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# Discovering Repeated Patterns From the Onsets in a Multidimensional Representation of Music

Paul Lascabettes<sup>1</sup>[0000-0001-7132-8642] and Isabelle Bloch<sup>2</sup>[0000-0002-6984-1532]

<sup>1</sup> STMS - Sorbonne Université, Ircam, CNRS, Ministère de la Culture, Paris, France  
`paul.lascabettes@ircam.fr`

<sup>2</sup> Sorbonne Université, CNRS, LIP6, Paris, France  
`isabelle.bloch@sorbonne-universite.fr`

**Abstract.** This article deals with the discovery of repeated patterns in a multidimensional representation of music using the theory of mathematical morphology. The main idea proposed here is to use the onsets to discover musical patterns. By definition, the morphological erosion of musical data by a musical pattern corresponds to its onsets. However, the erosion of musical data by the onsets is not always equal to the musical pattern. We propose a theorem which guarantees the equality if the musical pattern satisfies a topological condition. This condition is met when the patterns do not intersect, or only slightly, which is coherent in a musical context. Due to the importance of repetition in music, this idea proves to be relevant for the musical pattern discovery task.

**Keywords:** Pattern discovery · Mathematical morphology · Point-set algorithms · Geometric pattern discovery in music · Music analysis.

## 1 Introduction

While mathematical morphology has been widely applied to image processing, analysis and understanding, there are only few direct applications of this theory, in particular in its algebraic setting, to symbolic representations of music. This is partly due to the different nature of the patterns. On the one hand, the objects associated with digital image processing are defined on a discrete grid, endowed with a discrete connectivity, and are often connected sets. On the other hand, those associated with symbolic music are sparse, in the sense that they are often not connected, according to the underlying connectivity of the space of representation. It is therefore necessary to adapt the morphological tools and the expected results when applying this theory to symbolic representations of music. This has been started by adapting the basic operators of mathematical morphology to find a musical meaning, for example to extract harmonic components or to obtain musical transformations [6, 8]. Among the other existing applications, Karvonen et al. have developed automatic methods to discover approximate occurrences of a given pattern in symbolic musical databases [4, 5].

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We advocate in this article that mathematical morphology can provide relevant tools for the discovery of repeated patterns in a multidimensional representation of music, i.e. in a discrete set of points. Even if we focus on symbolic representations of music, the developed results can be applied to other types of discrete data composed of repeated patterns.

There are many algorithms for discovering repeated patterns from a multidimensional representation of music. Most of these algorithms are based on the SIA algorithm [12, 13], which stands for “Structure Induction Algorithm”. This algorithm, developed by Meredith et al., consists in discovering the largest translatable patterns in a multidimensional dataset. Among other fundamental algorithms, the SIATEC algorithm [10, 12, 13] reveals the repetitions of the largest translatable patterns discovered by SIA. However, these algorithms discover too many patterns, some of which are not musically relevant. Therefore, with the aim of improving the precision and efficiency of SIA, Collins et al. proposed the SIAR algorithm [1], which corresponds approximately to using a sliding window of size  $r$  in SIA in order to avoid discovering patterns that are too long, and the SIACT algorithm [1], to detect if a sub-pattern of the largest translatable pattern is musically more important. Finally, the method we propose here belongs to the category of algorithms that discover patterns in order to describe musical data, such as COSIATEC [12] (cover the dataset without overlaps), SIATECCompress [11] (cover the dataset with overlaps) or Forth’s algorithm [2] (cover not the entire dataset and with overlaps).

The originality of our approach comes from the use of a musical meaning to discover patterns by distinguishing the role of onsets or musical pattern. In particular, we provide various mathematical results related to musical problems. These results optimize the discovery of repeated patterns, and are the foundation for a new approach to discover musical patterns.

This article is organized as follows. Section 2 summarizes the principal operators of mathematical morphology and their properties used in this article. Section 3 presents the problem of discovering the musical patterns from their onsets. Section 4 provides a solution to this problem if the musical patterns satisfy a specific topological condition. Section 5 interprets this result from a musical point of view. Section 6 demonstrates how to use this result to optimize the discovery of musical patterns. Finally, Section 7 concludes this article and proposes some future work.

## 2 Notations and Background: Binary Mathematical Morphology

In this section, we recall the concepts of mathematical morphology used in this article, for it to be self-contained. In particular, we consider here the simple case of binary mathematical morphology, where the set  $E$  is equal to  $\mathbb{R}^n$ . In the remainder of this article,  $\mathcal{P}(E)$  is the power set of  $E$ . First, let  $S \in \mathcal{P}(E)$  and  $t \in E$ , we denote the *translate* of  $S$  by  $t$  as  $S_t = \{s + t \mid s \in S\}$ . The majority of morphological operators, in the deterministic setting, result from two basic

ones: the *dilation* resulting from the *Minkowski addition*  $\oplus$  [14] and the *erosion* from the *substraction*  $\ominus$  [3].

**Definition 1 (Dilation and Erosion).** Let  $S \in \mathcal{P}(E)$ , the dilation  $\delta_S$  and erosion  $\varepsilon_S$  by  $S$  are defined by:

$$\begin{aligned} \delta_S : \mathcal{P}(E) &\longrightarrow \mathcal{P}(E) \\ X &\longmapsto X \oplus S = \{x + s \mid x \in X, s \in S\} \end{aligned} \quad (1)$$

$$\begin{aligned} \varepsilon_S : \mathcal{P}(E) &\longrightarrow \mathcal{P}(E) \\ X &\longmapsto X \ominus S = \{x \in E \mid S_x \subseteq X\} \end{aligned} \quad (2)$$

In this case,  $S$  is called a *structuring element*. By composing these two operations, we obtain the two other fundamental operators of mathematical morphology: *opening* and *closing*.

**Definition 2 (Opening and Closing).** Given  $S \in \mathcal{P}(E)$ , the opening  $\gamma_S$  and the closing  $\varphi_S$  are defined by:

$$\begin{aligned} \gamma_S : \mathcal{P}(E) &\longrightarrow \mathcal{P}(E) \\ X &\longmapsto \delta_S(\varepsilon_S(X)) = \bigcup \{S_x \mid x \in E \wedge S_x \subseteq X\} \end{aligned} \quad (3)$$

$$\begin{aligned} \varphi_S : \mathcal{P}(E) &\longrightarrow \mathcal{P}(E) \\ X &\longmapsto \varepsilon_S(\delta_S(X)) \end{aligned} \quad (4)$$

These four operations satisfy many different properties. In particular, they are all *increasing* (i.e.  $X \subseteq X' \Rightarrow f_S(X) \subseteq f_S(X')$ , where  $f_S$  denotes one of the four operations). Also, the dilation and the closing are *increasing according to the structuring element* (i.e.  $S \subseteq S' \Rightarrow \forall X, \delta_S(X) \subseteq \delta_{S'}(X)$  and  $\varphi_S(X) \subseteq \varphi_{S'}(X)$ ). While the erosion and the opening are *decreasing according to the structuring element* (i.e.  $S \subseteq S' \Rightarrow \forall X, \varepsilon_{S'}(X) \subseteq \varepsilon_S(X)$  and  $\gamma_{S'}(X) \subseteq \gamma_S(X)$ ). Moreover, the position of the structuring element  $S$  with respect to the origin  $O_E$  of the space  $E$  has an impact on the result of the dilation or erosion. In particular, if the origin belong to  $S$ , the dilation is *extensive* (i.e.  $X \subseteq \delta_S(X)$ ) and the erosion is *anti-extensive* (i.e.  $\varepsilon_S(X) \subseteq X$ ), and conversely, these properties hold only if the origin belong to  $S$ . However, the position of  $S$  in relation to the origin does not change the result of the opening and the closing, only the shape is involved. In any case, the opening is anti-extensive (i.e.  $\gamma_S(X) \subseteq X$ ) and the closing is extensive (i.e.  $X \subseteq \varphi_S(X)$ ). In addition, it is easy to check that the dilation is *commutative* ( $\delta_S(S') = \delta_{S'}(S)$ ). Finally, if we consider that the origin is included in  $S$  and that  $X$  is a piece of music, among the four basic operations of mathematical morphology, erosion and opening are *analysis operators* (extracting musical data), while dilation and closing are *generation operators* (enriching the music).

### 3 Presentation of the Problem of Discovering the Musical Pattern From Its Onsets

We present here the problem of *discovering the musical pattern from its onsets*. We define the onsets of a pattern as the origins of its occurrences in musical data. This makes the connection with the morphological erosion.

**Definition 3 (Onsets).** *Let  $P, X \in \mathcal{P}(E)$ . The onsets  $O$  of  $P$  in  $X$  are defined by:*

$$O = \varepsilon_P(X) \tag{5}$$

The onsets of a pattern can be interpreted as the beginnings of this pattern in the musical data if the origin of the pattern is placed on its first note, assuming that the dimensions of space are oriented. For example, in Figure 1(a), musical data are represented by dots, which can be seen as a pattern repeated four times. In this two-dimensional representation, which is used in the rest of this article, the vertical axis refers to note pitch, while the horizontal axis indicates time. Repetitions of the pattern are indicated in the figure by dotted lines. The onsets of this pattern are therefore composed of four dots, which correspond to the beginnings of the repeated pattern. The interesting result is illustrated in Figure 1(b), by now considering the onsets as a pattern: the onsets of the onsets of the pattern are equal to the pattern. In other words, by computing the points where the onsets occur, we discover the pattern (the repetitions of the onsets are indicated by the dotted lines in the figure). This result is quite surprising, and we can formalize it with morphological operators as follows.

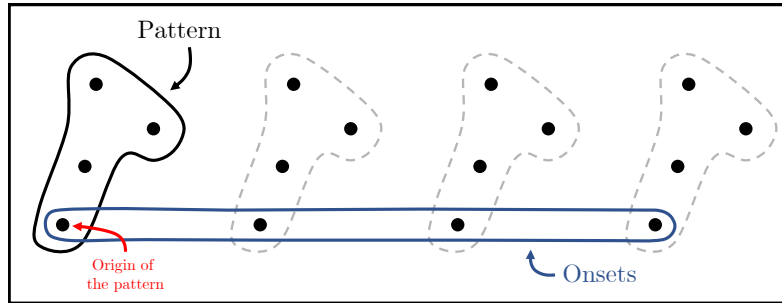
**Definition 4 (Discovering the Musical Pattern From Its Onsets).** *Let  $P, O, X \in \mathcal{P}(E)$ , where  $O = \varepsilon_P(X)$ . The problem of discovering the musical pattern from its onsets is to understand when the following equation is true:*

$$P = \varepsilon_O(X) \tag{6}$$

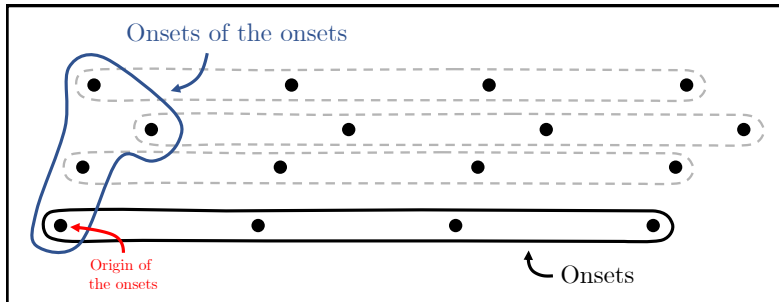
Let  $P, O, X \in \mathcal{P}(E)$  such that  $O = \varepsilon_P(X)$ . This can be interpreted as  $X$  the musical data,  $P$  a musical pattern and  $O$  its onsets. First of all, the problem of discovering the musical pattern from its onsets, i.e. the equality  $P = \varepsilon_O(X)$ , is not always true as shown in Figure 2. However, Lemma 1 allows us to state that the inclusion  $P \subseteq \varepsilon_O(X)$  is always true. Therefore,  $P$  is always included in the onsets of  $O$ . This implies that taking the onsets of the onsets of a pattern enlarges it, as can be seen in Figure 2 where  $P$ , represented in Figure 2(b), is included in  $\varepsilon_O(X)$ , represented in Figure 2(d). Note that the figures are 2D examples, for the sake of clarity, but all theoretical results apply for any finite dimension that can represent note duration, velocity or voice in a musical context.

**Lemma 1.** *Let  $P, O, X \in \mathcal{P}(E)$ , such that  $O = \varepsilon_P(X)$  and  $O \neq \emptyset$ . We have:*

$$P \subseteq \varepsilon_O(X) \tag{7}$$



(a) Representation of the pattern and its onsets, which are interpreted as the points where the pattern begins. Pattern repetitions are indicated by the dotted-line border. The origin of the pattern is chosen at the starting point of the pattern (assuming that the time axis is from left to right).



(b) The onsets of the onsets are equal to the pattern. Onsets repetitions are indicated by the dotted-line border. That is, the dots corresponding to where the onsets begin are the same as the pattern dots. As before, the origin of the onsets is chosen at its starting point.

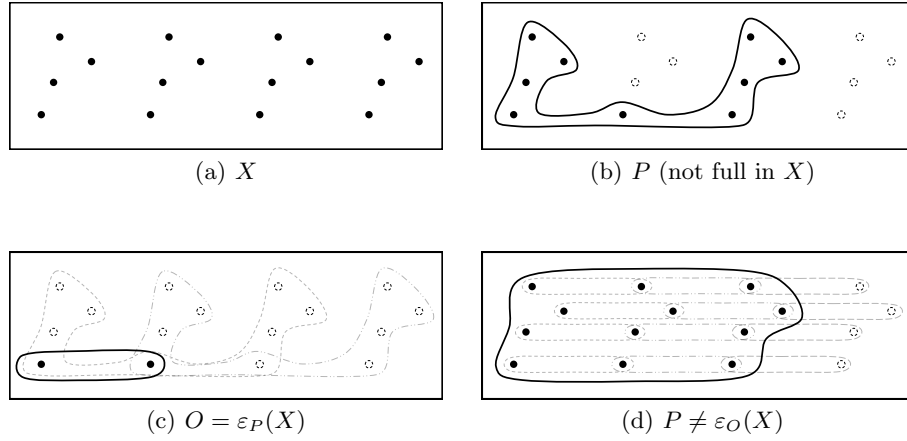
**Fig. 1.** Presentation of the problem of discovering the pattern from its onsets. In this example, the onsets of the onsets of the pattern are equal to the pattern, which shows that it is sufficient to know the onsets to discover the pattern.

*Proof (Lemma 1).* Let  $p \in P$  and  $o \in O$ . Because  $O = \varepsilon_P(X)$  and  $O \neq \emptyset$ ,  $\exists x \in X : p + o = x$ . This is true for all  $o \in O$ , consequently  $O_p \subseteq X$ , which leads to  $p \in \varepsilon_O(X)$ .  $\square$

In the remainder of this article, we prove that the problem of discovering the musical pattern from its onsets can be solved under some assumptions that are musically interpretable.

## 4 Main Results

Because the equality  $P = \varepsilon_O(X)$  is not always true, we need to add an additional condition on  $P$  and  $X$  to ensure this equality. First, we consider the particular case where the data  $X$  is composed of patterns  $P$  which are repeated with



**Fig. 2.** Example where  $P$  is not full in  $X$  (Definition 5) and  $P \neq \varepsilon_O(X)$ , however the inclusion  $P \subseteq \varepsilon_O(X)$  is satisfied (the origin is the first point on the left).

translations, that is to say  $\gamma_P(X) = X$ . Moreover, we introduce a new definition, illustrated in Figure 3, that is, to some extent, the counterpart of the connectivity (without holes) used in image processing. For  $P \in \mathcal{P}(E)$ , let us note  $\text{CH}(P)$  the *convex hull* of  $P$  defined by  $\text{CH}(P) = \{\sum_{i=1}^k \lambda_i p_i \mid k \in \mathbb{N}^* \wedge \sum_{i=1}^k \lambda_i = 1 \wedge \forall i \in \llbracket 1, k \rrbracket, p_i \in P, \lambda_i \geq 0\}$ , where  $\llbracket 1, k \rrbracket$  is the set of integers from 1 to  $k$  included.

**Definition 5 ( $P$  Full in  $X$ ).** Let  $P, X \in \mathcal{P}(E)$ .  $P$  is full in  $X$  if  $P_t \subseteq X$  for any  $t$  implies that  $\text{CH}(P_t)$  does not contain any point of  $X$  other than  $P_t$ , i.e.:

$$\forall t \in E, P_t \subseteq X \Rightarrow \text{CH}(P_t) \cap X = P_t \quad (8)$$

Under the condition that  $P$  is full in  $X$ , Theorem 1 provides the equality  $P = \varepsilon_O(X)$ . Therefore, it is possible, under the assumptions of the theorem, to discover the pattern  $P$  from its onsets  $O$ .

**Theorem 1.** Let  $P, X \in \mathcal{P}(E)$  such that:

- $\gamma_P(X) = X$ ,
- $P$  is full in  $X$ ,
- $X$  is bounded.

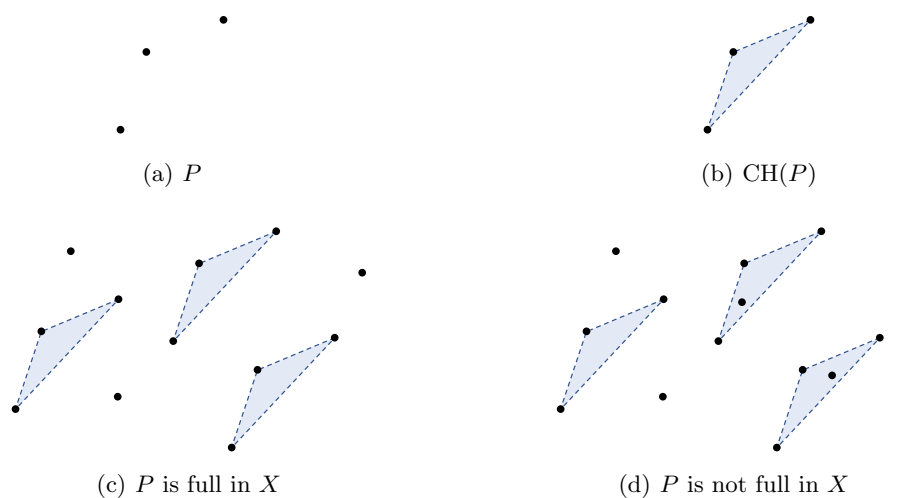
Then, by defining  $O = \varepsilon_P(X)$ , we have:

$$P = \varepsilon_O(X) \quad (9)$$

We provide another link between  $P$  and  $O$  using morphological operators with the following lemma, which is useful for the proof of Theorem 1.

**Lemma 2.** Let  $P, O, X \in \mathcal{P}(E)$ , such that  $O = \varepsilon_P(X)$  and  $\gamma_P(X) = X$ . We have:

$$\varepsilon_O(X) = \varphi_O(P) \quad (10)$$



**Fig. 3.** Musical pattern  $P$  (a) and its convex hull (b).  $P$  is full in  $X$  (set of dots) in (c) because there are no dots inside the convex hull where it is included in  $X$ . However, this is not the case for (d).

*Proof (Lemma 2).* Under the assumptions of the lemma, we have:

$$\begin{aligned}
 & \gamma_P(X) = X \\
 \Rightarrow & \delta_P(\varepsilon_P(X)) = X && \text{(definition of } \gamma_P) \\
 \Rightarrow & \delta_P(O) = X && \text{(definition of } O) \\
 \Rightarrow & \delta_O(P) = X && \text{(commutativity of } \oplus) \\
 \Rightarrow & \varepsilon_O(\delta_O(P)) = \varepsilon_O(X) && \text{(composing by } \varepsilon_O) \\
 \Rightarrow & \varphi_O(P) = \varepsilon_O(X) && \text{(definition of } \varphi_O)
 \end{aligned}$$

□

*Proof (Theorem 1).* Under the assumptions of the theorem, we can use Lemma 2 and Lemma 1. We prove that  $P = \varepsilon_O(X)$  with two inclusions.

$\subseteq$  The first inclusion comes directly from Lemma 1.

$\supseteq$  It has been proved by Serra that for every  $C \in \mathcal{P}(E)$  convex and  $S \in \mathcal{P}(E)$  bounded, we have:  $\varphi_S(C) = C$  (proposition IV-4 in [15]). Since  $X$  is bounded,  $O = \varepsilon_P(X)$  is also bounded and  $\text{CH}(P)$  is convex by definition. Consequently, we have:

$$\varphi_O(\text{CH}(P)) = \text{CH}(P) \tag{11}$$



Let  $t \in E$  such that  $P_t \subseteq X$  (such  $t$  exists because  $\gamma_P(X) = X$ ), therefore  $t \in O$  and we have the following implications:

$$\begin{aligned}
& P \subseteq \text{CH}(P) \\
\Rightarrow & \varphi_O(P) \subseteq \varphi_O(\text{CH}(P)) && (\varphi_O \text{ is increasing}) \\
\Rightarrow & \varphi_O(P) \subseteq \text{CH}(P) && (\varphi_O(\text{CH}(P)) = \text{CH}(P) \text{ Eq. (11)}) \\
\Rightarrow & \varepsilon_O(X) \subseteq \text{CH}(P) && (\varphi_O(P) = \varepsilon_O(X) \text{ Lemma 2}) \\
\Rightarrow & \varepsilon_O(X)_t \subseteq \text{CH}(P)_t && (\text{translation by } t) \\
\Rightarrow & \varepsilon_O(X)_t \subseteq \text{CH}(P_t) && (\text{CH}(P)_t = \text{CH}(P_t)) \\
\Rightarrow & \varepsilon_{O_{-t}}(X) \subseteq \text{CH}(P_t) && (\varepsilon_O(X)_t = \varepsilon_{O_{-t}}(X)) \\
\Rightarrow & \varepsilon_{O_{-t}}(X) \cap X \subseteq \text{CH}(P_t) \cap X && (\text{intersection with } X) \\
\Rightarrow & \varepsilon_{O_{-t}}(X) \subseteq \text{CH}(P_t) \cap X && (\varepsilon_{O_{-t}}(X) \subseteq X \text{ because } O_E \in O_{-t}) \\
\Rightarrow & \varepsilon_{O_{-t}}(X) \subseteq P_t && (\text{CH}(P_t) \cap X = P_t \text{ because } P \text{ is full}) \\
\Rightarrow & \varepsilon_O(X) \subseteq P && (\text{translation by } -t)
\end{aligned}$$

□

## 5 Musical Interpretations of the Theorem

The main result of this article, i.e. Theorem 1, is illustrated in Figure 4 with a piece of music  $X$  composed of a musical pattern  $P$  repeated several times (therefore  $\gamma_P(X) = X$ ). The assumptions of Theorem 1 are satisfied because  $X$  is bounded and  $P$  is full in  $X$ . By definition, the morphological erosion of  $X$  by the musical pattern  $P$  is equal to the onsets  $O$ . Moreover, Theorem 1 ensures that the other way around is true: we can obtain the pattern  $P$  by applying the erosion by the onsets  $O$ . All these morphological links between the piece  $X$ , the pattern  $P$  and its onsets  $O$  are summarized in Figure 4. In this case, we can describe the piece of music  $X$  by a morphological dilation between the pattern and its onsets:

$$X = P \oplus O \tag{12}$$

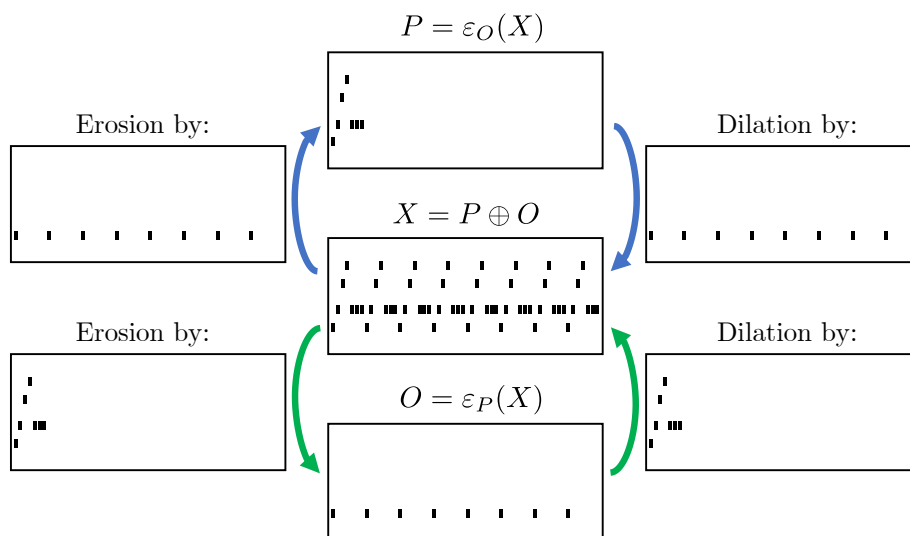
This last equality increases the relevance of the use of morphological operators applied to music.

*Remark 1 (About the assumptions in Theorem 1).* In Theorem 1, there are three assumptions which are:  $P$  is full in  $X$ ,  $\gamma_P(X) = X$  and  $X$  is bounded.

- $P$  is full in  $X$ : This assumption is coherent in a musical context. In the *Generative Theory of Tonal Music* (GTTM), the authors state several rules about overlaps in music [9]. In particular, the *Grouping Well-Formedness Rules 4* asserts:

“If a group  $G_1$  contains part of a group  $G_2$ , it must contain all of  $G_2$ ”

This proves that it is very rare for patterns to intersect in music. However, later in GTTM, in Section 3.4 *Grouping Overlaps*, it is mentioned that this



**Fig. 4.** Illustration using morphological operators of the relations between the musical pattern  $P$ , which is full in  $X$ , and its onsets  $O$ .

rule is not always true and that patterns can overlap in a specific case: if patterns overlap, it is only the first or last note of the pattern that can be part of two patterns. Therefore, if we assume that a pattern  $P$  satisfies the GTTM rules (i.e. patterns do not intersect or only share their first or last note) and that the data  $X$  is composed of  $P$  which is repeated with translations (i.e.  $\gamma_P(X) = X$ ), this leads to having a pattern  $P$  that is full in  $X$ . This reasoning shows that the property that a pattern is full is perfectly coherent in a musical context. Moreover, this property allows us to obtain the fundamental result of Theorem 1, and if we remove this assumption, the theorem is not always true, as shown in Figure 2.

- $\gamma_P(X) = X$ : In general, there are several musical patterns in a piece of music and we cannot always have  $\gamma_P(X) = X$ . However, to ensure this equality, we can restrict the analysis to musical sections composed of repeated patterns, and not the whole piece of music. Since musical sections do not intersect, it is therefore possible to apply Theorem 1 to temporal regions separately, and thus to apply Theorem 1 independently to each musical section and not to the whole piece. Therefore, to respect this assumption, we can consider the discovery of each pattern  $P_i$  separately and restrict  $X$  to  $\gamma_{P_i}(X)$ .
- $X$  is bounded: In our analysis, the case where  $X$  is not bounded never happens because  $X$  represents a piece of music which is by definition bounded. There is neither an infinite number of notes, nor an infinite value for pitch or other musical characteristics. Therefore, this assumption is necessary to ensure that Serra's proposition is true but is not a restriction for the musical applications.

## 6 Optimize the Discovery of Musical Patterns With the Onsets

In this article, we propose the novelty of having two methods for discovering the musical patterns in a multidimensional representation of music: one can discover either the musical patterns  $P$  or their onsets  $O$ . Using Theorem 1, it is enough to identify only one of the two to obtain the other one (if  $P$  is full in  $X$  which is usually the case in music). In each case, it is necessary to proceed differently, but the main idea is to use  $X$  to discover the musical patterns. The following lemma indicates that musical patterns or the onsets must be composed of sub-patterns in  $X$ , i.e. if a sub-pattern  $P'$  does not appear in  $X$  then it cannot be contained in musical patterns or the onsets.

**Lemma 3.** *Let  $P, P', X \in \mathcal{P}(E)$ . If  $\gamma_{P'}(X) = \emptyset$  then  $\gamma_{P \cup P'}(X) = \emptyset$ .*

*Proof (Lemma 3).* This is due to the decreasingness of the opening with respect to the structuring element, i.e:  $P' \subseteq P \cup P' \Rightarrow \gamma_{P \cup P'}(X) \subseteq \gamma_{P'}(X) = \emptyset$ .  $\square$

From the previous lemma, we can deduce that the points of the musical patterns or the onsets must belong to the set:

$$\{x_2 - x_1 \mid x_1, x_2 \in X \wedge x_1 \leq x_2\}, \quad (13)$$

where  $x_1 \leq x_2$  means that the temporal component of  $x_1$  is less than or equal to the temporal component of  $x_2$ . In other words, we can restrict ourselves to this set, rather than trying to cover the whole  $E$  set to discover musical patterns. This observation has already been made by Meredith et al. with the SIA algorithm to optimize the discovery of musical patterns [12]. However, we propose here to give a musical meaning to the discovered patterns, distinguishing the role of pattern  $P$  and onsets  $O$ . Because many elements of this set are not musically interesting, additional conditions have to be added to find the appropriate musical patterns. Depending on whether we want to discover  $P$  or  $O$ , we then add different constraints.

*Learning the Musical Patterns  $P$ .* To reduce the possibilities to be tested to discover musical patterns  $P$ , we need to add more constraints that remove irrelevant candidates. For example, to avoid obtaining musical patterns that are too long in time, Collins et al. proposed to add a constraint on the temporal length of the musical pattern with the SIAR algorithm [1]. However, the possibilities remains very large and we propose another approach to discover musical patterns, using the onsets.

*Learning the Onsets  $O$ .* One of the most important results of this article is the possibility of discovering musical patterns from the onsets. Under the assumptions mentioned in Theorem 1, it is possible to discover the musical pattern from the onsets with the morphological erosion. The major usefulness of this result

arises from the algorithmic complexity reduction: it can be faster to discover the onsets than the patterns because there are far fewer possible choices.

In order to discover *exact repetitive patterns*, it is possible to consider time-periodic onsets. For example, if we are looking for musical patterns  $P$  that repeat in time at regular intervals of length  $L > 0$ , the onsets  $O$  are:

$$O = \{(iL, 0, \dots, 0) \mid i \in \llbracket 0, m - 1 \rrbracket\}, \quad (14)$$

where the first coordinate  $iL$  represents the temporal component. In this case, we are looking for at least  $m$  exact repetitions of the musical pattern  $P$ . As a result, only one parameter  $L$  is required to discover patterns that repeat exactly at regular intervals. However, in the case where the patterns are not repeated exactly with a time translation, they can also be discovered with *transpositions*, which means that a pitch translation is also allowed. Up to an octave transposition up or down is enough to cover most cases, i.e. a pitch translation in  $\llbracket -12, 12 \rrbracket$ . Representing the pitch by the second coordinate, the onsets  $O$  are of the following form:

$$O = \{(iL, p_i, 0, \dots, 0) \mid i \in \llbracket 0, m - 1 \rrbracket \wedge p_i \in \llbracket -12, 12 \rrbracket\} \quad (15)$$

We can search for the values of  $L$  and  $p_i$  that are in the set defined in Equation 13, choosing those that maximize the number of notes after an opening by the onsets. Therefore, only a few parameters  $L$  and  $p_i$  are required to discover patterns that repeat at regular intervals with transpositions, where each parameter has a limited number of values.

## 7 Conclusion and Future Work

In this article, we have proposed an original approach based on mathematical morphology for discovering musical patterns using a multidimensional representation of music. The originality of our method is to provide a musical meaning to the discovered patterns, by distinguishing the role of onsets from the musical patterns. With this characterization, we introduced the problem of *discovering the musical pattern from its onsets*. We then proposed a solution to this problem if the patterns respect a topological property that is musically coherent, because patterns do not intersect in music. This reveals the possibility of discovering musical patterns from their onsets. We have demonstrated the relevance of this result in a musical context, as it optimizes the discovery of musical patterns due to the importance of repetition in musical data. An interesting feature is that we proved a mathematically strong result, with a relevant musical interpretation, based on simple morphological operations. In future work, we intend to apply the method developed in this article to music databases, and compare the obtained results with existing methods. In addition, we aim to determine the links between point-set algorithms and mathematical morphology [7], and produce other mathematical results to characterize the discovered pairs of musical patterns and their onsets. These directions of research are original in that they

produce mathematical results that come from a musical motivation, but which can be applied to other types of discrete data, e.g. in topological data analysis. Finally, based on this approach, we hope to extend the application of mathematical morphology to symbolic representations of music, which will allow us to develop additional theories and produce further original mathematical results.

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