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# Extremal determinants: the periodic one-dimensional essentially bounded case*

Jean-Baptiste Caillau† Yacine Chitour‡ Pedro Freitas§ Yannick Privat¶

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**Abstract**

The functional determinant of elliptic differential operators on the circle was introduced in [3]. In the present paper, optimisation of this determinant over essentially bounded functions is studied. In the one dimensional case, existence and uniqueness of maximisers and minimisers is proved.

## 1 Statement of the problem

The fundamental reference for spectral problems on the circle $\mathbb{S}^1$ (geometrisation of the periodic boundary conditions) is [3], more general than [4]. The latter reference, however, provides much more elementary arguments enabling one to establish links with the discrete setting. We will use the following notations to state the problem: let $n, N$ be two positive integers, we denote $M_N(\mathbb{R})$, $\mathrm{Id}_N$ and $SO(N)$ the set of $N \times N$ matrices with real coefficients, the corresponding identity matrix and the group of orthogonal matrices (matrices $U$ verifying $U^T U = \mathrm{Id}_N$), respectively. For $Q \in M_N(\mathbb{R})$ we use the Frobenius norm $\|Q\| = \mathrm{Tr}(Q^T Q)^{1/2}$ and recall it derives from the inner product on $M_N(\mathbb{R})$ given by

$$\langle Q_1, Q_2 \rangle = \mathrm{Tr}(Q_1^T Q_2), \quad Q_1, Q_2 \text{ in } M_N(\mathbb{R}). \tag{1}$$

We will also consider $[Q_1, Q_2]$, which denotes the Lie bracket between two square matrices of the same size. We use $\mathrm{Com}(M)$ to denote the comatrix of $M \in M_N(\mathbb{R})$. If $\nu^2 = 1$, we use $c_\nu(t)$ (respectively $s_\nu(t)$) to denote $\cosh(t)$ if $\nu = 1$ and $\cos(t)$ if $\nu = -1$ (respectively $\sinh(t)$ if $\nu = 1$ and $\sin(t)$ if $\nu = -1$). With these conventions, one also has for every $x \in \mathbb{R}$ that

$$c_\nu^2(x) - \nu s_\nu^2(x) = 1, \ \dot{c}_\nu(x) = \nu s_\nu(x), \ \dot{s}_\nu(x) = c_\nu(x), \tag{2}$$

$$c_\nu(2x) = 1 + 2\nu s_\nu^2(x), \ s_\nu(2x) = 2\nu s_\nu(x) c_\nu(x). \tag{3}$$

As a consequence, if $d$ is a positive real number, the solution of the linear second order equation $\ddot{y} = \nu d y$ is given by

$$y(t) = c_\nu(dt) y(0) + \frac{1}{d} s_\nu(dt) \dot{y}(0), \quad t \in \mathbb{R}. \tag{4}$$

We use $E(x)$ to denote the integer part of $x \in \mathbb{R}$ and $\mathrm{sgn}(x)$ to denote the sign of a non zero real number $x$.

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Following [3], we address optimisation issues relative to the determinant of a differential operator $A = \sum_{k=0}^{n} A_k D^k$ defined on $\mathbb{R}^N$-valued functions, where $D = -i\mathrm{d}/\mathrm{d}x$ is the derivation operator for such functions and the $A_k : \mathbb{S}^1 \to M_N(\mathbb{R})$, $0 \leq k \leq n$, are matrix-valued functions defined on $\mathbb{S}^1$. For the rest of the paper, we identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$ and functions on $\mathbb{S}^1$ with one-periodic functions. We will assume that

$$A = -\operatorname{Id}_N \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \tag{5}$$

*i.e.*, the maximal order of differentiation $n$ is equal to two, $A_2 = \operatorname{Id}_N$, $A_1 = 0$ and $A_0 = V$ is a Hill potential. The quantity to be optimised (*cf.* [3] on page 2) is

$$\det(A) := (-1)^N \det(\operatorname{Id}_{2N} - R(A)) \tag{6}$$

with $R(A)$ the monodromy operator. More precisely, $R(A)$ is equal to the fundamental matrix at time 1 (after normalising the length of the circle to be equal to $2\pi$) of the linear time-varying system on $\mathbb{R}^{2N}$

$$\left\{ \begin{array}{rcl} \dot{R}(x) & = & \mathcal{A}_{V(x)} R(x), \\ R(0) & = & \operatorname{Id}_{2N}, \end{array} \right. \tag{7}$$

where one sets

$$\mathcal{A}_Q = \begin{pmatrix} 0 & \operatorname{Id}_N \\ Q & 0 \end{pmatrix}, \text{ for every } Q \in M_N(\mathbb{R}).$$

**Remark 1.1** *In [3], the potential $V$ appears as $-V$ in* (7) *and we have changed notations in order to stick with previous optimisation literature [2].*

**Remark 1.2** *Since the trace of $\mathcal{A}_V$ is the zero function, it follows that* $\det R$ *remains constant equal to one along any trajectory of* (7), i.e., *$R$ entirely lies in $SL_{2N}(\mathbb{R})$, the subset of $M_{2N}(\mathbb{R})$ made of the elements with determinant equal to one.*

We are now in a position to properly define the optimisation problems discussed in the present paper. For every positive $M$, the set $\mathcal{V}_M$ of Hill potentials is given by the measurable functions $V$ so that

$$\mathcal{V}_M = \{V : [0,1] \to M_N(\mathbb{R}) \mid \operatorname{ess\,sup}_{x\in[0,1]} \|V(x)\| \leq M^2\}, \tag{8}$$

and we say that a potential $V$ satisfies an $L^\infty$-constraint if it belongs to some $\mathcal{V}_M$.

**Remark 1.3** *Note that $\mathcal{V}_M$ is convex and invariant by transposition and conjugation by orthogonal matrices, i.e. $V_{U(\cdot)} = U^T(\cdot)V(\cdot)U(\cdot)$ belongs to $\mathcal{V}_M$ if and only $V$ does, for any measurable $SO(N)$-valued $U(\cdot)$ defined on* $[0,1]$. *One could have defined equivalently $\mathcal{V}_M$ with potentials $V : \mathbb{R} \to M_N(R)$ periodic of period* 1 *and satisfying the same $L^\infty$ bound. In that case, $\mathcal{V}_M$ is clearly invariant by translation of $x_0 \in \mathbb{R}$,* i.e. *$V_{x_0}(\cdot) = V(\cdot + x_0)$ belongs to $\mathcal{V}_M$ if and only $V$ does.*

**Remark 1.4** *For $q \in [1,\infty)$, one could replace the $L^\infty$ constraint by the integral condition*

$$\int_0^1 \|V(x)\|^q \, dx \leq M^{2q}$$

*which is referred to as an $L^q$-constraint.*

The cost function associated to a potential $V$ is from now on denoted $\mathcal{C}(V)$ and is given by

$$\mathcal{C}(V) = (-1)^N \det\big(\operatorname{Id}_{2N} - R(1)\big), \tag{9}$$

where $R$ is defined in (7). We will study the following optimisation questions: for every $M > 0$,

$$\mathbf{Max-Det(M)}: \qquad \max_{V\in\mathcal{V}_M} \mathcal{C}(V) \text{ along (7)}, \tag{10}$$

$$\mathbf{Min-Det(M)}: \qquad \min_{V\in\mathcal{V}_M} \mathcal{C}(V) \text{ along (7)}. \tag{11}$$

To derive common statements for both optimisation problems, we use $\mathcal{C}_\varepsilon$ to denote $\varepsilon\mathcal{C}$ where $\varepsilon = \pm 1$ and in that way **Max-Det** becomes the minimisation of $\mathcal{C}_-$ while **Min-Det** is simply the minimisation of $\mathcal{C}_+$. That is we study, for a given $M > 0$,

$$\mathbf{Ext-Det}_\varepsilon\mathbf{(M)}: \quad \min_{V\in\mathcal{V}_M} \mathcal{C}_\varepsilon(V) \text{ along (7)}. \tag{12}$$

We begin our analysis in Section 2 by stating the necessary condition satisfied by optimisers (existence is clear). The problem can be formulated as an optimal control problem over the set of matrices with a matrix valued control, so Pontryagin maximum principle provides the appropriate information. We also obtain some additional properties of optimisers. In Section 3 we focus on the one-dimensional case. We prove existence an uniqueness of maximisers and minimisers for the determinant over a bounded set in $L^\infty(\mathbb{S}^1)$.

# 2 Optimality conditions, invariance and symmetries

## 2.1 Pontryagin maximum principle

In this section, we will derive the equations verified by the minimisers of $\mathbf{Ext-Det}_\varepsilon$ as well as their first properties. From now on, $M$ is an arbitrary positive number and $\varepsilon \in \{-1, 1\}$. First of all, since $\mathcal{V}_M$ is non empty and, for any $R \in M_{2N}(\mathbb{R})$, the set $\{\mathcal{A}_V \mid V \in M_N(\mathbb{R}),\ \|V\| \le M^2\}$ is compact and convex, then $\mathbf{Ext-Det}_\varepsilon\mathbf{(M)}$ admits minimisers according to Filippov theorem, *cf.* [1]. The state space of the problem is $M_{2N}(\mathbb{R})$. According to the Pontryagin maximum principle (PMP) (see, *e.g.*, [1, Theorem 12.4]), a solution $R$ of $\mathbf{Ext-Det}_\varepsilon\mathbf{(M)}$ with minimising potential $V$ is necessarily the projection of an *extremal*, *i.e.*, an integral curve $\lambda = (R, P) \in M^2_{2N}(\mathbb{R})$ of a Hamiltonian vector field on $M_{2N}(\mathbb{R})$ satisfying certain additional conditions. We hereby present a definition of extremal adapted to our setting. The fact that this is equivalent to the standard definition of normal extremal is the subject of Proposition 2.1 given below.

**Definition 2.1** *A curve $\lambda : [0, T] \to M_{2N}(\mathbb{R})^2$ is called* **extremal** *with respect to the control $V \in \mathcal{V}_M$ if:*

*(i) letting $\lambda = (R, P)$, it satisfies*

$$\dot{R}(x) = \mathcal{A}_{V(x)}R(x), \tag{13}$$

$$\dot{P}(x) = -\mathcal{A}^T_{V(x)}P(x). \tag{14}$$

*(ii) It holds that $R(0) = \mathrm{Id}_{2N}$ and the following transversality condition holds true*

$$P(1) = (-1)^N\varepsilon\ \mathrm{Com}\big(\mathrm{Id}_{2N} - R(1)\big). \tag{15}$$

*(iii) Assume moreover that there exists $H_0 \in \mathbb{R}$ such that*

$$H_0 = H\big(R(x), P(x), V(x)\big) = \max_{\|W\|\le M^2} H\big(R(x), P(x), W\big), \text{ for a.e. } x \in [0, 1], \tag{16}$$

*where $H$ is the Hamiltonian function defined on $M_{2N}(\mathbb{R})^2 \times M_N(\mathbb{R})$ by*

$$H\big(R, P, W\big) = \langle P, \mathcal{A}_W R\rangle = \langle \mathcal{A}^T_W P, R\rangle. \tag{17}$$

*such an extrermal is called* **strong extremal**.

**Remark 2.1** *Note that every potential $V$ admits a unique extremal (which is possibly strong).*

We then get the following.

**Proposition 2.1** *Let $R : [0,T] \to \mathcal{M}$ be an optimal trajectory of $\mathbf{Ext-Det}_\varepsilon(\mathbf{M})$ with minimising potential $V$. Then $R$ is the projection on $M_{2N}(\mathbb{R})$ of a unique strong extremal $\lambda = (R,P) : [0,T] \to M^2_{2N}(\mathbb{R})$.*

*Proof.* Let $V$ be a minimising potential of $\mathbf{Ext-Det}_\varepsilon(\mathbf{M})$ and $R$ the associated trajectory by (7). Pontryagin maximum principle implies that there exists a nontrivial pair $(p^0, P)$ where the cost multiplier $p^0$ is a nonpositive real number and the covector $P : [0,1] \to M_{2N}(\mathbb{R})$ is a Lipschitz function so that

1. $(R(x), P(x)) \in M_{2N}(\mathbb{R})^2$ satisfy on $[0,1]$ the adjoint equations:

$$\dot{R} = \frac{\partial H}{\partial P}, \tag{18}$$

$$\dot{P} = -\frac{\partial H}{\partial R}; \tag{19}$$

2. we have the maximality condition given by (16);

3. the following transversality condition holds true: $P(1) = p^0 \nabla \mathcal{C}_\varepsilon(V)$.

In addition, since $H$ does not depend on time, its value in (16) does not depend on time and is denoted by the constant real number $H_0$. As

$$\frac{\partial H}{\partial P} = \mathcal{A}_W R, \quad \frac{\partial H}{\partial R} = \mathcal{A}_W^T P, \quad \nabla_R \det(\mathrm{Id}_{2N} - R) = -\mathrm{Com}(\mathrm{Id}_{2N} - R),$$

the items of Proposition 2.1 follow at once, except the facts that $p^0$ can be taken equal to $-1$ and $\lambda$ is unique. To establish the first fact, it is enough to show that $p^0$ cannot be null. To show that, we argue by contradiction and, in that case, it follows that $P(1) = 0$. Since (21) is linear in $P$, one gets that $P$ is identically equal to zero on $[0,1]$. This contradicts the non triviality of the pair $(p^0, P)$ and hence $p^0 \neq 0$. Regarding the uniqueness of $\lambda$, note first that, given $M > 0$, trajectories of (7) are in one to one correspondence with potentials in $\mathcal{V}_M$, since to each such trajectory, there is a unique potential $V \in \mathcal{V}_M$ necessarily defined as the lower left $N \times N$ block of $\mathcal{A}_V = \dot{R}R^{-1}$ (recall that $R$ is absolutely continuous). By Item 3., $P(1)$ is determined by $R(1)$ and hence $P$ is computed from (19). The proof of Proposition 2.1 is complete.

□

To take advantage of the maximisation condition (16), after defining $q = PR^T$, we rewrite Proposition 2.1 only using $q$ and we deduce at once that

**Proposition 2.2** *Assume that a trajectory $R$ of $\mathbf{Ext-Det}_\varepsilon(\mathbf{M})$ with potential $V$ is the projection of a weak extremal trajectory $\lambda = (R,P)$. Define*

$$q = PR^T = \begin{pmatrix} Z_1 & \psi \\ \varphi & Z_2 \end{pmatrix}, \tag{20}$$

*where the various blocs are $N \times N$ matrices. Then, the dynamics of $q$ is given by*

$$\dot{q}(x) = \left[q(x), \mathcal{A}^T_{V(x)}\right], \text{ for a.e. } x \in [0,1], \quad q(1) = (-1)^N \varepsilon \, \mathrm{Com}\big(\mathrm{Id}_{2N} - R(1)\big) R^T(1), \tag{21}$$

*which yields, for a.e.* $x \in [0,1]$,

$$\begin{aligned} \dot{Z}_1 &= \psi - V^T\varphi, && (22)\\ \dot{\varphi} &= Z_2 - Z_1, && (23)\\ \dot{\psi} &= Z_1 V^T - V^T Z_2, && (24)\\ \dot{Z}_2 &= \varphi V^T - \psi. && (25) \end{aligned}$$

*The Hamiltonian function* $H$ *defined in* (17) *is equal to*

$$H\big(R,P,W\big) = \langle q, \mathcal{A}_W^T\rangle = \mathrm{Tr}(\psi) + \langle \varphi, W\rangle. \tag{26}$$

*Moreover, it holds*

$$\begin{aligned} q^T(x) &= R(x)q^T(1)R^{-1}(x), \text{ for every } x \in [0,1], && (27)\\ \ddot{\varphi} &= -2\psi + V^T\varphi + \varphi V^T \text{ for a.e. } x \in [0,1], && (28) \end{aligned}$$

*and in particular* $q(\cdot)$ *is periodic of period one.*

*Assume moreover* $\lambda = (R,P)$ *is a strong extremal. If* $\varphi(x) \neq 0$, *then* $V(x) = M^2 \frac{\varphi(x)}{\|\varphi(x)\|}$ *and, for every* $x \in [0,1]$, *it holds*

$$\begin{aligned} H_0 &= \mathrm{Tr}(\psi) + M^2\|\varphi(x)\|, && (29)\\ \mathrm{Tr}(\ddot{\varphi}) &= -2H_0 + 4M^2\|\varphi(x)\|. && (30) \end{aligned}$$

*Proof.* Most the above is immediate except (27). The latter follows from the fact that, for every $x \in [0,1]$,

$$q^T(x) = R(x)R^{-1}(1)q^T(1)R(1)R^{-1}(x).$$

The above equation then yields (27) after noticing that $R(1)$ and $q^T(1)$ commute. □

**Convention 2.1** *From now on, we call extremal the couple* $(R,q)$ *instead of* $(R,P)$.

**Remark 2.2** *In the light of Item (iii) of the above proposition, one can see that the potential* $V$ *is not (immediately) defined at a zero of* $\varphi$. *In the sequel, the latter function* $\varphi$ *is refereed to as the* **switching function** *and we single out a particular instance of zero of* $\varphi$, *namely that of* **switching time** *defining such a point* $x_* \in (0,1)$ *for which* $\varphi(x_*) = 0$ *and there exist two sequences* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *of two by two distinct points, both converging to* $x_*$ *such that* $\langle \varphi(x_n), \varphi(y_n)\rangle < 0$ *for* $n \in \mathbb{N}$. *Clearly, a zero of* $\varphi$ *in* $(0,1)$ *which is not a zero of* $\dot{\varphi}$ *is a switching time.*

**Remark 2.3** *Thanks to Remark 1.2, one could have taken* $SL_{2N}(\mathbb{R})$ *as state space of the problem and written the PMP in accordance. However, as explained next, we do not get any new information by proceeding in that way. To see that, we first of all recall some basic facts. We use* $sl_{2N}(\mathbb{R})$ *to denote the subset of* $M_{2N}(\mathbb{R})$ *made of the elements with trace equal to zero. Moreover* $SL_{2N}(\mathbb{R})$ *admits a structure of Lie group with* $sl_{2N}(\mathbb{R})$ *as Lie algebra. At every* $R \in SL_{2N}(\mathbb{R})$, *we represent* $T_RSL_{2N}(\mathbb{R})$, *the tangent space to* $SL_{2N}(\mathbb{R})$ *at* $R$, *as the following subset of* $M_{2N}(\mathbb{R})$

$$T_RSL_{2N}(\mathbb{R}) = \{rR \mid r \in M_{2N}(\mathbb{R}) \text{ such that } \mathrm{Tr}(r) = 0\}. \tag{31}$$

*Using now the inner product introduced in* (1), *one can identify* $T_R^*SL_{2N}(\mathbb{R})$, *the cotangent space to* $SL_{2N}(\mathbb{R})$ *at* $R$, *as the following subset of* $M_{2N}(\mathbb{R})$

$$T_R^*SL_{2N}(\mathbb{R}) = \{q(R^{-1})^T \mid q \in M_{2N}(\mathbb{R}) \text{ such that } \mathrm{Tr}(q) = 0\}. \tag{32}$$

*We next notice that the flow associated with* (21) *is isospectral (*cf. *for instance [?]), in particular the trace of $q$ is constant on* $[0,1]$ *equal to* $\mathrm{Tr}(q(1))$. *Define now*

$$\tilde{q}(x) = q(x) - \frac{\mathrm{Tr}(q(1)}{2N}\,\mathrm{Id}_{2N}, \quad \tilde{P}(x) = \tilde{q}(x)\big(R^T(x)\big)^{-1}, \text{ for } x \in [0,1].$$

*Clearly the curve $\tilde{\lambda} = (R, \tilde{P})$ takes values in $T^*SL_{2N}(\mathbb{R})$ and is an integral curve of the Hamiltonian vector field $\vec{H}$ on $T^*SL_{2N}(\mathbb{R})$ associated with $H$. (See for instance [1] for the Hamiltonian formalism.) Finally, when applying the PMP to $R$ within $T^*SL_{2N}(\mathbb{R})$, we claim that $\tilde{\lambda}$ turns out to be the required extremal with $R$ as projection onto $SL_{2N}(\mathbb{R})$: the dynamics of $\tilde{\lambda}$ has been described just previously,* i.e., $\dot{\tilde{\lambda}} = \vec{H}(\tilde{\lambda})$, *the maximality condition is exactly* (16) *and the tranversality condition* (15) *now says that $P(1) - p^0\nabla_R\mathcal{C}_\varepsilon(V)$ belongs to the normal cone at $T^*_{R(1)}SL_{2N}(\mathbb{R})$, where the gradient is projected on $T^*_{R(1)}SL_{2N}(\mathbb{R})$. Since that normal cone is equal to $\mathbb{R}(R^T(1))^{-1}$ and since one easily shows that $p^0 = -1$, one gets the claim regarding $\tilde{\lambda}$. It is therefore completely equivalent to use $\lambda$ or $\tilde{\lambda}$, meaning that nothing new is gained by writing the PMP within $T^*SL_{2N}(\mathbb{R})$ instead of $M_{2N}(R)$.*

We close this section by providing the following property regarding translated potentials.

**Lemma 2.1** *Let $R$ be a trajectory of* $\mathbf{Ext-Det}_\varepsilon(\mathbf{M})$ *associated with potential $V$ and corresponding extremal $(R,q)$. For $x_0 \in \mathbb{R}$, consider the potential $V_{x_0}$ translated from $V$ according to Remark 1.3. Then $V_{x_0}$ has same cost as $V$ with corresponding extremal $(R_{x_0}, q_{x_0})$ and one gets that*

$$q_{x_0}(x) = q(x+x_0), \quad \varphi_{x_0}(x) = \varphi(x+x_0), \quad \forall x \in \mathbb{R}. \tag{33}$$

*where $\varphi$ ($\varphi_{x_0}$, respectively) denotes the switching function associated with $V$ ($V_{x_0}$, respectively).*

*In case* $\mathrm{Id}_{2N} - R(1)$ *is invertible, then $q(\cdot)$ is*

*Proof.* It is immediate to derive that the trajectory $R_{x_0}$ of (7) associated with $V_{x_0}$ is given by

$$R_{x_0}(x) = R(x+x_0)R(x_0)^{-1}, \quad \forall x \in \mathbb{R}, \tag{34}$$

and, by periodicity of $V$, it follows that

$$R_{x_0}(1) = R(x_0)R(1)R(x_0)^{-1}. \tag{35}$$

Using the above equation, one gets that

$$\mathcal{C}_\varepsilon(V_{x_0}) = (-1)^N\varepsilon\det(\mathrm{Id}_{2N} - R_{x_0}(1)) = \mathcal{C}_\varepsilon(V),$$

and hence has same cost as $V$. Let $\lambda_{x_0} = (R_{x_0}, P_{x_0})$ be the unique extremal associated with $R_{x_0}$. Then, from (27), it holds

$$q^T_{x_0}(x) = R_{x_0}(x)\big(R_{x_0}(1)\big)^{-1}q^T_{x_0}(1)R_{x_0}(1)\big(R_{x_0}(x)\big)^{-1}, \quad \forall x \in [0,1],$$

and, from (21), one has

$$\begin{aligned} q_{x_0}(1) &= (-1)^N\varepsilon\ \mathrm{Com}\big(\mathrm{Id}_{2N} - R_{x_0}(1)\big)R^T_{x_0}(1) \\ &= (-1)^N\varepsilon\ \mathrm{Com}\Big(\mathrm{Id}_{2N} - R(x_0)R(1)R(x_0)^{-1}\Big)R^T_{x_0}(1) \\ &= (-1)^N\varepsilon\ \big(R(x_0)^T\big)^{-1}\mathrm{Com}\big(\mathrm{Id}_{2N} - R(1)\big)R(x_0)^TR(x_0)^{-1} \\ &= \big(R(x_0)^T\big)^{-1}q(1)R(x_0)^T. \end{aligned}$$

Using the above equation, (34) and (35), one gets (33). □

## 2.2 First properties and related optimal control problems

We first prove that there always exists potentials $V$ with negative costs, implying that minimal values for $\mathbf{Ext-Det}_\varepsilon(\mathbf{M})$ are always negative, which in particular, exclude the zero potential from optimality.

**Lemma 2.2** *The cost $\mathcal{C}_\varepsilon(0)$ associated with the zero potential is equal to zero. For every $N\times N$ diagonal matrix $D=\mathrm{diag}(\varepsilon_1 d_1^2,\cdots,\varepsilon_N d_N^2)$, where $\varepsilon_i^2=1$ and $d_i>0$ for $1\le i\le N$, let $\mathcal{C}_\varepsilon(D)$ be the cost associated with the constant potential equal to $D$. Then*

$$\mathcal{C}_\varepsilon(D)=(-2)^N\varepsilon\Pi_{i=1}^N\big(1-c_{\varepsilon_i}(d_i)\big). \tag{36}$$

*Moreover, for every $M>0$, $D\in\mathcal{V}_M$ if $\sum_{i=1}^N d_i^2\le M^2$ and then $\mathcal{C}_\varepsilon(D)<0$ if one chooses $\varepsilon_1\varepsilon=-1$, $\varepsilon_i=1$ for $2\le i\le N$ and $d_1$ not a multiple of $2\pi$ if $\varepsilon_1=-1$.*

***Proof:*** One clearly has that the trajectory $R_0$ of (7) associated with the zero potential is equal to

$$R_0(x)=\begin{pmatrix}\mathrm{Id}_N & x\,\mathrm{Id}_N\\ 0 & \mathrm{Id}_N\end{pmatrix}\ \text{for } x\in[0,1].$$

The conclusion follows at once.

Using (4), one easily deduces the value resolvent matrix $R_D$ associated with $D$ at $x=1$,

$$R_D(1)=\begin{pmatrix}\mathrm{diag}(c_{\varepsilon_1}(d_1),\cdots,c_{\varepsilon_N}(d_N)) & \mathrm{diag}(\frac{s_{\varepsilon_1}(d_1)}{d_1},\cdots,\frac{s_{\varepsilon_N}(d_N)}{d_N})\\ \mathrm{diag}(\varepsilon_1 d_1 s_{\varepsilon_1}(d_1),\cdots,\varepsilon_N d_N s_{\varepsilon_N}(d_N)) & \mathrm{diag}(c_{\varepsilon_1}(d_1),\cdots,c_{\varepsilon_N}(d_N))\end{pmatrix}. \tag{37}$$

An elementary computation yields (36) and the rest of the lemma follows.

■

We now derive basic informations regarding an optimal trajectory,

**Lemma 2.3** *Assume that $R$ is an optimal trajectory associated with a minimizing cost $V$ and let $H_0$ be the constant value of the Hamiltonian defined in (16). Then the following facts hold true.*

(*a*) *The cost $\mathcal{C}_\varepsilon(V)$ is negative and hence $\mathrm{Id}_{2N}-R(1)$ is invertible. Moreover, the switching function $\varphi$ is of class $C^2$, the matrix $q=RP^T$ defined in (20) is periodic of period one, it holds that*

$$q^T(1)=\mathcal{C}_\varepsilon(V)\big(\mathrm{Id}_{2N}-R(1)\big)^{-1}R(1)\ \text{and}\ q^T(x)=R(x)q^T(1)R^{-1}(x)\ \text{for every } x\in[0,1], \tag{38}$$

*and the following relation holds true*

$$H_0=2M^2\int_0^1\|\varphi(x)\|\,dx. \tag{39}$$

(*b*) *If $H_0=0$ then there exists an invertible $Z_*\in M_N(\mathbb{R})$ such that*

$$q\equiv\begin{pmatrix}Z_* & 0\\ 0 & Z_*\end{pmatrix}, \tag{40}$$

*and $(-1)^N\varepsilon$ is negative.*

(*c*) *If $H_0>0$, then $\varphi$ has a finite number of zeroes in $[0,1]$ at which either $\dot\varphi$ does not vanish or $\ddot\varphi$ is well defined and does not vanish.*

***Proof:*** From (28) and the expression of $V$ at points where $\varphi$ does not vanish, one deduces that $\varphi$ is of class $C^2$ on $[0,1]$. The one periodicity of $q$ is an immediate consequence of Lemma 2.2. In that case, one can simplify (21) and (27) to get (38). The latter equation implies that $R(1)$ and $q^T(1)$ commute, which implies by using (38) that $q(0) = q(1)$. Since $q$ is solution of a Cauchy problem (the ODE $\dot{q} = \left[q, \mathcal{A}_V^T\right]$ together with an initial condition), it follows that $q$ is periodic of period one. Finally, integrating (30) between $x = 0$ and $x = 1$ and using the periodicity of $\mathrm{Tr}(\dot{\varphi})$, one gets (39).

Assume $H_0 = 0$. From (39), it follows that $\varphi \equiv 0$ and then (28) implies that $\psi \equiv 0$ as well. The rest of the dynamics of $q$ clearly yields that $q$ is constant on $[0,1]$, verifying (40). By using the latter fact after taking the determinant in (38) it follows that

$$(\det Z_*)^2 = (-1)^N \varepsilon \left[\det\left(\mathrm{Id}_{2N} - R(1)\right)\right]^{2N-1} = \mathcal{C}_\varepsilon(V)^{2N-1},$$

and the last part of Item $(b)$ follows.

We next provide an argument for Item $(c)$. Arguing by contradiction, it would follow that there exists a sequence $(x_k)_{k\in\mathbb{N}}$ of two by two distinct times in $[0,1]$ so that $\lim_{k\to\infty} x_k = \bar{x}$ and $\varphi(x_k) = 0$ for $k \geq 0$. Since $\varphi$ is of class $C^1$, one has that $\varphi(\bar{x}) = 0$ by continuity of $\varphi$ and then

$$0 = \lim_{k\to\infty} \frac{\varphi(x_k) - \varphi(\bar{x})}{x_k - \bar{x}} = \dot{\varphi}(\bar{x}).$$

Since $V$ is bounded, one deduces from (28) that $\ddot{\varphi}$ is twice differentiable at $x = \bar{x}$. Moreover, $\ddot{\varphi}(\bar{x})$ is not zero since, from (**??**), it holds

$$\mathrm{Tr}(\ddot{\varphi}(\bar{x})) = -2H_0 < 0.$$

By a Taylor expansion at order two, one obtains that there exists an open interval $I$ centered at $\bar{x}$ so that $\varphi(x) = 0$ for $x \in I$ only if $x = \bar{x}$. That contradicts the existence of the sequence $(x_k)_{k\in\mathbb{N}}$.

■

We end the section by providing preliminary symmetry properties for a minimising potential. For that purpose we define the following matrices of $M_{2N}(\mathbb{R})$.

$$\begin{aligned} J &= \mathcal{A}_{\mathrm{Id}_{2N}} \text{ i.e. } J = \begin{pmatrix} 0 & \mathrm{Id}_N \\ \mathrm{Id}_N & 0 \end{pmatrix}, \quad A = \mathcal{A}_{-\mathrm{Id}_{2N}} \text{ i.e. } A = \begin{pmatrix} 0 & \mathrm{Id}_N \\ -\mathrm{Id}_N & 0 \end{pmatrix}, \\ \mathcal{U} &= \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}, \text{ for every } U \in SO(N), \quad \mathcal{B}_Q = \mathcal{A}_{Q^T}^T, \text{ for every } Q \in M_N(\mathbb{R}). \end{aligned}$$

Note that $J^2 = A^T A = \mathrm{Id}_{2N}$.

**Proposition 2.3** *Let $M > 0$, $V \in \mathcal{V}_M$ and $R$ the trajectory of* (7) *associated with $V$. The following items are equivalent.*

(1.) *$V$ is a minimising potential for $\mathbf{Ext} - \mathbf{Det}_\varepsilon(M)$ along* (7)*;*

(2.) *for every $U \in SO(N)$, $V_{\mathcal{U}} = \mathcal{U}^T V \mathcal{U}$ is a minimising potential for $\mathbf{Ext} - \mathbf{Det}_\varepsilon(M)$ along* (7) *with $\mathcal{U}^T R \mathcal{U}$ as associated trajectory;*

(3.) *$V$ is a minimising potential for $\mathbf{Ext} - \mathbf{Det}_\varepsilon(M)$ along trajectories of each of the following four dynamical systems*

$$\begin{cases} \dot{S}(x) = \mathcal{B}_{V(x)} S(x), \\ S(0) = \mathrm{Id}_N, \end{cases} \quad \begin{cases} \dot{S}(x) = -\mathcal{B}_{V(x)} S(x), \\ S(0) = \mathrm{Id}_N, \end{cases}$$

$$\begin{cases} \dot{S}(x) = S(x) \mathcal{B}_{V^T(x)}, \\ S(0) = \mathrm{Id}_N, \end{cases} \quad \begin{cases} \dot{S}(x) = -S(x) \mathcal{A}_{V(x)}, \\ S(0) = \mathrm{Id}_N, \end{cases}$$

*with $JRJ$, $A^T R A$, $R^T$ and $R^{-1}$ as associated optimal trajectories respectively and same value of the cost;*

(4.) *$V^T$ is a minimising potential for* $\mathbf{Ext}-\mathbf{Det}_\varepsilon(M)$ *along* (7) *with* $A^T(R^T)^{-1}A$ *as associated trajectory.*

*Proof.* Showing the several items is immediate once one notices that

$$J\mathcal{A}_Q J = \mathcal{B}_Q, \quad A^T\mathcal{A}_Q A = -\mathcal{B}_Q, \text{ for every } Q \in M_N(\mathbb{R}).$$

As for the equality of the costs, we just check the following

$$\det(\mathrm{Id}_{2N} - R^{-1}(1)) = \det\Big((R(1)-\mathrm{Id}_{2N})R^{-1}(1)\Big) = \det(\mathrm{Id}_{2N} - R(1)).$$

□

Idea for showing symmetry of minimising potentials.

For $\lambda \in [0,1]$ and $V \in \mathcal{V}_M$, set $V_\lambda = (1-\lambda)V + \lambda V^T$. We now define the optimal control problem $*\mathbf{Ext}*-\mathbf{Det}_\varepsilon(M)$ as

$$\text{for every } M>0, \quad *\mathbf{Ext}-\mathbf{Det}*_\varepsilon(\mathbf{M}): \quad \min_{V\in\mathcal{V}_M,\ \lambda\in[0,1]} \mathcal{C}_\varepsilon(V), \text{ along } \left\{\begin{array}{lcl} \dot{R}(x) & = & \mathcal{A}_{V_\lambda(x)}R(x), \\ R(0) & = & \mathrm{Id}_{2N}, \\ \dot{\lambda} & = & 0. \end{array}\right.$$

The new Hamiltonian function is now given by

$$H\big(R,P,\lambda,\mu,W\big) = \langle P, \mathcal{A}_{W_\lambda}R\rangle = \langle \mathcal{A}^T_{W_\lambda}P, R\rangle.$$

Setting as before $q = PR^T$ with (20), one also has

$$H\big(R,P,\lambda,\mu,W\big) = \langle q, \mathcal{A}_{W_\lambda}\rangle = \langle \varphi, W_\lambda\rangle = \langle \varphi, W\rangle + \lambda\langle \varphi, W^T - W\rangle.$$

Let $(V,\lambda_*)$ be a minimiser of $*\mathbf{Ext}*-\mathbf{Det}_\varepsilon(M)$. Then both $V$ and $V^T$ are minimisers of $\mathbf{Ext}-\mathbf{Det}_\varepsilon(M)$ and then necessarily $\langle \varphi, W^T - W\rangle \equiv 0$ along an extremal associated to $(V,\lambda_*)$ since otherwise $V$ or $V^T$ is excluded from the minimisation. I hope to deduce something. However, I am not sure that the reasoning is correct. JB???????

## 3 One-dimensional case

From now on $N=1$; $M$ is a positive number and

$$\mathcal{V}_M = \{V : [0,1] \to \mathbb{R} \mid V \text{ measurable and ess sup}_{x\in[0,1]}|V(x)| \le M^2\}. \tag{41}$$

From Item (*iii*) of Proposition 2.2, it holds that $V(x) \in \{-M^2, M^2\}$ as soon as $\varphi(x) \neq 0$ and this motivates the following definition.

**Definition 3.1** *Let $R$ be a trajectory of* (7) *associated to some $V \in \mathcal{V}_M$. A* **bang arc** *$\gamma : I \to M_2(\mathbb{R})$ is a piece of $R$ defined on some non empty subinterval $I \subset [0,1]$ such that $V = \nu M^2$ is constant on $I$, with $\nu \in \{-1,1\}$. A trajectory $R$ of* (7) *is said to be* **bang** *if it is made of a unique bang arc and* **bang-bang** *if it is the concatenation of bang arcs.*

We first examine the **Max-Det** problem. In dimension $N=1$, the cost to maximise is

$$\begin{array}{rcl} \mathcal{C}_V & = & -\det(I_2 - R(1)), \\ & = & -(1 - \mathrm{Tr}R(1) + \det R(1)), \\ & = & \mathrm{Tr}R(1) - 2 \end{array}$$

since the monodromy $R(1)$ belongs to $SL_2(\mathbb{R})$. Maximising $\mathcal{C}_V$ is so equivalent to maximising the trace of the monodromy

$$\mathrm{Tr}R(1) = z(1) + y'(1),$$

where $z$ and $y$ satisfy $-w'' + V(x)w = 0$ with respective initial conditions $(z(0), z'(0)) = (1,0)$ and $(y(0), y'(0)) = (0,1)$.

**Proposition 3.1** *Let $V_1$ and $V_2$ be two potentials in $L^1_{\rm loc}(\mathbb{R}_+)$, $V_1 \geq |V_2|$ a.e., and let $y_1$ and $y_2$ satisfy $-y_i'' + V_i(x)y_i = 0$, $i = 1, 2$. If $y_1(0) \geq |y_2(0)|$ and $y_1'(0) \geq |y_2'(0)|$, then $y_1(x) \geq |y_2(x)|$ and $y_1'(x) \geq |y_2'(x)|$ for all $x \geq 0$.*

*Proof.* (i) First assume $V_1$ and $V_2$ constant, $V_1 \equiv A$ and $V_2 \equiv B$ with $A$ and $B$ two reals such that $A \geq |B|$. One has

$$y_1(x) = y_1(0)\cosh(\alpha x) + xy_1'(0)\,\mathrm{sinhc}(\alpha x)$$

where $\alpha = \sqrt{A}$, and where we denote $\mathrm{sinhc}(x) = \sinh(x)/x$ if $x \neq 0$, $\mathrm{sinhc}(0) = 1$. If $B$ is nonnegative, let $\beta := \sqrt{B} \leq \alpha$; one has

$$\begin{aligned} |y_2(x)| &= |y_2(0)\cosh(\beta x) + xy_2'(0)\,\mathrm{sinhc}(\beta x)| \\ &\leq |y_2(0)|\cosh(\beta x) + x|y_2'(0)|\,\mathrm{sinhc}(\beta x) \\ &\leq y_1(0)\cosh(\alpha x) + xy_1'(0)\,\mathrm{sinhc}(\alpha x) = y_1(x) \end{aligned}$$

for $x \geq 0$ since both cosh and sinhc are nondecreasing functions on $\mathbb{R}_+$ (and $\beta \leq \alpha$). Similarly, for $x \geq 0$,

$$\begin{aligned} |y_2'(x)| &= |\beta y_2(0)\sinh(\beta x) + y_2'(0)\cosh(\beta x)| \\ &\leq \alpha y_1(0)\sinh(\alpha x) + y_1'(0)\cosh(\alpha x) = y_1'(x). \end{aligned}$$

If $B$ is negative, let $\beta := \sqrt{-B} \leq \alpha$; one has (denoting $\mathrm{sinc}(x) = \sin(x)/x$ if $x \neq 0$, $\mathrm{sinc}(0) = 1$)

$$\begin{aligned} |y_2(x)| &= |y_2(0)\cos(\beta x) + xy_2'(0)\,\mathrm{sinc}(\beta x)| \\ &\leq |y_2(0)|\cosh(\beta x) + x|y_2'(0)|\,\mathrm{sinhc}(\beta x) \\ &\leq y_1(x) \end{aligned}$$

for $x \geq 0$ since $|\cos| \leq \cosh$ and $|\,\mathrm{sinc}\,| \leq \mathrm{sinhc}$ on $\mathbb{R}_+$. Similarly, for $x \geq 0$,

$$\begin{aligned} |y_2'(x)| &= |-\beta y_2(0)\sin(\beta x) + y_2'(0)\cos(\beta x)| \\ &\leq \alpha y_1(0)\sinh(\alpha x) + y_1'(0)\cosh(\alpha x). \end{aligned}$$

(ii) Take now some positive $x$, and assume $V_1$ and $V_2$ are piecewise constant on $[0, x]$; there exists a common subdivision $0 = x_0 < x_1 < ... < x_N = x$, $N \geq 1$, such that on every $[x_i, x_{i+1}[$ both $V_1$ and $V_2$ are constant, with $V_1 \geq |V_2|$. A simple recurrence using step (i) allows to conclude that $y_1(x) \geq |y_2(x)|$ and $y_1'(x) \geq |y_2'(x)|$.

(iii) Consider eventually $V_1$ and $V_2$ locally integrable on $\mathbb{R}_+$, and fix $x > 0$. There exist two sequences $(V_{1,n})_n$, $(V_{2,n})_n$ of piecewise constant functions converging in $L^1(0, x)$ to $V_1$ and $V_2$, respectively. These sequences can be chosen such that $V_{1,n} \geq |V_{2,n}|$, $n \in \mathbb{N}$. Then according to point (ii), for all $n \in \mathbb{N}$, $y_{1,n}(x) \geq |y_{2,n}(x)|$ and $y_{1,n}'(x) \geq |y_{2,n}'(x)|$, where $y_{i,n}$ denotes the solution associated with $V_{i,n}$ and fixed initial conditions $(y_i(0), y_i'(0))$, $i = 1, 2$. Since, for any given initial condition $(y_0, y_0')$, the mapping $V \mapsto (y(x), y'(x))$ (where $y$ is the solution of $-y'' + Vy = 0$, $y(0) = y_0$, $y'(0) = y_0'$) is continuous from $L^1(0, x)$ to $\mathbb{R}^2$ (see, *e.g.*, Proposition 7 in [2]), passing to the limit one obtains that $y_1(x) \geq |y_2(x)|$ and $y_1'(x) \geq |y_2'(x)|$. As $x$ is arbitrary, the desired conclusion holds. □

**Corollary 3.1** *For $V$ in $L^\infty(0, 1)$, let $y$ and $z$ denote the solutions of*

$$-y'' + V(x)y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

$$-z'' + V(x)z = 0, \quad z(0) = 1, \quad z'(0) = 0.$$

*Then, for any positive bound $M$, the constant potential $V \equiv M^2$ is the unique function maximising both $y(1)$, $y'(1)$, $z(1)$ and $z'(1)$ over essentially bounded potentials such that $\|V\|_\infty \leq M^2$.*

**Theorem 3.1** *The unique solution of the* **Max-Det(M)** *problem in the periodic case is the constant potential equal to* $M^2$.

*Proof.* It is clear from the previous corollary that the constant potential $V \equiv M^2$ maximises $z(1) + y'(1)$ among essentially bounded potentials such that $\|V\|_\infty \leq M^2$. Let $V$ be a measurable function satisfying the same bound and such that $V$ is strictly inferior to $M^2$ on a positive measure subset of $[0, 1]$; a direct estimation allows to prove that the associated values of both $z(1)$ and $y'(1)$ (hence of their sum) are strictly smaller than the values obtained for the constant potential $V \equiv M^2$. □

We eventually handle **Min-Det**. In particular, we immediately derive the following result after Lemmas 2.3 and 2.2.

**Lemma 3.1** *Assume that $R$ is an optimal trajectory associated with a potential $V$ minimising $\mathcal{C}_1$. Then the following cases may occur.*

*(i) If $H_0 = 0$, then $V$ is equal to the constant potential $V_0 \equiv -M^2$ and $\varphi$ never vanishes on on $[0, 1]$. In that case, the minimal cost is equal to $\mathcal{C}_1(V_0) = -2\big(1 - c_-(M)\big)$;*

*(ii) if $H_0 \neq 0$, then $\varphi$ has a finite number of zeros in $[0, 1]$ and $V(x) = M^2\mathrm{sgn}(\varphi(x))$ outside a finite set made of the zeros of $\varphi$.*

*Hence, either $R$ is the bang trajectory $R_0$ associated with $V_0$ or it is a bang-bang trajectory with a finite number of bang arcs.*

*Proof.* From Lemma 2.2, we know that the minimal value of $\mathcal{C}_1$ is negative, and then, Item $(a)$ of Lemma 2.3 only leaves the possibility of $\varphi$ never vanishing on $[0, 1]$. Hence $V$ is constant equal $M$ or $-M$. Since $C_1(M) > 0$, Item $(i)$ follows at once. Item $(ii)$ is essentially a rewriting of Item $(b)$ of Lemma 2.3 together with Item $(iii)$ of Proposition 2.2. □

We focus now on strong extremals associated with $H_0 \neq 0$. We have the following two intermediate results.

**Lemma 3.2** *Let $(R, q)$ be a strong extremal projecting on an optimal trajectory $R$ which is associated with a potential $V$ minimising $\mathcal{C}_1$ with corresponding $H_0 \neq 0$. Assume furthermore that*

1. *$V$ is not identically equal to $-M^2$;*

2. *$x_0 < x_1$ are two consecutive zeros of $\varphi$ in $[0, 1]$, i.e., $|\varphi| > 0$ on $(x_0, x_1)$.*

*Set $T := x_1 - x_0 > 0$ and $\nu = \mathrm{sgn}(\varphi)$ on $(x_0, x_1)$. Then both $c_\nu(MT)$ and $s_\nu(MT)$ are non zero and the following holds:*

$$\varphi(x) = \frac{H_0}{M^2 c_\nu(MT)} s_\nu(M(x - x_0)) s_\nu(M(x_1 - x)), \text{ for } x \in [x_0, x_1]. \tag{42}$$

*In particular,*

$$\dot{\varphi}(x_0) = -\dot{\varphi}(x_1) = H_0 \frac{s_\nu(MT)}{c_\nu(MT)} \neq 0. \tag{43}$$

*Proof.* In the case $N = 1$ and using the notations of the lemma, one can rewrite (**??**) as

$$\ddot{\varphi} = 4\nu M^2 (\varphi - \frac{\nu H_0}{2M^2}) \text{ for } x \in [x_0, x_1]. \tag{44}$$

Integrating (44) yields that

$$\varphi(x) = \frac{\nu H_0}{2M^2}\big(1 - c_\nu(2M(x-x_0))\big) + Bs_\nu(2M(x-x_0)), \tag{45}$$

$$\dot\varphi(x) = 2M^2Bc_\nu(2M(x-x_0)) - H_0 s_\nu(2M(x-x_0)), \tag{46}$$

where $B$ is a constant satisfying

$$-\frac{\nu H_0}{2M^2}(1 - c_\nu(2MT)) = Bs_\nu(2MT). \tag{47}$$

From (46), one deduces that

$$\dot\varphi(x_0) = 2M^2B, \quad \dot\varphi(x_1) = 2M^2Bc_\nu(2MT) - H_0 s_\nu(2MT). \tag{48}$$

We prove next that $s_\nu(MT) \neq 0$. Arguing by contradiction, it would first imply that $\nu = -1$ and then $V = -M^2$, $c_\nu(2MT) = 1$, $s_\nu(2MT) = 0$ and, from (48), that $\dot\varphi(x_0) = \dot\varphi(x_1) = 2M^2B$. If $B \neq 0$, then $\mathrm{sgn}(B)\dot\varphi$ is positive in a right neighborhood of $x_0$ while it is negative in a left neighborhood of $x_1$, implying that $\varphi$ must vanish inside $(x_0, x_1)$. This contradicts Item 2., and therefore one deduces that $B = 0$ and then $\ddot\varphi(x_0) = \ddot\varphi(x_1) = -2H_0$, yielding that $H_0 > 0$ and $x_0$ and $x_1$ are not switching times. We claim that every zero of $\varphi$ is not a switching time and that $V \equiv -M^2$. Indeed, recall that a zero of $\varphi$ is isolated and there are a finite number of them. Consider then $x_2$ distinct from $x_0$ and $x_1$. Assume that it is consecutive to $x_1$, *i.e.* $|\varphi| > 0$ on $(x_1, x_2)$. Reproducing the reasoning done on $[x_0, x_1]$ with $x_1$ (respectively $x_2$) replacing $x_0$ (respectively $x_1$), we conclude from (48) that the corresponding $B$ is equal to zero and from (45) that $c_{\nu'}(2M(x_2 - x_1)) = 1$, *i.e.*, $\nu' = -1$ and $s_{\nu'}(M(x_2 - x_1)) = 0$. Being back to the previous situation, one deduces that $\dot\varphi(x_2) = 0$. Proceeding in that way step by step, one gets the claim. This contradicts Item 1. and finally one has proved that $s_\nu(MT) \neq 0$. From (47) and (3), one gets that

$$B = \frac{H_0}{2M^2}\frac{c_\nu(MT)}{s_\nu(MT)},$$

and direct computations finally yield (42) and (44). □

To state our subsequent results, one needs to define, for every positive real number $M$ the function $F_M : [0,1] \to \mathbb{R}_+$ by

$$F_M(x) = x + \frac{\pi - \arctan\big(\tanh(Mx)\big)}{M}. \tag{49}$$

The basic facts on this function are the following:

$$F_M(0) = \frac{\pi}{M},\ F_M(1) = 1 + \frac{\pi - \arctan\big(\tanh(M)\big)}{M}, F_M'(x) = \frac{2\tanh^2(Mx)}{1+\tanh^2(Mx)},\ \forall x \in [0,1]. \tag{50}$$

Hence $F_M$ is a $C^1$, strictly increasing bijection from $[0,1]$ to $[\frac{\pi}{M}, F_M(1)]$ and $F_M(1) > 1$. Our second intermediate result goes as follows.

**Lemma 3.3** *Let $(R, q)$ be a strong extremal projecting on an optimal trajectory $R$ which is associated with a potential $V$ minimising $\mathcal{C}_1$ with corresponding $H_0 \neq 0$. Assume furthermore that $R$ is not a bang trajectory. Then, up to a translation, $V$ is periodic of period $T_1 + T_2$ so that $V = M^2$ on $[0, T_1]$ and $V = -M^2$ on $[T_1, T_1 + T_2]$ where $T_1, T_2 \in (0,1)$ so that they satisfy*

$$T_2 = \frac{\pi - \arctan\big(\tanh(MT_1)\big)}{M}, \tag{51}$$

*and there exists a positive integer $l$ such that*

$$F_M(T_1) = \frac{1}{l}. \tag{52}$$

*Proof.* Notice that $R$ must have at least two distinct bang arcs and then at least two switching points. Moreover, all the zeros of $\varphi$ must be switching times according to (43). Thanks to Lemma 2.1, we can assume, up to translating the potential $V$, that 0 is a switching time and $\varphi > 0$ in a right neighborhood of zero (since both signs are taken on $[0,1]$). Since $\dot\varphi(0) \neq 0$, it must be positive and (43) yields that both $H_0$ and $\nu$ are positive. We first claim that $x = 1$ must be a switching time. For otherwise, $\varphi(1) \neq 0$ and hence $V$ has a constant sign in a left neighborhood of 1. If $V = M^2$ there, then for $a > 0$ small enough one has that $\varphi_{-a}(a) = \varphi(0) = 0$ and $\dot\varphi_{-a}(a) = \dot\varphi(0) \neq 0$, *i.e.*, $a$ is a switching time for $V_{-a}$. This is in contradiction with the fact that $V_{-a} = M$ in an open neighborhood of $a$. If now $V = -M^2$ in a left neighborhood of 1, let $x_r < 1$ be the largest zero of $\varphi$ in $[0,1]$. It turns out that $V_{x_r}$ changes sign at $x = 1 - x_r$ but this is in contradiction with the fact that $\varphi_{x_r}(1 - x_r) = \varphi(1) \neq 0$. We have proved the claim. Now we show that the last bang must correspond to $V = -M^2$. Indeed if it were not the case, then $V_a = M^2$ in an open neighborhood of some $a > 0$ small enough with $\varphi_a(a) = 0$, which is not possible. It means that $R$ is the concatenation of an even number of bang arcs, $\gamma_i$, $1 \leq i \leq 2l$, where on the $\gamma_{2j-1}$'s, $1 \leq j \leq l$, one has $V = M^2$ and on the $\gamma_{2j}$'s, $1 \leq j \leq l$, one has $V = -M^2$. Let $T_i > 0$ be the duration of each bang arc $\gamma_i$, for $1 \leq i \leq 2l$, and clearly

$$\sum_{i=1}^{2l} T_i = 1. \tag{53}$$

We next prove that $T_2 = F(T_1)$. Indeed, consider (43) written for $(x_0, x_1) = (0, T_1)$ and then $(x_0, x_1) = (T_1, T_1 + T_2)$. One deduces that

$$H_0 \tanh(MT_1) = \dot\varphi(0) = -\dot\varphi(T_1), \quad H_0 \tan(MT_2) = \dot\varphi(T_1) = -\dot\varphi(T_1 + T_2). \tag{54}$$

It follows at once that

$$\tanh(MT_1) = -\tan(MT_2) \in (0,1).$$

It follows that $MT_2 - k\pi \in (\frac{3\pi}{4}, \pi)$ for some non negative integer $k$. Then $k = 0$ otherwise, using (42), $\varphi$ would have another zero in $(T_1, T_1 + T_2)$, which is not possible. One deduces (51). We finally prove that

$$T_{2j-1} = T_1, \quad T_{2j} = T_2, \text{ for } 1 \leq j \leq l. \tag{55}$$

We only provide an argument for $T_3 = T_1$ since the other equalities are deduced in an identical manner. For that purpose, consider (43) written for $(x_0, x_1) = (T_1 + T_2, T_1 + T_2 + T_3)$. One deduces that

$$H_0 \tanh(MT_3) = \dot\varphi(T_1 + T_2) = -\dot\varphi(T_1 + T_2 + T_3).$$

Using (54), one gets that

$$\tanh(MT_3) = \frac{\dot\varphi(T_1 + T_2)}{H_0} = -\tan(MT_2) = \tanh(MT_1),$$

yielding that $T_1 = T_3$ and $V$ is $(T_1 + T_2)$-periodic. One deduces (52) from (53), which concludes the proof of Lemma 3.3. □

We are able to state the proposition providing a complete solution to **Min-Det** in the case $N = 1$.

**Theorem 3.2** *For every positive $M$, the optimal control problem* **Min-Det(M)** *admits a unique minimising potential $V_{min}$ in $L^\infty(\mathbb{S}^1)$ defined as follows.*

(*a*) *If $M \in (0, \pi]$, $V_{min} = V_0 \equiv -M^2$ and the minimal value for* **Min-Det(M)** *is equal to $\mathcal{C}_1(V_0) = -2\big(1 - c_-(M)\big)$;*

(*b*) *If $M > \pi$, $V_{min}$ is the potential $V_1$ equal to $M^2$ on $[0, t_1]$ and $-M^2$ on $[t_1, 1]$, with $F_M(t_1) = 1$ and the minimal value for* **Min-Det(M)** *is equal to $\mathcal{C}_1(V_1) = -2\big(1 - c_-(M(1 - t_1))c_+(t_1)\big)$.*

*Proof.* If $M \leq \pi$, then $F_M(x) > 1$ for every $x \in (0,1]$ and one deduces from (52) that there is no $T_1 \in (0,1)$ satisfying the properties required for the existence of a an optimal trajectory $R$ which is not a bang trajectory. Therefore, the only candidate left as minimising potential by Lemma 3.1 is $V = V_0$, *i.e.* Item $(a)$ holds true. Assume now that $M > \pi$. Define the positive integer $L := E(\frac{M}{\pi})$ and the $2L$ times

$$t_l = F_M^{-1}(\frac{1}{l}),\ s_l = \frac{1}{l} - t_l \quad 1 \leq l \leq L. \tag{56}$$

According to Lemma 3.3, there exists a bang-bang trajectory $R_l$ with $2l$ bang arcs and associated with the periodic potential $V_l$ of period $\frac{1}{l}$ so that $V_l = M^2$ on $[0, t_l]$ and $V_l = -M^2$ on $[t_l, t_l + s_l]$. Recall that $R_0$ is the trajectory of (7) associated with $V_0$. Then, one gets from Lemmas 3.1 and 3.3 that a minimising potential $V_{min}$ must be equal to $V_l$ for some integer $0 \leq l \leq L$. In order to conclude, one is left with the computation of the costs $\mathcal{C}_1(V_l)$, for positive integers $1 \leq l \leq L$. A lengthy but straightforward computation yields that

$$\begin{aligned} R_l(\frac{1}{l}) &= \begin{pmatrix} c_-(Ms_l) & \frac{s_-(Ms_l)}{M} \\ -Ms_-(Ms_l) & c_-(Ms_l) \end{pmatrix} \begin{pmatrix} c_+(Mt_l) & \frac{s_+(Mt_l)}{M} \\ Ms_+(Mt_l) & c_+(Mt_l) \end{pmatrix} \\ &= \begin{pmatrix} c_-(Ms_l)c_+(Mt_l) + s_-(Ms_l)s_+(Mt_l) & \frac{c_-(Ms_l)s_+(Mt_l)+s_-(Ms_l)c_+(Mt_l)}{M} \\ M(s_-(Ms_l)c_+(Mt_l) + c_-(Ms_l)s_+(Mt_l)) & c_-(Ms_l)c_+(Mt_l) - s_-(Ms_l)s_+(Mt_l) \end{pmatrix}, \end{aligned} \tag{57}$$

and one has that $\det(R_l(\frac{1}{l})) = 1$ and

$$\alpha_l = -\frac{\mathrm{Tr}(R_l(\frac{1}{l}))}{2} = -c_-(Ms_l)c_+(Mt_l),\ 1 \leq l \leq L. \tag{58}$$

We use $r_l, \frac{1}{r_l}$ in $\mathbb{C}$ to denote the eigenvalues of $R_l(\frac{1}{l})$. Since $V_l$ is $\frac{1}{l}$-periodic, one gets that $R_l(1) = R_l^l(\frac{1}{l})$ and hence

$$\mathcal{C}_1(V_l) = -\det\big(\mathrm{Id}_2 - R_l^l(\frac{1}{l})\big) = (-2)(1 - \frac{r_l^l + r_l^{-l}}{2}),\ 1 \leq l \leq L. \tag{59}$$

Recall that $Ms_l \in (\frac{3\pi}{4}, \pi)$ and hence, it holds, for $1 \leq l \leq L$ that

$$\begin{aligned} -c_-(Ms_l) &= -c_-\Big(\pi - \arctan\big(\tanh(Mt_l)\big)\Big) = c_-\Big(\arctan\big(\tanh(Mt_l)\big)\Big) \\ &= \frac{1}{\sqrt{1 + \tanh^2(Mt_l)}} = \frac{c_+(Mt_l)}{\sqrt{c_+^2(Mt_l) + s_+^2(MT_1)}}, \end{aligned}$$

and then

$$\alpha_l = \frac{c_+^2(Mt_l)}{\sqrt{2c_+^2(Mt_l) - 1}} > 1. \tag{60}$$

Let $\xi_l > 0$ such that $\alpha_l = c_+(\xi_l)$. Since $r_l$ and $\frac{1}{r_l}$ are the roots of the degree two polynomial $X^2 + 2c_+(\xi_l)X + 1$, one gets that $r_l = -e^{\xi_l}$ and finally it holds

$$\mathcal{C}_1(V_l) = (-2)\Big(1 - (-1)^l c_+(l\xi_l)\Big).$$

For even $l$'s, the cost is non negative, implying that $V_l$ cannot be minimising. For odd $l$'s, the cost is smaller than $-4$ and then smaller than $\mathcal{C}_1(V_0)$. It remains to show that $\mathcal{C}_1(V_l)$ reaches its minimal value for $l = 1$. For that, it is enough to prove that the mapping $G : l \mapsto l\xi_l$ is strictly decreasing for $l \in [1, L]$. A lengthy but straightforward computation yields that

$$G'(l) = Mt_l\Big(\frac{\xi_l}{Mt_l} - \frac{c_+(Mt_l)}{s_+(Mt_l)}\frac{F_M(Mt_l)}{t_l}\Big), \quad l \in [1, L].$$

Since $F_M(Mt_l) > t_l$, one would have that $G'(l) < 0$ if one shows that $\xi_l < Mt_l$. In turn, that last inequality is itself equivalent $\alpha_l < c_+(Mt_l)$, inequality which does hold true by (60). This concludes the proof of Theorem 3.2. □

## 4 Idea for Case $N \geq 1$

The goal of this section consists in proving the following proposition.

**Proposition 4.1** *let $V_0$ be a a minimising potential for $\mathbf{Ext}-\mathbf{Det}_\varepsilon(M)$ so that $H_0$, the constant value of the Hamiltonian defined in (16) is positive. Then there exists $v_0 \in M_N(\mathbb{R})$ with norm equal to $M^2$ so that $V_0$ is piecewise constant, taking values in $\{-v_0, v_0\}$ and with a finite number of discontinuities.*

The proof of the proposition requires several intermediate computations which are given next.

We start by providing the following lemma which gather several expressions for the first and second differential of the cost function.

**Lemma 4.1** *Consider a twice differentiable mapping $\xi \mapsto V_\xi$ defined for $\xi$ in a neighborhood of zero and taking values in $\mathcal{V}_M$. We use $(R_\xi, q_\xi)$ to denote the corresponding mapping of weak extremals and we assume that $\mathrm{Id}_{2N} - R_0(1,0)$ is invertible.*

One has to use further the space translations of an optimal potential.

Hence in the sequel, let $V_0$ be an optimal potential so that the constant value of the Hamiltonian $H_0$ is positive and let $q_0$ and $\varphi_0$ the associated covector and switching functions. According to Item $(c)$ of Lemma 2.3, the corresponding switching function admits a finite number of zeros that we subdivide in two (possibly empty) finite subsets $I_1(0)$ and $I_2(0)$, the first one gathering the zeros at which $\dot\varphi_0$ does not vanish and the second one gathering the zeros at which $\dot\varphi$ vanishes but $\ddot\varphi_0$ does not. Moreover, to fix ideas, we assume that $V_0$ is right-continuous and $\|V_0(x)\| = M^2$ for $x \notin I_1(0)$.

Note also that $V_0$ only admits discontinuities on $I_1(0)$ and, for every $x_0 \in I_1(0)$, it holds

$$-\lim_{x\to x_0^-} V_0(x) = \lim_{x\to x_0^+} V_0(x) = M^2 \frac{\dot\varphi_0(x_0)}{\|\dot\varphi_0(x_0)\|}. \tag{61}$$

For $\eta$ small enough, define the family of potentials $V_\eta(\cdot) = V_0(\eta + \cdot)$ which are simply the space translates of the periodic potential $V_0$. Recalling Lemma 2.1, we set

$$q_\eta(\cdot) = q_0(\eta + \cdot), \quad \varphi_\eta(\cdot) = \varphi_0(\eta + \cdot), \quad I_1(\eta) = I_1(0) - \eta, \quad I_2(\eta) = I_2(0) - \eta, \quad C(\eta) = C_\varepsilon(V_\eta),$$

and one has $C(\eta) \equiv C(0)$.

One aims at computing non trivial expressions of the first two derivatives of $\eta \mapsto C(\eta)$ in particular in terms of the first and second differentials of $V \mapsto C_\varepsilon(V)$ seen as functions of

$$\mathcal{S}_M = \{V : [0,1] \to M_N(\mathbb{R}) \mid \|V(x)\| = M^2 \text{ for a.e. } x \in [0,1]\}. \tag{62}$$

Note that for every $V \in \mathcal{S}_M$ the tangent space $T_V\mathcal{S}_M$ to $\mathcal{S}_M$ at $V$ is given by

$$T_V\mathcal{S}_M = \{W : [0,1] \to M_N(\mathbb{R}) \mid \langle V(x), W(x)\rangle = 0 \text{ for a.e. } x \in [0,1]\}. \tag{63}$$

We have the following computational result. We use $\delta_y$ to denote the Dirac distribution at $y \in [0,1]$.

**Lemma 4.2** *For $\eta$ small enough, one has*

$$\frac{dV_\eta}{d\eta} = \dot V_\eta + \mathbf{\Delta}^1_\eta \tag{64}$$

*where $\dot V_\eta(x)$ is a piecewise-continuous function admitting discontinuities on $I_1(\eta)$ only such that, for every $x_0 \in I_1(0)$,*

$$-\lim_{x\to x_0^-} \dot V_\eta(x-\eta) = \lim_{x\to x_0^+} \dot V_\eta(x-\eta) = M^2 \frac{z_0}{\|\dot\varphi(x_0)\|}, \tag{65}$$

*where* $z_0 = \ddot{\varphi}(x_0) - \langle \ddot{\varphi}(x_0), \frac{\dot{\varphi}(x_0)}{\|\dot{\varphi}(x_0)\|} \rangle \frac{\dot{\varphi}(x_0)}{\|\dot{\varphi}(x_0)\|}$, *and*

$$\mathbf{\Delta}_\eta^1 = 2M^2 \sum_{x_0 \in I_1(0)} \frac{\dot{\varphi}_0(x_0)}{\|\dot{\varphi}(x_0)\|} \delta_{x_0-\eta}. \tag{66}$$

*Moreover,*

$$\frac{d\dot{V}_\eta}{d\eta} = \ddot{V}_\eta + \mathbf{\Delta}_\eta^2 \tag{67}$$

*where* $\ddot{V}_\eta(x)$ *is a piecewise-continuous function admitting discontinuities on* $I_1(\eta)$ *only and*

$$\mathbf{\Delta}_\eta^2 = 2M^2 \sum_{x_0 \in I_1(0)} \frac{z_0}{\|\dot{\varphi}(x_0)\|} \delta_{x_0-\eta}. \tag{68}$$

***Proof:*** Computations.

■

We have the following result.

**Lemma 4.3** *For* $\eta$ *small enough, one has*

$$0 = \frac{dC(\eta)}{d\eta} = -\int_0^1 \langle \varphi_\eta(x), \dot{V}_\eta(x) \rangle \, dx, \tag{69}$$

*and*

$$0 = \frac{d^2C(\eta)}{d\eta^2} = -\int_0^1 \langle \dot{\varphi}_\eta(x), \dot{V}_\eta(x) \rangle \, dx - \int_0^1 \langle \varphi_\eta(x), \ddot{V}_\eta(x) \rangle \, dx. \tag{70}$$

***Proof:*** Some subsequent computations are a bit formal but can be rendered easily rigourous. One has

$$\frac{dC(\eta)}{d\eta} = -(-1)^N \varepsilon \det(Id_{2N} - R_\eta(1,0)) \mathrm{Tr}\big(\frac{dR_\eta(1,0)}{d\eta}(Id_{2N} - R_\eta(1,0))^{-1}\big).$$

On the other hand

$$\frac{dR_\eta(1,0)}{d\eta} = \int_0^1 R_\eta(1,x) \frac{dA_{V_\eta}}{d\eta}(x) R_\eta(x,0) \, dx.$$

(One has to pay attention that the previous formula holds only in the sense of distributions since $\frac{dA_{V_\eta}}{d\eta}$ is not a function.)

Combining the two equations and using the definition of $q_\eta$ in (38), one deduces that

$$\frac{dC(\eta)}{d\eta} = -\int_0^1 \langle q_\eta(x), \frac{dA_{V_\eta}}{d\eta}(x) \rangle \, dx = -\int_0^1 \langle \varphi_\eta(x), \frac{dV_\eta}{d\eta}(x) \rangle \, dx.$$

The key point to conclude is that

$$\langle \varphi_\eta, \mathbf{\Delta}_\eta^1 \rangle_{dis} = \langle \varphi_\eta, \mathbf{\Delta}_\eta^2 \rangle_{dis} = 0,$$

where the above has to be understood in the distributional sense.

■

**Remark 4.1** *In a certain sense, not only* $\dot{V}_\eta \in T_{V_\eta}\mathcal{S}_M$ *(which is obvious) but we have shown that* $\frac{dV_\eta}{d\eta}$ *is an admissible variation to* $V_\eta$ *i.e.,* $\frac{dV_\eta}{d\eta} \in T_{V_\eta}\mathcal{S}_M$.

The following formula is simply obtained by deriving twice the fact that $V_\eta$ has constant norm and using the expression of $V_0$ in terms of $\varphi_0$: one has

$$-\int_0^1 \langle \varphi_\eta(x), \ddot{V}_\eta(x)\rangle \, dx = \frac{1}{M^2}\int_0^1 \|\varphi_\eta\|\|\dot{V}_\eta(x)\|^2 \, dx. \tag{71}$$

Here comes the delicate point. Is it possible to relate

$$-\int_0^1 \langle \dot{\varphi}_0(x), \dot{V}_0(x)\rangle \, dx,$$

with $d^2C(V_0)(W,W)$ for some generalized variation $W$? Such a term is non negative and, using (70) one would get from (71) that $\dot{V}_0 = 0$ for almost every $x \in [0,1]$. the dream...

## References


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## 5 Appendix

In this appendix, we apply the Pontryagin Maximum Principle (PMP) to the control system (7), in order to derive necessary optimality conditions. These are essential to the proofs of Proposition 2.1.

For the Hamiltonian formalism used below, we refer to [1].

**Proposition 5.1** *Let $H \in C^\infty(T^*sl_{2N}(\mathbb{R})(\mathbb{R}))$ be an Hamiltonian function. Upon the identification of $T^*sl_{2N}(\mathbb{R})(\mathbb{R})$ given in (32), the corresponding Hamiltonian vector field $\vec{H}$ is defined as $\vec{H}(R,P) = (\vec{H}_P, -\vec{H}_R) \in T_{(R,P)}T^*sl_{2N}(\mathbb{R})(\mathbb{R})$ by*

$$\vec{H}_P = h_P R, \tag{72}$$
$$\vec{H}_R = (h_R + qh_P^T)(R^{-1})^T, \tag{73}$$

*where $q = PR^T$, $h_P$ and $h_R$ belong to $sl_{2N}(\mathbb{R})(\mathbb{R})$, and the two last matrices are given by*

$$h_P = \frac{\partial H}{\partial P}R^{-1} - \frac{1}{2N}\operatorname{Tr}(\frac{\partial H}{\partial P}R^{-1})\operatorname{Id}_{2N}, \tag{74}$$
$$h_R = \frac{\partial H}{\partial R}R^T - qh_P^T - \frac{1}{2N}\operatorname{Tr}\Big(\frac{\partial H}{\partial R}R^T - qh_P^T\Big)\operatorname{Id}_{2N} + \frac{1}{2N}\operatorname{Tr}\Big(\frac{\partial H}{\partial P}R^{-1}\Big)q^T. \tag{75}$$

***Proof:*** Upon the given identifications, we have for every $(R,P) \in T^*sl_{2N}(\mathbb{R})(\mathbb{R})$ with $q = PR^T$ that

$$T_{(R,P)}T^*sl_{2N}(\mathbb{R})(\mathbb{R}) = \left\{\left(rR, (p - qr^T)(R^{-1})^T\right) \mid (r,p) \in sl_{2N}(\mathbb{R})(\mathbb{R})^2\right\}. \tag{76}$$

It immediately follows that there exists $h_P, h_R$ in $sl_{2N}(\mathbb{R})(\mathbb{R})$ so that (72) and (73) hold true.

Letting $\left(\frac{\partial H}{\partial R}, \frac{\partial H}{\partial P}\right)$ in $M_{2N}(\mathbb{R})^2$ be the partial derivatives of $H$ at $(R,P) \in T^*sl_{2N}(\mathbb{R})(\mathbb{R})$, we have that

$$d_{(R,P)}H\big(rR, (p - qr^T)(R^{-1})^T\big) = \langle rR, \frac{\partial H}{\partial R}\rangle + \langle (p - qr^T)(R^{-1})^T, \frac{\partial H}{\partial P}\rangle, \quad \forall (r,p) \in sl_{2N}(\mathbb{R})(\mathbb{R})^2. \tag{77}$$

On the other hand, the Hamiltonian vector field $\vec{H}$ on $T^*sl_{2N}(\mathbb{R})(\mathbb{R})$, with components $\vec{H}(R,P) = (\vec{H}_P, -\vec{H}_R) \in T_{(R,P)}(T^*sl_{2N}(\mathbb{R})(\mathbb{R}))$, is the only vector field on $T^*sl_{2N}(\mathbb{R})(\mathbb{R})$ such that one has that

$$d_{(R,P)}H\big(rR, (p - qr^T)(R^{-1})^T\big) = \langle rR, \vec{H}_R\rangle + \langle (p - qr^T)(R^{-1})^T, \vec{H}_P\rangle, \quad \forall (r,p) \in sl_{2N}(\mathbb{R})(\mathbb{R})^2. \tag{78}$$

Substracting (77) from (78) and using the properties of the inner product, one deduces that

$$\langle r, \frac{\partial H}{\partial R}R^T - (h_R + qh_P) - (\frac{\partial H}{\partial P}R^{-1} - h_p)^T q^T\rangle + \langle p, \frac{\partial H}{\partial P}R^{-1} - h_p\rangle = 0, \quad \forall (r,p) \in sl_{2N}(\mathbb{R})(\mathbb{R})^2. \tag{79}$$

Choosing $r = 0$ yields that $\langle p, \frac{\partial H}{\partial P}R^{-1} - h_p\rangle = 0$ for every $p \in sl_{2N}(\mathbb{R})(\mathbb{R})$. Since the orthogonal of $sl_{2N}(\mathbb{R})(\mathbb{R})$ in $M_{2N}(R)$ with respect to $\langle\cdot,\cdot\rangle$ is equal to $\mathbb{R}\,\mathrm{Id}_{2N}$, one first deduces $\frac{\partial H}{\partial P}R^{-1} - h_p$ is parallel to $\mathrm{Id}_{2N}$ and, by taking the trace in that relation, (74) follows. Using that fact in the equation obtained from (79) with $p = 0$, one derives (75).

■