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# Ergodic pairs for degenerate pseudo Pucci's fully nonlinear operators

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## Abstract

We study the ergodic problem for fully nonlinear operators which may be singular or degenerate when at least one of the components of the gradient of solutions vanishes. We prove the convergence of both explosive solutions and solutions of Dirichlet problems for approximating equations. We further characterize the ergodic constant as the infimum of constants for which there exist bounded sub solutions. As intermediate results of independent interest, we prove a priori Lipschitz estimates depending only on the norm of the zeroth order term, and a comparison principle for equations having no zero order terms.

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## 1 Introduction

This article deals with the existence of solutions to the ergodic problem associated to the "pseudo Pucci's" operators.

The history of the ergodic problem begins with the seminal paper of Lasry and Lions in 1989, [22] which considers the Laplacian case . More precisely  $\Omega$  being an open bounded  $C^2$  domain in  $\mathbb{R}^N$ , for  $\beta \in ]1, 2]$  and  $f$  being continuous in  $\Omega$  and bounded,  $(u, c)$  is a solution of

the ergodic problem if

$$\begin{cases} -\Delta u + |\nabla u|^\beta = f + c & \text{in } \Omega \\ u = +\infty & \text{on } \partial\Omega \end{cases}$$

1 The results in [22] are extended to the case of the  $p$ Laplace operator by Leonori Porretta in  
 2 [23]. In [15] the authors consider the case where the Laplacian is replaced by  $-\operatorname{div}(A(x)\nabla u)$   
 3 where  $A$  is positively definite and regular enough. Recently in [13], we considered the case  
 4 where the leading term is Fully Non Linear elliptic, singular or degenerate, on the model of  
 5  $-|\nabla u|^\alpha F(D^2u)$ , where  $\alpha > -1$  and  $F$  is fully non linear elliptic and positively homogeneous  
 6 of degree 1.

7 In the present paper, we will assume that  
 -There exist some constants  $a < A$ , so that for any  $M \in S, N \in S, N \geq 0$ ,

$$atr(N) \leq F(M + N) - F(M) \leq AtrN, \quad (1.1) \text{fnl}$$

where  $S$  is the space of symmetric matrices on  $\mathbb{R}^N$ . We will also assume that  $F$  is positively  
 homogenous of degree 1. We define the operator

$$F(\nabla u, D^2u) = F(\Theta_\alpha(\nabla u)D^2u\Theta_\alpha(\nabla u)), \text{ where } \Theta_\alpha(p) := \operatorname{Diag}(|p_i|^{\frac{\alpha}{2}}) \quad (1.2) \text{defF}$$

8  $\alpha \geq 0$ , and  $F$  satisfies (1.1), and we are interested in the following :

Let  $\beta \in ]\alpha + 1, \alpha + 2]$ , find  $(u, c)$  which is a solution of the "ergodic problem "

$$\begin{cases} -F(\nabla u, D^2u) + |\nabla u|^\beta = f + c & \text{in } \Omega \\ u = +\infty & \text{on } \partial\Omega. \end{cases}$$

Note that this equation presents a new type of degeneracy, since the leading term degenerates  
 on every point where at least one derivative  $\partial_i u$  is zero. When  $F(X) = \operatorname{tr}X$ , the operator is  
 nothing else than the anisotropic  $p$ -Laplacian for  $p = \alpha + 2$  ( also called pseudo  $p$ Laplacian).  
 Let us recall that the equation of the "anisotropic  $p$ Laplacian, " called also " orthoptic  
 $p$ Laplacian" is

$$-\sum_i \partial_i (|\partial_i u|^{p-2} \partial_i u) = f. \quad (1.3) \text{plap}$$

9 This equation can easily be solvable, for convenient  $f$ , by standard methods in the calculus of  
 10 variations. But the regularity results are much more difficult to obtain. Lipschitz regularity is

1 proved in the singular case in [20] while the case  $p > 2$  is treated in [4] for a more degenerate  
 2 equation including the pseudo  $p$ Laplacian case. In [18], [10] the authors consider viscosity  
 3 solutions for the fully non linear extension of the pseudop-Laplacian, say the case where the  
 4 operator is  $-F(\Theta_\alpha(\nabla u)D^2u\Theta_\alpha(\nabla u))$ , and  $\alpha > 0$ . More general anisotropic fully non linear  
 5 degeneracy is treated in [19]. In the variational case, one important result can be found in  
 6 [3].

In [10] the Lipschitz interior regularity of the solutions is obtained as a corollary of the  
 following estimate between  $u$  sub-solution and  $v$  super-solution of the equation with some  
 eventually different right hand side, say if  $\Omega = B(0, 1)$ , for all  $r > 0$ ,  $r < 1$ , there exists  $c_r$  so  
 that for all  $(x, y) \in B(0, r)^2$

$$u(x) - v(y) \leq \sup(u - v) + c_r|x - y|.$$

7 This Lipschitz estimate is extended to the equations presenting an Hamiltonian of the form  
 8  $b(x)|\nabla u|^\beta$  with  $\beta \in [\alpha + 1, \alpha + 2]$  when the sub-and super-solutions are bounded. This is  
 9 done in Section 2. This estimate does not permit to prove existence's results for the ergodic  
 10 problem : This existence is generally obtained by passing to the limit in an equation with  
 11 boundary conditions coming to  $+\infty$ , and then requires Holder's or Lipschitz estimates for  
 12 globally unbounded solution. However, as in [22], [15], [13], the presence of an Hamiltonian  
 13 "superlinear" with a good sign, permits to get an interior Lipschitz estimates for the solutions,  
 14 which does not require that the solution be bounded, but that the zero order term be so.  
 15 This is done in Section 3.

16 The main result of this paper is then

(sympaetpas) **Theorem 1.1.** *Suppose that  $f$  is bounded and locally Lipschitz continuous in  $\Omega$ , and that  $F$   
 satisfies (1.2), and  $\alpha \geq 0$ . Consider the Dirichlet problems*

$$\begin{cases} -F(\nabla u, D^2u) + |\nabla u|^\beta = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4) \text{eq3}$$

and, for  $\lambda > 0$ ,

$$\begin{cases} -F(\nabla u, D^2u) + |\nabla u|^\beta + \lambda|u|^\alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.5) \text{eq45}$$

1 The following alternative holds.

2 1. Suppose that there exists a bounded sub solution of (1.4). Then the solution  $u_\lambda$  of  
 3 (1.5) satisfies:  $(u_\lambda)$  is bounded and uniformly converging up to a sequence  $\lambda_n \rightarrow 0$  to a  
 4 solution of (1.4).

5 2. Suppose that there is no solution for the Dirichlet problem (1.4). Suppose in addition  
 6 that  $\alpha \geq 2$ . Then,  $(u_\lambda)$  satisfies, up to a sequence  $\lambda_n \rightarrow 0$  and locally uniformly in  $\Omega$ ,

?(a)? (a)  $u_\lambda \rightarrow -\infty$ ;

?(b)? (b) there exists a constant  $c_\Omega \geq 0$  such that  $\lambda|u_\lambda|^\alpha u_\lambda \rightarrow -c_\Omega$ ;

?(c)? (c)  $c_\Omega$  is an ergodic constant and  $v_\lambda = u_\lambda + |u_\lambda|_\infty$  converges to a solution of the ergodic  
 problem

$$\begin{cases} -F(\nabla v, D^2 v) + |\nabla v|^\beta = f + c_\Omega & \text{in } \Omega \\ v = +\infty & \text{on } \partial\Omega \end{cases} \quad (1.6) \text{ergodic}$$

9 whose minimum is zero.

10 Note that, even when a sub-solution to (1.4) exists, there exists an ergodic pair, this will  
 11 be proved in Theorem 4.3, section 4.3

In a second time, we prove that the ergodic constant can be characterized by an inf-formula  
 analogous to the one which defines the principal eigenvalues for fully nonlinear operators.  
 Following [8], [25], we define

$$\mu^* = \inf\{\mu : \exists \varphi \in \mathcal{C}(\bar{\Omega}), -F(\nabla \varphi, D^2 \varphi) + |\nabla \varphi|^\beta \leq f + \mu\}.$$

For the following Theorem we introduce some new assumption :

$$C(x) = ((\gamma + 1)F(\nabla d, \nabla d(x) \otimes \nabla d(x)))^{\frac{1}{\beta - \alpha - 1}} \gamma^{-1} \text{ is } \mathcal{C}^2 \text{ in a neighborhood of } \partial\Omega. \quad (1.7) \text{C(x)}$$

12 In particular (1.7) is satisfied when the boundary is  $\mathcal{C}^3$  and  $F$  is  $\mathcal{C}^2$ . But it is automatically  
 13 satisfied in the case where  $F$  is one of the Pucci's operators. We will prove in Theorem 4.3  
 14 that under the assumption (1.7), any ergodic function is equivalent near the boundary to  
 15  $C(x)d^{-\gamma}$  and this allows to prove the uniqueness of the ergodic constant in Theorem 1.2 :

16 **Theorem 1.2.** Suppose that  $f$  is bounded and locally Lipschitz continuous in  $\Omega$ , and that  
 17  $F$  satisfies (1.2), (1.1) and (1.7). Suppose that  $\alpha \geq 2$ . Let  $c_\Omega$  be an ergodic constant for  
 1 problem (1.6); then:

- 2     1.  $c_\Omega$  is unique;
- 3     2.  $c_\Omega = \mu^*$ ;
- 4     3. the map  $\Omega \mapsto c_\Omega$  is nondecreasing with respect to the domain, and continuous;
- 5     4. if either  $\alpha = 0$  or  $\alpha \neq 0$  and  $\sup_\Omega f + c_\Omega < 0$ , then  $\mu^*$  is not achieved. Moreover, if  
6        $\Omega' \subset\subset \Omega$ , then  $c_{\Omega'} < c_\Omega$ .

7     Theorems 1.1 and 1.2 are obtained by means of several intermediate results, most of  
8     which are of independent interest. A first fundamental tool is an interior Lipschitz estimate  
9     for solutions of equation (1.5) that does not depend on the  $L^\infty$  norm of the solution but only  
10    on the norm of the zero order term. This is done in section 3.

11    The uniqueness in Theorem 1.2 is obtained, by using the results in Section 6 : In this  
12    part, we give a comparison theorem for sub- and super- solutions of equation (1.4), in which  
13    zero order terms are lacking. The change of equation that allows to prove the comparison  
14    principle of Theorem 6.1 is standard, it has already been employed in [23] and [13].

15    Let us finally remark that the question of uniqueness (up to constants) of the ergodic  
16    function is open. We recall that the usual proof for linear operators, see [22, 25], relies on  
17    the strong comparison principle, which does not hold for degenerate operators. Let us also  
18    recall that for  $p$ -laplacian operators the uniqueness of the ergodic function is obtained in [23]  
19    for  $p \geq 2$ , in [13] for  $p \leq 2$  and under the condition  $\sup_\Omega f + c < 0$ . The same result is  
20    obtained in [13]. In all these degenerate cases, the  $C^1$  regularity is a crucial step, see [14], in  
21    the case considered in [13]. In the present context of operators which degenerate as soon as  
22    one derivative  $\partial_i u$  is zero, the  $C^1$  regularity is known only in the case  $N = 2$ , [2], and more  
23    precisely only for equation(1.3) and for  $f = 0$ ,  $p = \alpha + 2$  . The method that the authors  
24    employ in [2] is very specific to the variational setting , since it relies essentially on very sharp  
1    Moser's iterations.

2 **Remark 1.3.** *In the equations considered here, we will use the euclidian norm for the gra-*  
 3 *dient term  $|\nabla u|^\beta$ , but other norms should lead to analogous results, with obvious changes.*

4 **Remark 1.4.** *The threshold value  $\alpha = 2$  appears in a lot of papers treating of these anisotropic*  
 5 *equations, let us cite in a non exhaustive manner [4], [26], The restriction  $\alpha \leq 2$  or in some*  
 6 *cases  $\alpha \geq 2$  has been in posterior papers relaxed. We are convinced that the results restricted*  
 7 *by this condition here, hold true without it, the fact that we cannot obtain them here is a*  
 8 *lacking of the method employed.*

## 9 2 Existence results for the Dirichlet problem

### 10 Notations

- 11 • We use  $d(x)$  to denote a  $\mathcal{C}^2$  positive function in  $\Omega$  with coincides with the distance  
 12 function from the boundary in a neighborhood of  $\partial\Omega$
- 13 • For  $\delta > 0$ , we set  $\Omega_\delta = \{x \in \Omega : d(x) > \delta\}$
- We denote by  $\mathcal{M}^+, \mathcal{M}^-$  the Pucci's operators with ellipticity constants  $a, A$ , namely,  
 for all  $M \in \mathcal{S}$ ,

$$\mathcal{M}^+(M) = A \operatorname{tr}(M^+) - a \operatorname{tr}(M^-)$$

$$\mathcal{M}^-(M) = a \operatorname{tr}(M^+) - A \operatorname{tr}(M^-)$$

and we often use that, as a consequence of (1.1), for all  $M, N \in \mathcal{S}$  one has

$$\mathcal{M}^-(N) \leq F(M + N) - F(M) \leq \mathcal{M}^+(N).$$

14 In some parts of the paper we will need the following properties of  $F$  which are an easy  
 1 consequence of the assumptions (1.1) and (1.2) and of the positive homogeneity:

( $P_1$ ) There exists  $c$  so that for any  $(p, q) \in \mathbb{R}^N$ , and  $M \in \mathcal{S}$  one has

$$|F(p, M) - F(q, M)| \leq c(|p|^{\frac{\alpha}{2}} + |q|^{\frac{\alpha}{2}}) \sum_1^N \left| |p_i|^{\frac{\alpha}{2}} - |q_i|^{\frac{\alpha}{2}} \right| |M|.$$

(P<sub>2</sub>) There exists  $c$  so that for any  $(p, q) \in \mathbb{R}^N$ , and  $M \in S$ , diagonal

$$|F(p, M) - F(q, M)| \leq c \sum_1^N ||p_i|^\alpha - |q_i|^\alpha| |M|.$$

2 The main result of this section is the following :

(exidir)  
3 **Theorem 2.1.** *Suppose that  $\alpha > 0$ ,  $\beta \leq \alpha + 2$  and  $F$  satisfies, that  $f$  and  $b$  are bounded .*  
4 *Suppose that  $\lambda > 0$ . Then there exists a unique  $u$  which satisfies*

$$\begin{cases} -F(\nabla u, D^2 u) + b(x)|\nabla u|^\beta + \lambda|u|^\alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

5 *Furthermore  $u$  is Lipschitz continuous, with some Lipschitz bound depending on  $|u|_\infty, |f|_\infty, |b|_\infty$*   
6 *in the case  $\beta < \alpha + 2$ , and requires that  $b$  be Lipschitz in the case  $\beta = \alpha + 2$ .*

7 It is classical that this existence's result is obtained by exhibiting convenient sub- and  
8 super-solutions, proving a Lipschitz estimate between them, a comparison result, and finally  
9 applying Perron's method adapted to the present context. We will not give the details of all  
10 the proofs, since the ideas here are a mixing of the arguments in [9], [12]. We enounce these  
11 results :

(theolip) **Theorem 2.2.** *Suppose that  $u$  is a USC bounded by above viscosity subsolution of*

$$-F(\nabla u, D^2 u) + b(x)|\nabla u|^\beta \leq g \quad \text{in } B_1$$

*and  $v$  is a LSC bounded by below viscosity supersolution of*

$$-F(\nabla v, D^2 v) + b(x)|\nabla v|^\beta \geq f \quad \text{in } B_1,$$

*with  $f$  and  $g$  continuous and bounded, and  $b$  is continuous, and Holder continuous when  $\beta = \alpha + 2$ . Then, for all  $r < 1$ , there exists  $c_r$  such that for all  $(x, y) \in B_r^2$*

$$u(x) - v(y) \leq \sup_{B_1} (u - v) + c_r |x - y|.$$

1 In order to prove Theorem 2.2 we first need the following Hölder's estimate:

(lem1) **Lemma 2.3.** *Under the hypothesis of Theorem 2.2, for any  $\gamma \in (0, 1)$ , there exists  $c_{r,\gamma} > 0$*   
*such that for all  $(x, y) \in B_r^2$*

$$u(x) - v(y) \leq \sup_{B_1} (u - v) + c_{r,\gamma} |x - y|^\gamma. \quad (2.1) \text{ holdest}$$



*Proof of Lemma 2.3.* We borrow ideas from [21], [5], [11], [10]. Fix  $x_o \in B_r$ , and define

$$\phi(x, y) = u(x) - v(y) - \sup_{B_1} (u - v) - M|x - y|^\gamma - L(|x - x_o|^2 + |y - x_o|^2)$$

with  $L = \frac{16(\sup u - \inf v)}{(1-r)^2}$  and  $M = \frac{4(\sup u - \inf v)}{\delta^\gamma}$ ,  $\delta$  will be chosen later small enough depending only on the data and on universal constants. We want to prove that  $\phi(x, y) \leq 0$  in  $B_1$  which will imply the result, taking first  $x = x_o$  and making  $x_o$  vary.

We argue by contradiction and suppose that  $\sup_{B_1} \phi(x, y) > 0$ . By the previous assumptions on  $M$  and  $L$  the supremum is achieved on  $(\bar{x}, \bar{y})$  which belongs to  $B_{\frac{1+r}{2}}$  and it is such that  $0 < |\bar{x} - \bar{y}| \leq \delta$ .

By Ishii's Lemma [16], [17], for all  $\epsilon > 0$  there exist  $X$  and  $Y$  in  $S$  such that  $(q^x, X) \in \bar{J}^{2,+} u(\bar{x})$ ,  $(q^y, -Y) \in \bar{J}^{2,-} v(\bar{y})$  with

$$\begin{aligned} q^x &= \gamma M |\bar{x} - \bar{y}|^{\gamma-2} (\bar{x} - \bar{y}) + 2L(\bar{x} - x_o), \\ q^y &= \gamma M |\bar{x} - \bar{y}|^{\gamma-2} (\bar{x} - \bar{y}) - 2L(\bar{y} - x_o), \end{aligned}$$

with

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 2 \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}$$

and  $B = D^2(|\cdot|^\gamma)$ . Hence

$$-F(q^x, X) + b(\bar{x})|q^x|^\beta \leq g(\bar{x}), \quad -F(q^y, -Y) + b(\bar{y})|q^y|^\beta \geq f(\bar{y}) \quad (2.2) \text{ ?equaz}$$

Using the computations in [10] one gets the existence of  $c_1$  so that

$$F(q^x, X) \leq F(q^y, -Y) - c_1 M^{1+\alpha} |\bar{x} - \bar{y}|^{(\gamma-1)(\alpha+1)-1}$$

So to conclude in the present case it is sufficient to obtain that for  $\delta$  small,  $|b(x)|q_x|^\beta - b(y)|q_y|^\beta$  is small with respect to  $M^{1+\alpha} |\bar{x} - \bar{y}|^{(\gamma-1)(\alpha+1)-1}$ . This is obtained using

1) If  $\beta < \alpha + 2$

$$\begin{aligned} |b(x) - b(y)||q_x|^\beta &\leq 2|b|_\infty M^\beta |\bar{x} - \bar{y}|^{(\gamma-1)\beta} \\ &\leq 2|b|_\infty |\bar{x} - \bar{y}|^{2+\alpha-\beta} (M^{1+\alpha} |\bar{x} - \bar{y}|^{(\gamma-1)(\alpha+1)-1}) \\ &\ll M^{1+\alpha} |\bar{x} - \bar{y}|^{(\gamma-1)(\alpha+1)-1}. \end{aligned}$$

2) If  $\beta = \alpha + 2$  we just use the continuity of  $b$

$$|b(x) - b(y)||q_x|^\beta \leq o(1) M^{1+\alpha} |\bar{x} - \bar{y}|^{(\gamma-1)(\alpha+1)-1}.$$

We use also

$$|b(y)(|q_x|^\beta - |q_y|^\beta)| \leq |b|_\infty |q_x + q_y| cM^{\beta-1} |\bar{x} - \bar{y}|^{(\gamma-1)(\beta-1)}$$

2 which is also small with respect to  $M^{1+\alpha} |\bar{x} - \bar{y}|^{(\gamma-1)(\alpha+1)-1}$ . We can then conclude to a  
3 contradiction, since one has

$$\begin{aligned} -g(\bar{x}) &\leq F(q^x, X) - b(\bar{x})|q^x|^\beta \\ &\leq F(q^y, Y) - cM^{1+\alpha} |\bar{x} - \bar{y}|^{\gamma-2+(\gamma-1)\alpha} - b(\bar{y})|q^y|^\beta \\ &\leq -f(\bar{y}) - c\delta^{-\gamma(1+\alpha)} |\bar{x} - \bar{y}|^{\gamma(\alpha+1)-(2+\alpha)} \\ &\leq -f(\bar{y}) - c\delta^{-(2+\alpha)}. \end{aligned}$$

4 This is a contradiction with the fact that  $f$  and  $g$  are bounded, as soon as  $\delta$  is small  
5 enough.  $\square$

6 *Proof.* of Theorem 2.2

For fixed  $\tau \in (0, \frac{\inf(1, \alpha)}{2})$ ,  $\tau < \alpha + 2 - \beta$  when  $\alpha + 2 - \beta > 0$  and  $\tau < \gamma_b$  where  $\gamma_b$  is some  
Holder's exponent for  $b$ , and  $\beta = \alpha + 2$ . Let  $s_o = (1 + \tau)^{\frac{1}{\tau}}$ , and define for  $s \in (0, s_o)$

$$\omega(s) = s - \frac{s^{1+\tau}}{2(1+\tau)}, \quad (2.3) \{?\}$$

7 which we extend continuously after  $s_o$  by a constant.

8 Note that  $\omega(s)$  is  $\mathcal{C}^2$  on  $s > 0$ ,  $s < s_o$ , satisfies  $\omega' > \frac{1}{2}$ ,  $\omega'' < 0$  on  $]0, 1[$ , and  $s > \omega(s) \geq \frac{s}{2}$ .

As before in the Hölder case, with  $L = \frac{16(\sup u - \inf v)}{(1-r)^2}$  and  $M = \frac{4(\sup u - \inf v)}{\delta}$ , we define

$$\phi(x, y) = u(x) - v(y) - \sup_{B_1} (u - v) - M\omega(|x - y|) - L(|x - x_o|^2 + |y - x_o|^2).$$

9 Classically, as before, we suppose that there exists a maximum point  $(\bar{x}, \bar{y})$  such that  $\phi(\bar{x}, \bar{y}) >$   
10  $0$ , then by the assumptions on  $M$ , and  $L$ ,  $\bar{x}, \bar{y}$  belong to  $B(x_o, \frac{1+r}{2})$ , hence they are interior  
11 points. This implies, using (2.1) in Lemma 2.3 with  $\gamma < 1$  such that  $\frac{\gamma}{2} > \frac{\tau}{\inf(1, \alpha)}$  that, for  
12 some constant  $c_r$ ,

$$L|\bar{x} - x_o|^2 \leq c_r |\bar{x} - \bar{y}|^\gamma. \quad (2.4) \boxed{\text{hh}}$$

1 and then one has  $|\bar{x} - x_o| \leq \left(\frac{c_r}{L}\right)^{\frac{1}{2}} |\bar{x} - \bar{y}|^{\frac{\gamma}{2}}$ .

Furthermore, there exist  $X$  and  $Y$  in  $S$  such that  $(q^x, X) \in \bar{J}^{2,+}u(\bar{x})$ ,  $(q^y, -Y) \in \bar{J}^{2,-}v(\bar{y})$  with

$$q^x = M\omega'(|\bar{x} - \bar{y}|)\frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} + L(\bar{x} - x_o), \quad q^y = M\omega'(|\bar{x} - \bar{y}|)\frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} - L(\bar{y} - x_o).$$

and

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 2 \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}$$

and  $B(x) = D^2(\omega(|x|))$ . Following the computations in [10] ( for this we need among other things (2.4)), one gets the existence of  $c_1$  so that

$$F(q^x, X) \leq F(q^y, -Y) \leq -c_1 M^{1+\alpha} |\bar{y} - \bar{x}|^{\tau-1}$$

2 So to conclude we need to prove that  $|b(\bar{x})|q_x|^\beta - b(\bar{y})|q_y|^\beta|$  is small with respect to  $M^{1+\alpha}|\bar{y} - \bar{x}|^{\tau-1}$ . This is obtained as in the Holder's case by using (the constant  $c$  can vary from one  
3  
4 line to another)

1) If  $\beta < \alpha + 2$

$$|b(\bar{x}) - b(\bar{y})||q_x|^\beta \leq 2|b|_\infty M^\beta \ll M^{1+\alpha} |\bar{x} - \bar{y}|^{\tau-1}$$

5 by the assumption  $\tau < 2 + \alpha - \beta$ .

2) If  $\beta = \alpha + 2$

$$|b(\bar{x}) - b(\bar{y})||q_x|^\beta \leq c|\bar{x} - \bar{y}|^{\gamma_b} M^{2+\alpha} \leq c|\bar{x} - \bar{y}|^{\gamma_b - \tau} (M^{1+\alpha} |\bar{y} - \bar{x}|^{\tau-1}).$$

We finally use

$$|b(\bar{y})| \left| |q^y|^\beta - |q^x|^\beta \right| \leq c|b|_\infty |q^y + q^x| M^{\beta-1} \leq cM^{\beta-1}.$$

6 So the expected result holds by the choice of  $M$  respectively to  $\delta$ . Once more as in the proof  
7 of lemma 2.3 one can conclude to a contradiction.

8

□

9 It is clear that Theorem 2.2 can be extended to the case where  $\Omega$  replaces  $B(0, 1)$  and  
10  $\Omega' \subset\subset \Omega$  replaces  $B(0, r)$ . Furthermore it is not difficult to see as in [12] that we have the  
1 following Lipschitz estimate up to the boundary :

If  $u$  is a sub-solution of

$$-|\nabla u|^\alpha F(D^2u) + b(x)|\nabla u|^\beta \leq f,$$

and  $v$  is a super-solution of

$$-|\nabla v|^\alpha F(D^2 v) + b(x)|\nabla v|^\beta \geq g,$$

and  $u \leq 0$ ,  $v \geq 0$  on  $\partial\Omega$ , then there exists  $c$  so that for any  $(x, y) \in \bar{\Omega}^2$

$$u(x) - v(y) \leq \sup(u - v) + c|x - y|.$$

(thcompar) **Theorem 2.4.** *Suppose that  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ . Suppose that  $\alpha, \beta$  are as above, as well as  $F$ . Let  $b$  be Hölder continuous. Let  $\gamma$  be a non decreasing continuous function such that  $\gamma(0) = 0$ . Suppose that  $u$  is a sub-solution of*

$$-F(\nabla u, D^2 u) + b(x)|\nabla u|^\beta + \gamma(u) \leq g$$

and

$$-F(\nabla v, D^2 v) + b(x)|\nabla v|^\beta + \gamma(v) \leq f$$

2 *with  $g \leq f$ , both of them being continuous and bounded. Then if  $g < f$  in  $\Omega$  or  $\gamma$  is increasing,*  
3 *if  $u \leq v$  on  $\partial\Omega$ ,  $u \leq v$  in  $\Omega$ .*

4 *Proof.* of Theorem 2.4

We use classically the doubling of variables. Suppose that  $u > v$  somewhere, then consider

$$\psi_j(x, y) = u(x) - v(y) - \frac{j}{2}|x - y|^2.$$

Then for  $j$  large enough the supremum of  $\psi_j$  is positive and achieved on a pair  $(x_j, y_j) \in \Omega^2$ , both of them converging to some maximum point  $\bar{x}$  for  $u - v$ . Since  $(x_j, y_j)$  converges to  $(\bar{x}, \bar{x})$ , both of them belong, for  $j$  large enough, to some  $\Omega' \subset\subset \Omega$ , independant on  $j$ . Furthermore, using the Lipschitz estimate proved in Theorem 2.2 :

$$\sup(u - v) \leq u(x_j) - v(y_j) - \frac{j}{2}|x_j - y_j|^2 \leq \sup(u - v) + c|x_j - y_j| - \frac{j}{2}|x_j - y_j|^2,$$

5 from this one derives that  $j|x_j - y_j|$  is bounded.

Using Ishii's lemma , [16], [17], there exist  $X_j$  and  $Y_j$  in  $S$  such that  $(j(x_j - y_j), X_j) \in \bar{J}^{2,+} u(x_j)$ ,  $(j(x_j - y_j), -Y_j) \in \bar{J}^{2,-} v(y_j)$  and  $X_j, Y_j$  satisfy

$$-3j \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} \leq 3j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

1 We obtain, denoting the modulus of continuity of  $b$  by  $\omega(b, \delta)$  :

$$\begin{aligned} g(x_j) - \gamma(u(x_j)) &\geq -F(j(x_j - y_j), X_j) + b(x_j)|j(x_j - y_j)|^\beta \\ &\geq -F(j(x_j - y_j), -Y_j) + b(y_j)|j(x_j - y_j)|^\beta + c\omega(b, |x_j - y_j|) \\ &\geq f(y_j) + o(1) - \gamma(v(y_j)). \end{aligned}$$

By passing to the limit, one gets on the point  $\bar{x}$  limit of a subsequence of  $x_j$

$$g(\bar{x}) - \gamma(u(\bar{x})) \geq f(\bar{x}) - \gamma(v(\bar{x}))$$

2 and in both cases we obtain a contradiction.

3

□

*Proof.* of Theorem 2.1 We just give the hints to emphasize the difference with the operators and the results in [12]. We begin by exhibit a sub- and a super-solution which are zero on the boundary. Suppose first that  $\beta < \alpha + 2$ . Let us choose some constant  $\kappa$  so that  $\lambda \log(1 + \kappa)^{1+\alpha} > |f|_\infty$ . Let us suppose  $d < \frac{\kappa}{C}$ , where  $C$  will be chosen large enough depending on  $|f|_\infty, |b|_\infty$  and on universal constants. We can assume that in  $d < \frac{\kappa}{C}$  the distance to the boundary is  $C^2$  and satisfies  $|\nabla d| = 1$ . Let us consider in  $Cd < \kappa$  the function

$$\varphi(x) = \log(1 + Cd(x)).$$

Then we have

$$-F(\nabla\varphi, D^2\varphi) + b(x)|\nabla\varphi|^\beta \geq C^{2+\alpha} \left( a \frac{\sum |\partial_i d|^{2+\alpha}}{(1 + Cd)^{2+\alpha}} - A \frac{|D^2 d|_\infty \sum |\partial_i d|^\alpha}{(1 + Cd)^{\alpha+1}} \right) - C^\beta \frac{|b|_\infty (\sum |\partial_i d|^2)^{\frac{\beta}{2}}}{(1 + Cd)^\beta}.$$

4 Using the inequalities

$$\sum_i |\partial_i d|^{\alpha+2} \leq \left( \sum_i |\partial_i d|^2 \right)^{\frac{\alpha+2}{2}},$$

and

$$\sum_i |\partial_i d|^\beta \leq \left( \sum_i |\partial_i d|^2 \right)^{\frac{\beta}{2}}, \text{ if } \beta > 2, \quad \sum_i |\partial_i d|^\beta \leq \left( \sum_i |\partial_i d|^2 \right)^{\frac{\beta}{2}} N^{1-\frac{\beta}{2}} \text{ if not,}$$

5 and analogous inequalities for  $\sum |\partial_i d|^\alpha$ , one gets that for  $Cd < \kappa$  there exist constants  
1  $\kappa_1, \kappa_2, \kappa_3$ , depending only on  $a, A$  and on universal constants, so that

$$\begin{aligned} -F(\nabla\varphi, D^2\varphi) + b(x)|\nabla\varphi|^\beta &\geq \kappa_1 \frac{C^{2+\alpha}}{(1 + Cd)^{\alpha+2}} \left( 1 - C^{\beta-\alpha-2} \kappa_2 (1 + Cd)^{\beta-\alpha-2} - AC^{-2} \kappa_3 (1 + Cd)^2 \right) \\ &\geq \frac{\kappa_1}{2} \frac{C^{2+\alpha}}{(1 + \kappa)^{\alpha+2}}. \end{aligned}$$

as soon as  $C$  is large enough, more precisely such that

$$C^{\beta-\alpha-2} \kappa_2 + AC^{-2} \kappa_3 (1 + \kappa)^2 < \frac{1}{2}$$

2 and then assuming also  $C$  so that  $\kappa_1 \frac{C^{2+\alpha}}{(1+\kappa)^{\alpha+2}} > 2|f|_\infty$ , we get that  $\varphi$  is a super-solution in  
3  $Cd < \kappa$ . Extending it by  $\log(1 + \kappa)$  in  $Cd > \kappa$  and using the fact that the infimum of two  
4 super-solutions is a super-solution, we have the result. To get a sub-solution take  $-\varphi$  and  
5 adapt the constant.

Note that in the case  $\beta = \alpha + 2$  the previous conclusion still holds if  $|b|_\infty$  is small enough depending on universal constants. Note now that if  $u$  is a supersolution of the equation

$$-F(\nabla u, D^2 u) + b(x)|\nabla u|^{\alpha+2} = \epsilon^{1+\alpha} f$$

Then  $u_\epsilon := \frac{u}{\epsilon}$  satisfies

$$-F(\nabla u_\epsilon, D^2 u_\epsilon) + \epsilon b |\nabla u_\epsilon|^{\alpha+2} = f$$

6 and then a solution for the second problem gives one for the first one.

7 The existence and uniqueness is then a direct consequence of the existence of these sub-  
8 and super-solutions and of Perron's method adapted to the context. We do not give the  
9 details.

10 □

<sup>(nonhom)</sup>  
11 **Remark 2.5.** *In the sequel we will use a variant of this existence's result, that is to say, the*  
12 *boundary condition will be  $R$  in place of  $0$ . This can be done by taking for the super-solution*  
13  *$R + \log(1 + Cd)$  in  $Cd < \kappa$  extended by  $R + \log(1 + \kappa)$  in  $Cd > \kappa$  and for the sub-solution*  
14 *by taking for  $k$  large enough  $R - k \log(1 + Cd)$  in  $Cd < \kappa$  extended by  $R - k \log(1 + \kappa)$  in*  
15  *$Cd > \kappa$ .*

### 16 **3 Uniform Lipschitz estimates when $\alpha \geq 2$ for unbounded** 17 **solutions**

<sup>(secLipuni)</sup>  
1 In this subsection we prove the following Lipschitz estimates :

<sup>(lipunif)</sup> **Proposition 3.1.** *Let  $F$  satisfy (1.2), (1.1), and is positively homogeneous of degree 1, and,*  
*for  $\lambda \geq 0$ ,  $\alpha \geq 2$  and  $\beta > \alpha + 1$ , let  $u$  and  $v$  be respectively a bounded by above sub-solution*  
*and a bounded from below super-solution of equation (1.5) in  $B$ , with  $f$  Lipschitz continuous*  
*in  $B$ . Then, for any positive  $p \geq \frac{(2+\alpha-\beta)^+}{\beta-\alpha-1}$ , there exists a positive constant  $M$ , depending*

only on  $p, \alpha, \beta, a, A, N, \|f - \lambda|u|^\alpha u\|_\infty$  and on the Lipschitz constant of  $f$ , such that, for all  $x, y \in B$  one has

$$u(x) - v(y) \leq \sup_B (u - v)^+ + M \frac{|x - y|}{(1 - |y|)^{\frac{\beta}{\beta - \alpha - 1}}} \left[ 1 + \left( \frac{|x - y|}{(1 - |x|)} \right)^p \right]$$

2 Due to the results in the previous subsection, in the case  $\beta \leq 2 + \alpha$ , the existence and  
 3 uniqueness of  $u_\lambda$  for equation (1.4) has been proved, and the Lipschitz bound on  $u_\lambda$  depends  
 4 on the  $L^\infty$  norm of  $u_\lambda$  (more precisely on the oscillation of  $u_\lambda$ ). The strength of Proposition  
 5 3.1 is that it provides bounds on  $u_\lambda$  independent on  $\lambda$ , as soon as  $f - \lambda u_\lambda$  is bounded. This  
 6 will allow to pass to the limit when  $\lambda$  goes to zero in the next sections.

7 As in [15], [13] it is sufficient to do the case  $\Omega = B(0, 1)$ .

Let us define a "distance" function  $d$  which equals  $1 - |x|$  near the boundary and is extended as a smooth function which has the properties

$$\begin{cases} d(x) = 1 - |x| & \text{if } |x| > \frac{1}{2} \\ \frac{1 - |x|}{2} \leq d(x) \leq 1 - |x| & \text{for all } x \in \bar{B} \\ |Dd(x)| \leq 1 & -c_1 Id \leq D^2 d(x) \leq 0 \text{ for all } x \in \bar{B} \end{cases}$$

8 for some constant  $c_1 > 0$ .

Let us define as in [15]  $\xi = \frac{|x - y|}{d(x)}$  and the function

$$\phi(x, y) = \frac{k}{d(y)^\tau} |x - y| (L + \xi^p) + \sup(u - v)$$

9 where  $L$  and  $k$  will be chosen large later, as well as  $p$  and  $\tau$ . It is clear that if we prove that  
 10 for such  $k$  and  $L$  one has for all  $(x, y) \in B^2$

$$u(x) - v(y) \leq \phi(x, y),$$

11 we are done.

12 So we suppose by contradiction that  $u(x) - v(y) - \phi(x, y) > 0$  somewhere, then necessarily  
 13 the supremum is achieved on a pair  $(x, y)$  with  $d(x) > 0$ ,  $d(y) > 0$  and  $x \neq y$ . Using Ishi's  
 14 lemma, [16], [17], one gets that on such a point, one has for all  $\epsilon > 0$  the existence of two  
 1 symmetric matrices  $X_\epsilon$  and  $Y_\epsilon$ , such that

$$(D_x\phi, X_\epsilon) \in J^{2,+}u(x), \quad (-D_y\phi, -Y_\epsilon) \in J^{2,-}v(y)$$

with

$$-\left(\frac{1}{\epsilon} + |D^2\phi|\right) I_{2N} \leq \begin{pmatrix} X_\epsilon & 0 \\ 0 & Y_\epsilon \end{pmatrix} \leq D^2\phi + \epsilon(D^2\phi)^2. \quad (3.1) \text{eqine}$$

Since  $u$  is a viscosity subsolution one has

$$-F(\Theta_\alpha(\nabla_x\phi)X_\epsilon\Theta_\alpha(\nabla_x\phi)) + |\nabla_x\phi|^\beta + \lambda|u|^\alpha u(x) \leq f(x),$$

while

$$-F(-\Theta_\alpha(\nabla_y\phi)Y_\epsilon\Theta_\alpha(\nabla_y\phi)) + |\nabla_y\phi|^\beta + \lambda|v|^\alpha v(y) \geq f(y).$$

Let us multiply (3.1) on the right by

$$\begin{pmatrix} \Theta_\alpha(\nabla_x\phi) & 0 \\ 0 & \Theta_\alpha(\nabla_y\phi) \end{pmatrix} \begin{pmatrix} \sqrt{1+t} I_N & 0 \\ 0 & I_N \end{pmatrix}$$

2 where  $t > 0$ , and  $I_N$  denotes the identity in  $\mathbb{R}^N$ , and on the left by its transpose, then we  
3 obtain that

$$\begin{aligned} & \begin{pmatrix} \sqrt{1+t} \Theta_\alpha(\nabla_x\phi) & 0 \\ 0 & \Theta_\alpha(\nabla_y\phi) \end{pmatrix} \begin{pmatrix} X_\epsilon & 0 \\ 0 & Y_\epsilon \end{pmatrix} \begin{pmatrix} \sqrt{1+t} \Theta_\alpha(\nabla_x\phi) & 0 \\ 0 & \Theta_\alpha(\nabla_y\phi) \end{pmatrix} \\ \leq & \begin{pmatrix} \sqrt{1+t} \Theta_\alpha(\nabla_x\phi) & 0 \\ 0 & \Theta_\alpha(\nabla_y\phi) \end{pmatrix} (D^2\phi) \begin{pmatrix} \sqrt{1+t} \Theta_\alpha(\nabla_x\phi) & 0 \\ 0 & \Theta_\alpha(\nabla_y\phi) \end{pmatrix} \\ + & \begin{pmatrix} \sqrt{1+t} \Theta_\alpha(\nabla_x\phi) & 0 \\ 0 & \Theta_\alpha(\nabla_y\phi) \end{pmatrix} (\epsilon(D^2\phi)^2) \begin{pmatrix} \sqrt{1+t} \Theta_\alpha(\nabla_x\phi) & 0 \\ 0 & \Theta_\alpha(\nabla_y\phi) \end{pmatrix} \end{aligned} \quad (3.2) \text{eqXY}$$

4 Note that by the positive homogeneity of  $F$  with respect to  $X$ , one has

$$\begin{aligned} F(t\Theta_\alpha(\nabla_x\phi)X_\epsilon\Theta_\alpha(\nabla_x\phi)) & - F((1+t)\Theta_\alpha(\nabla_x\phi)X_\epsilon\Theta_\alpha(\nabla_x\phi)) \\ & + F(-\Theta_\alpha(\nabla_y\phi)Y_\epsilon\Theta_\alpha(\nabla_y\phi)) + \lambda(|u|^\alpha u(x) - |v|^\alpha v(y)) + |\nabla_x\phi|^\beta \\ & - |\nabla_y\phi|^\beta - f(x) + f(y) \\ & \leq 0 \end{aligned}$$

1 and then using  $u(x) - v(y) > 0$

$$\begin{aligned} t|\nabla_x\phi|^\beta & \leq F(\nabla_x\phi, tX_\epsilon) - t\lambda|u|^\alpha u(x) + tf(x) \\ & \leq F(\nabla_x\phi, (1+t)X_\epsilon) - F(\nabla_y\phi, -Y_\epsilon) + |\nabla_y\phi|^\beta - |\nabla_x\phi|^\beta + t(f(x) - \lambda|u|^\alpha u(x))^+ \\ & + f(x) - f(y) \\ & \leq \mathcal{M}^+((1+t)\Theta_\alpha(\nabla_x\phi)X_\epsilon\Theta_\alpha(\nabla_x\phi) + \Theta_\alpha(\nabla_y\phi)Y_\epsilon\Theta_\alpha(\nabla_y\phi)) \\ & + |\nabla_y\phi|^\beta - |\nabla_x\phi|^\beta + t(f(x) - \lambda|u|^\alpha u(x))^+ + f(x) - f(y) \end{aligned} \quad (3.3) \text{?questi1?}$$



2 Suppose that we get an estimate of the form

$$\begin{aligned} \mathcal{M}^+((1+t)\Theta_\alpha(\nabla_x\phi)X_\epsilon\Theta_\alpha(\nabla_x\phi) &+ \Theta_\alpha(\nabla_y\phi)Y_\epsilon\Theta_\alpha(\nabla_y\phi)) \\ &\leq \psi(t, x, y, D\phi, D^2\phi) + c\epsilon(1+t)(|\Theta_\alpha(\nabla_x\phi)|^2 \\ &+ |\Theta_\alpha(\nabla_y\phi)|^2) |D^2\phi|^2, \end{aligned} \quad (3.4) \quad \boxed{\text{equesti}}$$

3 for some function  $\psi$ , then we will derive that

$$\begin{aligned} t|\nabla_x\phi|^\beta &\leq \psi(t, x, y, D\phi, D^2\phi) + c\epsilon(1+t)(|\Theta_\alpha(\nabla_x\phi)|^2 + |\Theta_\alpha(\nabla_y\phi)|^2)|D^2\phi|^2 \\ &+ |\nabla_y\phi|^\beta - |\nabla_x\phi|^\beta + t(f - \lambda|u|^\alpha u(x))^+ + f(x) - f(y), \end{aligned}$$

and then letting  $\epsilon$  go to 0, one gets

$$t|\nabla_x\phi|^\beta \leq \psi(t, x, y, D\phi, D^2\phi) + |\nabla_y\phi|^\beta - |\nabla_x\phi|^\beta + t(f - \lambda|u|^\alpha u(x))^+ + f(x) - f(y).$$

Let us recall some useful estimates on  $\psi$  in (3.4) : Using the computations and the estimates in [15], let us recall that :

$$D_x\phi = \frac{k}{d(y)^\tau} ((L + (1+p)\xi^p)\eta - p\xi^{p+1}Dd(x)),$$

where  $\eta = \frac{x-y}{|x-y|}$  and

$$D_y\phi = -\frac{k}{d(y)^\tau} (L + (1+p)\xi^p)\eta + \tau \frac{k|x-y|}{d(y)^{\tau+1}} (L + \xi^p) Dd(y).$$

Note that one has

$$|D_x\phi|, |D_y\phi| \leq ck \frac{L + \xi^{p+1}}{d(y)^{\tau+1}}$$

4 and always like in [15] we can choose  $L > 1$  and large enough in order that  $|D_x\phi| \geq$   
5  $ck \frac{(L+\xi^p)(1+\xi)}{d(y)^\tau}$ .

1 We can sum up  $D^2\phi$  as follows

$$\begin{aligned} D^2\phi &= \gamma_1 \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \gamma_2 \begin{pmatrix} T & -T \\ -T & T \end{pmatrix} \\ &+ \gamma_3 \begin{pmatrix} -(C + {}^tC) & C \\ {}^tC & 0 \end{pmatrix} + \gamma_4 \begin{pmatrix} 0 & -{}^tD \\ -D & (D + {}^tD) \end{pmatrix} \\ &+ \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \end{aligned} \quad (3.5) \quad \boxed{D2phi}$$

with  $B = I - \eta \otimes \eta, T = \eta \otimes \eta, C = \eta \otimes Dd(x), D = \eta \otimes Dd(y)$ , and where

$$\gamma_1 = \frac{k}{d(y)^\tau} \frac{L + (1+p)\xi^p}{|x-y|}, \quad \gamma_2 = \frac{k}{d(y)^\tau} p(1+p) \frac{\xi^p}{|x-y|}, \quad \gamma_3 = \frac{k}{d(y)^\tau} p(1+p) \frac{\xi^p}{d(x)},$$

$$\gamma_4 = \frac{k}{d(y)^\tau} \tau \frac{(L + (1+p)\xi^p)}{d(y)}$$

and

$$\begin{aligned} X_1 &= \frac{k}{d(y)^\tau} \left( \frac{p(p+1)\xi^{p+1}}{d(x)} Dd(x) \otimes Dd(x) - p\xi^{p+1} D^2 d(x) \right), \\ X_2 &= \frac{k}{d(y)^\tau} \frac{\tau p \xi^{p+1}}{d(y)} Dd(x) \otimes Dd(y), \quad X_3 = \frac{k}{d(y)^\tau} \frac{\tau p \xi^{p+1}}{d(y)} Dd(y) \otimes Dd(x) \\ X_4 &= \frac{k}{d(y)^\tau} \left( \frac{\tau(\tau+1)(L+\xi^p)|x-y|}{d(y)^2} Dd(y) \otimes Dd(y) - \frac{\tau|x-y|}{d(y)} (L+\xi^p) D^2 d(y) \right). \end{aligned}$$

2 Then multiplying ( 3.5) by  $\begin{pmatrix} \sqrt{1+t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Theta_\alpha(\nabla_x \phi) & 0 \\ 0 & \Theta_\alpha(\nabla_y \phi) \end{pmatrix}$  on the left and the  
3 right, one obtains

$$\begin{aligned} \psi(t, x, y, D\phi, D^2\phi) &:= \gamma_1 \begin{pmatrix} (1+t)\Theta_\alpha(\nabla_x \phi)B\Theta_\alpha(\nabla_x \phi) & -\sqrt{1+t}\Theta_\alpha(\nabla_x \phi)B\Theta_\alpha(\nabla_y \phi) \\ -\sqrt{1+t}\Theta_\alpha(\nabla_y \phi)B\Theta_\alpha(\nabla_x \phi) & \Theta_\alpha(\nabla_y \phi)B\Theta_\alpha(\nabla_y \phi) \end{pmatrix} \\ &+ \gamma_2 \begin{pmatrix} (1+t)\Theta_\alpha(\nabla_x \phi)T\Theta_\alpha(\nabla_x \phi) & -\sqrt{1+t}\Theta_\alpha(\nabla_x \phi)T\Theta_\alpha(\nabla_y \phi) \\ -\sqrt{1+t}\Theta_\alpha(\nabla_y \phi)T\Theta_\alpha(\nabla_x \phi) & \Theta_\alpha(\nabla_y \phi)T\Theta_\alpha(\nabla_y \phi) \end{pmatrix} \\ &+ \gamma_3 \begin{pmatrix} -(1+t)\Theta_\alpha(\nabla_x \phi)(C+{}^t C)\Theta_\alpha(\nabla_x \phi) & \sqrt{1+t}\Theta_\alpha(\nabla_x \phi) {}^t C \Theta_\alpha(\nabla_y \phi) \\ \sqrt{1+t}\Theta_\alpha(\nabla_y \phi)C\Theta_\alpha(\nabla_x \phi) & 0 \end{pmatrix} \\ &+ \gamma_4 \begin{pmatrix} 0 & -\sqrt{1+t}\Theta_\alpha(\nabla_x \phi)D\Theta_\alpha(\nabla_y \phi) \\ -\sqrt{1+t}\Theta_\alpha(\nabla_y \phi) {}^t D \Theta_\alpha(\nabla_x \phi) & \Theta_\alpha(\nabla_y \phi)(D+{}^t D)\Theta_\alpha(\nabla_x \phi) \end{pmatrix} \\ &+ \begin{pmatrix} (1+t)\Theta_\alpha(\nabla_x \phi)X_1\Theta_\alpha(\nabla_x \phi) & \sqrt{1+t}\Theta_\alpha(\nabla_x \phi)X_2\Theta_\alpha(\nabla_y \phi) \\ \sqrt{1+t}\Theta_\alpha(\nabla_y \phi)X_3\Theta_\alpha(\nabla_x \phi) & \Theta_\alpha(\nabla_y \phi)X_4\Theta_\alpha(\nabla_y \phi) \end{pmatrix} \end{aligned}$$

4 Multiplying the inequality ( 3.2) by  $({}^t v, {}^t v)$  on the left and  $\begin{pmatrix} v \\ v \end{pmatrix}$  on the right, where  $v$   
1 is any unit vector, one gets defining  $w_t = (\sqrt{1+t}\Theta_\alpha(\nabla_x \phi) - \Theta_\alpha(\nabla_y \phi))(v)$

$$\begin{aligned} &{}^t v((1+t)\Theta_\alpha(\nabla_x \phi)X\Theta_\alpha(\nabla_x \phi) + \Theta_\alpha(\nabla_y \phi)Y\Theta_\alpha(\nabla_y \phi))v \\ &\leq \gamma_1^t w_t B w_t + \gamma_2^t w_t T w_t + c\gamma_3 t (|C| + |{}^t C|)(|\Theta_\alpha(D_x \phi)|^2 + |\Theta_\alpha(D_y \phi)|^2) \\ &+ \gamma_4 t (|D| + |{}^t D|)(|\Theta_\alpha(D_x \phi)|^2 + |\Theta_\alpha(D_y \phi)|^2) \\ &+ \gamma_3 (|C| + |{}^t C|)|\Theta_\alpha(D_x \phi) - \Theta_\alpha(D_y \phi)| |\Theta_\alpha(D_x \phi)| + |\Theta_\alpha(D_y \phi)||v|^2 \\ &+ \gamma_4 (|D| + |{}^t D|)|\Theta_\alpha(D_x \phi) - \Theta_\alpha(D_y \phi)| (|\Theta_\alpha(D_x \phi)| + |\Theta_\alpha(D_y \phi)|)|v|^2 \\ &+ (\sqrt{1+t}(|X_1| + |X_2|) + |X_3| + |X_4|)(|\Theta_\alpha(D_x \phi)| + |\Theta_\alpha(D_y \phi)|)|v|^2. \end{aligned} \quad (3.6) \quad \boxed{\text{eigenvalue}}$$

2 Note that

$$\begin{aligned} |w_t|^2 &\leq 2((\sqrt{1+t}-1)^2|\Theta_\alpha(D_x\phi)|^2 + 2|\sum_i |\partial_{i,x}\phi|^{\frac{\alpha}{2}} - |\partial_{i,y}\phi|^{\frac{\alpha}{2}}|^2) \\ &\leq 2t^2|\Theta_\alpha(D_x\phi)|^2 + 2|\sum_i |\partial_{i,x}\phi|^{\frac{\alpha}{2}} - |\partial_{i,y}\phi|^{\frac{\alpha}{2}}|^2. \end{aligned}$$

Using (3.6), every eigenvalue of  $(1+t)X+Y$  satisfies  $\lambda_i((1+t)\Theta_\alpha(\nabla_x\phi)X\Theta_\alpha(\nabla_x\phi) + \Theta_\alpha(\nabla_y\phi)Y\Theta_\alpha(\nabla_y\phi)) \leq ct^2\Gamma_1 + t\Gamma_2 + (\gamma_1 + \gamma_2)\Gamma_4 + \Gamma_3$  for some universal constant  $c$ , where we have denoted

$$\Gamma_1 = (\gamma_1 + \gamma_2)|\Theta_\alpha(D_x\phi)|^2 \leq ck^{1+\alpha} \frac{(L + \xi^{p+1})^{1+\alpha}}{d(y)^{\tau+(\tau+1)\alpha}|x-y|}.$$

$$\Gamma_2 := (\gamma_3 + \gamma_4 + |X_1| + |X_2| + |X_3|)(|\Theta_\alpha(D_x\phi)|^2 + |\Theta_\alpha(D_y\phi)|^2) \leq ck^{1+\alpha} \frac{(L + \xi^{p+2})(L + \xi^{p+1})^\alpha}{d(y)^{(\tau+1)(1+\alpha)}}.$$

$$\Gamma_3 := \sum_1^4 |X_i| \leq k^{1+\alpha}|x-y| \frac{(L + \xi^{p+2})(L + \xi^{p+1})^\alpha}{d(y)^{2+\tau+(\tau+1)\alpha}}.$$

Finally

$$\Gamma_4 := |\sum_i |D_{i,x}\phi|^{\frac{\alpha}{2}} - |D_{i,y}\phi|^{\frac{\alpha}{2}}|^2.$$

To majorize  $\Gamma_4$  observe that if  $\xi \leq 1$ ,

$$\frac{|\xi|^{p+1}|Dd|k}{d(y)^\tau} \leq \frac{\xi^p|x-y|}{d(x)d(y)^\tau} k \leq 2k|x-y| \frac{\xi^p}{d(y)^{\tau+1}}$$

while if  $\xi \geq 1$ ,

$$\frac{\xi^{p+1}k}{d(y)^\tau} \leq 2k(1 + \xi^{p+1}) \frac{|x-y|}{d(y)^{\tau+1}}$$

3 As a consequence  $|D_{ix}\phi + D_{iy}\phi| \leq ck(1 + \xi^{p+1}) \frac{|x-y|}{d(y)^{\tau+1}}$ .

1 Then since  $\alpha \geq 2$  one has by the mean value's theorem

$$\begin{aligned} \Gamma_4 = \sum_i ||D_{ix}\phi|^{\frac{\alpha}{2}} - |D_{iy}\phi|^{\frac{\alpha}{2}}|^2 &\leq c \sum_i |D_{ix}\phi + D_{iy}\phi|^2 (|D_{ix}\phi|^{\frac{\alpha}{2}-1} + |D_{iy}\phi|^{\frac{\alpha}{2}-1})^2 \\ &\leq c \frac{k^\alpha|x-y|^2}{d(y)^{(\tau+1)\alpha}} (L + \xi^{p+1})^\alpha \end{aligned}$$

and then

$$\Gamma_4(\gamma_1 + \gamma_2) = |\sum_i (|D_{ix}\phi|^{\frac{\alpha}{2}} - |D_{iy}\phi|^{\frac{\alpha}{2}})|^2 k \frac{(L + \xi^{p+1})}{d(y)^\tau|x-y|} \leq ck^{\alpha+1}|x-y| \frac{(L + \xi^{p+1})^{\alpha+1}}{d(y)^{\tau(\alpha+1)+\alpha}}$$

2 We now choose  $\tau > \frac{\alpha+1}{\beta-1-\alpha}$ , and  $p > \frac{2\alpha+2}{\beta-(\alpha+1)}$ , which imply that by taking  $k$  and  $L$  large  
3 enough one has  $\Gamma_2 < \frac{|D_x\phi|^\beta}{2}$ . We have obtained that

$$\frac{t}{2}|D_x\phi|^\beta \leq t(f(x) - \lambda|u|^\alpha u(x)) + |D_y\phi|^\beta - |D_x\phi|^\beta + (\Gamma_1 t^2 + \Gamma_3 + (\gamma_1 + \gamma_2)\Gamma_4)$$

Note now that we can choose  $t$  optimal or equivalently  $t_o = \frac{|D_x\phi|^\beta}{4\Gamma_1}$  and with this value of  $t_o$  one has

$$|D_x\phi|^{2\beta} \leq 16\Gamma_1 t(f(x) - \lambda|u|^\alpha u(x)) + \Gamma_1(|D_y\phi|^\beta - |D_x\phi|^\beta) + \Gamma_3\Gamma_1 + \Gamma_1(\gamma_1 + \gamma_2)\Gamma_4$$

There remains to see that from this one derives a contradiction, indeed, the left hand side is greater than  $\left(\frac{L+\xi^{p+1}}{d(y)^\tau}\right)^{2\beta}$  while

$$\Gamma_1\Gamma_3 \leq ck^{2(1+\alpha)} \frac{(L^{2(1+\alpha)} + \xi^{2(p+2)(\alpha+1)})}{d(y)^{2(\tau+1)(\alpha+1)}}$$

4 which is negligible w.r.t.  $|D_x\phi|^{2\beta}$  by the choice of  $\tau$  and  $p$ .

5 Furthermore

$$\begin{aligned} (\gamma_1 + \gamma_2)\Gamma_4\Gamma_1 &\leq c \frac{k}{d(y)^\tau} \frac{L + (1+p)^2 \xi^p}{|x-y|} k^{1+\alpha} \frac{(L + \xi^{p+1})^{1+\alpha}}{d(y)^{\tau+(\tau+1)\alpha}|x-y|} k^\alpha \frac{|x-y|^2 (L + \xi^{p+1})^\alpha}{(d(y)^{\tau+1})^\alpha} \\ &\leq ck^{2(1+\alpha)} \frac{(L + \xi^{p+1})^{2(1+\alpha)}}{d(y)^{2\tau(\alpha+1)+2\alpha}} \end{aligned}$$

which is small w.r. t.  $|\nabla_x\phi|^{2\beta}$  by the choice of  $\tau$ ,  $p$ ,  $k$  and  $L$ . Finally

$$16\Gamma_1 t |f - \lambda|u|^\alpha|_\infty \leq 16|D_x\phi|^\beta |f - \lambda|u|^\alpha|_\infty$$

6 which is small with respect to  $|D_x\phi|^{2\beta}$  as soon as  $L$  and  $k$  are chosen large.

Furthermore

$$||D_x\phi|^\beta - |D_y\phi|^\beta| \leq |D_x\phi + D_y\phi| \left(ck \frac{L + \xi^{p+1}}{d(y)^{\tau+1}}\right)^{\beta-1} \leq ck^\beta \frac{|x-y|(L + \xi^{p+1})^\beta}{d(y)^{(1+\tau)\beta}}$$

1 and then

$$||D_x\phi|^\beta - |D_y\phi|^\beta|\Gamma_1 \leq k^{\beta+1+\alpha} \frac{(L + \xi^{p+1})^{\beta+1+\alpha}}{d(y)^{\tau(\alpha+1)+\alpha+(\tau+1)\beta}}$$

which is also small with respect to  $|D_x\phi|^{2\beta}$  using the assumptions on  $p$  and  $\tau$ . We have obtained a contradiction. Finally  $\phi(x, y) \leq 0$ , which implies some Lipschitz estimate. Arguing

as in [15], one can obtain an optimal behaviour of the gradient of  $u$  when  $u$  is a solution, in the form

$$|\nabla u_\lambda| \leq cd^{-\gamma-1}.$$

Furthermore, these estimates can easily be extended to a  $\mathcal{C}^2$  domain in place of a ball, using the interior sphere property, and replacing of course  $1 - |x|$  by  $d(x, \partial\Omega)$ .

## 4 Existence and behaviour near the boundary of ergodic function.

In this section we prove the existence of solutions of equation (1.4) blowing up at the boundary, which will be used in the proof of existence of ergodic pairs. In what follows we drop the assumption on the boundedness of the right hand side  $f$ , and we consider continuous functions in  $\Omega$ , possibly unbounded as  $d(x) \rightarrow 0$ .

**Theorem 4.1.** *Let  $\alpha \geq 0$ ,  $\beta \in (\alpha + 1, \alpha + 2]$ ,  $\lambda > 0$  and let  $F$  satisfy (1.1) and (1.2). Let further  $f \in \mathcal{C}(\Omega)$  be bounded from below and such that*

$$\lim_{d(x) \rightarrow 0} f(x)d(x)^{\frac{\beta}{\beta-1-\alpha}} = 0. \quad (4.1)$$

Then, the infinite boundary value problem

$$\begin{cases} -|\nabla u|^\alpha F(D^2u) + |\nabla u|^\beta + \lambda|u|^\alpha u = f & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \quad (4.2)$$

admits solutions, and any its solution  $u$  satisfies, for all  $x \in \Omega$ ,

$$\begin{aligned} \frac{c_0}{d(x)^\gamma} - \frac{D_1}{\lambda^{\frac{1}{\alpha+1}}} &\leq u(x) \leq \frac{C_0}{d(x)^\gamma} + \frac{D_1}{\lambda^{\frac{1}{\alpha+1}}} && \text{if } \gamma > 0, \\ c_0|\log d(x)| - \frac{D_1}{\lambda^{\frac{1}{\alpha+1}}} &\leq u(x) \leq C_0|\log d(x)| + \frac{D_1}{\lambda^{\frac{1}{\alpha+1}}} && \text{if } \gamma = 0, \end{aligned} \quad (4.3)$$

for positive constants  $c_0, C_0$  and  $D_1$  depending only on  $\alpha, \beta, a, A, |d|_{\mathcal{C}^2(\Omega)}$  and on  $f$ .

When  $F$  satisfies furthermore (1.7) one has a better estimate :

**Theorem 4.2.** *Let  $\beta \in (\alpha + 1, \alpha + 2]$ ,  $\lambda > 0$  and let  $F$  satisfy (1.2), (1.1) and (1.7). Let further  $f \in \mathcal{C}(\Omega)$  be bounded from below and such that*

$$\lim_{d(x) \rightarrow 0} f(x)d(x)^{\frac{\beta}{\beta-1-\alpha}-\gamma_0} = 0, \quad (4.4)$$

for some  $\gamma_0 \geq 0$ . Then, any solution  $u$  of (4.2) satisfies: for any  $\nu > 0$  and for any  $0 \leq \gamma_1 \leq \gamma_0$ , with  $\gamma_1 < \inf(1, \alpha)$ , and  $\gamma_1 < \gamma$  when  $\gamma > 0$ , there exists  $D = \frac{D_1}{\lambda^{1/(\alpha+1)}}$ , with  $D_1 > 0$  depending on  $\nu, \gamma_1, \alpha, \beta, a, A, |d|_{\mathcal{C}^2(\Omega)}, |C(\cdot)|_{\mathcal{C}^2(\Omega)}$  and on  $f$ , such that, for all  $x \in \Omega$ ,

$$\begin{aligned} \frac{C(x)}{d(x)^\gamma} - \frac{\nu}{d(x)^{\gamma-\gamma_1}} - D \leq u(x) \leq \frac{C(x)}{d(x)^\gamma} + \frac{\nu}{d(x)^{\gamma-\gamma_1}} + D \quad \text{if } \gamma > 0, \\ |\log d(x)| (C(x) - \nu d(x)^{\gamma_1}) - D \leq u(x) \leq |\log d(x)| (C(x) + \nu d(x)^{\gamma_1}) + D \quad \text{if } \gamma = 0. \end{aligned} \tag{4.5} \text{ ?bouest?}$$

2 Furthermore, the solution  $u$  is unique.

3 *Proof.* of Theorem 4.1 :

4 In almost all the results, we will just detail the case  $\gamma > 0$  and leave the case  $\gamma = 0$  to  
5 the reader. Let  $\delta$  be small enough in order that in  $d < 2\delta$  the distance is  $\mathcal{C}^2$ . We define

$$\varphi(x) = C_0 d(x)^{-\gamma},$$

then we have, by an easy computation, using the fact that  $d$  is  $\mathcal{C}^2$  near the boundary and the properties of  $F$  :

$$|F(\nabla\varphi, D^2\varphi) - \gamma^{1+\alpha}(\gamma+1)d^{-(\gamma+1)\alpha-\gamma-2}C_0^{1+\alpha}F(\nabla d, \nabla d \otimes \nabla d)| \leq cd^{-(\gamma+1)\alpha-\gamma-1},$$

and

$$|\nabla\varphi|^\beta = C_0^\beta d^{-(\gamma+1)\beta} |\nabla d|^\beta$$

so taking  $C_0$  conveniently large and using the asymptotic behaviour of  $f$ ,

$$-F(\nabla\varphi, D^2\varphi) + |\nabla\varphi|^\beta \geq f^+.$$

In particular for any positive constant  $D$ ,  $\varphi_1 := \varphi + D$  is also a supersolution of

$$-F(\nabla\varphi_1, D^2\varphi_1) + |\nabla\varphi_1|^\beta + \lambda|\varphi_1|^\alpha \varphi_1 \geq f^+.$$

6 We extend  $\varphi_1$  inside  $\Omega$  by taking

$$\bar{w} = \begin{cases} \varphi_1 & \text{in } d < \delta \\ \frac{C_0}{\delta^\gamma} e^{\frac{1}{d(x)^{-2\delta} + \frac{1}{\delta}}} + D, & \text{if } \delta < d \leq 2\delta \\ D & \text{if } d > 2\delta. \end{cases}$$

where, since  $\varphi_2$  is  $\mathcal{C}^2$ ,  $K_1$  being defined by

$$|F(\nabla\varphi_1, D^2\varphi_1)| + |\nabla\varphi_1|^\beta \leq K_1,$$

7 we have chosen  $D \geq \left( \frac{|f|_\infty \lambda + K_1}{\lambda} \right)^{\frac{1}{1+\alpha}}$ . Then  $\bar{w}$  is a convenient super-solution ( note that on  
1  $\delta < d \leq 2\delta$  we use the fact that the infimum of two super-solutions is still a super-solution).

Let us exhibit a convenient sub-solution : For  $s > 0$ ,  $c_0 = \gamma^{-1} \left( \frac{(\gamma+1)\alpha}{2} \right)^{\frac{1}{\beta-\alpha-1}}$  and  $x \in \Omega \setminus \Omega_{2\delta}$ , let us consider the function

$$\varphi^s(x) = c_0(d(x) + s)^{-\gamma}.$$

One has

$$-F(\nabla\varphi^s, D^2\varphi^s) + |\nabla\varphi^s|^\beta + \lambda(\varphi^s)^{1+\alpha} \leq -|\nabla\varphi^s|^\alpha \mathcal{M}^-(D^2\varphi^s) + |\nabla\varphi^s|^\beta + \lambda(\varphi^s)^{1+\alpha} \leq f(x),$$

2 for  $\delta$  and  $s$  sufficiently small, since  $f$  is bounded from below.

3 Moreover, for  $D \geq c_0\delta^{-\gamma} + \left( \frac{|f^-|_\infty}{\lambda} \right)^{\frac{1}{\alpha+1}}$ , the constant function  $c_0(\delta + s)^{-\gamma} - D$  is also a  
4 sub solution in  $\Omega$ .

Therefore, the function

$$\underline{w}^s(x) = \begin{cases} \varphi^s(x) - D & \text{in } \Omega \setminus \Omega_\delta \\ c_0(\delta + s)^{-\gamma} - D & \text{in } \Omega_\delta \end{cases}$$

5 is a convenient sub-solution.

Using Remark 2.5 after the existence Theorem 2.1, let  $u_R$  which satisfies

$$\begin{cases} -F(Du_R, D^2u_R) + |\nabla u_R|^\beta + \lambda|u_R|^\alpha u_R = f_R & \text{in } \Omega \\ u_R = R & \text{on } \partial\Omega, \end{cases} \quad (4.6) \quad \square$$

6 where  $f_R = \inf(f, R)$ . Observing, using the comparison principle in Theorem 2.4, one gets  
7 that  $\underline{w}^s \leq u_R \leq \bar{w}$ ,  $u_R$  is non decreasing, and since it is trapped between  $\underline{w}^s$  and  $\bar{w}^s$   
8 it is locally uniformly bounded, hence locally uniformly Lipschitz. By classical results for  
9 uniformly Lipschitz viscosity solutions, it converges to a solution of (4.2). The boundary  
10 behaviour follows by letting  $s$  go to zero.

11 Note that here we did not use the uniform Lipschitz estimates, so  $\alpha \geq 2$  is not needed.  
12 Furthermore, we do not have the precise estimate at the boundary, and then the uniqueness  
13 could fail.

1

□

2 *Proof.* of Theorem 4.2

We introduce for  $\delta > 0$  small

$$\varphi_1 = \left( \left( \frac{F(\nabla d, \nabla d \otimes \nabla d)(\gamma + 1)}{\gamma^{\beta - \alpha - 1}} \right)^{\frac{1}{\beta - \alpha - 1}} + \nu d^{\gamma_1} \right) d^{-\gamma} + D$$

and

$$\underline{w}_{\epsilon, \delta} = \left( \left( \frac{F(\nabla d, \nabla d \otimes \nabla d)(\gamma + 1)}{\gamma^{\beta - \alpha - 1}} \right)^{\frac{1}{\beta - \alpha - 1}} - \nu d^{\gamma_1} \right) (d + \delta)^{-\gamma} - D$$

3 where  $D$  is some constant to be chosen later . Recall that  $C(x) = \left( \frac{F(\nabla d, \nabla d \otimes \nabla d)(\gamma + 1)}{\gamma^{\beta - \alpha - 1}} \right)^{\frac{1}{\beta - \alpha - 1}}$ .

We prove that  $\varphi_1$  is a supersolution in  $\Omega_{2\delta} = \{x \in \Omega, d(x, \partial\Omega) \leq 2\delta\}$  for  $\delta$  small enough. We denote  $\varphi(x) = C(x)d^{-\gamma} + \nu d^{-\gamma + \gamma_1}$ . Then  $D\varphi = -\gamma C(x)d^{-\gamma - 1}\nabla d(1 + \nu \frac{\gamma - \gamma_1}{\gamma C(x)}d^{\gamma_1}) + DC(x)d^{-\gamma}$ ,

$$\begin{aligned} D^2\varphi(x) &= \gamma(\gamma + 1)C(x)d^{-\gamma - 2} \left( 1 + \nu \frac{(\gamma - \gamma_1)(\gamma - \gamma_1 + 1)}{\gamma(\gamma + 1)C(x)}d^{\gamma_1} \right) \nabla d \otimes \nabla d \\ &\quad - \gamma C(x)d^{-\gamma - 1} \left( 1 + \nu \frac{(\gamma - \gamma_1)}{\gamma C(x)}d^{\gamma_1} \right) D^2d - \gamma d^{-\gamma - 1} (\nabla d \otimes \nabla C + \nabla C \otimes \nabla d) + d^{-\gamma} D^2C. \end{aligned}$$

4 In particular

$$|D^2\varphi - C(x)\gamma(\gamma + 1)d^{-\gamma - 2}(\nabla d \otimes \nabla d)(1 + \nu \frac{(\gamma - \gamma_1)(\gamma + 1 - \gamma_1)}{\gamma(\gamma + 1)C(x)}d^{\gamma_1})| \leq cd^{-\gamma - 1}$$

5 which implies

$$\begin{aligned} |F(D\varphi, D^2\varphi) - C(x)\gamma(\gamma + 1)d^{-\gamma - 2}(1 + \nu \frac{(\gamma - \gamma_1)(\gamma + 1 - \gamma_1)}{\gamma(\gamma + 1)C(x)}d^{\gamma_1})F(\nabla\varphi, \nabla d \otimes \nabla d)| \\ \leq cd^{-(\gamma + 1)(1 + \alpha)} = o(d^{-(\gamma + 1)\alpha - \gamma - 2 + \gamma_1}). \end{aligned} \quad (4.7) \quad \boxed{\text{majFD2}}$$

6 Let  $\tau \in ]\frac{2\alpha}{\alpha}, 1[$ , by Property  $(P_2)$ , one has ( in the computations below  $c$  denotes always some  
1 universal constant which varies from one line to another) :

$$\begin{aligned} |F(\nabla\varphi, \nabla d \otimes \nabla d) - F(-\gamma C(x)d^{-\gamma - 1}\nabla d(1 + \nu \frac{\gamma - \gamma_1}{\gamma C(x)}d^{\gamma_1}), \nabla d \otimes \nabla d)| \\ \leq c|\nabla d|^2 d^{-(\gamma + 1)\alpha} \sum_{i=1}^{i=N} \left| \left| \gamma C(x)\partial_i d(1 + \nu \frac{\gamma - \gamma_1}{\gamma C(x)}d^{\gamma_1}) \right|^\alpha \right. \\ \left. - \left| \gamma C(x)\partial_i d(1 + \nu \frac{\gamma - \gamma_1}{\gamma C(x)}d^{\gamma_1}) + \partial_i C d \right|^\alpha \right| \\ \leq cd^{-(\gamma + 1)\alpha} d^{\tau\alpha} + c \sum_{i, |\partial_i d| > d^\tau} |\partial_i d|^\alpha \left( \left( 1 + \frac{\partial_i C}{\gamma C \partial_i d(1 + \nu \frac{\gamma - \gamma_1}{\gamma C(x)}d^{\gamma_1})} d \right)^\alpha - 1 \right) \end{aligned}$$



2

$$\begin{aligned}
&\leq cd^{-(\gamma+1)\alpha} \left( d^{\tau\alpha} + \sum_{i, |\partial_i d| > d^\tau} |\partial_i d|^\alpha \frac{cd}{|\partial_i d|} \right) \\
&\leq cd^{-(\gamma+1)\alpha} (d^{\tau\alpha} + d^{1-\tau(1-\alpha)^+}) \\
&\leq cd^{-(\gamma+1)\alpha} d^{\tau\alpha} = o(d^{-(\gamma+1)\alpha+\gamma_1}).
\end{aligned} \tag{4.8} \text{majphi}$$

3 Now observe that by using a Taylor expansion at order 2 and the properties of  $F$

$$\begin{aligned}
&F(-\gamma C(x) d^{-\gamma-1} \nabla d (1 + \nu \frac{\gamma - \gamma_1}{\gamma C(x)} d^{\gamma_1}), \nabla d \otimes \nabla d) \\
&= (\gamma C(x) d^{-\gamma-1})^\alpha (1 + \nu \alpha \frac{\gamma - \gamma_1}{\gamma C(x)} d^{\gamma_1}) F(\nabla d, \nabla d \otimes \nabla d) \\
&+ o(d^{-(\gamma+1)\alpha+2\gamma_1}).
\end{aligned} \tag{4.9} \text{TE}$$

4 ,

5 Gathering (4.7) (4.8) and (4.9) one obtains

$$\begin{aligned}
&|F(\nabla \varphi, D^2 \varphi) \\
&- (C(x) \gamma)^{1+\alpha} (\gamma + 1) d^{-\gamma-2} (1 + \nu \alpha \frac{\gamma - \gamma_1}{\gamma C(x)} d^{\gamma_1}) (1 + \nu \frac{(\gamma - \gamma_1)(\gamma + 1 - \gamma_1)}{\gamma(\gamma + 1)C(x)} d^{\gamma_1}) F(\nabla d, \nabla d \otimes \nabla d) \\
&= o(d^{-(\gamma+1)(1+\alpha)-1+\gamma_1}).
\end{aligned}$$

On the other hand one has, using a Taylor expansion at the order 2 and the mean value's Theorem

$$\left| |\nabla \varphi|^\beta - (d^{-\gamma-1} C(x) \gamma)^\beta (1 + \nu \frac{\beta(\gamma - \gamma_1)}{\gamma C(x)} d^{\gamma_1}) |\nabla d|^\beta \right| \leq cd^{-(\gamma+1)\beta+2\gamma_1} + d^{-(\gamma+1)\beta+1}. \tag{4.10} \text{beta}$$

6 Considering the term in  $d^{\gamma_1} d^{-\gamma-2-(\gamma+1)\alpha}$  in  $(C(x) \gamma)^{1+\alpha} (\gamma + 1)^\alpha d^{-\gamma-2} ((1 + \nu \alpha \frac{\gamma - \gamma_1}{\gamma C(x)} d^{\gamma_1})) (1 + \nu \frac{(\gamma - \gamma_1)(\gamma + 1 - \gamma_1)}{\gamma(\gamma + 1)C(x)} d^{\gamma_1})$  and in (4.10), and using the definition of  $C(x)$ , one has

1

$$\begin{aligned}
&-(C(x) \gamma)^{1+\alpha} (\gamma + 1) \left( \nu \alpha \frac{\gamma - \gamma_1}{\gamma C(x)} + \nu \frac{(\gamma - \gamma_1)(\gamma + 1 - \gamma_1)}{\gamma(\gamma + 1)C(x)} \right) F(\nabla d \otimes \nabla d) \\
&+ (C(x) \gamma)^\beta \beta \nu \frac{\gamma - \gamma_1}{\gamma C(x)} \\
&> (C(x) \gamma)^{\beta-1} \nu \frac{(\gamma - \gamma_1) \gamma_1}{(\gamma + 1)},
\end{aligned}$$

2 and then

$$-F(\nabla\varphi, D^2\varphi) + |\nabla\varphi|^\beta \geq d^{-(\gamma+1)\alpha-\gamma-2+\gamma_1} (C(x)\gamma)^{\beta-1} \nu \frac{(\gamma-\gamma_1)\gamma_1}{(\gamma+1)} + o(d^{-(\gamma+1)\alpha-\gamma-2+\gamma_1}).$$

Taking  $\delta$  small enough one gets that by the assumption on  $f$  and using  $\gamma_1 < \gamma_o$

$$-F(\nabla\varphi, D^2\varphi) + |\nabla\varphi|^\beta > f^+ \geq f,$$

and then

$$-F(\nabla\varphi_1, D^2\varphi_1) + |\nabla\varphi_1|^\beta + \lambda\varphi_1^{1+\alpha} \geq f.$$

We then consider in  $\delta < d \leq 2\delta$ , as in the proof of Theorem 4.1

$$\varphi_2 = \frac{C(x)(1 + \nu\delta^{\gamma_1})}{\delta^\gamma} e^{\frac{1}{d-2\delta} + \frac{1}{\delta}} + D$$

where, if  $K_1$  is so that  $|F(\nabla\varphi_2, D^2\varphi_2)| + |\nabla\varphi_2|^\beta \leq K_1$ , we have denoted  $D$  some constant so that  $D \geq \left(\frac{|f|_\infty + K_1}{\lambda}\right)^{\frac{1}{1+\alpha}}$ . We have obtained that for  $\delta$  small and  $d < \delta$ , the function

$$\bar{w} = \begin{cases} \varphi + D & \text{in } d < \delta \\ \varphi_2 + D & \text{in } \delta < d < 2\delta \\ D & \text{in } d > 2\delta \end{cases}$$

3 is a convenient super-solution.

In the same manner let

$$\underline{w}_\delta = \left( \left( \frac{F(\nabla d, \nabla d \otimes \nabla d)(\gamma+1)}{\gamma^{\beta-\alpha-1}} \right)^{\frac{1}{\beta-\alpha-1}} - \nu d^{\gamma_1} \right) (d+\delta)^{-\gamma} - D$$

4 Then

$$\begin{aligned} -F(\nabla\underline{w}_\delta, D^2\underline{w}_\delta) + |\nabla\underline{w}_\delta|^\beta + \lambda|\underline{w}_\delta|^\alpha \underline{w}_\delta - f & \\ \leq -(d+\delta)^{-(\gamma+1)\alpha-\gamma-2+\gamma_1} (C(x)\gamma)^{\beta-1} \nu \frac{(\gamma-\gamma_1)\gamma_1}{(\gamma+1)} (1 + o(1)) & \\ + c\lambda(d+\delta)^{-\gamma(1+\alpha)} - f & \\ \leq 0, & \end{aligned}$$

5 when  $d$  is small, and arguing as previously one can choose  $D$  in order that  $\underline{w}_\delta$  be a sub-solution  
1 also in  $d > \delta$ .

2 Now arguing as in the proof of the previous Theorem, more precisely taking  $u_R$  a solution  
3 of (4.6) one gets the existence of  $u$  which blows up on the boundary and now is so that  
4  $u \sim C(x)d^{-\gamma}$  near the boundary. The uniqueness can be shown as in [13].  
5 □

6 We can now prove the Theorem

(exiergo) **Theorem 4.3.** *Under the assumptions of Theorem 4.1, and assuming in addition that  $\alpha \geq 2$   
8 and  $f$  is Lipschitz continuous, there exists an ergodic pair  $(u, c)$ , furthermore  $u(x) \sim C(x)d^{-\gamma}$   
1 near the boundary.*

*Proof.* of Theorem 4.3 : By Theorem 4.1, for  $\lambda > 0$  there exists a solution  $U_\lambda$  of problem (4.2), which satisfies estimates (4.3). It then follows that  $\lambda|U_\lambda|^\alpha U_\lambda$  is locally bounded in  $\Omega$ , uniformly with respect to  $0 < \lambda < 1$ . Let us fix an arbitrary point  $x_0 \in \Omega$ . Then, there exists  $c \in \mathbb{R}$  such that, up to a sequence  $\lambda_n \rightarrow 0$ ,

$$\lambda|U_\lambda(x_0)|^\alpha U_\lambda(x_0) \rightarrow -c.$$

On the other hand, Proposition 3.1 yields that  $(U_\lambda)$  is locally uniformly Lipschitz continuous. Therefore, for  $x$  in a compact subset of  $\Omega$ , one has using again (4.3) and the mean value's Theorem,

$$\lambda||U_\lambda(x)|^\alpha U_\lambda(x) - |U_\lambda(x_0)|^\alpha U_\lambda(x_0)| \leq \lambda \frac{K}{\lambda^{\frac{\alpha}{\alpha+1}}} |U_\lambda(x) - U_\lambda(x_0)| \rightarrow 0.$$

It then follows that  $c$  does not depend on the choice of  $x_0$  and, up to a sequence and locally uniformly in  $\Omega$ , one has

$$\lambda|U_\lambda|^\alpha U_\lambda \rightarrow -c.$$

Moreover, the function  $V_\lambda(x) = U_\lambda(x) - U_\lambda(x_0)$  is locally uniformly bounded, locally uniformly Lipschitz continuous and satisfies

$$-F(\nabla V_\lambda, D^2 V_\lambda) + |\nabla V_\lambda|^\beta = f - \lambda|U_\lambda|^\alpha U_\lambda \quad \text{in } \Omega.$$

If  $V$  denotes the local uniform limit of  $V_\lambda$  for a sequence  $\lambda_n \rightarrow 0$ , then one has

$$-F(\nabla V, D^2 V) + |\nabla V|^\beta = f + c \quad \text{in } \Omega.$$

Let us define for arbitrary  $s > 0$  :

$$\begin{aligned} \phi(x) &= \frac{\sigma}{(d(x) + s)^\gamma} - \frac{\sigma}{(\delta_o + s)^\gamma} \quad \text{if } \gamma > 0, \\ \phi(x) &= -\sigma \log(d(x) + s) + \sigma \log(\delta_o + s) \quad \text{if } \gamma = 0, \end{aligned} \tag{4.11} \text{ ?phi?}$$

and  $\sigma = \left(\frac{\alpha}{2} + 1\right)^{\frac{1}{\beta - \alpha - 1}} \gamma^{-1}$  if  $\gamma > 0$ ,  $\sigma = \frac{\alpha}{2}$  if  $\gamma = 0$ . Using the computations in Theorem 4.1 and using Theorem 2.4, we have that, for some  $\delta_0 > 0$  sufficiently small,

$$V_\lambda \geq \phi + \min_{d(x)=\delta_0} V_\lambda \quad \text{in } \Omega \setminus \Omega_{\delta_0},$$

2 Letting  $\lambda, s \rightarrow 0$  we deduce that  $V(x) \rightarrow +\infty$  as  $d(x) \rightarrow 0$ . This shows that  $(c, V)$  is an  
 3 ergodic pair and concludes the proof. The asymptotic behaviour can be proved as in [13].  $\square$

## 4 **5 Proof of Theorem 1.1.**

5 We now prove Theorem 1.1.

*Proof of Theorem 1.1.* Let  $u_\lambda$  be a solution of (1.5). We begin by giving a bound that will be useful in the whole proof. Observe that  $u_\lambda^+$  is a sub solution of

$$-F(\nabla u_\lambda^+, D^2 u_\lambda^+) \leq |f|_\infty.$$

By the existence's Theorem in [10] let  $V$  be a solution of

$$\begin{cases} -F(\nabla V, D^2 V) = 2 & \text{in } \Omega \\ V = 0 & \text{on } \partial\Omega \end{cases}$$

Then  $V$  is bounded,  $|f|^{\frac{1}{1+\alpha}} V$  is then a super-solution and the comparison Theorem in [10] implies that

$$|u_\lambda^+|_\infty \leq |V|_\infty |f|_\infty^{\frac{1}{1+\alpha}} \leq c |f|_\infty^{\frac{1}{1+\alpha}}. \quad (5.1) \text{upper}$$

6 Let us consider first the case where there exists a sub-solution  $\varphi$  for (1.4). Then,  $\varphi - |\varphi|_\infty$  is  
 7 a sub-solution of equation (1.5), and by the comparison principle we deduce  $u_\lambda \geq \varphi - |\varphi|_\infty$ .  
 8 Thus, in this case  $(u_\lambda)$  is uniformly bounded in  $\Omega$ . The Lipschitz estimates in Theorem 2.2  
 9 then yield that  $u_\lambda$  is uniformly converging up to a sequence to a Lipschitz solution of problem  
 10 (1.4). Note that in this case we did not use  $\alpha \geq 2$ .

11 We now treat the second case, i.e. we suppose that (1.4) has no solutions. In particular  
 12  $|u_\lambda|_\infty$  diverges, since otherwise we could extract from  $(u_\lambda)$  a subsequence converging to a  
 13 solution of (1.4).

14 On the other hand, since  $-\left(\frac{|f|_\infty}{\lambda}\right)^{\frac{1}{1+\alpha}}$  is a sub solution of (1.5), by the comparison  
 1 principle we obtain  $u_\lambda^- \leq \left(\frac{|f|_\infty}{\lambda}\right)^{\frac{1}{1+\alpha}}$ , which, jointly with (5.1), yields  $\lambda |u_\lambda|_\infty^{1+\alpha} \leq c_1 |f|_\infty$ .

2 Hence, there exists  $(x_\lambda) \subset \Omega$  such that  $u_\lambda(x_\lambda) = -|u_\lambda|_\infty \rightarrow -\infty$  and there exists a constant  
 3  $c_\Omega \geq 0$  such that, up to a subsequence,  $\lambda|u_\lambda|_\infty^{1+\alpha} \rightarrow c_\Omega$ .

4 The rest of the proof follows the lines in [13].

5

□

6 *Proof.* of Theorem 1.2 We do not give the proof which follows the lines in [13].

□

## 7 6 A comparison principle for degenerate non linear el- 8 liptic equations without zero order terms

(compzero):

(2.4) **Theorem 6.1.** *Suppose that  $b$  is continuous and bounded on  $\Omega$  and that either  $\alpha = 0$  or  $\alpha \neq 0$  and  $f$  is a continuous function such that  $f \leq -m < 0$ . Suppose that  $u$  and  $v$  are respectively a sub-solution and a super solution of*

$$-F(\nabla u, D^2u) + b(x)|\nabla u|^\beta = f.$$

10 *Suppose that  $u$  or  $v$  is Lipschitz and both the two are bounded on  $\overline{\Omega}$ , that  $u \leq v$  on  $\partial\Omega$ . Then*  
 11  *$u \leq v$  in  $\Omega$ .*

12 *Proof.* Without loss of generality, we will suppose that  $u$  is Lipschitz continuous.

13 The case  $\alpha = 0$  is quite standard, and is done in [13], for the sake of shortness we do not  
 14 reproduce it here.

For the case  $\alpha \neq 0$  and  $f < 0$ , we use the change of function  $u = \varphi(z)$ ,  $v = \varphi(w)$  with

$$\varphi(s) = -\gamma_1(\alpha + 1) \log(\delta + e^{-\frac{s}{\alpha+1}}).$$

1 This function is used in [1], [6], [7], [23], [24].

We choose  $\delta$  small enough in order that the range of  $\varphi$  covers the ranges of  $u$  and  $v$ . The constant  $\gamma_1$  will be chosen small enough depending only on  $a$ ,  $\alpha$ ,  $\beta$ ,  $\inf_\Omega(-f)$  and  $|b|_\infty$ ; in this proof, any constant of this type will be called universal. Observe that  $\varphi' > 0$  while  $\varphi'' < 0$ . Let  $Z = \sum_i |\partial_i z|^{\frac{\alpha}{2}} \partial_i z$ ,  $W = \sum_i |\partial_i w|^{\frac{\alpha}{2}} \partial_i w$ . In the viscosity sense,  $z$  and  $w$  are respectively sub- and super- solution of

$$-F((\varphi')^{1+\alpha} \Theta_\alpha(\nabla z) D^2 z \Theta_\alpha(\nabla z) + (\varphi')^\alpha \varphi''(Z \otimes Z)) + b(x) \varphi'(z)^{\beta-\alpha-1} |\nabla z|^\beta - f \leq 0. \quad (6.1) \quad \boxed{\text{eq1}}$$

$$-F((\varphi')^{1+\alpha}\Theta_\alpha(\nabla w)D^2w\Theta_\alpha(\nabla w)+(\varphi')^\alpha\varphi''(W\otimes W))+b(x)\varphi'(w)^{\beta-\alpha-1}|\nabla w|^\beta-f \geq 0. \quad (6.2) \quad \boxed{\text{eq1}}$$

We define

$$H(x, s, p) = \frac{-a\varphi''(s)}{\varphi'(s)} \sum_i |p_i|^{2+\alpha} + b(x)\varphi'(s)^{\beta-\alpha-1}|p|^\beta + \frac{-f(x)}{\varphi'(s)^{\alpha+1}}.$$

The point is to prove that at  $\bar{x}$ , a maximum point of  $z - w$ ,  $\frac{\partial H(\bar{x}, s, p)}{\partial s} > 0$  for all  $p$ . This will be sufficient to get a contradiction. A simple computation gives

$$\varphi' = \frac{\gamma_1 e^{-\frac{s}{\alpha+1}}}{\delta + e^{-\frac{s}{\alpha+1}}}, \quad \varphi'' = \frac{-\gamma_1 \delta e^{-\frac{s}{\alpha+1}}}{(\alpha+1)(\delta + e^{-\frac{s}{\alpha+1}})^2}.$$

Hence

$$\left(\frac{-\varphi''}{\varphi'}\right)' = \frac{\delta}{(\alpha+1)^2} \frac{e^{-\frac{s}{\alpha+1}}}{(\delta + e^{-\frac{s}{\alpha+1}})^2} \text{ i.e. } \left(\frac{-\varphi''}{\varphi'}\right)' = -\frac{\varphi''}{(\alpha+1)\gamma_1} > 0.$$

Differentiating  $H$  with respect to  $s$  gives:

$$\partial_s H = a \sum_i |p_i|^{\alpha+2} \frac{-\varphi''}{(\alpha+1)\gamma_1} + (-f) \frac{-(\alpha+1)\varphi''}{(\varphi')^{\alpha+2}} + b(x)|p|^\beta(\beta-\alpha-1)(\varphi')^{\beta-\alpha-2}\varphi''.$$

2 Since  $-\varphi''$  is positive, we need to prove that

$$K := \frac{a \sum_i |p_i|^{\alpha+2}}{(\alpha+1)\gamma_1} + (-f) \frac{\alpha+1}{(\varphi')^{\alpha+2}} - |b|_\infty |p|^\beta(\beta-\alpha-1)(\varphi')^{\beta-\alpha-2} > 0.$$

1 We start by treating the case  $\beta < \alpha + 2$ .

Observe first that the boundedness of  $u$  and  $v$ , implies that there exists universal positive constants  $c_o$  and  $c_1$  such that

$$c_o \gamma_1 \leq \varphi' \leq c_1 \gamma_1.$$

Hence, it is easy to see that there exist three positive universal constants  $C'_1, C_i, i = 2, 3$  such that

$$K > \frac{C'_1 \sum_i |p_i|^{\alpha+2}}{\gamma_1} + \frac{C_2}{\gamma_1^{\alpha+2}} - \frac{C_3 (\sum_i |p_i|^2)^{\frac{\beta}{2}}}{\gamma_1^{\alpha+2-\beta}}.$$

We now observe that since  $\alpha > 0$   $C'_1 \sum_i |p_i|^{\alpha+2} \geq C'_1 N^{-\frac{\alpha}{2}} (\sum_i p_i^2)^{\frac{\alpha+2}{2}} := C_1 (\sum_i p_i^2)^{\frac{\alpha+2}{2}} = C_1 |p|^{\alpha+2}$ . We choose  $\gamma_1 = \min \left\{ 1, \left(\frac{C_3}{C_2}\right)^\beta, \left(\frac{C_3}{C_1}\right)^{\frac{1}{\alpha+1-\beta}} \right\}$ . With this choice of  $\gamma_1$ , for  $|p| \leq 1$ ,

$$\frac{C_1 |p|^{\alpha+2}}{\gamma_1} + \frac{C_2}{\gamma_1^{\alpha+2}} - \frac{C_3 |p|^\beta}{\gamma_1^{\alpha+2-\beta}} \geq \frac{C_2}{\gamma_1^{\alpha+2}} - \frac{C_3}{\gamma_1^{\alpha+2-\beta}} > 0;$$

while for  $|p| \geq 1$ ,

$$\frac{C_1|p|^{\alpha+2}}{\gamma_1} + \frac{C_2}{\gamma_1^{\alpha+2}} - \frac{C_3|p|^\beta}{\gamma_1^{\alpha+2-\beta}} \geq \frac{(C_1)|p|^{\alpha+2}}{\gamma_1} - \frac{C_3|p|^\beta}{\gamma_1^{\alpha+2-\beta}} > 0.$$

2 If  $\beta = \alpha + 2$ , just take  $\gamma_1 < \frac{a}{(\alpha+1)|b|_\infty}$ .

This gives that for  $\gamma_1$  small enough depending only on  $\min(-f)$ ,  $\alpha$ ,  $|b|_\infty$  and  $\beta$  one has, for some universal constant  $C$ ,

$$\partial_s H(x, s, p) \geq C > 0. \quad (6.3) \quad \boxed{\text{gamma}}$$

3 We now conclude the proof of the comparison principle. Suppose by contradiction that  
4  $\sup(z - w) > 0$ .

We introduce  $\psi_j(x, y) = z(x) - w(y) - \frac{j}{2}|x - y|^2$ ,

$$(p_j, X_j) \in \bar{J}^{2,+} z(x_j), (p_j, -Y_j) \in \bar{J}^{2,-} w(y_j), \text{ with } p_j = j(x_j - y_j)$$

and

$$\begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} \leq j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

On  $(x_j, y_j)$ , by a continuity argument, for  $j$  large enough one has

$$z(x_j) > w(y_j) + \frac{\sup(z - w)}{2}.$$

Note for later purposes that since  $z$  or  $w$  are Lipschitz,  $p_j = j(x_j - y_j)$  is bounded. Observe that the monotonicity of  $\frac{\varphi''}{\varphi'}$  implies that

$$N = p_j \otimes p_j \left( \frac{\varphi''(z(x_j))}{\varphi'(z(x_j))} - \frac{\varphi''(w(y_j))}{\varphi'(w(y_j))} \right) \leq 0.$$

5 Using the fact that  $z$  and  $w$  are respectively sub and super solutions of the equation (6.2),  
1 the estimate (6.3) and that  $H$  is decreasing in the second variable, one obtains:

$$\begin{aligned} 0 &\geq \frac{-f(x_j)}{(\varphi')^{\alpha+1}(z(x_j))} - F\left(p_j, X_j + \frac{\varphi''(z(x_j))}{\varphi'(z(x_j))} p_j \otimes p_j\right) + b(x_j) |p_j|^\beta \varphi'(z(x_j))^{\beta-\alpha-1} \\ &\geq \frac{-f(x_j)}{(\varphi')^{\alpha+1}(z(x_j))} - F(p_j, -Y_j + \frac{\varphi''(w(y_j))}{\varphi'(w(y_j))} p_j \otimes p_j) \\ &\quad + a \sum_i |(p_j)_i|^{2+\alpha} \left( \frac{\varphi''(w(y_j))}{\varphi'(w(y_j))} - \frac{\varphi''(z(x_j))}{\varphi'(z(x_j))} \right) + |p_j|^\beta b(x_j) \varphi'(z(x_j))^{\beta-\alpha-1} \\ &\geq \frac{f(y_j) - f(x_j)}{(\varphi'(w(y_j)))^{\alpha+1}} + (b(x_j) - b(y_j)) |p_j|^\beta \varphi'(w(y_j))^{\beta-\alpha-1} \\ &\quad + H(x_j, z(x_j), p_j) - H(x_j, w(y_j), p_j) \\ &\geq C(z(x_j) - w(y_j)) + \frac{o(1)}{\gamma_1^{\alpha+1}}. \end{aligned}$$

Here we have used the continuity of  $f$  and  $b$ , the boundedness of  $p_j$  and that

$$\psi(x_j, y_j) \geq \sup(\psi(x_j, x_j), \psi(y_j, y_j)).$$

2 Passing to the limit one gets a contradiction, since  $(x_j, y_j)$  converges to  $(\bar{x}, \bar{x})$  such that  
3  $z(\bar{x}) > w(\bar{z})$ . □

4 Theorem 6 enables us to prove, arguing as in [13], Theorem 1.2.

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