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# Shifted Legendre polynomials algorithm used for the numerical analysis of viscoelastic plate with a fractional order model

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## Abstract

An effective numerical algorithm is presented to analyse the fractional viscoelastic plate in the time domain for the first time in this paper. The viscoelastic behaviour of the plate is described with fractional Kelvin-Voigt (FKV) constitutive model in three-dimensional space. A governing equation with three independent variables is established. Ternary unknown function in the governing equation is solved by deriving integer and fractional order differential operational matrices of the shifted Legendre polynomials. Error analysis and mathematical example are presented to verify the effectiveness and accuracy of proposed algorithm. Finally, numerical analysis of the plate under different loading conditions is carried out. Effects of the damping coefficient on vibration amplitude of the viscoelastic plate are studied. The results obtained are consistent with the current reference and actual situation. It shows that shifted Legendre polynomials algorithm is suitable for numerical analysis of fractional viscoelastic plates.

*Keywords:* Viscoelastic plate, Fractional Kelvin-Voigt model, Governing equation, Shifted Legendre polynomial, operational matrix, Numerical analysis

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## 1. Introduction

Viscoelastic materials have attracted more and more attention due to their good vibration damping properties, especially in the field of mechanical, aerospace and civil engineering. Viscoelastic plates have been widely used as structural components due to their light weight and high strength [1]. These plates are subjected to static, quasi-static

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and dynamic loads. The system may exhibit great amplitude vibrations. With the development of fractional calculus, the material constitutive models with fractional order are widely used to describe their viscoelastic damping characteristics. A lot of research work is concentrated in the dynamic analysis of viscoelastic plate by using the fractional order constitutive models.

Nick et al. [2] used the Analog Equation Method to convert the three coupled nonlinear partial fractional differential equations of the viscoelastic material plate into three uncoupled linear equations. Several kinds of plates under different harmonic loads and different boundary conditions were analyzed, and the influence of the material's viscoelastic properties was studied. Hedrih [3] used fractional order constitutive law to describe the creep behavior of materials. The linear partial fractional differential equation was proposed to analyse the lateral vibration of the plate under transverse excitation. Rossikhin and Shitikova [4] proposed the Laplace integral transformation method to solve the transient forced vibration problem of linear viscoelastic plate. The damping features of the plate were characterised by fractional derivatives with respect to time. The effect of viscosity on the numerical solution was analysed. Katsikadelis and Babouskos [5] analysed the post-buckling response of a viscoelastic plate with all edges simply supported. The behavior of viscoelastic materials was described by fractional generalized Kelvin-Voigt model. The deflection and stress of plates under different linear loads were obtained, which demonstrated the efficiency and accuracy of the proposed solution procedure. Rouzegar et al. [6] used the fractional Voigt model to analyze the vibration of a fully simply supported viscoelastic plate. An analytical solution of the derived linear governing equation of the viscoelastic plate was proposed. The results were compared with the classical elastic and viscoelastic plate in the literature. The above research consisted to calculate the displacement of the viscoelastic plate by expanding the displacement function into infinite series. This approach is complicated and difficult to obtain a unique numerical solution directly in the time domain. In this paper, the governing equation of the viscoelastic plate under harmonic and constant loading conditions is solved directly in the time domain, which largely simplified the calculation procedure.

Orthogonal functions play an important role in finding numerical solutions of partial fractional differential equations in the time domain, such as Bernstein polynomials [7], Jacobi polynomials [8], Hybrid functions polynomials [9], Legendre polynomials [10], Chebyshev polynomials [11], Chebyshev wavelets [12], Haar wavelets [13] etc. Among them, shifted Legendre polynomials algorithm is easier to solve the partial fractional differential equations with different physical mechanisms and practical meanings. Abbas-

bandy et al. [14] proposed a Legendre polynomials algorithm to solve a time fractional convection-diffusion equation which was widely used in the numerical simulation in engineering and science. The use of the operational matrix of the fractional order Legendre functions simplified the solution of the fractional differential equation. Hashemizadeh and Ebrahimzadeh [15] solved the time fractional dissipative Klein-Gordon equations by using the shifted Legendre polynomials. The approximated solutions were computed by substituting the partial fractional differential equations into a set of algebraic equations. Cao et al. [16] used shifted Legendre polynomials algorithm to analyse the dynamic response of viscoelastic beam under various loading conditions. The displacement of viscoelastic PMMA beam was solved directly in time domain, which demonstrated the efficiency of the proposed method. These numerical algorithms are used to obtain the solution of the governing equations based on the two-dimensional viscoelastic material model. In this paper, shifted Legendre polynomials algorithm is used for the first time to investigate the governing equation of plate with three-dimensional fractional viscoelastic model in the time domain.

This paper is organized as follows: In Section 2, viscoelastic plate's partial differential governing equation under FKV model is derived in three-dimensional space. In Section 3 the differential operational matrices based on shifted Legendre polynomials are obtained. Error analysis of the proposed algorithm and a mathematical example are given in Section 4. In Section 5, transverse displacements of the viscoelastic plate are calculated and compared under different loading conditions in the time domain. Section 6 presents the conclusions.

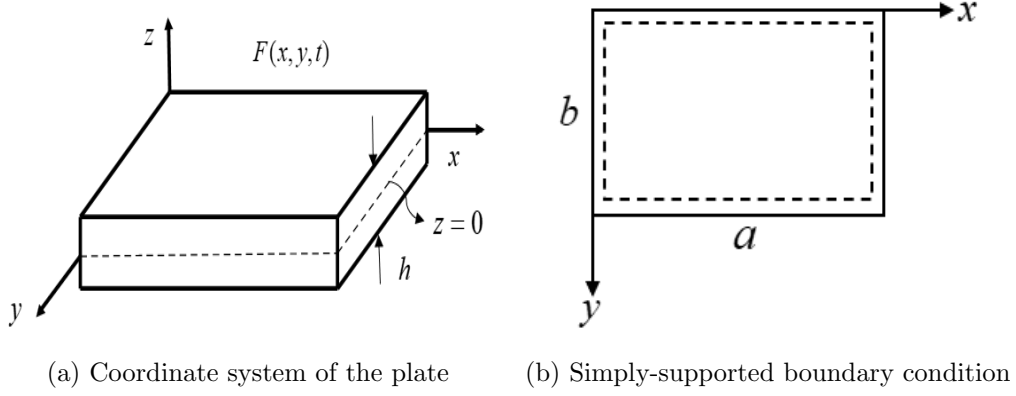
## 2. Establishment of governing equation of viscoelastic plate with FKV model

Consider a viscoelastic plate with length of  $a$ , width of  $b$  and thickness of  $h$  as shown in Fig. 1. The rectangular plate is simply-supported along the four edges. The transverse distributed load applied on the top surface of the plate is expressed by  $F(x, y, t)$ .

In this paper, the fractional model is used to describe the viscoelastic property of the material. Shemermergor is one of the first researchers to introduce a modification to the Kelvin-Voigt model of fractional derivative viscoelastic materials. The time derivative of the Kelvin-Voigt model is replaced by the  $\alpha$ -order fractional derivative [17], and the following definition of the FKV model can be obtained.

$$\sigma = (E + \eta D_t^\alpha) \varepsilon \quad (1)$$

where  $\sigma$  is stress,  $\varepsilon$  is strain.  $E$  represents Young's modulus of the plate and  $\eta$  is damping



**Fig. 1.** Configuration of simply-supported viscoelastic plate.

coefficient of the structure.  $D_t^\alpha(\cdot)$  is fractional derivative in the Caputo sense, which is defined as follows

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & \alpha > 0, m-1 \leq \alpha < m \\ \frac{d^m f(t)}{dt^m}, & \alpha = m, \end{cases} \quad (2)$$

where  $m \in N_+$  ( $N_+$  denotes positive integer),  $\Gamma(*)$  is Gamma function,  $\Gamma(*) = \int_0^\infty e^{-t} t^{*-1} dt$ .  $\alpha$  is fractional derivative order.

Based on the reference [3], the stress-strain component of the FKV model is expressed in three-dimensional space and the partial fractional differential governing equation is derived, the governing equation of the viscoelastic plate under the FKV model can be rewritten as

$$D \left( \frac{\partial^4 w(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y, t)}{\partial y^4} \right) + \eta D D_t^\alpha \left( \frac{\partial^4 w(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y, t)}{\partial y^4} \right) + \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} = F(x, y, t) \quad (3)$$

where  $D = Eh^3/12(1-\nu^2)$ ,  $\nu$  is Poisson's ratio,  $\rho$  is plate's density,  $w(x, y, t)$  is the displacement in the  $z$  direction.

The simply-supported boundary conditions are determined by

$$w(x, y, t)|_{x=0, a} = w(x, y, t)|_{y=0, b} = \frac{\partial^2 w(x, y, t)}{\partial x^2} \Big|_{x=0, a} = \frac{\partial^2 w(x, y, t)}{\partial y^2} \Big|_{y=0, b} = 0 \quad (4)$$

### 3. Numerical algorithm

#### 3.1. Shifted Legendre polynomials

Shifted Legendre polynomials  $l_n(x)$  of degree  $n$  defined on  $[0, 1]$  as following [18]

$$l_n(x) = \sum_{i=0}^n (-1)^{n+i} \frac{\Gamma(n+i-1)}{\Gamma(n-i+1)(\Gamma(i+1))^2} x^i \quad (5)$$

where  $i = 0, 1, \dots, n$ .

The shifted Legendre polynomials satisfy the orthogonality property

$$\int_0^1 l_i(x) l_j(x) dx = \begin{cases} 0, & i \neq j \\ \frac{1}{2i+1}, & i = j \end{cases} \quad (6)$$

where  $i, j = 0, 1, \dots, n$ .

Let  $\varphi_n(x)$  denotes a matrix formed by a series of shifted Legendre polynomials, it can be expressed by

$$\varphi_n(x) = [l_i(x), 0 \leq i \leq n]^T \quad (7)$$

In equivalent form

$$\varphi_n(x) = AP(x) \quad (8)$$

where  $P(x) = [1, x, \dots, x^n]^T$ ,

$$A = [a_{ij}]_{i,j=0}^n, [a_{ij}] = \begin{cases} 0, & i < j \\ (-1)^{i+j} \frac{\Gamma(i+j+1)}{\Gamma(i-j+1)(\Gamma(i+1))^2}, & i \geq j. \end{cases}$$

For practical application, the interval of shifted Legendre polynomials is extended to  $[0, H]$ . The shifted Legendre polynomials of degree  $n_1$  can be obtained.

$$\begin{aligned} L_{n_1}(x) &= \sum_{i=0}^{n_1} (-1)^{n_1+i} \frac{\Gamma(n_1+i+1)}{\Gamma(n_1-i+1)(\Gamma(i+1))^2} \left(\frac{x}{H}\right)^i \\ &= \sum_{i=0}^{n_1} (-1)^{n_1+i} \frac{\Gamma(n_1+i+1)}{\Gamma(n_1-i+1)(\Gamma(i+1))^2} \left(\frac{1}{H}\right)^i x^i \end{aligned} \quad (9)$$

where  $x \in [0, H]$ ,  $i = 0, 1, \dots, n_1$ .

$\varphi_{n_1}(x)$  can be converted into the following matrix product

$$\varphi_{n_1}(x) = MG(x) \quad (10)$$

where  $G(x) = [1, x, \dots, x^{n_1}]^T$ ,

$$M = [m_{ij}]_{i,j=0}^{n_1}, [m_{ij}] = \begin{cases} 0, & i < j \\ (-1)^{i+j} \frac{\Gamma(i+j+1)}{\Gamma(i-j+1)(\Gamma(i+1))^2} \left(\frac{1}{H}\right)^i, & i \geq j. \end{cases}$$

Similarly, they can be obtained

$$\varphi_{n_2}(y) = RG(y) \quad (11)$$

where  $y \in [0, S]$ ,  $G(y) = [1, y, \dots, y^{n_2}]^T$ .

$$R = [r_{ij}]_{i,j=0}^{n_2}, [r_{ij}] = \begin{cases} 0, & i < j \\ (-1)^{i+j} \frac{\Gamma(i+j+1)}{\Gamma(i-j+1)(\Gamma(i+1))^2} \left(\frac{1}{S}\right)^i, & i \geq j. \end{cases} \quad (12)$$

$$\varphi_{n_3}(t) = NG(t) \quad (13)$$

where  $t \in [0, K]$ ,  $G(t) = [1, t, \dots, t^{n_3}]^T$ .

$$N = [n_{ij}]_{i,j=0}^{n_3}, [n_{ij}] = \begin{cases} 0, & i < j \\ (-1)^{i+j} \frac{\Gamma(i+j+1)}{\Gamma(i-j+1)(\Gamma(i+1))^2} \left(\frac{1}{K}\right)^i, & i \geq j. \end{cases} \quad (14)$$

### 3.2. Approximation of ternary function

The displacement with time can be considered as a separable function according to the well-known vibration analysis procedure. The displacement function of Eq. (3) can be separated into [6]

$$w(x, y, t) = w(x, y)w(t) \quad (15)$$

where  $w(x, y) \in L^2([0, H] \times [0, S])$ ,  $w(t) \in L^2([0, K])$ .

Any one-variable function  $w(t) \in L^2([0, K])$  can be approximated by the shifted Legendre polynomials of finite terms.

$$w(t) \approx \sum_{k=0}^{n_3} c_k L_k(t) = C^T \varphi_{n_3}(t) \quad (16)$$

where  $n_3$  represents the number of terms of shifted Legendre polynomials,  $c_k = \langle w(t), L_k(t) \rangle$  represents the coefficient of shifted Legendre polynomials,  $C^T = [c_k]_{i=0}^{n_3}$ , and  $\varphi_{n_3}(t) = [L_k(t), 0 \leq k \leq n_3]^T$ .

Two-variable function  $w(x, y) \in L^2([0, H] \times [0, S])$  can be approximated as follows

$$w(x, y) \approx \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} w_{ij} L_i(x) L_j(y) = \varphi_{n_1}^T(x) W \varphi_{n_2}(y) \quad (17)$$

where  $n_1, n_2$  represent the number of terms of shifted Legendre polynomials,  $\varphi_{n_1}(x) = [L_i(x), 0 \leq i \leq n_1]^T$ ,  $\varphi_{n_2}(y) = [L_j(y), 0 \leq j \leq n_2]^T$ .  $w_{ij} = \langle w(x, y), L_i(x) L_j(y) \rangle$  is co-

efficient,  $W$  is the  $(n_1 + 1) \times (n_2 + 1)$  coefficient matrix to be calculated, which is defined

$$\text{as } W = \begin{bmatrix} w_{0,0} & w_{0,1} & \cdots & w_{0,n_2} \\ w_{1,0} & w_{1,1} & \cdots & w_{1,n_2} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n_1,0} & w_{n_1,1} & \cdots & w_{n_1,n_2} \end{bmatrix}.$$

By substituting Eq. (16) and Eq. (17) into Eq. (15), the following result can be obtained

$$w(x, y, t) = w(x, y)w(t) \approx \varphi_{n_1}^T(x)W\varphi_{n_2}(y)C^T\varphi_{n_3}(t) \quad (18)$$

### 3.3. Differential operational matrix

#### 3.3.1. Integer order differential operational matrix

**Definition 1** If a matrix  $D_x^m$  ( $m \in N_+$ ) such that  $\varphi_{n_1}^{(m)}(x) = D_x^m \varphi_{n_1}(x)$  exists, then  $D_x^m$  is called a  $m$ -order differential operational matrix of shifted Legendre polynomials.

When  $m = 1$ , it can be obtained

$$\varphi'_{n_1}(x) = (MG(x))' = MG'(x) = M \begin{bmatrix} 1' \\ x' \\ \vdots \\ (x^{n_1})' \end{bmatrix} = M \begin{bmatrix} 0 \\ 1 \\ \vdots \\ n_1 x^{n_1-1} \end{bmatrix} = MV_1 G(x) \quad (19)$$

where  $V_1 = [v_{ij}]_{i,j=0}^{n_1}$ ,  $v_{ij} = \begin{cases} 0, & i \neq j + 1 \\ i, & i = j + 1. \end{cases}$

Eq. (19) can be rewritten according to Eq. (10)

$$\varphi'_{n_1}(x) = D_x \varphi_{n_1}(x) = MV_1 M^{-1} \varphi_{n_1}(x) \quad (20)$$

where  $D_x = MV_1 M^{-1}$  is a first-order differential operational matrix of the shifted Legendre polynomials.

When  $m = 2$ , it can be obtained

$$\varphi''_{n_1}(x) = (\varphi'_{n_1}(x))' = MV_1 M^{-1} \varphi'_{n_1}(x) = (MV_1 M^{-1})^2 \varphi_{n_1}(x) = D_x^2 \varphi_{n_1}(x) \quad (21)$$

where  $D_x^2 = (MV_1 M^{-1})^2$  is second-order differential operational matrix of the shifted Legendre polynomials.

Above all, the  $m$ -order differential operational matrices of the shifted Legendre polynomials can be derived.

$$\varphi_{n_1}^{(m)}(x) = (MV_1 M^{-1})^m \varphi_{n_1}(x) = D_x^m \varphi_{n_1}(x) \quad (22)$$



$$\varphi_{n_2}^{(m)}(y) = (RV_2R^{-1})^m \varphi_{n_2}(y) = D_y^m \varphi_{n_2}(y) \quad (23)$$

$$\varphi_{n_3}^{(m)}(t) = (NV_3N^{-1})^m \varphi_{n_3}(t) = D_t^m \varphi_{n_3}(t) \quad (24)$$

where  $V_2 = [v_{ij}]_{i,j=0}^{n_2}$ ,  $V_3 = [v_{ij}]_{i,j=0}^{n_3}$ .

The partial differential equation terms in Eq. (3) can be written as

$$\begin{aligned} \frac{\partial^2 w(x, y, t)}{\partial x^2} &\approx \frac{\partial^2 (\varphi_{n_1}^T(x) W \varphi_{n_2}(y) C^T \varphi_{n_3}(t))}{\partial x^2} = \frac{\partial^2 \varphi_{n_1}^T(x)}{\partial x^2} W \varphi_{n_2}(y) C^T \varphi_{n_3}(t) \\ &= \varphi_{n_1}^T(x) (MV_1M^{-1})^2 W \varphi_{n_2}(y) C^T \varphi_{n_3}(t) \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{\partial^4 w(x, y, t)}{\partial x^4} &\approx \frac{\partial^4 (\varphi_{n_1}^T(x) W \varphi_{n_2}(y) C^T \varphi_{n_3}(t))}{\partial x^4} = \frac{\partial^4 \varphi_{n_1}^T(x)}{\partial x^4} W \varphi_{n_2}(y) C^T \varphi_{n_3}(t) \\ &= \varphi_{n_1}^T(x) (MV_1M^{-1})^4 W \varphi_{n_2}(y) C^T \varphi_{n_3}(t) \end{aligned} \quad (26)$$

$$\begin{aligned} \frac{\partial^2 w(x, y, t)}{\partial y^2} &\approx \frac{\partial^2 (\varphi_{n_1}^T(x) W \varphi_{n_2}(y) C^T \varphi_{n_3}(t))}{\partial y^2} = \varphi_{n_1}^T(x) W \frac{\partial^2 \varphi_{n_2}(y)}{\partial y^2} C^T \varphi_{n_3}(t) \\ &= \varphi_{n_1}^T(x) W (RV_2R^{-1})^2 \varphi_{n_2}(y) C^T \varphi_{n_3}(t) \end{aligned} \quad (27)$$

$$\begin{aligned} \frac{\partial^4 w(x, y, t)}{\partial y^4} &\approx \frac{\partial^4 (\varphi_{n_1}^T(x) W \varphi_{n_2}(y) C^T \varphi_{n_3}(t))}{\partial y^4} = \varphi_{n_1}^T(x) W \frac{\partial^4 \varphi_{n_2}(y)}{\partial y^4} C^T \varphi_{n_3}(t) \\ &= \varphi_{n_1}^T(x) W (RV_2R^{-1})^4 \varphi_{n_2}(y) C^T \varphi_{n_3}(t) \end{aligned} \quad (28)$$

$$\begin{aligned} \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} &\approx \frac{\partial^4 (\varphi_{n_1}^T(x) W \varphi_{n_2}(y) C^T \varphi_{n_3}(t))}{\partial x^2 \partial y^2} \\ &= \frac{\partial^4 (\varphi_{n_1}^T(x) W \varphi_{n_2}(y))}{\partial x^2 \partial y^2} C^T \varphi_{n_3}(t) \\ &= \varphi_{n_1}^T(x) (MV_1M^{-1})^2 W (RV_2R^{-1})^2 \varphi_{n_2}(y) C^T \varphi_{n_3}(t) \end{aligned} \quad (29)$$

$$\begin{aligned} \frac{\partial^2 w(x, y, t)}{\partial t^2} &\approx \frac{\partial^2 (\varphi_{n_1}^T(x) W \varphi_{n_2}(y) C^T \varphi_{n_3}(t))}{\partial t^2} \\ &= \varphi_{n_1}^T(x) W \varphi_{n_2}(y) C^T \frac{\partial^2 \varphi_{n_3}(t)}{\partial t^2} \\ &= \varphi_{n_1}^T(x) W \varphi_{n_2}(y) C^T (NV_3N^{-1})^2 \varphi_{n_3}(t) \end{aligned} \quad (30)$$

### 3.3.2. Fractional order differential operational matrix

**Definition 2** If a matrix  $K_t$  such that  $D_t^\alpha \varphi_{n_3}(t) = K_t \varphi_{n_3}(t)$  exists, then  $K_t$  is called a fractional order operational matrix of shifted Legendre polynomials.

Because the fractional derivative of  $\alpha$  order of  $t^{n_3}$  can be expressed as

$$D_t^\alpha t^{n_3} = \frac{\Gamma(n_3 + 1)}{\Gamma(n_3 + 1 - \alpha)} t^{n_3 - \alpha} \quad (31)$$

The following results can be obtained

$$\begin{aligned} D_t^\alpha \varphi_{n_3}(t) &= D_t^\alpha (NG(t)) = ND_t^\alpha G(t) = ND_t^\alpha \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{n_3} \end{bmatrix} \\ &= N \begin{bmatrix} 0 \\ \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha} \\ \vdots \\ \frac{\Gamma(n_3+1)}{\Gamma(n_3+1-\alpha)} t^{n_3-\alpha} \end{bmatrix} = NQG(t) \end{aligned} \quad (32)$$

where  $Q = [q_{ij}]_{i,j=0}^{n_3}$ ,  $q_{ij} = \begin{cases} 0, & \text{otherwise} \\ \frac{\Gamma(i)}{\Gamma(i+1-\alpha)} t^{-\alpha}, & i = j, i \neq 1. \end{cases}$

Eq. (32) can be rewritten according to Eq. (13)

$$D_t^\alpha \varphi_{n_3}(t) = K_t \varphi_{n_3}(t) = NQN^{-1} \varphi_{n_3}(t) \quad (33)$$

where  $K_t = NQN^{-1}$  is  $\alpha$  order differential operational matrix of  $\varphi_{n_3}$ .

The terms with fractional derivative in Eq. (3) can be obtained

$$\begin{aligned} D_t^\alpha \frac{\partial^4 w(x, y, t)}{\partial x^4} &\approx D_t^\alpha \left[ \varphi_{n_1}^T(x) (MV_1 M^{-1})^4 W \varphi_{n_2}(y) C^T \varphi_{n_3}(t) \right] \\ &= \varphi_{n_1}^T(x) (MV_1 M^{-1})^4 W \varphi_{n_2}(y) C^T D_t^\alpha \varphi_{n_3}(t) \\ &= \varphi_{n_1}^T(x) (MV_1 M^{-1})^4 W \varphi_{n_2}(y) C^T NQN^{-1} \varphi_{n_3}(t) \end{aligned} \quad (34)$$

$$\begin{aligned} D_t^\alpha \frac{\partial^4 w(x, y, t)}{\partial y^4} &\approx D_t^\alpha \left[ \varphi_{n_1}^T(x) W (RV_2 R^{-1})^4 \varphi_{n_2}(y) C^T \varphi_{n_3}(t) \right] \\ &= \varphi_{n_1}^T(x) W (RV_2 R^{-1})^4 \varphi_{n_2}(y) C^T D_t^\alpha \varphi_{n_3}(t) \\ &= \varphi_{n_1}^T(x) W (RV_2 R^{-1})^4 \varphi_{n_2}(y) C^T NQN^{-1} \varphi_{n_3}(t) \end{aligned} \quad (35)$$

$$\begin{aligned}
D_t^\alpha \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} &\approx D_t^\alpha \left[ \frac{\partial^2 (\varphi_{n_1}^T(x) W \varphi_{n_2}(y) C^T \varphi_{n_3}(t))}{\partial x^2 \partial y^2} \right] \\
&= \varphi_{n_1}^T(x) (MV_1 M^{-1})^2 W (RV_2 R^{-1})^2 \varphi_{n_2}(y) C^T D_t^\alpha \varphi_{n_3}(t) \\
&= \varphi_{n_1}^T(x) (MV_1 M^{-1})^2 W (RV_2 R^{-1})^2 \varphi_{n_2}(y) C^T N Q N^{-1} \varphi_{n_3}(t) \quad (36)
\end{aligned}$$

#### 3.4. Transformation of the governing equation of viscoelastic plate

Integer order and fractional order differential operators are substituted into the initial governing equation. Eq. (3) can be transformed into the following algebraic equation, where  $W$  and  $C$  are unknown.

$$\begin{aligned}
&D \varphi_{n_1}^T(x) (MV_1 M^{-1})^4 W \varphi_{n_2}(y) C^T \varphi_{n_3}(t) + D \varphi_{n_1}^T(x) W (RV_2 R^{-1})^4 \varphi_{n_2}(y) C^T \varphi_{n_3}(t) \\
&+ 2D \varphi_{n_1}^T(x) (MV_1 M^{-1})^2 W (RV_2 R^{-1})^2 \varphi_{n_2}(y) C^T \varphi_{n_3}(t) \\
&+ \eta D \varphi_{n_1}^T(x) (MV_1 M^{-1})^4 W \varphi_{n_2}(y) C^T N Q N^{-1} \varphi_{n_3}(t) \\
&+ \eta D \varphi_{n_1}^T(x) W (RV_2 R^{-1})^4 \varphi_{n_2}(y) C^T N Q N^{-1} \varphi_{n_3}(t) \\
&+ 2\eta D \varphi_{n_1}^T(x) (MV_1 M^{-1})^2 W (RV_2 R^{-1})^2 \varphi_{n_2}(y) C^T N Q N^{-1} \varphi_{n_3}(t) \\
&+ \rho h \varphi_{n_1}^T(x) W \varphi_{n_2}(y) C^T (NV_3 N^{-1})^2 \varphi_{n_3}(t) = F(x, y, t) \quad (37)
\end{aligned}$$

The boundary conditions in Eq. (4) are rewritten as

$$\begin{aligned}
w(0, y, t) &\approx \varphi_{n_1}^T(0) W \varphi_{n_2}(y) C^T \varphi_{n_3}(t) = 0 \\
w(a, y, t) &\approx \varphi_{n_1}^T(a) W \varphi_{n_2}(y) C^T \varphi_{n_3}(t) = 0 \\
w(x, 0, t) &\approx \varphi_{n_1}^T(x) W \varphi_{n_2}(0) C^T \varphi_{n_3}(t) = 0 \\
w(x, b, t) &\approx \varphi_{n_1}^T(x) W \varphi_{n_2}(b) C^T \varphi_{n_3}(t) = 0 \\
\frac{\partial^2 w(0, y, t)}{\partial x^2} &\approx \varphi_{n_1}^T(0) (MV_1 M^{-1})^2 W \varphi_{n_2}(y) C^T \varphi_{n_3}(t) = 0 \\
\frac{\partial^2 w(a, y, t)}{\partial x^2} &\approx \varphi_{n_1}^T(a) (MV_1 M^{-1})^2 W \varphi_{n_2}(y) C^T \varphi_{n_3}(t) = 0 \\
\frac{\partial^2 w(x, 0, t)}{\partial y^2} &\approx \varphi_{n_1}^T(x) W (RV_2 R^{-1})^2 \varphi_{n_2}(0) C^T \varphi_{n_3}(t) = 0 \\
\frac{\partial^2 w(x, b, t)}{\partial y^2} &\approx \varphi_{n_1}^T(x) W (RV_2 R^{-1})^2 \varphi_{n_2}(b) C^T \varphi_{n_3}(t) = 0 \quad (38)
\end{aligned}$$

Based on collection method, variable  $(x, t)$  is discretized into  $(x_i, t_j)$  by taking nodes.

Eq. (37) is transformed into a set of algebraic equations. Coefficient matrices  $W$  and  $C$  are determined by using the least square method. The numerical solutions of partial fractional differential equations can be obtained in the time domain.

#### 4. Error analysis and mathematical example

In this section, error analysis and mathematical example are proposed to verify the efficiency and accuracy of the shifted Legendre polynomials algorithm.

##### 4.1. Error analysis

The efficiency of shifted Legendre polynomials is proved for the approximation of ternary displacement function by following theorem. Lagrangian interpolation polynomial combined with shifted Chebyshev interpolation points are used to prove the theorem. The ternary function  $w(x, y, t)$  can be separated into  $w(x, y)w(t)$  by Eq. (15). Thus, the approximation theorem of  $w(t)$  is given. The same method applies to the approximation of  $w(x, y)$  with the proposed algorithm.

**Theorem 1** *Suppose  $w(t)(t \in [0, K])$  is a smooth function, and  $w_n(t)$  is the best approximation of  $w(t)$  given in Eq. (16). Then, it follows that*

$$\|w(t) - w_n(t)\|_\infty \leq C_1 \frac{(K/2)^{n+1}}{(n+1)!2^n} \quad (39)$$

where  $C_1$  is a non-negative constant.

**Proof** Let  $w(t)$  is a sufficiently smooth function on  $\Lambda$ . We assume that  $w_n(t) \in \Lambda_n$  is the best approximation of  $w(t)$  in this discussion.

There is an integrable and measurable function  $u_n(t) \in \Lambda_n$ , then according to the definition of the best approximation, it can be obtained that

$$\|w(t) - w_n(t)\|_\infty \leq \|w(t) - u_n(t)\|_\infty \quad (40)$$

If  $u_n(t)$  is the Lagrange interpolation polynomial at node points  $t_k (k = 0, 1, \dots, n)$ , where  $t_k$  are the roots of  $(n+1)$ -degree shifted Chebyshev polynomial in  $[0, K]$ . And  $u_n(t)$  also satisfies the above inequality. Then through the Lagrange interpolation polynomial formula and its error formula, we get

$$w(t) - u_n(t) = \frac{w^{(n+1)}(\theta)}{(n+1)!} \prod_{k=0}^n (t - t_k) \quad (41)$$

where  $\theta \in [0, K]$ , and hence it can be obtained that

$$\|w(t) - u_n(t)\|_\infty \leq \max_{0 \leq t \leq K} \left| w^{(n+1)}(\theta) \right| \frac{\left\| \prod_{k=0}^n (t - t_k) \right\|_\infty}{(n+1)!} \quad (42)$$

where  $(n + 1)!$  is a constant,  $w(t)$  is a smooth function on the interval  $[0, K]$ , therefore, there is a non-negative constant  $C_1$ , such that

$$\max_{0 \leq t \leq K} |w^{(n+1)}(\theta)| \leq C_1 \quad (43)$$

By substituting Eq. (43) into Eq. (42) and considering the estimation of Chebyshev interpolation nodes [19], we get

$$\|w(t) - u_n(t)\|_\infty \leq C_1 \frac{(K/2)^{n+1}}{(n+1)!2^n} \quad (44)$$

According to Eq. (40) and the above derivation, the following conclusion can be drawn

$$\|w(t) - w_n(t)\|_\infty \leq \|w(t) - u_n(t)\|_\infty \leq C_1 \frac{(K/2)^{n+1}}{(n+1)!2^n} \quad (45)$$

□

In summary, shifted Legendre polynomials algorithm is an effective algorithm for approximating ternary unknown function.

#### 4.2. Mathematical example

In this part, an abstract partial fractional differential equation is solved, which has the same structure as the partial differential equation of plate's motion. The objective is to solve this abstract partial fractional differential equation with the proposed numerical algorithm and compare it with its exact solution in order to verify the accuracy and efficiency of the algorithm.  $F(x, y, t)$  is obtained from algebraic solution.

The form of the mathematical example is as follows

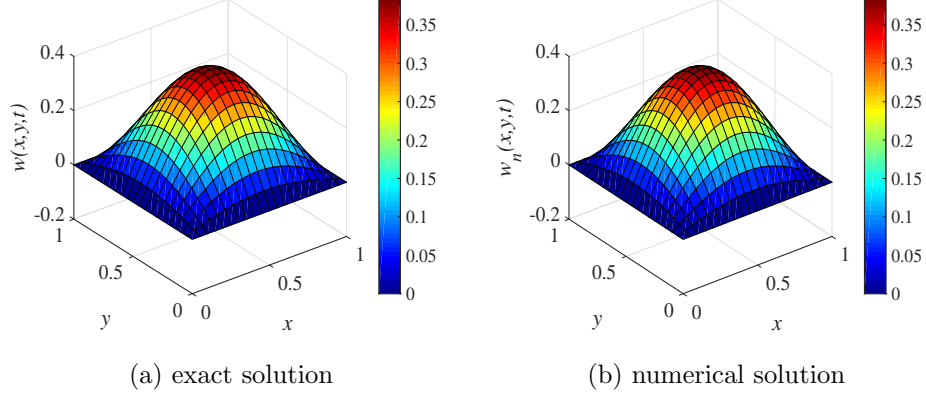
$$\begin{aligned} & 0.1 \left( \frac{\partial^4 w(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y, t)}{\partial y^4} \right) + 0.1 D_t^\alpha \left( \frac{\partial^4 w(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y, t)}{\partial y^4} \right) \\ & + 5.4 \frac{\partial^2 w}{\partial t^2} = F(x, y, t) \end{aligned} \quad (46)$$

where  $\alpha = 0.41$ ,  $F(x, y, t) = 0.16\pi^4 \sin(\pi x) \sin(\pi y) t^2 \left( 1 + \frac{\Gamma(3)}{\Gamma(3-\alpha)} t^{2-\alpha} \right) + 4.32 \sin(\pi x) \sin(\pi y)$ .

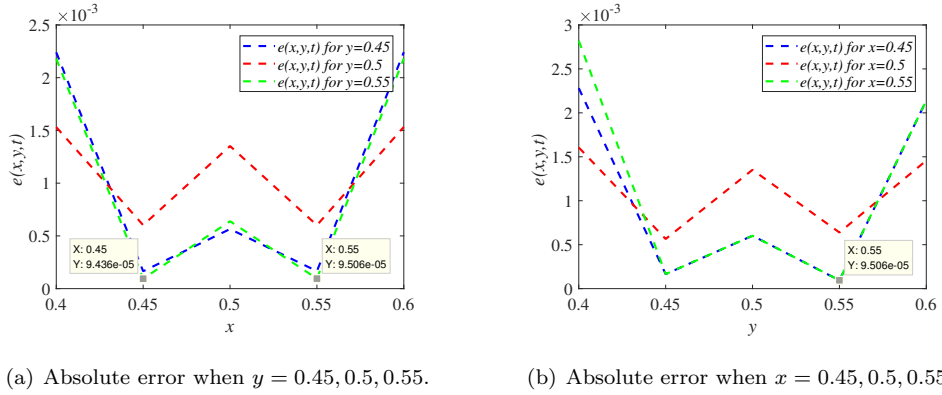
The boundary conditions are expressed as follows

$$w(x, y, t)|_{x=0,1} = w(x, y, t)|_{y=0,1} = \frac{\partial^2 w(x, y, t)}{\partial x^2} \Big|_{x=0,1} = \frac{\partial^2 w(x, y, t)}{\partial y^2} \Big|_{y=0,1} = 0 \quad (47)$$

The exact solution of Eq. (46) is  $w(x, y, t) = 0.4 \sin(\pi x) \sin(\pi y) t^2$ .



**Fig. 2.** Comparison of the exact and numerical solutions at different nodes when  $t = 1, (n_1, n_2, n_3) = (2, 2, 2)$ .



**Fig. 3.** Absolute error when  $t = 1, (n_1, n_2, n_3) = (2, 2, 2)$ .

The partial fractional differential equation is solved by shifted Legendre polynomials algorithm when  $(n_1, n_2, n_3) = (2, 2, 2)$ . The numerical solution and exact solution are denoted by  $w_n(x, y, t)$  and  $w(x, y, t)$ , respectively. It is obvious from the color maps as shown in Fig. 2 that the match between the numerical solution and the exact solution is excellent when  $t = 1$ . The absolute error  $e(x, y, t)$  is represented by  $e(x, y, t) = |w_n(x, y, t) - w(x, y, t)|$ . Absolute error is shown in Fig. 3(a) when  $y = 0.45, 0.5, 0.55, x \in [0.4, 0.6]$ . And the absolute error is shown in Fig. 3(b) when  $x = 0.45, 0.5, 0.55, y \in [0.4, 0.6]$ . As observed, the maximum absolute error value is almost less than  $3 \times 10^{-3}$  in all cases. The minimum absolute error value can reach  $9.436 \times 10^{-5}$ . The efficiency of the shifted Legendre polynomials algorithm for solving partial fractional differential equations of viscoelastic plates is verified.

## 5. Numerical results and discussions

In this part, numerical analysis of the viscoelastic plate is carried out based on the proposed algorithm. The viscoelastic plate under different loading conditions is studied, and the influence of **damping coefficient** on vibration amplitude of the plate is also studied. Geometric and material properties of the plate considered in the numerical computations are shown in Table. 1.

The boundary conditions are

$$w(x, y, t)|_{x=0,1} = w(x, y, t)|_{y=0,1} = \frac{\partial^2 w(x, y, t)}{\partial x^2} \Big|_{x=0,1} = \frac{\partial^2 w(x, y, t)}{\partial y^2} \Big|_{y=0,1} = 0 \quad (48)$$

Table 1: Geometrical and material properties of the plate considered in the numerical computations [20].

Physical quantity	Symbol	Value	Unit
Length	$a$	1	m
Width	$b$	1	m
Thickness	$h$	0.002	m
Young's Modulus	$E$	$2.1 \times 10^5$	MPa
Density	$\rho$	7850	kg/m <sup>3</sup>
Damping coefficient	$\eta$	1050	$Pa \cdot s^\alpha$
Poisson's ratio	$\nu$	0.3	–
Fractional derivative order	$\alpha$	0.25	–

### 5.1. Effect of loading time on the displacement of the plate

The response of viscoelastic plate under transverse simple harmonic load is first studied. The simple harmonic load form is  $F = 0.0001k\cos(\varpi t)$ , where  $k$  is the amplitude parameter and  $\varpi$  is the excitation frequency [2]. When  $k = 1$  m and  $\varpi = 1$  rad/s, the numerical solutions of the plate displacement are solved based on shifted Legendre polynomials algorithm at different time as shown in Fig. 4.

It can be seen from the figures that the plate has a certain deformation and produces transverse displacement. The displacement reaches the maximum at the center position of the plate ( $x = 0.5$  m,  $y = 0.5$  m). The displacement of the plate is symmetrical about the center of the plate. The displacement of the four edges of the viscoelastic plate is always zero and is not affected by time, which is consistent with the boundary conditions. In other positions, the transverse displacement of the viscoelastic plate increases with time.

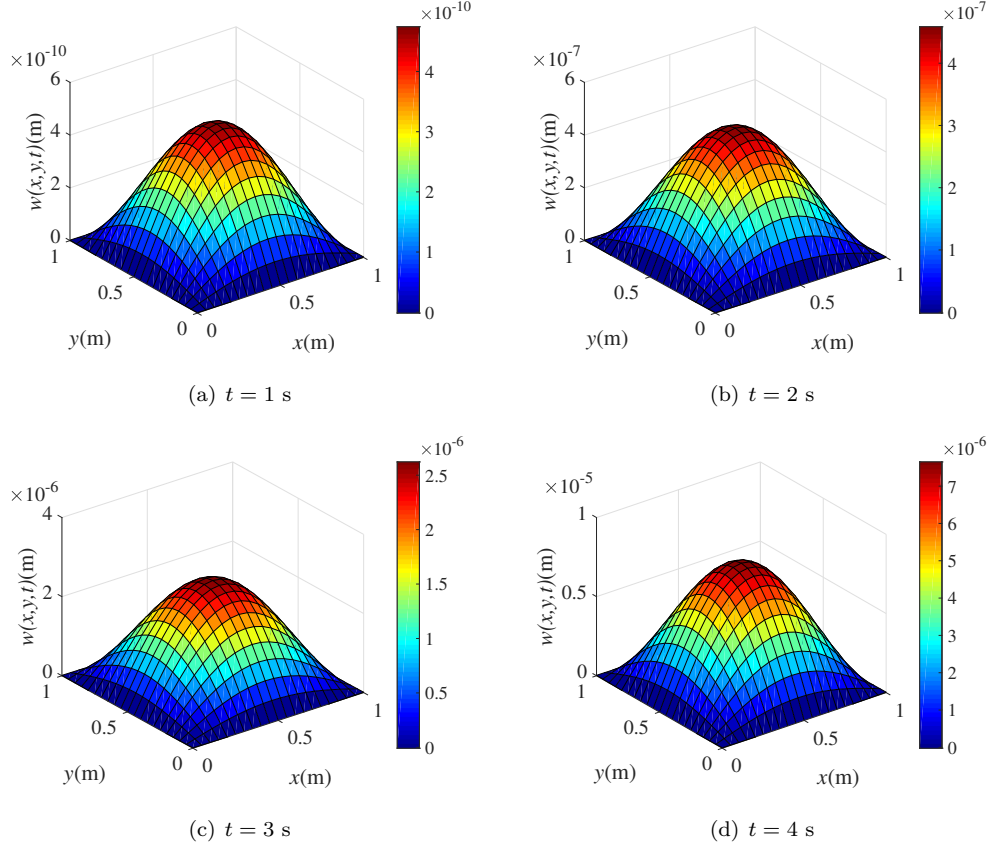


Fig. 4. Transverse displacements of viscoelastic plate under different time.

Through the computation of the proposed algorithm, numerical solutions of transverse displacements at different time can be directly obtained in the time domain. When  $t = 1$  s, the maximum displacement of the viscoelastic plate is  $4.744 \times 10^{-10}$  m, and when  $t = 4$  s, the maximum value reaches  $7.648 \times 10^{-6}$  m.

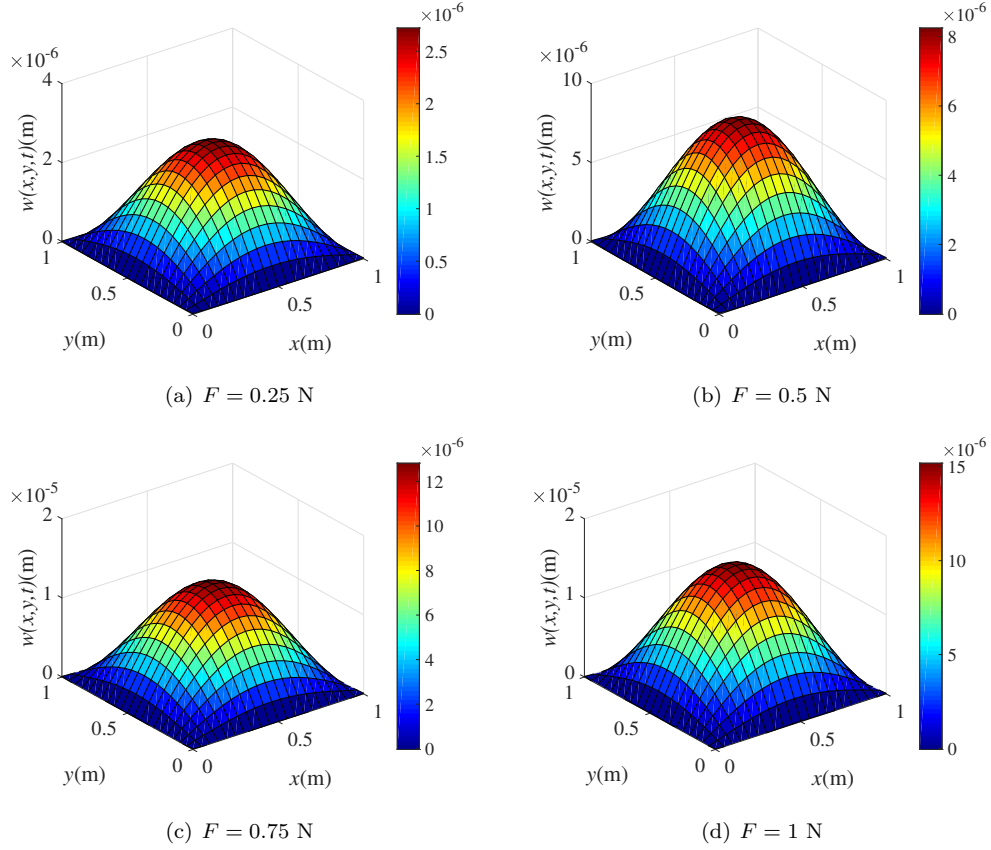
### 5.2. Effect of different loads on the displacement of the plate

When a constant transverse load is applied to the viscoelastic plate, the plate will deform and generate transverse displacements. Fig. 5 shows the transverse displacements of the plate at different positions under the same loading time under four loads: 0.25, 0.5, 0.75 and 1 N. It can be seen from the figures that the viscoelastic plate's displacement gradually increases with the increase of the applied transverse load, and the maximum value is obtained when  $F = 1$  N.

By using the proposed numerical algorithm, the time domain displacement solutions of the viscoelastic plate under different loads are obtained successfully. The material properties shown by the calculated results are consistent with the actual material prop-



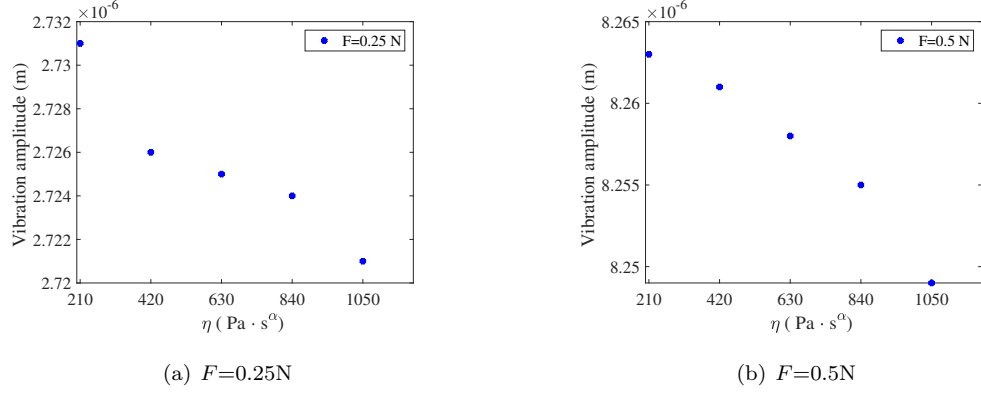
erties, which reflects the high accuracy of this algorithm. The algorithm can provide a theoretical basis for the development and performance prediction of viscoelastic material plates.



**Fig. 5.** Transverse displacements of viscoelastic plate under different loads.

### 5.3. Effects of vibration amplitude

The proposed algorithm is used to calculate and study vibration amplitude of the viscoelastic plate in this subsection. Vibration amplitude of the plate are calculated for a point located at  $x = 0.5 \text{ m}$  and  $y = 0.5 \text{ m}$  from the left corner of the plate when  $t = 1 \text{ s}$ . In numerical computations, the fractional order derivative  $\alpha$  is taken as 0.25. When the damping coefficient takes different values, the vibration amplitude of the viscoelastic plate under different loads ( $F = 0.25 \text{ N}$  and  $F = 0.5 \text{ N}$ ) is shown in the Fig. 6.



**Fig. 6.** The effect of damping coefficient on vibration amplitude under different loads.

As observed, increasing the damping coefficient reduces the vibration amplitude of the plate. The stiffness of the plate increases with the increase of  $\eta$ , thus reducing the amplitude of vibration. The conclusions obtained are consistent with the conclusions of the Ref. [21]. The Galerkin method is used in Ref. [21] to solve the partial fractional differential governing equation of the plate. The algorithm proposed in this paper provides a new idea for this research. Shifted Legendre polynomials algorithm can be used to solve and study vibration amplitudes of viscoelastic plate in the time domain.

## 6. Conclusions

In this paper, FKV constitutive model is applied to establish the viscoelastic plate's governing equation in three-dimensional space. Shifted Legendre polynomials algorithm is proposed to solve directly the equation in the time domain. The numerical analysis of viscoelastic plate is effectuated with different loading conditions. The influence of damping coefficient on vibration amplitude is investigated in the time domain. The results show that the algorithm has high accuracy and certain practical value. The results are as follows

1. FKV model reduces the number of physical parameters and describes the damping characteristics of viscoelastic plates well in three dimensional space.
2. Shifted Legendre polynomials algorithm can accurately solve ternary displacement functions of viscoelastic plates in the time domain.
3. Numerical solutions of viscoelastic plate's displacement under different loading conditions are compared and analyzed.
4. When the fractional derivative order is fixed, increasing the damping coefficient will increase the damping properties of the plate, resulting in reduced vibration amplitude.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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