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Incremental stabilization and multi-agent synchronization of discrete-time nonlinear systems

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Abstract

In this paper, we propose a novel distributed state-feedback design for robust synchronization of networks of identical discrete-time nonlinear agents under generic time-invariant communication graphs. We focus on the class of almost differentiable (possibly time-varying) dynamics that are linear in the input. By generalizing results on synchronization of linear agents, we build strong links between the solution to the synchronization problem in the linear and nonlinear framework. This is also enabled by the introduction of new results on design of incrementally stabilizing controllers based on contraction analysis. Finally, we propose numerically tractable sufficient conditions for the synchronization of networks of non-smooth Lur'e systems.

Keywords: Discrete-time, nonlinear systems, synchronization, multi-agent systems, contraction, incremental stability.

1. Introduction

Multi-agent systems control has attracted a lot of attention from our community. Many modern control problems can be formalized as a network of interacting agents aiming at achieving some sort of agreement [19]. In this paper, we focus on the problem of state synchronization of a network of homogeneous systems (namely, all the agents are identical), described by a time-varying discrete-time nonlinear system that is linear in the control input. The problem of synchronization presents mature results in the linear framework, especially for continuous-time agent dynamics [26, 20]. For discrete-time systems, major contributions can be found in [30, 25, 6]. However, in this setting, further investigations are still required to understand the relationship between the *structure* of a network (i.e., the eigenvalues and eigenvectors associated with its representation) and the possibility of finding a suitable synchronizing control. As a matter of fact, the structure of the communication graph has a significant impact on discrete-time networks compared to continuous-time ones. Only recently a novel implicit formulation allowed overcoming some of these structural limitations, proposing a result comparable to the continuous-time framework [3].

Avoiding implicit computations, graph normalization approaches like [13, 30] have proven to be effective, yet they may not always be feasible. Additionally, existing solutions for nonlinear dynamics are limited to specific agent structures, such as Lur'e system forms [28] or linear systems with saturated inputs [4]. Moreover, they commonly employ observer design [14, 22] or data-based optimization techniques [10].

In this work, we propose solutions to the synchronization problem based on discrete-time contraction analysis [33, 17, 16, 18] and incremental stability [9, 1, 32, 28, 12, 31]. There are two main motivations for this choice. Firstly, contraction analysis allows us to study nonlinear systems via linear systems-like arguments. Hence, we can take inspiration from the well-established linear systems literature [30] and provide a link between the two scenarios. Secondly, incremental stability easily translates to synchronization of homogeneous networks. As a matter of fact, trajectories originated from incrementally stable dynamics “forget” their initial conditions, while offering strong robustness properties. In particular, in a network where the agents are homogeneous, each one can be considered as a singular trajectory of the same system starting from different initial conditions. As distances between trajectories of incrementally stable systems asymptotically decrease to zero, by designing a distributed controller making the network dynamics incrementally stable we indirectly obtain robust state synchronization¹. The three main contributions of this paper can be summarized as follows:

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¹ It is worth noting that in the context of synchronization it is

- (i) We provide new results on robust synchronization of identical discrete-time linear systems, arising from the solution of a modified Algebraic Riccati Inequality (MARI) [29, 4, 34, 40]. In comparison with the results in [30, 25, 13], we do not rely on the normalized Laplacian or different scalar gains for each agent. In practice, we assume that all the controllers have to be designed equally and no further local degree of freedom is available. Our sufficient conditions include the normalized Laplacian case as a sub-case. In doing so, we obtain bounds linking the connectivity properties of the graph with the simultaneous stabilizability properties of the agents.
- (ii) We revisit the problem of incremental stabilization via feedback of a nonlinear system and we provide a new *closed-form* state-feedback controller guaranteeing uniform exponential incremental stability properties for the closed-loop system. The proposed feedback is a direct generalization of a linear LQR design. The conditions are based on contraction analysis and apply to dynamics that are time-varying and not differentiable everywhere, generalizing existing results [9, 33, 7, 18]. We also highlight that some other works addressed a similar problem, relying on control contraction metrics, see, e.g. [35, 36]. However, such approaches are not constructive nor explicit since the solution relies on nested optimizations at each time-instant.
- (iii) We exploit the new results on contraction analysis and incremental stability to provide new controller designs for robust synchronization of identical discrete-time nonlinear agents. This is done under the assumptions of almost differentiable dynamics, linear input vector field, and generic connected communication graphs (i.e., possibly directed and weighted). As in the linear case, we relate the connectivity properties of the graph to the simultaneous stabilizability properties of the agents.

The paper is organized as follows. Section 2 introduces some preliminary concepts of graph theory and presents the problem of exponential synchronization. Section 3 presents the results for the linear framework. Section 4 extends the solution to nonlinear agents. Section 5 provides an LMI-based method for controller design and a numerical example. Finally, Section 6 concludes the paper while the Appendix contains the proofs of our results.

Notation. We denote by \mathbb{R} the set of real numbers, by $\mathbb{R}_{\geq 0}$ the set of non-negative real numbers, by \mathbb{N} the set of non-negative integers and by \mathbb{C} the set of complex numbers. Given a complex number $z \in \mathbb{C}$, we use $\Re(z)$ and $\Im(z)$ to identify its real and imaginary parts respectively.

more natural to invoke the notion of incremental stability than the one of convergent system, e.g., [33, 16]. Indeed, state synchronization may be achieved on an unbounded trajectory.

$|\cdot|$ denotes the standard Euclidean norm, $|x|_{\mathcal{M}}$ is a distance function between any point $x \in \mathbb{R}^n$ and a closed set $\mathcal{M} \subset \mathbb{R}^n$, namely, $|x|_{\mathcal{M}} := \inf_{z \in \mathcal{M}} |x - z|$. Given two symmetric matrices $A, B \in \mathbb{R}^{n \times n}$, we say $A \preceq B$ if the matrix $A - B$ is negative semidefinite. Similarly, we say $A \succ B$ if $A - B$ is positive definite. Moreover, we denote by $\mathbb{S}_{>0}^n$ (resp. $\mathbb{S}_{\geq 0}^n$) the set of symmetric positive definite (resp. semi-definite) real matrices of dimension n . Given a set of vectors x_1, \dots, x_N with $N \in \mathbb{N}$ and arbitrary dimensions, we define $\text{col}(x_1, \dots, x_N) = [x_1^\top \cdots x_N^\top]^\top$. Given a square matrix $A \in \mathbb{R}^{n \times n}$, we use $\text{spec } A$ to represent its spectrum. Given a set of square matrices A_1, \dots, A_N with $N \in \mathbb{N}$ we identify by $\text{diag}(A_1, \dots, A_N)$ the block diagonal matrix whose diagonal blocks are A_1, \dots, A_N respectively. Given a positive integer $n \in \mathbb{N}$, we identify by \mathbf{I}_n the identity matrix of dimension n . We use $\mathbf{1}$, resp. $\mathbf{0}$, to identify a column vector of 1s, resp. 0s, of appropriate dimension. The symbol \otimes denotes the Kronecker product, which satisfies $(A \otimes B)^\top = A^\top \otimes B^\top$, $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ and for A, B, C symmetric, $A \preceq B \implies (A \otimes C) \preceq (B \otimes C)$ and $(C \otimes A) \preceq (C \otimes B)$. For a square matrix $A \in \mathbb{R}^{n \times n}$, we define $\text{Sym}(A) := \frac{1}{2}(A^\top + A)$. Finally, we denote with co the convex hull.

2. Problem statement

In this section, we first present some preliminary concepts related to graph theory and multi-agent networks. Then, we formulate the problem of exponential synchronization.

2.1. Highlights on graph theory

In multi-agent systems, a communication graph is typically described by a triplet $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ where $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ is a set of $N \in \mathbb{N}$ *vertices* (or nodes), $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of *edges* ϵ_{jh} modeling the interconnection between such nodes, and $\mathcal{A} \in \mathbb{R}^{N \times N}$ is the *adjacency matrix*, whose entries $a_{jh} \geq 0$ *weight* the flow of information from vertex j to vertex h . We denote by $L \in \mathbb{R}^{N \times N}$ the *Laplacian matrix* of the graph, defined as

$$\ell_{jh} = -a_{jh} \quad \text{for } j \neq h, \quad \ell_{jh} = \sum_{i=1}^N a_{ji} \quad \text{for } j = h,$$

where ℓ_{jh} is the (j, h) -th entry of L . We denote with \mathcal{N}_i the set of in-neighbors of node i , i.e. the set $\mathcal{N}_i := \{j \in \{1, \dots, N\} \mid \epsilon_{ij} \in \mathcal{E}\}$. In this paper, we consider directed, weighted graphs. As such, the Laplacian matrix is not assumed to be diagonalizable and admits complex eigenvalues. We identify the Laplacian of the network as

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \quad (1)$$

where L_{11} is a scalar, L_{12} is a $N - 1$ row vector, L_{21} is a $N - 1$ column vector and L_{22} is a $(N - 1) \times (N - 1)$ matrix. We also suppose the graph contains a directed spanning

tree. Based on these assumptions, we recall the following result from the literature, see e.g., [24, 15].

Lemma 1 ([24]) *The Laplacian matrix L has a simple eigenvalue 0 and all the other eigenvalues have positive real parts if and only if the directed graph has a directed spanning tree.*

2.2. The problem of multi-agent synchronization

Consider a homogeneous network of discrete-time agents, where the dynamics of each node is described by a nonlinear, possibly time-varying, difference equation, linear in the input, of the form

$$x_i^+ = f(x_i, \mathbf{t}) + Bu_i + w_i, \quad i = 1, \dots, N, \quad (2)$$

where $f : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$ is full column rank, $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $w_i \in \mathbb{R}^n$ represent the state, the control input and an unmeasured disturbance of node i at timestep $\mathbf{t} \in \mathbb{N}$, respectively, and $x_i^+ \in \mathbb{R}^n$ represents the state of node i at timestep $\mathbf{t} + 1$. Note that $w_i \in \mathbb{R}^n$ can be any (not necessarily bounded) input sequence and can represent, for instance, an external disturbance affecting the dynamics or some (bounded) model uncertainty. We define the state of the entire network $\mathbf{x} \in \mathbb{R}^{Nn}$ and the entire disturbance $\mathbf{w} \in \mathbb{R}^{Nn}$ as

$$\mathbf{x} := \text{col}(x_1, \dots, x_N), \quad \mathbf{w} := \text{col}(w_1, \dots, w_N). \quad (3)$$

Our objective is to design a distributed state-feedback control law of the form

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij} [\alpha(x_i, \mathbf{t}) - \alpha(x_j, \mathbf{t})] = \sum_{j=1}^N \ell_{ij} \alpha(x_j, \mathbf{t}) \quad (4)$$

for all $i = 1, \dots, N$, where the function $\alpha : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^m$, stabilizes the dynamics (2) on the so-called *synchronization manifold* \mathcal{M} defined as

$$\mathcal{M} := \{\mathbf{x} \in \mathbb{R}^{Nn} \mid x_i = x_j, \text{ for all } i, j \in \{1, \dots, N\}\}, \quad (5)$$

where the states of all the agents of the network agree with each other. Note that the i -th agent can only use the state x_j of its neighbors $j \in \mathcal{N}_i$ alongside its own local information x_i . Moreover, the control action u_i is equal to zero on the synchronization manifold in the absence of disturbances, i.e. when $\mathbf{w} = 0$. In other words, when synchronization is achieved, no correction term is needed for each individual agent. Hence, in general stabilizing independently all the agents on an *agreed* equilibrium point is not a valid solution. We formalize our synchronization problem as follows.

Problem 1 (Robust network synchronization) *The distributed feedback control law (4) solves the robust synchronization problem for the network (3) if there exist a function $\alpha : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^m$ and real numbers $c \geq 1$, $\rho \in (0, 1)$ and $\gamma \geq 0$ such that, for any initial condition*

$(\mathbf{x}(\mathbf{t}_0), \mathbf{t}_0) \in \mathbb{R}^{Nn} \times \mathbb{N}$, and any disturbance sequences $\mathbf{t} \mapsto w_i(\mathbf{t})$, $i \in \{1, \dots, N\}$, the solutions to the closed-loop system

$$x_i^+ = f(x_i, \mathbf{t}) + B \sum_{j=1}^N \ell_{ij} \alpha(x_j, \mathbf{t}) + w_i, \quad i = 1, \dots, N.$$

satisfy for all $\mathbf{t} \geq \mathbf{t}_0$

$$|\mathbf{x}(\mathbf{t})|_{\mathcal{M}} \leq c \rho^{\mathbf{t} - \mathbf{t}_0} |\mathbf{x}(\mathbf{t}_0)|_{\mathcal{M}} + \sup_{\substack{t \in [\mathbf{t}_0, \mathbf{t}] \\ i, j \in \{1, N\}}} \gamma |w_i(t) - w_j(t)|, \quad (6)$$

where \mathcal{M} is defined in (5).

Remark 1 *It is worth noting that while (4) assumes that each agent has access to its own and its neighbors complete state information, an output-feedback scenario with the inclusion of an observer and reduction-type arguments [8] could be considered due to the robust convergence required by (6). For the sake of clarity and ease in notation, we focus on the full-state information problem.*

3. Synchronization of Linear Systems

It is well known that the problem of linear synchronization boils down to a problem of simultaneous stabilization, see, e.g., [30, 39]. In this section, we reinterpret these results into our framework and present the discrete synchronization problem to non-diagonalizable Laplacian matrices. We start by recalling necessary and sufficient conditions for state synchronization. Then, we derive novel conditions for multi-agent synchronization under any Laplacian whose eigenvalues belong to a given compact set by exploiting the gain margin properties of Riccati-based designs. These results will act as a foundation for the nonlinear analysis in Section 4, which constitutes our main contribution.

We start by presenting a general result for network synchronization for linear systems. It is shown that the existence of a common control law for systems associated with each non-zero eigenvalues of the Laplacian is equivalent to solving the synchronization problem. Similar results have appeared multiple times in the literature, e.g., [20, Theorem 13], [13, Lemma 1], [30, Lemma 1] for the normalized Laplacian scenario and [38, Theorem 3.2]. However, we present a reformulated proof that will be instrumental for the analysis of the nonlinear framework.

Theorem 1 *Consider a network of $N \in \mathbb{N}$ agents described by dynamics*

$$x_i^+ = Ax_i + Bu_i + w_i \quad (7)$$

The diffusive control law $u_i = K \sum_{j=1}^N \ell_{ij} x_j$ solves Problem 1 if and only if

(i) the interconnection graph (possibly directed, weighted) \mathcal{G} has a directed spanning tree or matrix A is Schur-Cohn stable², and

(ii) the gain K is such that matrix $(A + \lambda_i BK)$ is Schur-Cohn stable for any $\lambda_i \in \text{spec } L \setminus \{0\}$.

The reformulated proof of Theorem 1 can be found in Appendix A.1.

The results of Theorem 1 are not constructive. To obtain a set of constructive conditions, we first recall the design procedure for pencil matrices stabilization. Then, in Section 3.2, we apply this result to the case of networks.

3.1. The pencil matrix for discrete-time linear systems

Consider a discrete-time linear system described by

$$x^+ = Ax + Bu, \quad u = Kx \quad (8)$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and, without loss of generality, B is assumed to be full-column rank. The goal is to find a gain matrix $K \in \mathbb{R}^{m \times n}$ such that the complex closed-loop matrix $(A + \lambda BK)$ is Schur-Cohn for some complex numbers λ . Inspired by [27, Definition 3.13], we formally define this notion as follows.

Definition 1 (Complex gain margin for LTI systems)

The matrix K is said to have a complex gain margin with radius $r > 0$ and center c if $A + \lambda BK$ is Schur-Cohn for any λ in $\{\lambda \in \mathbb{C} : |\lambda - c| \leq r\}$.

To find the complex gain margin of a matrix K , we propose a solution based on the discrete-time Modified Algebraic Riccati Inequality (MARI) [29, 4, 34] defined as

$$A^\top PA - \sigma A^\top PB(R + B^\top PB)^{-1} B^\top PA \preceq \rho P, \quad (9)$$

where $R \in \mathbb{S}_{>0}^m$, $P \in \mathbb{S}_{>0}^n$ and generally $\sigma \in (0, 1]$, $\rho \in (0, 1)$. A differential version of the MARI (9) will be the fundamental tool allowing the derivation of our non-linear results. Therefore, we briefly discuss some of the properties of such an inequality. Note that, since B is assumed to be full column rank, the matrix $R + B^\top PB$ is positive definite and, consequently, invertible. The main difference between the MARI (9) and the more common discrete-time Algebraic Riccati Inequality (DARI)

$$A^\top PA - A^\top PB(R + B^\top PB)^{-1} B^\top PA + Q \prec P \quad (10)$$

lies in the presence of the scalar σ . First, note that if $R \succ 0$, then the positive semi-definite matrix Q can be embedded in the right-hand side of (9) by exploiting $\rho P \prec P - Q$, which holds for a suitable $\rho \in (0, 1)$ as long as $P - Q \succ 0$. Inequality $P - Q \succ 0$ holds when $R \in \mathbb{S}_{>0}^m$ because one

can rearrange (10) as $A^\top [P - PB(R + B^\top PB)^{-1} B^\top P] A \prec P - Q$ and applying the block matrix inversion identity

$$(A + \mathcal{B}\mathcal{D}^{-1}\mathcal{C})^{-1} = \mathcal{A}^{-1} - \mathcal{A}^{-1}\mathcal{B}(\mathcal{D} + \mathcal{C}\mathcal{A}^{-1}\mathcal{B})^{-1}\mathcal{C}\mathcal{A}^{-1}, \quad (11)$$

with $\mathcal{A}^{-1} = P$, $\mathcal{D} = R$, $\mathcal{C} = B^\top$, $\mathcal{B} = B$, yields to

$$0 \prec A^\top (P^{-1} + BR^{-1}B^\top)^{-1} A \prec P - Q. \quad (12)$$

Therefore, the DARI (10) is a special case of the MARI (9) when $\sigma = 1$ and R is positive definite. As such, the MARI allows for an extra degree of freedom. Its role is to weigh the impact of the input on the solution to the inequality. In other words, the smaller the σ , the less we can rely on the input to stabilize the system.

We next reformulate [40, Theorem 2], showing that the degree of freedom offered by the MARI (9) allows obtaining sufficient conditions for the existence of a state feedback gain K solving the simultaneous stabilization problem. This result expands on the findings of [4, Theorem 1], by taking into account the interplay between R and P in the MARI (10), thus allowing for larger certified gain margins.

Proposition 1 *Let the pair (A, B) be stabilizable and $R \in \mathbb{S}_{>0}^m$. Let $P \in \mathbb{S}_{>0}^n$ be a solution to the MARI (9) for some $\sigma \in (0, 1]$ and for some $\rho \in (0, 1)$. Then the matrix*

$$K = -(R + B^\top PB)^{-1} B^\top PA, \quad (13)$$

has a complex gain margin center and radius

$$c = 1 + \frac{\lambda_{\min}(R)}{\lambda_{\max}(B^\top PB)}, \quad r = \sqrt{c(c - \sigma)}. \quad (14)$$

Moreover, the following bound holds

$$c - \sqrt{c(c - \sigma)} \leq 1 - \sqrt{1 - \sigma} \leq 1 + \sqrt{1 - \sigma} \leq c + \sqrt{c(c - \sigma)} \quad (15)$$

The proof of Proposition 1 is omitted as it can be derived from [40, Theorem 2].

Remark 2 *We highlight an interesting difference between the continuous and discrete-time frameworks. In the continuous-time case it is possible to obtain infinite-gain margin properties³. This is due to the fact that continuous-time stability is associated to the eigenvalues of the closed-loop matrix being in the left-half of the complex plane. Therefore, any gain stabilizing the eigenvalue with most positive real part stabilizes all the others too. This does not hold in the discrete-time framework, since discrete-time stability demands for the eigenvalue to lay inside the unit disc. Therefore, only finite gain margin properties are typically available to discrete-time controllers, e.g. [23].*

²A matrix is said to be Schur-Cohn stable, or simply Schur stable, if all of its eigenvalues are inside the unitary disk.

³Namely, if K stabilizes the pair (A, B) , then κK is stabilizing feedback for any $\kappa \geq 1$ [27, 11].

3.2. Main result on robust linear synchronization

By relying on the result of Proposition 1, we now provide a set of constructive conditions for Theorem 1. These conditions will be generalized to the nonlinear systems framework in Section 4 and thus provide an essential intuition on the proposed synchronization strategy.

We consider a network of systems (7) and combine the results of Theorem 1 and Proposition 1 to design the state-feedback gain K inducing synchronization over general time-invariant graphs. To this end, define the following quantities:

$$\eta_i := \left(\frac{|\lambda_i|}{\Re(\lambda_i)} \right)^2 = 1 + \left(\frac{\Im(\lambda_i)}{\Re(\lambda_i)} \right)^2, \quad i = 2, \dots, N, \quad (16a)$$

$$\bar{\eta} := \max\{\eta_2, \dots, \eta_N\}, \quad \underline{\eta} := \min\{\eta_2, \dots, \eta_N\}, \quad (16b)$$

$$\bar{\lambda} := \max_{i \in \{2, \dots, N\}} \Re(\lambda_i), \quad \underline{\lambda} := \min_{i \in \{2, \dots, N\}} \Re(\lambda_i), \quad (16c)$$

for the non-zero eigenvalues λ_i , $i = 2, \dots, N$, of a Laplacian matrix L . Our MARI-based design is effective whenever the following inclusion holds for the graph-induced quantities (16) and the MARI parameters $\sigma \in (0, 1]$ and c as in (14) :

$$\frac{\bar{\eta}\sigma}{c} \in \left(0, 1 - \frac{(\bar{\eta}\bar{\lambda} - \underline{\eta}\underline{\lambda})^2}{(\bar{\eta}\bar{\lambda} + \underline{\eta}\underline{\lambda})^2} \right). \quad (17)$$

The next lemma establishes a useful implication of (17).

Lemma 2 *Consider the quantities $\underline{\eta}$, $\bar{\eta}$, $\underline{\lambda}$, $\bar{\lambda}$ in (16), c in (14) and let $\sigma \in (0, 1]$. The following interval of the real axis*

$$\mathcal{K} := \left[\frac{c - \sqrt{c(c - \bar{\eta}\sigma)}}{\underline{\eta}\underline{\lambda}}, \frac{c + \sqrt{c(c - \bar{\eta}\sigma)}}{\bar{\eta}\bar{\lambda}} \right] \quad (18)$$

is nonempty if and only if (17) holds.

The proof of Lemma 2 is given in Appendix A.2.

We are ready to present the main result on robust synchronization of linear systems.

Theorem 2 *Consider the network (7) and suppose that L is a Laplacian matrix describing a directed and weighted communication graph with a directed spanning tree. Let $R \in \mathbb{S}_{>0}^m$ and suppose that there exists $P \in \mathbb{S}_{>0}^n$ such that (9) holds for a selection of σ satisfying (17) with $\bar{\eta}$, $\underline{\eta}$, $\bar{\lambda}$, $\underline{\lambda}$ defined in (16). Then, the distributed control law u_i in (4), with $\alpha(x) = \kappa Kx$ and K selected as in (13), solves Problem 1 for any scalar gain $\kappa \in \mathcal{K}$ as defined in (18).*

The proof of Theorem 2 is presented in Appendix A.3.

We now highlight the importance of bounds (17) on σ in Proposition 2. First, differently from the continuous-time scenario [15], the bounds on the scalar gain κ depend on the imaginary part of the Laplacian eigenvalues via $\underline{\eta}$ and $\bar{\eta}$. This is expected, as discrete-time stability requires

the eigenvalues to lay inside the unit disc, thus imposing bounds on both their real and imaginary parts. Differently, continuous-time stability requires the eigenvalue to lay in the negative half-plane, which constraints their real parts only. Hence, in the case where there is at least one complex eigenvalue, definitions (16) imply $\bar{\eta} > 1$.

As a consequence, it is necessary that $\sigma < c\bar{\eta}^{-1}$ for a real solution to the square roots in (18) to exist, thus possibly excluding the DARI scenario $\sigma = 1$. In the case of real eigenvalues, $\eta_i = 1$ for all $i = 1, \dots, N$. Then, (17) and (18) simplify to

$$\frac{\sigma}{c} \in \left(0, 1 - \frac{(\bar{\lambda} - \underline{\lambda})^2}{(\bar{\lambda} + \underline{\lambda})^2} \right)$$

$$\kappa \in \left[\frac{c - \sqrt{c(c - \sigma)}}{\underline{\lambda}}, \frac{c + \sqrt{c(c - \sigma)}}{\bar{\lambda}} \right],$$

By (15), this last bound includes the results in [6], where all eigenvalues of L are supposed to be real. Finally, we highlight that smaller values of σ lead to robust synchronization over a broader range of graphs.

Remark 3 *We remark that in the continuous-time case, thanks to the infinite-gain margin property (see Remark 2), the communication graph imposes only a lower bound to the gain κ , see e.g. [15, 21].*

4. Synchronization of Nonlinear Systems

As shown in Section 3, the problem of synchronization can be interpreted as a simultaneous stabilization problem by means of an opportune change of coordinate. In order to extend the results of Theorem 1 to the nonlinear case, in this section, we first focus on the general properties a single nonlinear system has to fulfill. Then, we exploit these properties to recast the results of Section 3.2 to networks of nonlinear agents.

4.1. The case of single nonlinear discrete agents

We consider time-varying discrete-time nonlinear system of the form

$$x^+ = \varphi(x, \mathbf{t}) + w \quad (19)$$

where the function $\varphi : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^n$ is such that the following mild property holds.

Property 1 *Function $\varphi : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^n$ is continuous in its first argument and there exists a (possibly unbounded) set of matrices $\mathcal{D}\varphi \subset \mathbb{R}^{n \times n}$ such that, for each $x_a, x_b \in \mathbb{R}^n$ and each $\mathbf{t} \in \mathbb{N}$, there exists an integrable function $\psi : [0, 1] \rightarrow \mathbb{R}^{n \times n}$ satisfying*

$$\varphi(x_a, \mathbf{t}) - \varphi(x_b, \mathbf{t}) = \int_0^1 \psi(s) ds (x_a - x_b) \quad (20a)$$

$$\psi(s) \in \mathcal{D}\varphi, \quad \forall s \in [0, 1]. \quad (20b)$$

The above definition allows considering a wide class of dynamical systems. First, note that when $n = 1$, Property 1 boils down to the requirement of φ being absolutely continuous. Trivially, such a class of systems includes continuously differentiable ones with $\mathcal{D}\varphi$ containing all of their Jacobians. Moreover, Property 1 includes functions that are differentiable almost everywhere (i.e. everywhere but on a set of measure zero), such as piecewise smooth and Lipschitz functions. In this case, $\mathcal{D}\varphi$ contains all the possible Clarke generalized gradients [5]. As a particular case, for linear systems of the form (8), $\mathcal{D}\varphi = \{A\}$. When moving to nonlinear systems, this allows the inclusion of some useful nonlinearities, such as saturations and arctangents, by selecting $\mathcal{D}\varphi$ as the vertices of the convex hull of all possible Jacobians. Finally, in the time-varying scenario, the set $\mathcal{D}\varphi$ contains *all* possible generalized Jacobians of ψ for *all* times. Hence, this set is possibly unbounded and the function ψ does not depend on \mathbf{t} .

We now explore the design of stabilizers showing gain margin properties in the nonlinear framework by means of incremental input-to-state stability (δ ISS) arguments. Typically, δ ISS is obtained via incremental Lyapunov functions [31, 32, 1, 7, 9]. However, in [33, Theorem 15], the equivalence between uniform global exponentially δ ISS and global contractivity is shown for continuously differentiable discrete-time dynamics. Hence, we aim at exploiting contraction to obtain δ ISS.

4.2. Sufficient conditions for exponential δ ISS

While the first in-depth analysis of the relation between contraction, incremental stability and convergence in discrete-time appeared in [33], to the authors' knowledge the first results date back to [9]. We now generalize these existing results to the framework of non-smooth dynamics whose vector fields satisfy Property 1. Moreover, we extend recent advances of [7] to time-varying dynamics. We start by recalling the definition of δ ISS for discrete-time systems, see e.g. [1, 32, 7].

Definition 2 (Exponential δ ISS) *System (19) is globally uniformly Incrementally Input-to-State Stable with exponential convergence rate (exponentially δ ISS) if there exist $c, \gamma > 0$ and $\rho \in [0, 1)$ such that, for all $\mathbf{t} \geq \mathbf{t}_0$ with $\mathbf{t}_0 \in \mathbb{N}$ and for any initial states $x_1(\mathbf{t}_0), x_2(\mathbf{t}_0)$ and any pair of disturbance sequences $\mathbf{t} \mapsto w_1(\mathbf{t}), \mathbf{t} \mapsto w_2(\mathbf{t})$, the resulting solutions $x_1(\mathbf{t}), x_2(\mathbf{t})$ of (19) satisfy*

$$|x_1(\mathbf{t}) - x_2(\mathbf{t})| \leq c\rho^{\mathbf{t}-\mathbf{t}_0}|x_1(\mathbf{t}_0) - x_2(\mathbf{t}_0)| + \sup_{t \in [\mathbf{t}_0, \mathbf{t}]} \gamma|w_1(t) - w_2(t)|. \quad (21)$$

We now introduce a sufficient δ ISS condition that extends the results in [7, Theorem 2], [33, Theorem 14], [9, Theorem 6.1] to the case of time-varying non-smooth vector fields satisfying Property 1.

Lemma 3 *Consider system (19) and suppose that φ satisfies Property 1 with a specific set-valued map $\mathcal{D}\varphi$. Moreover, suppose that there exists $P \in \mathbb{S}_{>0}^n$ and $\rho \in (0, 1)$ satisfying*

$$J^\top P J \preceq \rho^2 P, \quad \forall J \in \mathcal{D}\varphi. \quad (22)$$

Then, system (19) is exponentially δ ISS according to Definition 2.

The proof of Lemma 3 is given in Appendix B.1.

Equipped with sufficient conditions for contraction of non-smooth dynamics, we conclude this subsection with some pedagogical examples, providing a useful insight into the applicability of the result.

Eexample 1 *Consider a system of the form*

$$x^+ = \varphi(x) = \text{sat}_r(Ax), \quad (23)$$

where $r \in \mathbb{R}_{>0}^n$ and the vector saturation function $\text{sat}(\cdot)$ has components $\text{sat}_i(\cdot) := \max(\min(\cdot, r_i), -r_i)$. It can be easily verified that, for all $x \in \mathbb{R}^n$, the generalized Jacobian of φ [5] satisfies

$$\partial\varphi(x) \subset \text{co}\{\Delta A, \Delta \in \Delta\},$$

where $\Delta := \{\Delta = \text{diag}(\delta_1, \dots, \delta_n) : \delta_i \in \{0, 1\}, \forall i = 1, \dots, n\}$ is a finite set of matrices representing the vertices of a polytope. More generally, let \mathcal{V} be a set of matrices $\mathcal{V} := \{A_1, \dots, A_v\}$ with $v \in \mathbb{N}$, such that $\partial\varphi(x) \in \text{co}\{\mathcal{V}\}$ for all $x \in \mathbb{R}^n$. Then, it suffices to verify (22) on \mathcal{V} and convexity of the equivalent formulation

$$\begin{pmatrix} \rho^2 P & J^\top \\ J & P^{-1} \end{pmatrix} \succeq 0$$

(obtained via a Schur complement) ensures that (22) holds for $\mathcal{D}\varphi = \text{co}\{\mathcal{V}\}$. Similar reasonings can be followed for smooth monotone saturation-like functions, such as arctangents or hyperbolic tangents.

Eexample 2 *The study of incremental stability properties of neural networks is gaining attention in the research community, e.g. [7, 2, 37]. The presented contraction analysis tools can be valuable to derive such properties. For example, a multilayer perceptron with $L \in \mathbb{N}$ layers and ReLU activation functions can be described by the following dynamics*

$$\begin{cases} x^+ = y_L \\ y_\ell = W_\ell \nu(y_{\ell-1}) + b_\ell, \quad \ell = 1, \dots, L \\ y_0 = x \end{cases} \quad (24)$$

with the ReLU function $\nu(\cdot)$ applied component-wise, i.e., $\nu(x)$ has components $\nu_i(x_i) := \max(0, x_i)$, $y_\ell, b_\ell \in \mathbb{R}^{n_\ell}$ and $W_\ell \in \mathbb{R}^{n_\ell \times n_{\ell-1}}$. Denoting by $x \mapsto \varphi(x)$ the function

satisfying $y_L = \varphi(x)$ recursively defined in (24), by the chain rule [5, Theorem 2.6.6], for all $x \in \mathbb{R}^n$, we have

$$\partial\varphi(x) \subset \text{co}\{W_L \Delta_{L-1} W_{L-1} \dots \Delta_1 W_1, \\ \Delta_i \in \mathbf{\Delta}, \forall i = 1, \dots, L-1\},$$

with $\mathbf{\Delta}$ defined as in Example 1. Proceeding as in Example 1, we can conclude exponential δ ISS properties of (24) by checking (26) on a set \mathcal{V} satisfying $\partial\varphi(x) \in \text{co}\{\mathcal{V}\}$ for all $x \in \mathbb{R}^n$. Similar results extend to more complex recurrent neural networks, as shown in [7].

4.3. Nonlinear robust feedback with gain margin design

Paralleling the linear derivation in Theorem 1, we now exploit the results of Lemma 3 to design a feedback stabilizers $u = \alpha(x, t)$ inducing exponential δ ISS (as defined in Definition 2) with respect to w and for a nonlinear system of the form

$$\dot{x}^+ = f(x, t) + Bu + w, \quad (25)$$

where $f : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^n$ satisfies Property 1 and B is full column rank. Again, the result we obtain establishes a gain margin property as defined below. We emphasize the similarity with (2), anticipating that this construction will be used in Section 4.4 to solve the robust nonlinear synchronization problem.

Definition 3 (Gain margin for nonlinear systems)

A function $\alpha : \mathbb{R}^n \times \mathbb{N} \mapsto \mathbb{R}^m$ is a δ ISS feedback with gain margin of center $c \in \mathbb{R}$ and radius $r > 0$ if, for any real number $\kappa \in [c - r, c + r]$, system (25) with $u = \kappa\alpha(x, t)$ is exponentially δ ISS with respect to w .

The following proposition can be seen as a non-smooth nonlinear counterpart of Proposition 1. We now state our first result for the nonlinear framework.

Proposition 2 Let $R \in \mathbb{S}_{\geq 0}^m$ and assume that f in (25) satisfies Property 1 for some $Df \subset \mathbb{R}^{n \times n}$. Moreover, suppose that there exists $P \in \mathbb{S}_{> 0}^n$ satisfying

$$J^\top Q J \preceq \rho^2 P, \quad \forall J \in Df, \quad (26a)$$

$$Q := P - \sigma P B (R + B^\top P B)^{-1} B^\top P, \quad (26b)$$

for some $\rho \in (0, 1)$ and $\sigma \in (0, 1]$. Then for the system (25) the function

$$u = \alpha(x, t) = -\kappa (R + B^\top P B)^{-1} B^\top P f(x, t) \quad (27)$$

is a δ ISS feedback with gain margin of center and radius

$$c = 1 + \frac{\lambda_{\min}(R)}{\lambda_{\max}(B^\top P B)}, \quad r = \sqrt{c(c - \sigma)}. \quad (28)$$

The proof of Proposition 2 can be found in Appendix B.2. Note that for linear dynamics, the feedback (27) coincides with (13). Similar to the linear case, taking into account the relation between the choice of R and the solution P

of (26) via the value c in (28) allows for improved gain margins. We also remark that condition (26) is *difficult to handle* for two main reasons: i) it represents an infinite set of inequalities, ii) it is nonlinear in the variable P . We postpone to Section 5 a discussion on possible workarounds to these issues.

Remark 4 Consider the role of the parameter σ in (26b). On the one hand, a strictly positive σ implies that the system can be made exponentially δ ISS with the addition of an input acting in the correct directions. On the other hand, $\sigma = 0$ implies that the autonomous system $\dot{x}^+ = f(x, t)$ is already contracting, while a negative σ would mean that the autonomous system is sufficiently robust to withstand inputs in the wrong directions. Then, the parameter σ in (26b) plays a similar role to the one in the MARI (9). As a consequence, inequality (26) can be interpreted as a nonlinear version of the MARI (9).

Remark 5 Our nonlinear δ ISS results parallel the linear ones on robust stabilization via MARI-based design. Indeed, mimicking the linear scenario, nonlinear feedbacks designed via Proposition 2 present a gain margin property, since any gain $\kappa \in [c - \sqrt{c(c - \sigma)}, c + \sqrt{c(c - \sigma)}]$ is a good controller for system (25). This parallels the continuous-time scenario in [11], where contractive laws based on Riccati-like design show infinite-gain margin properties. Finally, similarly to the MARI-based approach, the allowable range for κ is centered at $c \geq 1$ for any σ . Then, if $\sigma = 1$ and R is singular, the range degenerates to a single point $\kappa = 1$, and the gain margin property is lost.

Remark 6 Note that (26b) implies $Q \in \mathbb{S}_{> 0}^n$ if $R \in \mathbb{S}_{> 0}^m$. Indeed, by (26b) and (11) with $\mathcal{A} = P$, $\mathcal{B} = B$, $\mathcal{C} = B^\top$ and $\mathcal{D} = R$, the invertibility of R yields

$$Q = (1 - \sigma)P + \sigma(P^{-1} + BR^{-1}B^\top)^{-1}.$$

Since $\sigma \in (0, 1]$ and $P \succ 0$, matrix Q is a σ -governed linear interpolation between positive definite matrices and, thus, it is positive.

4.4. Main result on nonlinear synchronization

We now exploit the δ ISS results of Proposition 2 to derive a solution to the nonlinear multi-agent robust synchronization problem. The idea is to exploit contraction properties to show convergence to the synchronization manifold. The link between Proposition 1 and Proposition 2 is evident. However, the Jordan transformation which is the basis of the proof of Theorem 1 and, consequently, of Theorem 2 cannot be easily applied in the nonlinear scenario. Thus, in order to exploit Proposition 2 for synchronization, we introduce the following technical lemma, which will be used in the proof to design an appropriate transformation.

Lemma 4 Let the weighted graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ be directed, with Laplacian L and L_{11}, L_{12} defined as in (1).

Moreover, suppose \mathcal{G} has a directed spanning tree. Then, there exist $M \in \mathbb{S}_{>0}^{N-1}$ and constants $\underline{m}, \bar{m}, \underline{\mu}, \bar{\mu} > 0$, $\rho_M \in (0, 1]$ such that

$$\underline{m} \mathbf{I}_{N-1} \preceq M \preceq \bar{m} \mathbf{I}_{N-1}, \quad \rho_M \leq \frac{\underline{m}}{\bar{m}} \quad (29a)$$

$$M(L_{22} - \mathbf{1} L_{12}) + (L_{22} - \mathbf{1} L_{12})^\top M \succeq 2\underline{\mu} M \quad (29b)$$

$$(L_{22} - \mathbf{1} L_{12})^\top M (L_{22} - \mathbf{1} L_{12}) \preceq \bar{\mu}^2 M. \quad (29c)$$

The proof of Lemma 4 is given in Appendix B.3.

We now present the following main result on network synchronization of nonlinear systems.

Theorem 3 Consider the network (2) and suppose that f satisfies Property 1 for some $\mathcal{D}f \subset \mathbb{R}^{n \times n}$ and L is a Laplacian matrix describing a directed and weighted communication graph with a directed spanning tree. Let $\rho \in (0, 1)$ and $\sigma \in (0, 1]$ satisfy

$$\rho \leq \rho_M, \quad \sigma \leq \frac{c}{\varsigma}, \quad \varsigma := \left(\frac{\bar{\mu}}{\underline{\mu}} \right)^2 \quad (30)$$

with $\rho_M, \underline{\mu}, \bar{\mu}$ as in Lemma 4 and c as in (28). If, for some $R \in \mathbb{S}_{\geq 0}^m$, there exists $P \in \mathbb{S}_{>0}^n$ satisfying (26a), (26b), then, the distributed control law $u_i = \sum_{j=1}^N \ell_{ij} \alpha(x_j, t)$ in (4) with α defined as in (27) and κ satisfying

$$\kappa \in \left[\frac{c - \sqrt{c(c - \varsigma\sigma)}}{\varsigma \underline{\mu}}, \frac{c + \sqrt{c(c - \varsigma\sigma)}}{\varsigma \underline{\mu}} \right], \quad (31)$$

solves Problem 1 for the network (2), namely, (6) holds.

The proof of Theorem 3 can be found in Appendix B.4.

The difference between the allowed set of gains (31) and (18) arises from the Lyapunov-like conditions (29). Indeed, while Theorem 3 builds on Lemma 4, (18) follows from spectral arguments. Nevertheless, while more conservative, the matrix inequalities (29) can be related to spectral properties of L , as presented in the next remark.

Remark 7 Note that the contraction inequality (26a) in the context of Theorem 3 is tightly related to the structure of the Laplacian matrix and its eigenvalues. To appreciate this link, consider the network graph to be undirected and leader-connected. Under these conditions, the Laplacian L in (1) satisfies $L_{11} = 0$, $L_{12} = \mathbf{0}$. Then, $L_{22} = L_{22}^\top \succ 0$ and we can select $M = \mathbf{I}_{N-1}$ and $\underline{\mu}$ (resp. $\bar{\mu}$) in Lemma 4 as the smallest (resp. largest) eigenvalue of L_{22} , thus recovering a result similar to the linear scenario of Lemma 2. Consequently, the admissible values of σ are related to the condition number of L_{22} . Moreover, by picking $M = \mathbf{I}_{N-1}$, the contraction rate ρ disentangles from the network structure, as $\underline{m} = \bar{m} = 1$. Indeed, from (29a) we can select $\rho_M = 1$ and, consequently, condition (30) in Theorem 3 imposes no constraints on $\rho \in (0, 1)$, as in Lemma 3.

5. GEVPs for exponential δ ISS

We now discuss numerically efficient formulations of the results of Section 4, and apply them to specific classes of systems. LMI-based conditions for robust stabilization are a valuable tool for control design for discrete-time nonlinear systems, see e.g. [28, 41, 12, 18]. Hence, inspired by these works and recent LMI approaches for solving MARI inequalities [34, 29], we propose LMI-based conditions to obtain the solution of the MARI-like inequality (26). Our result provides a viable solution to the design problem of robustly synchronizing controllers.

First, we introduce an equivalent formulation for (26). The parameters of the proposed reformulation can be obtained by solving a generalized eigenvalue problem (GEVP). Then, we focus our attention on the case where the set of possible open-loop Jacobians of the system dynamics (25) is polytopic. Finally, we target the specific case of Lur'e systems.

5.1. Formulation as a GEVP

We start by reformulating Proposition 2 as an LMI problem. This provides convex analysis conditions for constructing matrix P . Given this new formulation, the convergence rate ρ and the parameter σ in (26) can be estimated as part of a GEVP.

Proposition 3 Let $\sigma \in (0, 1]$, $\rho \in (0, 1)$. The following LMIs in the decision variables W, Σ and U

$$W \succ 0, \quad \Sigma \succ 0, \quad U \succ 0, \quad \begin{pmatrix} \rho W & WJ^\top \\ JW & \rho \Sigma \end{pmatrix} \succeq 0, \quad (32a)$$

$$\begin{pmatrix} W + \sigma BUB^\top - \Sigma & BUB^\top \\ BUB^\top & \frac{1}{\sigma} \left(\frac{W}{1-\sigma} + BUB^\top \right) \end{pmatrix} \succeq 0, \quad (32b)$$

hold if and only if conditions (26) hold with $P = W^{-1} \succ 0$ and $R = U^{-1} \in \mathbb{S}_{>0}^m$. Moreover, (32) is a generalized eigenvalue problem in (σ, ρ) , namely, if it is feasible for $(\bar{\sigma}, \bar{\rho})$, then it is feasible for any (σ, ρ) such that $\sigma \geq \bar{\sigma}$, $\rho \geq \bar{\rho}$ and, conversely, if it is infeasible for $(\underline{\sigma}, \underline{\rho})$, then it is infeasible for any (σ, ρ) such that $\sigma \leq \underline{\sigma}$, $\rho \leq \underline{\rho}$.

The proof of Proposition 3 can be found in Appendix B.5.

Combined with Proposition 2, Proposition 3 requires the satisfaction of (32) for all $J \in \mathcal{D}f$. This may turn out to be impracticable, as $\mathcal{D}f$ could be infinite dimensional. However, under some additional assumptions on system (25), we can follow a polytopic approach similar to the one in Examples 1 and 2. Hence, we propose the following result addressing the case where the open-loop system Jacobian belongs to a polytopic set defined by a finite number of vertices.

Corollary 1 Let $R \in \mathbb{S}_{>0}^m$ and assume that f in (25) satisfies Property 1 for some $\mathcal{D}f \subset \mathbb{R}^{n \times n}$. Moreover, suppose there exists a finite set of matrices $\mathcal{V} := \{A_1, \dots, A_v\} \subset \mathbb{R}^{n \times n}$ such that $\mathcal{D}f \subseteq \text{co}\{\mathcal{V}\}$. If there exist matrices

$W, \Sigma \in \mathbb{S}_{>0}^n$ and scalars $\rho \in (0, 1)$, $\sigma \in (0, 1]$ satisfying (32) for all $J \in \mathcal{V}$, the control law $u = \alpha(x, t)$ with α defined in (27) and $P = W^{-1}$ makes the closed-loop exponentially δ ISS with respect to w with gain margin of center and radius (28).

5.2. Lur'e systems

We further specialize our result to the case of Lur'e systems. Namely, we now consider nonlinear discrete-time systems of the form

$$x^+ = f(x) + Bu = Ax + F\phi(Cx) + Bu, \quad (33)$$

where $C \in \mathbb{R}^{p \times n}$, $F \in \mathbb{R}^{n \times p}$ and the square nonlinearity $\phi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is a pool of p , possibly different, feedback nonlinear elements $\phi(y) := \text{diag}(\phi_1(y_1), \dots, \phi_p(y_p))$ whose components $\phi_i, i = 1, \dots, p$, satisfy Property 1 for some intervals $\mathcal{D}\phi_i \subset \mathbb{R}$, $i = 1, \dots, p$. We assume that each function ψ_i belongs to an incremental sector $[0, \omega_i]$, with $\omega_i \geq 0$, in the following classical sense:

$$(\phi_i(s_1) - \phi_i(s_2))(\phi_i(s_1) - \phi_i(s_2) - \omega_i(s_1 - s_2)) \leq 0, \quad (34)$$

for all $s_1, s_2 \in \mathbb{R}$. By the non-smooth mean value theorem [5, Theorem 2.3.7], we may combine bounds (34) into

$$2\text{Sym}\{J_\phi S(J_\phi - \Omega)\} \leq 0 \quad (35)$$

which holds for all diagonal $J_\phi \in \mathcal{D}\phi = \text{diag}(\mathcal{D}\phi_1, \dots, \mathcal{D}\phi_p)$, for any diagonal $S \in \mathbb{S}_{\geq 0}^p$ and for some diagonal $\Omega = \text{diag}(\omega_1, \dots, \omega_p) \in \mathbb{S}_{\geq 0}^p$. We then have the following result.

Proposition 4 *Let $R \in \mathbb{S}_{>0}^m$ and suppose that ϕ in (33) satisfies (34) for some $\Omega = \text{diag}(\omega_1, \dots, \omega_p)$. If there exist symmetric matrices W, Σ , a diagonal matrix $S \in \mathbb{S}_{\geq 0}^p$ and scalars $\rho \in (0, 1)$, $\sigma \in (0, 1]$ satisfying (32b) and*

$$W \succ 0, \quad \Sigma \succ 0, \quad \begin{pmatrix} \rho W & WA^\top & WC^\top \\ AW & \rho \Sigma & -F\Omega^\top S \\ CW & -S\Omega F^\top & 2S \end{pmatrix} \succeq 0, \quad (36)$$

then the control law $u = \alpha(x, t)$ with α defined in (27) and $P = W^{-1}$ makes the closed-loop (33)-(27) exponentially δ ISS with respect to w with gain margin of center and radius (28).

The proof of Proposition 4 can be found in Appendix B.6.

5.3. Numerical example

In what follows, we propose a simple numerical example to validate the results of Theorem 3 with the construction in Proposition 4. Consider a network of $N = 6$ agents connected according to the weighted, directed graph in Figure 1 and evolving according to the planar Lur'e dynamics (33) as follows

$$x_i^+ = f(x_i) + Bu_i = Ax_i + F\phi(Cx_i) + Bu_i, \quad i = 1, \dots, 6,$$

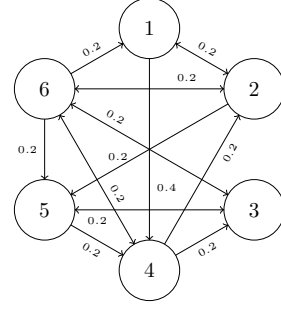


Figure 1: Communication graph considered in Section 5.3

where

$$A = \begin{pmatrix} 1.1 & 0.1 \\ -0.3 & 0.5 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}, \\ C = (1 \quad -1), \quad F^\top = (-0.1 \quad 0.6),$$

and $\phi(\cdot) = \text{sat}_3(\cdot) = \max(\min(\cdot, 3), -3)$. It is simple to verify that $J_\phi \in \{0, 1\}$ and the Laplacian matrix is

$$L = 0.2 * \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & 0 & -1 & 0 & -1 \\ 0 & 0 & 3 & -1 & -1 & -1 \\ -2 & 0 & 0 & 4 & -1 & -1 \\ 0 & -1 & -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 0 & -1 & 3 \end{pmatrix}.$$

Conditions (35), (36) and (32b) are solved with $\rho = \rho_M - 0.001 = 0.999$, $\sigma = \zeta^{-1} + 0.001 = 0.101$, $S = \Omega = 1$ and provide $U = 2.8981$ and

$$W = \begin{pmatrix} 0.0455 & -0.2400 \\ -0.2400 & 1.3805 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.0458 & -0.2396 \\ -0.2396 & 1.3807 \end{pmatrix}.$$

Then, we select a control law of the form (4) with α as in (27) with $\kappa = 0.3402$. As $c = 1.0205$, it can be verified that the recovered values satisfy (30) and (31). To show the robustness of the proposed approach, we simulate perturbed agents. The perturbed dynamics include unmodeled nonlinear time-varying components and noise, resulting in

$$x_i^+ = f(x_i) + Bu_i + 0.2 \sin(t) \sin(x_i) + w_i, \quad i = 1, \dots, 6,$$

with w_i sampled from a Gaussian distribution according to $w_i \sim \mathcal{N}(0, 0.25)$. The simulation results are shown in Figure 2. Robust exponential convergence to a non-trivial trajectory with initial conditions sampled from a uniform distribution $x_i(0) \in [-50, 50]$ is presented in Figure 2a and Figure 2b. As expected, Figure 2c shows an exponential decrease of the average error between the agents, which converges to a bounded value in the presence of additive noise.

6. Conclusions

In this paper, we explored the design of incrementally stabilizing feedbacks for discrete-time nonlinear systems

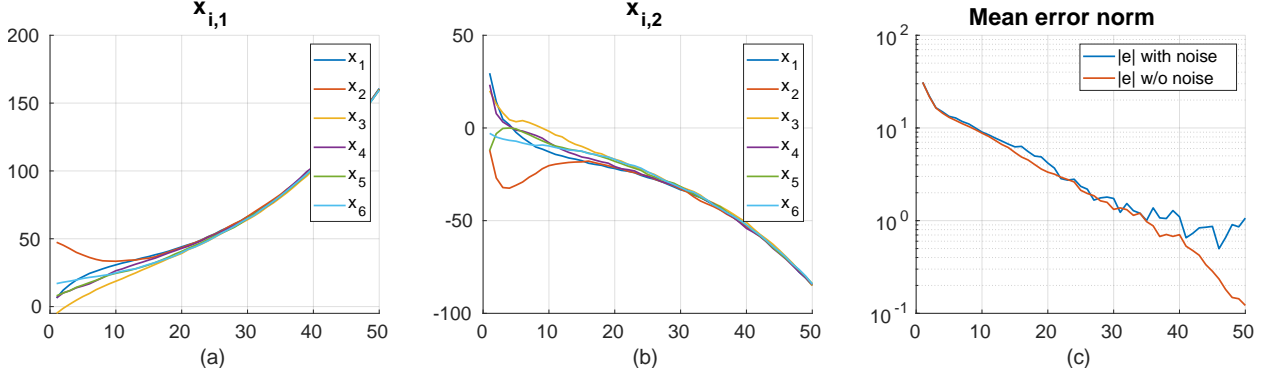


Figure 2: Trajectories of the perturbed systems: a-b) state components with noise. c) mean error wrt agent 1 with and without noise.

and their application to multiagent synchronization under generic connected communication graphs. Starting from the linear scenario, we provided constructive designs for robust stabilization and sufficient convex conditions for network synchronization. We exploited new contraction analysis results and focused on Euclidean metrics. The analysis is focused on input-linear systems. An interesting direction that will be the subject of future research is the generalization to different input-vector fields, by means of non-quadratic Lyapunov functions or Riemannian metrics. Also, extensions to include more precise robustness bounds or learning-based techniques can be envisioned to relax the assumption of model knowledge. Similarly, a complete analysis of the closed-loop behavior under the inclusion of discrete-time observers to relax full-state information assumptions will be the subject of future research.

Acknowledgement. We thank Mattia Giaccagli for providing the main steps of the proof of Lemma 4.

Appendix A. Proofs of the Linear results

Appendix A.1. Proof of Theorem 1

Sufficiency. Using the Kronecker notation, the closed-loop network dynamics can be written as

$$\mathbf{x}^+ = ((I_N \otimes A) + (L \otimes BK))\mathbf{x} + \mathbf{w}, \quad (\text{A.1})$$

with \mathbf{x} and \mathbf{w} defined in (3). To show convergence to the synchronization manifold \mathcal{M} , define a virtual leader z as the root node of the directed spanning tree.

Without loss of generality, assume $z := x_1$. Recalling the Laplacian structure (1), define $N-1$ error coordinates $\mathbf{e} := \text{col}(e_2, \dots, e_N) \in \mathbb{R}^{Nn}$ with $e_i := x_i - z$. Compactly, this reads as

$$\begin{pmatrix} z \\ \mathbf{e} \end{pmatrix} := (T \otimes I_n)\mathbf{x} = \begin{pmatrix} 1 & \mathbf{0}^\top \\ -\mathbf{1} & I_{N-1} \end{pmatrix} \otimes I_n \mathbf{x}, \quad (\text{A.2})$$

where we observe that $T^{-1} = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{1} & I_{N-1} \end{pmatrix}$ so that, according to the partitioning in (1), we have

$$\begin{pmatrix} 1 & \mathbf{0}^\top \\ -\mathbf{1} & I_{N-1} \end{pmatrix} L \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{1} & I_{N-1} \end{pmatrix} = \begin{pmatrix} 0 & L_{12} \\ \mathbf{0} & L_{22} - \mathbf{1} L_{12} \end{pmatrix}.$$

Exploiting the structure of T , we have the error dynamics

$$\begin{aligned} \mathbf{e}^+ &= \begin{pmatrix} \mathbf{0} & I_{N-1} \end{pmatrix} \\ &\times \left(((TT^{-1} \otimes A) + (TLT^{-1} \otimes BK)) \begin{pmatrix} z \\ \mathbf{e} \end{pmatrix} + (T \otimes I_n)\mathbf{w} \right) \\ &= A_{\text{cl}} \mathbf{e} + \tilde{\mathbf{w}}, \end{aligned} \quad (\text{A.3a})$$

where we defined the closed-loop matrix A_{cl} as

$$A_{\text{cl}} := (I_{N-1} \otimes A) + ((L_{22} - \mathbf{1} L_{12}) \otimes BK) \quad (\text{A.3b})$$

and $\tilde{\mathbf{w}} := \text{col}(\tilde{w}_2, \dots, \tilde{w}_N) \in \mathbb{R}^{Nn}$ with $\tilde{w}_i := w_i - w_1$. If A_{cl} is Schur-Cohn stable, the use of standard arguments for linear systems yields

$$|\mathbf{e}(t)| \leq c \rho^{t-t_0} |\mathbf{e}(t_0)| + \sup_{t \in [t_0, t]} \gamma |\tilde{\mathbf{w}}(t)|, \quad (\text{A.4})$$

for some $c, \gamma > 0$ and $\rho \in (0, 1)$. Since T in (A.2) has a bounded norm, we have, for some $c_1 > 0$,

$$\begin{aligned} |\mathbf{e}|^2 &= \left| \begin{pmatrix} z \\ \mathbf{e} \end{pmatrix} - \begin{pmatrix} z \\ \mathbf{0} \end{pmatrix} \right|^2 = \inf_{z^* \in \mathbb{R}^n} \left| \begin{pmatrix} z \\ \mathbf{e} \end{pmatrix} - \begin{pmatrix} z^* \\ \mathbf{0} \end{pmatrix} \right|^2 \\ &= \inf_{\mathbf{x}^* \in \mathcal{M}} |(T \otimes I_n)\mathbf{x} - (T \otimes I_n)\mathbf{x}^*|^2 \\ &\leq \inf_{\mathbf{x}^* \in \mathcal{M}} c_1^{-1} |\mathbf{x} - \mathbf{x}^*|^2 = c_1^{-1} |\mathbf{x}|_{\mathcal{M}}^2 \end{aligned}$$

Also, since T is invertible, for some $c_2 > 0$,

$$\begin{aligned} |\mathbf{x}|_{\mathcal{M}}^2 &= \inf_{\mathbf{x}^* \in \mathcal{M}} |\mathbf{x} - \mathbf{x}^*|^2 \\ &= \inf_{\mathbf{x}^* \in \mathcal{M}} |(T^{-1} \otimes I_n)(T \otimes I_n)(\mathbf{x} - \mathbf{x}^*)|^2 \\ &\leq \inf_{z^* \in \mathbb{R}^n} c_2 \left| \begin{pmatrix} z \\ \mathbf{e} \end{pmatrix} - \begin{pmatrix} z^* \\ \mathbf{0} \end{pmatrix} \right|^2 = c_2 |\mathbf{e}|^2. \end{aligned}$$

Then, we obtain the following relations

$$\sqrt{c_1} |\mathbf{e}| \leq |\mathbf{x}|_{\mathcal{M}} \leq \sqrt{c_2} |\mathbf{e}|, \quad |\tilde{\mathbf{w}}| \leq \sup_{i,j \in [1, N]} c_3 |w_i - w_j|, \quad (\text{A.5})$$

for some $c_1, c_2, c_3 > 0$. As a consequence, if A_{cl} is Schur-Cohn stable, one obtains robust synchronization as in

Problem 1. Therefore, in the rest of the proof we set $\tilde{\mathbf{w}} = 0$ and we show that A_{cl} in (A.3b) is Schur-Cohn stable. Let $T_J \in \mathbb{C}^{(N-1) \times (N-1)}$ be a transformation such that $\mathcal{L} = T_J(L_{22} - \mathbf{1}L_{12})T_J^{-1}$ is in Jordan canonical form. By defining the resulting closed-loop matrix after change of coordinates

$$\hat{A}_{\text{cl}} = (T_J \otimes \mathbf{I}_{N-1})A_{\text{cl}}(T_J^{-1} \otimes \mathbf{I}_{N-1}),$$

Schur-Cohn stability of \hat{A}_{cl} implies Schur-Cohn stability of A_{cl} . By the properties of the Kronecker product and (A.3b), we have

$$\hat{A}_{\text{cl}} = (\mathbf{I}_{N-1} \otimes A) + (\mathcal{L} \otimes BK). \quad (\text{A.6})$$

Since \mathcal{L} is in its Jordan form, the former matrix is block triangular with diagonal block equal to $(A + \lambda BK)$ with λ in $\text{spec } \mathcal{L}$.

Hence, Schur-Cohn stability of \hat{A}_{cl} holds if and only if the complex matrix $(A + \lambda BK)$ is Schur-Cohn stable for all $\lambda \in \text{spec } \mathcal{L}$. Due to the similarity transformations, $\text{spec } \mathcal{L} = \text{spec } L_{22} - \mathbf{1}L_{12} = \text{spec } L \setminus \{\lambda_1\}$, where $\lambda_1 = 0$ is the eigenvalue associated to the eigenvector $\mathbf{1}$. Consider now the case in item (i) where the graph has a directed spanning tree. Then, by Lemma 1, L has only one zero eigenvalue and the gain K is such that $(A + \lambda BK)$ is Schur-Cohn stable for all $\lambda \in \text{spec } \mathcal{L}$, which implies Schur-Cohn stability of A_{cl} in (A.3b). If instead A is Schur-Cohn in item (ii), then $(A + \lambda BK)$ is Schur-Cohn stable for all eigenvalues λ of L (including the zero ones) and A_{cl} in (A.3b) is exponentially stable.

Necessity. Consider a Laplacian matrix of the form (1). Following the lines of the sufficiency proof, the error dynamics between agents and a virtual leader are described by (A.3a). We first study the connectivity requirement. Suppose that synchronization is achieved, A is unstable and at least one agent is not connected. Without loss of generality, assume x_1 to be such a node. Since it is not connected, the Laplacian takes the form

$$L = \begin{pmatrix} 0 & \mathbf{0}^\top \\ \mathbf{0} & L' \end{pmatrix},$$

where L' is the Laplacian matrix of the connected portion of the graph. Then, by (A.3a) with $\tilde{\mathbf{w}} = 0$, we have

$$\mathbf{e}^+ = ((\mathbf{I}_N \otimes A) + (L' \otimes BK))\mathbf{e}.$$

Notice that L' describes a graph with a directed spanning tree. Then, by Lemma 1, it has one zero eigenvalue. By performing similar steps to the ones in the sufficiency proof, we define the transformed closed-loop matrix

$$\hat{A}_{\text{cl}} = (\mathbf{I}_{N-1} \otimes A) + (\mathcal{L}' \otimes BK),$$

where \mathcal{L}' is in Jordan form. Note that \hat{A}_{cl} is Schur-Cohn stable if and only if the complex matrix $(A + \lambda BK)$ is Schur-Cohn stable for all $\lambda \in \text{spec } \mathcal{L}'$. However, $\text{spec } \mathcal{L}'$ includes a zero eigenvalue. Hence, \hat{A}_{cl} is stable if and

only if A is Schur-Cohn stable, showing the first item by establishing contradiction.

We now prove the necessity of item (ii). If the agents are synchronized, the \mathbf{e} subsystem in (A.3a) is asymptotically stable and the matrix \hat{A}_{cl} in (A.6) is Schur-Cohn. Since \mathcal{L} contains all the nonzero eigenvalues of L and \hat{A}_{cl} is block-upper triangular, item (ii) must hold, and this concludes the proof.

Appendix A.2. Proof of Lemma 2

It is trivial that (17) implies $\bar{\eta}\sigma \leq c$ and then the square root in (2) is well-defined. To complete the proof, we show that the bound

$$\frac{c - \sqrt{c(c - \bar{\eta}\sigma)}}{\eta\underline{\lambda}} \leq \frac{c + \sqrt{c(c - \bar{\eta}\sigma)}}{\bar{\eta}\bar{\lambda}} \quad (\text{A.7})$$

holds if and only if (17) is satisfied, namely if and only if

$$\frac{\bar{\eta}\sigma}{c} \leq 1 - \frac{(\bar{\eta}\bar{\lambda} - \eta\underline{\lambda})^2}{(\bar{\eta}\bar{\lambda} + \eta\underline{\lambda})^2} = \frac{4\bar{\eta}\eta\bar{\lambda}\underline{\lambda}}{(\bar{\eta}\bar{\lambda} + \eta\underline{\lambda})^2} = \left(\frac{2\bar{\eta}\bar{\lambda}}{\bar{\eta}\bar{\lambda} + \eta\underline{\lambda}} \right)^2 \frac{\eta\underline{\lambda}}{\bar{\eta}\bar{\lambda}},$$

which, due to the positivity of the squared term, is equivalent to

$$\left(\frac{\bar{\eta}\bar{\lambda} + \eta\underline{\lambda}}{2\bar{\eta}\bar{\lambda}} \right)^2 \bar{\eta}\sigma - c \frac{\eta\underline{\lambda}}{\bar{\eta}\bar{\lambda}} \leq 0. \quad (\text{A.8})$$

Thus, we must show that (A.7) \iff (A.8). By the lower bound of (17), $\bar{\eta}\sigma > 0$. Then, multiplying (A.8) by $\bar{\eta}\sigma$ paired with addition and subtraction of $c(c - \bar{\eta}\sigma)$ at the right-hand side yields the equivalent inequality

$$\begin{aligned} c(c - \bar{\eta}\sigma) &\geq \left(\frac{\bar{\eta}\bar{\lambda} + \eta\underline{\lambda}}{2\bar{\eta}\bar{\lambda}} \right)^2 (\bar{\eta}\sigma)^2 + c^2 - \left(1 + \frac{\eta\underline{\lambda}}{\bar{\eta}\bar{\lambda}} \right) \bar{\eta}\sigma c \\ &\geq \left(c - \frac{\bar{\eta}\bar{\lambda} + \eta\underline{\lambda}}{2\bar{\eta}\bar{\lambda}} \bar{\eta}\sigma \right)^2. \end{aligned} \quad (\text{A.9})$$

By taking the square root, (A.9) is equivalent to

$$\sqrt{c(c - \bar{\eta}\sigma)} \geq c - \frac{\bar{\eta}\bar{\lambda} + \eta\underline{\lambda}}{2\bar{\eta}\bar{\lambda}} \bar{\eta}\sigma, \quad (\text{A.10})$$

where the right-hand side is non-negative because $\eta\underline{\lambda} \leq \bar{\eta}\bar{\lambda}$ and $\bar{\eta}\sigma \leq c$. Exploiting the expansion $c\bar{\eta}\sigma = (c - \sqrt{c(c - \bar{\eta}\sigma)})(c + \sqrt{c(c - \bar{\eta}\sigma)}) > 0$, inequality (A.10) is equivalent to

$$\begin{aligned} \frac{\eta\underline{\lambda}}{\bar{\eta}\bar{\lambda}} &\geq \left(c - \sqrt{c(c - \bar{\eta}\sigma)} \right) \frac{2}{\bar{\eta}\sigma} - 1 \\ &\geq \frac{c^2 + c(c - \bar{\eta}\sigma) - 2c\sqrt{c(c - \bar{\eta}\sigma)}}{c\bar{\eta}\sigma} \\ &\geq \frac{\left(c - \sqrt{c(c - \bar{\eta}\sigma)} \right)^2}{(c - \sqrt{c(c - \bar{\eta}\sigma)})(c + \sqrt{c(c - \bar{\eta}\sigma)})} \\ &\geq \frac{c - \sqrt{c(c - \bar{\eta}\sigma)}}{c + \sqrt{c(c - \bar{\eta}\sigma)}}, \end{aligned}$$

which coincides with (A.7), thus completing the proof.

Appendix A.3. Proof of Theorem 2

By Theorem 1, Problem 1 is solved (equivalently, (6) holds) if the matrices $(A + \lambda_i \kappa BK)$ are Schur-Cohn for all $\lambda_i \in \text{spec } L \setminus 0$. By Proposition 1, each one of these matrices is Schur-Cohn if

$|\kappa \lambda_i - c|^2 \leq c(c - \sigma)$. By expanding the norm, we conclude that the closed-loop matrix associated to λ_i is Schur-Cohn if $c\sigma - 2c\kappa \Re(\lambda_i) + \kappa^2 |\lambda_i|^2 \leq 0$. Solving for κ and recalling the definition of η_i in (16), we obtain robust synchronization if

$$\kappa \in \left[\frac{c - \sqrt{c(c - \eta_i \sigma)}}{\eta_i \Re(\lambda_i)}, \frac{c + \sqrt{c(c - \eta_i \sigma)}}{\eta_i \Re(\lambda_i)} \right] \quad \forall i = 2, \dots, N, \quad (\text{A.11})$$

because we simultaneously stabilize all the closed-loop matrices. First, note that from (16c) we have $\eta_i \geq 1$. Moreover, $\eta_i < \infty$, because all eigenvalues λ_i have positive real part. Then, since $\sigma > 0$ and $c \geq 1$, for any $i = 2, \dots, N$ it holds that

$$\frac{c - \sqrt{c(c - \bar{\eta} \sigma)}}{\bar{\eta} \lambda} \leq \frac{c - \sqrt{c(c - \eta_i \sigma)}}{\eta_i \Re(\lambda_i)} \leq \frac{c - \sqrt{c(c - \bar{\eta} \sigma)}}{\eta \lambda}.$$

Consequently, for any $\kappa \in \mathcal{K}$ as per (18), condition (A.11) holds and (6) holds, as to be proven.

Appendix B. Proofs of the nonlinear results

Appendix B.1. Proof of Lemma 3

Consider the candidate Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ defined as

$$V(x_1, x_2) := (x_1 - x_2)^\top P(x_1 - x_2),$$

for any two states $x_1, x_2 \in \mathbb{R}^n$. Given any selection of x_1, x_2 and $w_1, w_2 \in \mathbb{R}^n$, define function $\tilde{\Phi} : \mathbb{R} \rightarrow \mathbb{R}^n$ as

$$\tilde{\Phi}(s) = \varphi(sx_1 + (1-s)x_2, \mathbf{t}) + sw_1 + (1-s)w_2.$$

We have

$$x_1^+ - x_2^+ = \tilde{\Phi}(1) - \tilde{\Phi}(0) = \varphi(x_1, \mathbf{t}) - \varphi(x_2, \mathbf{t}) + w_1 - w_2.$$

In view of (20), we obtain

$$\begin{aligned} V^+ &= V(x_1^+, x_2^+) \\ &= (x_1^+ - x_2^+)^\top P \left[\int_0^1 \psi(s) ds (x_1 - x_2) + w_1 - w_2 \right] \end{aligned} \quad (\text{B.1})$$

for some $\psi(s) \in \mathcal{D}\varphi$, for all $s \in [0, 1]$. Then, adding and subtracting $\rho^2 V(x_1, x_2)$ and $V(x_1^+, x_2^+)$ to the right-hand

side of (B.1) yields

$$\begin{aligned} &V(x_1^+, x_2^+) - \rho^2 V(x_1, x_2) \\ &= 2(x_1^+ - x_2^+)^\top P \left[\int_0^1 \psi(s) ds (x_1 - x_2) + w_1 - w_2 \right] \\ &\quad - (x_1^+ - x_2^+)^\top P(x_1^+ - x_2^+) \int_0^1 ds \\ &\quad - \rho^2 (x_1 - x_2)^\top P(x_1 - x_2) \int_0^1 ds \\ &= \int_0^1 \xi^\top \Upsilon(s) \xi ds + 2(x_1^+ - x_2^+)^\top P(w_1 - w_2) \end{aligned}$$

where we defined $\xi := \text{col}(x_1 - x_2, x_1^+ - x_2^+)$ and

$$\Upsilon(s) := \begin{pmatrix} -\rho^2 P & \psi^\top(s)P \\ P\psi(s) & -P \end{pmatrix}.$$

Applying Schur complement and inequality (22) we obtain $\Upsilon(s) \preceq 0$ for all $s \in [0, 1]$. As a consequence, since $\rho \in (0, 1)$, we obtain

$$V(x_1^+, x_2^+) - \rho^2 V(x_1, x_2) \leq 2(x_1^+ - x_2^+)^\top P(w_1 - w_2).$$

By the generalized Young's inequality and by considering the decomposition $P = \sqrt{P}^\top \sqrt{P}$, we have

$$\begin{aligned} &2(x_1^+ - x_2^+)^\top P(w_1 - w_2) \leq \\ &(1 - \rho)V(x_1^+, x_2^+) + \frac{1}{1 - \rho}(w_1 - w_2)^\top P(w_1 - w_2). \end{aligned}$$

Then, by combining the previous inequalities we obtain

$$V(x_1^+, x_2^+) - \rho V(x_1, x_2) \leq \frac{|P|}{\rho(1 - \rho)} |w_1 - w_2|^2.$$

As $\rho \in (0, 1)$ and $P \succ 0$, the function V is a dissipation-form incremental Lyapunov function [32, Definition 7]. Then, the result follows by [32, Theorem 8]. Finally, by using standard arguments (i.e. [33, Theorem 14]) one can conclude the exponential behavior of solutions.

Appendix B.2. Proof of Proposition 2

For the sake of compactness, let us start by defining

$$Y = Y^\top := (R + B^\top P B)^{-1}, \quad \Omega = I_n - \kappa B Y B^\top P.$$

Since B is assumed to be full column rank, the matrix $R + B^\top P B$ is invertible, and Y exists. Then, Lemma 3 states that the closed-loop (25), (27), which can be written as (19) with

$$\varphi(x, \mathbf{t}) = f(x, \mathbf{t}) + B\alpha(x, \mathbf{t}) = \Omega f(x, \mathbf{t}), \quad (\text{B.2})$$

is exponentially δ ISS if

$$J^\top \Omega^\top P \Omega J \preceq \rho^2 P, \quad \forall J \in \mathcal{D}f. \quad (\text{B.3})$$

By expanding the left-hand side in (B.3) and by adding and subtracting $\sigma J^\top PBYB^\top PJ$, due to (B.3) we obtain the equality

$$J^\top \Omega^\top P \Omega J = J^\top P J - \sigma J^\top PBYB^\top PJ + (\sigma - 2\kappa)J^\top PBYB^\top PJ + \kappa^2 J^\top PBYB^\top PBYB^\top PJ,$$

where we note that the first two terms at the right-hand side coincide with $J^\top QJ$.

Note that, since $R \succeq 0$ and B is full rank, by selecting c as in (28), we have $Y^{-1} \succeq cB^\top PB$. Then inequality (26) implies, for all $J \in \mathcal{D}f$

$$\begin{aligned} J^\top \Omega^\top P \Omega J &\leq \rho^2 P + \\ &J^\top PBY ((\sigma - 2\kappa)Y^{-1} + \kappa^2 B^\top PB) YB^\top PJ \\ &\leq \rho^2 P + (c^{-1}\kappa^2 - 2\kappa + \sigma)J^\top PBYB^\top PJ. \end{aligned}$$

Since Y is positive definite, (B.3) holds and Lemma 3 applies if $\kappa^2 - 2c\kappa + c\sigma \leq 0$, which holds if and only if

$$c - \sqrt{c(c - \sigma)} \leq \kappa \leq c + \sqrt{c(c - \sigma)},$$

concluding the proof.

Appendix B.3. Proof of Lemma 4

Since the graph has a directed spanning tree, Lemma 1 ensures that the Laplacian L , as in (1), has one zero eigenvalue and $N - 1$ eigenvalues with positive real part. Consider the transformation

$$T = T^{-1} := \begin{pmatrix} 1 & 0 \\ \mathbf{1} & -\mathbf{I}_{N-1} \end{pmatrix}$$

and consider the change of coordinates on the Laplacian defined by

$$\mathcal{L} := TLT^{-1} = T \begin{pmatrix} 0 & -L_{12} \\ \mathbf{0} & -L_{22} \end{pmatrix} = \begin{pmatrix} 0 & -L_{12} \\ \mathbf{0} & L_{22} - \mathbf{1}L_{12} \end{pmatrix},$$

where we exploited $L\mathbf{1} = \mathbf{0}$ (see Section 2.1). Since T is full rank, by similarity transformation $\text{spec } \mathcal{L} = \text{spec } L$, namely, it has one zero eigenvalue and $N - 1$ eigenvalues with positive real part. Then, due to the block-triangular structure of \mathcal{L} , all the eigenvalues of $L_{22} - \mathbf{1}L_{12}$ have positive real part. Define $\tilde{L} := \mathbf{1}L_{12} - L_{22}$. Since all eigenvalues of \tilde{L} have negative real part, by the Lyapunov equation there exists $M \in \mathbb{S}_{>0}^{N-1}$ satisfying

$$M\tilde{L} + \tilde{L}^\top M = -\mathbf{I}_{N-1} \prec 0.$$

In turn, this implies $M(-\tilde{L}) + (-\tilde{L})^\top M \succ 0$. As a consequence, there exists a sufficiently small scalar $\underline{\mu} > 0$ such that (29b) holds. We now move to the other inequalities in Lemma 4. Since $M \in \mathbb{S}_{>0}^{N-1}$, (29a) and (29c) trivially hold with \underline{m} and \bar{m} being the smallest and largest eigenvalues of M respectively, $\rho_M = \underline{m}\bar{m}^{-1}$ and a sufficiently large $\bar{\mu}$.

Appendix B.4. Proof of Theorem 3

Mimicking the linear framework, we show convergence to the synchronization manifold \mathcal{M} by focusing our analysis on the error between agents. If these error dynamics are robustly stable (ISS) with respect to the perturbation signal generated by the errors between the different signals w_i , then Problem 1 is solved. Bearing in mind the steps of the proof of Theorem 1, without loss of generality, we define a virtual leader $z = x_1$ and define $N - 1$ error coordinates with respect to such a leader node, $\mathbf{e} := \text{col}(e_2, \dots, e_N) \in \mathbb{R}^{Nn}$ with $e_i := x_i - z$ for all $i = 1, \dots, N$. Similarly, we define the incremental disturbance $\tilde{\mathbf{w}} := \text{col}(\tilde{w}_2, \dots, \tilde{w}_N)$ with $\tilde{w}_i := w_i - w_1$. The error dynamics are described, for all $i = 2, \dots, N$, by

$$e_i^+ = f(z + e_i, \mathbf{t}) - f(z, \mathbf{t}) + B \sum_{j=1}^N (\ell_{ij} - \ell_{1j}) \alpha(z + e_j, \mathbf{t}) + \tilde{w}_i. \quad (\text{B.4})$$

Since by definition of the Laplacian entries $\sum_{j=1}^N \ell_{ij} = 0$ for any i , we can subtract $B \sum_{j=1}^N (\ell_{ij} - \ell_{1j}) \alpha(z, \mathbf{t}) = 0$ from the right-hand side so that (B.4) becomes

$$e_i^+ = \tilde{f}(e_i, \mathbf{t}) + B \sum_{j=2}^N \tilde{\ell}_{ij} \tilde{\alpha}(e_j, \mathbf{t}) + \tilde{w}_i$$

$$\tilde{f}(e_i, \mathbf{t}) := f(z + e_i, \mathbf{t}) - f(z, \mathbf{t}) \quad (\text{B.5})$$

$$\tilde{\alpha}(e_j, \mathbf{t}) := \alpha(z + e_j, \mathbf{t}) - \alpha(z, \mathbf{t})$$

$$\tilde{\ell}_{ij} := \ell_{ij} - \ell_{1j}.$$

Overall, the closed-loop system can be written in compact form as

$$\mathbf{e}^+ = \boldsymbol{\varphi}(\mathbf{e}, \mathbf{t}) + \tilde{\mathbf{w}}, \quad (\text{B.6})$$

where we defined

$$\boldsymbol{\varphi}(\mathbf{e}, \mathbf{t}) := \begin{pmatrix} \tilde{f}(e_2, \mathbf{t}) + B \sum_{j=2}^N \tilde{\ell}_{2j} \tilde{\alpha}(e_j, \mathbf{t}) \\ \vdots \\ \tilde{f}(e_N, \mathbf{t}) + B \sum_{j=2}^N \tilde{\ell}_{Nj} \tilde{\alpha}(e_j, \mathbf{t}) \end{pmatrix}. \quad (\text{B.7})$$

Now, select the following candidate Lyapunov function

$$V(\mathbf{e}) = \mathbf{e}^\top (M \otimes P) \mathbf{e}, \quad (\text{B.8})$$

with M defined in Lemma 4. Note that, due to the properties of the Kronecker product, since $M \succ 0$ and $P \succ 0$, $M \otimes P$ is symmetric and positive-definite. Now, for each value of $z = x_1$ and $\mathbf{e} = \text{col}(e_2, \dots, e_N)$, define the function $F_t : \mathbb{R} \rightarrow \mathbb{R}^{Nn}$ as

$$F_t(s) := \begin{pmatrix} \tilde{f}_s(s, e_2, \mathbf{t}) + B \sum_{j=2}^N \tilde{\ell}_{2j} \tilde{\alpha}_s(s, e_j, \mathbf{t}) \\ \vdots \\ \tilde{f}_s(s, e_N, \mathbf{t}) + B \sum_{j=2}^N \tilde{\ell}_{Nj} \tilde{\alpha}_s(s, e_j, \mathbf{t}) \end{pmatrix}, \quad (\text{B.9})$$

parametrized by $\mathbf{t} \in \mathbb{N}$, with the definitions

$$\tilde{f}_s(s, e_i, \mathbf{t}) := f(z + se_i, \mathbf{t}) - f(z, \mathbf{t}), \quad (\text{B.10a})$$

$$\tilde{\alpha}_s(s, e_j, \mathbf{t}) := \alpha(z + se_j, \mathbf{t}) - \alpha(z, \mathbf{t}), \quad (\text{B.10b})$$

$$= -\kappa YB^\top P \tilde{f}_s(s, e_i, \mathbf{t}),$$

where we used (27) and $Y := (R + B^\top PB)^{-1}$. From (B.9)-(B.10) we have $F_t(0) = 0$ and from (20) we get

$$\varphi(\mathbf{e}, \mathbf{t}) = F_t(1) = F_t(1) - F_t(0) = \int_0^1 \partial F(s) ds \mathbf{e}, \quad (\text{B.11})$$

where

$$\partial F(s) := [\mathbf{I}_{(N-1)n} - \kappa((L_{22} - \mathbf{1}L_{12}) \otimes BYB^\top P)]\Psi(s), \quad (\text{B.12})$$

$$\Psi(s) := \text{diag}(\tilde{\psi}_2(s) \dots, \tilde{\psi}_N(s)), \quad (\text{B.13})$$

$$\tilde{\psi}_i(s) \in \mathcal{D}f, \quad \forall i = 2, \dots, N \quad (\text{B.14})$$

are obtained from (B.9) by proceeding as in (A.3). Since $V(\mathbf{e}^+) = 2(\mathbf{e}^+)^\top (M \otimes P)\mathbf{e}^+ - V(\mathbf{e}^+)$, subtracting $\rho V(\mathbf{e})$ on both sides and combining (B.6) with (B.11), we obtain

$$\begin{aligned} V(\mathbf{e}^+) - \rho V(\mathbf{e}) &= 2(\mathbf{e}^+)^\top (M \otimes P) \int_0^1 \partial F(s) ds \mathbf{e} \\ &\quad - [(\mathbf{e}^+)^\top (M \otimes P)\mathbf{e}^+ + \rho \mathbf{e}^\top (M \otimes P)\mathbf{e}] \int_0^1 ds \\ &\quad + 2(\mathbf{e}^+)^\top (M \otimes P)\tilde{\mathbf{w}}. \end{aligned}$$

Then, by collecting everything under the integral and defining the extended error vector $\boldsymbol{\xi} = \text{col}(\mathbf{e}, \mathbf{e}^+)$ we obtain

$$V(\mathbf{e}^+) - \rho V(\mathbf{e}) = - \int_0^1 \boldsymbol{\xi}^\top \Upsilon(s) \boldsymbol{\xi} ds + 2(\mathbf{e}^+)^\top (M \otimes PB)\tilde{\mathbf{w}}, \quad (\text{B.15a})$$

$$\Upsilon(s) := \begin{pmatrix} \rho(M \otimes P) & -\partial F^\top(s)(M \otimes P) \\ -(M \otimes P)\partial F(s) & (M \otimes P) \end{pmatrix}. \quad (\text{B.15b})$$

Since $P \succ 0$, $M \succ 0$, and $M \otimes P$ is invertible, we can study the positive definiteness of $\Upsilon(s)$ via its Schur complement

$$\hat{\Upsilon}(s) = \rho(M \otimes P) - \partial F^\top(s)(M \otimes P)\partial F(s).$$

By using the definition of $\partial F(s)$ in (B.12) and the properties of Kronecker products, we obtain

$$\hat{\Upsilon}(s) = \rho(M \otimes P) - \Psi(s)^\top (T_a + 2\text{Sym}\{T_b\} + T_c)\Psi(s) \quad (\text{B.16})$$

where we defined

$$T_a := M \otimes P,$$

$$T_b := -\kappa(M(L_{22} - \mathbf{1}L_{12}) \otimes PBYB^\top P),$$

$$T_c := \kappa^2[(L_{22} - \mathbf{1}L_{12})^\top M(L_{22} - \mathbf{1}L_{12}) \otimes PBYB^\top PBYB^\top P].$$

For T_b , by the properties of the Kronecker product, since $PBYB^\top P$ is symmetric we obtain

$$2\text{Sym}\{T_b\} = -\kappa(2\text{Sym}\{M(L_{22} - \mathbf{1}L_{12})\} \otimes PBYB^\top P).$$

Consequently, by Lemma 4 and using again the properties of the Kronecker product, the following holds

$$2\text{Sym}\{T_b\} \preceq -2\kappa\mu(M \otimes PBYB^\top P). \quad (\text{B.17})$$

Similarly, by exploiting the Kronecker product and by using again Lemma 4, we get

$$T_c \preceq \kappa^2\bar{\mu}^2(M \otimes PBYB^\top PBYB^\top P). \quad (\text{B.18})$$

Note that, since $R \succeq 0$ and B is full rank, there always exists a scalar $c \geq 1$ such that $Y^{-1} \succeq cB^\top PB$, specifically c as in (28). Hence, bound (B.18) leads to

$$T_c \preceq \kappa^2\bar{\mu}^2c^{-1}(M \otimes PBYB^\top P). \quad (\text{B.19})$$

Using (B.17) and (B.19), matrix $\hat{\Upsilon}$ in (B.16) can be bounded as

$$\begin{aligned} \hat{\Upsilon}(s) &\succeq \rho(M \otimes P) - \Psi(s)^\top (M \otimes \bar{P})\Psi(s) \\ \bar{P} &= P + (c^{-1}\kappa^2\bar{\mu}^2 - 2\kappa\mu)PBYB^\top P \end{aligned} \quad (\text{B.20})$$

Now, consider \bar{P} . By addition and subtraction, it can be rewritten as

$$\bar{P} = P - \sigma PBYB^\top P + (\kappa^2\bar{\mu}^2c^{-1} - 2\kappa\mu + \sigma)PBYB^\top P.$$

Then, proceeding as in the proof of Theorem 2 (see equation (A.11)), if $\kappa^2\bar{\mu}^2 - 2\kappa\mu + \sigma \leq 0$, namely if

$$\frac{\mu}{\bar{\mu}^2} \left(c - \sqrt{c^2 - \frac{\bar{\mu}^2}{\mu^2}\sigma c} \right) \leq \kappa \leq \frac{\mu}{\bar{\mu}^2} \left(c + \sqrt{c^2 - \frac{\bar{\mu}^2}{\mu^2}\sigma c} \right),$$

which holds due to the selection in (31), we obtain

$$\bar{P} \preceq P - \sigma PBYB^\top P = Q, \quad (\text{B.21})$$

with Q defined in (26b). Using (29a) from Lemma 4 and (B.21), $\hat{\Upsilon}(s)$ in (B.20) satisfies

$$\hat{\Upsilon}(s) \succeq \underline{m}\rho(\mathbf{I}_{N-1} \otimes P) - \bar{m}\Psi(s)^\top (\mathbf{I}_{N-1} \otimes Q)\Psi(s).$$

Recalling from (B.13) the block-diagonal structure of $\Psi(s)$ and exploiting (26), we obtain

$$\begin{aligned} \hat{\Upsilon}(s) &\succeq \underline{m}\rho(\mathbf{I}_{N-1} \otimes P) - \bar{m} \text{diag}(\{\tilde{\psi}_i(s)^\top Q \tilde{\psi}_i(s)\}_{i=2}^N) \\ &\succeq \underline{m}\rho(\mathbf{I}_{N-1} \otimes P) - \bar{m} \text{diag}(\{\rho^2 P\}_{i=2}^N) \\ &= \underline{m}\rho(\mathbf{I}_{N-1} \otimes P) - \bar{m}\rho^2(\mathbf{I}_{N-1} \otimes P) \\ &\succeq \rho(\underline{m} - \bar{m}\rho)(\mathbf{I}_{N-1} \otimes P) \\ &\succeq \rho(\underline{m} - \bar{m}\rho_M)(\mathbf{I}_{N-1} \otimes P) \succeq 0, \end{aligned}$$

where we used $0 < \rho \leq \rho_M = \underline{m}\bar{m}^{-1}$. Since $\hat{\Upsilon}(s) \succeq 0$ for each $s \in [0, 1]$, we conclude that also $\Upsilon(s)$ defined in (B.15b) satisfies $\Upsilon(s) \succeq 0$ for all $s \in [0, 1]$, and (B.15a) implies

$$V(\mathbf{e}^+) - \rho V(\mathbf{e}) \leq 2(\mathbf{e}^+)^\top (M \otimes P)\tilde{\mathbf{w}} \quad (\text{B.22})$$

By the generalized Young's inequality and by considering the factorization $M \otimes P = \sqrt{M \otimes P}^\top \sqrt{M \otimes P} = (\sqrt{M \otimes P})^2$ (with $\sqrt{M \otimes P}$ denoting the unique positive square root of $M \otimes P \succ 0$), we have

$$\begin{aligned} 2(\mathbf{e}^+)^\top (M \otimes P)\tilde{\mathbf{w}} &= 2(\mathbf{e}^+)^\top (\sqrt{M \otimes P})^2 \tilde{\mathbf{w}} \\ &\leq (1 - \sqrt{\rho})V(\mathbf{e}^+) + \frac{1}{1 - \sqrt{\rho}} \tilde{\mathbf{w}}^\top (M \otimes P)\tilde{\mathbf{w}}. \end{aligned}$$

Then, since $\rho \in (0, 1)$, inequality (B.22) implies

$$V(\mathbf{e}^+) - \sqrt{\rho}V(\mathbf{e}) \leq \frac{1}{\sqrt{\rho}(1 - \sqrt{\rho})} \tilde{\mathbf{w}}^\top (M \otimes P) \tilde{\mathbf{w}},$$

thus proving exponential ISS properties of the \mathbf{e} dynamics due to the quadratic form of (B.8). Finally, similarly to the linear scenario of Theorem 1, relations (A.5) hold and robust synchronization as in Problem 1 is obtained, thus concluding the proof.

Appendix B.5. Proof of Proposition 3

Consider the last LMI of (32). Since $W \succ 0$ and $U \succ 0$, then its (2, 2) entry is positive definite. Then, by the Schur complement, (32b) holds if and only if

$$\begin{aligned} & W + \sigma BUB^\top - \Sigma \succeq \\ & \sigma(1 - \sigma)BUB^\top(W + (1 - \sigma)BUB^\top)^{-1}BUB^\top, \end{aligned}$$

which can be rearranged as

$$\begin{aligned} \Sigma \preceq & W + \sigma B(U - (1 - \sigma)UB^\top \\ & \times (W + (1 - \sigma)BUB^\top)^{-1}BU)B^\top. \end{aligned} \quad (\text{B.23})$$

By the matrix inversion lemma (11) with $\mathcal{A} = U^{-1}$, $\mathcal{B} = \sqrt{1 - \sigma}B^\top$, $\mathcal{C} = \sqrt{1 - \sigma}B$, $\mathcal{D} = W$, inequality (B.23) is equivalent to

$$\Sigma \preceq W + \sigma B(U^{-1} + (1 - \sigma)B^\top W^{-1}B)^{-1}B^\top.$$

Left and right multiplication of both sides by $W^{-1} \succ 0$ yields the equivalent condition

$$\begin{aligned} W^{-1}\Sigma W^{-1} \preceq & W^{-1} - \sigma W^{-1}B(-U^{-1} + B^\top W^{-1}B) \\ & + \sigma B^\top W^{-1}B)^{-1}B^\top W^{-1}. \end{aligned} \quad (\text{B.24})$$

Once again, by the matrix inversion lemma (11) applied with $\mathcal{A} = W$, $\mathcal{B} = \sqrt{\sigma}B$, $\mathcal{C} = \sqrt{\sigma}B^\top$, $\mathcal{D} = -(U^{-1} + B^\top W^{-1}B)$, inequality (B.24) is equivalent to

$$W^{-1}\Sigma W^{-1} \preceq (W - \sigma B(U^{-1} + B^\top W^{-1}B)^{-1}B^\top)^{-1}.$$

By left and right multiplying both sides by W , we obtain the equivalent inequality

$$\Sigma \preceq W(W - \sigma B(U^{-1} + B^\top W^{-1}B)^{-1}B^\top)^{-1}W. \quad (\text{B.25})$$

Since $(\mathcal{A}\mathcal{B}\mathcal{C})^{-1} = \mathcal{C}^{-1}\mathcal{B}^{-1}\mathcal{A}^{-1}$ for any invertible matrices \mathcal{A} , \mathcal{B} , \mathcal{C} , inequality (B.25) is equivalent to

$$\begin{aligned} \Sigma \preceq & (W^{-1}(W - \sigma B(U^{-1} + B^\top W^{-1}B)^{-1}B^\top)W^{-1})^{-1} \\ \preceq & (P - \sigma PB(R + B^\top PB)^{-1}B^\top P)^{-1} = Q^{-1}, \end{aligned} \quad (\text{B.26})$$

where we used $P = W^{-1}$, $R = U^{-1}$ and the definition of Q in (26b). Consider now the right LMI in (32a). By left and right multiplication by the matrix

$$T = \begin{pmatrix} W^{-1} & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & I_n \end{pmatrix},$$

we have

$$\begin{pmatrix} \rho P & J^\top \\ J & \rho \Sigma \end{pmatrix} = \begin{pmatrix} \rho W^{-1} & J^\top \\ J & \rho \Sigma \end{pmatrix} \succeq 0.$$

Then, by the Schur complement, we obtain the equivalent condition $\rho W^{-1} - \rho^{-1}J^\top \Sigma^{-1}J \succeq 0$, which, with the selection $P = W^{-1}$, can be written as

$$J^\top \Sigma^{-1}J \preceq \rho^2 P. \quad (\text{B.27})$$

Summarizing, we proved the equivalence of (32) with the four inequalities $P \succ 0$, $\Sigma \succ 0$, (B.26) and (B.27), where we emphasize that, under (32), Q^{-1} exists due to the positive definiteness of Q implied by $R \succ 0$, as established in Remark 6. More specifically, (32) is equivalent to

$$P \succ 0, \quad \Sigma \succ 0, \quad \Sigma^{-1} \succeq Q, \quad J^\top \Sigma^{-1}J \preceq \rho^2 P. \quad (\text{B.28})$$

To complete the first part of the proof, we show that (B.28) is equivalent to (26). If (26) holds, then $Q \succ 0$ and (B.28) holds with $\Sigma = Q^{-1}$. If (B.28) holds, then

$$J^\top QJ \preceq J^\top \Sigma^{-1}J \preceq \rho^2 P,$$

thus completing the first part of the proof. To prove that (32) is a GEVP in (σ, ρ, R) , let us denote by $\bar{W}, \bar{\Sigma}$ the solution of (32) with $(\bar{\sigma}, \bar{\rho}, \bar{R})$. If $\sigma \geq \bar{\sigma} > 0$, $0 \prec R \preceq \bar{R}$, we obtain

$$\begin{aligned} \bar{Q} &= \bar{P} - \bar{\sigma} \bar{P} B (\bar{R} + B^\top \bar{P} B)^{-1} B^\top \bar{P} \\ &\succeq \bar{P} - \sigma \bar{P} B (R + B^\top \bar{P} B)^{-1} B^\top \bar{P} = Q, \end{aligned}$$

where $\bar{P} = \bar{W}^{-1} \succ 0$. Then, by (B.28) and since $\bar{\Sigma}^{-1} \succeq \bar{Q}$ and $\rho \geq \bar{\rho}$, the following inequalities hold

$$\bar{P} \succ 0, \quad \bar{\Sigma} \succ 0, \quad \bar{\Sigma}^{-1} \succeq Q, \quad J^\top \bar{\Sigma}^{-1}J \preceq \bar{\rho}^2 \bar{P}.$$

Due to the equivalence between inequalities (B.28) and (32), we conclude that $P = \bar{P}$ is solution to (32) with (σ, ρ, R) . Similar reasoning proves infeasibility of (B.28) for any (σ, ρ, R) such that $\sigma \leq \underline{\sigma}$, $\rho \leq \underline{\rho}$, $R \succeq \underline{R}$ if (B.28) is infeasible for (σ, ρ, R) .

Appendix B.6. Proof of Proposition 4

Due to [5, Theorem 2.3.7], (35) holds for any diagonal $S \in \mathbb{S}_{\geq 0}^p$ and any $J_\phi \in \mathcal{D}\phi$. Define the matrix $\Lambda^\top := \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & FJ_\phi \end{pmatrix}$ with any diagonal $J_\phi \in \mathcal{D}\phi$. It is easy to verify that (35) implies

$$\Lambda^\top \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -F\Omega^\top S \\ 0 & -S\Omega F^\top & 2S \end{pmatrix} \Lambda = \Lambda^\top \Pi \Lambda \preceq 0. \quad (\text{B.29})$$

Consider now (36), which implies

$$\Lambda^\top \begin{pmatrix} \rho W & WA^\top & WC^\top \\ AW & \rho \Sigma & -F\Omega^\top S \\ CW & -S\Omega F^\top & 2S \end{pmatrix} \Lambda = \Lambda^\top (\Xi + \Pi) \Lambda \succeq 0,$$

where we defined

$$\Xi := \begin{pmatrix} \rho W & WA^\top & WC^\top \\ AW & \rho\Sigma & 0 \\ CW & 0 & 0 \end{pmatrix}. \quad (\text{B.30})$$

By (B.29), we then have

$$\Lambda^\top \Xi \Lambda \succeq \Lambda^\top (\Xi + \Pi) \Lambda \succeq 0,$$

thus showing $\Lambda^\top \Xi \Lambda \succeq 0$. Then, the expansion of the product leads to

$$\Lambda^\top \Xi \Lambda = \begin{pmatrix} \rho W & W(A + FJ_\phi C)^\top \\ (A + FJ_\phi C)W & \rho\Sigma \end{pmatrix} \succeq 0,$$

for all $J_\phi \in \mathcal{D}\phi$. By Proposition 3, the assumption (32b) implies that conditions (26) hold with $P = W^{-1} \succ 0$ for any $J \in \mathcal{D}f = A + F\mathcal{D}\phi C$. The proof is concluded by Proposition 2.

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