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1 Propagation of elastic waves in a fluid-loaded anisotropic 2 functionally graded waveguide: Application to 3 ultrasound characterization

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8 Non-destructive evaluation of heterogeneous materials is of major interest not only in industrial but
9 also in biomedical fields. In this work, the studied structure is a three-layered one: A laterally
10 heterogeneous anisotropic solid layer is sandwiched between two acoustic fluids. An original
11 method is proposed to solve the wave equation in such a structure without using a multilayered
12 model for the plate. This method is based on an analytical solution, the matricant, explicitly
13 expressed under the Peano series expansion form. This approach is validated for the study of a
14 fluid-loaded anisotropic and homogeneous plane waveguide with two different fluids on each side.
15 Then, original results are given on the propagation of elastic waves in an asymmetrically
16 fluid-loaded waveguide with laterally varying properties. This configuration notably corresponds to
17 the axial transmission technique to the ultrasound characterization of cortical bone *in vivo*.

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21 I. INTRODUCTION

22 A lot of natural media have unidirectional varying elas-
23 tic properties. The mantle crust, the oceans, and cortical bone
24 are some of these functionally graded media. Scientists fo-
25 cused on the advantages presented by this type of materials
26 in terms of mechanical behavior, and since the 1980s, they
27 developed industrial functionally graded materials (FGMs)
28 particularly exploited in high-technology and biomedical ap-
29 plications. Consequently, the non-destructive evaluation of
30 these materials is a key issue. Surface and guided waves play
31 a major role in non-destructive testing and evaluation of
32 complex structures. Several studies are dedicated to the leaky
33 Lamb wave propagation in fluid-loaded plates (Chimenti and
34 Nayfeh, 1990; Chimenti and Rokhlin, 1990; Deschamps and
35 Poncelet, 2000). In all these studies, the media are homoge-
36 neous or multilayered. In this work, we introduce a general
37 method to take into account the continuity of the property
38 variation in an anisotropic waveguide. This method is based
39 on the knowledge of an analytical solution of the wave equa-
40 tion, the matricant, explicitly expressed via the Peano series
41 (Peano, 1888). The accuracy of the numerical evaluation of
42 this solution and its validity domain are perfectly managed
43 (Baron, 2005; Youssef and El-Arabawi, 2007). Because it
44 deals with an analytical solution, all the wave propagation
45 parameters are controlled. This represents an advantage with
46 respect to numerical methods such as finite-element and
47 finite-difference methods for which the problem treated is a
48 global one, making difficult to analyze and interpret the ex-
49 perimental data, which result from the interaction and cou-
50 pling of numerous physical phenomena. To the best of our
51 knowledge, this is the first method to evaluate the mechani-
52 cal behavior of a fluid-loaded anisotropic waveguide with

continuously varying properties without modeling the FGM 53
plate by a multilayered plate. Consequently, in the context of 54
real materials with continuously varying properties (such as 55
bone, bamboo, or manufactured FGM), this method could 56
allow assessing the solution to a more realistic model with a 57
controlled accuracy and without an increase in the 58
computation-time. 59

In this work, we first present the method and its setup 60
with fluid-structure interaction; then we proceed to the vali- 61
dation of the method by comparing our results to the disper- 62
sion curves obtained from classical schemes on homoge- 63
neous waveguides (isotropic and anisotropic). Two 64
advantages of the method are underlined: (i) An asymmetric 65
fluid-loading may be taken into account without modifying 66
the scheme for the numerical solution, and (ii) the influence 67
of the property gradient on the ultrasonic response of the 68
waveguide may be investigated via the frequency spectrum 69
of the reflection coefficient modulus. Finally, we get onto the 70
relevancy of this model applied to the ultrasound character- 71
ization of cortical bone by the axial transmission technique. 72

73 II. BACKGROUND

Contemporary work efforts over the last 2 decades illus- 74
trate some of the technology interest on guided waves to 75
non-destructive evaluation. Namely, Rose (2002) gave a re- 76
view of ultrasonic guided wave inspection potential. A lot of 77
papers deal with the interaction between guided waves and a 78
solid plate immersed in a fluid or embedded between two 79
different fluids. Guided modes in an infinite elastic isotropic 80
plate in vacuum were first treated by Rayleigh (1885) then 81
by Lamb (1917). The Lamb wave problem is reserved, 82
strictly speaking, for wave propagation in a traction-free ho- 83
mogeneous isotropic plate. To deal with guided modes in a 84
fluid-loaded plate, we use the term “leaky Lamb waves” as 85
the energy is partly radiated in the fluids on both sides of the 86
plate. For the basic Lamb problem—plate in vacuum—the 87

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88 solutions of the dispersion equation are real whereas in the
89 case of a plate bounded by media on both sides, the disper-
90 sion equation has complex solutions. In 1961, [Worlton](#)
91 (1961) gave an experimental confirmation of the theoretical
92 work of [Lamb \(1917\)](#), by obtaining experimentally the dis-
93 persion curves of aluminum and zirconium plates, asserting
94 that water loading has little effect on the behavior of waves
95 in plates. In 1976, [Pitts et al. \(1976\)](#) presented some numeri-
96 cal test results on the relationship between real part of the
97 reflection coefficient poles and the phase velocity of leaky
98 Lamb modes in a homogeneous isotropic brass plate in wa-
99 ter. [Folds and Loggins \(1977\)](#) proposed analytical expres-
100 sions of the reflection and transmission coefficients for plane
101 waves at oblique incidence on a multilayered isotropic plate
102 immersed in water based on [Brekhovskikh's \(1980\)](#) analysis.
103 They found good agreement with their theoretical results and
104 experimental data. Few years later, [Fiorito et al. \(1979\)](#) de-
105 veloped a resonance formalism for the fluid-loaded elastic
106 plate and gave some theoretical and numerical results for an
107 isotropic steel plate immersed in the water. This formalism
108 was generalized to the interactions of acoustic plane waves
109 with an asymmetrically fluid-loaded elastic plate by [Franklin](#)
110 [et al. \(2001\)](#). [Nayfeh, Chimenti, and Rokhlin](#) produced a lot
111 of works on wave propagation in anisotropic media and par-
112 ticularly in fiber composite plates immersed in a fluid ([Chi-
113 menti and Nayfeh, 1986, 1990; Chimenti and Rokhlin, 1990;
114 Nayfeh and Chimenti, 1988, 1989; Rokhlin and Wang,
115 2002](#)). Based on their formalism, [Deschamps and Poncelet](#)
116 (2000) placed the emphasis on the difference between what
117 they called transient Lamb wave-solutions of the character-
118 istic equation of the plate for complex frequency and real
119 slowness (time attenuation), and heterogeneous Lamb waves
120 for which the slowness is complex and the frequency is real
121 (spatial attenuation). These two ways of resolution of the
122 dispersion equation have two different physical meanings—
123 space or time attenuation—and consequences of this differ-
124 ence are developed in their paper ([Deschamps and Poncelet,
125 2000](#)). A critical point is the validity of the Cremer's coinci-
126 dence hypothesis: The real couples (angular frequency ω and
127 phase velocity v_φ), such that the reflection coefficient is
128 minimum, may be identified as velocity dispersion of plate
129 waves. Experimentally checked in a lot of configurations, it
130 appears to be not well satisfied in several cases (for instance,
131 graphite-epoxy plates when the ratio between fluid and plate
132 mass densities is not “small”) ([Chimenti and Nayfeh, 1986;
133 Nayfeh and Chimenti, 1988](#)). The results obtained by [Des-
134 champs and Poncelet \(2000\)](#) on fluid-loaded plate show a
135 good correlation between dispersion curves obtained in com-
136 plex frequency and the minima of the reflection coefficient,
137 which suggests that the Cremer's coincidence principle is
138 still valid considering time attenuation. All these studies
139 show the evidence that the wave propagation in fluid-loaded
140 elastic plate emerges as a very delicate problem, which needs
141 cautious treatment.

142 III. GENERAL FORMULATION OF THE PROBLEM

143 We consider a three-dimensional multilayer system com-
144 posed of one elastic solid layer sandwiched between two

acoustic fluid layers. Let $\mathbf{R}(O, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ be the Cartesian
frame of reference where O is the origin of the space and
 $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is an orthonormal basis for this space. The coor-
dinate of the generic point \mathbf{x} in \mathbf{R} is specified by (x_1, x_2, x_3) .
The acoustic fluid layers occupy an open unbounded domain.
Both fluids f_1 and f_2 are supposed perfect, of respective mass
densities ρ_{f_1} and ρ_{f_2} ; the constant speeds of sound in each
fluid are c_{f_1} and c_{f_2} , respectively. The thickness of the solid
layer is denoted by d and its mass density by ρ . The inter-
faces between the fluids and the solid layer are infinite planes
parallel to the $(\mathbf{x}_1, \mathbf{x}_2)$ -plane. The \mathbf{x}_3 -axis is oriented down-
ward and the origin O is located at the interface between the
upper fluid f_1 and the solid layer. Therefore, we assume that
the structure is a two-dimensional one and that the guided
waves travel in the plane $x_2=0$; in the following parts, this
coordinate is implicit and is omitted in the mathematical ex-
pressions. Moreover, the solid layer will be so-called plate.

The elastic plate is supposed to be anisotropic and is
liable to present continuously varying properties along its
thickness (\mathbf{x}_3 -axis). These mechanical properties are repre-
sented by the stiffness fourth-order tensor $\mathbf{C}=\mathbf{C}(x_3)$ and the
mass density $\rho=\rho(x_3)$.

A. System equations

1. The wave equation in the fluid f_n (for $n=1$ or 2)

In the fluid f_n and the context of the linear acoustic
theory, the linearized Euler equation and the constitutive
equations are written as

$$-\frac{\partial p^{(n)}}{\partial x_j} = \rho_{f_n} \frac{\partial^2 u_j^{(n)}}{\partial t^2},$$

$$p^{(n)} = K_{f_n} \operatorname{div} \mathbf{u}^{(n)},$$

where $\mathbf{u}^{(n)}$ and $p^{(n)}$, respectively, represent the displacement
vector and the pressure in the fluid f_n ; its compressibility and
the speed of sound in the fluid at equilibrium are, respec-
tively, K_{f_n} and $c_{f_n} = \sqrt{K_{f_n}/\rho_{f_n}}$. The operator div is the diver-
gence.

The solutions of system (1) for the fluid f_n are sought
under the form

$$\mathbf{f}_n(x_1, x_2; t) = \mathbf{A}_n(x_3) \exp i(k_1 x_1 + k_3^{(n)} x_3 - \omega t),$$

where k_1 and $k_3^{(n)}$ are the wavenumbers, respectively, along
the \mathbf{x}_1 -axis and \mathbf{x}_3 -axis in the fluid f_n ; the angular frequency
is noted ω and i is the imaginary unit.

We consider an incident wave reaching the plate at an
angle θ_1 from the \mathbf{x}_3 -axis in the fluid f_1 . The incident
displacement-field is defined in the following form, assum-
ing that its amplitude is normalized:

$$\mathbf{u}_I^{(1)} = \begin{pmatrix} \sin \theta_1 \\ 0 \\ \cos \theta_1 \end{pmatrix} \exp i(k_1 x_1 + k_3^{(1)} x_3 - \omega t),$$

with $k_1 = (\omega/c_{f_1}) \sin \theta_1$ and $k_3^{(1)} = (\omega/c_{f_1}) \cos \theta_1$. From this, the
expressions of the reflected displacement-field $\mathbf{u}_R^{(1)}$ in f_1 and
of the transmitted displacement-field $\mathbf{u}_T^{(2)}$ in f_2 are deduced as
follows:

$$\mathbf{u}_R^{(1)} = R \begin{pmatrix} \sin \theta_1 \\ 0 \\ -\cos \theta_1 \end{pmatrix} \exp i(k_1 x_1 - k_3^{(1)} x_3 - \omega t),$$

$$\mathbf{u}_T^{(2)} = T \frac{c_{f_2}}{c_{f_1}} \begin{pmatrix} \sin \theta_1 \\ 0 \\ \cos \theta_1 \end{pmatrix} \exp i(k_1 x_1 + k_3^{(2)} x_3 - \omega t). \quad (4)$$

where $R=R(x_1, x_3; t)$ and $T=T(x_1, x_3; t)$, respectively, represent the reflection and transmission coefficients, which will be expressed explicitly in the sequel. The incident, reflected, and transmitted pressure fields, respectively, noted $p_I^{(1)}$, $p_R^{(1)}$, and $p_T^{(2)}$, are deduced from expressions (3) and (4), and the second equation of the system (1)

$$p_I^{(1)} = -i\omega \times \rho_{f_1} c_{f_1} \times \exp i(k_1 x_1 - k_3^{(1)} x_3 - \omega t),$$

$$p_R^{(1)} = -i\omega \times \rho_{f_1} c_{f_1} \times R \times \exp i(k_1 x_1 - k_3^{(1)} x_3 - \omega t),$$

$$p_T^{(2)} = -i\omega \times \rho_{f_2} c_{f_2} \times T \times \exp i(k_1 x_1 + k_3^{(2)} x_3 - \omega t). \quad (5)$$

B. The wave equation in the plate waveguide

The body forces in the solid plate are neglected. The balance equation of linear momentum associated with the constitutive law of linear elasticity (Hooke's law) gives the following equations:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2},$$

$$\sigma_{ij} = \frac{1}{2} C_{ijkl} \left(\frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_k} \right), \quad (6)$$

where u_i (for $i=1, \dots, 3$) and σ_{ij} (for $i, j=1, \dots, 3$), respectively, represent the components of the displacement-field \mathbf{u} and of the stress $\boldsymbol{\sigma}$. In system (6), the Einstein convention of summation on repeated indices is used. The solutions are sought for the vectors of displacement \mathbf{u} and traction σ_{i3} (for $i=1, \dots, 3$) (assumed to be harmonic in time t and space along the \mathbf{x}_1 -axis) under the form

$$\mathbf{f}(x_1, x_3; t) = \mathbf{A}(x_3) \exp i(k_1 x_1 - \omega t). \quad (7)$$

C. Fluid-loading interface conditions

The conditions at both interfaces $x_3=0$ and $x_3=d$ are the continuity of the normal displacement and the one of the normal stress. We consider that the fluids f_1 and f_2 are perfect; consequently, the shear stresses are zero at the interfaces [$\sigma_{13}(x_1, 0; t) = \sigma_{13}(x_1, d; t) = 0$ and $\sigma_{23}(x_1, 0; t) = \sigma_{23}(x_1, d; t) = 0$]. The following relations are obtained:

$$u_3(x_1, 0; t) = u_3^{(1)}(x_1, 0; t), \quad u_3(x_1, d; t) = u_3^{(2)}(x_1, d; t),$$

$$\sigma_{33}(x_1, 0; t) = -p^{(1)}(x_1, 0; t), \quad \sigma_{33}(x_1, d; t) = -p^{(2)}(x_1, d; t), \quad (8)$$

with

$$u_3^{(1)} = \mathbf{u}_I^{(1)} \cdot \mathbf{x}_3 + \mathbf{u}_R^{(1)} \cdot \mathbf{x}_3, \quad u_3^{(2)} = \mathbf{u}_T^{(2)} \cdot \mathbf{x}_3 \quad \text{and} \quad (9)$$

$$p^{(1)} = p_I^{(1)} + p_R^{(1)}, \quad p^{(2)} = p_T^{(1)}. \quad (9)$$

D. A closed-form solution: The matricant

Introducing expression (7) in Eq. (6), we obtain the wave equation under the form of a second-order differential equation with non-constant coefficients. For particular forms of profiles, this equation has analytical solutions expressed with special functions (Bessel or Hankel functions) (Vlasie and Rousseau, 2004). But, in the general case, there is no analytical solution to the problem thus formulated. The most current methods to solve the wave equation in unidirectionally heterogeneous media are derived from the Thomson–Haskell method (Haskell, 1953; Thomson, 1950). These methods are appropriate for multilayered media (Hosten and Castaings, 2003; Kenneth, 1982; Lévesque and Piché, 1992; Wang and Rokhlin, 2001). But, for continuously varying media, these techniques mean to replace the continuous profiles of properties with step-wise functions. Thereby, the studied problem becomes an approximate one, even before the resolution step; the accuracy of the solution as its validity domain is hard to evaluate. Moreover, the multilayered model of the waveguide creates some “virtual” interfaces likely to induce artifacts. In order to deal with the exact problem, it is to keep the continuity of the properties' variation, the wave equation is re-written under the form of an ordinary differential equation system with non-constant coefficients for which an analytical solution exists: the matricant (Baron, 2005).

1. Hamiltonian form of the wave equation

We consider that the plate presents material symmetries that allow decoupling the pressure–shear vertical (P-SV) waves, polarized in the propagation plane ($\mathbf{x}_1, \mathbf{x}_3$) and the shear horizontal (SH) waves polarized along \mathbf{x}_2 -axis. The incident medium f_1 is a perfect fluid; only the P-SV waves travel in the plate. Applying a spatio-temporal Fourier transform on (x_1, t) of the displacement-field [noted $\hat{\mathbf{u}}(k_1, x_3; \omega)$ after Fourier transform] and on the traction field [noted $\hat{\sigma}_{i3}(k_1, x_3; \omega)$ for $i=1, \dots, 3$] and using the Voigt notation (C_{ijkl} for $i, j, k, \ell=1, \dots, 3$ is replaced with c_{IJ} for $I, J=1, \dots, 6$), Eq. (6) leads to Stroh's (1962) sextic plate formalism (Hamiltonian formulation of the wave equation)

$$\frac{\partial \hat{\sigma}_{13}}{\partial x_3} = \rho(\omega)^2 \hat{u}_1 - ik_1 \hat{\sigma}_{11}, \quad \frac{\partial \hat{\sigma}_{33}}{\partial x_3} = \rho(\omega)^2 \hat{u}_3 - ik_1 \hat{\sigma}_{13}, \quad (10)$$

$$\hat{\sigma}_{11} = ik_1 c_{11} \hat{u}_1 + c_{33} \frac{\partial \hat{u}_3}{\partial x_3}, \quad \hat{\sigma}_{13} = c_{55} \left(\frac{\partial \hat{u}_1}{\partial x_3} + ik_1 \hat{u}_3 \right), \quad (11)$$

$$\hat{\sigma}_{33} = ik_1 c_{13} \hat{u}_1 + c_{33} \frac{\partial \hat{u}_3}{\partial x_3}. \quad (11)$$

274 According to Eqs. (10) and (11), $\hat{\sigma}_{11}$ is function of \hat{u}_1
 275 and $\hat{\sigma}_{33}$. The wave equation becomes a matrix system ex-
 276 pressed using the Thomson–Haskell parametrization of
 277 Stroh’s (1962) sextic plate formalism

$$\frac{d}{dx_3} \eta(x_3) = i\omega \mathbf{Q}(x_3) \eta(x_3), \quad (12) \quad 278$$

that is, 279

$$\frac{d}{dx_3} \begin{pmatrix} i\omega \hat{u}_1 \\ i\omega \hat{u}_3 \\ \hat{\sigma}_{13} \\ \hat{\sigma}_{33} \end{pmatrix} = i\omega \begin{pmatrix} 0 & s_1 & 1/c_{55}(x_3) & 0 \\ -s_1 c_{13}(x_3)/c_{33}(x_3) & 0 & 0 & 1/c_{33}(x_3) \\ \rho(x_3) - s_1^2 \zeta(x_3) & 0 & 0 & -s_1 c_{13}(x_3)/c_{33}(x_3) \\ 0 & \rho(x_3) & -s_1 & 0 \end{pmatrix} \begin{pmatrix} i\omega \hat{u}_1 \\ i\omega \hat{u}_3 \\ \hat{\sigma}_{13} \\ \hat{\sigma}_{33} \end{pmatrix}, \quad (13)$$

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286 with the relations

$$\zeta(x_3) = c_{11}(x_3) - \frac{c_{13}^2(x_3)}{c_{33}(x_3)}, \quad k_1 = \omega s_1, \quad (14)$$

287 where s_1 is the x_1 -component of the slowness. The matrix \mathbf{Q}
 288 includes all the information about the heterogeneity of the
 289 waveguide because it is expressed from the plate mechanical
 290 properties ($\rho(x_3), \mathbb{C}(x_3)$) and from two acoustical parameters
 291 (s_1, ω).

293 2. Explicit solution: The Peano expansion of the 294 matricant

295 The wave equation thus formulated has an analytical
 296 solution expressed between a reference point $(x_1, 0, x_3^0)$ and
 297 some point of the plate $(x_1, 0, x_3)$ in the propagation plane.
 298 This solution is called the matricant and is explicitly written
 299 under the form of the Peano series expansion (Gantmacher,
 300 1959; Peano, 1888; Pease, 1965):

$$\mathbf{M}(x_3, x_3^0) = \mathbf{I} + (i\omega) \int_{x_3^0}^{x_3} \mathbf{Q}(\xi) d\xi + (i\omega)^2 \int_{x_3^0}^{x_3} \int_{x_3^0}^{\xi} \mathbf{Q}(\xi_1) d\xi_1 d\xi + \dots, \quad (15)$$

301
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303 where \mathbf{I} is the identity matrix of dimension (4,4). If the ma-
 304 trix components $\mathbf{Q}(x_3)$ are bounded in the study interval,
 305 these series are always convergent (Baron, 2005). The com-
 306 ponents of the matrix \mathbf{Q} are continuous in x_3 and the study
 307 interval is bounded (thickness of the waveguide); conse-
 308 quently the hypothesis is always verified. We underline that
 309 the $i\omega$ -factorization leads up to a polynomial form of the
 310 matricant. The $i\omega$ -polynomial coefficients are matrices inde-
 311 pendent of ω .

312 3. Boundary conditions: Fluid-structure interaction

313 Using the propagator property of the matricant through
 314 the plate thickness, the state-vector [defined in Eq. (13)] at
 315 the second interface $\eta(d)$ is evaluated from the state-vector
 316 at the first interface $\eta(0)$ as follows:

$$\eta(d) = \mathbf{M}(d, 0) \eta(0). \quad (16)$$

317

The fluid-structure interaction sets the conditions of zero
 shear stresses (see Sec. III C), used after a spatio-temporal
 Fourier transform on (x_1, t) . Equation (16) becomes

$$\begin{pmatrix} i\omega \hat{u}_1(k_1, d; \omega) \\ i\omega \hat{u}_3(k_1, d; \omega) \\ 0 \\ \hat{\sigma}_{33}(k_1, d; \omega) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12} & M_{22} & M_{23} & M_{24} \\ M_{13} & M_{23} & M_{33} & M_{34} \\ M_{14} & M_{24} & M_{34} & M_{44} \end{pmatrix} \begin{pmatrix} i\omega \hat{u}_1(k_1, 0; \omega) \\ i\omega \hat{u}_3(k_1, 0; \omega) \\ 0 \\ \hat{\sigma}_{33}(k_1, 0; \omega) \end{pmatrix}. \quad (17)$$

The condition to obtain a nontrivial solution to Eq. (17)
 leads to the following relation:

$$i\omega \hat{u}_1(k_1, 0; \omega) \times M_{13} + i\omega \hat{u}_3(k_1, 0; \omega) \times M_{32} + \hat{\sigma}_{33}(k_1, 0; \omega) \times M_{34} = 0, \quad (18)$$

where M_{ij} (for $i, j = 1, \dots, 4$) represent the components of the
 matrix \mathbf{M} . The displacement component $\hat{u}_1(k_1, 0; \omega)$ can be
 expressed as a linear combination of $\hat{u}_3(k_1, 0; \omega)$ and
 $\hat{\sigma}_{33}(k_1, 0; \omega)$; thus system (16) of dimension 4 is reduced to a
 matrix system of dimension 2

$$\eta(d) = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \eta(0) \quad \text{where} \quad \eta(x_3) = \begin{pmatrix} i\omega \hat{u}_3 \\ \hat{\sigma}_{33} \end{pmatrix}, \quad (19)$$

with the relations 333

$$P_1 = M_{22} - M_{21} \frac{M_{32}}{M_{31}}, \quad P_2 = M_{24} - M_{21} \frac{M_{34}}{M_{31}},$$

$$P_3 = M_{42} - M_{41} \frac{M_{32}}{M_{31}}, \quad P_4 = M_{44} - M_{41} \frac{M_{34}}{M_{31}}. \quad (20)$$

Interface conditions (8) are transformed in the Fourier
 domain (k_1, ω) . The expressions of the displacement and the
 pressure in the fluids [see Eqs. (3)–(5)], so that the one of the
 displacement and traction fields in the solid plate [see Eq.
 (19)], are substituted in the transformed interface conditions.

341 Setting $\eta(0) = (\alpha_1, \alpha_2)^T \exp i(k_1 x_1 - \omega t)$, where the super-
 342 script T designates the transpose operator, we obtain the fol-
 343 lowing matrix equation:

$$344 \begin{pmatrix} i\omega s_3^{(1)} c_{f_1} & 1 & 0 & & 0 \\ -i\omega \rho_{f_1} c_{f_1} & 0 & 1 & & 0 \\ 0 & P_1 & P_2 & -i\omega s_3^{(2)} c_{f_2} \exp(i\omega s_3^{(2)} d) & \\ 0 & P_3 & P_4 & -i\omega \rho_{f_2} c_{f_2} \exp(i\omega s_3^{(2)} d) & \end{pmatrix} \begin{pmatrix} \hat{R} \\ \alpha_1 \\ \alpha_2 \\ \hat{T} \end{pmatrix} = \begin{pmatrix} i\omega s_3^{(1)} c_{f_1} \\ i\omega \rho_{f_1} c_{f_1} \\ 0 \\ 0 \end{pmatrix}, \quad (21)$$

345 where $\mathbf{s}^{(n)} = \mathbf{k}^{(n)}/\omega$ is the slowness-vector in the fluid f_n (n
 346 = 1 or 2). The quantities \hat{R} and \hat{T} are, respectively, the reflec-
 347 tion and transmission coefficients expressed in the Fourier
 348 domain: $\hat{R} = \hat{R}(k_1, x_3; \omega)$ and $\hat{T} = \hat{T}(k_1, x_3; \omega)$. The two first
 349 lines of system (21) express the boundary conditions at the
 350 first interface ($x_3=0$) and the two last lines those at the sec-
 351 ond interface ($x_3=d$) introducing the Fourier transform of
 352 expressions (3)–(5) in the following relations:

$$354 \eta(0) - \begin{pmatrix} i\omega \hat{u}_{3R}^{(1)} \\ -\hat{p}_R \end{pmatrix}_{x_3=0} = \begin{pmatrix} i\omega \hat{u}_{3I}^{(1)} \\ -\hat{p}_I \end{pmatrix}_{x_3=0},$$

$$355 \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \eta(0) - \begin{pmatrix} i\omega \hat{u}_{3T}^{(2)} \\ -\hat{p}_T \end{pmatrix}_{x_3=d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (22)$$

356 where $\hat{u}_{3R}^{(1)}$, $\hat{u}_{3I}^{(1)}$, and $\hat{u}_{3T}^{(2)}$ are the components along \mathbf{x}_3 -axis of
 357 $\hat{\mathbf{u}}_R^{(1)}$, $\hat{\mathbf{u}}_I^{(1)}$, and $\hat{\mathbf{u}}_T^{(2)}$ vectors, respectively. Note the equality
 358 between the quantities $u_{3T}^{(2)}$ and $u_3^{(2)}$ where this last is defined
 359 in Eq. (9).

360 4. Expression of the reflection and transmission 361 coefficients

362 From Eq. (21), we deduce the analytical expressions of
 363 the reflection and transmission complex coefficients

$$364 \hat{R}(s_1, x_3; \omega) = \frac{(P_3 - P_1 Z_2 + P_4 Z_1 - P_2 Z_1 Z_2)}{(P_3 - P_1 Z_2 - P_4 Z_1 + P_2 Z_1 Z_2)},$$

$$365 \hat{T}(s_1, x_3; \omega) = -\frac{2Z_2(\rho_{f_1} c_{f_1} / \rho_{f_2} c_{f_2})(P_1 P_4 - P_2 P_3)}{(P_3 - P_1 Z_2 - P_4 Z_1 + P_2 Z_1 Z_2)} \times \exp(-i\omega s_3^{(2)} d), \quad (23)$$

367 with $Z_n = \rho_{f_n} / \sqrt{1/c_{f_n}^2 - s_1^2}$ (for $n=1$ or 2) and $k_1 = \omega s_1$.

368 IV. VALIDATION OF THE METHOD

369 The aim of this section is to check that the Peano expan-
 370 sion of the matricant is well-adapted to study fluid-loaded
 371 waveguides. We take into account the fluid-structure interac-
 372 tion in different configurations of homogeneous plates com-

TABLE I. Elastic properties of transversely isotropic plate (with $c_{23}=c_{22}-2c_{44}$).

ρ (g cm ⁻³)	c_{11} (GPa)	$c_{22}=c_{33}$ (GPa)	$c_{12}=c_{13}$ (GPa)	c_{44} (GPa)	$c_{66}=c_{55}$ (GPa)
1.85	23.05	15.1	8.7	3.25	4.7

paring the results obtained from the numerical implementa-
 tion of the Peano expansion of the matricant to results taken
 from literature.

The numerical evaluation of $P_1, P_2, P_3,$ and P_4 requires
 us to truncate the Peano series and to numerically calculate
 the integrals. Thus, the error can be estimated and controlled
 (Baron, 2005). We retained 70 terms in the series and evalu-
 ate the integrals over 100 points using Simpson's rule
 (fourth-order integration method). These choices ensure the
 convergence of the solution and the accuracy of the results
 for a reasonable computation-time (never exceeding few
 minutes on a desktop computer). Expressions (23) give the
 frequency spectrum (modulus and phase) of the reflection
 coefficient for different incidences (s_1 varies from zero-
 normal incidence to $1/c_{f_1}$ corresponding to the critical inci-
 dence in the fluid f_1).

A lot of works detailed the relationship between the
 poles and the zeros of the reflection coefficient and the leaky
 Lamb wave dispersion curves (Chimenti and Rokhlin, 1990;
 Deschamps and Poncelet, 2000). The results of Sec. IV A
 compare the dispersion curves obtained by seeking the poles
 of reflection coefficient (23) and the results taken from lit-
 erature or from closed-form solution.

A. Validation for a homogeneous and isotropic or anisotropic fluid-loaded plate

The method is tested by plotting the dispersion curves
 (variation of the phase velocity versus frequency-thickness
 product) for an isotropic aluminum plate immersed in water.
 The data in the paper of Chimenti and Rokhlin (1990) are
 used. The results obtained (not shown) by the present method
 are in perfect agreement with the results presented by them
 (Chimenti and Rokhlin, 1990).

As mentioned by Chimenti and Rokhlin (1990), there
 are few differences between the zeros' loci and the poles' loci
 for a plate immersed in a fluid whose the mass density is
 lower than the plate mass density. As underlined by several
 authors, fluid-load does have just a weak influence on guided
 wave traveling in the plate immersed in water.

Taking into account the anisotropy does not change the
 scheme for the numerical solution of wave equation with the
 matricant. We consider a transverse isotropic plate immersed
 in water ($\rho_f = 1$ g cm⁻³, $c_f = 1.485$ mm μ s⁻¹) whose proper-
 ties are reported in Table I. For that configuration, Nayfeh
 and Chimenti (1989) developed a method to obtain an ana-
 lytical expression of the reflection coefficient. By using the
 data from this paper, the results obtained (see Fig. 1) with the
 present method are in perfect agreement with theirs. The
 curves presented in Figs. 1(a) and 1(b) are superimposed and
 need to be presented separately.

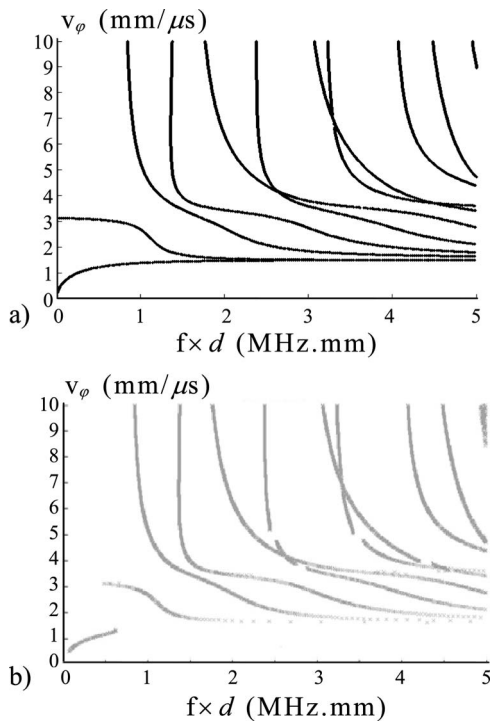


FIG. 1. Dispersion curves for a transversely isotropic plate immersed in water; comparison between (a) analytical results from Nayfeh and Chimenti (1989) and (b) results obtained via the Peano series of the matricant.

422 B. Validation for an asymmetrically loaded 423 homogeneous isotropic plate ($f_1 \neq f_2$)

424 The formalism presented here to solve the wave equa-
425 tion in an unidirectionally graded medium presents two main
426 advantages: Without changing the scheme to obtain the nu-
427 merical solution we can take into account (i) an asymmetric
428 loading and (ii) the unidirectional continuous heterogeneity.

429 The mechanical behavior of the plate is different for
430 symmetric and asymmetric loadings. For example, in the
431 symmetric loading case, there is a unique critical frequency
432 and a unique phase velocity value v_φ in the plate, which
433 corresponds to the propagation velocity in the fluid ($v_\varphi = c_{f_1}$
434 $= c_{f_2}$), for which the displacements and the stresses at the
435 interfaces are quasi-null; whereas in the asymmetric loading
436 ($f_1 \neq f_2$), there are two critical frequencies and two values of
437 the phase velocity in the plate for which the structure does
438 not respond (Dickey et al., 1995). The validation is done on
439 an isotropic aluminum plate with the following properties:
440 $\rho = 2.79 \text{ g cm}^{-3}$; the longitudinal and transverse wave
441 velocities are, respectively, $v_L = 6.38 \text{ mm } \mu\text{s}^{-1}$ and $v_T =$
442 $3.10 \text{ mm } \mu\text{s}^{-1}$. The characteristic properties of the fluid f_1
443 correspond to those of water (see Sec. IV A); the character-
444 istic properties of the fluid f_2 correspond to glycerine: ρ_{f_2}
445 $= 1.26 \text{ g cm}^{-3}$ and $c_{f_2} = 1.920 \text{ mm } \mu\text{s}^{-1}$. This configuration
446 is the same as the one studied by Franklin et al. (2001). The
447 modulus of the reflection coefficient versus the incident
448 angle is plotted in Fig. 2 for a fixed frequency-thickness
449 product ($f \times d = 4.7 \text{ MHz mm}$).

450 This figure shows the perfect agreement between our
451 results and the ones presented by Franklin et al. (2001).

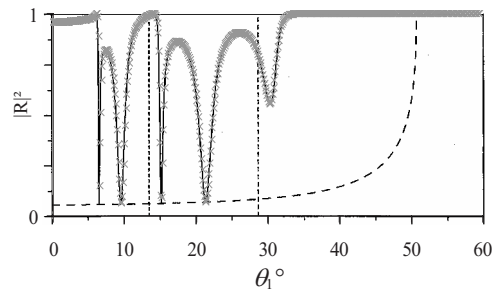


FIG. 2. Reflection coefficient modulus for an asymmetrically fluid-loaded aluminum plate (water and glycerine) versus incident angle: in dark lines results published in Franklin et al. (2001), and in gray crosses the reflection coefficient modulus calculated from Eq. (23). The vertical lines (dashed) represent the two critical angles for longitudinal waves and transverse waves. The dashed curve corresponds to the resonant amplitudes (Franklin et al., 2001).

V. RELEVANCY OF THE METHOD FOR ULTRASOUND CHARACTERIZATION OF CORTICAL BONE

452 Cortical bone is a kind of hard tissue found at the edges
453 of long bones and supports most of the load of the body.
454 Several studies demonstrated the heterogeneous nature of the
455 cortical bone, particularly they show evidence the gradual
456 variation of the volumetric porosity (ratio between pores and
457 total volume) along the cortical thickness. Yet, the porosity is
458 intrinsically linked to the macroscopic mechanical behavior
459 of the cortical bone (Baron et al., 2007). Therefore, the con-
460 tinuous variation of porosity induces a continuous variation
461 of material properties. Taking into account the gradient
462 should prove itself to be essential in the context of diagnosis
463 and therapeutic monitoring of osteoporosis. Indeed, the gra-
464 dient characterization would allow assessing geometrical
465 (cortex thickness) and material (elastic coefficients variation)
466 information, which are fundamental parameters to evaluate
467 the bone fragility. For several years, the quantitative ultra-
468 sonography (by axial and transverse transmissions) proved
469 itself to be a hopeful alternative technique to evaluate the
470 fracture risk (Marin et al., 2006). However, the inter-
471 individual and inter-site variations of bone mechanical prop-
472 erties make the standardization of the protocol of fracture
473 risk evaluation by ultrasound very delicate.

474 The focus is set on a configuration closed to the axial
475 transmission device for *in vivo* conditions. In this context,
476 the relevancy of studying the head wave propagation has
477 been demonstrated (Bossy et al., 2004a, 2004b; Camus et al.,
478 2000). As a consequence all the reflection coefficients pre-
479 sented in this paper were calculated for an incident angle
480 corresponding to the grazing-angle for longitudinal waves
481 [critical angle of longitudinal wave propagation in the plate
482 at the first interface ($x_3 = 0$)].

483 The surrounding media in the *in vivo* configuration of
484 ultrasound characterization of cortical bone are the muscle
485 for the upper fluid f_1 ($c_{f_1} = 1.54 \text{ mm } \mu\text{s}^{-1}$ and ρ_{f_1}
486 $= 1.07 \text{ g cm}^{-3}$) and the marrow for the lower fluid f_2 (c_{f_2}
487 $= 1.45 \text{ mm } \mu\text{s}^{-1}$ and $\rho_{f_2} = 0.9 \text{ g cm}^{-3}$) (Burlew et al., 1980;
488 Hill et al., 1986). We are interested in the influence of the
489 continuous gradient of the mechanical properties on the ul-
490 trasonic response in the configuration of *in vivo* cortical bone
491 characterization.

494 **A. Determination of a realistic range of variation of**
 495 **elastic bone properties**

496 In order to define numerical values for a realistic value
 497 of the gradient of the different material properties, it is nec-
 498 essary to determine the limiting values reached by each elas-
 499 tic property. Our approach consists in considering *in vitro*
 500 measurements published in Dong and Guo, 2004 and per-
 501 formed in 18 samples. It is assumed that these limiting val-
 502 ues for elastic properties are relevant for physiologic ranges
 503 of variations.

504 We assume that cortical bone is transverse isotropic.
 505 Transverse isotropy has been shown experimentally by dif-
 506 ferent authors (Dong and Guo, 2004; Reilly and Burnstein,
 507 1974; Rho, 1996) to be a realistic approximation of cortical
 508 bone degree of anisotropy.

509 Dong and Guo (2004) measured the homogenized bone
 510 properties by performing tensile and torsional tests with a
 511 mechanical testing system on 18 different human femoral
 512 bone specimens. The authors measured the values of the lon-
 513 gitudinal and transverse Young's moduli (E_L and E_T , respec-
 514 tively) as well as the values of the longitudinal shear modu-
 515 lus G_L . From these measurements and by assuming constant
 516 values of Poisson's ratio, the values of the different compo-
 517 nents of the stiffness-tensor corresponding to the values of
 518 E_L , E_T , and G_L measured in Dong and Guo, 2004 were ob-
 519 tained following the relationships given in the Appendix.

520 The value of the longitudinal Poisson's ratio ν_L is taken
 521 equal to 0.37 for all computations because it corresponds to
 522 the average value found in Dong and Guo, 2004. The value
 523 of the transverse Poisson's ratio ν_T is taken equal to 0.45
 524 following Eq. (A3) of the Appendix. The values of the stiff-
 525 ness coefficients corresponding to the mean values of the
 526 bone mechanical properties are referred to as "reference" set
 527 of parameters in what follows. The maximum and minimum
 528 values of the stiffness coefficients are obtained by consider-
 529 ing, respectively, the maximum and minimum values of E_L
 530 and E_T within the range of variation measured in Dong and
 531 Guo, 2004, which is a simple way to obtain a realistic range
 532 of variation for the stiffness coefficients in cortical bone.
 533 Furthermore, the elastic properties deduced from the ap-
 534 proach reported above were constrained to verify the thermo-
 535 dynamical stability conditions given in the Appendix by Eq.
 536 (A4).

537 We choose reference value of mass density ρ equal to
 538 1.722 g cm^{-3} , following the value taken in Macocco *et al.*,
 539 2006. In order to derive a realistic range of variation for mass
 540 density, we assume that the reference value is given by a
 541 porosity of 7%, which corresponds approximately to the
 542 mean porosity at the radius (Baron *et al.*, 2007). The porosity
 543 was assumed to vary between 3% and 15% (Bousson *et al.*,
 544 2001; Dong and Guo, 2004) and a rule of mixture leads to
 545 the range of variation of mass density.

546 **B. Modeling a gradient of material property**

547 The impact of a controlled gradient vector δ of any in-
 548 vestigated material property S on the response of the struc-
 549 ture studied is assessed. The scalar S corresponds to one of
 550 the stiffness coefficients c_{ij} of C or to mass density ρ . In each

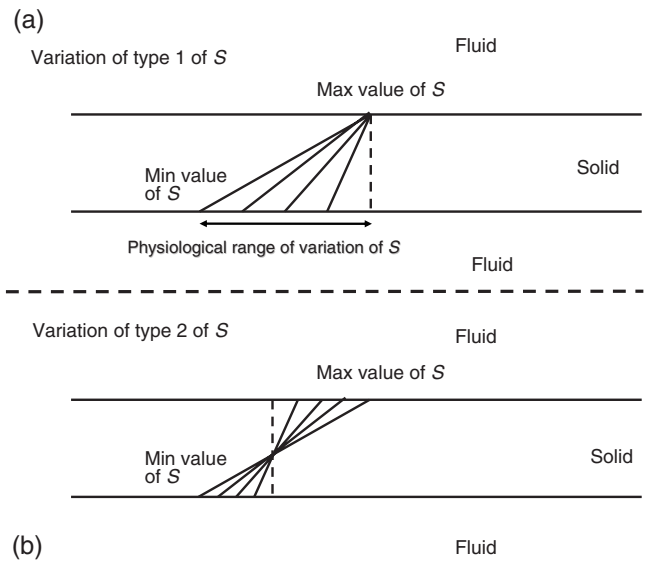


FIG. 3. Schematic representation of the two types of spatial variation considered for the material property S corresponding to the stiffness coefficient and to mass density. The solid lines in the solid layer indicate the spatial dependence of S . The dotted line in the solid layer indicates homogeneous material properties corresponding to the reference material properties. The variation of type 1 shown in (a) corresponds to a constant value at the bone-soft tissues' interface. The variation of type 2 shown in (b) corresponds to a constant value in the middle of the bone.

set of simulations, all the material properties are constant and equal to their reference value while S is subjected to the defined gradient.

The gradient vector $\delta = \text{grad } S = \delta x_3$ is assumed to be independent of x_1 in all cases, where x_3 is a unit vector along x_3 -axis and grad is the gradient operator acting on a scalar field. The quantity δ is always taken positive because the porosity is known to be higher in the endosteal part ($x_3 = d$) than in the periosteal part ($x_3 = 0$) of the bone. Moreover, only the simple situation of affine spatial variations of S is considered, corresponding to a constant value of δ . This affine spatial variation of S is chosen because the actual physiological spatial dependence of S remains unknown. Two different affine spatial dependencies of the studied material property are considered and are illustrated in Fig. 3. Associated gradient δ will be referred to as type 1 or 2.

Type 1. The gradient of type 1 is such that the physical property S takes the same value S_M at the upper interface $x_3 = 0$ of the solid plate for all values of the gradient δ . The quantity $S(x_3)$ is therefore given by

$$S(x_3) = S_M + \delta \times x_3, \quad (24)$$

where S_M is given by the maximal value of the material property S considered. The maximal value δ_M of δ is chosen so that $S(d)$ is equal to S_m , where S_m is given by the minimal value of S . The gradient δ_M is given by

$$\delta_M = \frac{(S_m - S_M)}{d}. \quad (25)$$

Type 2. The gradient of type 2 is such that the physical property S takes the same value at the middle $x_3 = d/2$ of the solid plate for all values of gradient δ . Furthermore, the

TABLE II. Mean value, maximum, and minimum values of the homogenized longitudinal and transversal Young moduli, of the four elastic constants and of mass density affecting the ultrasonic propagation in the framework of the model. These values are taken from [Dong and Guo \(2004\)](#).

Mechanical quantity	E_L (GPa)	E_T (GPa)	c_{11} (GPa)	c_{13} (GPa)	c_{33} (GPa)	$c_{55}=G_L$ (GPa)	ρ (g cm ⁻³)
Mean value (reference)	16.6	9.5	23.1	8.7	15.1	4.7	1.722
Minimum	13.4	6.5	17.6	5.1	9.1	3.3	1.66
Maximum	20.6	12.8	29.6	15.9	25.9	5.5	1.753

mean value of the property S is identical for all δ . The quantity $S(x_3)$ is given by

$$S(x_3) = \frac{(S_m + S_M)}{2} + \delta \times \left(x_3 - \frac{d}{2} \right). \quad (26)$$

The maximal value of δ is also given by Eq. (25), so that all values of $S(x_3)$ are again always comprised between S_m and S_M . Again, the maximal value δ_M of δ is given by Eq. (25).

Gradient of type 2 leads for all magnitudes of δ to a constant value of the spatial average of the gradient.

For both types of spatial variations, five different values of δ regularly distributed between 0 and δ_M are arbitrarily considered in the thickness.

Table II recalls the maximum, minimum, and mean measured values of E_L , E_T , and G_L as given by [Dong and Guo \(2004\)](#). Table II also shows the maximum, minimum, and mean values of the four components (c_{11} , c_{13} , c_{33} , and c_{55}) of the stiffness-tensor \mathbb{C} affecting wave propagation derived from Eq. (A3) of the Appendix.

In Table III, the minimal and maximal values of each variable corresponding to the realistic range of variation obtained (i) by considering the reference values of Table II and (ii) by verifying that the thermodynamical stability conditions are fulfilled. Values resulting from the stability conditions are marked with an asterisk.

In the simulations, the values of S_M and S_m are those reported in Table III.

C. Results and discussion

First of all, the method allows investigating the influence of the fluids on the ultrasonic response. In the case of the characterization of cortical bone, the two fluids f_1 and f_2 are different, which corresponds to an asymmetrical loading (see Sec. IV B): The fluid f_1 has been considered as muscle ($c_{f_1} = 1540$ m s⁻¹ and $\rho_{f_1} = 1.07$ g cm⁻³) and fluid f_2 as marrow ($c_{f_2} = 1450$ m s⁻¹ and $\rho_{f_2} = 0.9$ g cm⁻³). The properties of these two fluids are very close to those of water. The frequency spectrum of the reflection coefficient modulus has

been plotted for the *in vivo* configuration and compared with the result obtained for a cortical bone plate immersed in water for the ten profiles of mechanical properties (figure not shown). For homogeneous plates as for linearly graded plates, the two curves are very close; however, the modulus of the reflection coefficient at null-frequency is not null and the minimum values are greater than for water but obtained for the same frequency-thickness products. That is why all the following results have been calculated for a cortical bone plate immersed in water.

The reflection coefficient calculated with the Peano series of the matricant is sensitive to the variation of the properties gradient. As we consider that the osteoporosis entails a trabecularization of cortical bone from the endosteal side, the characterization of the gradient of the properties between the endosteal and periosteal regions may be an element of the diagnosis of the osteoporosis progress and of the therapeutic follow-up.

It is known that the gradient of the properties along the cortical thickness is due to the continuous variation of the porosity growing progressively from the periosteal to the endosteal region. From previous work, we know that the porosity influences all the stiffness coefficients ([Baron et al., 2007](#)). The frequency spectrum of the reflection coefficient has been plotted for the ten profiles presented in Fig. 3 applied to all the stiffness coefficients implied (c_{11} , c_{13} , c_{33} , and c_{55} and to the mass density ρ) (see Table III). The reflection coefficients have been calculated at an incident angle corresponding to the grazing-angle (critical angle for the longitudinal waves in the bone plate). For the two types of gradients, differences appear between all the gradients and the homogenized plates (corresponding to the maximum value for type 1 and to the average value for type 2) particularly on the location of the extrema values of the reflection coefficient modulus (see Fig. 4).

The increase in the gradient of properties shifts the minimum and maximum values forward high frequency-thickness products. However, the results (see Fig. 4) put on evidence that the behavior of the reflection coefficient modulus is sensibly the same for frequency-thickness products be-

TABLE III. The minimal and maximal values of each variable corresponding to the realistic range of variation obtained (i) by considering the reference values of Table II and (ii) by verifying that the thermodynamical stability conditions are fulfilled. Values resulting from the stability conditions are marked with an asterisk.

Material property S	c_{11} (GPa)	c_{13} (GPa)	c_{33} (GPa)	$c_{55}=G_L$ (GPa)	ρ (g cm ⁻³)
Realistic range [S_m, S_M] (reference)	[17.6, 29.6]	[5.1, 11.1*]	[11.8*, 25.9]	[3.3, 5.5]	[1.66, 1.753]

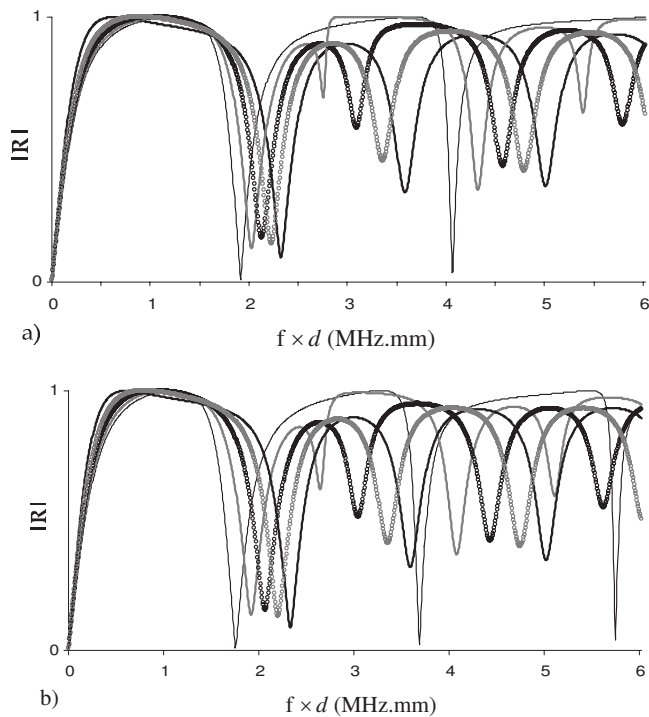


FIG. 4. Frequency spectrum of the reflection coefficient modulus for different properties' variations corresponding to (a) the five profiles of type 1 [Fig. 3(a)] and (b) the five profiles of type 2 [Fig. 3(b)].

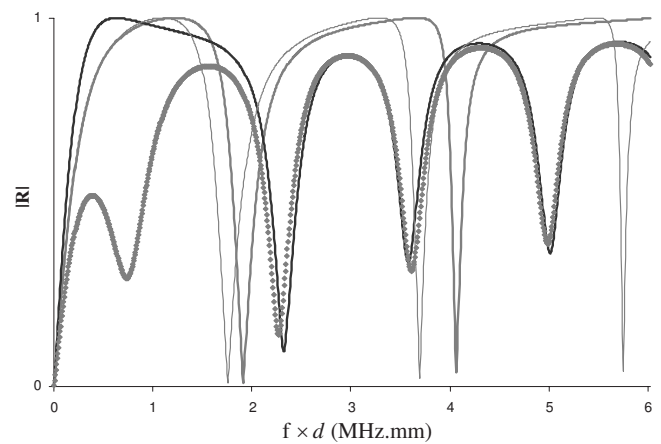


FIG. 5. Frequency spectrum of the reflection coefficient modulus for different properties' variations at grazing-incidence $\theta_1 = \theta_c$: for a homogeneous plate whose properties are those at $x_3=0$ at $\theta_c=21.1^\circ$ (thick gray line), for a homogeneous plate whose properties are the average of the properties through the plate at $\theta_c=23.6^\circ$ (thin gray line), for a plate with all its properties linearly varying at $\theta_c=21.1^\circ$ (black line), and for a plate with an affine variation of the c_{11} only at $\theta_c=21.1^\circ$ (gray lozenges).

656 tween 0 and 1.5 MHz mm. Beyond this value, the behavior is
657 clearly different. At sufficiently high frequency, the wave-
658 length is smaller and more sensitive to the affine variation of
659 the material properties. It is noteworthy that for a heteroge-
660 neous waveguide, the minima of the reflection coefficient
661 magnitude do not reach zero (except for null-frequency) but
662 end at a finite value and the changes in phase (not shown) are
663 not so rapid, which means that total transmission does not
664 take place in this situation.

665 The influence of the variation of each parameter (stiff-
666 ness coefficients) on the frequency spectrum of the reflection
667 coefficient has been investigated. This analysis has been led
668 for an incidence angle corresponding to the longitudinal
669 wave critical angle in the plate at $x_3=0$. This incidence cor-
670 responds to the generation and the propagation of the head
671 wave.

672 It appears that each of them has an impact on the reflec-
673 tion wave, but the leading term is c_{11} . The frequency spec-
674 trum of the reflection coefficient for a varying c_{11} in an affine
675 way is very close to the frequency spectrum of the reflection
676 coefficient obtained for the affine variation of all the material
677 properties (c_{ij} and ρ) and is the most different to the results
678 from homogenized plates (average value or maximum value)
679 compared to the frequency spectrum calculated for the *one-*
680 *parameter variation* of the other elastic parameters (c_{13} , c_{33} ,
681 and c_{55} , and ρ) (see Fig. 5). It is noteworthy that c_{11} is the
682 stiffness coefficient associated with the axial direction and
683 determining the speed of the head wave. So, it seems that the
684 head wave would be the indicator of the c_{11} gradient. It is
685 important to note that for homogenized plates (extremum
686 value or average) the frequency spectrum of the reflection
687 coefficient is really different from that of the plate with con-

tinuously varying properties. We infer that the approximation
by a homogeneous plate of cortical bone and all the more so
for the osteoporotic cortical bone (for which the gradient
would be greater) may induce bad interpretation of the ultra-
sonic response.

The authors do not know any results in literature about
the measurement of the variation of the porosity within the
cortical thickness. The assumption of an affine gradient is a
first step; other gradients may be investigated and the method
presented in this article would be applied in the same way for
non-affine gradients (Shuvalov *et al.*, 2005).

VI. CONCLUSION AND PERSPECTIVES

Stroh's (1962) sextic plate formalism has been employed
for analyzing the leaky Lamb waves in anisotropic heteroge-
neous plates immersed in fluids. This formalism and espe-
cially the polynomial form of the solution [see Eq. (15)]
presents several analytical and numerical advantages. First,
the low-frequency asymptotics are naturally assessed evalu-
ating only two or three terms in the series (Shuvalov *et al.*,
2005). The information thus collected is of major interest in
the analysis of the elastic behavior of waveguides (Baron *et*
al., 2008). Second, the polynomial form makes the numerical
evaluation of the solution faster. Indeed, the polynomial co-
efficients are independent of the frequency, so they are cal-
culated for a fixed slowness value and stored. When the fre-
quency varies, there is no need to recalculate the polynomial
coefficients, it comes to a polynomial evaluation whose co-
efficients are perfectly known, which is time-saving.

The Peano series of the matricant is a method that keeps
the continuity of the profiles and so, the authenticity of the
problem. One of the key points for methods based on multi-
layered media to deal with FGM is to relevantly discretize
the properties' profiles. The choice of the discretization may
lead to some errors especially in the evaluation of the reso-
nances.

723 This elegant mathematical tool is also very adaptative to
724 different physical problems. In the case studied—a fluid-
725 loaded plane waveguide—the anisotropy, the heterogeneity
726 (continuous or discontinuous variation of properties) and the
727 asymmetric fluid-loading are taken into account without
728 changing the resolution scheme.

729 Further work needs to be done to relate the results pre-
730 sented in this paper to dispersion curves and propagation of
731 transient and heterogeneous waves in a fluid-loaded continu-
732 ously heterogeneous waveguide.

733 Furthermore, from this study, the transient response of a
734 fluid-loaded plate is considered. The frequency spectrum of
735 the reflection coefficient is calculated for incidences be-
736 tween the normal and critical incidences for compression
737 waves in the fluid f_1 . Thus, the plate transfer function is
738 calculated in the Fourier domain (x_1 -wavenumber, fre-
739 quency): $\hat{R}(k_1, x_3; \omega)$. A double inverse Fourier transform on
740 (k_1, ω) is applied on $\hat{R}(k_1, x_3; \omega)$ to transform into the space-
741 time domain; the temporal signals can be obtained at differ-
742 ent points along the propagation x_3 -axis: $R(x_1, x_3; t)$.

743 Lastly, the formalism presented here is well-adapted to
744 deal with wave propagation in anisotropic tubes with radial
745 property gradients (Shuvalov, 2003). The wave equation
746 keeps the same form as Eq. (12), the state-vector is expressed
747 from the displacement and traction components in the cylin-
748 drical basis, and the matrix \mathbf{Q} depends on the radial position
749 r ($\mathbf{Q}=\mathbf{Q}(r)$). In cylindrical homogeneous structures, taking
750 into account an anisotropy more important than transverse
751 isotropy is fussy because there is no analytical solution to the
752 “classical” wave equation (second-order differential equa-
753 tion). Stroh’s (1962) formalism, upon which the Peano ex-
754 pansion of the matricant is based, is a promising alternative
755 solution that allows considering altogether the geometry
756 (cylinder), the anisotropy, and the heterogeneity (radial prop-
757 erty gradients) of a structure.

758 APPENDIX: THERMODYNAMICS STABILITY 759 CONDITIONS AND STIFFNESS COEFFICIENTS

760 Hooke’s law is written under the form $\sigma_{ij}=C_{ijkl}\epsilon_{kl}$ for
761 ($i, j, k, \ell=1, \dots, 3$), where σ is the stress-tensor, ϵ is the
762 strain-tensor, and C is the fourth-order stiffness-tensor. In the
763 transversely isotropic case, with $(\mathbf{x}_2, \mathbf{x}_3)$ as isotropic plane,
764 the stiffness-tensor is expressed as a stiffness matrix (using
765 Voigt’s notation)

$$766 \quad \mathbf{C} = \begin{pmatrix} c_{11} & c_{13} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{33} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{55} \end{pmatrix}. \quad (A1)$$

767 We introduce the matrix \mathbf{S} , the inverse of the matrix \mathbf{C} . It is
768 expressed by

$$769 \quad \mathbf{S} = \begin{pmatrix} 1/E_L & -\nu_L/E_L & -\nu_L/E_L & 0 & 0 & 0 \\ -\nu_L/E_L & 1/E_T & -\nu_T/E_T & 0 & 0 & 0 \\ -\nu_L/E_L & -\nu_T/E_T & 1/E_L & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_L & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_L \end{pmatrix}. \quad (A2)$$

with $E_{L,T}$, the longitudinal (L) and transverse (T) Young’s
770 moduli; $\nu_{L,T}$, the longitudinal (L) and transverse (T) Pois-
771 son’s ratios; and $G_{L,T}$, the longitudinal (L) and transverse (T)
772 shear moduli. By inverting Eq. (A1) and identifying it with
773 Eq. (A2), we obtain the following relations: 774

$$775 \quad E_L = \frac{c_{11}c_{33} - 2c_{13}^2 + c_{11}c_{23}}{c_{33} + c_{23}}, \quad \nu_L = \frac{c_{13}}{c_{33} + c_{23}},$$

$$776 \quad E_T = \frac{c_{11}(c_{33}^2 - c_{23}^2) + 2c_{13}^2(c_{23} - c_{33})}{c_{11}c_{33} - c_{13}^2}, \quad \nu_T = \frac{c_{11}c_{23} - c_{13}^2}{c_{11}c_{33} - c_{13}^2},$$

$$777 \quad G_T = c_{44}, \quad G_L = c_{55}. \quad (A3)$$

Knowing the stiffness coefficient values, we can verify
778 if the thermodynamical stability conditions are satisfied as
779 follows: 780

$$781 \quad E_L > 0, \quad E_T > 0, \quad -1 < \nu_T < 1,$$

$$782 \quad \frac{(1 - \nu_T)E_L}{2} - \nu_L^2 > 0, \quad G_L > 0. \quad (A4)$$

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