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INTERVAL MAPS WITH DENSE PERIODICITY

JOZEF BOBOK, JERNEJ ČINČ, PIOTR OPROCHA, AND SERGE TROUBETZKOY

ABSTRACT. We consider the class of interval maps with dense set of periodic points CP and its closure \overline{CP} equipped with the metric of uniform convergence. Besides studying basic topological properties and density results in the spaces CP and \overline{CP} we prove that \overline{CP} is dynamically characterized as the set of interval maps for which every point is chain-recurrent. Furthermore, we prove that a strong topological expansion property called topological exactness (or leo property) is attained on the open dense set of maps in CP and on a residual set in \overline{CP} . Moreover, we show that every second category set in CP and \overline{CP} is rich in a sense that it contains uncountably many conjugacy classes. An analogous conclusion also holds in the setting of interval maps preserving any fixed non-atomic probability measure with full support. Finally, we give a detailed description of the structure of periodic points of generic maps in CP and \overline{CP} and show that generic maps in CP and \overline{CP} satisfy the shadowing property.

1. INTRODUCTION

Many definitions of chaos in a compact metric space X involve an assumption that the set of periodic points is dense in X . In the current article we put ourselves in the context of X being a unit interval and study dynamical, topological and dimension properties of the set of interval maps with dense set of periodic points CP and its closure \overline{CP} using the metric of uniform convergence. Our line of study can be thought as a continuation of Barge and Martin's study [2], where the authors gave a nice dynamical characterization of the maps from CP .

Let $I := [0, 1]$ and let $C(I)$ denote the set of all continuous interval maps. We call $f \in C(I)$ a chain recurrent map if f is chain recurrent in every point of I . The notion of a chain recurrent map is one of the standard notions in topological dynamics. In [3] Block and Coven provided a dynamical description of chain recurrent maps. In complement to these results, our first main result shows that a map $f \in C(I)$ is chain-recurrent if and only if $f \in \overline{CP}$ (Theorem 3.1). Theorem 3.1 implies that the maps id and up to conjugacy

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the map $1 - \text{id}$ are the only maps with zero topological entropy from \overline{CP} (Corollary 3.2). A map $f \in C(I)$ is called *turbulent* if there exist nondegenerate closed intervals $J, K \subset I$ with at most one point in common, such that $J \cup K \subset f(J) \cap f(K)$.

Note that Barge and Martin in [2] proved that if f has a dense set of periodic points, then either f^2 is the identity map or f^2 is turbulent. Theorem 3.1 together with results from [3] also implies that if $f \in \overline{CP}$ such that $h_{\text{top}}(f) > 0$ then f^2 is turbulent. Therefore $h_{\text{top}}(f) \geq \log 2/2$ for every map $f \in \overline{CP}$ except for the id or maps conjugate to $1 - \text{id}$.

Our second main result is that CP and \overline{CP} are uniformly locally arcwise connected (see the beginning of Subsection 4.2 and Theorem 4.10). In particular, uniform local arcwise connectedness implies local connectedness. In the same section we also prove that the set of Lebesgue measure-preserving interval maps C_λ is arcwise connected (Corollary 4.8), which, in particular, answers on a question of Michał Misiurewicz asked during a conference in Barcelona in 2020. These results fit in the context of study initiated by Kolyada, Misiurewicz and Snoha [21] who proved that the spaces of transitive, piecewise linear transitive and piecewise monotone transitive interval maps are locally connected and arcwise connected.

Next we turn our attention towards studying dynamical properties which hold on dense sets in the sets CP and \overline{CP} . We prove another fundamental result that locally eventually onto (or leo for short, sometimes also called topologically exact) maps form an open dense collection of maps in CP (Theorem 5.3). This, in particular, gives another aspect to the dynamical characterization of maps from CP from the paper of Barge and Martin [2]. Theorem 5.3 implies that there is a residual (but not open) set of leo maps in \overline{CP} as well. To put these results in the context, it was proven by the first and the last author in [6] that generic interval maps of C_λ are also leo. The proof of Theorem 5.3 also yields that leo maps are open and dense in C_λ . As an application we get that maps satisfying the periodic specification property form an open dense collection in CP (Corollary 5.7).

Let \mathcal{X} be a standard Borel space and ν a non-atomic probability measure on \mathcal{X} . Let $\text{Aut}(\nu)$ denote the topological group of all measurable, measure-preserving bijections of \mathcal{X} . A classical theorem of Rokhlin says that the conjugacy classes on $\text{Aut}(\nu)$ are meager (see [19, Theorem 2.5]). Moreover, Foreman and Weiss [16] proved that equivalence relation of conjugacy among the generic element in $\text{Aut}(\nu)$ is non-classifiable. This line of research culminated recently in the paper of Solecki [26] who answered the question of Glasner and Weiss about the structure of the closure of subgroups of $\text{Aut}(\nu)$ generated by iterates of elements $T \in \text{Aut}(\nu)$.

We study analogous questions in the setting of interval map where the conjugating maps are homeomorphisms. For $f \in C(I)$ we consider its *conjugacy class* defined by

$$\{\psi^{-1} \circ f \circ \psi \in C(I) : \psi \text{ is a homeomorphism of } I\}.$$

Before studying other typical properties of the set \overline{CP} we ask ourselves a fundamental question motivated by the preceding paragraph about the number of conjugacy classes in the residual sets in \overline{CP} . A beautiful result of Kechris and Rosendal [20] shows that there is a residual set in the space of all homeomorphisms of the Cantor set such that each pair of its elements are conjugate (so-called *residual conjugacy class*). Moreover, Bernardes and Darji [4] proved there is a residual conjugacy class in the space of all self-maps of the Cantor set. In the case of \overline{CP} all the standard invariants (entropy (see Proposition 5.22), mixing,...) do not exclude the possibility that such a class exists.

Our fourth main result, Theorem 5.8, is that every conjugacy class in \overline{CP} is nowhere dense. This implies that a residual conjugacy class cannot exist in \overline{CP} , it also implies that every second category set $G \subset \overline{CP}$ contains uncountably many conjugacy classes (Corollary 5.9). An analogous conclusion also holds in the setting of interval maps preserving any fixed non-atomic probability measure with full support and in CP . Corollary 5.9 also has an application to the results in [10]; it implies that there are uncountably many dynamically non-equivalent attractors in the neighborhood of any attractor in the construction of a parameterized family of planar homeomorphisms with pseudo-arc attractors in [10].

Finally we study typical properties of maps from CP as well as maps from \overline{CP} . A result in this direction was given in the literature recently in [10], where the second and third author proved that a very strong condition on the values of interval maps called the δ -crookedness is satisfied for some iterate of any generic map from \overline{CP} . This result can be interpreted through the notion of inverse limits; namely, the inverse limit of a generic map in \overline{CP} is the pseudo-arc (see [10, Theorem 1.3]). Similarly as in the context of C_λ in [7] we obtain that the shadowing property is typical in CP and \overline{CP} . Furthermore, we prove that these results are optimal in CP and \overline{CP} since there is a dense set of maps in CP resp. \overline{CP} which do not have the shadowing property (see Theorem 5.16). We also provide a study of the structure and dimension of periodic points of generic maps in CP and \overline{CP} (related results in the setting of interval continuous map have been given by Agronsky, Bruckner and Laczkovicz [1]). Namely, in Theorem 5.19 we completely describe the structure of fixed points, periodic points and the union of all periodic points as well as their Hausdorff, upper box and lower box dimension. Finally, we also show in Theorem 5.21 that the set of leo maps in CP whose periodic points have full Lebesgue measure and

whose periodic points of period k have positive measure for each $k \geq 1$ is dense in CP to give a counterpart of Theorem 5.19.

Let us give a brief outline of the paper. In Section 2 we start with introducing the basic notions that are used throughout the article. In Section 3 we prove Theorem 3.1 which gives a dynamical characterization of \overline{CP} . We start Section 4 with Subsection 4.1 where we prove structure statements that CP is a residual subset of \overline{CP} and that $\overline{CP} \setminus CP$ is dense in \overline{CP} . We continue with Subsection 4.2 where we prove the results about topological structure of spaces CP , \overline{CP} and C_λ , namely Theorem 4.7, Corollary 4.8 and Theorem 4.10. In Section 5 we address denseness properties of CP and \overline{CP} . First, in Subsection 5.2 we prove in Theorem 5.3 that the set of leo maps is open and dense in CP and then in Subsection 5.3 we show through Corollary 5.9 the result about conjugacy classes in \overline{CP} . In Subsection 5.1 we introduce the technique called window perturbations which is used in the rest of the article. We finish the article with Subsections 5.4 and 5.5 where we prove that shadowing is a typical property in \overline{CP} (Theorem 5.16) and describe the structure and dimension of periodic points of generic maps in \overline{CP} (Theorem 5.19).

2. PRELIMINARIES

Let $I := [0, 1]$ denote the unit interval, d denote the Euclidean distance on I , and $C(I)$ denote the set of all surjective continuous interval maps. A point x is called *periodic of period* $N \in \mathbb{N}$, if $f^N(x) = x$ and $f^i(x) \neq x$ for $1 \leq i < N$. Periodic points of $f \in C(I)$ of period N are denoted by $\text{Per}(f, N)$ and the set of periodic points of $f \in C(I)$ by $\text{Per}(f)$. Let λ denote the Lebesgue measure on the unit interval $I := [0, 1]$. We denote by $C_\lambda \subset C(I)$ the family of all continuous Lebesgue measure-preserving functions of I and more generally by $C_\mu \subset C(I)$ the family of all continuous interval maps preserving a non-atomic probability measure μ on I with full support. For μ a non-atomic probability measure on I with full support the map $\psi: I \rightarrow I$ defined as

$$\psi(x) = \mu([0, x])$$

is a homeomorphism of I ; moreover, if f preserves μ then $\psi \circ f \circ \psi^{-1} \in C_\lambda$ (see Remark in [7]).

Remark 2.1. *The map $f \mapsto \psi \circ f \circ \psi^{-1}$ is a homeomorphism of C_μ with C_λ .*

Let $CP \subset C(I)$ denote the set of continuous maps with a dense set of periodic points, i.e., all maps $f \in C(I)$ such that the closure $\overline{\text{Per}(f)} = I$. We consider the uniform metric on $C(I)$ defined by

$$\rho(f, g) := \sup_{x \in [0, 1]} |f(x) - g(x)|.$$

Let $B(x, \xi)$ denote the open ball of radius ξ centered at the point x in a metric space X and for a set $U \subset X$ we shall denote

$$B(U, \xi) := \bigcup_{x \in U} B(x, \xi).$$

We call a set containing a dense G_δ set *residual* and call a property *generic* if it is attained on at least a residual set of the Baire space on which we work.

Definition 2.2. *We say a map $f \in C(I)$ is*

- *transitive if for all nonempty open sets $U, V \subset I$ there exists $n \geq 0$ so that $f^n(U) \cap V \neq \emptyset$,*
- *topologically mixing if for all nonempty open sets $U, V \subset I$ there exists $n_0 \geq 0$ so that $f^n(U) \cap V \neq \emptyset$ for every $n \geq n_0$,*
- *locally eventually onto (leo) (also known as topologically exact) if for every nonempty open set $U \subset I$ there exists $n \in \mathbb{N}$ so that $f^n(U) = I$.*

3. A DYNAMICAL CHARACTERIZATION OF \overline{CP}

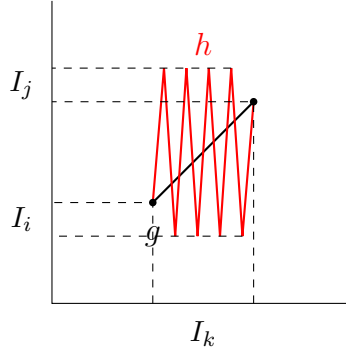
For $f \in C(I)$, $x, y \in I$ and $\varepsilon > 0$ an ε -chain from x to y is a finite sequence $x = x_0, \dots, x_n = y$ where $n > 0$ and $|f(x_i) - x_{i+1}| < \varepsilon$ for $0 < i < n - 1$. We say that x is *chain-recurrent (for f)* if for every $\varepsilon > 0$ there is an ε -chain from x to itself. We will call f *chain-recurrent* if every point in I is chain-recurrent for f . A point $x \in I$ is called *non-wandering* if for any open set U containing x there exists $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. It follows from the definitions that every non-wandering point is chain-recurrent and that any chain-recurrent map is onto.

Theorem 3.1. *A map $f \in C(I)$ is chain-recurrent if and only if $f \in \overline{CP}$.*

Proof. (\Rightarrow) Fix such a map f . The map f has to be onto. Fix $\varepsilon > 0$. By our assumption on f , there exist points $0 = x_0^1, \dots, x_0^m = 1 \in I$ and finitely many disjoint $\varepsilon/6$ -chains $\mathcal{A}_\ell := \{x_0^\ell, x_1^\ell, x_2^\ell, \dots, x_{n(\ell)-1}^\ell, x_{n(\ell)}^\ell = x_0^\ell\}$ from x_0^ℓ to itself. Since \mathcal{A}_ℓ are disjoint, the set $\mathcal{A} = \bigcup_{1 \leq \ell \leq m} \mathcal{A}_\ell$ can be written spatially as $\mathcal{I} = \{0 = i_0 < i_1 < \dots < i_{n-1} = 1\}$. We can assume that

- (i) $|f(u) - f(v)| < \varepsilon/6$ for each $k \in \{0, \dots, n-2\}$ and points $i_k \leq u < v \leq i_{k+1}$.
- (ii) $|i_k - i_{k+1}| < 2\varepsilon/3$ for each $k \in \{0, \dots, n-2\}$.

Let $\varphi: \mathcal{A} \rightarrow \mathcal{A}$ satisfy $\varphi(x_i^\ell) = x_{i+1}^\ell \pmod{n(\ell)}$. Consider the extension $g: I \rightarrow I$ defined as the connect-the-dots map of (\mathcal{A}, φ) . Clearly, \mathcal{A} is a union of $n(\ell)$ -cycles for g , $1 \leq \ell \leq m$. From (i) and the fact that \mathcal{A} is a union of

FIGURE 1. The perturbation h of g .

$\varepsilon/6$ -chains for f we obtain for each $x \in (i_k, i_{k+1})$

$$(1) \quad |g(i_k) - g(x)| \leq |g(i_k) - g(i_{k+1})| \leq |g(i_k) - f(i_k)| + |f(i_k) - f(i_{k+1})| + |f(i_{k+1}) - g(i_{k+1})| < \varepsilon/6 + \varepsilon/6 + \varepsilon/6 < \varepsilon/2,$$

hence

$$|f(x) - g(x)| \leq |f(x) - f(i_k)| + |f(i_k) - g(i_k)| + |g(i_k) - g(x)| < < \varepsilon/6 + \varepsilon/6 + \varepsilon/2 = 5\varepsilon/6;$$

it means that

$$(2) \quad \rho(f, g) < 5\varepsilon/6.$$

Denote $I_k := [i_k, i_{k+1}]$, $0 \leq k \leq n-2$. We can consider a regular piecewise affine perturbation $h: I \rightarrow I$ of g satisfying (see Figure 1).

- (iii) $g|_{\mathcal{A}} = h|_{\mathcal{A}}$,
- (iv) for every non-empty interval J , either $h(J) \supset I_k$ for some k or $\lambda(h(J)) > \alpha\lambda(J)$ for some $\alpha > 1$,
- (v) for each k , $h(I_k)$ contains \mathcal{I} -neighbors of $g(I_k)$,
- (vi) $|g(x) - h(x)| < 7\varepsilon/6$, $x \in I$.

The property (vi) can be fulfilled since (ii) and (1). We get from (2) and (vi)

$$(3) \quad |f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)| < 5\varepsilon/6 + 7\varepsilon/6 = 2\varepsilon, \quad x \in I.$$

Let us prove that h is leo. Fix a non-empty interval J , Property (iv) implies that after a finite number m of iterations $h^m(J) \supset I_k$ for some k . Remembering that $n = \Pi_\ell n(\ell)$ is the common period of our cycles, properties (iii) and (v) imply that applying h^n to this I_k must contain I_k and its \mathcal{I} -neighbors. Since there are finitely many intervals we eventually cover the whole space. The claim is proven, h is a leo map, in particular h is a map from the set CP . Since ε is arbitrary we have $f \in \overline{CP}$.

(\Leftarrow) Fix a map $f \in \overline{CP}$ and $\varepsilon > 0$. Since f is uniformly continuous, there is $\delta > 0$ such that $|f(u) - f(v)| < \varepsilon/2$ whenever $|u - v| < \delta$. Fix an arbitrary point $x \in I$. Taking a map $g \in CP$ such that $\rho(f, g) < \varepsilon/2$ and a g -periodic point p of period n satisfying $|x - p| < \delta$, we put

$$x_0 = x, \quad x_i = g^i(p), \quad 1 \leq i \leq n-1, \quad x_n = x.$$

Then

$$|f(x_0) - x_1| \leq |f(x) - f(p)| + |f(p) - g(p)| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for each $1 \leq i \leq n-2$,

$$|f(x_i) - x_{i+1}| = |f(g^i(p)) - g(g^i(p))| < \varepsilon/2 < \varepsilon;$$

choosing $\delta < \varepsilon/2$, since $p = g^n(p)$ and $x_n = x$, also

$$|f(x_{n-1}) - x_n| \leq |f(x_{n-1}) - p| + |p - x_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, the finite sequence $x_0 = x, x_1 = g(p), \dots, x_{n-1} = g^{n-1}(p), x_n$ is an ε -chain from x to itself. \square

A map $f \in C(I)$ is called *turbulent* if there exist nondegenerate closed intervals $J, K \subset I$ with at most one point in common, such that $J \cup K \subset f(J) \cap f(K)$.

Corollary 3.2. *The following hold:*

- (1) *The maps id and up to conjugacy the map $1 - \text{id}$ are the only maps with zero topological entropy from \overline{CP} .*
- (2) *$f \in \overline{CP}$ such that $h_{\text{top}}(f) > 0 \implies f^2$ turbulent. Therefore, either $h_{\text{top}}(f) = 0$ or $h_{\text{top}}(f) \geq \log 2/2$. In the latter case f has a periodic point of period 6.*

Proof. The statements follow from [3], where the authors prove that interval map f with chain-recurrent set I is either such that $f^2 = \text{id}$ or f^2 is turbulent. Furthermore, turbulent maps have topological entropy at least $\frac{1}{2} \log 2$. But if $f^2 = \text{id}$ then either $f = \text{id}$ or f is topologically conjugate to $1 - \text{id}$. By the definition of turbulence, the second iterate of f has a horseshoe and thus also the final of the claims above follows. \square

Theorem 3.3. *If a map $f \in C(I)$ satisfies $\psi \circ f = g \circ \psi \in C(I)$ where map $g \in CP$ and where ψ is (not necessarily strictly) monotone, but the ψ -preimages of g -periodic points are singletons, then f is chain recurrent. In particular, $f \in \overline{CP}$.*

Proof. Fix any $x \in I$. If x is in a singleton fibre of ψ then it is approximated by periodic points since $g \in CP$ and hence belongs to a chain recurrent set. So assume that $x \in J = \psi^{-1}(z)$ for some z where $\text{diam}(J) > 0$. Since z is not a periodic point, for any $\varepsilon > 0$ there is k and sets $J_i \subset I$, $i = -k, \dots, k$ such that $\text{diam}(J_{-k}) < \varepsilon$, $\text{diam}(J_k) < \varepsilon$ and $f(J_i) = J_{i+1}$ for $i = -k, \dots, k-1$ and

$x \in J_0 \subset J$. But then there is also a sequence of periodic points p_n such that $\lim_n p_n \in J_{-k}$. Then it is not hard to see that there is a 2ε chain from x to x . Indeed, we iterate x until reaching J_k , then we move onto orbit of $f^{2k}(p_n)$ and then return from p_n to J_{-k} hitting a point in $f^{-k}(x) \cap J_{-k} \neq \emptyset$. This can be done for every $\varepsilon > 0$ showing that x is chain-recurrent. \square

4. TOPOLOGICAL STRUCTURE OF THE SETS CP AND \overline{CP}

4.1. Basic properties of CP , \overline{CP} and $\overline{CP} \setminus CP$. By definition \overline{CP} is a complete metric space, thus a Baire space. The next proposition implies that CP equipped with uniform metric is also a Baire space, see [17, p. 10].

Lemma 4.1. *CP is a residual subset of \overline{CP} .*

Proof. Fix a sequence $(J_m)_m$ of rational subintervals of $[0, 1]$ and a sequence $(f_n)_n$ of maps from CP dense in \overline{CP} . By [7, Remark p. 2] each map f_n from CP has an invariant measure μ_n with the full support $[0, 1]$. In this proof we will use the notation $B_{\overline{CP}}(\cdot, \cdot)$ to denote an open ball in \overline{CP} . Using the increasing homeomorphism ψ_n of $[0, 1]$ defined by $\psi_n(x) = \mu_n([0, x])$, we get that $g_n = \psi_n \circ f_n \circ \psi_n^{-1} \in C_\lambda$ and

$$U_n := \{\psi_n \circ g \circ \psi_n^{-1} : g \in B_{\overline{CP}}(f_n, 1/n)\}$$

is an open neighborhood of g_n in \overline{CP} . Since $C_\lambda \subset CP$ the set $U_n \cap CP$ contains a small C_λ neighborhood of g_n . Repeating the construction from [7, Lemma 12], we can consider a piecewise affine map $g \in U_n \cap C_\lambda$ having a transversal periodic point in $\psi_n(J_m)$. Then for a sufficiently small δ positive, all maps in the ball $B_{\overline{CP}}(g, \delta) \subset U_n$ have a periodic point in $\psi_n(J_m)$. Put

$$V_{m,n} := \{\psi_n^{-1} \circ g' \circ \psi_n : g' \in B_{\overline{CP}}(g, \delta)\}.$$

Clearly $V_{m,n} \subset B_{\overline{CP}}(f_n, 1/n)$ is open and all maps in $V_{m,n}$ have a periodic point in J_m . Moreover, the set $V_m := \bigcup_n V_{m,n}$ is open and dense in \overline{CP} containing maps with periodic points in J_m . It follows that any map from $\bigcap_m V_m$ contains periodic points in all J_m , and thus is in CP . At the same time $\bigcap_m V_m$ is a dense G_δ in \overline{CP} . \square

Corollary 4.2. *If A is a residual subset in CP , then A is a residual subset of \overline{CP} .*

Remark 4.3. *Transitive interval maps have dense periodic points, thus all maps from $\overline{CP} \setminus CP$ are not transitive.*

Lemma 4.4. *The set of piecewise affine leo maps is dense in CP .*

Proof. Fix $f \in CP$, choose a homeomorphism ψ of I such that $\hat{f} := \psi \circ f \circ \psi^{-1} \in C_\lambda$ (see [7, p. 2]). Proposition 8 from [6] states that piecewise affine leo

maps are dense in C_λ . Choose a sequence (\hat{f}_n) of piecewise affine maps in C_λ converging to \hat{f} and choose a sequence of piecewise affine homeomorphism ψ_n converging to ψ . Let $f_n := \psi_n^{-1} \circ \hat{f}_n \circ \psi_n$. The maps f_n are clearly piecewise affine, they are leo since the leo property is preserved by conjugation by a homeomorphism. Since leo maps have dense periodic points $f_n \in CP$. Finally $f_n = \psi_n^{-1} \circ \hat{f}_n \circ \psi_n \rightarrow \psi^{-1} \circ \hat{f} \circ \psi = f$ since $\psi_n \rightarrow \psi$ and $\hat{f}_n \rightarrow \hat{f}$. \square

Theorem 4.5. $\overline{CP} \setminus CP$ is dense in \overline{CP} .

Proof. Start with any piecewise affine topologically mixing map T and $\eta \in (0, 1/2)$. Note that any piecewise monotone topologically mixing map is leo (see Proposition 2.34 in [25]). We will construct a non-transitive map with no flat parts in $B(T, \eta)$ which can be uniformly approximated by maps from CP . Once we have constructed such a map, this completes the proof since piecewise affine leo maps are dense in CP (Lemma 4.4). More precisely we will perform a construction from [22, Example 4.6] resembling a Denjoy map [14, Example 14.9].

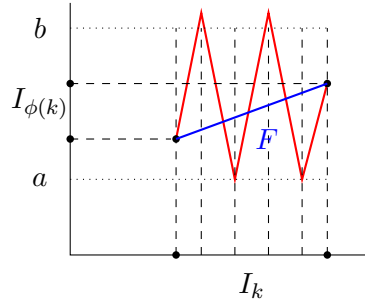
First we choose a point $z \in (0, 1)$ with a dense orbit under map T , which exists since T is leo. Denote by $D_0 := \{z, T(z)\} \cup T^{-1}(\{z\})$ and inductively set $D_{n+1} := T(D_n) \cup T^{-1}(D_n)$. Let

$$D := \bigcup_{n=1}^{\infty} D_n.$$

Note that D is a dense and countable set such that $T(D) = D$. Since the set of points with dense orbit is residual and the space is perfect, it is uncountable. Denote the set of turning points of T by \mathcal{T}_T . Since T is a piecewise affine map \mathcal{T}_T is finite and hence we can replace z by another point when necessary, ensuring this way that also $D \cap (\{0, 1\} \cup \mathcal{T}_T) = \emptyset$. Enumerate its elements by $D = \{x_i : i \in \mathbb{N}\}$ where $x_i \neq x_j$ for $i \neq j$. Furthermore observe that if $T^n(x_i) = x_j$ for some $n > 0$ then $i \neq j$ and $x_i \notin \{x_j, T(x_j), T^2(x_j), \dots\}$, if not z would be an eventually periodic point. By definition, both D and $[0, 1] \setminus D$ are invariant under T . Define a function $\phi: \mathbb{N} \rightarrow \mathbb{N}$ so that $T(x_i) = x_{\phi(i)}$.

We follow the standard Denjoy construction, see [14] for details. At the i -th step we remove the point x_i from the interval and replace the hole with an interval I_i of length η^{i+1} (the new interval is longer, so it naturally redefines the set $\{x_j : j > i\}$). This way a new continuous map F is defined on the extended space so that each interval I_i is mapped by an affine map onto $I_{\phi(i)}$ in such a way that F is continuous.

Note that F is semi-conjugate to T by collapsing the intervals I_i back to single points as explained below. As the domain of F is isometric to $[0, 1 + \gamma]$, where $\gamma = \sum_{i \in \mathbb{N}} \eta^{i+1} < \eta$ we view $F: [0, 1 + \gamma] \rightarrow [0, 1 + \gamma]$. In this way every interval I_i becomes some interval $[a_i, b_i] \subset (0, 1 + \gamma)$ and there is a quotient map

FIGURE 2. Construction of F_n . Note that $b - a = \text{diam}(I_k)$.

$\pi: [0, 1 + \gamma] \rightarrow [0, 1]$ that does not increase distance, collapses every interval $[a_i, b_i]$ into a single point x_i , and has the property that $T \circ \pi = \pi \circ F$. If we fix i, j , so that $x_i \notin \{x_j, T(x_j), T^2(x_j), \dots\}$ then $F^n((a_j, b_j)) \cap (a_i, b_i) = \emptyset$ for all $n > 0$.

Thus, the map F is not transitive, and is non-constant on any open interval. Nonetheless we will show that it can be obtained as uniform limit of maps from CP . To do so, let F_n be the map obtained from F in the following way:

- (1) outside of $\bigcup_{k \geq n} I_k$ we have $F_n(x) = F(x)$.
- (2) If $|k| \geq n$ then $F_n|_{I_k}$ is defined as the map with 5 pieces of monotonicity, constant slope, and a 3-fold map constructed using the interior 3 pieces of monotonicity with the properties that on each piece of monotonicity J we have $F_n(J) \supset I_{\phi(k)} = F(I_k)$ and

$$\gamma > \text{diam} F_n(J) > \max\{\text{diam} I_k, \text{diam} I_{\phi(k)}\}.$$

The remaining two pieces of the monotonicity with one endpoint contained in the set of endpoints of I_k are used to make the map F_n continuous. We can additionally require that the slope on each piece of I_k is at least 3 because the pieces of monotonicity do not need to be of equal length.

It is clear that F_n converges uniformly to F . We claim that each F_n is topologically mixing. Indeed, fix any interval J and any $\varepsilon > 0$. Note that if an interval $I \subset I_k$ for some $|k| \geq n$ and $F_n(I) \subset I_{\phi(k)}$ and $|\phi(k)| \geq n$ then either $\text{diam} F_n(I) > \text{diam}(I)$ or $F_n(I)$ contains two consecutive critical points in $I_{\phi(k)}$ and then $F_n^2(I) \not\supseteq I_{\phi^2(k)}$. But since diameter of an interval can not grow to infinity, it means that for J there is some iterate m such that $F_n^m(J) \not\subset I_k$ for any k . But since T is mixing and $\pi(F_n^m(J))$ is a nondegenerate interval, for any $\delta > 0$ there are points $u, v \in \pi(F_n^m(J))$ and $j > 0$ such that $T^j(u) < \delta$ and $T^j(v) > 1 - \delta$. But since δ is arbitrarily small, there are $p, q \notin \bigcup_k I_k \cap F_n^m(J)$ such that $F_n^j(p) < \varepsilon$ and $F_n^j(q) > 1 + \gamma - \varepsilon$. This means that $[\varepsilon, 1 + \gamma - \varepsilon] \subset F_n^{m+j}(J)$, because $F^j(p) = F_n^j(p)$ and $F^j(q) = F_n^j(q)$.

showing that F_n is mixing. This proves the claim, and as a consequence each F_n has dense periodic points.

To finish the proof, let $\zeta: [0, 1] \rightarrow [0, 1 + \gamma]$ be a linear onto map with $\zeta(0) = 0$ and let $G = \zeta^{-1} \circ F \circ \zeta$. Note that if we fix any $z \in [0, 1 + \gamma]$ then there is $z' \in [0, 1 + \gamma]$ such that z' is either endpoint of I_k for some k or $z' = z$ and $z \notin I_k$ for any k . In any of these cases $T(\pi(z)) \leq F(z') \leq T(\pi(z)) + \gamma$. But $|F(z') - F(z)| < \sup_k \text{diam} F(I_k) \leq \gamma$. This means that $T(\pi(z)) - 2\gamma \leq F(z) \leq T(\pi(z)) + 2\gamma$. On the other hand, for any $x \in [0, 1]$ there is $z \in [0, 1 + \gamma]$ such that $\pi(z) = x$ and so $T(x) - 2\gamma \leq F(z) \leq T(x) + 2\gamma$. Fix any $x \in [0, 1]$ and let $z = \zeta(x)$ and $y = \pi(z)$. Note that $d(x, z) < \gamma$ and $d(y, z) \leq \gamma$ so $d(x, z) < \gamma$. This gives

$$\begin{aligned} d(T(x), G(x)) &= d(T(x), \zeta^{-1}(T(x))) + d(\zeta^{-1}(T(x)), \zeta^{-1}(F(z))) \leq \\ &\leq \gamma + d(T(x), F(z)) \leq 3\gamma + d(T(x), T(y)). \end{aligned}$$

But if $\gamma \rightarrow 0$ then $d(T(x), T(y)) \rightarrow 0$, hence taking small γ we can make $\rho(T, G)$ arbitrarily small. This completes the proof. \square

Proposition 4.6. *Sets \overline{CP} and CP are nowhere dense in $C(I)$.*

Proof. First note that the map $x \mapsto x^2 \notin \overline{CP}$ and the same is true for any map in its small neighbourhood in $C(I)$. For what follows we refer the reader to the proof of Theorem 4.5 where a procedure of blowing up the points is described in more detail. Fix $\varepsilon > 0$, $f \in CP$ and take any $g \in C(I)$ that is topologically mixing and piecewise affine, has finite set of local extrema (i.e., there is no interval with constant value) and ε -close to f . Let p be a fixed point of g and $D = \cup_{n \geq 0} g^{-n}(p)$. Observe that D is countable and dense in I . We enumerate it $D = \{a_0 = p, a_1, \dots\}$. Now take a collection of intervals $\{I_n\}_{n \geq 0}$ such that $\sum_{n \geq 0} \lambda(I_n) = \gamma$ where $\gamma > 0$ is small. As in the proof of Theorem 4.5 we modify g to a continuous map G by blowing up each point a_n to an interval I_n , and if $g(a_n) = a_m$ ($n > 0$) then we define $I_m = G(I_n)$ putting a piecewise affine map properly assigning the values of endpoints so that continuity and monotonicity are preserved; and $G: I_0 \rightarrow I_0$ is a properly rescaled version of $x \mapsto x^2$, i.e., rescaled so that $G(I_0) = I_0$. Since $\bigcup I_m$ is dense in $[0, 1 + \gamma]$ the map G is uniquely defined on its complement. Interior points of I_0 cannot be periodic points of G , thus any map sufficiently close to G can not be in CP . Furthermore, if we take $\varepsilon \rightarrow 0$ and $\gamma \rightarrow 0$ the resulting maps G converge to f . \square

4.2. CP and \overline{CP} are uniformly locally arcwise connected. The first step is to prove the following theorem that answers a question Michał Misiurewicz asked at during a talk at the “Workshop on topological and combinatorial dynamics” in April 2021 at the CRM in Montreal.

Recall that $f, g \in C(I)$ are homotopic, if there is a family of maps $\{h_t\}_{t \in [0,1]}$ such that $h_0 := f$, $h_1 = g$ and the map $(x, t) \mapsto h_t(x)$ is continuous from $I \times [0, 1]$ to I .

Theorem 4.7. *Every map $f \in C_\lambda$ is homotopic to Id.*

Proof. We first show how to reduce to the case when $f(0) = 0$. Suppose $f(0) > 0$ and define $\varepsilon := \inf\{x \in [0, 1] : f(x) = 0\}$. For $a \in [0, \varepsilon]$ we define f_a as a two-fold perturbation on the interval $[0, a]$. $\{f_a\}_{a \in [0, \varepsilon]}$ is a continuous family of maps in C_λ and $f_\varepsilon(0) \in \{0, 1\}$. Let $g := f_\varepsilon$. Next, for every $\alpha \in [0, 1]$ we define $g_\alpha : I \rightarrow I$ by:

$$g_\alpha(x) := \begin{cases} x, & x \leq \alpha, \\ \alpha + (1 - \alpha)g\left(\frac{x - \alpha}{1 - \alpha}\right), & x > \alpha. \end{cases}$$

We have $g_\alpha \in C_\lambda$ for all $\alpha \in [0, 1]$, and they form a continuous family of maps and $g_1 = \text{Id}$. \square

Corollary 4.8. *The space C_λ is arcwise connected.*

A metric space (X, d) is called *uniformly locally arcwise connected* if for any $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < d(x, y) < \delta$, then x and y are joined by an arc of diameter smaller than ε (see [18]). For the following definitions and statements see [23, Subsections 5.10 and 5.22]. A metric space (X, d) is *connected im kleinen* at $x \in X$ if for each open set U containing x , there is a component of U which contains x in its interior. The space X is *locally connected at x* if for each open set U containing x there is a connected open set V such that $x \in V \subset U$. We say a space is *locally connected* if it is locally connected at every $x \in X$. Note that connectedness im kleinen at every $x \in X$ implies local connectedness. However, uniformly locally arcwise connectedness implies connectedness im kleinem at every $x \in X$ since such a point will be always contained in the interior of some arc. Thus uniformly locally arcwise connectedness implies local connectedness. In [21] it is proven that each of the three spaces: all transitive maps, all piecewise monotone transitive maps and all piecewise linear transitive maps are uniformly locally arcwise connected.

The fact that transitive maps have dense sets of periodic points allows us to restate Lemma 2.6 from [21], which will help us to prove arcwise connectedness of \overline{CP} .

Lemma 4.9. *For any transitive maps $f, g \in CP$ there is an arc $A \subset CP$ joining f and g such that its diameter is smaller than or equal to $5d(f, g)$.*

Theorem 4.10. *CP and \overline{CP} are uniformly locally arcwise connected.*

Proof. Fix any maps $f, g \in CP$. Let $\delta = d(f, g)$. Since by Lemma 4.4 leo maps are dense in CP , there are sequences $f_n \rightarrow f$ and $g_n \rightarrow g$ of leo maps. We

may also assume that $d(g_n, g) < \delta/2^{n+1}$, so in particular $d(g_n, g_{n+1}) < \delta/2^{n+1}$ and $d(g_0, f_0) < 2\delta$. Let A_0 be the arc joining f_0, g_0 , A_n arc joining g_{n-1}, g_n and A_{-n} an arc joining f_{n-1}, f_n . Furthermore, we assume that these arcs are obtained by application of Lemma 4.9, so $\text{diam}A_0 < 10\delta$ and $\text{diam}A_n \leq 5\delta/2^{|n|}$. It is also clear that if we denote $J_n = \cup_{i=-n}^n A_i$ then it is an arc, and $d(J_n, J_m) < 5\delta/2^n$ for any $m > n$, so it is a Cauchy sequence. Then the limit $J = \lim_n J_n$ exists. But it is also clear that $\lim_n \text{diam}(\sup_{i \geq n} A_i) = 0$ and $\lim_n \text{diam}(\sup_{i \geq n} A_{-i}) = 0$, so $\cup_n J_n$ compactifies with two points, which in fact are f, g as limits of endpoints of J_n . This shows J is an arc and also that $J \subset CP$. By our construction

$$\text{diam}J \leq \sum_{n=0}^{\infty} 10\delta/2^n \leq 20\delta.$$

This proves uniform local arcwise connectedness of CP .

The proof that \overline{CP} is uniformly locally arcwise connected follows the same lines as for CP with the only difference that in place of the sequence of leo maps we now can use elements of CP and its uniform local arcwise connectedness. \square

We do not know if an analog of Lemma 4.9 is true for C_λ . More precisely, it would be interesting to know the answer of the following question. Fix any $f, g \in C_\lambda$. Is there a constant $\eta > 0$ and an arc $A \subset C_\lambda$ connecting f with g such that $\text{diam}A \leq \eta d(f, g)$?

5. DENSENESS PROPERTIES IN CP AND \overline{CP}

5.1. Window perturbations in CP . The proofs in this section use window perturbations as a tool. Let $J \subset I$ be an interval, m an odd positive integer and $\{J_i \in I : 1 \leq i \leq m\}$ a finite collection of intervals satisfying $\cup_{i=1}^m J_i = J$ and $\text{Int}(J_i) \cap \text{Int}(J_j) = \emptyset$ when $i \neq j$. We will refer to this as a *partition* of J .

In our previous articles we considered the following notion in C_λ . Fix $f \in C_\lambda$ an interval $J \subset I$ and a partition $\{J_i\}$ of J . A map $g \in C_\lambda$ is an *m -fold window perturbation of f with respect to J* and the partition $\{J_i\}$ if

- $g|_{J^c} = f|_{J^c}$
- for each $1 \leq i \leq m$ the map $g|_{J_i}$ is an affinely scaled copy of $f|_J$ with $g|_{J_1}$ having the same orientation as $f|_J$, $g|_{J_2}$ having the opposite orientation to $f|_J$, and then continuing with the orientations alternating.

The essence of this definition is illustrated by Figure 3.

We will apply this in CP in the following way. We fix $f \in CP$ and conjugate f to a map $\hat{f} = \psi^{-1} \circ f \circ \psi \in C_\lambda$. We can apply the above procedure to \hat{f}

to obtain an m -fold window perturbation \hat{g} of \hat{f} , and then we call the map $g := \psi \circ \hat{g} \circ \psi^{-1}$ an m -fold window perturbation of f .

We call an m -fold window perturbation regular if the intervals $\psi(J_i)$ all have the same length.

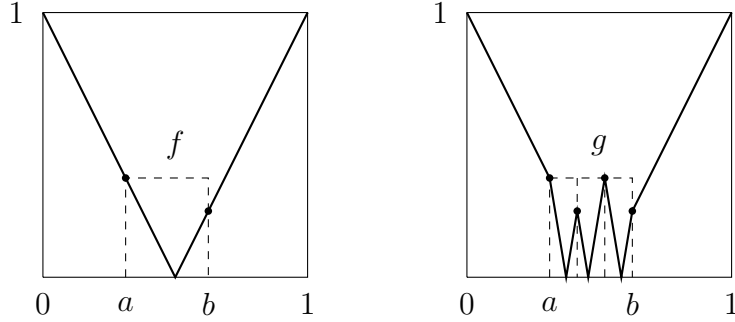


FIGURE 3. For $f \in C_\lambda$ shown on the left, we show on the right the graph of g which is a 3-fold piecewise window perturbation of f on the interval $[a, b]$.

5.2. Leo maps form an open dense collection in CP . We state a part of the Barge-Martin result [2] which we use later.

Proposition 5.1. *Suppose $f \in CP$. The following assertions hold.*

- (a) *There is a collection (perhaps finite or empty) $\mathcal{J} = \mathcal{J}(f) = \{J_1, J_2, \dots\}$ of closed subintervals of $[0, 1]$ with mutually disjoint interiors, such that for each i , $f^2(J_i) = J_i$, and there is a point $x_i \in J_i$ such that $\{f^{4n}(x_i) : n \geq 0\}$ is dense in J_i .*
- (b) *If $x \in (0, 1) \setminus \bigcup_{i \geq 1} \overset{\circ}{J}_i$, then $f^2(x) = x$.*
- (c) *For each $J \in \mathcal{J}$, the map $f^2|_J$ is topologically mixing.*

At the first glance, one might naively expect that a similar characterization holds in \overline{CP} , namely, for a map in $\overline{CP} \setminus CP$ one could hope to partition I based on restricting to a different ω -limit sets. The following example shows this is not always possible. First note that the map f from Figure 4 is in $\overline{CP} \setminus CP$. Indeed, f is not in CP since periodic points are not dense in $[1 - \varepsilon, 1]$ for $\varepsilon > 0$ small. However $f \in \overline{CP}$ since one can approximate the leftmost critical value of f with maps that have critical values slightly above $3/4$ and thus these perturbations will all be transitive and therefore in CP . On the other hand, we cannot expect a Barge-Martin decomposition for f . Namely, the set $[0, 3/4]$ is f -invariant, and there exists a Cantor set of points in $[3/4, 1 - \varepsilon]$, for small $\varepsilon > 0$, which get mapped to $[0, 3/4]$. Therefore, we cannot separate the ω -limit set of $f|_{[0, 3/4]}$ from the ω -limit set of $f|_{[3/4, 1]}$.

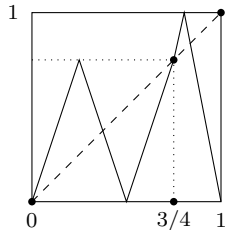


FIGURE 4. An example explaining that Barge-Martin-like decomposition in \overline{CP} is not possible.

The following definition is crucial in the proof of Theorem 5.3.

Definition 5.2. For $f \in CP$, an interval or a point $J \subsetneq [0, 1]$ f -splits I if

- either $J = f(J)$
- or J and $f(J)$ have disjoint interiors and $f^2(J) = J$

and each nonempty connected component C of $I \setminus (J \cup f(J))$ satisfies $f^2(\overline{C}) = \overline{C}$.

Theorem 5.3. Leo maps form an open dense collection in CP .

Proof. Let

$$M := \{f \in CP : f(\{0, 1\}) \cap \{0, 1\} = \emptyset\}.$$

Clearly M is open in CP . We claim M is also dense in CP . Fix $f \in CP$ and $\varepsilon > 0$. By Remark from [7] there is a non-atomic invariant measure μ for f with full support on I and a homeomorphism $\psi : I \rightarrow I$ such that $g := \psi^{-1} \circ f \circ \psi \in C_\lambda$. If $g(0) = 0$ or $g(1) = 1$ then we can perform a 2-fold window perturbation of size at most ε on the arc $[0, \varepsilon]$ to destroy the fixed points and similarly we can destroy $f(1) = 0$ or $f(0) = 1$. We obtain a map $G \in C_\lambda$ with $\rho(G, g) < \varepsilon$ such that $G(0), G(1) \notin \{0, 1\}$. Note that $G \in CP$, and thus the map $F := \psi \circ G \circ \psi^{-1} \in CP$ as well. Since this is true for every ε there exists a map $F \in CP$ arbitrarily close to f with $F(0), F(1) \notin \{0, 1\}$.

Notice that if $f \in M$ then $f^2 \neq \text{Id}$. If $f^2 \neq \text{Id}$, the set of intervals \mathcal{J} in Proposition 5.1 is non-empty.

We claim that the set

$$N := \{g \in M : \text{a periodic interval or point of period } p \in \{1, 2\} \text{ } g\text{-splits } [0, 1]\}$$

coincides with the set of maps in M that are not leo. The only if direction is clear, thus suppose that $f \in M$ is not leo. Suppose first that $\#\mathcal{J} = 1$. We claim that $J \subsetneq I$. Suppose the contrary, i.e., $J = I$. By Proposition 5.1(c) f is topologically mixing, but for any interval (a, b) we have $f^n(a, b)$ is an interval and $\lambda(f^n(a, b)) \rightarrow 1$. Since $f \in CP$ it is surjective and since $f \in M$

we have some $0 < c < d < 1$ such that $f(c) = 0$ and $f(d) = 1$ (or vice versa). Combining these two fact yields that $f^n(a, b) = I$ for sufficiently large n , contradicting that f is not leo. We have shown $J \subsetneq I$. By Proposition 5.1(b) $f^2(\overline{C}) = \overline{C}$ for each connected component C of J^c and thus J f -splits I .

If $J_1, J_2 \in \mathcal{J}$ then again by Proposition 5.1(b) $f^2(\overline{C}) = \overline{C}$ for each nonempty connected component C of $(J_1 \cup J_2)^c$ and thus either of the J_i f -splits I . This finishes the proof of the claim.

Let us prove that N is closed in M . Suppose that $\{f_n\} \subset N$ converges to f . Since I is compact we can choose a subsequence of n_i 's such that the corresponding intervals J_{n_i} converge to an interval J , which is possibly degenerate to a point or $J = [0, 1]$. Since $f_n^2(J_n) = J_n$ and the f_n are continuous we get $f^2(J) = J$.

If $J \neq [0, 1]$ and $J \cap (0, 1) \neq \emptyset$ then J f -splits I and so $f \in N$. If $J = \{0\}$ then $f(0) = 0$ or $f(0) = 1$ and so $f \notin M$, and similarly if $J = \{1\}$ we have $f \notin M$.

Finally we turn to the case $J = [0, 1]$. But in this case it follows from Proposition 5.1(a) and the fact $f_n \rightarrow f$ that $f(0) = 0$ or $f(0) = 1$ (and similarly for $f(1)$). Therefore, also in this case $f \notin M$.

To conclude the proof note that the set N is nowhere dense in M since the set of piecewise affine leo maps is dense in M by Lemma 4.4. \square

Remark 5.4. *The set of leo piecewise affine interval maps f preserving λ such that $f(\{0, 1\}) \cap \{0, 1\} = \emptyset$ forms a dense collection in C_λ , see [6, Proposition 8]. Thus the proof of Theorem 5.3 works without change in the setting of C_λ . Suppose μ is a non-atomic invariant measure μ with full support. By Remark 2.1 leo maps are open and dense in C_μ as well.*

Remark 5.5. *In fact the proof of Theorem 5.3 shows more, namely that the set of leo maps f such that $f(\{0, 1\}) \cap \{0, 1\} = \emptyset$ is an open and dense subset of CP . Suppose μ is a non-atomic invariant measure μ with full support. By Remark 5.4 the space C_μ contains an open dense set of leo maps f such that $f(\{0, 1\}) \cap \{0, 1\} = \emptyset$.*

Applying Corollary 4.2 and Theorem 4.5 yields

Corollary 5.6. *The set of leo maps is residual in \overline{CP} but is not open.*

Blokh showed that topological mixing implies the periodic specification property [25, Theorem 3.4] (see also [8, Appendix A]), thus we have

Corollary 5.7. *Maps satisfying the periodic specification property form an open dense collection in CP , C_λ and C_μ .*

5.3. Conjugacy classes in \overline{CP} . A beautiful result of Kechris and Rosendal [20] shows that there is residual set in the space of all homeomorphisms of the

Cantor set such that each pair of its elements are conjugate (so-called *residual conjugacy class*). Moreover, Bernardes and Darji [4] proved there is a residual conjugacy class in the space of all self-maps of the Cantor set. In the case of \overline{CP} (or C_μ and $C(I)$) all the standard invariants (entropy, mixing, ...) do not exclude the possibility that such a class exists. Our next result shows that in \overline{CP} a residual conjugacy class does not exist.

Let $\mathcal{H}(I)$ denote the set of all homeomorphism (increasing or decreasing) of I and for $f \in \overline{CP}$ put

$$G_f = \{\psi^{-1} \circ f \circ \psi : \psi \in \mathcal{H}(I)\}.$$

Theorem 3.1 directly implies that the set $\overline{G_f}$ is a subset of \overline{CP} .

As usually, $\text{sgn}: \mathbb{R} \rightarrow \{-1, 0, 1\}$ is defined as

$$\text{sgn}(x) := \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Theorem 5.8. *For every $f \in \overline{CP}$, the set G_f is nowhere dense in \overline{CP} .*

Proof. Let us fix $f \in \overline{CP}$. For every $\psi \in \mathcal{H}(I)$ define $f_\psi := \psi^{-1} \circ f \circ \psi \in \overline{CP}$. We start with three basic but useful observations:

- (A) $\forall \psi \in \mathcal{H}(I)$, $\text{sgn}(f(0) - f(1)) = \text{sgn}(f_\psi(0) - f_\psi(1))$.
- (B) If $f_{\psi_n} \rightrightarrows g$ then $f_{\psi_n}^m \rightrightarrows g^m$ for each $m \in \mathbb{N}$ (by induction over m).
- (C) For each $\psi \in \mathcal{H}(I)$ the map $H_\psi: \overline{CP} \rightarrow \overline{CP}$ defined as $H_\psi(f) = \psi^{-1} \circ f \circ \psi$, $f \in \overline{CP}$, is a homeomorphism of \overline{CP} .

Assume to the contrary that there exists a nonempty open set $U \subset \overline{G_f} \subset \overline{CP}$. Notice that by (C) $H_\psi(U)$ is an open set in \overline{CP} . Using [7, Remark p.2] there exist a homeomorphism $\psi \in \mathcal{H}(I)$ and a piecewise affine leo map α such that $\alpha \in H_\psi(U) \cap C_\lambda$ and each of the points 0 and 1 have a preimage outside the set $\{0, 1\}$ using Remark 5.4. Fix $\varepsilon > 0$ so small that $B(\alpha, \varepsilon) \subset H_\psi(U)$. There is a small $\delta > 0$ such that each window perturbation of α on the intervals $[0, \delta]$, $[1 - \delta, 1]$ remains in $B(\alpha, \varepsilon)$, see Subsection 5.1. For $u \in [0, 1]$ and $n \in \mathbb{N}$ denote

$$O_\alpha^{-n}(u) := \{x \in I : \alpha^n(x) = u\}.$$

Since 0 and 1 have α -preimages outside the set $\{0, 1\}$, for each positive integer n there is $x \in O_\alpha^{-n}(u)$, $u \in \{0, 1\}$, such that $f^i(x) \notin \{0, 1\}$ for $i = 0, 1, \dots, n-1$. Because α is leo, there exists n so large that one can choose x_i, y_i for $i \in \{0, 1\}$

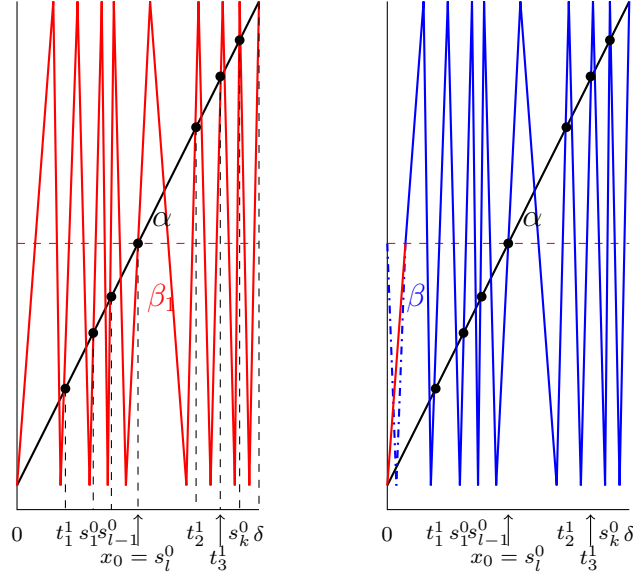


FIGURE 5. The maps α (black) and β constructed in two steps (β_1 in red, then modified to β in blue) from the proof of Theorem 5.8 on the interval $[0, \delta]$ with $x_0 = s_\ell^0$. On the interval $[1 - \delta, 1]$ the redefinition of S_0 and T_1 is analogous.

such that

$$\begin{aligned} x_0 &\in O_\alpha^{-n}(0) \cap (0, \delta), \\ x_1 &\in O_\alpha^{-n}(1) \cap (0, \delta), \\ y_0 &\in O_\alpha^{-n}(0) \cap (1 - \delta, 1), \\ y_1 &\in O_\alpha^{-n}(1) \cap (1 - \delta, 1) \end{aligned}$$

and define

$$\begin{aligned} S_0 &:= \{s_1^0 < s_2^0 \cdots < s_{k_0}^0\} = \{x_0, \alpha(x_0), \dots, \alpha^{n-1}(x_0)\} \cap ([0, \delta] \cup [1 - \delta, 1]), \\ S_1 &:= \{s_1^1 < s_2^1 \cdots < s_{k_1}^1\} = \{x_1, \alpha(x_1), \dots, \alpha^{n-1}(x_1)\} \cap ([0, \delta] \cup [1 - \delta, 1]), \\ T_0 &:= \{t_1^0 < t_2^0 \cdots < t_{\ell_0}^0\} = \{y_0, \alpha(y_0), \dots, \alpha^{n-1}(y_0)\} \cap ([0, \delta] \cup [1 - \delta, 1]), \\ T_1 &:= \{t_1^1 < t_2^1 \cdots < t_{\ell_1}^1\} = \{y_1, \alpha(y_1), \dots, \alpha^{n-1}(y_1)\} \cap ([0, \delta] \cup [1 - \delta, 1]). \end{aligned}$$

As mentioned above these four sets are disjoint from the set $\{0, 1\}$.

Now we distinguish two cases:

I. $\text{sgn}(f^n(0) - f^n(1)) \in \{1, 0\}$. As it is shown in Figure 5 in this case there exists a window perturbation β of α on the intervals $[0, \delta]$, $[1 - \delta, 1]$ such that

α and β coincide on S_0 and T_1 , hence

$$\begin{aligned}\beta^i(0) &= \alpha^i(x_0) && \text{for each } i \in \{1, \dots, n\} \text{ and also} \\ \beta^i(1) &= \alpha^i(y_1) && \text{for each } i \in \{1, \dots, n\};\end{aligned}$$

then

$$\operatorname{sgn}(f^n(0) - f^n(1)) \neq \operatorname{sgn}(\beta^n(0) - \beta^n(1)) = -1$$

hence from (A),(B) follows $\beta \notin B(\alpha, \varepsilon)$, a contradiction.

II. $\operatorname{sgn}(f^n(0) - f^n(1)) \in \{0, -1\}$. In this case there exists a window perturbation β of α on the intervals $[0, \delta]$, $[1 - \delta, 1]$ such that α and β coincide on S_1 and T_0 , hence

- $\beta^i(0) = \alpha^i(x_1)$ for each $i \in \{1, \dots, n\}$ and also
- $\beta^i(1) = \alpha^i(y_0)$ for each $i \in \{1, \dots, n\}$;

then

$$\operatorname{sgn}(f^n(0) - f^n(1)) \neq \operatorname{sgn}(\beta^n(0) - \beta^n(1)) = 1$$

and from (A),(B) one again gets $\beta \notin B(\alpha, \varepsilon)$, a contradiction.

We have shown that the set \overline{G}_f has an empty interior, i.e., G_f is nowhere dense. \square

As an immediate consequence of Theorem 5.8 we obtain.

Corollary 5.9. *Every second category set $G \subset \overline{CP}$ contains uncountably many conjugacy classes.*

Proof. By Theorem 5.8 each union of countably many conjugacy classes is a set of the first category. \square

Remark 5.10. *Adapting the preceding two proofs easily leads to results analogous to Theorem 5.8 and Corollary 5.9 in the setting of C_λ or CP . By Remark 2.1 the conclusions of Theorem 5.8 and Corollary 5.9 hold also for C_μ .*

In [10] the authors constructed a parameterized family of planar homeomorphisms arising from a residual set \mathcal{T} of functions from C_λ . The maps from \mathcal{T} share several dynamical properties (shadowing, topological exactness, δ -crookedness, the same structure of the sets of periodic points, etc.). In this context the question if \mathcal{T} could be contained in a single conjugacy class naturally arose. Corollary 5.9 and Remark 5.10 provide a negative answer to this question.

5.4. **Shadowing is generic in \overline{CP} .** Now let us recall the definition of shadowing and periodic shadowing. For some $\delta > 0$, a sequence $(x_n)_{n \in \mathbb{N}_0} \subset I$ is called δ -pseudo orbit of $f \in C(I)$ if $d(f(x_n), x_{n+1}) < \delta$ for every $n \in \mathbb{N}_0$. A *periodic δ -pseudo orbit* is a δ -pseudo orbit for which there exists $N \in \mathbb{N}_0$ such that $x_{n+N} = x_n$, for all $n \in \mathbb{N}_0$. We call the sequence $(x_n)_{n \in \mathbb{N}_0}$ an *asymptotic pseudo orbit* if $\lim_{n \rightarrow \infty} d(f(x_n), x_{n+1}) = 0$.

Definition 5.11. *We say that a map $f \in C(I)$ has the:*

- shadowing property *if for every $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following condition: given a δ -pseudo orbit $\mathbf{y} := (y_n)_{n \in \mathbb{N}_0}$ we can find a corresponding point $x \in I$ which ε -traces \mathbf{y} , i.e.,*

$$d(f^n(x), y_n) < \varepsilon \text{ for every } n \in \mathbb{N}_0.$$

- periodic shadowing property *if for every $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following condition: given a periodic δ -pseudo orbit $\mathbf{y} := (y_n)_{n \in \mathbb{N}_0}$ we can find a corresponding periodic point $x \in I$, which ε -traces \mathbf{y} .*

Lemma 5.12 (Lemma 22 from [7]). *For every $\varepsilon > 0$ and every map $f \in C_\lambda$ there are $\delta < \frac{\varepsilon}{2}$ and $F \in C_\lambda$ such that:*

- (1) *F is piecewise affine and $\rho(f, F) < \frac{\varepsilon}{2}$,*
- (2) *if $g \in C_\lambda$ and $\rho(F, g) < \delta$ then every δ -pseudo orbit $\mathbf{x} := \{x_i\}_{i=0}^\infty$ for g is ε -traced by a point $z \in I$. Furthermore, if \mathbf{x} is a periodic sequence, then z can be chosen to be a periodic point.*

In fact, a close study of the proof of Lemma 22 from [7] reveals that in statement (2) we may consider wider neighborhood of maps $g \in C(I)$, $\rho(F, g) < \delta$. We explain this in more detail in the following paragraph.

For an interval $J \subset I$ let $\text{diam}(J) := \sup\{d(x, y) : x, y \in J\}$. Take any $g \in C_\lambda$ such that $\rho(F, g) < \delta$ and let $\mathbf{x} := \{x_i\}_{i=0}^\infty$ be a δ -pseudo orbit for g . In the proof we claim that there is a sequence of closed intervals J_i such that

- (a) $\text{diam} J_i \leq \gamma$ and if $i > 0$ then $J_i \subset g(J_{i-1})$,
- (b) $\text{dist}(x_i, J_i) < \gamma$,
- (c) for every i there is p such that $F(J_i) = F([a_p, a_{p+1}])$ and $x_i \in [a_p, a_{p+1}]$

provided that $g \in C_\lambda$ and $\rho(F, g) < \delta$, where points a_i are endpoints of a partition of I and $\gamma > 0$ is a sufficiently small constant. The claim is proved simply by the definition of uniform metric and covering of intervals (by continuity of F), so in fact the conditions hold for any $g \in C(I)$, $\rho(F, g) < \delta$, in particular for $g \in \overline{CP}$. Now the tracing point is obtained as any $z \in \cap g^{-n}(J_n) \neq \emptyset$.

Therefore perturbing the original map $f \in \overline{CP}$ to a map $\hat{f} \in CP$, passing through conjugacy of \hat{f} from CP into C_λ , applying Lemma 5.12 (with

sufficiently decreased constants) and then passing back to CP (which gives coverings satisfying (a)) we obtain the following adaptation of the result to the CP case.

Lemma 5.13. *For every $\varepsilon > 0$ and every map $f \in \overline{CP}$ there are $\delta < \frac{\varepsilon}{2}$ and $F \in CP$ such that:*

- (1) $\rho(f, F) < \frac{\varepsilon}{2}$,
- (2) *if $g \in \overline{CP}$ and $\rho(F, g) < \delta$ then every δ -pseudo orbit $\mathbf{x} := \{x_i\}_{i=0}^\infty$ for g is ε -traced by a point $z \in I$. Furthermore, if \mathbf{x} is a periodic sequence, then z can be chosen to be a periodic point.*

In what follows, we will need a result by Chen [9], which connects the following definition with the shadowing property for interval maps.

Definition 5.14. *Let $f \in C(I)$ and fix any $\varepsilon > 0$. A point $x \in I$ is ε -linked to a point $y \in I$ by f if there exists an integer $m > 1$ and a point $z \in B(x, \varepsilon)$ such that $f^m(z) = y$ and $d(f^j(x), f^j(z)) < \varepsilon$ for $0 < j < m$. We say $x \in I$ is linked to $y \in I$ by f if x is ε -linked to y by f for every $\varepsilon > 0$. We say $C \subset I$ is linked by f if every point $c \in C$ is linked to some point in C . We say that f has the linking property if the set C consisting of critical points together with endpoints of I is linked by f .*

The following result is a consequence of [9] and [24].

Theorem 5.15. *For a topologically mixing piecewise monotone map of the interval, the shadowing property and the linking property are equivalent.*

Theorem 5.16. *Shadowing is generic in CP and in \overline{CP} and there is a dense set of maps in CP which do not have the shadowing property.*

Proof. The proof that shadowing property is generic follows exactly the same arguments as the proof of Theorem 3 in [7] with Lemma 5.12 replaced by Lemma 5.13.

To see that shadowing property is not dense property in \overline{CP} , fix any map $f \in CP$ and conjugate it by a homeomorphism ψ with a map $F \in C_\lambda$. Since by Remark 5.4 C_λ contain an open and dense subset U of leo maps and piecewise affine maps form a dense subset of C_λ , we can choose a piecewise affine leo map G in U which is in an arbitrarily small neighborhood of F . Now, we can perturb map G by a 2-fold perturbation in a small neighborhood of 0 to a piecewise affine map \hat{G} such that 0 is eventually periodic with periodic orbit P and P is disjoint from the set of turning points of \hat{G} . That is indeed possible, because the set of preimages of any point p under G is dense in I , since G is leo. If the perturbation is sufficiently small then \hat{G} remains in U and so it is leo. But it is clear that \hat{G} does not have the linking property, so also

does not have the shadowing property by Theorem 5.15. If \hat{G} is sufficiently close to F then its conjugate g by ψ^{-1} is sufficiently close to f . Observe that by definition both \hat{G} and g belong to CP . This proves the density of maps without the shadowing property, completing the proof. \square

Remark 5.17. *Using Lemma 5.12 instead of Lemma 5.13 we obtain with analogous arguments as in the proof of Theorem 5.16 that there is a dense set of maps in C_λ which do not have the shadowing property.*

5.5. Periodic point structure in \overline{CP} . For $f \in C(I)$ denote

$$\text{Fix}(f, k) := \{x : f^k(x) = x\}$$

$$\text{Per}(f, k) := \{x : f^k(x) = x \text{ and } f^i(x) \neq x \text{ for all } 1 \leq i < k\}$$

$$k(x) := k \text{ for } x \in \text{Per}(f, k)$$

and

$$\text{Per}(f) := \bigcup_{k \geq 1} \text{Per}(f, k) = \bigcup_{k \geq 1} \text{Fix}(f, k).$$

Definition 5.18. *We call a periodic point $p \in \text{Per}(f, k)$ transverse if there exist three adjacent intervals $A = (a_1, a_2)$, $B = [a_2, c_1]$, $C = (c_1, c_2)$, with $p \in B$, B possibly reduced to a point, so that (1) $f^k(x) = x$ for all $x \in B$ and either (2.a) $f^k(x) > x$ for all $x \in A$ and $f^k(x) < x$ for all $x \in C$ or (2.b) $f^k(x) < x$ for all $x \in A$ and $f^k(x) > x$ for all $x \in C$.*

Theorem 5.19. *For a generic map $f \in \overline{CP}$ and for each $k \geq 1$:*

- (1) *the set $\text{Fix}(f, k)$ is a Cantor set,*
- (2) *$\text{Per}(f, k)$ is a relatively open dense subset of $\text{Fix}(f, k)$,*
- (3) *the set $\text{Fix}(f, k)$ has Hausdorff dimension and lower box dimension zero. In particular, $\text{Per}(f, k)$ has Hausdorff dimension and lower box dimension zero. As a consequence, the Hausdorff dimension of $\text{Per}(f)$ is also zero.*
- (4) *the set $\text{Per}(f, k)$ has upper box dimension one. Therefore, $\text{Fix}(f, k)$ has upper box dimension one as well.*

Proof. First note that the last part of (3) follows from the first part of (3) since

$$\dim_H(\text{Per}(f)) \leq \sup_{k \geq 1} \dim_H(\text{Fix}(f, k)) \leq \sup_{k \geq 1} \underline{\dim}_{\text{Box}}(\text{Fix}(f, k)) = 0.$$

For the proofs of 1), 2), 3) and 4) it suffices to prove the result for a fixed $k \in \mathbb{N}$. Fix a countable dense set of maps $\{f_i\}$ in CP . Let ψ_i be a homeomorphism of I such that $\hat{f}_i = \psi_i^{-1} \circ f_i \circ \psi_i$ belongs to C_λ . Let PA_λ denote the set of piecewise affine Lebesgue measure-preserving interval maps. The set of maps in PA_λ having

- a) no interval with slope ± 1 and
- b) having at least one periodic point of period k , and
- c) all periodic points of period k transverse

is dense in C_λ (Definition 10 and Lemma 12 in [7]). The set $\text{Fix}(\hat{g}, k)$ is finite for such a map \hat{g} , since \hat{g} does not have slope ± 1 . Thus we can choose $\hat{g}_i \in \text{PA}_\lambda$ satisfying a), b) and c) close enough to \hat{f}_i and piecewise affine homeomorphisms $\hat{\psi}_i$ of I sufficiently close to ψ_i such that

- (i) each periodic point of g_i is a point where ψ_i is differentiable and
- (ii) the set of piecewise affine maps $\{g_i := \hat{\psi}_i \circ \hat{g}_i \circ \hat{\psi}_i^{-1}\}$ is a dense subset of CP .

Notice that all points in $\text{Fix}(g_i, k)$ are also transverse. The advantage of transversality is that for each point in $\text{Fix}(g_i, k)$, there is at least one corresponding periodic point in $\text{Fix}(g, k)$ if the perturbed map g is sufficiently close to g_i .

After these modifications the rest of the proof follows exactly the same arguments as the proof of Theorem 1 in [7]. For convenience we repeat and streamline the proof, correcting an error in the proof of (2) of Theorem 1 in [7] along the way.

Since $\hat{g}_i \in \text{PA}_\lambda$ does not have slope ± 1 the set $\text{Fix}(\hat{g}_i, k)$ is finite, and thus the set $\text{Fix}(g_i, k)$ is finite as well. Suppose it consists of ℓ_i disjoint orbits and the set $\text{Per}(g_i, k)$ consists of $\bar{\ell}_i \leq \ell_i$ distinct orbits. In particular

$$(4) \quad \ell_i \leq \#\text{Fix}(g_i, k) \leq k\ell_i \text{ and } \text{Per}(g_i, k) = k\bar{\ell}_i.$$

Choosing one point from each of the orbits in $\text{Fix}(g_i, k)$ defines the set $\{x_{l,i} : 1 \leq l \leq \ell_i\} \subset \text{Fix}(g_i, k)$. Let $k(x_{l,i})$ denote the minimal period of $x_{l,i}$. Let $\hat{x}_{l,i} := \psi_i^{-1}(x_{l,i})$.

The construction in the proof depends on integers $n_i \geq 1$ which will be defined in the proof, for most of the estimates it suffices to have $n_i = 1$, but for the upper box dimension estimates we will need n_i growing sufficiently quickly. We consider a very small number \hat{a}_i (several restrictions will be introduced progressively) and define new maps $h_i := \psi_i \circ \hat{h}_i \circ \psi_i^{-1}$ where $\hat{\psi}_i \in \text{PA}_\lambda$ is obtained by applying a regular $2n_i + 1$ -fold window perturbation of \hat{g}_i with respect to $\hat{I}_{l,i} := (\hat{x}_{l,i} - \hat{a}_i, \hat{x}_{l,i} + \hat{a}_i)$. Finally set $I_{l,i} := \psi_i(\hat{I}_{l,i})$, and let a_i be half the length of this interval.

We choose \hat{a}_i so small that $\{h_i\}$ satisfies the following properties:

- i) $\psi_i|_{I_{l,i}}$ is affine.
- ii) $2a_i \leq (k\ell_i)^{-i}$. This implies $a_i \rightarrow 0$ and thus the collection $\{h_i\}_{i \geq 1}$ is dense in CP .

- iii) (Disjointness) $I_{l,i} \cap I_{l',i} = \emptyset$ if $l \neq l'$ and for all $1 \leq l < \ell_i$ and all $0 \leq j < k$ the images $h_i^j(I_{l,i})$ are mutually disjoint,
- iv) (Regularity) For each $1 \leq j \leq k(x_{l,i})$ the map h_i^j restricted to the interval $I_{l,i}$ is composed of $2n_i + 1$ full monotone branches with widths equal to $2a_i/(2n_i + 1)$. This implies
 - a) The map $h_i^{k(x_{l,i})}$ has exactly $2n_i + 1$ fixed points in the interval $I_{l,i}$ and these points have period $k(x_{l,i})$,
 - b) The map h_i^k has $(2n_i + 1)^{k/k(x_{l,i})}$ fixed points in this interval.
 - c) The full branches of $h_i^{k/k(x_{l,i})}$ have width $b_{l,i} := 2a_i/((2n_i + 1)^{k/k(x_{l,i})})$, thus each subinterval of $I_{l,i}$ of length $2b_{l,i}$ contain at least one full branch and at most parts of three branches, and thus at least one fixed point and at most 3 fixed points of $h_i^{k/k(x_{l,i})}$.
 - d) The map h_i has a point of period k in each interval $I_{l,i}$.
 - e) The total number $N_{l,i}$ of fixed points of h_i^k arising from the orbit of $x_{l,i}$ satisfies

$$N_{l,i} = (2n_i + 1)^{k/k(x_{l,i})} k(x_{l,i}).$$

Summing over the points $x_{l,i}$ and using $1 \leq k(x_{l,i}) \leq k$ yields

$$\#\text{Fix}(h_i, k) = \sum_{l=1}^{\ell_i} N_{l,i} \leq (2n_i + 1)^k k \ell_i.$$

Since we have ℓ_i different points $x_{l,i}$ and among them there is a fixed point of g_i , so there is at least one l with $k(x_{l,i}) = 1$, yielding

$$\max((2n_i + 1)\ell_i, (2n_i + 1)^k) \leq \#\text{Fix}(h_i, k).$$

- v) new periodic points obtained by perturbations of each $x_{l,i} \in \text{Per}(g_i, k)$ have to visit all k disjoint intervals $h_i^j(I_{l,i})$ ($0 \leq j < k$). Thus for each $x_{l,i} \in \text{Per}(g_i, k)$ the $N_{l,i} = (2n_i + 1)k$ points are not only in $\text{Fix}(h_i, k)$ but also in $\text{Per}(h_i, k)$; and so $\#\text{Per}(h_i, k) \geq (2n_i + 1)k\ell_i$.

Consider $\delta_i > 0$ and

$$\mathcal{G} := \bigcap_{j \in \mathbb{N}} \bigcup_{i \geq j} B(h_i, \delta_i),$$

where the ball is taken in \overline{CP} . By construction the set \mathcal{G} is a dense G_δ subset of \overline{CP} , by Lemma 4.1 the set $\mathcal{G} \cap CP$ is a G_δ subset of CP .

(1) First we claim that if $\delta_i > 0$ goes to zero sufficiently quickly then $\text{Fix}(f, k)$ is a Cantor set for each $f \in \mathcal{G}$.

By its definition the set $\text{Fix}(f, k)$ is closed for any continuous map f . The set $\text{Fix}(h_i, k)$ is finite. We choose δ_i so small that if $g \in B(h_i, \delta_i)$ then $\text{Fix}(g, k) \subset B(\text{Fix}(h_i, k), a_i)$; in particular the set $\text{Fix}(g, k)$ can not contain an interval

whose length is longer than $2a_i$. Suppose $f \in \mathcal{G}$, thus $f \in B(h_{i_j}, \delta_{i_j})$ for some subsequence i_j . Since $a_{i_j} \rightarrow 0$ the set $\text{Fix}(f, k)$ can not contain an interval.

It remains to show that there are no isolated points in $\text{Fix}(f, k)$. By (iib) each interval $I_{l,i}$ contains exactly $(2n_i + 1)^{k/k(x_{l,i})}$ points of $\text{Fix}(h_i, k)$. We choose δ_i so small that for each $g \in B(h_i, \delta_i)$ each interval $I_{l,i}$ contains at least $(2n_i + 1)^{k/k(x_{l,i})}$ points of $\text{Fix}(g, k)$ and $\text{Fix}(g, k) \subset \cup_{l, 0 \leq j \leq k(x_{l,i})} h_i^j(I_{l,i})$. Now we apply this to $f \in \mathcal{G}$ and $x \in \text{Fix}(f, k)$. Let l_j be such that $x \in I_{l_j, i_j}$. Each of these intervals contains other points of $\text{Fix}(f, k)$, and their lengths tend to zero, which show that x is not isolated and completes the proof of (1).

(2) Fix $f \in \mathcal{G}$ and an i such that $f \in B(h_i, \delta_i)$. As we already saw in the proof of (1) we have $\text{Fix}(f, k) \subset B(\text{Fix}(h_i, k), a_i)$.

But (ii(d)) implies that $\text{Fix}(h_i, k) \subset B(\text{Per}(h_i, k), 2a_i)$. Combining these two facts yields $\text{Fix}(f, k) \subset B(\text{Fix}(h_i, k), a_i) \subset B(\text{Per}(h_i, k), 3a_i)$.

Next we claim that δ_i can be chosen so that $\text{Per}(h_i, k) \subset B(\text{Per}(f, k), a_i)$. Once the claim is proven we will have shown that $\text{Per}(f, k)$ is dense in $\text{Fix}(f, k)$.

Condition (iia) tells us that if $x_{l,i} \in \text{Per}(h_i, k)$ then the $(2n_i + 1)$ fixed points of h_i^k in the interval $I_{l,i}$ all have period k . We have chosen δ_i so small that each $I_{l,i}$ contains at least as many points of $\text{Fix}(g, k)$ for any $g \in B(h_i, \delta_i)$ and $\text{Fix}(g, k) \subset \cup_{l, 0 \leq j \leq k(x_{l,i})} h_i^j(I_{l,i})$. We additionally require that δ_i is so small that a similar statements hold for periodic points, namely that if $k(x_{l,i}) = k$ then each $I_{l,i}$ contains at least as many points of $\text{Per}(g, k)$ for any $g \in B(h_i, \delta_i)$ and $\text{Per}(g, k) \subset \cup_{l: k(x_{l,i})=k, 0 \leq j \leq k} h_i^j(I_{l,i})$. This implies $\text{Per}(h_i, k) \subset B(\text{Per}(g, k), a_i)$ as well, and finishes the proof of the claim.

Finally notice that $\text{Per}(f, k) = \text{Fix}(f, k) \setminus \cup_{1 \leq \ell < k: \ell | k} \text{Fix}(f, \ell)$. But this finite union is closed, thus $\text{Per}(f, k)$ is relatively open subset of $\text{Fix}(f, k)$. This completes the proof of (2).

3) It suffices to prove the lower box dimension statement for $\text{Fix}(f, k)$, the other statements follow since $\text{Per}(f, k) \subset \text{Fix}(f, k)$ and since the Hausdorff dimension of a set is smaller than its lower box dimension. We will show that if $\delta_i > 0$ goes to zero sufficiently quickly then the lower box dimension of $\text{Fix}(f, k)$ is zero for any $f \in \mathcal{G}$.

Consider the open cover $\mathcal{C}_i := \{(x - a_i, x + a_i) : x \in \text{Fix}(h_i, k)\}$ of $\text{Fix}(h_i, k)$, notice that the intervals are pairwise disjoint by assumption (iii) and that \mathcal{C}_i covers the domain $\{I_{l,i} : 1 \leq l \leq l_i\}$ of our perturbations. Choose δ_i sufficiently small so that for each $f \in B(h_i, \delta_i)$ the collection \mathcal{C}_i covers the set $\text{Fix}(f, k)$.

Fix $f \in \mathcal{G}$, thus $f \in B(h_{i_j}, \delta_{i_j})$ for some subsequence i_j . Let $N(\varepsilon)$ denote the number of intervals of length $\varepsilon > 0$ needed to cover $\text{Fix}(f, k)$, combining the previous discussion with Equation (4) yields $N(2a_{i_j}) \leq k l_{i_j} = \#\mathcal{C}_i$. Applying

(ii) yields

$$\frac{\log(N(2a_{i_j}))}{\log(1/2a_{i_j})} \leq \frac{\log(k\ell_{i_j})}{\log(1/2a_{i_j})} \leq \frac{1}{i_j}$$

and thus the lower box dimension of $\text{Fix}(f, k)$ defined by

$$\liminf_{\varepsilon \rightarrow 0} \frac{\log(N(\varepsilon))}{\log(1/\varepsilon)}$$

is 0.

(4) Instead of covering $\text{Fix}(h_i, k)$ by intervals of length a_i we cover it by intervals of length $2b_i$. By (iic) each such interval covers at most three points of $\text{Fix}(h_i, k)$. Thus we need at least $(\#\text{Fix}(h_i, k))/3$ such intervals to cover $\text{Fix}(h_i, k)$; so by (iie) we need at least $(2n_i + 1)^k/3$ such intervals to cover $\text{Fix}(h_i, k)$. Fix such a covering and choose δ_i sufficiently small so that all periodic points of period k of any $f \in B(h_i, \delta_i)$ are contained in the covering intervals.

Thus

$$(5) \quad \frac{\log(N(2b_i))}{\log(1/2b_i)} \geq \frac{\log((2n_i + 1)^k/3)}{\log(1/2b_i)} = \frac{\log((2n_i + 1)^k) - \log(3)}{\log((2n_i + 1)^k) - \log(2a_i)}.$$

The sequence a_i has been fixed above, if n_i grows sufficiently quickly the last term in (5) approaches one. We can not cover $\text{Fix}(h_i, k)$ by fewer intervals, and thus we can not cover $\text{Fix}(f, k)$ by fewer intervals for any $f \in B(h_i, \delta_i)$.

By the above discussion, if we fix any $f \in \mathcal{G}$ then $f \in B(h_{i_j}, \delta_{i_j})$ for some subsequence i_j and therefore, the upper box dimension of $\text{Fix}(f, k)$ defined as

$$\limsup_{\varepsilon \rightarrow 0} \frac{\log(N(\varepsilon))}{\log(1/\varepsilon)}$$

is 1. □

Remark 5.20. *The statements of Theorem 5.19 also hold for generic maps in CP by Lemma 4.1.*

Theorem 5.21. *The set of leo maps in CP whose periodic points have full Lebesgue measure and whose periodic points of period k have positive measure for each $k \geq 1$ is dense in CP .*

Proof. Fix a map $\alpha \in CP$. By [7, Remark, p. 2536] the map α is from C_μ for some nonatomic probability measure μ with $\text{supp } \mu = I$. Moreover, the map $\psi: I \rightarrow I$ defined by $\psi(x) = \mu([0, x])$ is a homeomorphism for which $\beta = \psi \circ \alpha \circ \psi^{-1} \in C_\lambda$. By [7, Theorem 2] the statement of Theorem 5.21 is

true in C_λ . So there exists a map $\gamma \in C_\lambda$ for which

$$(6) \quad \begin{aligned} \rho(\beta, \gamma) &< \varepsilon_1, \quad \lambda(\text{Per}(\gamma, k)) > 0 \text{ for each } k, \\ \lambda(\text{Per}(\gamma)) &= \lambda\left(\bigcup_{k \geq 1} \text{Per}(\gamma, k)\right) = 1. \end{aligned}$$

Obviously there is a piecewise affine homeomorphism $\ell: I \rightarrow I$ such that

$$(7) \quad \rho(\psi, \ell) + \rho(\psi^{-1}, \ell^{-1}) < \varepsilon_1.$$

Choose $\varepsilon_2 > 0$ arbitrary. If ε_1 is sufficiently small, from (6) and (7) one has for $\delta = \ell^{-1} \circ \gamma \circ \ell$, $\text{Per}(\delta, k) = \ell^{-1}(\text{Per}(\gamma, k))$ for each k hence also $\text{Per}(\delta) = \ell^{-1}(\text{Per}(\gamma))$,

$$\rho(\alpha, \delta) = \rho(\psi^{-1} \circ \beta \circ \psi, \ell^{-1} \circ \gamma \circ \ell) < \varepsilon_2,$$

and since ℓ is piecewise affine,

$$\lambda(\text{Per}(\delta, k)) > 0 \text{ for each } k \text{ and } \lambda(\text{Per}(\delta)) = 1. \quad \square$$

For completeness we also show the following proposition which is well known in many situations. Let $CP_{\text{entr}=\infty}$ denote the set of maps from CP with topological entropy ∞ .

Proposition 5.22. *The set of maps $CP_{\text{entr}=\infty}$ is a residual set in CP and thus in \overline{CP} .*

Proof. Every map $f \in CP \setminus \{\text{id}\}$ has a fixed point b where the graph of f is transverse to the diagonal at b . This is indeed true since it holds for maps in $C_\lambda \setminus \{\text{id}\}$ and any map $f \in CP$ is conjugate to a map $\hat{f} = \psi^{-1} \circ f \circ \psi \in C_\lambda$; therefore also $f \in CP \setminus \{\text{id}\}$ have a transverse fixed point. Using an $(n+2)$ -fold window perturbation on a neighborhood of b , and again passing through C_λ , we can create a map $g \in CP$ arbitrarily close to f with a horseshoe with entropy $\log(n)$ in the window. Since horseshoes are stable under perturbations around stable fixed points, there is an open ball $B(g, \delta)$ in CP such that each h in this ball has topological entropy at least $\log(n)$ for any $n \geq 1$. The result also hold for \overline{CP} by Lemma 4.1. \square

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