial_X"] \arrow[d, "\partial"] \arrow[dr, "\gamma"] & \mathbb{H}^0_c(X \setminus X_v, j_{!*}{}^p\mathbf{L}) \arrow[d, "\alpha_X"] \\ \mathbb{H}^0_{X_v}(X, j_{!*}{}^p\mathbf{L}) \arrow[r, "A"] & \mathbb{H}^0(X, j_{!*}{}^p\mathbf{L}) \end{tikzcd} \tag{10.18}$$

where α_X and A are natural, ∂_X (resp. ∂) is a connecting morphism defined by the first (resp. the second) triangle, and $\gamma = \alpha_X \circ \partial_X = A \circ \partial$.

Lemma 10.8. *i) We have:*
$Gr^W_{{}^pw}\, {}^p\tau_0\mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) \simeq Gr^W_{{}^pw} Gr_0^{{}^p\tau}\mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) \simeq Gr^W_{{}^pw} H^{-1}(B^*_v, \operatorname{Ker}\rho_0)$.
ii) For all elements $u \in Gr^W_{{}^pw}\, {}^p\tau_0\mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L})$, the image $\gamma(u)$ vanish:

$$\gamma(u) = 0 \in {}^p\tau_0 Gr^W_{{}^pw}\mathbb{H}^0(X, j_{!*}{}^p\mathbf{L}) = {}^p\tau_0\mathbb{H}^0(X, j_{!*}{}^p\mathbf{L}).$$

i) Under the decomposition by induction $D(n, V^*)$ on V^* (see Lemma 10.1), we apply the decomposition in equation 10.13 to deduce a decomposition over B^*_v:

$$k^*\, {}^p\tau_{\leq 0}K \simeq \oplus_{i\leq -1} k^*\, {}^p\mathcal{H}^i(K) \oplus k^*\operatorname{Ker}\rho_0 \oplus k^*\operatorname{Im} G_0 \tag{10.19}$$

Since (Equation 10.8) is proved for $i < 0$, we deduce :
$Gr^W_{{}^pw}\mathbb{H}^{-1}(B^*_v,\, {}^p\mathcal{H}^i(K)) = 0$ for $i < 0$ and $j = -1$: $w = {}^pw + i - 1 < {}^pw$.
If $i = 0$, since (Equation 10.8) is proved for $\operatorname{Im} G_0$: $Gr^W_{{}^pw}\mathbb{H}^{-1}(B^*_v, \operatorname{Im} G_0) = 0$.
We deduce:

$$Gr^W_{{}^pw}\, {}^p\tau_{\leq 0}\mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) \simeq Gr^W_{{}^pw}\mathbb{H}^{-1}(B^*_v,\, {}^p\tau_{\leq 0}K) \simeq Gr^W_{{}^pw}\mathbb{H}^{-1}(B^*_v, \operatorname{Ker}\rho_0) \tag{10.20}$$

ii) We suppose the element u in $Gr^W_{{}^pw}\mathbb{H}^{-1}(B^*_v, \operatorname{Ker}\rho_0)$ and prove first:
$\gamma(u)$ *is primitive*: Let $\rho : \mathbb{H}^0(X, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^0(H, j_{!*}{}^p\mathbf{L})$ denote the restriction to a general hyperplane section H, we prove:

$$\rho \circ \alpha_X \circ \partial_X(u) = 0 \text{ for } u \in {}^p\tau_0 Gr^W_{{}^pw}\mathbb{H}^0(X, j_{!*}{}^p\mathbf{L}).$$

Let $H^*_v := B^*_v \cap H, X^* := X \setminus X_v, V^* := V \setminus v$ in the following diagram

$$\begin{array}{ccccc} {}^p\tau_{\leq 0}\mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) & \xrightarrow{\sim r_1} & \mathbb{H}^{-1}(B^*_v,\, {}^p\tau_{\leq 0}K) & \xrightarrow{\rho^*_v} & \mathbb{H}^{-1}(H^*_v, K_H[1]) \\ \downarrow \partial_X & & \downarrow \partial_V & & \downarrow \partial_H \\ {}^p\tau_{\leq 0}\mathbb{H}^0_c(X^*, j_{!*}{}^p\mathbf{L}) & \xrightarrow{\sim r_2} & \mathbb{H}^0_c(V^*,\, {}^p\tau_{\leq 0}K) & \xrightarrow{\rho^*_c} & \mathbb{H}^0_c(H^*,\, {}^p\tau_{\leq 0}K_H[1]) \\ \downarrow \alpha_X & & \downarrow \alpha_V & & \downarrow \alpha_H \\ {}^p\tau_{\leq 0}\mathbb{H}^0(X, j_{!*}{}^p\mathbf{L}) & \xleftarrow{l_X} & \mathbb{H}^0(V,\, {}^p\tau_{\leq 0}K) & \xrightarrow{\rho} & \mathbb{H}^0(H,\, {}^p\tau_{\leq 0}K_H[1]) \\ \downarrow \rho & & \downarrow \rho & & \downarrow\simeq \\ {}^p\tau_{\leq 0}\mathbb{H}^0(H, j_{!*}{}^p\mathbf{L}) & \xleftarrow{\sim l_H} & {}^p\tau_{\leq 0}\mathbb{H}^0(H, j_{!*}{}^p\mathbf{L}) & \xrightarrow{\sim} & {}^p\tau_{\leq 0}\mathbb{H}^0(H, j_{!*}{}^p\mathbf{L}) \end{array} \tag{10.21}$$

where the restriction $\rho : K \to K_H[1]$ induces the various morphisms: ρ^*_v, ρ^*_c, ρ, the natural morphisms $r_1, r_2,\ l_H$ are isomorphisms by the decomposition

assumption on the complement of v or on H, and l_X is a morphism by definition of ${}^p\tau_{\leq 0}$.

All the morphisms in the diagram are commutative as the columns represent $\rho \circ \gamma$ respectively on X, on V and γ on H.
We are interested by the level $Gr^W_{{}^pw}$ of all the spaces in the diagram.

Since $u \in Gr^W_{{}^pw}\mathbb{H}^{-1}(B^*_v, \operatorname{Ker}\rho_0)$ (Equation 10.20) is killed by ρ:
$Gr^W_{{}^pw}(\rho^*_v \circ r_1)(u) = 0$ on the first line. By the commutativity of the diagram:
$Gr^W_a(\rho \circ \alpha_X \circ \partial_X)(u) = Gr^W_{{}^pw}(\alpha_H \circ \partial_H \circ \rho^*_v \circ r_1)(u) = 0$
where the isomorphism on the last line gives the equality.

iii) To prove $\gamma(u) = 0$, we just apply Poincaré duality to the diagram

$$\begin{tikzcd}
 & Gr^W_{{}^pw}\mathbb{H}^0(X, j_{!*}{}^p\mathbf{L}) \arrow[dr, "A^*"] & \\
Gr^W_{{}^pw}\mathbb{H}^0_{X_v}(X, j_{!*}{}^p\mathbf{L}) \arrow[ur, "A"] \arrow[rr, "I"] & & Gr^W_{{}^pw}\mathbb{H}^0(X_v, j_{!*}{}^p\mathbf{L})
\end{tikzcd}$$

where A^* is the dual of A, $I := A^* \circ A$.

Poincaré duality is expressed by a scalar product P on $\mathbb{H}^0(X, j_{!*}{}^p\mathbf{L})$ and a non-degenerate pairing P_v

$$P_v : Gr^W_{{}^pw}\mathbb{H}^0_{X_v}(X, j_{!*}{}^p\mathbf{L}) \otimes Gr^W_{{}^pw}\mathbb{H}^0(X_v, j_{!*}{}^p\mathbf{L}) \to \mathbb{C}.$$

while the duality between A and A^* is defined by the relation

$$P(Ab, c) = P_v(b, A^*c).$$

for all $b \in Gr^W_{{}^pw}\mathbb{H}^0_{X_v}(X, j_{!*}{}^p\mathbf{L})$ and $c \in Gr^W_{{}^pw}\mathbb{H}^0(X, j_{!*}{}^p\mathbf{L})$.

Let C be the Weil operator defined by the HS on $Gr^W_a\mathbb{H}^0_{X_v}(X, j_{!*}{}^p\mathbf{L})$, respectively on $Gr^W_{{}^pw}\mathbb{H}^0(X, j_{!*}{}^p\mathbf{L})$ and $u \in Gr^W_{{}^pw}\mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L})$, then

$$P(C.A(\partial u), A(\overline{\partial u})) = P_v(C\partial u, A^* \circ A(\overline{\partial u})) = P_v(C.\partial(u), I(\overline{\partial u})) = 0 \tag{10.22}$$

since $I \circ \partial = 0$ by the long exact sequence defined by the second triangle in the equation 10.17. Since the polarization Q on the primitive part of $\mathbb{H}^0(X, j_{!*}{}^p\mathbf{L})$ is defined by the relation $Q(a, b) := P(Ca, \overline{b})$, we deduce: $\gamma(u) = A(\partial u) = 0$ by polarization as $\gamma(u)$ is already primitive.

iv) The end of the proof of the crucial case $i = 0, j = 0, w = {}^pw$ is based on two lemmas.

Lemma 10.9. *The morphism* $Gr^W_{{}^pw}\mathbb{H}^0_c(X \setminus X_v, j_{!*}{}^p\mathbf{L}) \overset{\alpha_X}{\to} \mathbb{H}^0(X, j_{!*}{}^p\mathbf{L})$ *is injective.*

In the long exact sequence $H^{-1}(X_v, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^0_c(X \setminus X_v, j_{!*}{}^p\mathbf{L}) \overset{\alpha_X}{\to} \mathbb{H}^0(X, j_{!*}{}^p\mathbf{L})$, the weight w of $H^{-1}(X_v, j_{!*}{}^p\mathbf{L})$ satisfy $w < {}^pw$ since X_v is closed, hence the morphism α_X in the lemma is injective since $Gr^W_{{}^pw}\mathbb{H}^0(X, j_{!*}{}^p\mathbf{L}) = \mathbb{H}^0(X, j_{!*}{}^p\mathbf{L})$.

Lemma 10.10. *i) The connecting morphism ∂_X in the diagram 10.18 induces an injective morphism:*
$Gr^W_{{}^pw} Gr_0^{{}^p\tau} \mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) \xrightarrow{\partial_X} Gr^W_{{}^pw} Gr_0^{{}^p\tau} \mathbb{H}^0_c(X \setminus X_v, j_{!*}{}^p\mathbf{L})$.

ii) Moreover $Gr^W_{{}^pw}\, {}^p\tau_0 \mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) \xrightarrow{\partial_X} Gr^W_{{}^pw} Gr_0^{{}^p\tau} \mathbb{H}^0_c(X \setminus X_v, j_{!*}{}^p\mathbf{L})$ is injective.

Let $V^* := V \setminus v$. We prove first:
Sub-lemma: 1) $\mathbb{H}^1_c(V^*, {}^p\mathcal{H}^0(K))$ is pure of weight ${}^pw + 1$.
2) By duality, $\mathbb{H}^{-1}(V^*, {}^p\mathcal{H}^0(K))$ is pure of weight ${}^pw - 1$.

Proof of the sub-lemma. 1) Let H_1 be a general hyperplane section of V not containing v intersecting the strata transversally with non singular inverse image $X_1 \subset X$.
Let $k : V \setminus H_1 \to V$. The long exact sequence of hypercohomology with coefficients in $k_! k^*\, {}^p\mathcal{H}^0(K)$:

$$\mathbb{H}^1_{H_1}(V, k_! k^*\, {}^p\mathcal{H}^0(K)) \xrightarrow{\varphi} \mathbb{H}^1(V, k_! k^*\, {}^p\mathcal{H}^0(K)) \to \mathbb{H}^1(V \setminus H_1, k_! k^*\, {}^p\mathcal{H}^0(K))$$

where $\mathbb{H}^1(V \setminus H_1, k_! k^*\, {}^p\mathcal{H}^0(K)) = 0$, since $V \setminus H_1$ is affine and $k_! k^*\, {}^p\mathcal{H}^0(K) \in {}^pD_c^{\leq 0}$ ([BBD 83] Théorème 4.1.1); hence φ is surjective.

Since $v \notin V^*$, the decomposition apply to $X_1 \to H_1$ and to $(X \setminus X_v) \to V^*$. Then, Thom isomorphism on $X \setminus X_v$ corresponds to a Thom isomorphism

$$\mathbb{H}^{-1}(H_1, {}^p\mathcal{H}^0(K))(-1) \xrightarrow{\sim} \mathbb{H}^1_{H_1}(V, {}^p\mathcal{H}^0(K)) = \mathbb{H}^1_{H_1}(V, k_! k^*\, {}^p\mathcal{H}^0(K))$$

of pure HS of weight ${}^pw + 1$. We deduce that the middle term:
$\mathbb{H}^1(V, k_! k^*\, {}^p\mathcal{H}^0(K)) = \mathbb{H}^1_c(V^*, {}^p\mathcal{H}^0(K))$ is pure of weight ${}^pw + 1$.
2) By duality, $\mathbb{H}^{-1}(V^*, {}^p\mathcal{H}^0(K))$ is pure of weight ${}^pw - 1$.

Proof. i) We deduce from the sub-lemma and the long the long exact sequence:

$$\mathbb{H}^{-1}(V^*, {}^p\mathcal{H}^0(K)) \to \mathbb{H}^{-1}(B^*_v, {}^p\mathcal{H}^0(K)) \xrightarrow{\partial_V} \mathbb{H}^0_c(V^*, {}^p\mathcal{H}^0(K))$$

that the kernel of ∂_V has weight $< {}^pw$, hence:

$$Gr^W_{{}^pw} \mathbb{H}^{-1}(B^*_v, {}^p\mathcal{H}^0(K)) \xrightarrow{\partial_V} Gr^W_{{}^pw} \mathbb{H}^0_c(V^*, {}^p\mathcal{H}^0(K))$$

which translates by the decomposition over $V^* := V \setminus v$ into the statement i) of the lemma.

ii) follows by the isomorphism (Equation 10.20):

$$Gr^W_{{}^pw}\, {}^p\tau_0 \mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) \simeq Gr^W_{{}^pw} Gr_0^{{}^p\tau} \mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}).$$

Corollary 10.3. $Gr^W_{{}^pw} Gr_0^{{}^p\tau} \mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) = 0$.

Proof. Let $u \in Gr^W_{{}^pw}\, {}^p\tau_0 \mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) = Gr^W_{{}^pw} Gr_0^{{}^p\tau} \mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L})$ by (Equation 10.20). We consider the sequence

$$Gr^W_{{}^pw}\, {}^p\tau_0 \mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) \xrightarrow{\partial_X} Gr^W_{{}^pw}\, {}^p\tau_0 \mathbb{H}^0_c(X - X_v, j_{!*}{}^p\mathbf{L}) \xrightarrow{\alpha_X} \mathbb{H}^0(X, j_{!*}{}^p\mathbf{L})$$

Since $\gamma(u) = 0$ (Lemma 10.8) and α_X is injective on $Gr^W_{{}^pw}$ by Lemma (10.9), we deduce that

$$\partial_X u = 0 \in Gr^W_{{}^pw}\, {}^p\tau_0 \mathbb{H}^0_c(B^*_{X_v}, j_{!*}{}^p\mathbf{L}).$$

Since the class $\overline{\partial_X u} = 0 \in Gr_0^{{}^p\tau} Gr_{{}^pw}^W \mathbb{H}_c^0(B_{X_v}^*, j_{!*}{}^p\mathbf{L})$, we deduce by Lemma (10.10) $u = 0 \in Gr_{{}^pw}^W {}^p\tau_0 \mathbb{H}^{-1}(B_{X_v}^*, j_{!*}{}^p\mathbf{L}) \simeq Gr_{{}^pw}^W Gr_0^{{}^p\tau} \mathbb{H}^{-1}(B_{X_v}^*, j_{!*}{}^p\mathbf{L})$ which ends the proof of the corollary and the the crucial case.

Example 10.2. ([Na 85] Proposition 5.1) In the case of isolated singularity $x \in X$, let $\widetilde{X}$ denote a desingularization of X of dimension n such that the inverse image Z of x is a NCD. Local purity at x has interesting cohomological consequences on Z:

If $k < n$, the MHS on $H_x^k(X, \mathbb{Q})$ is of weight $< k$ and the morphism $H_Z^k(\widetilde{X}, \mathbb{Q}) \to H^k(Z, \mathbb{Q})$ is injective.

The MHS on $H_x^n(X, \mathbb{Q})$ is of weight $< n$ and the morphism $H_Z^k(\widetilde{X}, \mathbb{Q}) \to H^k(Z, \mathbb{Q})$ is an isomorphism.

If $k > n$, the MHS on $H_x^k(X, \mathbb{Q})$ is of weight $\geq k$ and the morphism $H_Z^k(\widetilde{X}, \mathbb{Q}) \to H^k(Z, \mathbb{Q})$ is surjective.

10.3 The decomposition theorem at v

We complete the proof of the decomposition in the case of a non signlar complex variety X, of dimension $n := \dim X$, $1 \leq \dim V \leq n$.

We suppose $D(n, V^*)$ satisfied on $V - V_0$ (Lemma 10.1) and the relative local purity at isolated strata in V_0 (Proposition 10.1). Since the proof is local, we suppose V_0 reduced to an isolated stratum $v \in V$.

In this section, we extend the decomposition to v. The case of a singular variety X follows by desingularization (see §10.3.3).

10.3.1 Extension of the perverse cohomology across v

The description below of the extensions of perverse cohomology sheaves across v is based on properties of the intermediate extension across v and in general along a stratification ([BBD 83], 2.1.11 and ch.2, §2.2).

Lemma 10.11. *Let S be a fixed stratum of V of dimension l, $\mathbf{L}'$ a local system on S, ${}^p\mathbf{L}' := \mathbf{L}'[l], i_v : v \to V$ a closed point in the closure of S, $k : (V \setminus v) \to V$ and $j : S \to V$. i)*

1. *We have a short exact sequence of perverse sheaves*

$$0 \to j_{!*}{}^p\mathbf{L}' \to {}^p\mathcal{H}^0(k_*k^*j_{!*}{}^p\mathbf{L}') \xrightarrow{h} R^0k_*k^*j_{!*}{}^p\mathbf{L}' \to 0$$

2. *${}^p\mathcal{H}^i(k_*k^*j_{!*}{}^p\mathbf{L}') = 0$ if $i < 0$, ${}^p\mathcal{H}^i(k_*k^*j_{!*}{}^p\mathbf{L}') = R^ik_*k^*j_{!*}{}^p\mathbf{L}'$ if $i > 0$.*

In particular: $H^0({}^p\mathcal{H}^0(k_*k^*j_{!*}{}^p\mathbf{L}')) \xrightarrow{\sim} R^0k_*(k^*j_{!*}{}^p\mathbf{L}')$.

ii) Let $j_{!*}\,{}^p\mathbf{L}^j, j \in \mathbb{Z}$, be a family of intermediate extensions of local systems on S and let $K' := \oplus_j j_{!*}\,{}^p\mathbf{L}^j[-j]$ denote their direct sum.
We have short exact sequences of perverse sheaves for $i \in \mathbb{Z}$

$$0 \to j_{!*}\,{}^p\mathbf{L}^i \to {}^p\mathcal{H}^i(k_*k^*K') \xrightarrow{h} R^ik_*k^{*p}\tau_{\leq i}K' \to 0 \tag{10.23}$$

where the last term $R^ik_*k^{*p}\tau_{\leq i}K' \simeq \oplus_{j\leq i}R^{i-j}k_*k^*j_{!*}\,{}^p\mathbf{L}^j$ is a sum of vector spaces supported on v in degee zero.

iii) There exist isomorphisms of the cohomology in degree 0

$$H^0({}^p\mathcal{H}^i(k_*k^*K')) \simeq R^ik_*k^{*p}\tau_{\leq i}K' \xrightarrow{\sim} \oplus_{j\leq i}R^ik_*(k^*j_{!*}\,{}^p\mathbf{L}^j[-j]). \tag{10.24}$$

iv) A morphism of perverse sheaves $\varphi : \mathcal{P} \to {}^p\mathcal{H}^i(k_*k^*K')$ factors through $j_{!*}\,{}^p\mathbf{L}^i$ if and only if $h \circ \varphi = 0$.

Proof. We recall that S is a non singular locally closed subvariety but V may be singular along S.

i) The perverse long exact sequence defined by the distinguished triangle:

$$\tau_{\leq -1}k_*k^*j_{!*}\,{}^p\mathbf{L}' \to k_*k^*j_{!*}\,{}^p\mathbf{L}' \to \tau_{\geq 0}k_*k^*j_{!*}\,{}^p\mathbf{L}' \tag{10.25}$$

is reduced to the statement in i) since: $j_{!*}\,{}^p\mathbf{L}' := \tau_{\leq -1}k_*k^*j_{!*}\,{}^p\mathbf{L}'$ is perverse, ${}^p\mathcal{H}^i(j_{!*}\,{}^p\mathbf{L}') = 0$ for $i \neq 0$ and $\tau_{\geq 0}k_*k^*j_{!*}\,{}^p\mathbf{L}'$ is supported by v.

ii) The statement follows from the case of a unique local system in i) since
${}^p\mathcal{H}^i(k_*k^*K') = {}^p\mathcal{H}^i(k_*k^{*p}\tau_{\leq i}K') \simeq \oplus_{j\leq i}\,{}^p\mathcal{H}^{i-j}(k_*k^*j_{!*}\,{}^p\mathbf{L}^j)$
as for $j > i$: ${}^p\mathcal{H}^i(k_*k^*j_{!*}\,{}^p\mathbf{L}^j[-j]) = {}^p\mathcal{H}^{i-j}(k_*k^*j_{!*}\,{}^p\mathbf{L}^j) = R^{i-j}k_*k^*j_{!*}\,{}^p\mathbf{L}^j$.

The statements iii) and iv) are clear.

Instead of the triangle in equation 10.25, we could refer to the distinguished triangle

$$i_{v*}i_v^!j_{!*}\,{}^p\mathbf{L}' \to j_{!*}\,{}^p\mathbf{L}' \to Rk_*k^*j_{!*}\,{}^p\mathbf{L}' \xrightarrow{[1]} \tag{10.26}$$

since in the statement i): $R^0k_*k^*j_{!*}\,{}^p\mathbf{L}' \simeq H^1(i_{v*}i_v^!j_{!*}\,{}^p\mathbf{L}')$.

Remark 10.2. [Dual statements for $k_!k^*$] We deduce from the long exact sequence of perverse cohomology defined by the triangle dual to (10.26):

$$k_!k^*j_{!*}\,{}^p\mathbf{L}' \to j_{!*}\,{}^p\mathbf{L}' \to i_{v*}i_v^*j_{!*}\,{}^p\mathbf{L}' \xrightarrow{[1]} \tag{10.27}$$

i) A short exact sequence:

$$0 \to i_{v*}H^{-1}(i_v^*j_{!*}\,{}^p\mathbf{L}') \to {}^p\mathcal{H}^0(k_!k^*j_{!*}\,{}^p\mathbf{L}') \to j_{!*}\,{}^p\mathbf{L}' \to 0$$

${}^p\mathcal{H}^i(k_!k^*j_{!*}\,{}^p\mathbf{L}') = i_{v*}H^{i-1}(i_v^*j_{!*}\,{}^p\mathbf{L}')$ for $i < 0$, ${}^p\mathcal{H}^i(k_!k^*j_{!*}\,{}^p\mathbf{L}') = 0$ for $i > 0$.
Moreover: $i_{v*}H^{-1}(i_v^*j_{!*}\,{}^p\mathbf{L}') \xrightarrow{\sim} H^0(i_v^*\,{}^p\mathcal{H}^0(k_!k^*j_{!*}\,{}^p\mathbf{L}'))$.

ii) ${}^p\mathcal{H}^i(k_!k^*K') = \oplus_{j\geq i}\,{}^p\mathcal{H}^{i-j}(k_!k^*j_{!*}\,{}^p\mathbf{L}^j)$ and we have short exact sequences

$$0 \to \oplus_{j\geq i} i_{v*} H^{i-j-1}(i_v^* j_{!*}\, {}^p\mathbf{L}^j) \xrightarrow{h'} {}^p\mathcal{H}^i(k_! k^* K') \to j_{!*}\, {}^p\mathbf{L}^i \to 0$$

where the first term is supported on v in degee zero.

iii) A morphism of perverse sheaves φ defined on ${}^p\mathcal{H}^i(k_! k^* K')$ factors through $j_{!*}\, {}^p\mathbf{L}^i$ if and only if $\varphi \circ h' = 0$.

10.3.2 Proof of the decomposition at v

The proof of the decomposition at v in the case $\dim X = n$ and X_v is a NCD follows from $D(n, V^*)$ (Lemma 10.1). The result is a consequence of the properties of the cohomology of the NCD above v following the relative local purity (Proposition 10.1).

Theorem 10.1. *With the Hypothesis of §10.0.2, let:*
$K := f_* j_{!*} {}^p\mathbf{L}, k : V^* \to V, i_v : \{v\} \to V$.
i) The decomposition of ${}^p\mathcal{H}^i(k^*K)$ (Equations 10.1 and 10.2) extends into a decomposition over V, including the term ${}^p\mathbf{L}_v^i$ (ch. 1, Equation 1.7)

$$ {}^p\mathcal{H}^i(K) \simeq {}^p\mathbf{L}_v^i \oplus k_{!*} k^*\, {}^p\mathcal{H}^i(K) \tag{10.28}$$

ii) Hard Lefschetz: Let $\eta : {}^p\mathcal{H}^i(K) \to {}^p\mathcal{H}^{i+2}(K)$ denote the cup-product with the class of a relative hyperplane section of X. The Lefschetz isomorphisms $k^*\eta^i$ on the open subset $V \setminus v$, extend to Lefschetz isomorphisms η^i on V:

$$k^*\eta^i : {}^p\mathcal{H}^{-i}(K)_{|V\setminus v} \xrightarrow{\sim} {}^p\mathcal{H}^i(K)_{|V\setminus v} \Longrightarrow \eta^i : {}^p\mathcal{H}^{-i}(K) \xrightarrow{\sim} {}^p\mathcal{H}^i(K) \tag{10.29}$$

Proof. Given the distinguished triangle $i_{v*} i_v^! K \xrightarrow{\alpha} K \xrightarrow{\rho} k_* k^* K \xrightarrow{[1]}$, let

$$ {}^p\mathcal{H}^{i-1}(k_* k^* K) \xrightarrow{{}^p\delta^{i-1}} {}^p\mathcal{H}^i(i_{v*} i_v^! K) \xrightarrow{{}^p\alpha^i} {}^p\mathcal{H}^i(K) \xrightarrow{{}^p\rho^i} {}^p\mathcal{H}^i(k_* k^* K) \xrightarrow{{}^p\delta^i} \tag{10.30}$$

be the associated long exact sequence of perverse cohomology on V

Lemma 10.12. *Under the decomposition $k^*\, {}^p\mathcal{H}^i(K)[-i] \simeq \oplus_{S\subset V\setminus v} i_{S!*}\, {}^p\mathbf{L}_S^i$ on $V \setminus v$. The perverse image $\operatorname{Im} {}^p\rho^i$ (Equation 10.30) is isomorphic to:*

$$\operatorname{Im} {}^p\rho^i \xrightarrow{\sim} k_{!*} k^*\, {}^p\mathcal{H}^i(K) \xrightarrow{\sim} \oplus_S i_{S!*}\, {}^p\mathbf{L}_S^i.$$

Proof of the lemma. We use the description of the intermediate extension of a perverse sheaf P on $V \setminus v$ in terms of the functor $k_!$ (extension by 0) and the derived direct image k_*: $k_{!*}P = \operatorname{Im}\,({}^p\mathcal{H}^0(k_! P) \to {}^p\mathcal{H}^0(k_* P))$ is the image in the abelian category of perverse sheaves ([BBD 83], section 2.1.7).

We fix a Whitney stratification $\mathcal{S}$ of V, morphisms $\sigma_j : k^*\, {}^p\mathcal{H}^j(K) \to k^*K[j]$ and a decomposition $k^*K \simeq \oplus_j k^*\, {}^p\mathcal{H}^j(K)[-j]$ on $V \setminus v$.

We apply (Lemma 10.11) in the case of a direct sum of local systems over all strata $S \neq v$ of to prove

1) *There exists a monomorphism:* $k_{!*}k^{*}\,{}^{p}\mathcal{H}^{i}(K) \subset {}^{p}\mathcal{H}^{i}(k_{*}k^{*}K)$:
The morphism of functors $\phi : k_{!} \to k_{*}$ induces the following morphisms

$$k_{!*}k^{*}\,{}^{p}\mathcal{H}^{i}(K) = \mathrm{Im}\left({}^{p}\mathcal{H}^{0}(k_{!}k^{*}\,{}^{p}\mathcal{H}^{i}(K)) \xrightarrow{{}^{p}\phi_{i}^{i}} {}^{p}\mathcal{H}^{0}(k_{*}k^{*}\,{}^{p}\mathcal{H}^{i}(K))\right) =$$
$$\oplus_{j}\mathrm{Im}\left({}^{p}\mathcal{H}^{i}(k_{!}\,{}^{p}\mathcal{H}^{j}(k^{*}K)[-j]) \xrightarrow{{}^{p}\phi_{i}^{j}} {}^{p}\mathcal{H}^{i}(k_{*}\,{}^{p}\mathcal{H}^{j}(k^{*}K)[-j])\right) =$$
$$\mathrm{Im}\left({}^{p}\mathcal{H}^{i}(k_{!}k^{*}K) \xrightarrow{{}^{p}\phi_{i}} {}^{p}\mathcal{H}^{i}(k_{*}k^{*}K)\right) \subset {}^{p}\mathcal{H}^{i}(k_{*}k^{*}K)$$

The first equality follows by definition of $k_{!*}$. The second equality follows as in the direct sum over j, only the term for $j = i$ is significant since: ${}^{p}\mathcal{H}^{i}(k_{!}k^{*}i_{S!*}\,{}^{p}\mathbf{L}_{S}^{j}[-j]) = 0$ for $j < i$, and ${}^{p}\mathcal{H}^{i}(k_{*}k^{*}i_{S!*}\,{}^{p}\mathbf{L}_{S}^{j}[-j]) = 0$ for $j > i$. The last equality follows from the decomposition of $k^{*}K$ (Equations 10.1 and 10.2).

2) $k_{!*}k^{*}\,{}^{p}\mathcal{H}^{i}(K) \subset \mathrm{Im}\,{}^{p}\rho^{i}$:

We deduce from the (non canonical) morphism $\sigma_{i} : k^{*}\,{}^{p}\mathcal{H}^{i}(K) \to k^{*}K[i]$ on $V \setminus v$ and from the composition of morphisms $k_{!}k^{*}\,{}^{p}\mathcal{H}^{i}(K) \to {}^{p}\mathcal{H}^{i}(K) \to k_{*}k^{*}\,{}^{p}\mathcal{H}^{i}(K)$:

$${}^{p}\mathcal{H}^{0}(k_{!}k^{*}\,{}^{p}\mathcal{H}^{i}(K)) \xrightarrow{a} {}^{p}\mathcal{H}^{i}(K) \xrightarrow{{}^{p}\mathcal{H}^{0}(\rho^{i})} {}^{p}\mathcal{H}^{0}(k_{*}k^{*}\,{}^{p}\mathcal{H}^{i}(K)) \overset{\sigma_{i}^{0}}{\hookrightarrow} {}^{p}\mathcal{H}^{0}(k_{*}k^{*}K[i])$$

where ${}^{p}\phi_{i}^{i} = {}^{p}\mathcal{H}^{0}(\rho^{i})\circ a$, ${}^{p}\rho^{i} = \sigma_{i}^{0}\circ {}^{p}\mathcal{H}^{0}(\rho^{i})$ since ${}^{p}\mathcal{H}^{0}(k_{*}k^{*}K[i]) = {}^{p}\mathcal{H}^{i}(k_{*}k^{*}K)$ and σ_{i}^{0} is an isomorphism by the decomposition hypothesis on $V \setminus v$. Hence:

$$k_{!*}k^{*}\,{}^{p}\mathcal{H}^{i}(K) = \sigma_{i}^{0}(\mathrm{Im}\,{}^{p}\phi_{i}^{i}) \subset \mathrm{Im}({}^{p}\rho^{i}) \subset {}^{p}\mathcal{H}^{0}(k_{*}k^{*}K[i])$$

3) $\mathrm{Im}\,{}^{p}\rho^{i} = k_{!*}k^{*}\,{}^{p}\mathcal{H}^{i}(K) = \oplus_{S_{l}\subset V\setminus v} i_{S!*}\,{}^{p}\mathbf{L}_{S}^{i}$:
By (Lemma 10.11 iv), it is enough to prove that the induced morphism below $\rho_{0}^{i} := H^{0}(i_{v}^{*}\,{}^{p}\rho^{i})$ vanish in degree 0:

Lemma 10.13. *The morphism ρ_{0}^{i} induced by $i_{v}^{*}\,{}^{p}\rho^{i}$ on the cohomology in degree zero*

$$H^{0}(i_{v}^{*}\,{}^{p}\mathcal{H}^{i}(K)) \xrightarrow{\rho_{0}^{i}=0} H^{0}(i_{v}^{*}\,{}^{p}\mathcal{H}^{i}(k_{*}k^{*}K)) = R^{i}k_{*}(k^{*}{}^{p}\tau_{\leq i}K)$$

vanish.

Proof of the lemma. This result involves the intricate relation between decomposition and local purity. The equality with the last term on the right follows by (Lemma 10.11 iii) and (Equation 10.24) applied to various strata.

i) Let $B_{v} \subset V$ be a small ball with center v. We prove:

$$H^{0}(i_{v}^{*}\,{}^{p}\mathcal{H}^{i}(K)) = \mathbb{H}^{0}(B_{v},\,{}^{p}\mathcal{H}^{i}(K)) \simeq \mathbb{H}^{0}(B_{v},\,{}^{p}\tau_{\leq 0}(K[i])) = \mathbb{H}^{i}(B_{v},\,{}^{p}\tau_{\leq i}K) \tag{10.31}$$

Indeed, we deduce from the distinguished triangle:

$${}^{p}\tau_{<0}(K[i]) \to {}^{p}\tau_{\leq 0}(K[i]) \to {}^{p}\mathcal{H}^{i}(K) \xrightarrow{[1]}$$

an exact sequence:

$$\mathbb{H}^0(B_v,\, {}^p\tau_{<0}(K[i])) \to \mathbb{H}^0(B_v,\, {}^p\tau_{\leq 0}K[i]) \to \mathbb{H}^0(B_v,\, {}^p\mathcal{H}^i(K)) \to \mathbb{H}^1(B_v,\, {}^p\tau_{<0}K[i])$$

where the two terms $\mathbb{H}^r(B_v,\, {}^p\tau_{<0}(K[i])) = 0$ for $r = 0, 1$ by definition of ${}^p\tau_{<0}$.

ii) Since $(R^i k_*(k^* {}^p\tau_{\leq i}K))_v = \mathbb{H}^i(B_v \setminus v, {}^p\tau_{\leq i}K) = {}^p\tau_{\leq i}\mathbb{H}^i(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L})$ by the decomposition of the restriction of $K := f_* j_{!*}{}^p\mathbf{L}$ to $V \setminus v$, the proof of the lemma is reduced to the following interpretation in terms of $B_v \subset V$ and $B_{X_v} := f^{-1}(B_v)$:

$$\mathbb{H}^i(B_v,\, {}^p\tau_{\leq i}K) \xrightarrow{\rho_0^i = 0} {}^p\tau_{\leq i}\mathbb{H}^i(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L}) \tag{10.32}$$

We lift the problem to B_{X_v} and deduce a factorization of the morphism ρ_0^i through the morphism ρ' below:

$$\mathbb{H}^i(B_v,\, {}^p\tau_{\leq i}K) \to {}^p\tau_{\leq i}\mathbb{H}^i(B_{X_v}, j_{!*}{}^p\mathbf{L}) \xrightarrow{\rho'} {}^p\tau_{\leq i}\mathbb{H}^i(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L})$$

where ${}^p\tau_{\leq i}\mathbb{H}^i(B_{X_v}, j_{!*}{}^p\mathbf{L}) := \mathrm{Im}(\mathbb{H}^i(B_v,\, {}^p\tau_{\leq i}K) \to \mathbb{H}^i(B_{X_v}, j_{!*}{}^p\mathbf{L})$.

Any element u of $\mathbb{H}^i(B_{X_v}, j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^i(X_v, j_{!*}{}^p\mathbf{L})$ is of weight $w \leq {}^pw + i$

since X_v is closed (§9.5.0.1, Corollary 9.4), while by the relative local purity (§9.1.0.3, Theorem 9.2) and (Proposition 10.1):

The weight w of ${}^p\tau_{\leq i}\mathbb{H}^i(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L})$ satisfy: $w > {}^pw + i$

hence $\rho'(u) = 0$ and $\rho_0^i = 0$, which ends the proof of the lemmas 10.13 and 10.12.

Remark 10.3. The space ${}^p\tau_{\leq i}\mathbb{H}^i(B_{X_v}, j_{!*}{}^p\mathbf{L})$ is a sub-MHS. In the above proof this result is not needed yet and will be proved later (Lemma 10.17). However, there exists an independent proof (Proposition 11.3).

Lemma 10.14. *Let ${}^p\mathbf{L}_v^i := \mathrm{Im}(\mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^i(X_v, j_{!*}{}^p\mathbf{L}))$, then:* $\mathrm{Im}\, {}^p\alpha^i \simeq i_*{}^p\mathbf{L}_v^i$.

Proof. By (Equation 10.30): $\mathrm{Ker}\, {}^p\delta^{i-1} = \mathrm{Im}\, {}^p\rho^{i-1} = \oplus_{S \subset V \setminus v} i_{S!*}\, {}^p\mathbf{L}_S^{i-1}$ and by (Equation 10.23), we extract from the exact sequence (Equation 10.30) the sub-exact sequence of perverse sheaves :

$$0 \to R^{i-1}k_*k^*({}^p\tau_{\leq i-1}K) \xrightarrow{{}^p\delta^{i-1}} \mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L}) \xrightarrow{{}^p\alpha^i} {}^p\mathcal{H}^i(K) \xrightarrow{{}^p\rho^i} \tag{10.33}$$

where the perverse cohomology ${}^p\mathcal{H}^i(i_v^! K)$ of the derived complex of vector spaces $i_v^!(K)$ coincides with its cohomology $\mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L})$ and ${}^p\delta^{i-1}$ induces an isomorphism onto $\mathrm{Ker}\, {}^p\alpha^i$. On the other hand, by definition, ${}^p\mathbf{L}_v^i$ is the image of I^i in the exact sequence

$$H^{i-1}(i_v^* k_* k^* K) \xrightarrow{\delta^{i-1}} \mathbb{H}^i_v(V, K) \xrightarrow{I^i} H^i(i_v^* K) \xrightarrow{\rho^i} H^i(i_v^* k_* k^* K). \tag{10.34}$$

If we prove $\mathrm{Im}\, \delta^{i-1} = \mathrm{Im}\, {}^p\delta^{i-1}$ up to the isomorphism $\mathbb{H}^i_v(V, K) \simeq \mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L})$ in (10.34) and (10.33), we deduce the result in the form of an exact sequence

$$0 \to i_{v,*}\, {}^p\mathbf{L}_v^i \xrightarrow{{}^p\alpha^i_{|}} {}^p\mathcal{H}^i(K) \xrightarrow{{}^p\rho^i_{|}} \oplus_{S \subset V \setminus v} i_{S!*}\, {}^p\mathbf{L}_S^j \to 0. \tag{10.35}$$

Again, by the decomposition theorem on $V \setminus v$, we have:
$R^{i-1}k_*k^{*p}\tau_{\leq i-1}K = \mathbb{H}^{i-1}(B_v \setminus v, {}^p\tau_{\leq i-1}K) = {}^p\tau_{\leq i-1}\mathbb{H}^{i-1}(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L})$ in (Equation 10.33) and since $H^{i-1}(i_v^*k_*k^*K) = \mathbb{H}^{i-1}(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L}))$ in (Equation 10.34), we need to prove:

Sub-lemma. $\delta^{i-1}({}^p\tau_{\leq i-1}\mathbb{H}^{i-1}(B_{X_v}\setminus X_v, j_{!*}{}^p\mathbf{L})) = \delta^{i-1}(\mathbb{H}^{i-1}(B_{X_v}\setminus X_v, j_{!*}{}^p\mathbf{L}))$

The proof is based again on the relative local purity theorem since then, the quotient space $\mathbb{H}^{i-1}(B_{X_v} \setminus X_v, j_{!*}{}^p\mathbf{L}))/{}^p\tau_{\leq i-1}$ has weight $w < {}^pw + i$, while $\mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L})$ has weight $w \geq {}^pw + i$ (ch. 9, Corollary 9.4), hence ${}^p\delta^{i-1}$ and δ^{i-1} have the same image in $\mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L})$, which ends the proof of (Lemma 10.14).

Proof of the splitting in (Equation 10.28: Dually, we introduce the distinguished triangle $k_!k^*K \xrightarrow{\beta} K \xrightarrow{\gamma} i_{v*}i_v^*K$, its exact sequence

$$ {}^p\mathcal{H}^i(k_!k^*K) \overset{{}^p\beta_i}{\to} {}^p\mathcal{H}^i(K) \overset{{}^p\gamma_i}{\to} {}^p\mathcal{H}^i(i_{v*}i_v^*K) \overset{{}^p\delta_i}{\to} {}^p\mathcal{H}^{i+1}(k_!k^*K) \tag{10.36} $$

and the sequence of morphisms: $\mathbb{H}^i_v(V, K) \xrightarrow{{}^p\alpha^i} {}^p\mathcal{H}^i(K) \xrightarrow{{}^p\gamma_i} \mathbb{H}^i(i_v^*K)$.

Lemma 10.15. *The restriction* ${}^p\gamma_{i|} : \operatorname{Im}{}^p\alpha^i \xrightarrow{\sim} \operatorname{Im}{}^p\gamma_i$ *is an isomorphism and the perverse cohomology* ${}^p\mathcal{H}^i(K)$ *splits on* V *into a direct sum:*

$$ \begin{gathered} {}^p\mathcal{H}^i(K) \xrightarrow{\sim} \operatorname{Im}{}^p\alpha^i \oplus \operatorname{Im}{}^p\beta_i = {}^p\mathbf{L}^i_v \oplus k_{!*}k^*\,{}^p\mathcal{H}^i(K) = \operatorname{Ker}{}^p\rho^i \oplus \operatorname{Im}{}^p\beta_i \\ \operatorname{Ker}{}^p\gamma_i = \operatorname{Im}{}^p\beta_i := \operatorname{Im}({}^p\beta_i : {}^p\mathcal{H}^i(Rk_!k^*K) \to {}^p\mathcal{H}^i(K)) \simeq k_{!*}k^*\,{}^p\mathcal{H}^i(K) \\ \operatorname{Ker}{}^p\rho^i \simeq \operatorname{Im}{}^p\alpha^i \simeq \operatorname{Im}{}^p\gamma_i \circ {}^p\alpha^i \simeq \operatorname{Im} I^i := i_{v,*}{}^p\mathbf{L}^i_v \end{gathered} \tag{10.37} $$

By (Remark 10.2) and by an argument dual to (Lemma 10.13), we deduce that ${}^p\beta_i$ factors through the morphism s_i in the sequence:

$$ {}^p\mathcal{H}^i(k_!k^*K) \to \oplus_{S\subset V\setminus v}\, i_{S!*}\,{}^p\mathbf{L}^i_S \xrightarrow{s_i} {}^p\mathcal{H}^i(K) \overset{{}^p\rho^i_|}{\to} \operatorname{Im}{}^p\rho^i \subset {}^p\mathcal{H}^i(k_*k^*K) \tag{10.38} $$

such that ${}^p\rho^i_| \circ s_i$ induces an isomorphism onto $\operatorname{Im}{}^p\rho^i = \oplus_{S_l\subset V\setminus v} i_{S!*}\,{}^p\mathbf{L}^i_S$. Hence s_i *defines a section of* ${}^p\rho^i_|$ in (Equation 10.38).

Since $\operatorname{Im}{}^p\alpha^i = \operatorname{Ker}{}^p\rho^i$ in the exact sequence (Equation 10.30), we deduce the splitting. Moreover, we deduce from the the relation $\operatorname{Ker}{}^p\gamma_i = \operatorname{Im}{}^p\beta_i$ (Equation 10.36) that ${}^p\gamma_i$ induces an isomorphism:

$$ \operatorname{Im}{}^p\alpha^i \xrightarrow{\sim} \operatorname{Im}{}^p\gamma_i \circ {}^p\alpha^i := i_{v*}{}^p\mathbf{L}^i_v \subset H^i(i_v^*K). $$

Remark 10.4. We extract from the sequence (Equation 10.30) an exact subsequence

$$ \begin{gathered} 0 \to \oplus_{S\subset V\setminus v, i-j>0} R^{i-j-1}k_*k^*i_{S!*}\,{}^p\mathbf{L}^j_S \overset{{}^p\delta^{i-1}}{\to} \mathbb{H}^i_v(X, K) \overset{{}^p\alpha^i}{\to} i_{v,*}\,{}^p\mathbf{L}^i_v \to 0 \\ \operatorname{Im}{}^p\delta^{i-1} \simeq \oplus_{S_l\subset V\setminus v, i-j>0} H^{i-j}_v(i_{S!*}\,{}^p\mathbf{L}^j_S) \simeq \operatorname{Ker}{}^p\alpha^i \simeq \operatorname{Ker} I^i \simeq \operatorname{Im}\delta^{i-1} \end{gathered} \tag{10.39} $$

10.3.2.1 Hard Lefschetz

The duality between ${}^p\mathcal{H}^i(K))$ and ${}^p\mathcal{H}^{-i}(K)$ follows from the auto-duality of $j_{!*}{}^p\mathbf{L}$ by the general statement of Verdier duality for proper morphisms.

Lemma 10.16 (Hard Lefschetz). *By induction, let the restriction of the iterated cup-product $\eta^i : {}^p\mathcal{H}^{-i}(K) \to {}^p\mathcal{H}^i(K)$ for $i \geq 0$ to $V \setminus v$ be an isomorphism and set*

$$ {}^p\mathbf{L}_v^i := \mathrm{Im}\,(\mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L}) \xrightarrow{I^i} \mathbb{H}^i(X_v, j_{!*}{}^p\mathbf{L})). $$

i) The HS on ${}^p\mathbf{L}_v^i$ is Poincaré dual to ${}^p\mathbf{L}_v^{-i}$ for $i \geq 0$ and the iterated cup-product induces isomorphisms $\eta^i : {}^p\mathbf{L}_v^{-i} \to {}^p\mathbf{L}_v^i$ for $i > 0$.

ii) The morphism η^i for $i \geq 0$ extends across v to an isomorphism on V.

Since the restriction of η^i on $k^*\,{}^p\mathcal{H}^{-i}(K)$ to $V \setminus v$ is an isomorphism, it remains an isomorphism on the intermediate extension $k_{!*}k^*\,{}^p\mathcal{H}^i(K)$ across the point $v \in V_0$. Hence it remains to prove Hard Lefschetz on the family of components ${}^p\mathbf{L}_v^i$. i) The duality between ${}^p\mathbf{L}_v^i$ and ${}^p\mathbf{L}_v^{-i}$ is deduced from Verdier duality between the functors $i^!_{X_v}$ and $i^*_{X_v}$ in the definition of ${}^p\mathbf{L}_v^i$.

The proof is by induction on a general hyperplane section H transversal to the NCD Y and X_v. Let i_H denote the closed embedding of H in X; the morphism η is equal to the composition of Gysin and restriction morphisms:

$$ j_{!*}{}^p\mathbf{L} \xrightarrow{\rho} i_{H*}i_H^* j_{!*}{}^p\mathbf{L} \xrightarrow{G} j_{!*}{}^p\mathbf{L}[2]. $$

We consider the induced morphisms on perverse cohomology. By (Corollary 10.1) it remains to consider the case $i = 1$ in i). Let

$$ {}^p\mathbf{L}(H)_v^0 := \mathrm{Im}(\mathbb{H}^0_{X_v \cap H}(X, j_{!*}{}^p\mathbf{L}[-1]) \xrightarrow{I^0} \mathbb{H}^0(X_v \cap H, j_{!*}{}^p\mathbf{L}[-1])) $$

be a polarized HS by induction on H and set $K_H = f_* i_{H*} i_H^* j_{!*}{}^p\mathbf{L}[-1]$. Recall the diagram

$$ {}^p\mathcal{H}^{-1}(K) \xrightarrow{\rho_{-1}} {}^p\mathcal{H}^0(K_H) \xrightarrow{G_1} {}^p\mathcal{H}^1(K)(1) \tag{10.40} $$

where ρ_{-1} is a monomorphism, G_1 an epimorphism. We consider the sequence

$$ {}^p\mathbf{L}_v^{-1} \xrightarrow{\rho_{-1}} {}^p\mathbf{L}(H)_v^0 \xrightarrow{G_1} {}^p\mathbf{L}_v^1(1). $$

By induction ${}^p\mathbf{L}(H)_v^0$ is a polarized HS, while ${}^p\mathbf{L}_v^{-1}$ and ${}^p\mathbf{L}_v^1(1)$ are HS. Hence, the image $\mathrm{Im}\,\rho_{-1}$ is a polarized sub-HS of ${}^p\mathbf{L}(H)_v^0$ and by duality $\ker G_1$ is the sub-HS orthogonal to $\mathrm{Im}\,\rho_{-1}$.

It follows that η is injective and by duality surjective. The lemma follows since the isomorphism extends at v on the intermediate extension $k_{!*}k^*\,{}^p\mathcal{H}^i(K)$ Recall that this concept of proof has been used for the first time by Deligne in (Weil II [De 80], théorème 4.1.1).

Lemma 10.17 (Compatibility with MHS). *i) The filtration ${}^p\tau^*$ on $\mathbb{H}^j(X_v, j_{!*}{}^p\mathbf{L})$ induced by the filtration ${}^p\tau$ on $f_* j_{!*}{}^p\mathbf{L}$ is a filtration by mixed Hodge sub-structures.*

ii) Equivalently the induced filtrations W and F on $i_v^* {}^p\mathcal{H}^i(f_* j_{!*} {}^p\mathbf{L}), W, F)$ define a graded polarized MHS.

Proof. The filtration is defined as in (§10.1.0.1) on the link. The proof is also similar to (Lemma 10.2). Let $B^j := \mathbb{H}^j(B_{X_v}, j_{!*} {}^p\mathbf{L}, W, F)$ with the same definition of the filtration ${}^p\tau$ on $B := \oplus_j B^j$. Then $\eta : B \to B(1)$ defined by cup-product with the hyperplane class is a morphism of MHS. We define an increasing filtration A on B by the same (Equation 10.7). We deduce: $\eta^i : \mathrm{Gr}_i^A = \mathrm{Gr}_{-i}^{{}^p\tau} B^{-i} \simeq \mathrm{Gr}_i^{{}^p\tau} B^i = \mathrm{Gr}_{-i}^A$. Hence A_i is the unique increasing filtration defined by the nilpotent endomorphism η on the MHS on B. Then A is necessarily a filtration by MHS.

Corollary 10.4 (Splitting of F). *The decomposition (Equation 10.28) is compatible with the induced filtrations F:*

$$(i_{v,*} {}^p\mathbf{L}_v^i, F) \oplus (k_{!*} k^* {}^p\mathcal{H}^i(K), F) \overset{\varphi \sim}{\to} ({}^p\mathcal{H}^i(K), F) \tag{10.41}$$

The induced filtration F on $i_{v,*} {}^p\mathbf{L}_v^i$ is a HS embedded in the MHS on $i_v^* {}^p\mathcal{H}^i(f_* j_{!*} {}^p\mathbf{L}), W, F)$ and the decomposition is in the category of MHS.

Example 10.3. In the case of the desingularization $f : X \to V$ of an isolated singularity $v \in V$, the decomposition is described in (Eexample 10.2) in terms of $H^i_{X_v}(X, \mathbb{Q}[\dim X])$ for $i < 0$ and $H^i(X_v, \mathbb{Q}[\dim X])$ for $i \geq 0$, since the morphism $H^i_{X_v}(X, \mathbb{Q}[\dim X]) \to H^i(X_v, \mathbb{Q}[\dim X])$ is injective for $i < 0$, surjective for $i > 0$ and an isomorphism for $i = 0$.

Corollary 10.5. *The perverse filtration ${}^p\tau$ on $(\mathbb{H}^{i+j}(X, j_{!*} {}^p\mathbf{L}), F)$ is compatible with Hodge filtration and the following decomposition:*

$$Gr_i^{{}^p\tau} \mathbb{H}^{i+j}(X, j_{!*} {}^p\mathbf{L}) \simeq \mathbb{H}^j(V, {}^p\mathcal{H}^i(f_* j_{!*} {}^p\mathbf{L})) \xrightarrow{\sim} \oplus_{S \subset V} \mathbb{H}^j(V, i_{S!*} {}^p\mathbf{L}_S^i).$$

consists of a direct sum of polarized HS of weight $a + i + j$

Proof. The proof is similar to (Lemma 10.2). Let $B^j := \mathbb{H}^j(X, j_{!*} {}^p\mathbf{L}, F)$ with the same definition of the filtration ${}^p\tau$ on $B := \oplus_j B^j$.

Then $\eta : B \to B(1)$ defined by cup-product with the hyperplane class is a morphism of HS. We define an increasing filtration A on B by the same (Equation 10.7). We deduce: $\eta^i : \mathrm{Gr}_i^A = \mathrm{Gr}_{-i}^{{}^p\tau} B^{-i} \simeq \mathrm{Gr}_i^{{}^p\tau} B^i = \mathrm{Gr}_{-i}^A$.

Hence A_i is the unique increasing filtration defined by the nilpotent endomorphism η on the MHS on B. Then A is necessarily a filtration by HS.

The HS on $\mathbb{H}^j(V, {}^p\mathcal{H}^i(f_* j_{!*} {}^p\mathbf{L}))$ is isomorphic to $Gr_i^{{}^p\tau} \mathbb{H}^{i+j}(X, j_{!*} {}^p\mathbf{L})$. Moreover, Since the decomposition of ${}^p\mathcal{H}^i(f_* j_{!*} {}^p\mathbf{L})$ is naturally compatible with the filtration F, the decomposition of $\mathbb{H}^j(V, {}^p\mathcal{H}^i(f_* j_{!*} {}^p\mathbf{L}))$ is also compatible with HS.

Remark 10.5. Let V be projective. The decomposition into primitive subspaces may be carried in two steps:

1) We deduce a scalar product on ${}^p\mathcal{H}^{-i}(f_*j_{!*}{}^p\mathbf{L})$ from Lefschetz isomorphism η^i and the duality ${}^p\mathcal{H}^{-i}(f_*j_{!*}{}^p\mathbf{L}) \simeq D\,{}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L})$. The primitive subperverse sheaf is defined by $\operatorname{Ker}\eta^{i+1} \subset {}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L})$ [De 93].

2) Let η_V denote the cup product with the class $c_1(V)$ of an hyperplane section H of the projective variety V. The scalar product is defined by Hard Lefschetz isomorphisms on V defined by iteration of η_V and duality. The polarized primitive sub-HS is defined by $\operatorname{Ker}\eta_V^{j+1} : \mathbb{H}^{-j}(V, \operatorname{Ker}\eta^{i+1}) \to \mathbb{H}^{j+2}(V, \operatorname{Ker}\eta^{i+1})$. The proof of polarization, by induction on the hyperplane section H_V of V is similar to Deligne's original proof (Weil II [De 80], théorème 4.1.1) (see also [CaMi 5]).

10.3.2.2 Octahedron diagram

With the notations of (Theorem 10.1), set $K := f_*j_{!*}{}^p\mathbf{L}$ and $I := \gamma \circ \alpha$ the composition map $I : i_{v*}i_v^!K \xrightarrow{\alpha} K \xrightarrow{\gamma} i_{v*}i_v^*K$. We consider the following upper and lower 'cap' diagrams ([BBD 83], 1.1.6)

$$\begin{tikzcd}
i_{v*}i_v^!K \arrow[dd, "I"'] \arrow[dr, "\alpha"] & & k_*k^*K \arrow[ll, "\delta[1]"'] \\
& K \arrow[ur, "\rho"] \arrow[dl, "\gamma"] & \\
i_{v*}i_v^*K \arrow[rr, "\delta'[1]"] & & k_!k^*K \arrow[ul, "\beta"] \arrow[uu, "\beta'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
i_{v*}i_v^*K \arrow[dd, "+1"'] \arrow[dr, "c'"] & & i_{v*}i_v^!K \arrow[ll, "I"'] \\
& i_{v*}i_v^*k_*k^*K \arrow[ur, "d\ \delta'''[1]"] \arrow[dl, "\delta''[1]"] & \\
k_!k^*K \arrow[rr, "\beta'"] & & k_*k^*K \arrow[ul, "\gamma'"] \arrow[uu, "+1"']
\end{tikzcd}
\tag{10.42}$$

where $i_v : v \to V$ and $k : V \setminus v \to V$, $i_{v*}i_v^*k_*k^*K$ is isomorphic to the cone of $I : i_{v*}i_v^!K \to i_{v*}i_v^*K$: $i_{v*}i_v^*k_*k^*K \simeq C(I) \simeq C(\beta')$.

Corollary 10.6. *i) The long perverse exact sequence of four terms defined by the common boundary of the diagrams is exact:*

$$\to {}^p\mathcal{H}^i(i_{v*}i_v^!K) \xrightarrow{{}^pI_i} {}^p\mathcal{H}^i(i_{v*}i_v^*K) \xrightarrow{{}^p\delta'_i} {}^p\mathcal{H}^{i+1}(k_!k^*K) \xrightarrow{{}^p\beta'_{i+1}} {}^p\mathcal{H}^{i+1}(k_*k^*K) \xrightarrow{{}^p\delta_{i+1}} \tag{10.43}$$

ii) The morphism ${}^p\gamma_i$ in (Equation 10.36) induces an isomorphism:

$$\operatorname{Im}{}^p\alpha^i \xrightarrow{\sim} \operatorname{Im}{}^pI^i := \operatorname{Im}(\mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L}) \to \mathbb{H}^i(X_v, j_{!*}{}^p\mathbf{L})) \simeq \mathbf{L}^i_v \tag{10.44}$$

Proof. i) In the case of the decomposition: ${}^p\mathcal{H}^i(K) \xrightarrow{\sim} \operatorname{Im}{}^p\alpha^i \oplus \operatorname{Ker}{}^p\gamma_i$ (Lemma 10.15) in the left commutative triangle of the upper cap, the exact sequence (Equation 10.43) follows by general properties of octahedron diagrams.

ii) The isomorphism in (Equation 10.44) is induced by ${}^p\gamma_i$.

10.3.3 Singular variety X

The decomposition theorem has been proved in the previous chapters in the case of non singular variety X.

We present here a short review of the case where X is projective singular by reduction to a desingularization of X.

1) The decomposition theorem of $f : X \to V$ is deduced directly by Deligne's argument on the composition of f with the desingularization morphism $\pi : X' \to X$ ([De 68] Proposition 2.16). We refer to (§6.2.2.2) for this method.

2) However, the theorem may be deduced again by the previous induction $D(n)$ applied in this setting.

3) The relative local purity theorem apply for all projective morphisms f (of singular varieties).

10.3.3.1 Hodge structure on $\mathbb{H}^*(X, j_{!*}{}^p\mathbf{L})$ for singular X

Let X be a projective complex variety maybe singular, $j : U \to X$ the embedding of a smooth Zariski open subset of X and ${}^p\mathbf{L}$ defined on U.

Corollary 10.7 (HS on $\mathbb{H}^*(X, j_{!*}{}^p\mathbf{L})$). *There exists a HS on $\mathbb{H}^i(X, j_{!*}{}^p\mathbf{L})$ independent of the choice of a desingularization.*

Proof. Let $\pi : X' \to X$ be a non singular modification of X, $j' : U \to X'$ the embedding of U in X' and $j'_{!*}{}^p\mathbf{L}$ the intermediate extension in X'. Let $\mathcal{S}$ be a Thom-Whitney stratification of π compatible with $j'_{!*}{}^p\mathbf{L}$.

There exists a decomposition on X of ${}^p\mathcal{H}^i(\pi_* j'_{!*}{}^p\mathbf{L}) \simeq \oplus_{S\subset X} i_{S!*}{}^p\mathbf{L}^i_S$ into a direct sum consisting of intermediate extensions of polarized VHS. We deduce a decomposition into a direct sum of polarized HS of weight $a + i + j$

$$Gr_i^{{}^p\tau}\mathbb{H}^{i+j}(X', j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^j(X, {}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L})) \xrightarrow{\sim} \oplus_{S\subset X}\ \mathbb{H}^j(X, i_{S!*}{}^p\mathbf{L}^i_S).$$

where for each stratum S, the hypercohomology is a direct sum of polarized primitive subspaces of the hypercohomology on X with coefficients into the primitive decomposition of $i_{S!*}{}^p\mathbf{L}^i_S$.

Since on the big strata U of X, we have $j_{!*}{}^p\mathbf{L} = i_{U!*}{}^p\mathbf{L}^0_U$, we deduce the HS on $\mathbb{H}^i(X, j_{!*}{}^p\mathbf{L})$ as a sub-quotient HS of $\mathbb{H}^i(X', j'_{!*}{}^p\mathbf{L})$, precisely a polarized sub-HS of $Gr_0^{{}^p\tau}\,\mathbb{H}^i(X', j'_{!*}{}^p\mathbf{L})$.

The uniqueness is deduced by the method of comparison of two non singular modifications X'_1 and X'_2 with a non singular modification X' of the fiber product $X'_1 \times_X X'_2$ such that $X' \to X_1$ (resp. $X' \to X_2$) is a composition of non singular modifications, then the HS on $\mathbb{H}^i(X, j_{!*}{}^p\mathbf{L})$ deduced from X'_1 or X'_2 coincide with the HS deduced from X'.

10.3.4 Decomposition of ${}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L})$

Independently of Deligne's argument (Equation 6.22), a direct proof by induction $D(n)$ (§10.0.0.1) applies. The case $\dim V = 0$ as initial step, is proved (§10.3.3.1).

The assumption $D(n, V^*)$ on the complement of isolated singularity follows by general normally embedded sections of V, which reduces the proof to isolated strata of V.

Let $\pi : X' \to X$ be a desingularization, ${}^p\tau'$ the perverse truncation of $K' := \pi_* j'_{!*}{}^p\mathbf{L}$ on X and ${}^p\tau_V$ the perverse truncation of $K := (f \circ \pi)_* j'_{!*}{}^p\mathbf{L} = f_*K'$ on V. We deduce from ${}^p\tau'$ a filtration ${}^p\tau_X := f_*({}^p\tau')$ on K

$$K := (f \circ \pi)_* j'_{!*}{}^p\mathbf{L} = f_*K',\ {}^p\tau_V,\ {}^p\tau_X := f_*({}^p\tau'), F := (f \circ \pi)_* F \qquad (10.45)$$

The filtrations ${}^p\tau_X$, ${}^p\tau_V$ and F are strict on K since the respective spectral sequence degenerates at rank one.

The induced filtration ${}^p\tau_X$ on ${}^p\mathcal{H}^i(K) := Gr_i^{{}^p\tau_V} K$ is defined by

$$({}^p\tau_X)_j\, {}^p\mathcal{H}^i(K) := \mathrm{Im}({}^p\mathcal{H}^i(f_*({}^p\tau_X)_j K) \to {}^p\mathcal{H}^i(K))$$

such that $Gr_j^{{}^p\tau_X}\, {}^p\mathcal{H}^i(K) \simeq {}^p\mathcal{H}^i(f_* Gr_j^{{}^p\tau'} K') \simeq {}^p\mathcal{H}^i(f_*\, {}^p\mathcal{H}^j(K'))$.

In terms of the decomposition of ${}^p\mathcal{H}^0(K') \simeq j_{!*}{}^p\mathbf{L} \oplus_{S \subset X, \dim S < \dim X} i_{S!*}{}^p\mathbf{L}^0_S$:

$$Gr_0^{{}^p\tau_X}\, {}^p\mathcal{H}^i(K) \simeq {}^p\mathcal{H}^i(f_*\, {}^p\mathcal{H}^0(K')) \simeq {}^p\mathcal{H}^i(f_* j_{!*}{}^p\mathbf{L}) \oplus_S {}^p\mathcal{H}^i(f_* i_{S!*}{}^p\mathbf{L}^0_S) \quad (10.46)$$

where the sum is over all strata S of $X : \dim S < \dim X$.

10.3.4.1 Relative local purity for all projective morphisms f

Let V_0 denote the union of the strata of dimension zero of V and $k : (V \backslash V_0) \to V$. We suppose V_0 reduced to a point v to simplify the notations.

Corollary 10.8. *The weight w of the space $H^i(i_v^* k_* k^*\, {}^p\mathcal{H}^j(f_* j_{!*}{}^p\mathbf{L}))$, isomorphic to the Intersection cohomology $\mathbb{H}^i(B_v \setminus \{v\}, {}^p\mathcal{H}^j(f_* j_{!*}{}^p\mathbf{L}))$, satisfies the relations:*

$$w \leq {}^p w + j + i \text{ if } i \leq -1 \text{ and } w > {}^p w + j + i \text{ if } i \geq 0 \qquad (10.47)$$

The inequalities on the weights w of the space $\mathbb{H}^i(B_v - \{v\}, {}^p\mathcal{H}^j(K))$ in the equation 10.47 are satisfied since $f \circ \pi$ is a stratification by NCD over the isolated strata as proved in (§10.6). Since $\mathbb{H}^i(B_v \setminus \{v\}, Gr_0^{{}^p\tau_X}\, {}^p\mathcal{H}^j(K))$ is a sub-quotient MHS of $\mathbb{H}^i(B_v \setminus \{v\}, {}^p\mathcal{H}^j(K))$, the inequalities on the weight w are also satisfied .

Finally the corollary follows since ${}^p\mathcal{H}^j(f_*j_{!*}{}^p\mathbf{L})$ is a summand of $Gr_0^{{}^p\tau_X}\,{}^p\mathcal{H}^j(K)$ (Equation 10.46) and there exists necessarily a decomposition of ${}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L})$ (Equation 6.23) into intermediate extensions of $({}^p\mathbf{L})^i_S$ as in theorem (ch.1, Theorem 1.2).

Corollary 10.9. *Let V_m denote the union of maximal strata of V. We have the decomposition*

$$ {}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L}) \simeq \oplus j_{V!*}(R^if_*j_{!*}{}^p\mathbf{L}_{|V_m}[\dim V]) \oplus_{S\subset\mathcal{S},\,\dim S<\dim V} i_{S!*}{}^p\mathbf{L}^i_S \tag{10.48}$$

10.3.4.2 Duality and polarization

The duality between ${}^p\mathbf{L}^{-k}_S$ and ${}^p\mathbf{L}^k_S$ follows from the auto-duality of $j_{!*}{}^p\mathbf{L}$ by Verdier duality formula for the proper morphism f.

Since Hard Lefschetz isomorphisms ${}^p\mathbf{L}^{-k}_S \simeq {}^p\mathbf{L}^k_S$ are also satisfied, we deduce that the VHS ${}^p\mathbf{L}^k_S$ are direct sum of polarized primitive sub-VHS.

For each strata S of V, the local system ${}^p\mathbf{L}^i_S$ defined by $f_*j_{!*}{}^p\mathbf{L}$ is a direct sum polarized variation of Hodge structure on the smooth variety S.

Corollary 10.10. *Let $f : X \to V$ be a morphism of projective varieties where X is non singular. The decomposition consists of a direct sum of polarized HS of weight $a + i + j$*

$$Gr_i^{{}^p\tau}\mathbb{H}^{i+j}(X, j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^j(V, {}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L})) \xrightarrow{\sim} \oplus_{S\subset V}\ \mathbb{H}^j(V, i_{S!*}{}^p\mathbf{L}^i_S).$$

Remark 10.6. The proof of the existence of HS on the hypercohomology of a singular complex variety X with coefficients in intermediate extensions of polarized VHS is reduced to the case where X is projective, by the decomposition theorem applied to a birational morphism $\pi : X' \to X$ where X' is projective and non singular (§6.2.2.1).

10.4 Relative local purity and Intersection morphisms

The relative local purity theorem may be interpreted in terms of the intersection morphism.

In the case of an isolated singularity $v \in V$, the local purity theorem has consequences on the weights of the cohomology with support $\mathbb{H}^i_{X_v}(X, \mathbb{Q})$. A direct proof of some consequences of the relative local purity appears in the work of Navarro Aznar ([Na 85] Proposition 5.1), where the results are treated in terms of the exceptional NCD in a desingularization and the MHS of $\mathbb{H}^i_{X_v}(X, \mathbb{Q})$.

In the case of constant coefficients, the intersection morphism is studied in ([CaMi 5] section 6.3) where a generalized Grauert contractibility criterion is proved. In fact, the statement is equivalent to the purity theorem.

10.4.1 Intersection morphism

Let $f : X \to V$ be a projective morphism of complex algebraic varieties, $v \in V$, $X_v := f^{-1}(v) \subset X$ and $B_{X_v} := f^{-1}(B_v)$ the inverse image of a small ball $B_v \subset V$. With the notations of Lemma 10.8, the perverse filtration ${}^p\tau$ on $K := f_* j_{!*} {}^p\mathbf{L}$ on V induces a filtration ${}^p\tau$ on $i_v^! K$ defined functorially by $i_v^! \, {}^p\tau K$ and computed by $i_v^!$-acyclic filtered representative of $(K, {}^p\tau)$.

Similarly, we define ${}^p\tau$ on $i_v^* K$ and $i_v^* k_* k^* K$, from which we deduce increasing filtrations ${}^p\tau$ on $\mathbb{H}^i_{X_v}(X, j_{!*} {}^p\mathbf{L})$, $\mathbb{H}^i(X_v, j_{!*} {}^p\mathbf{L})$ and $\mathbb{H}^i(B^*_{X_v}, j_{!*} {}^p\mathbf{L})$ (Lemma 10.2). The following proposition is equivalent to the local purity theorem.

Proposition 10.2 (Intersection morphism). *i) The morphisms induced by the intersection morphism I (ch. 2, Equation 2.1.3.1)*

$$\mathrm{Gr}_j^{{}^p\tau} \mathbb{H}^i_{X_v}(X, j_{!*} {}^p\mathbf{L}) \xrightarrow{{}^p\tau I_i^j} \mathrm{Gr}_j^{{}^p\tau} \mathbb{H}^i(X_v, j_{!*} {}^p\mathbf{L}) \tag{10.49}$$

are isomorphisms for each $i = j$ and vanish for all indices $i \neq j$.

ii) Conversely, if the intersection morphism ${}^p\tau I_i^j$ is an isomorphism in the case $i = j$, the inequalities on the weights in the local purity theorem hold.

We start the proof with a general result

Lemma 10.18. *The induced perverse filtrations ${}^p\tau$ have the properties:*

$$ {}^p\tau_i \mathbb{H}^i_{X_v}(X, j_{!*} {}^p\mathbf{L}) = \mathbb{H}^i_{X_v}(X, j_{!*} {}^p\mathbf{L}) \quad \textit{and} \quad {}^p\tau_{i-1} \mathbb{H}^i(X_v, j_{!*} {}^p\mathbf{L}) = 0. $$

Proof. Let $K := f_* j_{!*} {}^p\mathbf{L}$. We have convergent spectral sequences associated to the increasing induced filtrations ${}^p\tau$

$$\begin{aligned} {}_{p\tau}E_{v1}^{pq} &:= \mathbb{H}_v^{2p+q}(V, {}^p\mathcal{H}^{-p}(K)) \Rightarrow \mathrm{Gr}_{-p}^{{}^p\tau} \mathbb{H}^{p+q}_{X_v}(X, j_{!*} {}^p\mathbf{L}) \\ {}_{p\tau}E_1^{pq}(v) &:= \mathbb{H}^{2p+q}(v, {}^p\mathcal{H}^{-p}(K)) \Rightarrow \mathrm{Gr}_{-p}^{{}^p\tau} \mathbb{H}^{p+q}(X_v, j_{!*} {}^p\mathbf{L}). \end{aligned} \tag{10.50}$$

Since ${}^p\mathcal{H}^{-p}(K)$ is perverse, we have

$\mathbb{H}_v^i(V, {}^p\mathcal{H}^{-p}(K)) = 0$ for $i < 0$ (resp. $\mathbb{H}^i(v, {}^p\mathcal{H}^{-p}(K)) = 0$ for $i > 0$)

then: ${}_{p\tau}E_{v1}^{pq} = 0$ for $2p + q < 0$, (resp. ${}_{p\tau}E_1^{pq}(v) = 0$ for $2p + q > 0$), and

$$\begin{aligned} &\mathrm{Gr}_j^{{}^p\tau} \mathbb{H}^i_{X_v}(X, j_{!*} {}^p\mathbf{L}) = 0 \text{ for } i < j : \ {}^p\tau_i \, \mathbb{H}^i_{X_v}(X, j_{!*} {}^p\mathbf{L}) = \mathbb{H}^i_{X_v}(X, j_{!*} {}^p\mathbf{L}), \\ &\mathrm{Gr}_j^{{}^p\tau} \mathbb{H}^i(X_v, j_{!*} {}^p\mathbf{L}) = 0 \text{ for } i > j : \ {}^p\tau_{i-1} \, \mathbb{H}^i(X_v, j_{!*} {}^p\mathbf{L}) = 0 \end{aligned} \tag{10.51}$$

Proof of the proposition. The long exact sequence

$\mathbb{H}^{i-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) \xrightarrow{\partial_i} \mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L}) \xrightarrow{I_i} \mathbb{H}^i(X_v, j_{!*}{}^p\mathbf{L}) \xrightarrow{\gamma_i} \mathbb{H}^i(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) \xrightarrow{[1]}$

is deduced from the distinguished triangle:

$Ri^!_{X_v} j_{!*}{}^p\mathbf{L} \xrightarrow{I} i^*_{X_v} j_{!*}{}^p\mathbf{L} \longrightarrow i^*_{X_v} Rj_{X_v *} j^*_{X_v} j_{!*}{}^p\mathbf{L}$. By the decomposition theorem, the morphisms of the exact sequence are strictly compatible with the induced ${}^p\tau$ filtrations. We deduce the exact sequence (on the next two lines)

$$\mathrm{Gr}_j^{{}^p\tau}\mathbb{H}^{i-1}(X_v, j_{!*}{}^p\mathbf{L}) \xrightarrow{{}^p\tau\gamma^j_{i-1}} \mathrm{Gr}_j^{{}^p\tau}\mathbb{H}^{i-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) \xrightarrow{{}^p\tau\partial^j_i} \mathrm{Gr}_j^{{}^p\tau}\mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L})$$
$$\xrightarrow{{}^p\tau I^j_i} \mathrm{Gr}_j^{{}^p\tau}\mathbb{H}^i(X_v, j_{!*}{}^p\mathbf{L}) \xrightarrow{{}^p\tau\gamma^j_i} \mathrm{Gr}_j^{{}^p\tau}\mathbb{H}^i(B^*_{X_v}, j_{!*}{}^p\mathbf{L}) \xrightarrow{{}^p\tau\partial^j_{i+1}} Gr_j^{{}^p\tau}\mathbb{H}^{i+1}_{X_v}(X, j_{!*}{}^p\mathbf{L})$$

By the lemma, the morphism ${}^p\tau I^j_i$ vanishes if $i \neq j$.

For $j = i$, by the relative local purity theorem, the weights w of the term $\mathrm{Gr}_i^{{}^p\tau}\mathbb{H}^{i-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L})$ satisfy $w \leq {}^pw + i - 1$ and those of the term $\mathrm{Gr}_i^{{}^p\tau}\mathbb{H}^i(B^*_{X_v}, j_{!*}{}^p\mathbf{L})$ satisfy $w > {}^pw + i$. Moreover the weights w of the term $\mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L})$ satisfy: $w \geq {}^pw + i$, hence: ${}^p\tau\partial^i_i = 0$.

We have ${}^p\tau\gamma^i_i = 0$ since the weights w of the term $\mathrm{Gr}_i^{{}^p\tau}\mathbb{H}^i(X_v, j_{!*}{}^p\mathbf{L})$ satisfy $w \leq {}^pw + i$. Hence ${}^p\tau I^i_i : Gr_i^{{}^p\tau}\mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L}) \to Gr_i^{{}^p\tau}\mathbb{H}^i(X_v, j_{!*}{}^p\mathbf{L})$

is an isomorphism. Remark that by the decomposition theorem:

$$\mathrm{Gr}_i^{{}^p\tau}\mathbb{H}^i_{X_v}(X, j_{!*}{}^p\mathbf{L}) \simeq H^0(i^!_v\, {}^p\mathcal{H}^i(K)) \simeq \mathbf{L}^i_v \simeq \mathrm{Gr}_i^{{}^p\tau}\mathbb{H}^i(X_v, j_{!*}{}^p\mathbf{L}) \simeq H^0(i^*_v\, {}^p\mathcal{H}^i(K)).$$

Conversely, we prove the crucial step of the relative local purity theorem (ch. 11, Equation 11.5): $\mathrm{Gr}^W_{{}^pw}\mathrm{Gr}_0^{{}^p\tau}\mathbb{H}^{-1}((B^*_{X_v}, j_{!*}{}^p\mathbf{L}) = 0$. We consider the exact sequence:

$$\mathrm{Gr}_0^{{}^p\tau}\mathbb{H}^{-1}_{X_v}(X, j_{!*}{}^p\mathbf{L}) \xrightarrow{{}^p\tau I^0_{-1}} \mathrm{Gr}_0^{{}^p\tau}\mathbb{H}^{-1}(X_v, j_{!*}{}^p\mathbf{L}) \xrightarrow{{}^p\tau\gamma^0_{-1}} \mathrm{Gr}_0^{{}^p\tau}\mathbb{H}^{-1}(B^*_{X_v}, j_{!*}{}^p\mathbf{L})$$
$$\xrightarrow{{}^p\tau\partial^0_0} \mathrm{Gr}_0^{{}^p\tau}\mathbb{H}^0_{X_v}(X, j_{!*}{}^p\mathbf{L}) \xrightarrow{{}^p\tau I^0_0} \mathrm{Gr}_0^{{}^p\tau}\mathbb{H}^0(X_v, j_{!*}{}^p\mathbf{L})$$

By the above lemma $\mathrm{Gr}_0^{{}^p\tau}\mathbb{H}^{-1}_{X_v}(X, j_{!*}{}^p\mathbf{L}) = 0$, hence ${}^p\tau I^0_{-1} = 0$ and ${}^p\tau\gamma^0_{-1}$ is injective. If the morphism ${}^p\tau I^0_0$ is an isomorphism, then ${}^p\tau\partial^0_0 = 0$ and ${}^p\tau\gamma^0_{-1}$ is surjective, hence an isomorphism. By compatibility of ${}^p\tau$ with MHS, ${}^p\tau\gamma^0_{-1}$ induces an isomorphism

$$\mathrm{Gr}^W_w\mathrm{Gr}_0^{{}^p\tau}\mathbb{H}^{-1}(X_v, j_{!*}{}^p\mathbf{L}) \simeq \mathrm{Gr}^W_w\mathrm{Gr}_0^{{}^p\tau}\mathbb{H}^{-1}((B^*_{X_v}, j_{!*}{}^p\mathbf{L})$$

Since X_v is closed, the weights of $\mathbb{H}^{-1}(X_v, j_{!*}{}^p\mathbf{L})$ are $\leq a - 1$, which proves $\mathrm{Gr}^W_{{}^pw}\mathrm{Gr}_0^{{}^p\tau}\mathbb{H}^{-1}(X_v, j_{!*}{}^p\mathbf{L}) = \mathrm{Gr}^W_{{}^pw}\mathrm{Gr}_0^{{}^p\tau}\mathbb{H}^{-1}((B^*_{X_v}, j_{!*}{}^p\mathbf{L}) = 0$.

Remark 10.7. We deduce from the intersection isomorphism above and the duality map $D : \mathrm{Gr}_{-r}^{{}^p\tau}\mathbb{H}^{-r}_{X_v}(X, j_{!*}{}^p\mathbf{L}) \xrightarrow{\sim} \mathrm{Gr}_r^{{}^p\tau}\mathbb{H}^r(X_v, j_{!*}{}^p\mathbf{L})^*$ a non degenerate intersection form

$$\mathrm{Gr}_r^{{}^p\tau}\mathbb{H}^r_{X_v}(X, j_{!*}{}^p\mathbf{L}) \otimes \mathrm{Gr}_{-r}^{{}^p\tau}\mathbb{H}^{-r}_{X_v}(X, j_{!*}{}^p\mathbf{L}) \to \mathbb{Q},\ (a, b) \mapsto D(b)(I(a))$$

When X_v is a NCD on a smooth surface and in the case of intersection cohomology with rational coefficients viewed as homology, Grauert's contractibility theorem states that the intersection form defined on $H_2(X_v)$ is negative definite if and only if X_v contracts to a point v on a normal surface (see also [CaMi 5], [BPV 84] Theorem 2.1).

Chapter 11
Related results

We present in this chapter, in the first two sections interesting results that has been originally used in the text but were not strictly necessary. In the third section a tentative combinatorial description of the weight filtration on the logarithmic complex is given.

11.1 Hodge theory on perverse cohomology

Let $f : X \to V$ be a projective morphism on a non singular complex algebraic variety X of dimension $n+1$, $j_{!*}{}^p\mathbf{L}$ on X, $K := f_* j_{!*}{}^p\mathbf{L}$, $v \in V$ a closed point such that $Z = X_v := f^{-1}(v)$ is a NCD in X and B_v^* a punctured small ball with deleted center v.

An independent direct proof that ${}^p\tau^*\mathbb{H}^r(X_{B_v^*}, j_{!*}{}^p\mathbf{L})$ and ${}^p\tau^*\mathbb{H}^r(X_{B_v}, j_{!*}{}^p\mathbf{L})$ are filtrations by sub-MHS (Proposition 11.3 below) may be deduced from the description of the perverse filtration directly on X ([CaMi 10]). In chapter 10, the filtration ${}^p\tau^*$ has been deduced after the proof of the decomposition theorem (Lemma 10.17 and 10.2).

11.1.1 Preliminaries

Let V be a quasi-projective variety and $K \in D_c^b(V, \mathbb{Q})$ a complex with constructible cohomology sheaves. The topological middle perversity truncations on K on V define an increasing *perverse filtration* ${}^p\tau$ on K.

A corresponding increasing filtration ${}^p\tau$ is defined on the hypercohomology of K on open subsets U of V

$$ {}^p\tau_i\mathbb{H}^k(U, K) := \operatorname{Im}\left\{\mathbb{H}^k(U, {}^p\tau_i K) \to \mathbb{H}^k(U, K)\right\}. \tag{11.1} $$

This perverse filtration ${}^p\tau$ on the hypercohomology groups $\mathbb{H}^k(V,K)$ is described by algebraic-geometric techniques in [CaMi 10] as follows.

Given an affine embedding $V \subset \mathbb{P}^N$ and $n := \dim V$, we consider two families of hyperplanes $\Lambda_* := \Lambda_i$ and $\Lambda'_* := \Lambda'_i$ for $1 \le i \le n$ in $\mathbb{P}^N$, defining on V two families H_* and W_* of increasing closed sub-varieties of V:
$H_{-r} := \cap_{1\le i\le r}\Lambda_i \cap V$ and $W_{-r} := \cap_{1\le i\le r}\Lambda'_i \cap V$

$$H_* : V = H_0 \supset H_{-1} \supset \ldots \supset H_{-n}, \quad W_* : V = W_0 \supset W_{-1} \supset \ldots \supset W_{-n} \tag{11.2}$$

and $H_{-n-1} = \emptyset = W_{-n-1}$. Let $h_i : (V \setminus H_{i-1}) \to V$ denote the open embeddings with indices in $[-n, 0]$. The decreasing filtration δ:

$$\delta^p\mathbb{H}^*(V,K) := \mathrm{Im}\Big(\oplus_{i+j=p}\mathbb{H}^*_{W_{-j}}(V,(h_i)_!h_i^*K) \to \mathbb{H}^*(V,K)\Big) \tag{11.3}$$

([CaMi 10], Remark 3.6.6) determines ${}^p\tau$ and is well adapted to computation in Hodge theory.

Proposition 11.1 ([CaMi 10], Theorem 4.2.1).
For an affine embedding of V into a projective space and for a general choice of both families H_* and W_* depending on K and the embedding of V, the filtration δ is equal to the perverse filtration ${}^p\tau$ up to a change in indices. Precisely

$$\delta^i\mathbb{H}^*(V,K) = {}^p\tau_{-i+l}\mathbb{H}^l(V,K)$$

11.1.2 Hodge structure on the perverse filtration

We consider two cases: $U := V \setminus W$ is quasi-projective, complement of a closed subvariety $W \subset V$ (resp. U is a punctured small ball B_v^* with deleted center v).

Proposition 11.2 (MHS on perverse hypercohomology).
Let $f : X \to V$ be a projective morphism where X is smooth, V projective, ${}^p\mathbf{L}$ a shifted admissible variation of MHS on $X \setminus Y$ the complement of a normal crossing divisor Y and $j : X \setminus Y \to X$.

Let $W \subset V$ be a closed algebraic subset of V such that $X_W := f^{-1}(W)$ is a sub-normal crossing divisor of Y. The perverse filtration ${}^p\tau$ on the hypercohomology $\mathbb{H}^*(X \setminus X_W, j_{!*}{}^p\mathbf{L})$ (resp. $\mathbb{H}^*_c(X \setminus X_W, j_{!*}{}^p\mathbf{L})$ by duality) is a filtration by sub-MHS.

Proof. The proof is in terms of logarithmic complexes. Let $Z := f^{-1}W$ the inverse image of W and $j_Z : (X - Z) \to X$. We apply (Proposition 11.1) to $K := f_*j_{Z*}j_Z^*j_{!*}{}^p\mathbf{L}$ (including $Z = Y$ or $Z = \emptyset$) (the case $f_*j_{Z!}j_Z^*j_{!*}{}^p\mathbf{L}$ is dual).

The perverse filtration ${}^p\tau$ on $\mathbb{H}^k(X-Z, j_{!*}{}^p\mathbf{L})$ is defined via the isomorphism $\mathbb{H}^k(X\setminus Z, j_{!*}{}^p\mathbf{L}) \simeq \mathbb{H}^k(V\setminus W, f_*j_{!*}\mathcal{L})$:

$$ {}^p\tau_i\mathbb{H}^k(X\setminus Z, j_{!*}\mathcal{L}) := {}^p\tau_i\mathbb{H}^k(V\setminus W, f_*j_{!*}\mathcal{L}). $$

We check that the filtration ${}^p\tau$ is a filtration by sub-MHS on $\mathbb{H}^k(X\setminus Z, j_{!*}{}^p\mathbf{L})$ using (Proposition 11.1). The system of truncations morphisms:

$\cdots \to {}^p\tau_{\leq -i}f_*IC^*\,{}^p\mathcal{L}(Log\,Z) \to {}^p\tau_{\leq -i+1}f_*IC^*\,{}^p\mathcal{L}(Log\,Z)\cdots$ is defined over $\mathbb{Q}$ and it is isomorphic in $D^b_c(V,\mathbb{C})$ to a system of inclusions maps:

$\cdots \to P^iK \to P^{i-1}K\cdots$ where $P^iK \simeq {}^p\tau_{\leq -i}f_*IC^*\,{}^p\mathcal{L}(Log\,Z)$ is decreasing and where we suppose K, all P^iK and Gr^i_PK injective complexes ([BBD 83], 3.1.2.7) defined over $\mathbb{Q}$.

The filtration P on K is defined up to unique isomorphism in the category of derived filtered complexes. Moreover, we represent the filtrations W and F by acyclic resolutions for the global sections functor $\Gamma(V,*)$ on the same complex (K,P,W,F) representative of $(f_*IC^*\,{}^p\mathcal{L}(Log\,Z),P,W,F)$.
To prove that $R\Gamma(V, Gr^P_iK, W, F)$ is a MHC, hence $\mathbb{H}^r(V, Gr^P_iK, W, F)$ is a MHS, it is enough to check the following lemma:

Lemma 11.1. *Let $f: X\to V$ be a fibration by NCD over the strata, and W a closed subvariety of V such that $Z := f^{-1}(W)$ is a NCD in X.*

The induced filtration δ (11.3) on $\mathbb{H}^j(X\setminus Z, j_{!*}{}^p\mathbf{L})$ corresponds to a filtration δ' by sub-MHS on $\mathbb{H}^j(X, IC^*\,{}^p\mathcal{L}(Log\,Z))$. Consequently the perverse filtration ${}^p\tau$ on $\mathbb{H}^j(X\setminus Z), j_{!*}{}^p\mathbf{L})$ is compatible with the MHS.

Proof. Let $U := V\setminus W$. We reduce the proof to the case of an affine embedding of U. Let $\pi : \widetilde{V}\to V$ be the blowing up of W such that the embedding of $U\hookrightarrow \widetilde{V}$ is affine. Since $f^{-1}(W)$ is a NCD in X, the morphism $f: X\to V$ factors as $f=\pi\circ g$ with $g: X\to\widetilde{V}$.

We apply the Proposition 11.1 to the embedding of U in $\widetilde{V}$, to construct the families H_* and W_* in $\widetilde{V}$ as intersection with two general families of hyperplanes Λ_* and Λ'_* in the ambient projective space with inverse image $H'_* := g^{-1}H_*$ and $W'_* := g^{-1}W_*$ in X, such that the various intersections $H'_i\cap W'_j$ are *transversal and intersect transversally* the various strata in X. Let $h'_i : (X\setminus H'_i)\to X$, $i_{H'_i} : H'_i\to X$ and $K' := IC^*\,{}^p\mathcal{L}(Log\,Z)$. The filtration in the formula 11.3 is written on X as

$$ \delta^p\mathbb{H}^*(X,K') := \mathrm{Im}(\oplus_{i+j=p}\mathbb{H}^*_{W'_{-j}}(X,(h'_i)_!h_i^{'*}K')\to\mathbb{H}^*(X,K')) \qquad (11.4) $$

Since the inverse image H'_* and W'_* are transversal in X and not critical for $\mathbf{L}$, the construction of the intermediate extension $j_{!*}{}^p\mathbf{L}$, commute with the restriction to each H'_i for various indices i.

By (ch 9, §9.5), there exists a structure of bifiltered complex on $h'_{i!}h_i^{'*}K'$ (resp. on $i^!_{W'_{-j}}h'_{i!}h_i^{'*}K'$) such that $\delta^p\mathbb{H}^*(X,K')$ is defined by sub-MHS.

Remark 11.1. The filtrations may be constructed as mixed cones over morphisms of logarithmic complexes.

In particular, $(h'_i)_! h_i'^* K'$ is the mixed cone $C(\rho')$ over the canonical morphism $\rho' : K' \to i_{H'_i *} i^*_{H'_i} K'$.

To construct $i^!_{W'_{-j}} h'_{i!} h_i'^* K'$, since W'_{-j} is not an hypersurface, we introduce the blowing up $\pi_{-j} : \widetilde{X} \to X$ of W'_{-j} in X such that $\widetilde{W}'_{-j} := \pi^{-1}_{-j}(W'_{-j})$ is a smooth hypersurface. We consider the bifiltered logarithmic complex $K'' := IC^* {}^p\mathcal{L}(Log(\pi^{-1}_{-j} Z) \cup \widetilde{W}'_{-j})$ on $\widetilde{X}$ and the cone $C(\rho'')$ over the morphism $\rho'' : \pi_{-j*} K'' \to i_{W'_{-j}*} i^*_{W'_{-j}} K'$. Then, $i^!_{W'_{-j}} K' \simeq C(\rho'')$. This construction applies similarly to the term $i_{H'_i *} i^*_{H'_i} K'$ instead of K', hence it applies also to the cone $C(\rho')$ to define the structure of bifiltered logarithmic complex on $i^!_{W'_{-j}} h'_{i!} h_i'^* K'$.

11.1.3 Mixed Hodge structure (MHS) on the Link

Let $f : X \to V$, X non singular, $v \in V$ a closed point such that $X_v := f^{-1}(v)$ and $X_v \cup Y$ are NCD in X and $i : X_v \to X$. The bifiltered complex $(K', W, F) := (i^* j_{X_v *}(j_{!*} {}^p\mathbf{L})_{|(X - X_v)}, W, F)$ (ch. 9, §9.5.1, Definition 9.35) defines a structure of MHC on the space of global sections $R\Gamma(X_v, K', W, F)$.

The topological interpretation of this intrinsic cohomology space in terms of balls with center v is more suggestive. A small ball $B_v \subset V$ with center v is defined as the trace of a ball $B \in \mathbb{C}^N$ with center 0. The link at a point $v \in V$ is a topological invariant represented by the boundary of a ball B_v of V. As B^*_v retracts on the boundary, we use the hypercohomology of B^*_v.

The inductive limit of the hypercohomology when the radius of B_v is small is an intrinsic group representing the hypercohomology of the link. The link at v corresponds to the tubular neighborhood of $X_v := f^{-1}(v)$.

In terms of $X_{B^*_v} = X_{B_v} \setminus X_v$, there exists an isomorphism

$$\mathbb{H}^r(X_{B^*_v}, j_{!*} {}^p\mathbf{L}) \xrightarrow{\sim} \mathbb{H}^r((X_v, i^*(j_{X_v})_* j^*_{X_v} j_{!*} {}^p\mathbf{L}) \tag{11.5}$$

Set $k : V \setminus \{v\} \to V, j_{X_v} : X \setminus Z \to X$ and $i_v : v \to V$. We denote again by W and F the filtrations $(f_{|X_v})_* W, (f_{|X_v})_* F$ on $(f_{|X_v})_* K'$. The isomorphism $(f_{|X_v})_* K' \simeq i_{v*} k_*((f_* j_{!*} {}^p\mathbf{L})_{|(V - v)})$, defines by transport of structure, a filtration ${}^p\tau^*$ on $(f_{|X_v})_* K'$ deduced from the perverse filtration ${}^p\tau$ on $(f_* j_{!*} {}^p\mathbf{L})_{|(V \setminus v)}$

$$ {}^p\tau^*(f_{|X_v})_* K' \simeq {}^p\tau^*(i^*_v k_*(f_* j_{!*} {}^p\mathbf{L})_{|(V \setminus v)}) := i^*_v k_*({}^p\tau(f_* j_{!*} {}^p\mathbf{L})_{|(V \setminus v)}) \tag{11.6}$$

Thus we have three filtrations W, F, ${}^p\tau^*$ on the complex $(f_{|X_v})_* K'$ inducing filtrations W, F and ${}^p\tau^*$ on the cohomology space $\mathbb{H}^j(X_v, i^* j_{X_v *}(j_{!*} {}^p\mathbf{L})_{|(X \setminus X_v)})$.

Proposition 11.3 (Compatibility with MHS). *i)The induced filtration ${}^p\tau^*$ on $\mathbb{H}^j(X_v, K')$ is a filtration by mixed Hodge substructures.*

ii) The induced filtrations W and F on $Gr_i^{{}^p\tau^*}(f_{|X_v})_*K'$ define a MHS compatible with the isomorphism:

$$H^j(v, \mathrm{Gr}_i^{{}^p\tau^*}(f_{|X_v})_*K', W, F) \simeq \mathrm{Gr}_i^{{}^p\tau^*}\mathbb{H}^{i+j}(X_v, K', W, F) \tag{11.7}$$

The filtration ${}^p\tau^*$ in the proposition corresponds to the perverse filtration ${}^p\tau$ on the hypercohomology of $B^*_{X_v}$ (Equation 11.5)

Corollary 11.1. *If the restriction of f to $X_{B_v^*}$ satisfy the decomposition theorem, there exists a MHS on $\mathbb{H}^j(B_v^*, {}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L}))$ with filtrations W and F satisfying the isomorphism*

$$\mathbb{H}^j(B_v^*, {}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L}), W, F) \simeq Gr_i^{{}^p\tau}\mathbb{H}^{i+j}(B^*_{X_v}, j_{!*}{}^p\mathbf{L}, W, F) \tag{11.8}$$

Once the above MHS is defined, we can express the notion of local purity and give a meaning to the purity theorem [DeG 81].

The proof of the proposition is reduced to the global result in [CaMi 10]. To construct ${}^p\tau^* := i_{v*}k_*{}^p\tau$ on $(f_*K', {}^p\tau^*, W, F)$ ($i_v^*\,{}^p\mathcal{H}^i(K) \neq {}^p\mathcal{H}^i(i_v^*K)$ in general), we introduce the blowing up of v in the diagram

$$\begin{array}{ccccc} Z := X_v & \overset{i}{\hookrightarrow} & X & \overset{j_Z}{\hookleftarrow} & X^* \\ g_v \downarrow & & \downarrow g & & \downarrow g' \\ \widetilde{Z} & \overset{\widetilde{i}}{\hookrightarrow} & \widetilde{V} & \overset{\widetilde{k}}{\hookleftarrow} & \widetilde{V}^* \\ \pi_v \downarrow & & \downarrow \pi & & \simeq\downarrow \pi' \\ v & \overset{i_v}{\hookrightarrow} & V & \overset{k}{\hookleftarrow} & V^* \end{array} \tag{11.9}$$

where $V^* := V \setminus \{v\}$ and $X^* := X \setminus X_v$, $\pi : \widetilde{V} \to V$ is the blowing up of $v \in V$ such that the embedding of V^* in $\widetilde{V}$ is affine.

Since $Z := f^{-1}(v)$ is a divisor, the morphism f to X, factors through $g : X \to \widetilde{V}$. Let $P := (j_{!*}{}^p\mathbf{L})_{|X^*}$ and $K' := i^*j_{Z,*}P$:

$$\mathbb{H}^j(Z, K') \simeq \mathbb{H}^j(f_{|Z})_*K') \simeq \mathbb{H}^j(i_v^*k_*((f_*j_{!*}{}^p\mathbf{L})_{|V^*})).$$

Let $(f_{|Z}) = \pi_v \circ g_v$. The complex $R\Gamma(Z, K')$ is a MHC (see Definition 9.35), endowed with a weight filtration W. Let ${}^p\mathcal{H}^i(g'_*P)$ denote the perverse cohomology on $\widetilde{V}^*$. Since $\pi'_*\,{}^p\mathcal{H}^i(g'_*P) \simeq k^*\,{}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L})$, we have:

$$i_v^*k_*k^*\,{}^p\mathcal{H}^i(f_*j_{!*}{}^p\mathbf{L}) \simeq \pi_{v*}\widetilde{i}^*\widetilde{k}_*\,{}^p\mathcal{H}^i(g'_*P) \simeq R\Gamma(\widetilde{Z}, \widetilde{k}_*\,{}^p\mathcal{H}^i(g'_*P)) \tag{11.10}$$

where $\widetilde{k}_*\,{}^p\mathcal{H}^i(g'_*P)$ is perverse. We apply (ch. 3, Equation 3.24) to write $\widetilde{i}^*\widetilde{k}_*\,{}^p\mathcal{H}^i(g'_*P)$ as a cone over the morphism can : $\widetilde{k}_!\,{}^p\mathcal{H}^i(g'_*P) \to \widetilde{k}_*\,{}^p\mathcal{H}^i(g'_*P)$. Hence $R\Gamma(\widetilde{Z}, \widetilde{k}_*\,{}^p\mathcal{H}^i(g'_*P))$ carry the structure of a mixed cone of two MHC by (Proposition 11.2).

Remark 11.2. A covering of $B_v^* \subset \mathbb{C}^n$ by Stein open subsets complements of linear hyperplanes through 0 in $\mathbb{C}^n$ corresponds by blowing up to a covering of $\widetilde{Z}$.

11.2 Fibration by normal crossing divisors

The proof of the decomposition by induction assume the reduction to a fibration by NCD over the strata S, locally on S. It is possible once for all to define a class of fibrations by NCD (Definition 11.1) over the strata. In this case we can rely on logarithmic complexes in all arguments based on Hodge theory.

Definition 11.1 (topological fibration by NCD over the strata).

A morphism $f : X \to V$ is a topological fibration by NCD over the strata of a stratification $\mathcal{S} = (S_\alpha)$ of V underlying a Thom-Whitney stratification of f, if X is smooth and the spaces $V_l = \cup_{\dim S_\alpha \leq l} S_\alpha$ satisfy the following properties:

1) The subspaces $X_{V_i} := f^{-1}(V_i)$ with indices $0 \leq i \leq \dim V$ are successive sub-NCD embedded in X.

2) The restriction of f to $X_S := f^{-1}(S)$ over each stratum S of $\mathcal{S}$ is a topological fibration: $f_| : X_S \to S$.

3) For each point $v \in V_i \setminus V_{i+1}$ (hence V_i is smooth at v) and a general normal section $\mathcal{N}_v$ to V_i at v, the subvariety $f^{-1}(\mathcal{N}_v)$ is smooth in X and intersects the NCD X_{V_i} transversally.

Sometimes we need to have the fibration by NCD to be adapted to a subspace Y of X or to a subspace Z of V, hence it is convenient to introduce:

Definition 11.2. i) The fibration is adapted to a NCD Y in X, or to a local system $\mathcal{L}$ defined on a Zariski open algebraic set in the complement of Y in X, if in addition:

Y is a union of strata of the underlying whitney stratification $\mathcal{S}$ of X, the union of the sub-spaces $X_{V_i} \cup Y$ are relative NCD over the strata of V for each $1 \leq i \leq n$ and the intermediate extension $j_{!*}\mathbf{L}$ of $\mathcal{L}$ is constructible with respect to the strata.

ii) The fibration is adapted to a subspace Z of V, if Z is a union of strata of the underlying whitney stratification $\mathcal{S}$ of V.

11.2.1 Fibration by NCD over the strata

Let X be smooth, and $v \in V_i \setminus V_{i-1} \neq \emptyset$ (hence V_i is smooth at v); the inverse image of a normal section N_v to V_i at v in general position, is a smooth subvariety $f^{-1}(N_v)$ of X intersecting the NCD X_{V_i} transversally, then $X_{V_i} \cap f^{-1}(N_v) = f^{-1}(v)$ is a NCD in $f^{-1}(N_v)$.

We say for simplification, that $X_{V_i \setminus V_{i-1}} := f^{-1}(V_i \setminus V_{i-1})$ is a relative NCD, and that the stratification $\mathcal{S}$ is admissible for f.

Proposition 11.4. *Let $f : X \to V$ be a projective morphism of complex algebraic varieties and Y a closed algebraic strict subspace containing the singularities of X.*

There exists a diagram $X \overset{\pi'}{\leftarrow} X' \overset{f'}{\to} V$ where π' is a non-singular modification of X, and Thom-Whitney stratifications of f, π' and $f' := f \circ \pi'$ simultaneously, satisfying the following properties.

i) The morphism π' (resp. f') is a fibration by relative NCD over the strata of the underlying Whitney stratification of X (resp. V).

ii) The inverse image $Y' := \pi'^{-1}(Y) \subset X'$ is a NCD and there exists a non singular Zariski open subset $U \subset f(X)$ dense in the image $f(X)$ of f such that f' is smooth on $U' := f'^{-1}(U)$ and π' induces an isomorphism:

$$U' \setminus U' \cap Y' \xrightarrow{\sim} \Omega := f^{-1}(U) \setminus f^{-1}(U) \cap Y.$$

iii) The fibrations are adapted to Y' over U: $U' \cap Y'$ is a strict relative NCD over U in the smooth fibers of f' (called horizontal or strict in the fibers, eventually empty).

Remark 11.3. i) We need the open subset Ω to be embedded in X and in X' such that we can lift a local system on $\Omega \subset X$ to a dense open subset in X'.

ii) The NCD $f'^{-1}(U) \cap Y'$ is called horizontal when at each point v in a maximal stratum of U the intersection $f'^{-1}(v) \cap Y'$ is a NCD in the smooth fiber X'_v, while at each point v in a stratum S of $V - U$ the fiber itself is a NCD in the inverse image of a general normal section at v.

We prove the Proposition 11.4 in three steps to simplify the exposition: *First we transform the morphism f (Lemma 11.2), then the desingularization $\pi : X' \to X$ (Corollary 11.2), then f and π simultaneously.*

11.2.2 Existence of fibrations by NCD over the strata

Let $\mathcal{S} = (S_\alpha)$ an algebraic Whitney stratification by a family of strata S_α of an algebraic variety V. The subspaces $V_l = \cup_{\dim S_\alpha \leq l} S_\alpha$ form an increasing family of closed algebraic subsets of V of dimension $\leq l$, with index $l \leq n$, where n is the dimension of V,

$$V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

Let $f : X \to V$ be an algebraic map. The inverse of a subspace $Z \subset V$, is denoted $X_Z := f^{-1}(Z)$. We are interested in the inverses $X_S = f^{-1}(S)$ of strata S of the stratification $\mathcal{S}$.

Definition 11.3. A Thom-Whitney stratification of an algebraic map $f : X \to V$ consists of two Whitney stratifications one for V and one for X such that the inverse X_S of each stratum S of V is a union of connected strata of X, the restriction of f to each stratum of X_S has maximal rank $\dim S$ and the restriction $f_| : X_S \to S$ is a locally trivial topological fibration.

Using Thom-Mather first isotopy theorem ([Mat 12]), one can prove the following property:

- (W) The link at any point of a stratum is a locally constant topological invariant of the stratum [Mat 12], [LeT 83].

If f is endowed with a Thom-Whitney stratification, the restriction of f to X_S is a topological fibration. (See [GMacP 88], [CaMi 5] subsection 3.5 for some consequences and [DeK 73], subsection 1.3.5).

The following lemma is based on the simple idea that the intersection of a NCD by a general non-singular subspace gives a family of relative NCD. This lemma is important for the proof of Proposition 11.4.

Lemma 11.2. *Let $f : X \to V$ be a projective morphism of algebraic varieties. We consider Y a closed algebraic strict subset of X containing the singularities of X and eventually the singularities of a local system on X. There exists a commutative diagram:*

$$\begin{array}{ccccccccccc} X & \xleftarrow{\pi_1} & X_1 & \cdots & \xleftarrow{\pi_i} & X_i & \cdots & \xleftarrow{\pi_k} & X_k & \xleftarrow{\pi_{k+1}} & X_{k+1} \\ f\downarrow & & f_1\downarrow & & & f_i\downarrow & & & f_k\downarrow & & f_{k+1}\downarrow \\ V & \xleftarrow{id} & V & \cdots & \xleftarrow{id} & V & \cdots & \xleftarrow{id} & V & \xleftarrow{id} & V \end{array}$$

where $\rho_i := \pi_1 \circ \ldots \circ \pi_i$, for $1 \leq i \leq k+1$, is a desingularization of X, π_1 is given by modifications over Y and $Y_i := \rho_i^{-1}(Y)$ is a NCD in X_i for $i \geq 1$ such that:

1. *there exists $V^1 \subset V$ and inductively a decreasing sequence V^i, for $0 < i \leq k$, of closed algebraic subspaces of V of dimension $d_i > 0$;*
2. *for $1 \leq j \leq i$, the inverse image $f_{i+1}^{-1}(V^j)$ is a NCD in the non-singular variety X_{i+1},*
3. *$f_{i+1}^{-1}(V^j) \setminus f_{i+1}^{-1}(V^{j+1})$ is a relative NCD over $V^j \setminus V^{j+1}$.*

The morphisms π_{i+1} are modifications over $f_i^{-1}(V^i)$, for all $i \geq 1$, obtained by blowing-ups over $f_i^{-1}(V^i)$ in X_i and the space $f_i^{-1}(V^i)$ is a NCD in X_i such that π_{i+1} induces an isomorphism:

$$(\pi_{i+1})_| : (X_{i+1} \setminus f_{i+1}^{-1}(V^i)) \xrightarrow{\sim} X_i \setminus f_i^{-1}(V^i).$$

Since ρ_i are desingularizations , the X_i are smooth for $i \geq 1$ and the commutativity of the diagram above gives $f_{i+1} := f_i \circ \pi_{i+1}$. Moreover, there exists a Whitney stratification $\mathcal{S}^i$ of V adapted to V^j for $j \leq i$, satisfying the following property: the strata of $\mathcal{S}^i$ and of $\mathcal{S}^{i-1}$ coincide outside V^i.

The open subsets $\Omega_i := f_i^{-1}(V \setminus V^1) \subset X_i$ (resp. $\Omega := f^{-1}(V \setminus V^1) \subset X$) are dense in X_i (resp. X) and we can suppose that, for $k+1 \geq i \geq 1$, ρ_i induces an isomorphism $(\Omega_i \setminus Y_i) \simeq (\Omega \setminus Y)$, and that, for $k+1 \geq i \geq 1$, the morphisms f_i induce on $\rho_i^{-1}(Y \cap \Omega)$ a fibration by relative NCD over the strata of $V \setminus V^1$.

Definition 11.4. An algebraic morphism $f : X \to V$ of complex varieties is an admissible fibration if X is a smooth variety and there exists a Thom-Whitney stratification $\mathcal{S}$ of f satisfying the following conditions

1. Let V_i denote the union of the strata of V of dimension $\leq i$, then $f^{-1}(V_i)$ is a NCD in X.
2. f induces a fibration by relative NCD of $f^{-1}(V_i)\backslash f^{-1}(V_{i-1})$ over $V_i\backslash V_{i-1}$.

Moreover, let Y be a strict NCD in X. Let S_0 the strata of maximal dimension in V. We say that f is an admissible fibration with respect to Y if $Y \cap f^{-1}(S_0)$ is a relative NCD over S_0.

Remark 11.4. 1. The open subset Ω is embedded in all X_i such that one can lift a local system $\mathcal{L}$ defined on Ω. The intersection $\rho_i^{-1}(Y \cap \Omega)$ in X_i is a horizontal relative NCD (remark 11.3).
2. Consider the stratification $\mathcal{S}^k$ of V and let $d_i := \dim V^i$, V_{d_i} the union of strata of the stratification $\mathcal{S}^k$ of dimension $\leq d_i$. Then, $f_{k+1}^{-1}(V_{d_i} \setminus V_{d_{i+1}})$ is a NCD in X_{k+1} fibred by NCD over $V^i \setminus V^{i+1} = V_{d_i} \setminus V_{d_{i+1}}$.
3. Moreover, we can suppose that the family of subspaces V^i is maximal in the following sense: the dimension of V^i is $n - i$ for $0 < i \leq k = n$.
4. From Lemma 11.2 we obtain that the morphism $f_{k+1} : X_{k+1} \to V$ is an admissible fibration.

Proof of Lemma 11.2. Since f is projective, we can always suppose $V = f(X)$. By Hironaka's resolution result [Hiro 64] we have a desingularization morphism $\pi_1 : X_1 \to X$ of X with NCD as exceptional divisor such that $Y_1 := \pi_1^{-1}(Y)$, as well as the inverse images of the irreducible components of Y, are sub-NCD of Y_1 in X_1.

Let $f_1 := f \circ \pi_1$. Since X_1 is smooth, there exists an open subset $U \subset V$ such that the restriction of f_1 to $f_1^{-1}(U)$ is smooth.

If the dimension of $f(Y)$ is strictly smaller than $n := \dim V$, let $U \subset V \setminus f(Y)$ such that the restriction of f_1 to $f_1^{-1}(U)$ is smooth and let $V^1 := V \setminus U$.

In the case $f(Y) = V$, since Y_1 is a NCD, there exists U such that the restriction of f_1 to $Y_1 \cap f_1^{-1}(U)$ is a fibration by relative NCD over U. Let $V^1 := V \setminus U$, since V is irreducible $d_1 := \dim V^1$ is strictly smaller than $\dim V = n$. We can always choose U, hence V^1, such that $d_1 = n - 1$.

We consider a Thom-Whitney stratification of the morphism $f_1 : X_1 \to V$; in particular the image by f_1 of a stratum of X_1 is a stratum of V. We suppose also that the stratification of X_1 is compatible with the divisor Y_1. Let $\mathcal{S}^0$ denotes the underlying stratification of V.

We shall construct the algebraic subspaces V^i of V, the morphisms $f_i = X_i \to V$, and $\pi_i : X_i \to X_{i-1}$ in the lemma by descending induction on $\dim V^i$. Let $\mathcal{S}^1$ be a Whitney stratification of V compatible with V^1 which is a refinement of $\mathcal{S}^0$. Let $D^1 := f_1^{-1}(V^1)$; we introduce the horizontal divisor (remark 11.3):

$$Y_h^1 := \overline{Y_1 \cap f_1^{-1}(V \setminus V^1)}.$$

Notice that $Y_h^1 \setminus D^1$ is a NCD. We construct $\pi_2 : X_2 \to X_1$ by blowing-ups over D^1 such that the inverse image $D^2 := \pi_2^{-1}(D^1)$ of D^1 and the union $D^2 \cup Y_h^2$ with $Y_h^2 := \pi_2^{-1}(Y_h^1)$ are NCD in the smooth variety X_2. We can do this construction without modification of $X_1 \setminus D^1$ because $Y_h^1 \setminus D^1$ is already a NCD.

Let $f_2 := f_1 \circ \pi_2$, hence $D^2 := f_2^{-1}(V^1)$, the next argument allows us to construct a subspace V^2 such that $D^2 \cap f_2^{-1}(V^1 \setminus V^2) = f_2^{-1}(V^1 \setminus V^2)$ is a relative NCD over $V^1 - V^2$.

The next lemma provides the argument used in the inductive construction of the diagram in lemma 11.2

Lemma 11.3. *Let $f : X \to Z$ be a projective morphism on a non-singular space X, T an algebraic subspace of Z such that $D := f^{-1}(T)$ is a NCD in X as well the inverse image of each irreductible component of T.*

Then, there exists a non singular algebraic subset S_0 in T, such that the dimension of $T \setminus S_0$ is strictly smaller than $\dim T$ and f induces a relative NCD over $T \setminus S_0$.

Proof. i) Let D_{β_i} denote the components of the divisor D and $D_{\beta_1,\ldots,\beta_j}$ the intersection of $D_{\beta_1}, \ldots, D_{\beta_j}$. Let $d := \dim T$ be the dimension of T; there exists an open subset $S_{\beta_1,\ldots,\beta_j}$ complementary of an algebraic subspace of T of dimension strictly smaller than d, such that $f^{-1}(S_{\beta_1,\ldots,\beta_j}) \cap D_{\beta_1,\ldots,\beta_j}$ is either empty, or a topological fibration over $S_{\beta_1,\ldots,\beta_j}$. Over the open subset $S_0' := \cap_j(\cap_{\beta_1,\ldots,\beta_j} S_{\beta_1,\ldots,\beta_j})$, the divisor $D \cap f^{-1}(S_0')$ is a topological fibration.

This property is required to obtain the big stratum in T of a Whitney stratification over which we have a fibration by NCD and which satisfies the assertion (2) of the definition 11.1 above.

ii) Still, we need to check the assertion (3) of the definition 11.1 on a dense open subset of S_0'. Let T_1 be an irreductible component of T of dimension d. Since the inverse image of T_1 is also a NCD contained in D, there exists a dense open subset S_1' in T_1 with inverse image a sub-NCD of D.

As the argument is local, we consider an open affine subset $\mathcal{U}$ of Z with non empty intersection $S_1 := T_1 \cap \mathcal{U} \subset S_1'$ and a projection $q : \mathcal{U} \to \mathbb{C}^d$ whose restriction to S_1 is a finite projection. There exists an open affine dense subset U_1 of $\mathbb{C}^d$ such that $q \circ f$ induces a smooth morphism over U_1.

Considering S_0' defined in i) above, we notice that f induces over $q^{-1}(U_1) \cap S_0'$ a fibration by NCD: indeed, let x be a point in $q^{-1}(U_1) \cap S_0' \cap S_1$. There exists an open neighborhood $\mathcal{U}_x$ of x in $(q^{-1}(U_1) \cap S_0') \subset \mathcal{U}$, small enough such that the restriction of q to $\mathcal{U}_x \cap S_1$ is not ramified on its image.

On the other hand, for all $y \in \mathcal{U}_x$, the space $f^{-1}(q^{-1}(q(y)) \cap \mathcal{U}_x)$ is smooth, since $q(y) \in U_1$. The dimension of $f^{-1}(q^{-1}(q(y) \cap \mathcal{U}_x)$ is $\dim X - d$. As x is in S_0' the dimension of $f^{-1}(y) \cap D$ is $\dim X - 1 - d$ and it is a divisor with normal crossings in $f^{-1}(q^{-1}(q(y) \cap \mathcal{U}_x)$ which has dimension: $\dim X - d$.

We remark that $q^{-1}(q(x)) \cap \mathcal{U}_x$ is a normal section of $q^{-1}(U_1) \cap S_0'$ at x in Z. Hence f induces on $q^{-1}(U_1) \cap S_0'$ a fibration by NCD lying in the normal sections.

The open subset S_0 is the union of open subsets $q^{-1}(U_1) \cap S_0'$ for the various irreductible components T_1 of T of maximal dimension d. The strata which are the connected component of S_0 satisfy the property (3) of the definition 11.1 and this ends the proof of the Lemma 11.3.

End of the proof of Lemma 11.2. Let d_1 be the dimension of V^1. According to Lemma 11.3 there exists an open algebraic subset S_0 of V^1, over which the restriction of f_2 induces a relative NCD over S_0, moreover, the dimension d_2 of the algebraic set $V^2 := V^1 \setminus S_0$ is $d_2 < d_1$. If $d_1 = n-1$, we can always choose S_0 such that $d_2 = n-2$.

We define a new Whitney stratification $\mathcal{S}^2$ of V, compatible with V^2 which coincides with $\mathcal{S}^1$ outside V^1, by keeping the same strata outside V^1 and by adding Thom-Whitney strata adapted to V^1 and to the connected components of the open algebraic subset S_0 that we have just defined.

We complete the proof by repeating this argument for the closed algebraic subspace V^2 of V^1.

This process is the beginning of an inductive argument. The initial step is the construction of V^1 above.

Induction hypothesis: Given projective morphisms $f_j : X_j \to V$ for $1 \leq j \leq i$ as in the diagram above, and $\pi_j : X_j \to X_{j-1}$, $f_j := f_{j-1} \circ \pi_j$ where X_j is non-singular, a Whitney stratification $\mathcal{S}^i$ of V compatible with a family of algebraic subspaces $V^j \subset \cdots \subset V^1 \subset V$ for $j \leq i$ such that there exists a Whitney stratification on X_i stratifying f_i, and such that $f_{i-1}^{-1}(V^j)$ are NCD in X_i, as well as the inverse image of each irreducible component of V^j for $1 \leq j < i$, and moreover f_i induces an admissible morphism over $V \setminus V^i$. Also, $Y_i := \rho_i^{-1}(Y)$ is a NCD divisor of X_i such that $f_i^{-1}(V^1) \cup Y_i$ is a NCD and $Y_i \cap (X_i \setminus f_i^{-1}(V^1))$ is a horizontal relative NCD over $V \setminus V_1$.

Inductive step: A sequence of blowings-up centered over $f_i^{-1}(V^i)$ leads to the construction of $\pi_{i+1} : X_{i+1} \to X_i$ such that X_{i+1} is non-singular and the inverse images of V^j, as well as its irreductible components, by $f_{i+1} := f_i \circ \pi_{i+1}$, for $1 \leq j \leq i$, and their union with $Y_{i+1} := \pi_{i+1}^{-1}(Y_i)$ are NCD in X_{i+1}. Moreover $Y_{i+1} \setminus f_{i+1}^{-1}(V^1)$ is a fibration by NCD over the strata of $V \setminus V^1$.

The stratification $\mathcal{S}^i$ of V underlies a Thom-Whitney stratification of f_{i+1}. It follows from Lemma 11.3 that in each maximal stratum S of V^i in $\mathcal{S}^i$, there exists an open dense subset $S_0^i(S)$ over which π_{i+1} induces a relative NCD. Let V_0^i be the union of all $S_0^i(S)$. The complement $V^i \setminus V_0^i$ of V_0^i is a closed algebraic strict subspace V^{i+1} of V^i, and f_{i+1} is admissible over $V \setminus V^{i+1}$ (see definition in Remark 11.4 (3)). We construct a Thom-Whitney stratification $\mathcal{S}^{i+1}$ of V compatible with V^{i+1}, keeping the same strata of the stratification $\mathcal{S}^i$ outside V^i and introducing as new strata subsets of the connected components of the open subset V_0^i of V^i.

The inductive argument ends when $V^{k+1} = \emptyset$, which occurs after a finite number of steps since the family V^i is strictly decreasing.

Corollary 11.2. *i) Let $f : X \to V$ be an algebraic morphism and Y a strict subvariety containing the singularities of X, there exists a desingularization $X' \xrightarrow{\pi} X$ of X such that $Y' = \pi^{-1}Y$ is a NCD and $f \circ \pi$ is admissible with respect to Y'.*

ii) Let X be an algebraic variety and Y a strict algebraic subspace, there exists a desingularization $X' \xrightarrow{\pi} X$ of X such that $Y' = \pi^{-1}Y$ is a NCD and π is admissible with respect to Y'.

Indeed by Lemma 11.2, the morphism f_{k+1} is a fibration by NCD over the strata, moreover we can suppose $k = n$.

To construct π as a fibration by NCD, we apply the lemma to the case $f = Id : X \to X$.

By the corollary we can choose π or $f \circ \pi$ to be admissible. To show that we can choose both to be admissible we shall need the following improved version of lemma 11.2

Lemma 11.4 (Fibrations relative to a subspace). *With the notations of Lemma 11.2, let Z be a strict algebraic subspace of V. There exists a modification π' of X*

$$X \xleftarrow{\pi'} X' \xrightarrow{f'} V, \quad f' := f \circ \pi'$$

and a Thom-Whitney stratification of f' compatible with Z such that $f' := f \circ \pi'$ is admissible and $f'^{-1}(Z) = \pi'^{-1}(f^{-1}(Z))$ is a relative NCD over the strata in Z.

Proof. The proof is a slight modification of the proof of Lemma 11.2 such that we construct stratifications compatible with Z.

Let ℓ be the dimension of Z. Since V is a variety and Z is a strict algebraic subspace, we have $\dim V > \ell$. In the proof of lemma 11.2, if the dimension d_1 of V^1 is $\geq \ell$ we may suppose that $V^1 \supset Z$. Inductively, as long as $d_j \geq \ell$, we may suppose that $V^j \supset Z$.

If we reach an index s such that $\dim V^s = d_s = \ell$, we may suppose $Z \subset V^s$, then by blowing-ups over $f_s^{-1}(V^s)$ we can assume that $f_{s+1}^{-1}(V^s) \subset X_{s+1}$ and $f_{s+1}^{-1}(Z)$ are transformed into NCD. Thus $f_{s+1}^{-1}(V^s)$ (resp. $f_{s+1}^{-1}(Z)$) induces a fibration over a open subset U of V_s (resp. Z) such that $\dim(V^s - U) < \dim V^s$. Then we suppose the stratification $\mathcal{S}^{s+1}$ of V compatible with Z and we continue similarly as in the above proof of the Lemma 11.2.

Otherwise, if we reach the situation where $d_s > \ell > d_{s+1}$, we consider the subspace $W^{s+1} := Z \cup V^{s+1}$ of dimension l and define the space X_{s+1} by blowing-ups over the subspace $f_s^{-1}(W^{s+1}) \subset X_s$ such that the inverse image of $f_{s+1}^{-1}(Z)$ is a NCD and we continue with Whitney stratifications of V compatible with Z. Then, by proceeding as in the end of the proof of Lemma 11.2, we obtain our proof.

Proof of the Proposition 11.4. We need to construct a diagram:

$$X' \xrightarrow{\pi} X \xrightarrow{f} V$$

such that π and $f \circ \pi$ are simultaneously fibrations by NCD over the strata.

Going back to the inductive argument for f in the Lemma 11.2, we apply at each step of the induction the corollary 11.2 to construct π.

The *initial step* of the induction starts with the desingularization by an admissible modification.

Induction hypothesis. We suppose there exists:

1) A diagram $\mathcal{D}_i$ of morphisms where π'_i is admissible:

$$X \xleftarrow{\pi'_i} X_i \xrightarrow{f_i} V, \quad f_i = f \circ \pi'_i$$

2) A decreasing family of algebraic subspaces $V^i \subset V^j$ for $j < i$ with inverse image $f_i^{-1}(V^j)$ consisting of NCD in X_i for $j < i$, and a stratification $\mathcal{S}^i$ of V compatible with the family V^j, such that the restriction of f_i to $X_i - f_i^{-1}(V^i)$ over $V - V^i$ is a fibration by relative NCD over the strata.

Inductive step. Let d_i (resp. $n - i$) be the dimension of V^i. We want to define a sub-space $V^{i+1} \subset V^i$ of dimension strictly smaller $d_{i+1} < d_i$ (resp. $n - i - 1$) and to extend the diagram $\mathcal{D}_i$ over the open subset $V^i - V^{i+1}$, that is, to construct a diagram $\mathcal{D}_{i+1}$ of morphisms: $X \xleftarrow{\pi'_{i+1}} X_{i+1} \xrightarrow{f_{i+1}} V$ where $f_{i+1} = f \circ \pi'_{i+1}$ such that:

1) π'_{i+1} is admissible and defined as a composition map $\pi'_{i+1} : X_{i+1} \xrightarrow{\pi_{i+1}} X_i \xrightarrow{\pi'_i} X$ of $\pi'_i : X_i \to X$ with a modification π_{i+1} inducing an isomorphism:

$$X_{i+1} \setminus f_{i+1}^{-1}(V^i) \xrightarrow{\sim} X_i - f_i^{-1}(V^i).$$

2) $f_{i+1}^{-1}(V^i)$ is a relative NCD over the open subset $V^i \setminus V^{i+1}$.

To achieve this step, we remark that the variety $f_i^{-1}(V^i)$ in X_i is over $f^{-1}(V^i)$ in X. Then, we apply the lemma 11.4 to the morphism $\pi'_i : X_i \to X$ (instead of $f : X \to V$ in the lemma), and $f^{-1}(V^i) \subset X$ (instead of $Z \subset V$ in the lemma) to construct π_{i+1} such that $\pi'_{i+1} := \pi'_i \circ \pi_{i+1}$ is an admissible morphism:

$$f^{-1}(V^i) \subset X \xleftarrow{\pi'_i} X_i \xleftarrow{\pi_{i+1}} X_{i+1}, \quad \pi'_{i+1} := \pi'_i \circ \pi_{i+1} : X_{i+1} \to X$$

That is we develop the constructions of the lemma 11.2 to construct π'_{i+1} over X *by modification only of subspaces over* $f_i^{-1}(V^i)$ to transform $f_i^{-1}(V^i)$ into a NCD, hence X_{i+1} in the diagram $\mathcal{D}_{i+1}$ differs from X_i in the diagram $\mathcal{D}_i$ only over $f_i^{-1}(V^i)$. The stratification of X is modified inside the subset $f^{-1}(V^i)$, hence we modify adequately the stratification of V inside the subset V^i.

There exists an open algebraic subset $U \subset V_i$ in the big strata of V_i over which f_{i+1} induces a fibration by NCD (Lemma 11.3). Let $V^{i+1} := V_i \setminus U$ of dimension strictly smaller than d_i. Then, the Thom-Whitney stratifications of

f and π'_i outside V_i extend over U into Thom-whitney stratifications of f and π'_{i+1} . Hence, we deduce a diagram $\mathcal{D}_{i+1}$ which extend the diagram $\mathcal{D}_i$ such that f_{i+1} (resp. π'_{i+1}) is a fibration by NCD over the strata of $f_{i+1}^{-1}(V^i \setminus V^{i+1})$ (resp. $f^{-1}(V^i \setminus V^{i+1})$.

Thus we complete the construction of the diagram $\mathcal{D}_{i+1}$ and the inductive step. At the end we define π'_m and f_m both admissible for some index m.

Corollary 11.3. *The decomposition theorem for f can be deduced from both cases π' and f' in the proposition 11.4.*

Let $\mathcal{L}$ be a local system on $X-Y$, there exists an open algebraic set $\Omega \subset X \setminus Y$ dense in X such that $\Omega' := \pi'^{-1}(\Omega) \subset X'$ is isomorphic to Ω, which carry the local system $\mathcal{L}$. Let $j' : \Omega' \to X'$, then the decomposition theorem for $j_{!*}\mathcal{L}$ with respect to the orginal proper algebraic morphism f follows from both cases of f' and π' ([De 68] Proposition 2.16).

11.3 Combinatorial description of the weight

We resume the notations of (§3.1). The intersection complex may be described by a combinatorial complex [KaK 86]. Such description leads to a definition of the weight filtration W on a combinatorial complex quasi-isomorphic to the de Rham complex $\Omega^*\mathcal{L}$.

First we take some time to associate to the set I of indices of the irreducible components of the NCD Y, a category $S(I)$ whose objects are indices of the combinatorial complex (still denoted $\Omega^*\mathcal{L}$).

The combinatorial weights W_0 or W_{-1} on $\Omega^*\mathcal{L}$ describe a combinatorial sub-complex $IC^*\mathcal{L}$ quasi- isomorphic to the intermediate extension $j_{!*}\mathcal{L}$.

To study the local properties of the graded complexes $\mathrm{Gr}_r^W \Omega^*\mathcal{L}$ for $r \in \mathbb{Z}$, we introduce locally the complexes $C_r^{KM}L$ for $K \subset M \subset I$. In particular, the main task is to show that for $K = M$ the complex $C_r^K L$ has a unique non zero cohomology $H^{|K|}C_r^K L$ for $r > 0$ (resp. $H^{|K|-1}C_r^K L$ for $r > 0$) with a pure polarized HS.

The result is a computation of the decomposition of $\mathrm{Gr}_r^W \Omega^*\mathcal{L}$ into intermediate extensions of the VHS defined by $H^{|K|}C_r^K L$ on the non singular intersections of components Y_K of the NCD.

11.3.0.1 Complexes with indices in the category $S(I)$

The objects of the category $S(I)$ attached to a set I, consist of sequences of decreasing subsets of I of the following form:

$$(s_.) = (I = s_1 \underset{\neq}{\supset} s_2 \dots \underset{\neq}{\supset} s_p \neq \emptyset),\ (p > 0)$$

Subtracting a subset s_i defines a morphism $\delta_i(s.) : (s. \setminus s_i) \to s.$ and more generally $Hom(s.', s.)$ is equal to one element iff $(s.') \leq (s.)$ that is $(s.')$ is obtained from $(s.)$ by deleting some subsets. We write $s. \in S(I)$ and define its degree or length $|s.|$ as the number of subsets s_i in $(s_.)$.

Correspondence with an open simplex. If $I = \{1, \ldots, n\}$ is finite, $S(I)$ can be realized as a barycentric subdivision of the open simplex Δ_{n-1} of dimension $n-1$. A subset s_λ corresponds to the barycenter b_{s_λ} of the vertices of s_λ and a sequence of subsets s_λ corresponds to the oriented sub-simplex of Δ_{n-1} defined by the vertices b_{s_λ}.
For example, for $I = \{1, 2\}$, the barycenter $\{3/2\}$ of $]1, 2[$ is defined by $\{1, 2\}$, and the open simplices $]1, 3/2[$, $]3/2, 2[$ are defined resp. by the sequences $\{1, 2\} \supset \{1\}$ and $\{1, 2\} \supset \{2\}$, plus the integer $\{3/2\}$ corresponding to $\{1, 2\}$.

Since all sequences contain I, all corresponding simplices must have the barycenter of Δ_{n-1} defined by I as vertex and the sub-simplices form a partition of the open simplex $\Delta^*_{n-1} = \Delta_{n-1} \setminus \partial\Delta_{n-1}$. In this way we define an incidence relation $\varepsilon(s., s.')$ between two adjacent sequences equal to $+1$ or -1.

Combinatorial objects of an abelian category with indices in $S(I)$ are defined as functors on $S(I)$.
We need the following construction. An algebraic or analytic variety over a fixed variety X with indices in $S(I)$ is a contravariant functor defined by $\Pi(s.) : X_{(s.)} \to X$ and denoted by Π with morphisms $\Pi(s.' \leq s.) : X_{s.} \to X_{s.'}$ over X.

A complex $\mathcal{F}$ of abelian sheaves $\mathcal{F}_{s.}$ over $X_{s.}$ is defined with functorial morphisms $\varphi(s.' \leq s.) : \mathcal{F}_{s.} \to \mathcal{F}_{s.'}$ for $(s.') \leq (s.)$. For each $s_\lambda \neq I \in (s_.)$ and the couple $\delta_{s_\lambda} := ((s_.) \setminus s_\lambda < (s.))$ corresponds a morphism $\Pi_{\delta_{s_\lambda}} : X_{s.} \to X_{(s_.)\setminus s_\lambda}$ and a morphism $\varphi(\delta_{s_\lambda} : \Pi_*\mathcal{F}_{s.} \to \Pi_*\mathcal{F}_{(s_.)\setminus s_\lambda}$ with a sign $\epsilon_{\delta_{s_\lambda}}$ defined by the incidence relation in the open simplex interpretation.
The direct image of a complex of abelian sheaves over Π is the simple complex complex associated to a double complex on X:

$$s(\mathcal{F}_{s.})_{s.\in S} = \oplus_{s.\in S}(\Pi_*\mathcal{F}_{s.})[|s.| - |I|],\ d = \Sigma_{\delta_{s_\lambda}} \varepsilon(\delta_{s_\lambda})\varphi(\delta_{s_\lambda})$$

denoted by $\Pi_*\mathcal{F}$ or preferably $s(\mathcal{F}_{s.})_{s.\in S(I)}$.

Example 11.1. In the case of the constant variety $X_{s.} = X$ defined by a variety X, the constant sheaf $\mathbb{Z}$ lifts to a sheaf on $X_{s.}$ such that the diagonal morphism:

$$\mathbb{Z}_X \to \oplus_{|(s.)|=|I|}\mathbb{Z}_{X_{s.}} : n \in \mathbb{Z} \to (\ldots, n_{s.} = n, \ldots) \in \oplus_{|(s.)|=|I|}\mathbb{Z}$$

defines a quasi-isomorphism $\mathbb{Z}_X \cong \Pi_*\Pi^*(\mathbb{Z}_X)$, as the barycentric subdivision of an open simplex of dimension $|I| - 1$ shows.

11.3.0.2 Local definition of the weight filtration

If $Y \subset X$ is a NCD with indices $i \in I$ and $M \subset I$, we consider the category $S(M)$ attached to M whose objects consist of sequences of decreasing subsets of M of the form $(s_.) = (M = s_1 \underset{\neq}{\supset} s_2 \ldots \underset{\neq}{\supset} s_p \neq \emptyset)$, $p > 0$.. Geometrically $S(M)$ corresponds to the case of a normal section to $Y_M^* = \cap_{i \in M} Y_i$ in X.

Let $(L, (N_i)_{i \in M}, F, m)$ be a polarized nilpotent orbit of weight m with monodromy weight filtrations $W^J := W(\sum_{i \in J} N_i)$ for $J \subset M$.

We introduce a local definition of the weight filtration on the family of de Rham complexes $(\Omega^* L)_{s. \in S(M)}$.

Notations. For each $s. \in S(M)$ let $W^{s_\lambda} = W(\sum_{i \in s_\lambda} N_i)$ be centered at 0. For $J \subset M$ and an integer r, we define $a_{s_\lambda}(J, r) = |s_\lambda| - 2|s_\lambda \cap J| + r$, and the functorial filtered vector space

$$W_r(J, s.)L := \bigcap_{s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r)} L, \quad F^r(J, s.) := F^{r-|J|} L \tag{11.11}$$

The following definition reduces for $r = -1$ to the construction of the intersection complex by Kashiwara and Kawai [KaK 86].

Definition 11.5. The weight W (centered at zero) and Hodge F filtrations on the combinatorial de Rham complex $\Omega^* L := s(\Omega^* L)_{s. \in S(M)}$ are defined by summing over J and $s.$:

$$W_r(\Omega^* L) := s(\bigcap_{s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r)} L)_{(J \subset M, s. \in S(M))} \tag{11.12}$$

where $a_{s_\lambda}(J, r) = |s_\lambda| - 2|s_\lambda \cap J| + r$ and $F^r(\Omega^* L) := s(F^{r-|J|} L)_{(J \subset M, s. \in S(M))}$.

We denote the bifiltered sum complex by

$$(\Omega^* L, W, F)$$

The filtrations can be constructed in two times, first by summing over J to get the sub-complexes:
weight: $W_r(s.) = s(W_r(J, s.))_{J \subset M}$ and Hodge: $F^r(s.) := s(F^r(J, s.))_{J \subset M}$.

Example 11.2. in dimension 2.
Let $W^{1,2} = W(N_1 + N_2), W^1 = W(N_1)$ and $W^2 = W(N_2)$, the weight W_r is a double complex:
$W_r(\{1,2\} \supset 1) \oplus W_r(\{1,2\} \supset 2) \to W_r(\{1,2\})$
where the first line is the direct sum of $W_r(\{1,2\} \supset 1) =$

$$(W^{1,2}_{r+2} \cap W^1_{r+1} \overset{(N_1,\, N_2)}{\longrightarrow} W^{1,2}_r \cap W^1_{r-1} \oplus W^{1,2}_r \cap W^1_{r+1} \overset{(-N_2,\, N_1)}{\longrightarrow} W^{1,2}_{r-2} \cap W^1_{r-1})$$

and $W_r(\{1,2\} \supset 2) =$

$$(W^{1,2}_{r+2} \cap W^2_{r+1} \overset{(N_1,\, N_2)}{\longrightarrow} W^{1,2}_r \cap W^2_{r+1} \oplus W^{1,2}_r \cap W^2_{r-1} \overset{(-N_2,\, N_1)}{\longrightarrow} W^{1,2}_{r-2} \cap W^2_{r-1})$$

The second line for $\{1,2\}$ is $W_r(\{1,2\}) =$

$$(W^{1,2}_{r+2} \overset{(N_1,\, N_2)}{\longrightarrow} W^{1,2}_r \oplus W^{1,2}_r \overset{(-N_2,\, N_1)}{\longrightarrow} W^{1,2}_{r-2}).$$

Lemma 11.5 (Kashiwara and Kawai [KaK 86]). *There exists a quasi-isomorphism $IC^*L \simeq W_{-1}(\Omega^*L)$.*

11.3.0.3 The complexes $C_r^{KM}L$ and $C_r^K L$

To study the graded part of the weight, we need to introduce the following full sub-categories:
For each subset $K \subset M$, let $S_K(M) = \{s. \in S(M) : K \in s.\}$ (that is $\exists \lambda : K = s_\lambda$). We remark the isomorphism of categories:

$$S(K) \times S(M-K) \overset{\sim}{\to} S_K(M),\ (s., s.') \to (K \cup s.', s.)$$

We consider the vector spaces with indices $J \subset M, s. \subset\in S_K M$,
$C_r^{KM}L(J, s.) := \bigcap_{K \neq s_\lambda \in s.} W^{s_\lambda}_{a_\lambda(J,r-1)} Gr^{W^K}_{a_{K(J,r)}} L$ and for each $(s.)$ the associated complex obtained by summing over J (resp. over $(s.)$:
$C_r^{KM}L(s.) := s(C_r L(J, s.))_{J \in M^+}$, $C_r^{KM}L := s(C_r^{KM}L(s.))_{(s.) \in S_K M}$.
We write $C_r^K L(J, s.)$, $C_r^K L(s.)$ and $C_r^K L$ when $K = M$.

Definition 11.6. For $K \subseteq M$ the complex $C_r^{KM}L$ is defined by summing over J and $(s.)$

$$C_r^{KM}L := s\,(\bigcap_{K \neq s_\lambda \in s.} W^{s_\lambda}_{a_\lambda(J,r-1)} Gr^{W^K}_{a_{K(J,r)}} L)_{\,J \subset M, s. \in S_K(M)} \tag{11.13}$$

In the case $K = M$ we write $C_r^K L$

$$C_r^K L = s((\cap_{K \underset{\neq}{\supset} s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} Gr^{W^K}_{a_{K(J,r)}} L)_{J \subset K, s. \in S(K)} \tag{11.14}$$

Example 11.3. For $n = 1$, K and M reduces to one element 1 and the theorem reduces to

$$Gr_r^W(\Omega^*L) \simeq C_r^1 L := Gr^{W^1}_{r+1} L \overset{N_1}{\to} Gr^{W^1}_{r-1} L$$

By the elementary properties of the weight filtration of N_1, it is quasi-isomorphic to $Gr^{W^1}_{r-1}(L/N_1 L)[-1]$ if $r > 0$, $Gr^{W^1}_{r+1}(ker N_1 : L \to L)$ if $r < 0$ and $C_0^1 L \simeq 0$.

Remark 11.5. The above combinatorial filtration W on the combinatorial de Rham complex $\Omega^*\mathcal{L} \simeq s(\Omega^*\mathcal{L})_{s. \in S(I)}$ where I is the set of indices of the NCD $Y \subset X$, has been studied in [E 08].

The decomposition of $\mathrm{Gr}_*^W \Omega^*\mathcal{L}$ depends on the fact that the cohomology of the complex $C_r^K L$ vanishes in all but the degree $|K|$ for $r > 0$ and $C_0^K L \simeq 0$. There is a mistake in the proof which occurs on ([E 08] Lemma A.12 iii).

We give below a correction in the case of surfaces.

Proposition 11.5. *Let (L, N_1, N_2) be an IMHS and let $K = M = \{1, 2\}$. The non zero cohomology of $C_r^K L$ is $H^2(C_r^K L)$ for $r \geq 0$ and $C_0^K L = 0$.*

Proof. Let L_1 resp. L_2, L_3 denote the following complexes

$$0 \to \mathrm{Gr}_{r+2}^{W^{12}} W_{r+1}^1 \xrightarrow{N_1,N_2} \mathrm{Gr}_r^{W^{12}} W_{r-1}^1 \oplus \mathrm{Gr}_r^{W^{12}} W_{r+1}^1 \xrightarrow{N_2,N_1} \mathrm{Gr}_{r-2}^{W^{12}} W_{r-1}^1 \to 0$$

$$0 \to \mathrm{Gr}_{r+2}^{W^{12}} W_{r+1}^2 \xrightarrow{N_1,N_2} \mathrm{Gr}_r^{W^{12}} W_{r+1}^2 \oplus \mathrm{Gr}_r^{W^{12}} W_{r-1}^2 \xrightarrow{N_2,N_1} \mathrm{Gr}_{r-2}^{W^{12}} W_{r-1}^2 \to 0$$

$$0 \to \mathrm{Gr}_{r+2}^{W^{12}} L \xrightarrow{N_1,N_2} \mathrm{Gr}_r^{W^{12}} L \oplus \mathrm{Gr}_r^{W^{12}} L \xrightarrow{N_2,N_1} \mathrm{Gr}_{r-2}^{W^{12}} L \to 0$$

The complex $C_r^K L$ is the sum of the double complex:

$$C_r^K L := (0 \to L_1 \oplus L_2 \xrightarrow{i_1,i_2} L_3 \to 0)$$

with natural embeddings (i_1, i_2).

The sum of $L_1 \xrightarrow{i_1} L_3$ is a complex quasi-isomorphic to $L_4[-1]$ where $L_4 := (L_3/L_1)$ is:

$$\mathrm{Gr}_{r+2}^{W^{12}}(L/W_{r+1}^1) \xrightarrow{N_1,N_2} \mathrm{Gr}_r^{W^{12}}(L/W_{r-1}^1) \oplus \mathrm{Gr}_r^{W^{12}}(L/W_{r+1}^1) \xrightarrow{N_2,N_1} \mathrm{Gr}_{r-2}^{W^{12}}(L/W_{r-1}^1)$$

We compute first the cohomology of L_2 and L_4.

Lemma 11.6. *i) The cohomology of L_2 for $r \geq 0$ is equal to:*
$H^0(L_2) = 0$, $H^1(L_2) = \mathrm{Gr}_r^{W^{12}} \mathrm{Ker}\,(N_2 : L/N_1 \to L/N_1 L)$ *and*
$H^2(L_2) = \mathrm{Gr}_{r-2}^{W^{12}} \mathrm{Coker}\,(N_2 : L/N_1 \to L/N_1 L)$.
ii) The cohomology of L_4 for $r \geq 0$ is equal to:
$H^0(L_4) = H^2(L_2) = 0$, $H^1(L_4) = \mathrm{Gr}_r^{W^{12}} \mathrm{Ker}\,(N_2 : L/N_1 \to L/N_1 L)$.

The computation of the cohomology is based on Kashiwara's canonical decomposition [KaK 86]: $\mathrm{Gr}_r^{W^{12}} L \simeq \oplus_k \mathrm{Gr}_r^{W^{12}} \mathrm{Gr}_k^{W^i} L$ for $i = 1, 2$, applied to the subspace $\mathrm{Ker}\, N_j \subset L, j = 1, 2$.
Sub-lemma. Let $\mathrm{Ker}\, N_2 := \mathrm{Ker}\, N_2 : L \to L$.
i) $\mathrm{Gr}_r^{W^{12}} \mathrm{Ker}\, N_2 = \oplus_{k \leq 0} \mathrm{Gr}_r^{W^{12}} \mathrm{Gr}_k^{W^2}(\mathrm{Ker}\, N_2)$,
$\mathrm{Gr}_r^{W^{12}} \mathrm{Coker}\, N_2 = \oplus_{k \geq 0} \mathrm{Gr}_r^{W^{12}} \mathrm{Gr}_k^{W^2}(\mathrm{Coker}\, N_2)$.
ii) $\mathrm{Gr}_r^{W^{12}} \mathrm{Ker}\, N_2 = \oplus_{k > r-1} \mathrm{Gr}_r^{W^{12}} \mathrm{Gr}_k^{W^1}(\mathrm{Ker}\, N_2)$.

Since the morphism $N_j, j = 1, 2$ is strict for the filtration $W^i, i = 1, 2$, an element $a \in L$ satisfy: $N_j(a) = 0$ if and only if its class in $\mathrm{Gr}_k^{W^i} L$ is in the kernel of the restriction $\widetilde{N}_j$ of N_j to $\mathrm{Gr}_k^{W^i} L$ (an element $c \in \mathrm{Gr}_k^{W^i} L$ such that $N_j c = 0$ is the class of an element $a_i \in W_k^i \mathrm{Ker}\, N_j$).

For i) we remark that $\mathrm{Ker}\, N_2$ is of weights W_k^2 for $k \leq 0$, while ii) follows from the isomorphism: $\mathrm{Gr}_r^{W^{12}} \mathrm{Gr}_k^{W^1} L \simeq \mathrm{Gr}_{r-k}^{\widetilde{W}^2} \mathrm{Gr}_k^{W^1} L$ where $\widetilde{W}^2 := W(\widetilde{N}_2)$ is the monodromy filtration of the restriction $\widetilde{N}_2$ to $\mathrm{Gr}_k^{W^1} L$.

In fact $\widetilde{N}_2 : \mathrm{Gr}_{r-k}^{\widetilde{W}^2}\mathrm{Gr}_k^{W^1} \to \mathrm{Gr}_{r-k-2}^{\widetilde{W}^2}\mathrm{Gr}_k^{W^1}$ is injective for $r-k \geq 1$ and surjective for $r-k \leq 1$ which proves the sub-lemma.

Proof continued. We write L_2 as the sum of a double complex $L'_2 \xrightarrow{N_2} L''_2$:

$L'_2 = (\mathrm{Gr}_{r+2}^{W^{12}} W^2_{r+1} \xrightarrow{N_1} \mathrm{Gr}_r^{W^{12}} W^2_{r+1}), L''_2 := (\mathrm{Gr}_r^{W^{12}} W^2_{r-1} \xrightarrow{N_1} \mathrm{Gr}_{r-2}^{W^{12}} W^2_{r-1})$

Let $\widetilde{N}_1$ be induced by N_1 then:

$L'_2 = \oplus_{k\leq r+1}(\mathrm{Gr}_{r+2-k}^{W(\widetilde{N}_1)}\mathrm{Gr}_k^{W^2} L \xrightarrow{\widetilde{N}_1} \mathrm{Gr}_{r-k}^{W(\widetilde{N}_1)}\mathrm{Gr}_k^{W^2} L)$

$L''_2 = \oplus_{k\leq r+1}(\mathrm{Gr}_{r+2-k}^{W(\widetilde{N}_1)}\mathrm{Gr}_{k-2}^{W^2} L \xrightarrow{\widetilde{N}_1} \mathrm{Gr}_{r-k}^{W(\widetilde{N}_1)}\mathrm{Gr}_{k-2}^{W^2} L)$,

hence $\widetilde{N}_1$ is injective on L'_2 and L''_2 since $r-k \geq -1$. We deduce

$L'_2 = \oplus_{k\leq r} Prim\ \mathrm{Gr}_{r-k}^{W(\widetilde{N}_1)}\mathrm{Gr}_k^{W^2} L[-1] := \oplus_{k\leq r} Prim^1_{r-k}\mathrm{Gr}_k^{W^2} L[-1]$

$L''_2 = \oplus_{k\leq r} Prim\ \mathrm{Gr}_{r-k-2}^{W(\widetilde{N}_1)}\mathrm{Gr}_{k-2}^{W^2} L[-1] := \oplus_{k\leq r} Prim^1_{r-k-2}\mathrm{Gr}_{k-2}^{W^2} L[-1]$

In particular $H^0(L'_2) = H^0(L''_2) = 0$. Let

$$L_2(k) := (Prim^1_{r-k}\mathrm{Gr}_k^{W^2} L \xrightarrow{N'_2} Prim^1_{r-k-2}\mathrm{Gr}_{k-2}^{W^2} L)[-1] \tag{11.15}$$

where N'_2, induced by N_2, is injective for $k \geq 1$ and surjective for $k-2 \leq -1$, hence we consider the cases $k \in [1, r]$ where $r \geq 0$, $k < 1$, $k = 1$ or $r+1$.

The complex L_2 splits into: $(\oplus_{k<1} L_2(k)) \oplus (\oplus_{k\in[1,r]} L_2(k))$ where:

$$\begin{aligned} &\oplus_{k<1} L_2(k) \simeq \mathrm{Gr}_r^{W^{12}}\mathrm{Ker}\, N_2 : (L/N_1L) \to L/N_1L)[-1], \\ &\oplus_{k\in[1,r]} L_2(k) \simeq \mathrm{Gr}_{r-2}^{W^{12}}\mathrm{Coker}\, N_2 : (L/N_1L) \to (L/N_1L)[-2] \end{aligned} \tag{11.16}$$

Since $k \leq r$, the formula does not cover all $\mathrm{Coker}\, N_2$ but covers $\mathrm{Gr}_{r-2}^{W^{12}}$, and $L_2 \simeq 0$ for $k = 1$ or $r+1$. In particular:

$H^1(L_2) \simeq \mathrm{Gr}_r^{W^{12}}\mathrm{Ker}\, N_2 : (L/N_1L) \to L/N_1L)$,

$H^2(L_2) \simeq \oplus_{k>1}\mathrm{Coker}\, N'_2 \simeq \mathrm{Gr}_{r-2}^{W^{12}}\mathrm{Coker}\, N_2 : (L/N_1L) \to (L/N_1L)$.

Sub-lemma . $H^1(L_4) \simeq \mathrm{Gr}_r^{W^{12}}\mathrm{Ker}\, N_2 : (L/N_1L) \to L/N_1L)$,

$H^i(L_4) = 0$ *if* $i \neq 1$.

We apply a similar argument to L_4: $\widetilde{N}_1 : \oplus_{k>r+1}\mathrm{Gr}_{r+2}^{W^{12}}\mathrm{Gr}_k^{W^1} \to \mathrm{Gr}_r^{W^{12}}\mathrm{Gr}_{k-2}^{W^1}$ is injective on $\mathrm{Gr}_k^{W^1}$ since $k > 1$ as $r \geq 0$, hence

$L'_4[1] = \oplus_{k>r+1}\mathrm{Gr}_r^{W^{12}} Prim\, \mathrm{Gr}_{k-2}^{W^1}$, $L''_4[1] = \oplus_{k>r+1}\mathrm{Gr}_{r-2}^{W^{12}} Prim\, \mathrm{Gr}_{k-2}^{W^1}$

then $\widetilde{N}_2 : \mathrm{Gr}_{r-k+2}^{W(\widetilde{N}_2)} Prim\, \mathrm{Gr}_{k-2}^{W^1} \to \mathrm{Gr}_{r-k}^{W(\widetilde{N}_2)} Prim\, \mathrm{Gr}_{k-2}^{W^1}$ is surjective since $r-k < -1$, hence $H^i(L_4) = 0$ for $i \neq 1$, while:

$H^1(L_4) = \mathrm{Gr}_r^{W^{12}}\mathrm{Ker}\,(N_2 : L/N_1L \to L/N_1L)$

Corollary 11.4. *For $r = 0$, $C_0^K L$ is acyclic. For $r \geq 1$, $H^i(C_r^K L) = 0$ if $i \neq 2$ and $H^2(C_r^K L) \simeq \mathrm{Gr}_{r-2}^{W^{12}}(L/(N_1L + N_2L))$,*

Indeed, $i_2 : H^1(L_2) \to H^1(L_4)$ is an isomorphism.

From now on, we assume:

$$L_r^K := H^{|K|}(C_r^K L),\ C_r^K L \simeq L_r^K[-|K|],\ C_0^K L \simeq 0 \tag{11.17}$$

The complex $C_r^{KM}L$ is related to $C_r^K L$ by the following formula

$$C_r^{KM}L \simeq IC^*(C_r^K L) \simeq IC^*(L_r^K)$$

where $IC^*(C_r^K L)$ is the simple complex defined by application of IC^* to each term of $C_r^K L$.

Lemma 11.7 (decomposition). *let $(L, N_i, i \in M)$ be a nilpotent orbit. For all subsets $K \subset M$ there exist canonical embeddings of $C_r^{KM}L$ in $Gr_r^M \Omega^* L$ inducing a quasi-isomorphism (decomposition as a direct sum in the derived category)*

$$Gr_r^W(\Omega^* L) \cong \oplus_{K \subset M} C_r^{KM}L,\ Gr_0^W(\Omega^* L) \simeq 0 \tag{11.18}$$

The proof is based on the following elementary remark:

Let W^i for $i = 1, 2$ be two increasing filtrations on an object V of an abelian category and a_i two integers, then we have an exact sequence:

$0 \to W^2_{a_2-1}Gr^{W^1}_{a_1} \oplus W^1_{a_1-1}Gr^{W^2}_{a_2} \to W^1_{a_1} \cap W^2_{a_2}/W^1_{a_1-1} \cap W^2_{a_2-1} \to Gr^{W^1}_{a_1}Gr^{W^2}_{a_2} \to 0$

The proof of the lemma is by induction by repeating such exact sequences on the terms of the combinatorial sum ([E 08], Theorem A.30).

11.3.0.4 Mixed case

In the case of a VMHS $(\mathcal{L}, W^0, F)$ on $X \setminus Y$ we define a structure of MHC on $\Omega^*\mathcal{L}$. The global construction is reduced to the following local case expressed in terms of an IMHS $(L, W^0, F, N_i, i \in M)$ attached to a point $x \in Y_M^*$.

In the definition (11.5) we consider now for each $s_\lambda \in S(M)$ the relative monodromy filtration on L $M^{s_\lambda} := W(N_{s_\lambda}, W^0))$ where $N_{s_\lambda} := \sum_{i \in s_\lambda} N_i$ with respect to the constant filtration W^0 defined on L by Deligne's extension of the locally constant filtration W^0 on $\mathcal{L}$.

Let $a_{s_\lambda}(J, r) = |s_\lambda| - 2|s_\lambda \cap J| + r$, we consider on the combinatorial de Rham complex $\Omega^* L \simeq s(\Omega^* L)_{s. \in S(M)}$ various filtrations

$$\begin{aligned}
&M_r(J, s.) = \bigcap_{s_\lambda \in s.} M^{s_\lambda}_{a_{s_\lambda}(J,r)}L, \quad W_r(J, s.)L = W_r^0 \cap M_r(J, s.)L \\
&M_r(\Omega^* L) = s(M_r(J, s.))_{J \subset M, s. \in S(M)}, F^r(\Omega^* L) = s(F^{r-|J|}L)_{J \subset M, s. \in S(M)} \\
&W_r(\Omega^* L) = s(W_r(J, s.))_{J \subset M, s. \in S(M)}
\end{aligned} \tag{11.19}$$

Definition 11.7. The bifiltered complex $(\Omega^* L, W^0, W, F)$ sum of the combinatorial de Rham complex $\Omega^* L \simeq s(\Omega^* L)_{s. \in S(M)}$ is endowed with

1. the constant filtration W^0 defined by Deligne's extension of the locally constant filtration W^0 on $\mathcal{L}$,

2. the weight filtration W (Equation 11.19) and Hodge filtration F.

Lemma 11.8 (decomposition). *let $(L, W^0, N_i, i \in M)$ be an IMHS. There exist a canonical quasi-isomorphism (decomposition as a direct sum in terms of $Gr_i^{W^0} L$ in the pure case*

$$Gr_r^W(\Omega^* L) \cong IC^*(Gr_r^{W^0} L) \oplus_{k<r} \mathrm{Gr}_{r-k}^W(\Omega^*(Gr_k^{W^0} L) \tag{11.20}$$

Proof. We use the exact sequence for each $J \subset M, (s.) \subset S(M)$
$0 \to M(J, s.)_{r-1} Gr_r^{W^0} \oplus W_{r-1}^0 Gr_r^{M(J,s.)} \to \mathrm{Gr}_r^{W(J,s.)} \to Gr_r^{M(J,s.)} Gr_r^{W^0} \to 0$
The first term may be expressed in terms of $W(J, s.)$ of the pure case $Gr_r^{W^0} L$

$$M(J, s.)_{r-1} Gr_r^{W^0} L = W(J, s.)_{-1}(Gr_r^{W^0} L)$$

(Equation 11.11). By (Lemma 11.5) the sum is the intersection complex

$$s(M(J, s.)_{r-1} Gr_r^{W^0} L)_{J \subset M, s. \in S(M)} \simeq IC^*(Gr_r^{W^0} L)$$

The second term decomposes into the sum of pure cases

$$Gr_r^{M(J,s.)} W_{r-1}^0 \simeq \oplus_{k<r} \mathrm{Gr}_r^{M(J,s.)} Gr_k^{W^0} L = \oplus_{k<r} \mathrm{Gr}_{r-k}^{W(J,s.)}(Gr_k^{W^0} L)$$

The sum over $J \subset M, s. \in S(M)$ decomposes as a sum of pure cases

$$s(Gr_r^{M(J,s.)} W_{r-1}^0)_{J \subset M, s. \in S(M)} \simeq \oplus_{k<r} \mathrm{Gr}_{r-k}^W \Omega^*(Gr_k^{W^0} L)$$

While the sum of the last term vanish by equation 11.18:

$$s(Gr_r^{M(J,s.)} Gr_r^{W^0} L)_{J \subset M, s. \in S(M)} = \mathrm{Gr}_0^W \Omega^*(Gr_r^{W^0} L) \simeq 0$$

Example 11.4. For $n = 1$, x a singular point of $\mathcal{L}$, $K = M = x$

$$\Omega^* L = (L \xrightarrow{N_1} L), W_r = (M_{r+1}^1 \cap W_r^0 L \xrightarrow{N_1} M_{r-1}^1 \cap W_r^0 L)$$
$$\mathrm{Gr}_r^W(\Omega^* L) \simeq (\mathrm{Gr}_{r+1}^{M^1} W_{r-1}^0 \xrightarrow{N_1} \mathrm{Gr}_{r-1}^{M^1} W_{r-1}^0) \oplus (M_r^1 \mathrm{Gr}_r^{W^0} \xrightarrow{N_1} M_{r-2}^1 \mathrm{Gr}_r^{W^0})$$
$$C_r^1 L = (Gr_{r+1}^{M^1} W_{r-1}^0 L \xrightarrow{N_1} Gr_{r-1}^{M^1} W_{r-1}^0 L)$$
$$C_r^{\emptyset M} = (M_r^1 \mathrm{Gr}_r^{W^0} \xrightarrow{N_1} M_{r-2}^1 \mathrm{Gr}_r^{W^0})$$

By the basic property of M^1: $\mathrm{Gr}_{r+1}^{M^1} W_{r-1}^0 L \simeq \oplus_{k<r} \mathrm{Gr}_r^{M^1} \mathrm{Gr}_k^{W^0} L$,
the complex $Gr_r^W C_r^1 L \simeq \oplus_{k \le r-1} C_{r,k}^1 L$ splits into a sum where

$$C_{r,k}^1 L = (\mathrm{Gr}_{r+1}^{M^1} \mathrm{Gr}_k^{W^0} L \xrightarrow{N_1} Gr_{r-1}^{M^1} \mathrm{Gr}_k^{W^0} L) \simeq$$
$$(\mathrm{Gr}_{r-k+1}^{W(\tilde{N}_1)} \mathrm{Gr}_k^{W^0} L \xrightarrow{N_1} Gr_{r-1-k}^{W(\tilde{N}_1)} \mathrm{Gr}_k^{W^0} L)$$

Since $r - k - 1 \ge 0, C_{r,k}^1 L[1] \simeq PrimGr_{r-1-k}^{W(\tilde{N}_1)} \mathrm{Gr}_k^{W^0} L$ is a polarized HS of weight $r - 1$. Moreover $C_r^{\emptyset M} \simeq \mathrm{Ker}\, N_1 : M_r^1 \mathrm{Gr}_r^{W^0} \to M_{r-2}^1 \mathrm{Gr}_r^{W^0})$ coincides with $IC^* L$.

The sheaf with restriction to the complement of x is $\mathrm{Gr}_r^{W^0} \mathcal{L}$ and with fiber $C_r^{\emptyset M}$ coincides with $j_{!*}\mathbf{L}$.

11.3.0.5 Global weight filtration on $\Omega^*\mathcal{L}$

Let $(\mathbf{L}, W^0, F)$ be a graded admissible polarized variation of MHS on $X^* := X \setminus Y$. There exists a filtration W on $\Omega^*\mathcal{L}$, extending W^0 on $\mathcal{L}$, which coincides locally with the various local weight filtrations at points $x \in Y$ (Definition 11.7), by the isomorphism $\Omega^* L \simeq (\Omega^*\mathcal{L})_x$. We deduce:

Theorem 11.1. *Let $(\mathbf{L}, W^0, F)$ be a graded polarized admissible VMHS on $X \setminus Y$. There exists a weight filtration W by perverse sub-sheaves on the logarithmic complex with coefficients $\mathcal{L}_X$, and a Hodge filtration F by complexes of analytic sub-sheaves such that the bi-filtered complex*

$$\Omega^*\mathcal{L} := (\Omega^*_X(Log\, Y) \otimes \mathcal{L}_X, W, F) \tag{11.21}$$

induces a MHS on the cohomology groups $\mathbb{H}^i(X \setminus Y, \mathbf{L})$.

Let $\mathrm{Gr}_k^{W^0}\mathbf{L}$ denote the graded polarized VHS defined by $(\mathbf{L}, W^0)$ on $X \setminus Y$ and let W be the combinatorial weight filtration defined on $\Omega^*(\mathrm{Gr}_k^{W^0}\mathcal{L})$ in (§11.3), we deduce from (Lemma 11.8):

$$Gr_r^W(\Omega^*\mathcal{L}) \cong IC^*(Gr_r^{W^0}\mathcal{L}) \oplus_{k<r} \mathrm{Gr}_{r-k}^W \Omega^*(Gr_k^{W^0}\mathcal{L}) \tag{11.22}$$

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