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# 1 Propagation of elastic waves in a fluid-loaded anisotropic 2 functionally graded waveguide: Application to 3 ultrasound characterization

4 Cécile Baron and Salah Naili<sup>a)</sup>

5 *Laboratoire de Mécanique Physique, Faculté des Sciences et Technologie, Université Paris 12–Val de*  
6 *Marne, 61 Avenue du Général de Gaulle, 94010 Créteil Cedex, France*

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8 Non-destructive evaluation of heterogeneous materials is of major interest not only in industrial but  
9 also in biomedical fields. In this work, the studied structure is a three-layered one: A laterally  
10 heterogeneous anisotropic solid layer is sandwiched between two acoustic fluids. An original  
11 method is proposed to solve the wave equation in such a structure without using a multilayered  
12 model for the plate. This method is based on an analytical solution, the matricant, explicitly  
13 expressed under the Peano series expansion form. This approach is validated for the study of a  
14 fluid-loaded anisotropic and homogeneous plane waveguide with two different fluids on each side.  
15 Then, original results are given on the propagation of elastic waves in an asymmetrically  
16 fluid-loaded waveguide with laterally varying properties. This configuration notably corresponds to  
17 the axial transmission technique to the ultrasound characterization of cortical bone *in vivo*.

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Pages: 1–XXXX

## 21 I. INTRODUCTION

22 A lot of natural media have unidirectional varying elas-  
23 tic properties. The mantle crust, the oceans, and cortical bone  
24 are some of these functionally graded media. Scientists fo-  
25 cused on the advantages presented by this type of materials  
26 in terms of mechanical behavior, and since the 1980s, they  
27 developed industrial functionally graded materials (FGMs)  
28 particularly exploited in high-technology and biomedical ap-  
29 plications. Consequently, the non-destructive evaluation of  
30 these materials is a key issue. Surface and guided waves play  
31 a major role in non-destructive testing and evaluation of  
32 complex structures. Several studies are dedicated to the leaky  
33 Lamb wave propagation in fluid-loaded plates (Chimenti and  
34 Nayfeh, 1990; Chimenti and Rokhlin, 1990; Deschamps and  
35 Poncelet, 2000). In all these studies, the media are homoge-  
36 neous or multilayered. In this work, we introduce a general  
37 method to take into account the continuity of the property  
38 variation in an anisotropic waveguide. This method is based  
39 on the knowledge of an analytical solution of the wave equa-  
40 tion, the matricant, explicitly expressed via the Peano series  
41 (Peano, 1888). The accuracy of the numerical evaluation of  
42 this solution and its validity domain are perfectly managed  
43 (Baron, 2005; Youssef and El-Arabawi, 2007). Because it  
44 deals with an analytical solution, all the wave propagation  
45 parameters are controlled. This represents an advantage with  
46 respect to numerical methods such as finite-element and  
47 finite-difference methods for which the problem treated is a  
48 global one, making difficult to analyze and interpret the ex-  
49 perimental data, which result from the interaction and cou-  
50 pling of numerous physical phenomena. To the best of our  
51 knowledge, this is the first method to evaluate the mechani-  
52 cal behavior of a fluid-loaded anisotropic waveguide with

continuously varying properties without modeling the FGM 53  
plate by a multilayered plate. Consequently, in the context of 54  
real materials with continuously varying properties (such as 55  
bone, bamboo, or manufactured FGM), this method could 56  
allow assessing the solution to a more realistic model with a 57  
controlled accuracy and without an increase in the 58  
computation-time. 59

In this work, we first present the method and its setup 60  
with fluid-structure interaction; then we proceed to the vali- 61  
dation of the method by comparing our results to the disper- 62  
sion curves obtained from classical schemes on homoge- 63  
neous waveguides (isotropic and anisotropic). Two 64  
advantages of the method are underlined: (i) An asymmetric 65  
fluid-loading may be taken into account without modifying 66  
the scheme for the numerical solution, and (ii) the influence 67  
of the property gradient on the ultrasonic response of the 68  
waveguide may be investigated via the frequency spectrum 69  
of the reflection coefficient modulus. Finally, we get onto the 70  
relevancy of this model applied to the ultrasound character- 71  
ization of cortical bone by the axial transmission technique. 72

## 73 II. BACKGROUND

Contemporary work efforts over the last 2 decades illus- 74  
trate some of the technology interest on guided waves to 75  
non-destructive evaluation. Namely, Rose (2002) gave a re- 76  
view of ultrasonic guided wave inspection potential. A lot of 77  
papers deal with the interaction between guided waves and a 78  
solid plate immersed in a fluid or embedded between two 79  
different fluids. Guided modes in an infinite elastic isotropic 80  
plate in vacuum were first treated by Rayleigh (1885) then 81  
by Lamb (1917). The Lamb wave problem is reserved, 82  
strictly speaking, for wave propagation in a traction-free ho- 83  
mogeneous isotropic plate. To deal with guided modes in a 84  
fluid-loaded plate, we use the term “leaky Lamb waves” as 85  
the energy is partly radiated in the fluids on both sides of the 86  
plate. For the basic Lamb problem—plate in vacuum—the 87

<sup>a)</sup>Author to whom correspondence should be addressed. Electronic mail:  
naili@univ-paris12.fr

88 solutions of the dispersion equation are real whereas in the  
89 case of a plate bounded by media on both sides, the disper-  
90 sion equation has complex solutions. In 1961, [Worlton](#)  
91 (1961) gave an experimental confirmation of the theoretical  
92 work of [Lamb \(1917\)](#), by obtaining experimentally the dis-  
93 persion curves of aluminum and zirconium plates, asserting  
94 that water loading has little effect on the behavior of waves  
95 in plates. In 1976, [Pitts et al. \(1976\)](#) presented some numeri-  
96 cal test results on the relationship between real part of the  
97 reflection coefficient poles and the phase velocity of leaky  
98 Lamb modes in a homogeneous isotropic brass plate in wa-  
99 ter. [Folds and Loggins \(1977\)](#) proposed analytical expres-  
100 sions of the reflection and transmission coefficients for plane  
101 waves at oblique incidence on a multilayered isotropic plate  
102 immersed in water based on [Brekhovskikh's \(1980\)](#) analysis.  
103 They found good agreement with their theoretical results and  
104 experimental data. Few years later, [Fiorito et al. \(1979\)](#) de-  
105 veloped a resonance formalism for the fluid-loaded elastic  
106 plate and gave some theoretical and numerical results for an  
107 isotropic steel plate immersed in the water. This formalism  
108 was generalized to the interactions of acoustic plane waves  
109 with an asymmetrically fluid-loaded elastic plate by [Franklin](#)  
110 [et al. \(2001\)](#). [Nayfeh, Chimenti, and Rokhlin](#) produced a lot  
111 of works on wave propagation in anisotropic media and par-  
112 ticularly in fiber composite plates immersed in a fluid ([Chi-  
113 menti and Nayfeh, 1986, 1990; Chimenti and Rokhlin, 1990;  
114 Nayfeh and Chimenti, 1988, 1989; Rokhlin and Wang,  
115 2002](#)). Based on their formalism, [Deschamps and Poncelet](#)  
116 (2000) placed the emphasis on the difference between what  
117 they called transient Lamb wave-solutions of the character-  
118 istic equation of the plate for complex frequency and real  
119 slowness (time attenuation), and heterogeneous Lamb waves  
120 for which the slowness is complex and the frequency is real  
121 (spatial attenuation). These two ways of resolution of the  
122 dispersion equation have two different physical meanings—  
123 space or time attenuation—and consequences of this differ-  
124 ence are developed in their paper ([Deschamps and Poncelet,  
125 2000](#)). A critical point is the validity of the Cremer's coinci-  
126 dence hypothesis: The real couples (angular frequency  $\omega$  and  
127 phase velocity  $v_\varphi$ ), such that the reflection coefficient is  
128 minimum, may be identified as velocity dispersion of plate  
129 waves. Experimentally checked in a lot of configurations, it  
130 appears to be not well satisfied in several cases (for instance,  
131 graphite-epoxy plates when the ratio between fluid and plate  
132 mass densities is not “small”) ([Chimenti and Nayfeh, 1986;  
133 Nayfeh and Chimenti, 1988](#)). The results obtained by [Des-  
134 champs and Poncelet \(2000\)](#) on fluid-loaded plate show a  
135 good correlation between dispersion curves obtained in com-  
136 plex frequency and the minima of the reflection coefficient,  
137 which suggests that the Cremer's coincidence principle is  
138 still valid considering time attenuation. All these studies  
139 show the evidence that the wave propagation in fluid-loaded  
140 elastic plate emerges as a very delicate problem, which needs  
141 cautious treatment.

### 142 III. GENERAL FORMULATION OF THE PROBLEM

143 We consider a three-dimensional multilayer system com-  
144 posed of one elastic solid layer sandwiched between two

acoustic fluid layers. Let  $\mathbf{R}(O, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$  be the Cartesian  
frame of reference where  $O$  is the origin of the space and  
 $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$  is an orthonormal basis for this space. The coor-  
dinate of the generic point  $\mathbf{x}$  in  $\mathbf{R}$  is specified by  $(x_1, x_2, x_3)$ .  
The acoustic fluid layers occupy an open unbounded domain.  
Both fluids  $f_1$  and  $f_2$  are supposed perfect, of respective mass  
densities  $\rho_{f_1}$  and  $\rho_{f_2}$ ; the constant speeds of sound in each  
fluid are  $c_{f_1}$  and  $c_{f_2}$ , respectively. The thickness of the solid  
layer is denoted by  $d$  and its mass density by  $\rho$ . The inter-  
faces between the fluids and the solid layer are infinite planes  
parallel to the  $(\mathbf{x}_1, \mathbf{x}_2)$ -plane. The  $\mathbf{x}_3$ -axis is oriented down-  
ward and the origin  $O$  is located at the interface between the  
upper fluid  $f_1$  and the solid layer. Therefore, we assume that  
the structure is a two-dimensional one and that the guided  
waves travel in the plane  $x_2=0$ ; in the following parts, this  
coordinate is implicit and is omitted in the mathematical ex-  
pressions. Moreover, the solid layer will be so-called plate.

The elastic plate is supposed to be anisotropic and is  
liable to present continuously varying properties along its  
thickness ( $\mathbf{x}_3$ -axis). These mechanical properties are repre-  
sented by the stiffness fourth-order tensor  $\mathbf{C}=\mathbf{C}(x_3)$  and the  
mass density  $\rho=\rho(x_3)$ .

#### A. System equations

##### 1. The wave equation in the fluid $f_n$ (for $n=1$ or $2$ )

In the fluid  $f_n$  and the context of the linear acoustic  
theory, the linearized Euler equation and the constitutive  
equations are written as

$$-\frac{\partial p^{(n)}}{\partial x_j} = \rho_{f_n} \frac{\partial^2 u_j^{(n)}}{\partial t^2},$$

$$p^{(n)} = K_{f_n} \operatorname{div} \mathbf{u}^{(n)},$$

where  $\mathbf{u}^{(n)}$  and  $p^{(n)}$ , respectively, represent the displacement  
vector and the pressure in the fluid  $f_n$ ; its compressibility and  
the speed of sound in the fluid at equilibrium are, respec-  
tively,  $K_{f_n}$  and  $c_{f_n} = \sqrt{K_{f_n}/\rho_{f_n}}$ . The operator  $\operatorname{div}$  is the diver-  
gence.

The solutions of system (1) for the fluid  $f_n$  are sought  
under the form

$$\mathbf{f}_n(x_1, x_2; t) = \mathbf{A}_n(x_3) \exp i(k_1 x_1 + k_3^{(n)} x_3 - \omega t),$$

where  $k_1$  and  $k_3^{(n)}$  are the wavenumbers, respectively, along  
the  $\mathbf{x}_1$ -axis and  $\mathbf{x}_3$ -axis in the fluid  $f_n$ ; the angular frequency  
is noted  $\omega$  and  $i$  is the imaginary unit.

We consider an incident wave reaching the plate at an  
angle  $\theta_1$  from the  $\mathbf{x}_3$ -axis in the fluid  $f_1$ . The incident  
displacement-field is defined in the following form, assum-  
ing that its amplitude is normalized:

$$\mathbf{u}_I^{(1)} = \begin{pmatrix} \sin \theta_1 \\ 0 \\ \cos \theta_1 \end{pmatrix} \exp i(k_1 x_1 + k_3^{(1)} x_3 - \omega t),$$

with  $k_1 = (\omega/c_{f_1}) \sin \theta_1$  and  $k_3^{(1)} = (\omega/c_{f_1}) \cos \theta_1$ . From this, the  
expressions of the reflected displacement-field  $\mathbf{u}_R^{(1)}$  in  $f_1$  and  
of the transmitted displacement-field  $\mathbf{u}_T^{(2)}$  in  $f_2$  are deduced as  
follows:

$$\begin{aligned}
194 \quad \mathbf{u}_R^{(1)} &= R \begin{pmatrix} \sin \theta_1 \\ 0 \\ -\cos \theta_1 \end{pmatrix} \exp i(k_1 x_1 - k_3^{(1)} x_3 - \omega t), \\
195 \quad \mathbf{u}_T^{(2)} &= T \frac{c_{f_2}}{c_{f_1}} \begin{pmatrix} \sin \theta_1 \\ 0 \\ \cos \theta_1 \end{pmatrix} \exp i(k_1 x_1 + k_3^{(2)} x_3 - \omega t). \quad (4)
\end{aligned}$$

196 where  $R=R(x_1, x_3; t)$  and  $T=T(x_1, x_3; t)$ , respectively, repre-  
197 sent the reflection and transmission coefficients, which will  
198 be expressed explicitly in the sequel. The incident, reflected,  
199 and transmitted pressure fields, respectively, noted  $p_I^{(1)}$ ,  $p_R^{(1)}$ ,  
200 and  $p_T^{(2)}$ , are deduced from expressions (3) and (4), and the  
201 second equation of the system (1)

$$\begin{aligned}
202 \quad p_I^{(1)} &= -i\omega \times \rho_{f_1} c_{f_1} \times \exp i(k_1 x_1 - k_3^{(1)} x_3 - \omega t), \\
203 \quad p_R^{(1)} &= -i\omega \times \rho_{f_1} c_{f_1} \times R \times \exp i(k_1 x_1 - k_3^{(1)} x_3 - \omega t), \\
204 \quad p_T^{(2)} &= -i\omega \times \rho_{f_2} c_{f_2} \times T \times \exp i(k_1 x_1 + k_3^{(2)} x_3 - \omega t). \quad (5)
\end{aligned}$$

## 205 B. The wave equation in the plate waveguide

206 The body forces in the solid plate are neglected. The  
207 balance equation of linear momentum associated with the  
208 constitutive law of linear elasticity (Hooke's law) gives the  
209 following equations:

$$\begin{aligned}
210 \quad \frac{\partial \sigma_{ij}}{\partial x_j} &= \rho \frac{\partial^2 u_i}{\partial t^2}, \\
211 \quad \sigma_{ij} &= \frac{1}{2} C_{ijkl} \left( \frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_k} \right), \quad (6)
\end{aligned}$$

212 where  $u_i$  (for  $i=1, \dots, 3$ ) and  $\sigma_{ij}$  (for  $i, j=1, \dots, 3$ ), respec-  
213 tively, represent the components of the displacement-field  $\mathbf{u}$   
214 and of the stress  $\boldsymbol{\sigma}$ . In system (6), the Einstein convention of  
215 summation on repeated indices is used. The solutions are  
216 sought for the vectors of displacement  $\mathbf{u}$  and traction  $\sigma_{i3}$  (for  
217  $i=1, \dots, 3$ ) (assumed to be harmonic in time  $t$  and space  
218 along the  $\mathbf{x}_1$ -axis) under the form

$$219 \quad \mathbf{f}(x_1, x_3; t) = \mathbf{A}(x_3) \exp i(k_1 x_1 - \omega t). \quad (7)$$

## 220 C. Fluid-loading interface conditions

221 The conditions at both interfaces  $x_3=0$  and  $x_3=d$  are the  
222 continuity of the normal displacement and the one of the  
223 normal stress. We consider that the fluids  $f_1$  and  $f_2$  are per-  
224 fect; consequently, the shear stresses are zero at the inter-  
225 faces [ $\sigma_{13}(x_1, 0; t) = \sigma_{13}(x_1, d; t) = 0$  and  $\sigma_{23}(x_1, 0; t)$   
226  $= \sigma_{23}(x_1, d; t) = 0$ ]. The following relations are obtained:

$$\begin{aligned}
227 \quad u_3(x_1, 0; t) &= u_3^{(1)}(x_1, 0; t), \quad u_3(x_1, d; t) = u_3^{(2)}(x_1, d; t), \\
228 \quad \sigma_{33}(x_1, 0; t) &= -p^{(1)}(x_1, 0; t), \quad \sigma_{33}(x_1, d; t) = -p^{(2)}(x_1, d; t), \quad (8)
\end{aligned}$$

229 with

$$u_3^{(1)} = \mathbf{u}_I^{(1)} \cdot \mathbf{x}_3 + \mathbf{u}_R^{(1)} \cdot \mathbf{x}_3, \quad u_3^{(2)} = \mathbf{u}_T^{(2)} \cdot \mathbf{x}_3 \quad \text{and} \quad 230$$

$$p^{(1)} = p_I^{(1)} + p_R^{(1)}, \quad p^{(2)} = p_T^{(1)}. \quad (9) \quad 231$$

## D. A closed-form solution: The matricant 232

Introducing expression (7) in Eq. (6), we obtain the 233  
wave equation under the form of a second-order differential 234  
equation with non-constant coefficients. For particular forms 235  
of profiles, this equation has analytical solutions expressed 236  
with special functions (Bessel or Hankel functions) (Vlasie 237  
and Rousseau, 2004). But, in the general case, there is no 238  
analytical solution to the problem thus formulated. The most 239  
current methods to solve the wave equation in unidirection- 240  
ally heterogeneous media are derived from the Thomson- 241  
Haskell method (Haskell, 1953; Thomson, 1950). These 242  
methods are appropriate for multilayered media (Hosten and 243  
Castaings, 2003; Kenneth, 1982; Lévesque and Piché, 1992; 244  
Wang and Rokhlin, 2001). But, for continuously varying me- 245  
dia, these techniques mean to replace the continuous profiles 246  
of properties with step-wise functions. Thereby, the studied 247  
problem becomes an approximate one, even before the reso- 248  
lution step; the accuracy of the solution as its validity do- 249  
main is hard to evaluate. Moreover, the multilayered model 250  
of the waveguide creates some "virtual" interfaces likely to 251  
induce artifacts. In order to deal with the exact problem, that 252  
is to keep the continuity of the properties' variation, the wave 253  
equation is re-written under the form of an ordinary differ- 254  
ential equation system with non-constant coefficients for 255  
which an analytical solution exists: the matricant (Baron, 256  
2005). 257

### 1. Hamiltonian form of the wave equation 258

We consider that the plate presents material symmetries 259  
that allow decoupling the pressure-shear vertical (P-SV) 260  
waves, polarized in the propagation plane ( $\mathbf{x}_1, \mathbf{x}_3$ ) and the 261  
shear horizontal (SH) waves polarized along  $\mathbf{x}_2$ -axis. The 262  
incident medium  $f_1$  is a perfect fluid; only the P-SV waves 263  
travel in the plate. Applying a spatio-temporal Fourier trans- 264  
form on  $(x_1, t)$  of the displacement-field [noted  $\hat{\mathbf{u}}(k_1, x_3; \omega)$  265  
after Fourier transform] and on the traction field [noted 266  
 $\hat{\sigma}_{i3}(k_1, x_3; \omega)$  for  $i=1, \dots, 3$ ] and using the Voigt notation 267  
( $C_{ijkl}$  for  $i, j, k, \ell=1, \dots, 3$  is replaced with  $c_{IJ}$  for  $I, J$  268  
 $= 1, \dots, 6$ ), Eq. (6) leads to Stroh's (1962) sextic plate for- 269  
malism (Hamiltonian formulation of the wave equation) 270

$$\frac{\partial \hat{\sigma}_{13}}{\partial x_3} = \rho(\omega)^2 \hat{u}_1 - ik_1 \hat{\sigma}_{11}, \quad \frac{\partial \hat{\sigma}_{33}}{\partial x_3} = \rho(\omega)^2 \hat{u}_3 - ik_1 \hat{\sigma}_{13}, \quad (10) \quad 271$$

$$\hat{\sigma}_{11} = ik_1 c_{11} \hat{u}_1 + c_{33} \frac{\partial \hat{u}_3}{\partial x_3}, \quad \hat{\sigma}_{13} = c_{55} \left( \frac{\partial \hat{u}_1}{\partial x_3} + ik_1 \hat{u}_3 \right), \quad 272$$

$$\hat{\sigma}_{33} = ik_1 c_{13} \hat{u}_1 + c_{33} \frac{\partial \hat{u}_3}{\partial x_3}. \quad (11) \quad 273$$

274 According to Eqs. (10) and (11),  $\hat{\sigma}_{11}$  is function of  $\hat{u}_1$   
 275 and  $\hat{\sigma}_{33}$ . The wave equation becomes a matrix system ex-  
 276 pressed using the Thomson–Haskell parametrization of  
 277 Stroh’s (1962) sextic plate formalism

$$\frac{d}{dx_3} \eta(x_3) = i\omega \mathbf{Q}(x_3) \eta(x_3), \quad (12) \quad 278$$

that is, 279

$$\frac{d}{dx_3} \begin{pmatrix} i\omega \hat{u}_1 \\ i\omega \hat{u}_3 \\ \hat{\sigma}_{13} \\ \hat{\sigma}_{33} \end{pmatrix} = i\omega \begin{pmatrix} 0 & s_1 & 1/c_{55}(x_3) & 0 \\ -s_1 c_{13}(x_3)/c_{33}(x_3) & 0 & 0 & 1/c_{33}(x_3) \\ \rho(x_3) - s_1^2 \zeta(x_3) & 0 & 0 & -s_1 c_{13}(x_3)/c_{33}(x_3) \\ 0 & \rho(x_3) & -s_1 & 0 \end{pmatrix} \begin{pmatrix} i\omega \hat{u}_1 \\ i\omega \hat{u}_3 \\ \hat{\sigma}_{13} \\ \hat{\sigma}_{33} \end{pmatrix}, \quad (13)$$

280  
 281  
 282  
 283  
 284  
 285

286 with the relations

$$\zeta(x_3) = c_{11}(x_3) - \frac{c_{13}^2(x_3)}{c_{33}(x_3)}, \quad k_1 = \omega s_1, \quad (14)$$

288 where  $s_1$  is the  $x_1$ -component of the slowness. The matrix  $\mathbf{Q}$   
 289 includes all the information about the heterogeneity of the  
 290 waveguide because it is expressed from the plate mechanical  
 291 properties ( $\rho(x_3), \mathbb{C}(x_3)$ ) and from two acoustical parameters  
 292 ( $s_1, \omega$ ).

## 293 2. Explicit solution: The Peano expansion of the 294 matricant

295 The wave equation thus formulated has an analytical  
 296 solution expressed between a reference point  $(x_1, 0, x_3^0)$  and  
 297 some point of the plate  $(x_1, 0, x_3)$  in the propagation plane.  
 298 This solution is called the matricant and is explicitly written  
 299 under the form of the Peano series expansion (Gantmacher,  
 300 1959; Peano, 1888; Pease, 1965):

$$\mathbf{M}(x_3, x_3^0) = \mathbf{I} + (i\omega) \int_{x_3^0}^{x_3} \mathbf{Q}(\xi) d\xi + (i\omega)^2 \int_{x_3^0}^{x_3} \int_{x_3^0}^{\xi} \mathbf{Q}(\xi_1) d\xi_1 d\xi + \dots, \quad (15)$$

303 where  $\mathbf{I}$  is the identity matrix of dimension (4,4). If the ma-  
 304 trix components  $\mathbf{Q}(x_3)$  are bounded in the study interval,  
 305 these series are always convergent (Baron, 2005). The com-  
 306 ponents of the matrix  $\mathbf{Q}$  are continuous in  $x_3$  and the study  
 307 interval is bounded (thickness of the waveguide); conse-  
 308 quently the hypothesis is always verified. We underline that  
 309 the  $i\omega$ -factorization leads up to a polynomial form of the  
 310 matricant. The  $i\omega$ -polynomial coefficients are matrices inde-  
 311 pendent of  $\omega$ .

## 312 3. Boundary conditions: Fluid-structure interaction

313 Using the propagator property of the matricant through  
 314 the plate thickness, the state-vector [defined in Eq. (13)] at  
 315 the second interface  $\eta(d)$  is evaluated from the state-vector  
 316 at the first interface  $\eta(0)$  as follows:

$$\eta(d) = \mathbf{M}(d, 0) \eta(0). \quad (16)$$

317

The fluid-structure interaction sets the conditions of zero  
 shear stresses (see Sec. III C), used after a spatio-temporal  
 Fourier transform on  $(x_1, t)$ . Equation (16) becomes

$$\begin{pmatrix} i\omega \hat{u}_1(k_1, d; \omega) \\ i\omega \hat{u}_3(k_1, d; \omega) \\ 0 \\ \hat{\sigma}_{33}(k_1, d; \omega) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12} & M_{22} & M_{23} & M_{24} \\ M_{13} & M_{23} & M_{33} & M_{34} \\ M_{14} & M_{24} & M_{34} & M_{44} \end{pmatrix} \begin{pmatrix} i\omega \hat{u}_1(k_1, 0; \omega) \\ i\omega \hat{u}_3(k_1, 0; \omega) \\ 0 \\ \hat{\sigma}_{33}(k_1, 0; \omega) \end{pmatrix}. \quad (17)$$

The condition to obtain a nontrivial solution to Eq. (17)  
 leads to the following relation:

$$i\omega \hat{u}_1(k_1, 0; \omega) \times M_{13} + i\omega \hat{u}_3(k_1, 0; \omega) \times M_{32} + \hat{\sigma}_{33}(k_1, 0; \omega) \times M_{34} = 0, \quad (18)$$

where  $M_{ij}$  (for  $i, j = 1, \dots, 4$ ) represent the components of the  
 matrix  $\mathbf{M}$ . The displacement component  $\hat{u}_1(k_1, 0; \omega)$  can be  
 expressed as a linear combination of  $\hat{u}_3(k_1, 0; \omega)$  and  
 $\hat{\sigma}_{33}(k_1, 0; \omega)$ ; thus system (16) of dimension 4 is reduced to a  
 matrix system of dimension 2

$$\eta(d) = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \eta(0) \quad \text{where} \quad \eta(x_3) = \begin{pmatrix} i\omega \hat{u}_3 \\ \hat{\sigma}_{33} \end{pmatrix}, \quad (19)$$

with the relations

$$P_1 = M_{22} - M_{21} \frac{M_{32}}{M_{31}}, \quad P_2 = M_{24} - M_{21} \frac{M_{34}}{M_{31}},$$

$$P_3 = M_{42} - M_{41} \frac{M_{32}}{M_{31}}, \quad P_4 = M_{44} - M_{41} \frac{M_{34}}{M_{31}}. \quad (20)$$

Interface conditions (8) are transformed in the Fourier  
 domain  $(k_1, \omega)$ . The expressions of the displacement and the  
 pressure in the fluids [see Eqs. (3)–(5)], so that the one of the  
 displacement and traction fields in the solid plate [see Eq.  
 (19)], are substituted in the transformed interface conditions.

341 Setting  $\eta(0) = (\alpha_1, \alpha_2)^T \exp i(k_1 x_1 - \omega t)$ , where the super-  
 342 script  $T$  designates the transpose operator, we obtain the fol-  
 343 lowing matrix equation:

$$344 \begin{pmatrix} i\omega s_3^{(1)} c_{f_1} & 1 & 0 & 0 \\ -i\omega \rho_{f_1} c_{f_1} & 0 & 1 & 0 \\ 0 & P_1 & P_2 & -i\omega s_3^{(2)} c_{f_2} \exp(i\omega s_3^{(2)} d) \\ 0 & P_3 & P_4 & -i\omega \rho_{f_2} c_{f_2} \exp(i\omega s_3^{(2)} d) \end{pmatrix} \begin{pmatrix} \hat{R} \\ \alpha_1 \\ \alpha_2 \\ \hat{T} \end{pmatrix} = \begin{pmatrix} i\omega s_3^{(1)} c_{f_1} \\ i\omega \rho_{f_1} c_{f_1} \\ 0 \\ 0 \end{pmatrix}, \quad (21)$$

345 where  $\mathbf{s}^{(n)} = \mathbf{k}^{(n)}/\omega$  is the slowness-vector in the fluid  $f_n$  ( $n$   
 346 = 1 or 2). The quantities  $\hat{R}$  and  $\hat{T}$  are, respectively, the reflec-  
 347 tion and transmission coefficients expressed in the Fourier  
 348 domain:  $\hat{R} = \hat{R}(k_1, x_3; \omega)$  and  $\hat{T} = \hat{T}(k_1, x_3; \omega)$ . The two first  
 349 lines of system (21) express the boundary conditions at the  
 350 first interface ( $x_3=0$ ) and the two last lines those at the sec-  
 351 ond interface ( $x_3=d$ ) introducing the Fourier transform of  
 352 expressions (3)–(5) in the following relations:

$$354 \eta(0) - \begin{pmatrix} i\omega \hat{u}_{3R}^{(1)} \\ -\hat{p}_R \end{pmatrix}_{x_3=0} = \begin{pmatrix} i\omega \hat{u}_{3I}^{(1)} \\ -\hat{p}_I \end{pmatrix}_{x_3=0},$$

$$355 \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \eta(0) - \begin{pmatrix} i\omega \hat{u}_{3T}^{(2)} \\ -\hat{p}_T \end{pmatrix}_{x_3=d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (22)$$

356 where  $\hat{u}_{3R}^{(1)}$ ,  $\hat{u}_{3I}^{(1)}$ , and  $\hat{u}_{3T}^{(2)}$  are the components along  $\mathbf{x}_3$ -axis of  
 357  $\hat{\mathbf{u}}_R^{(1)}$ ,  $\hat{\mathbf{u}}_I^{(1)}$ , and  $\hat{\mathbf{u}}_T^{(2)}$  vectors, respectively. Note the equality  
 358 between the quantities  $u_{3T}^{(2)}$  and  $u_3^{(2)}$  where this last is defined  
 359 in Eq. (9).

#### 360 4. Expression of the reflection and transmission 361 coefficients

362 From Eq. (21), we deduce the analytical expressions of  
 363 the reflection and transmission complex coefficients

$$364 \hat{R}(s_1, x_3; \omega) = \frac{(P_3 - P_1 Z_2 + P_4 Z_1 - P_2 Z_1 Z_2)}{(P_3 - P_1 Z_2 - P_4 Z_1 + P_2 Z_1 Z_2)},$$

$$365 \hat{T}(s_1, x_3; \omega) = -\frac{2Z_2(\rho_{f_1} c_{f_1} / \rho_{f_2} c_{f_2})(P_1 P_4 - P_2 P_3)}{(P_3 - P_1 Z_2 - P_4 Z_1 + P_2 Z_1 Z_2)} \times \exp(-i\omega s_3^{(2)} d), \quad (23)$$

367 with  $Z_n = \rho_{f_n} / \sqrt{1/c_{f_n}^2 - s_1^2}$  (for  $n=1$  or 2) and  $k_1 = \omega s_1$ .

#### 368 IV. VALIDATION OF THE METHOD

369 The aim of this section is to check that the Peano expan-  
 370 sion of the matricant is well-adapted to study fluid-loaded  
 371 waveguides. We take into account the fluid-structure interac-  
 372 tion in different configurations of homogeneous plates com-

TABLE I. Elastic properties of transversely isotropic plate (with  $c_{23}=c_{22}-2c_{44}$ ).

$\rho$ (g cm <sup>-3</sup> )	$c_{11}$ (GPa)	$c_{22}=c_{33}$ (GPa)	$c_{12}=c_{13}$ (GPa)	$c_{44}$ (GPa)	$c_{66}=c_{55}$ (GPa)
1.85	23.05	15.1	8.7	3.25	4.7

paring the results obtained from the numerical implementa-  
 tion of the Peano expansion of the matricant to results taken  
 from literature.

The numerical evaluation of  $P_1, P_2, P_3,$  and  $P_4$  requires  
 us to truncate the Peano series and to numerically calculate  
 the integrals. Thus, the error can be estimated and controlled  
 (Baron, 2005). We retained 70 terms in the series and evalu-  
 ate the integrals over 100 points using Simpson's rule  
 (fourth-order integration method). These choices ensure the  
 convergence of the solution and the accuracy of the results  
 for a reasonable computation-time (never exceeding few  
 minutes on a desktop computer). Expressions (23) give the  
 frequency spectrum (modulus and phase) of the reflection  
 coefficient for different incidences ( $s_1$  varies from zero-  
 normal incidence to  $1/c_{f_1}$  corresponding to the critical inci-  
 dence in the fluid  $f_1$ ).

A lot of works detailed the relationship between the  
 poles and the zeros of the reflection coefficient and the leaky  
 Lamb wave dispersion curves (Chimenti and Rokhlin, 1990;  
 Deschamps and Poncelet, 2000). The results of Sec. IV A  
 compare the dispersion curves obtained by seeking the poles  
 of reflection coefficient (23) and the results taken from lit-  
 erature or from closed-form solution.

#### A. Validation for a homogeneous and isotropic or anisotropic fluid-loaded plate

The method is tested by plotting the dispersion curves  
 (variation of the phase velocity versus frequency-thickness  
 product) for an isotropic aluminum plate immersed in water.  
 The data in the paper of Chimenti and Rokhlin (1990) are  
 used. The results obtained (not shown) by the present method  
 are in perfect agreement with the results presented by them  
 (Chimenti and Rokhlin, 1990).

As mentioned by Chimenti and Rokhlin (1990), there  
 are few differences between the zeros' loci and the poles' loci  
 for a plate immersed in a fluid whose the mass density is  
 lower than the plate mass density. As underlined by several  
 authors, fluid-load does have just a weak influence on guided  
 wave traveling in the plate immersed in water.

Taking into account the anisotropy does not change the  
 scheme for the numerical solution of wave equation with the  
 matricant. We consider a transverse isotropic plate immersed  
 in water ( $\rho_f = 1$  g cm<sup>-3</sup>,  $c_f = 1.485$  mm  $\mu$ s<sup>-1</sup>) whose proper-  
 ties are reported in Table I. For that configuration, Nayfeh  
 and Chimenti (1989) developed a method to obtain an ana-  
 lytical expression of the reflection coefficient. By using the  
 data from this paper, the results obtained (see Fig. 1) with the  
 present method are in perfect agreement with theirs. The  
 curves presented in Figs. 1(a) and 1(b) are superimposed and  
 need to be presented separately.

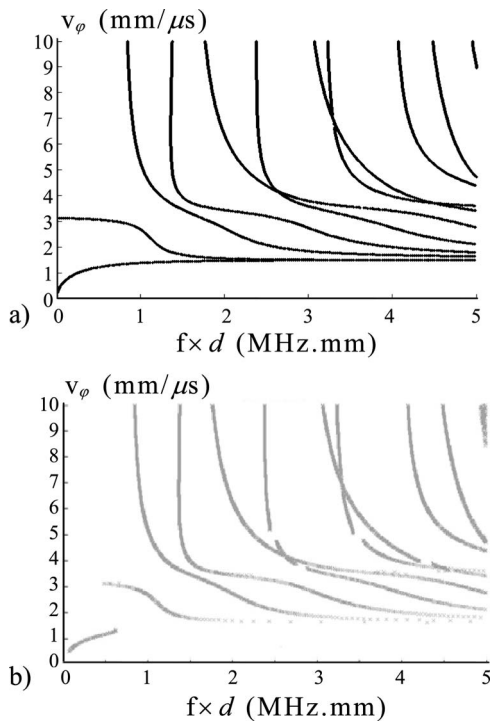


FIG. 1. Dispersion curves for a transversely isotropic plate immersed in water; comparison between (a) analytical results from Nayfeh and Chimenti (1989) and (b) results obtained via the Peano series of the matricant.

#### 422 B. Validation for an asymmetrically loaded 423 homogeneous isotropic plate ( $f_1 \neq f_2$ )

424 The formalism presented here to solve the wave equa-  
425 tion in an unidirectionally graded medium presents two main  
426 advantages: Without changing the scheme to obtain the nu-  
427 merical solution we can take into account (i) an asymmetric  
428 loading and (ii) the unidirectional continuous heterogeneity.

429 The mechanical behavior of the plate is different for  
430 symmetric and asymmetric loadings. For example, in the  
431 symmetric loading case, there is a unique critical frequency  
432 and a unique phase velocity value  $v_\varphi$  in the plate, which  
433 corresponds to the propagation velocity in the fluid ( $v_\varphi = c_{f_1}$   
434  $= c_{f_2}$ ), for which the displacements and the stresses at the  
435 interfaces are quasi-null; whereas in the asymmetric loading  
436 ( $f_1 \neq f_2$ ), there are two critical frequencies and two values of  
437 the phase velocity in the plate for which the structure does  
438 not respond (Dickey et al., 1995). The validation is done on  
439 an isotropic aluminum plate with the following properties:  
440  $\rho = 2.79 \text{ g cm}^{-3}$ ; the longitudinal and transverse wave  
441 velocities are, respectively,  $v_L = 6.38 \text{ mm } \mu\text{s}^{-1}$  and  $v_T =$   
442  $3.10 \text{ mm } \mu\text{s}^{-1}$ . The characteristic properties of the fluid  $f_1$   
443 correspond to those of water (see Sec. IV A); the character-  
444 istic properties of the fluid  $f_2$  correspond to glycerine:  $\rho_{f_2}$   
445  $= 1.26 \text{ g cm}^{-3}$  and  $c_{f_2} = 1.920 \text{ mm } \mu\text{s}^{-1}$ . This configuration  
446 is the same as the one studied by Franklin et al. (2001). The  
447 modulus of the reflection coefficient versus the incident  
448 angle is plotted in Fig. 2 for a fixed frequency-thickness  
449 product ( $f \times d = 4.7 \text{ MHz mm}$ ).

450 This figure shows the perfect agreement between our  
451 results and the ones presented by Franklin et al. (2001).

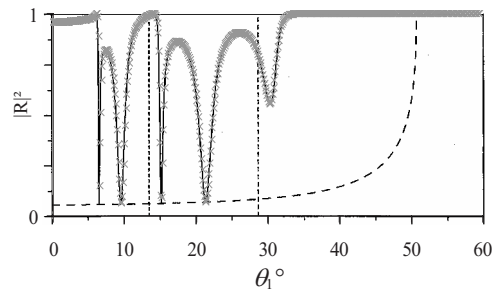


FIG. 2. Reflection coefficient modulus for an asymmetrically fluid-loaded aluminum plate (water and glycerine) versus incident angle: in dark lines results published in Franklin et al. (2001), and in gray crosses the reflection coefficient modulus calculated from Eq. (23). The vertical lines (dashed) represent the two critical angles for longitudinal waves and transverse waves. The dashed curve corresponds to the resonant amplitudes (Franklin et al., 2001).

#### V. RELEVANCY OF THE METHOD FOR ULTRASOUND CHARACTERIZATION OF CORTICAL BONE

452 Cortical bone is a kind of hard tissue found at the edges  
453 of long bones and supports most of the load of the body.  
454 Several studies demonstrated the heterogeneous nature of the  
455 cortical bone, particularly they show evidence the gradual  
456 variation of the volumetric porosity (ratio between pores and  
457 total volume) along the cortical thickness. Yet, the porosity is  
458 intrinsically linked to the macroscopic mechanical behavior  
459 of the cortical bone (Baron et al., 2007). Therefore, the con-  
460 tinuous variation of porosity induces a continuous variation  
461 of material properties. Taking into account the gradient  
462 should prove itself to be essential in the context of diagnosis  
463 and therapeutic monitoring of osteoporosis. Indeed, the gra-  
464 dient characterization would allow assessing geometrical  
465 (cortex thickness) and material (elastic coefficients variation)  
466 information, which are fundamental parameters to evaluate  
467 the bone fragility. For several years, the quantitative ultra-  
468 sonography (by axial and transverse transmissions) proved  
469 itself to be a hopeful alternative technique to evaluate the  
470 fracture risk (Marin et al., 2006). However, the inter-  
471 individual and inter-site variations of bone mechanical prop-  
472 erties make the standardization of the protocol of fracture  
473 risk evaluation by ultrasound very delicate.

474 The focus is set on a configuration closed to the axial  
475 transmission device for *in vivo* conditions. In this context,  
476 the relevancy of studying the head wave propagation has  
477 been demonstrated (Bossy et al., 2004a, 2004b; Camus et al.,  
478 2000). As a consequence all the reflection coefficients pre-  
479 sented in this paper were calculated for an incident angle  
480 corresponding to the grazing-angle for longitudinal waves  
481 [critical angle of longitudinal wave propagation in the plate  
482 at the first interface ( $x_3 = 0$ )].

483 The surrounding media in the *in vivo* configuration of  
484 ultrasound characterization of cortical bone are the muscle  
485 for the upper fluid  $f_1$  ( $c_{f_1} = 1.54 \text{ mm } \mu\text{s}^{-1}$  and  $\rho_{f_1}$   
486  $= 1.07 \text{ g cm}^{-3}$ ) and the marrow for the lower fluid  $f_2$  ( $c_{f_2}$   
487  $= 1.45 \text{ mm } \mu\text{s}^{-1}$  and  $\rho_{f_2} = 0.9 \text{ g cm}^{-3}$ ) (Burlew et al., 1980;  
488 Hill et al., 1986). We are interested in the influence of the  
489 continuous gradient of the mechanical properties on the ul-  
490 trasonic response in the configuration of *in vivo* cortical bone  
491 characterization.

494 **A. Determination of a realistic range of variation of**  
 495 **elastic bone properties**

496 In order to define numerical values for a realistic value  
 497 of the gradient of the different material properties, it is nec-  
 498 essary to determine the limiting values reached by each elas-  
 499 tic property. Our approach consists in considering *in vitro*  
 500 measurements published in Dong and Guo, 2004 and per-  
 501 formed in 18 samples. It is assumed that these limiting val-  
 502 ues for elastic properties are relevant for physiologic ranges  
 503 of variations.

504 We assume that cortical bone is transverse isotropic.  
 505 Transverse isotropy has been shown experimentally by dif-  
 506 ferent authors (Dong and Guo, 2004; Reilly and Burnstein,  
 507 1974; Rho, 1996) to be a realistic approximation of cortical  
 508 bone degree of anisotropy.

509 Dong and Guo (2004) measured the homogenized bone  
 510 properties by performing tensile and torsional tests with a  
 511 mechanical testing system on 18 different human femoral  
 512 bone specimens. The authors measured the values of the lon-  
 513 gitudinal and transverse Young's moduli ( $E_L$  and  $E_T$ , respec-  
 514 tively) as well as the values of the longitudinal shear modu-  
 515 lus  $G_L$ . From these measurements and by assuming constant  
 516 values of Poisson's ratio, the values of the different compo-  
 517 nents of the stiffness-tensor corresponding to the values of  
 518  $E_L$ ,  $E_T$ , and  $G_L$  measured in Dong and Guo, 2004 were ob-  
 519 tained following the relationships given in the Appendix.

520 The value of the longitudinal Poisson's ratio  $\nu_L$  is taken  
 521 equal to 0.37 for all computations because it corresponds to  
 522 the average value found in Dong and Guo, 2004. The value  
 523 of the transverse Poisson's ratio  $\nu_T$  is taken equal to 0.45  
 524 following Eq. (A3) of the Appendix. The values of the stiff-  
 525 ness coefficients corresponding to the mean values of the  
 526 bone mechanical properties are referred to as "reference" set  
 527 of parameters in what follows. The maximum and minimum  
 528 values of the stiffness coefficients are obtained by consider-  
 529 ing, respectively, the maximum and minimum values of  $E_L$   
 530 and  $E_T$  within the range of variation measured in Dong and  
 531 Guo, 2004, which is a simple way to obtain a realistic range  
 532 of variation for the stiffness coefficients in cortical bone.  
 533 Furthermore, the elastic properties deduced from the ap-  
 534 proach reported above were constrained to verify the thermo-  
 535 dynamical stability conditions given in the Appendix by Eq.  
 536 (A4).

537 We choose reference value of mass density  $\rho$  equal to  
 538  $1.722 \text{ g cm}^{-3}$ , following the value taken in Macocco *et al.*,  
 539 2006. In order to derive a realistic range of variation for mass  
 540 density, we assume that the reference value is given by a  
 541 porosity of 7%, which corresponds approximately to the  
 542 mean porosity at the radius (Baron *et al.*, 2007). The porosity  
 543 was assumed to vary between 3% and 15% (Bousson *et al.*,  
 544 2001; Dong and Guo, 2004) and a rule of mixture leads to  
 545 the range of variation of mass density.

546 **B. Modeling a gradient of material property**

547 The impact of a controlled gradient vector  $\delta$  of any in-  
 548 vestigated material property  $S$  on the response of the struc-  
 549 ture studied is assessed. The scalar  $S$  corresponds to one of  
 550 the stiffness coefficients  $c_{ij}$  of  $C$  or to mass density  $\rho$ . In each

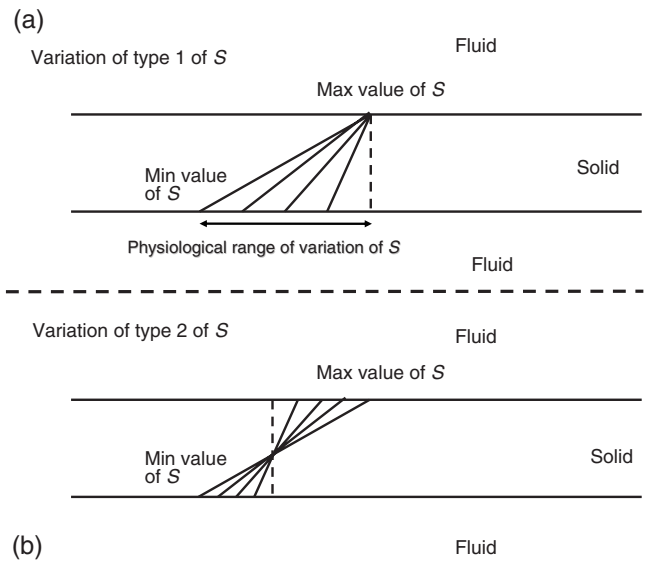


FIG. 3. Schematic representation of the two types of spatial variation con- sidered for the material property  $S$  corresponding to the stiffness coefficient and to mass density. The solid lines in the solid layer indicate the spatial dependence of  $S$ . The dotted line in the solid layer indicates the spatial dependence of  $S$ . The variation of type 1 shown in (a) corresponds to a constant value at the bone-soft tissues' interface. The variation of type 2 shown in (b) corresponds to a constant value in the middle of the bone.

set of simulations, all the material properties are constant and equal to their reference value while  $S$  is subjected to the defined gradient.

The gradient vector  $\delta = \text{grad } S = \delta \mathbf{x}_3$  is assumed to be independent of  $x_1$  in all cases, where  $\mathbf{x}_3$  is a unit vector along  $\mathbf{x}_3$ -axis and grad is the gradient operator acting on a scalar field. The quantity  $\delta$  is always taken positive because the porosity is known to be higher in the endosteal part ( $x_3=d$ ) than in the periosteal part ( $x_3=0$ ) of the bone. Moreover, only the simple situation of affine spatial variations of  $S$  is considered, corresponding to a constant value of  $\delta$ . This affine spatial variation of  $S$  is chosen because the actual physiological spatial dependence of  $S$  remains unknown. Two different affine spatial dependencies of the studied material property are considered and are illustrated in Fig. 3. Associated gradient  $\delta$  will be referred to as type 1 or 2.

*Type 1.* The gradient of type 1 is such that the physical property  $S$  takes the same value  $S_M$  at the upper interface  $x_3=0$  of the solid plate for all values of the gradient  $\delta$ . The quantity  $S(x_3)$  is therefore given by

$$S(x_3) = S_M + \delta \times x_3, \quad (24)$$

where  $S_M$  is given by the maximal value of the material property  $S$  considered. The maximal value  $\delta_M$  of  $\delta$  is chosen so that  $S(d)$  is equal to  $S_m$ , where  $S_m$  is given by the minimal value of  $S$ . The gradient  $\delta_M$  is given by

$$\delta_M = \frac{(S_m - S_M)}{d}. \quad (25)$$

*Type 2.* The gradient of type 2 is such that the physical property  $S$  takes the same value at the middle  $x_3=d/2$  of the solid plate for all values of gradient  $\delta$ . Furthermore, the



TABLE II. Mean value, maximum, and minimum values of the homogenized longitudinal and transversal Young moduli, of the four elastic constants and of mass density affecting the ultrasonic propagation in the framework of the model. These values are taken from [Dong and Guo \(2004\)](#).

Mechanical quantity	$E_L$ (GPa)	$E_T$ (GPa)	$c_{11}$ (GPa)	$c_{13}$ (GPa)	$c_{33}$ (GPa)	$c_{55}=G_L$ (GPa)	$\rho$ (g cm <sup>-3</sup> )
Mean value (reference)	16.6	9.5	23.1	8.7	15.1	4.7	1.722
Minimum	13.4	6.5	17.6	5.1	9.1	3.3	1.66
Maximum	20.6	12.8	29.6	15.9	25.9	5.5	1.753

580 mean value of the property  $S$  is identical for all  $\delta$ . The quan-  
581 tity  $S(x_3)$  is given by

$$582 \quad S(x_3) = \frac{(S_m + S_M)}{2} + \delta \times \left( x_3 - \frac{d}{2} \right). \quad (26)$$

583 The maximal value of  $\delta$  is also given by Eq. (25), so that  
584 all values of  $S(x_3)$  are again always comprised between  $S_m$   
585 and  $S_M$ . Again, the maximal value  $\delta_M$  of  $\delta$  is given by Eq.  
586 (25).

587 Gradient of type 2 leads for all magnitudes of  $\delta$  to a  
588 constant value of the spatial average of the gradient.

589 For both types of spatial variations, five different values  
590 of  $\delta$  regularly distributed between 0 and  $\delta_M$  are arbitrarily  
591 considered in the thickness.

592 Table II recalls the maximum, minimum, and mean mea-  
593 sured values of  $E_L$ ,  $E_T$ , and  $G_L$  as given by [Dong and Guo](#)  
594 (2004). Table II also shows the maximum, minimum, and  
595 mean values of the four components ( $c_{11}$ ,  $c_{13}$ ,  $c_{33}$ , and  $c_{55}$ ) of  
596 the stiffness-tensor  $\mathbb{C}$  affecting wave propagation derived  
597 from Eq. (A3) of the Appendix.

598 In Table III, the minimal and maximal values of each  
599 variable corresponding to the realistic range of variation ob-  
600 tained (i) by considering the reference values of Table II and  
601 (ii) by verifying that the thermodynamical stability condi-  
602 tions are fulfilled. Values resulting from the stability condi-  
603 tions are marked with an asterisk.

604 In the simulations, the values of  $S_M$  and  $S_m$  are those  
605 reported in Table III.

### 606 C. Results and discussion

607 First of all, the method allows investigating the influence  
608 of the fluids on the ultrasonic response. In the case of the  
609 characterization of cortical bone, the two fluids  $f_1$  and  $f_2$  are  
610 different, which corresponds to an asymmetrical loading (see  
611 Sec. IV B): The fluid  $f_1$  has been considered as muscle ( $c_{f_1}$   
612 = 1540 m s<sup>-1</sup> and  $\rho_{f_1}$  = 1.07 g cm<sup>-3</sup>) and fluid  $f_2$  as marrow  
613 ( $c_{f_2}$  = 1450 m s<sup>-1</sup> and  $\rho_{f_2}$  = 0.9 g cm<sup>-3</sup>). The properties of  
614 these two fluids are very close to those of water. The fre-  
615 quency spectrum of the reflection coefficient modulus has

616 been plotted for the *in vivo* configuration and compared with  
617 the result obtained for a cortical bone plate immersed in wa-  
618 ter for the ten profiles of mechanical properties (figure not  
619 shown). For homogeneous plates as for linearly graded  
620 plates, the two curves are very close; however, the modulus  
621 of the reflection coefficient at null-frequency is not null and  
622 the minimum values are greater than for water but obtained  
623 for the same frequency-thickness products. That is why all  
624 the following results have been calculated for a cortical bone  
625 plate immersed in water.

626 The reflection coefficient calculated with the Peano se-  
627 ries of the matricant is sensitive to the variation of the prop-  
628 erties gradient. As we consider that the osteoporosis entails a  
629 trabecularization of cortical bone from the endosteal side, the  
630 characterization of the gradient of the properties between the  
631 endosteal and periosteal regions may be an element of the  
632 diagnosis of the osteoporosis progress and of the therapeutic  
633 follow-up.

634 It is known that the gradient of the properties along the  
635 cortical thickness is due to the continuous variation of the  
636 porosity growing progressively from the periosteal to the en-  
637 dosteal region. From previous work, we know that the po-  
638 rosity influences all the stiffness coefficients ([Baron et al.](#),  
639 2007). The frequency spectrum of the reflection coefficient  
640 has been plotted for the ten profiles presented in Fig. 3 ap-  
641 plied to all the stiffness coefficients implied ( $c_{11}$ ,  $c_{13}$ ,  $c_{33}$ , and  
642  $c_{55}$  and to the mass density  $\rho$ ) (see Table III). The reflection  
643 coefficients have been calculated at an incident angle corre-  
644 sponding to the grazing-angle (critical angle for the longitu-  
645 dinal waves in the bone plate). For the two types of gradi-  
646 ents, differences appear between all the gradients and the  
647 homogenized plates (corresponding to the maximum value  
648 for type 1 and to the average value for type 2) particularly on  
649 the location of the extrema values of the reflection coefficient  
650 modulus (see Fig. 4).

651 The increase in the gradient of properties shifts the mini-  
652 mum and maximum values forward high frequency-  
653 thickness products. However, the results (see Fig. 4) put on  
654 evidence that the behavior of the reflection coefficient modu-  
655 lus is sensibly the same for frequency-thickness products be-

TABLE III. The minimal and maximal values of each variable corresponding to the realistic range of variation obtained (i) by considering the reference values of Table II and (ii) by verifying that the thermodynamical stability conditions are fulfilled. Values resulting from the stability conditions are marked with an asterisk.

Material property $S$	$c_{11}$ (GPa)	$c_{13}$ (GPa)	$c_{33}$ (GPa)	$c_{55}=G_L$ (GPa)	$\rho$ (g cm <sup>-3</sup> )
Realistic range [ $S_m, S_M$ ] (reference)	[17.6, 29.6]	[5.1, 11.1*]	[11.8*, 25.9]	[3.3, 5.5]	[1.66, 1.753]

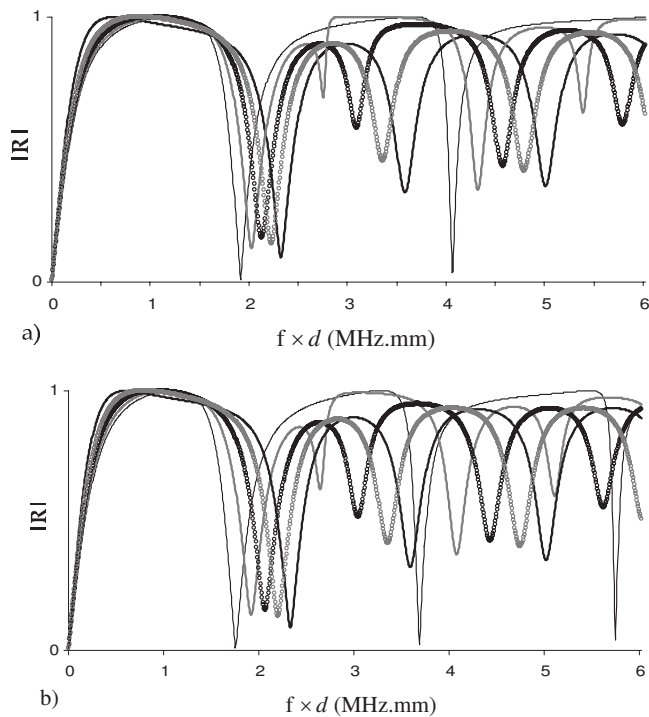


FIG. 4. Frequency spectrum of the reflection coefficient modulus for different properties' variations corresponding to (a) the five profiles of type 1 [Fig. 3(a)] and (b) the five profiles of type 2 [Fig. 3(b)].

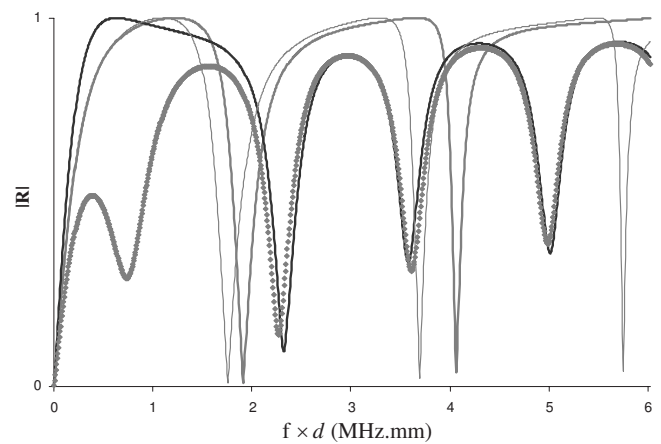


FIG. 5. Frequency spectrum of the reflection coefficient modulus for different properties' variations at grazing-incidence  $\theta_1 = \theta_c$ : for a homogeneous plate whose properties are those at  $x_3=0$  at  $\theta_c=21.1^\circ$  (thick gray line), for a homogeneous plate whose properties are the average of the properties through the plate at  $\theta_c=23.6^\circ$  (thin gray line), for a plate with all its properties linearly varying at  $\theta_c=21.1^\circ$  (black line), and for a plate with an affine variation of the  $c_{11}$  only at  $\theta_c=21.1^\circ$  (gray lozenges).

656 tween 0 and 1.5 MHz mm. Beyond this value, the behavior is  
657 clearly different. At sufficiently high frequency, the wave-  
658 length is smaller and more sensitive to the affine variation of  
659 the material properties. It is noteworthy that for a heteroge-  
660 neous waveguide, the minima of the reflection coefficient  
661 magnitude do not reach zero (except for null-frequency) but  
662 end at a finite value and the changes in phase (not shown) are  
663 not so rapid, which means that total transmission does not  
664 take place in this situation.

665 The influence of the variation of each parameter (stiff-  
666 ness coefficients) on the frequency spectrum of the reflection  
667 coefficient has been investigated. This analysis has been led  
668 for an incidence angle corresponding to the longitudinal  
669 wave critical angle in the plate at  $x_3=0$ . This incidence cor-  
670 responds to the generation and the propagation of the head  
671 wave.

672 It appears that each of them has an impact on the reflec-  
673 tion wave, but the leading term is  $c_{11}$ . The frequency spec-  
674 trum of the reflection coefficient for a varying  $c_{11}$  in an affine  
675 way is very close to the frequency spectrum of the reflection  
676 coefficient obtained for the affine variation of all the material  
677 properties ( $c_{ij}$  and  $\rho$ ) and is the most different to the results  
678 from homogenized plates (average value or maximum value)  
679 compared to the frequency spectrum calculated for the *one-*  
680 *parameter variation* of the other elastic parameters ( $c_{13}$ ,  $c_{33}$ ,  
681 and  $c_{55}$ , and  $\rho$ ) (see Fig. 5). It is noteworthy that  $c_{11}$  is the  
682 stiffness coefficient associated with the axial direction and  
683 determining the speed of the head wave. So, it seems that the  
684 head wave would be the indicator of the  $c_{11}$  gradient. It is  
685 important to note that for homogenized plates (extremum  
686 value or average) the frequency spectrum of the reflection  
687 coefficient is really different from that of the plate with con-

tinuously varying properties. We infer that the approximation  
by a homogeneous plate of cortical bone and all the more so  
for the osteoporotic cortical bone (for which the gradient  
would be greater) may induce bad interpretation of the ultra-  
sonic response.

The authors do not know any results in literature about  
the measurement of the variation of the porosity within the  
cortical thickness. The assumption of an affine gradient is a  
first step; other gradients may be investigated and the method  
presented in this article would be applied in the same way for  
non-affine gradients (Shuvalov *et al.*, 2005).

## VI. CONCLUSION AND PERSPECTIVES

Stroh's (1962) sextic plate formalism has been employed  
for analyzing the leaky Lamb waves in anisotropic heteroge-  
neous plates immersed in fluids. This formalism and espe-  
cially the polynomial form of the solution [see Eq. (15)]  
presents several analytical and numerical advantages. First,  
the low-frequency asymptotics are naturally assessed evalu-  
ating only two or three terms in the series (Shuvalov *et al.*,  
2005). The information thus collected is of major interest in  
the analysis of the elastic behavior of waveguides (Baron *et*  
*al.*, 2008). Second, the polynomial form makes the numerical  
evaluation of the solution faster. Indeed, the polynomial co-  
efficients are independent of the frequency, so they are cal-  
culated for a fixed slowness value and stored. When the fre-  
quency varies, there is no need to recalculate the polynomial  
coefficients, it comes to a polynomial evaluation whose co-  
efficients are perfectly known, which is time-saving.

The Peano series of the matricant is a method that keeps  
the continuity of the profiles and so, the authenticity of the  
problem. One of the key points for methods based on multi-  
layered media to deal with FGM is to relevantly discretize  
the properties' profiles. The choice of the discretization may  
lead to some errors especially in the evaluation of the reso-  
nances.

723 This elegant mathematical tool is also very adaptative to  
724 different physical problems. In the case studied—a fluid-  
725 loaded plane waveguide—the anisotropy, the heterogeneity  
726 (continuous or discontinuous variation of properties) and the  
727 asymmetric fluid-loading are taken into account without  
728 changing the resolution scheme.

729 Further work needs to be done to relate the results pre-  
730 sented in this paper to dispersion curves and propagation of  
731 transient and heterogeneous waves in a fluid-loaded continu-  
732 ously heterogeneous waveguide.

733 Furthermore, from this study, the transient response of a  
734 fluid-loaded plate is considered. The frequency spectrum of  
735 the reflection coefficient is calculated for incidences be-  
736 tween the normal and critical incidences for compression  
737 waves in the fluid  $f_1$ . Thus, the plate transfer function is  
738 calculated in the Fourier domain ( $x_1$ -wavenumber, fre-  
739 quency):  $\hat{R}(k_1, x_3; \omega)$ . A double inverse Fourier transform on  
740 ( $k_1, \omega$ ) is applied on  $\hat{R}(k_1, x_3; \omega)$  to transform into the space-  
741 time domain; the temporal signals can be obtained at differ-  
742 ent points along the propagation  $x_3$ -axis:  $R(x_1, x_3; t)$ .

743 Lastly, the formalism presented here is well-adapted to  
744 deal with wave propagation in anisotropic tubes with radial  
745 property gradients (Shuvalov, 2003). The wave equation  
746 keeps the same form as Eq. (12), the state-vector is expressed  
747 from the displacement and traction components in the cylin-  
748 drical basis, and the matrix  $\mathbf{Q}$  depends on the radial position  
749  $r$  ( $\mathbf{Q}=\mathbf{Q}(r)$ ). In cylindrical homogeneous structures, taking  
750 into account an anisotropy more important than transverse  
751 isotropy is fussy because there is no analytical solution to the  
752 “classical” wave equation (second-order differential equa-  
753 tion). Stroh’s (1962) formalism, upon which the Peano ex-  
754 pansion of the matricant is based, is a promising alternative  
755 solution that allows considering altogether the geometry  
756 (cylinder), the anisotropy, and the heterogeneity (radial prop-  
757 erty gradients) of a structure.

## 758 APPENDIX: THERMODYNAMICS STABILITY 759 CONDITIONS AND STIFFNESS COEFFICIENTS

760 Hooke’s law is written under the form  $\sigma_{ij}=C_{ijkl}\epsilon_{kl}$  for  
761 ( $i, j, k, \ell=1, \dots, 3$ ), where  $\sigma$  is the stress-tensor,  $\epsilon$  is the  
762 strain-tensor, and  $C$  is the fourth-order stiffness-tensor. In the  
763 transversely isotropic case, with  $(\mathbf{x}_2, \mathbf{x}_3)$  as isotropic plane,  
764 the stiffness-tensor is expressed as a stiffness matrix (using  
765 Voigt’s notation)

$$766 \quad \mathbf{C} = \begin{pmatrix} c_{11} & c_{13} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{33} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{55} \end{pmatrix}. \quad (\text{A1})$$

767 We introduce the matrix  $\mathbf{S}$ , the inverse of the matrix  $\mathbf{C}$ . It is  
768 expressed by

$$769 \quad \mathbf{S} = \begin{pmatrix} 1/E_L & -\nu_L/E_L & -\nu_L/E_L & 0 & 0 & 0 \\ -\nu_L/E_L & 1/E_T & -\nu_T/E_T & 0 & 0 & 0 \\ -\nu_L/E_L & -\nu_T/E_T & 1/E_L & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_L & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_L \end{pmatrix}. \quad (\text{A2})$$

with  $E_{L,T}$ , the longitudinal ( $L$ ) and transverse ( $T$ ) Young’s  
moduli;  $\nu_{L,T}$ , the longitudinal ( $L$ ) and transverse ( $T$ ) Poi-  
son’s ratios; and  $G_{L,T}$ , the longitudinal ( $L$ ) and transverse ( $T$ )  
shear moduli. By inverting Eq. (A1) and identifying it with  
Eq. (A2), we obtain the following relations:

$$775 \quad E_L = \frac{c_{11}c_{33} - 2c_{13}^2 + c_{11}c_{23}}{c_{33} + c_{23}}, \quad \nu_L = \frac{c_{13}}{c_{33} + c_{23}},$$

$$776 \quad E_T = \frac{c_{11}(c_{33}^2 - c_{23}^2) + 2c_{13}^2(c_{23} - c_{33})}{c_{11}c_{33} - c_{13}^2}, \quad \nu_T = \frac{c_{11}c_{23} - c_{13}^2}{c_{11}c_{33} - c_{13}^2},$$

$$777 \quad G_T = c_{44}, \quad G_L = c_{55}.$$

Knowing the stiffness coefficient values, we can verify  
if the thermodynamical stability conditions are satisfied as  
follows:

$$781 \quad E_L > 0, \quad E_T > 0, \quad -1 < \nu_T < 1,$$

$$782 \quad \frac{(1 - \nu_T)E_L}{2} - \nu_L^2 > 0, \quad G_L > 0. \quad (\text{A4})$$

- Baron, C. (2005). “Peano expansion of the matricant to study elastic wave  
propagation in continuously heterogeneous media,” University Bordeaux 1  
(in French).
- Baron, C., Poncelet, O., Shuvalov, A., and Deschamps, M. (2008). “Propa-  
gation in continuous stratification environment,” in *Materials and Acous-  
tics Handbook*, edited by M. Bruneau and C. Potel (ISTE Ltd. and Wiley,  
London, UK), Chap. 6.
- Baron, C., Talmant, M., and Laugier, P. (2007). “Effect of porosity on ef-  
fective diagonal stiffness coefficients ( $c_{ii}$ ) and anisotropy of cortical at 1  
MHz: A finite-difference time domain study,” *J. Acoust. Soc. Am.* **122**,  
1810–1817.
- Bossy, E., Talmant, M., and Laugier, P. (2004a). “Three-dimensional simu-  
lations of ultrasonic axial transmission velocity measurement on cortical  
bone models,” *J. Acoust. Soc. Am.* **115**, 2314–2324.
- Bossy, E., Talmant, M., Peyrin, F., Akrouf, L., Cloetens, P., and Laugier, P.  
(2004b). “An *in vitro* study of the ultrasonic axial transmission technique  
at the radius: 1-MHz velocity measurements are sensitive to both miner-  
alization and intracortical porosity,” *J. Bone Miner. Res.* **19**, 1548–1556.
- Bousson, V., Meunier, A., Bergot, C., Vicaut, E., Rocha, M. A., Morais, M.  
H., Laval-Jeantet, A. M., and Laredo, J. D. (2001). “Distribution of intra-  
cortical porosity in human midfemoral cortex by age and gender,” *J. Bone  
Miner. Res.* **16**, 1308–1317.
- Brekhovskikh, L. M. (1980). *Waves in Layered Media* (Academic, New  
York).
- Burlew, M., Madsen, E., Zagzebski, J., Banjavic, R., and Sum, S. (1980). “A  
new ultrasound tissue-equivalent material,” *Radiology* **134**, 517–520.
- Camus, E., Talmant, M., Berger, G., and Laugier, P. (2000). “Analysis of the  
axial transmission technique for the assessment of skeletal status,” *J.  
Acoust. Soc. Am.* **108**, 3058–3065.
- Chimenti, D. E., and Nayfeh, A. H. (1986). “Anomalous ultrasonic disper-  
sion in fluid-coupled, fibrous composite plates,” *Appl. Phys. Lett.* **49**, 492–  
493.

- 815 Chimenti, D. E., and Nayfeh, A. H. (1990). "Ultrasonic reflection and  
816 guided waves in fluid-coupled composite laminates," J. Nondestruct. Eval.  
817 9, 51–69.
- 818 Chimenti, D. E., and Rokhlin, S. (1990). "Relationship between leaky Lamb  
819 modes and reflection coefficient zeroes for a fluid-coupled elastic layer," J.  
820 Acoust. Soc. Am. 88, 1603–1611.
- 821 Deschamps, M., and Poncelet, O. (2000). "Transient Lamb waves: Compari-  
822 son between theory and experiment," J. Acoust. Soc. Am. 107, 3120–  
823 3129.
- 824 Dickey, J., Maidanik, G., and Uberall, H. (1995). "The splitting of disper-  
825 sion curves for the fluid-loaded plate," J. Acoust. Soc. Am. 98, 2365–2367.
- 826 Dong, X. N., and Guo, X. E. (2004). "The dependence of transversely iso-  
827 tropic elasticity of human femoral cortical bone on porosity," J. Biomech.  
828 37, 1281–1287.
- 829 Fiorito, R., Madigosky, W., and Überall, H. (1979). "Resonance theory of  
830 acoustic waves interacting with an elastic plate," J. Acoust. Soc. Am. 66,  
831 1857–1866.
- 832 Folds, D. L., and Loggins, C. D. (1977). "Transmission and reflection of  
833 ultrasonic waves in layered media," J. Acoust. Soc. Am. 62, 1102–1109.
- 834 Franklin, H., Danila, E., and Conoir, J.-M. (2001). "S-matrix theory applied  
835 to acoustic scattering by asymmetrically fluid-loaded elastic isotropic  
836 plates," J. Acoust. Soc. Am. 110, 243–253.
- 837 Gantmacher, F. (1959). *The Theory of Matrices* (Wiley, New York).
- 838 Haskell, N. A. (1953). "The dispersion of surface waves on multilayered  
839 media," Bull. Seismol. Soc. Am. 43, 377–393.
- 840 Hill, C., Bamber, J., and Ter Haar, G. (1986). *Physical Principles of Medical*  
841 *Ultrasonics* (Wiley, New York).
- 842 Hosten, B., and Castaings, M. (2003). "Surface impedance matrices to  
843 model the propagation in multilayered media," Ultrasonics 41, 501–507.
- 844 Kenneth, E. G. (1982). "A propagator matrix method for periodically strati-  
845 fied media," J. Acoust. Soc. Am. 73, 137–142.
- 846 Lamb, H. (1917). "On waves in an elastic plate," Proc. R. Soc. London Ser.  
847 A 93, 114–128.
- 848 Lévesque, D., and Piché, L. (1992). "A robust transfer matrix simulation for  
849 ultrasonic response of multilayered absorbing media," J. Acoust. Soc. Am.  
850 92, 452–467.
- 851 Macocco, K., Grimal, Q., Naili, S., and Soize, C. (2006). "Elastoacoustic  
852 model with uncertain mechanical properties for ultrasonic wave velocity  
853 prediction; application to cortical bone evaluation," J. Acoust. Soc. Am.  
854 119, 729–740.
- 855 Marin, F., Gonzales-Macias, J., Diez-Perez, A., Palma, S., and Delgado-  
856 Rodriguez, M. (2006). "Relationship between bone quantitative ultrasound  
857 and fractures: A meta-analysis," J. Bone Miner. Res. 21, 1126–1135.
- 858 Nayfeh, A. H., and Chimenti, D. E. (1988). "Ultrasonic wave reflection from  
liquid-coupled orthotropic plates with application to fibrous composites,"  
ASME J. Appl. Mech. 55, 863–870. 859 860
- Nayfeh, A. H., and Chimenti, D. E. (1989). "Free wave propagation in  
plates of general anisotropic media," ASME J. Appl. Mech. 56, 881–886. 861 862
- Peano, G. (1888). "Integration by series of linear differential equations (in  
French)," Math. Ann. 32, 450–456. 863 864
- Pease, M. (1965). *Methods of Matrix Algebra* (Academic, New York). 865
- Pitts, L. E., Plona, T. J., and Mayer, W. G. (1976). "Theoretical similarities  
of Rayleigh and Lamb modes of vibration," J. Acoust. Soc. Am. 60, 374–  
377. 866 867 868
- Rayleigh, J. W. S. (1885). "On wave propagating along the plane surface of  
an elastic solid," Proc. London Math. Soc. s1-17, 4–11. 869 870
- Reilly, D. T., and Burnstein, A. H. (1974). "The mechanical properties of  
cortical bone," J. Bone Jt. Surg., Am. Vol. 56, 1001–1022. 871 872
- Rho, J. Y. (1996). "An ultrasonic method for measuring the elastic proper-  
ties of human tibial cortical and cancellous bone," Ultrasonics 34, 777–  
783. 873 874 875
- Rokhlin, S. I., and Wang, L. (2002). "Stable recursive algorithm for elastic  
wave propagation in layered anisotropic media: Stiffness matrix method,"  
J. Acoust. Soc. Am. 112, 822–834. 876 877 878
- Rose, J. L. (2002). "A baseline and vision of ultrasonic guided wave inspec-  
tion potential," J. Pressure Vessel Technol. 124, 273–282. 879 880
- Shuvalov, A. (2003). "A sextic formalism for three-dimensional elastody-  
namics of cylindrically anisotropic radially inhomogeneous materials,"  
Proc. R. Soc. London, Ser. A 459, 1611–1639. 881 882 883
- Shuvalov, A., Poncelet, O., Deschamps, M., and Baron, C. (2005). "Long-  
wavelength dispersion of acoustic waves in transversely inhomogeneous  
anisotropic plates," Wave Motion 42, 367–382. 884 885 886
- Stroh, A. N. (1962). "Steady state problems in anisotropic elasticity," J.  
Math. Phys. 41, 77–103. 887 888
- Thomson, W. T. (1950). "Transmission of elastic waves through a stratified  
solid medium," J. Appl. Phys. 21, 89–93. 889 890
- Vlasie, V., and Rousseau, M. (2004). "Guided modes in a plane elastic layer  
with gradually continuous acoustic properties," NDT & E Int. 37, 633–  
644. 891 892 893
- Wang, L., and Rokhlin, S. I. (2001). "Stable reformulation of transfer matrix  
method for wave propagation in layered anisotropic media," Ultrasonics  
39, 413–424. 894 895 896
- Worlton, H. (1961). "Experimental confirmation of Lamb waves at mega-  
cycle frequencies," J. Appl. Phys. 32, 967–971. 897 898
- Youssef, I., and El-Arabawi, H. (2007). "Picard iteration algorithm com-  
bined with Gauss-Seidel technique for initial value problems," Appl.  
Math. Comput. 190, 345–355. 899 900 901