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***Abstract*— In this paper, we propose a novel distributed state-feedback design for robust synchronization of networks of identical discrete-time nonlinear agents under generic time-invariant communication graphs. We focus on the class of almost differentiable (possibly time-varying) dynamics that are linear in the input. By generalizing results on synchronization of linear agents, we build strong links between the solution to the synchronization problem in the linear and nonlinear framework. This is also enabled by the introduction of new results on design of incrementally stabilizing controllers based on contraction analysis. Finally, we propose numerically tractable sufficient conditions for the synchronization of networks of non-smooth Lur'e systems.**

***Index Terms*— Discrete-time, nonlinear systems, synchronization, multi-agent systems, contraction, incremental stability.**

I. Introduction

Multi-agent systems control has attracted a lot of attention from our community. Many modern control problems can be formalized as a network of interacting agents aiming at achieving some sort of agreement [1]. In this paper, we focus on the problem of state synchronization of a network of homogeneous systems (namely, all the agents are identical), described by a time-varying discrete-time nonlinear system that is linear in the control input.

The problem of synchronization presents mature results in the linear framework, especially for continuous-time agent dynamics [2], [3]. For discrete-time systems, major contributions can be found in [4]–[6]. However, in this setting, further investigations are still required to understand the relationship between the *structure* of a network (i.e., the eigenvalues and eigenvectors associated with its representation) and the possibility of finding a suitable synchronizing control. As a matter of fact, the structure of the communication graph has a significant impact on discrete-time networks compared to continuous-time ones and, while graph normalization approaches like [4], [7] have proven to be effective, they may not always be feasible. Additionally, existing solutions for nonlinear dynamics are limited to specific agent structures, such as Lur'e system forms [8] or linear systems with saturated inputs [9]. Moreover, they commonly employ observer design [10], [11] or data-based optimization techniques [12].

In this work, we propose solutions to the synchronization problem based on discrete-time contraction analysis [13]–[16] and incremental stability [8], [17]–[20]. There are two main motivations for this choice. Firstly, contraction analysis allows us to study nonlinear systems via linear systems-like arguments. Hence, we can take inspiration from the well-established linear systems literature [4] and provide a link between the two scenarios. Secondly, incremental stability easily translates to synchronization of homogeneous networks.

As a matter of fact, trajectories of incrementally stable dynamics "forget" their initial conditions, while offering strong robustness properties. In particular, in a network where the agents are homogeneous, each one can be considered as a singular trajectory of the same system starting from different initial conditions. As distances between trajectories of incrementally stable systems asymptotically decrease to zero, by designing a distributed controller making the network dynamics incrementally stable we indirectly obtain robust state synchronization. We remark that, differently from convergent systems as in [15], incremental stability does not require the final trajectory to remain bounded [13, Section 4.3]. The three main contributions of this paper can be summarized as follows:

(i) We provide new results on simultaneous stabilization and robust synchronization of identical discrete-time linear systems, arising from the solution of a modified Algebraic Riccati Inequality (MARI) [9], [21], [22]. In comparison with the results in [4], [5], [7], we do not rely on the normalized Laplacian. In practice, we assume that all the controllers have to be designed equally and no further local degree of freedom is available. In other words, we assume that the agents have access to aggregate information solely, without any knowledge about their neighbors in the graph nor their *degree*, which is the case in many applications, e.g. [23], [24]. Thus, we provide conditions that include normalized Laplacian as a subcase. In doing so, we obtain bounds linking the connectivity properties of the graph with the simultaneous stabilizability properties of the agents.

(ii) With respect to the works on incremental stability and contraction analysis [13], [16], [17], [25], we extend the current results to dynamics that are time-varying and not differentiable everywhere. We also propose quasi-convex conditions to tackle the feedback-design problem with numerically efficient methods. In addition, compared to [16], we build on these results to propose *closed-form* state-feedback controllers providing uniform exponential incremental stability properties for the closed-loop system. Recent works exploited control contraction metrics [26], [27] for designing controllers for input-affine nonlinear discrete-time systems based on optimization routines [28], [29]. However, we highlight that they do not provide a constructive design approach, as the solution relies on nested optimizations at each time-instant.

(iii) We exploit the new results on contraction analysis and incremental stability to provide new controller designs for robust synchronization of identical discrete-time nonlinear agents. This is done under the assumptions of almost differentiable dynamics, linear input vector field, and generic connected communication graphs (i.e., possibly directed and weighted). As in the linear case, we relate the connectivity properties of the graph to the simultaneous stabilizability properties of the agents.

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The paper is organized as follows. Section II introduces some preliminary concepts of graph theory, presents the problem of exponential synchronization, and further motivates the paper by highlighting important differences between the continuous and the discrete-time synchronization scenario. Section III presents the results for the linear framework. Section IV extends the solution to nonlinear agents. Finally, Section VI concludes the paper while the Appendix contains the proofs of some technical lemmas.

Notation. We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}_{\geq 0}$ the set of non-negative real numbers, by $\mathbb{N}$ the set of non-negative integers and by $\mathbb{C}$ the set of complex numbers. Given a complex number $z \in \mathbb{C}$, we use $\mathfrak{Re}(z)$ and $\mathfrak{Im}(z)$ to identify its real and imaginary parts respectively. $|x|_{\mathcal{M}}$ is a distance function between any point $x \in \mathbb{R}^n$ and a closed set $\mathcal{M} \subset \mathbb{R}^n$, namely, $|x|_{\mathcal{M}} := \inf_{z \in \mathcal{M}} |x - z|$. Given two symmetric matrices $A, B \in \mathbb{R}^{n \times n}$, we say $A \preceq B$ if the matrix $A - B$ is negative semidefinite. Similarly, we say $A \succ B$ if $A - B$ is positive definite. Moreover, we denote by $\mathbb{S}^n_{\succ 0}$ (resp. $\mathbb{S}^n_{\succeq 0}$) the set of symmetric positive definite (resp. semi-definite) real matrices of dimension n. Given a set of vectors $x_1, \ldots, x_N$ with $N \in \mathbb{N}$ and arbitrary dimensions, we define $\mathrm{col}(x_1, \ldots, x_N) = [x_1^\top \cdots x_N^\top]^\top$. Given a square matrix $A \in \mathbb{R}^{n \times n}$, we use $\mathrm{spec}(A)$ to represent its spectrum. Given a set of square matrices $A_1, \ldots, A_N$ with $N \in \mathbb{N}$ we identify by by $\mathrm{diag}(A_1, \ldots, A_N)$ the block diagonal matrix whose diagonal blocks are $A_1, \ldots, A_N$ respectively. Given a positive integer $n \in \mathbb{N}$, we identify by I_n the identity matrix of dimension n. We use $\mathbf{1}$, resp. $\mathbf{0}$, to identify a column vector of 1s, resp. 0s, of appropriate dimension. The symbol $\otimes$ denotes the Kronecker product, which satisfies $(A \otimes B)^\top = A^\top \otimes B^\top$, $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ and for A, B, C symmetric, $A \preceq B \implies (A \otimes C) \preceq (B \otimes C)$ and $(C \otimes A) \preceq (C \otimes B)$. For a square matrix $A \in \mathbb{R}^{n \times n}$, we define $\mathrm{He}\{A\} := A^\top + A$. Finally, we denote with **co** the convex hull.

II. Problem Statement

In this section, we first present some preliminary concepts related to graph theory and multi-agent networks. Then, we formulate the problem of exponential synchronization. Finally, we highlight the main differences between synchronization of continuous-time and discrete-time agents. As explained later, solvability of the discrete-time framework with a common control law requires extra conditions related to the communication graph.

A. Highlights on graph theory

In multi-agent systems, a communication graph is typically described by a triplet $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ where $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ is a set of $N \subset \mathbb{N}$ *vertices* (or nodes), $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of *edges* ϵ_{jh} modeling the interconnection between such nodes, and $\mathcal{A} \in \mathbb{R}^{N \times N}$ is the *adjacency matrix*, whose entries $a_{jh} \geq 0$ *weight* the flow of information from vertex j to vertex h. We denote by $L \in \mathbb{R}^{N \times N}$ the *Laplacian matrix* of the graph, defined as

$$\ell_{jh} = -a_{jh} \quad \text{for } j \neq h, \qquad \ell_{jh} = \sum_{i=1}^{N} a_{ji} \quad \text{for } j = h,$$

where ℓ_{jh} is the (j,h)-th entry of L. We denote with $\mathcal{N}_i$ the set of in-neighbors of node i, i.e. the set $\mathcal{N}_i := \{j \in \{1, \ldots, N\} \,|\, \epsilon_{ij} \in \mathcal{E}\}$. In this paper, we consider directed, weighted graphs. As such, the Laplacian matrix is not assumed to be diagonalizable and admits complex eigenvalues. We identify the Laplacian of the network as

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \tag{1}$$

where L_{11} is a scalar, L_{12} is a $N-1$ row vector, L_{21} is a $N-1$ column vector and L_{22} is a $(N-1) \times (N-1)$ matrix. We define connectivity following [30, Definition 5.1].

Definition 1 (Connected graph) *A graph $\mathcal{G}$ is connected if there is a node v such that, for any other node $v_k \in \mathcal{V} \setminus \{v\}$, there is a path from v to v_k.*

Based on this definition, we recall the following property from the literature, see e.g., [30]–[32].

Lemma 1 *A directed weighted graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is connected if and only if its Laplacian matrix L has only one trivial eigenvalue $\lambda_1 = 0$ and all other eigenvalues $\lambda_2, \ldots, \lambda_N$ have positive real parts.*

B. The problem of multi-agent synchronization

Consider a homogeneous network of discrete-time agents, where the dynamics of each node is described by a nonlinear, possibly time-varying, difference equation, linear in the input, of the form

$$x_i^+ = f(x_i, \mathfrak{t}) + Bu_i + w_i, \qquad i = 1, \ldots, N, \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$ is full column rank, $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $w_i \in \mathbb{R}^n$ represent the state, the control input and a disturbance affecting the dynamics of node i at timestep $\mathfrak{t} \in \mathbb{N}$, respectively, and $x_i^+ \in \mathbb{R}^n$ represents the state of node i at timestep $\mathfrak{t}+1$. We define the state of the entire network $\mathbf{x} \in \mathbb{R}^{Nn}$ and the entire disturbance $\mathbf{w} \in \mathbb{R}^{Nn}$ as

$$\mathbf{x} := \mathrm{col}(x_1, \ldots, x_N), \quad \mathbf{w} := \mathrm{col}(w_1, \ldots, w_N). \tag{3}$$

Our synchronization objective is to design a distributed feedback control law of the form

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij} \Big[\alpha(x_i, \mathfrak{t}) - \alpha(x_j, \mathfrak{t})\Big] = \sum_{j=1}^{N} \ell_{ij} \alpha(x_j, \mathfrak{t}) \tag{4}$$

for all $i = 1, \ldots, N$, for some function $\alpha : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m$, that stabilizes the dynamics (2) on the so-called *synchronization manifold* $\mathcal{M}$ defined as

$$\mathcal{M} := \{\mathbf{x} \in \mathbb{R}^{Nn} \mid x_i = x_j, \ \text{ for all } i, j \in \{1, \ldots, N\}\}, \tag{5}$$

where the states of all the agents of the network agree with each other. Furthermore, we require the control action u_i to be equal to zero on the synchronization manifold in the absence of disturbances, namely when $\mathbf{w} = 0$. In other words, when synchronization is achieved, no correction term is needed for each individual agent. As a consequence, independently stabilizing all the agents on a desired equilibrium point is not a valid solution in general. Note that the control law (4) satisfies this constraint by construction, due to the properties of the Laplacian matrix. We consider also a full-state information problem. Differently put, the i-th agent can use the complete state information x_j of its neighbors $j \in \mathcal{N}_i$ alongside its own local information x_i. We formalize our synchronization problem as follows.

Problem 1 (Robust network synchronization) *The distributed feedback control law* (4) *solves the robust synchronization problem for the network* (3) *if there exist a function $\alpha : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m$ and real numbers $c \geq 1$, $\rho \in (0,1)$ and $\gamma \geq 0$ such that, for any initial condition $(\mathbf{x}(\mathfrak{t}_0), \mathfrak{t}_0) \in \mathbb{R}^{Nn} \times \mathbb{N}$, and any disturbance sequences $\mathfrak{t} \mapsto w_i(\mathfrak{t})$, $i \in \{1, \ldots, N\}$, the solutions to the closed-loop system*

$$x_i^+ = f(x_i, \mathfrak{t}) + B \sum_{j=1}^{N} \ell_{ij} \alpha(x_j, \mathfrak{t}) + w_i, \quad i = 1, \ldots, N.$$

satisfy for all $\mathfrak{t} \geq \mathfrak{t}_0$

$$|\mathbf{x}(\mathfrak{t})|_{\mathcal{M}} \leq c\,\rho^{\mathfrak{t}-\mathfrak{t}_0}\,|\mathbf{x}(\mathfrak{t}_0)|_{\mathcal{M}} + \sup_{\substack{t\in[\mathfrak{t}_0,\,\mathfrak{t}]\\ i,j\in[1,N]}} \gamma|w_i(t)-w_j(t)|, \quad (6)$$

where $\mathcal{M}$ *is defined in* (5).

C. Linear synchronization and gain margin

In order to understand the challenges of Problem 1, we start from the linear scenario and recall a few fundamental synchronization results. Consider a network of $N \in \mathbb{N}$ continuous-time linear systems of the form

$$\dot{x}_i = Ax_i + Bu_i, \quad (7)$$

with A in $\mathbb{R}^{n\times n}$, B in $\mathbb{R}^{n\times m}$ and u_i, selected, according to (4), as a (linear) diffusive coupling of the form

$$u_i = K\sum_{j=1}^{N}\ell_{ij}x_j. \quad (8)$$

It is well known (see, e.g. [30]) that the solution to the synchronization problem only requires two assumptions: the stabilizabilty of the pair (A, B) and the connectivity of the graph. This can be shown with a a suitable change of coordinate recasting the synchronization problem as a stabilization one. As a matter of fact, the synchronization problem can be viewed as a simultaneous stabilization problem for the complex valued matrices

$$A + \lambda_i BK, \qquad i = 2,\dots,N \quad (9)$$

with λ_i being the (non-zero) eigenvalues of the Laplacian matrix L [30]. In continuous-time, the simultaneous stabilization of (9) via a unique gain K can be achieved by exploiting the so-called *infinite-gain margin property*[1] [33]. In few words, stabilization of the complex matrix $(A + \underline{\lambda}BK)$, with $\underline{\lambda}$ being the eigenvalue of L with the smallest (non-zero) real part, leads to the stabilization of all other complex matrices, and so to synchronization.

It is natural to hope that such a result can be extended to the discrete-time framework. Therefore, we consider a network of $N \in \mathbb{N}$ homogeneous agents whose dynamics can be described by a difference equation of the form

$$x_i^+ = Ax_i + Bu_i \quad (10)$$

with control input u_i defined as in (8). Similarly to the continuous-time framework, the discrete-time synchronization problem can be seen as the $N-1$ simultaneous stabilization problems of (9) (see [7]). However, in the discrete-time framework, the infinite gain margin property does not hold. Hence, the simultaneous stabilization of (9) may not be achievable for arbitrary λ_i, thus imposing conditions on the solvability of the synchronization problem, e.g. [4], [7]. In practical terms, stabilization of the pencil matrix associated to (9) in the discrete-time framework is achievable only for a set of eigenvalues whose norm lies inside a compact set.

A possible workaround to the aforementioned limitations is presented in [4], [5]. The authors consider a *normalized* Laplacian matrix, namely each row of the Laplacian matrix is differently normalized based on the *in-degree* of each node. Such a normalization allows containing the eigenvalues of L inside a suitable set where a solution to the $N-1$ simultaneous stabilization problems (9) exists.

In this paper, we aim at exploiting properties similar to the ones in continuous-time to provide general conditions on the spectrum of the Laplacian under which Problem 1 can be solved by controllers of the form (8). In the linear case, the resulting design is independent of any normalization, thus being prone for extensions to a broad class of networked problems, such as *open networks* [34] or *switching networks* [35]. Normalization-based conditions can be recovered by restricting the considered Laplacians to the set of normalized ones. Furthermore, the proposed approach can be extended to the nonlinear case.

[1]Namely, if K stabilizes the pair (A, B), then κK is stabilizing feedback for any $\kappa \geq 1$.

III. Synchronization of Linear Systems

Motivated by the limitation highlighted in Section II-C, we exploit simultaneous stabilization tools for solving the linear multi-agent synchronization problem. We revisit the results in [4] by providing a solution with non-normalized information exchange. Our solution allows for non-diagonalizable Laplacian matrices, thus recovering the results of [36] as a corollary. We start by presenting necessary and sufficient conditions for state synchronization. Then, we propose a Riccati-based design allowing for simultaneous stabilization under any Laplacian whose eigenvalues belong to a given compact set. We also show that such a design possesses gain margin properties. Finally, we specialize these results to the multi-agent framework.

A. Necessary and sufficient conditions

We start by presenting a general result for network synchronization for linear systems. It is shown that the existence of a common control law for systems associated with each non-zero eigenvalues of the Laplacian is equivalent to solving the synchronization problem. Hence, we extend [4, Lemma 1] to the case of general Laplacian matrices. While this section focuses on the full-state information case, we remark that it can be straightforwardly extended to the static output-feedback scenario.

Theorem 1 *For a network of* $N \in \mathbb{N}$ *agents described by dynamics*

$$x_i^+ = Ax_i + Bu_i + w_i \quad (11)$$

The diffusive control law u_i *defined in* (8) *solves Problem 1 if and only if*

(i) the interconnection graph (possibly directed, weighted) $\mathcal{G}$ *is connected or matrix* A *is Schur-Cohn stable*[2]*, and*

(ii) the gain K *is such that matrix* $(A+\lambda BK)$ *is Schur-Cohn stable for any* $\lambda \in spec(L) \setminus \{0\}$.

Proof: Sufficiency. Using the Kronecker notation, the closed-loop network dynamics can be written as

$$\mathbf{x}^+ = ((\mathrm{I}_N \otimes A) + (L \otimes BK))\mathbf{x} + \mathbf{w}, \quad (12)$$

with $\mathbf{x}$ and $\mathbf{w}$ defined in (3). To show convergence to the synchronization manifold $\mathcal{M}$, define a virtual leader as the node providing connectivity characterized in Definition 1. Without loss of generality, assume $z := x_1$ to be such a node. Recalling the Laplacian structure (1), define $N-1$ error coordinates $\mathbf{e} := \mathrm{col}(e_2,\dots,e_N) \in \mathbb{R}^{Nn}$ with $e_i := x_i - z$. Compactly, this reads as

$$\begin{pmatrix} z \\ \mathbf{e}\end{pmatrix} := (T\otimes \mathrm{I}_n)\mathbf{x} = \left(\begin{pmatrix} 1 & \mathbf{0}^\top \\ -\mathbf{1} & \mathrm{I}_{N-1}\end{pmatrix}\otimes \mathrm{I}_n\right)\mathbf{x},. \quad (13)$$

where we observe that $T^{-1} = \left(\begin{smallmatrix} 1 & \mathbf{0}^\top \\ \mathbf{1} & \mathrm{I}_{N-1}\end{smallmatrix}\right)$ so that, according to the partitioning in (1), we have

$$\begin{pmatrix} 1 & \mathbf{0}^\top \\ -\mathbf{1} & \mathrm{I}_{N-1}\end{pmatrix} L \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{1} & \mathrm{I}_{N-1}\end{pmatrix} = \begin{pmatrix} 0 & L_{12} \\ \mathbf{0} & L_{22} - \mathbf{1}\,L_{12}\end{pmatrix}.$$

[2]A matrix is said to be Schur-Cohn stable, or simply Schur stable, if all of its eigenvalues are inside the unitary disk.

Exploiting the structure of T, we obtain the error dynamics

$$\begin{aligned}\mathbf{e}^+ &= \begin{pmatrix}\mathbf{0} & \mathrm{I}_{N-1}\end{pmatrix} \\ &\times \left(((TT^{-1}\otimes A) + (TLT^{-1}\otimes BK))\begin{pmatrix} z \\ \mathbf{e}\end{pmatrix} + (T\otimes \mathrm{I}_n)\mathbf{w}\right) \\ &= A_{\mathrm{cl}}\,\mathbf{e} + \widetilde{\mathbf{w}}\,, \end{aligned} \tag{14a}$$

where we defined the closed-loop matrix A_{cl} as

$$A_{\mathrm{cl}} := (\mathrm{I}_{N-1}\otimes A) + ((L_{22} - \mathbf{1}\,L_{12})\otimes BK) \tag{14b}$$

and $\widetilde{\mathbf{w}} := \mathrm{col}(\widetilde{w}_2,\dots,\widetilde{w}_N) \in \mathbb{R}^{Nn}$ with $\widetilde{w}_i := w_i - w_1$. If A_{cl} is Schur-Cohn stable, the use of standard arguments for linear systems yields

$$|\mathbf{e}(\mathrm{t})| \le c\,\rho^{\mathrm{t}-\mathrm{t}_0}\,|\mathbf{e}(\mathrm{t}_0)| + \sup_{t\in[\mathrm{t}_0,\mathrm{t}]}\gamma|\widetilde{\mathbf{w}}(t)|, \tag{15}$$

for some $c,\gamma>0$ and $\rho\in(0,1)$. Since T in (13) has a bounded norm, we have, for some $c_1>0$,

$$\begin{aligned}|\mathbf{e}|^2 &= \left|\begin{pmatrix} z \\ \mathbf{e}\end{pmatrix} - \begin{pmatrix} z \\ \mathbf{0}\end{pmatrix}\right|^2 = \inf_{z^\star\in\mathbb{R}^n}\left|\begin{pmatrix} z \\ \mathbf{e}\end{pmatrix} - \begin{pmatrix} z^\star \\ \mathbf{0}\end{pmatrix}\right|^2 \\ &= \inf_{\mathbf{x}^\star\in\mathcal{M}}|(T\otimes\mathrm{I}_n)\mathbf{x} - (T\otimes\mathrm{I}_n)\mathbf{x}^\star|^2 \\ &\le \inf_{\mathbf{x}^\star\in\mathcal{M}}|(T\otimes\mathrm{I}_n)|^2|\mathbf{x}-\mathbf{x}^\star|^2 \\ &\le \inf_{\mathbf{x}^\star\in\mathcal{M}} c_1^{-1}|\mathbf{x}-\mathbf{x}^\star|^2 = c_1^{-1}|\mathbf{x}|_{\mathcal{M}}^2\end{aligned}$$

Also, since T is invertible, for some $c_2>0$,

$$\begin{aligned}|\mathbf{x}|_{\mathcal{M}}^2 &= \inf_{\mathbf{x}^\star\in\mathcal{M}}|\mathbf{x}-\mathbf{x}^\star|^2 = \inf_{\mathbf{x}^\star\in\mathcal{M}}|(T^{-1}\otimes\mathrm{I}_n)(T\otimes\mathrm{I}_n)(\mathbf{x}-\mathbf{x}^\star)|^2 \\ &\le \inf_{\mathbf{x}^\star\in\mathcal{M}} c_2|(T\otimes\mathrm{I}_n)\mathbf{x} - (T\otimes\mathrm{I}_n)\mathbf{x}^\star|^2 \\ &\le \inf_{z^\star\in\mathbb{R}^n} c_2\left|\begin{pmatrix} z \\ \mathbf{e}\end{pmatrix} - \begin{pmatrix} z^\star \\ \mathbf{0}\end{pmatrix}\right|^2 = c_2|\mathbf{e}|^2.\end{aligned}$$

Then, we obtain the following relations

$$\sqrt{c_1}|\mathbf{e}| \le |\mathbf{x}|_{\mathcal{M}} \le \sqrt{c_2}|\mathbf{e}|, \quad |\widetilde{\mathbf{w}}| \le \sup_{i,j\in[1,N]} c_3|w_i - w_j|, \tag{16}$$

for some $c_1,c_2,c_3>0$. As a consequence, if A_{cl} is Schur-Cohn stable, one obtains robust synchronization as in Problem 1. Therefore, in the rest of the proof we set $\widetilde{\mathbf{w}}=0$ and we show that A_{cl} in (14b) is Schur-Cohn stable. Let $T_J\in\mathbb{C}^{(N-1)\times(N-1)}$ be a transformation such that $\mathcal{L} = T_J(L_{22}-\mathbf{1}\,L_{12})T_J^{-1}$ is in Jordan canonical form. By defining the resulting closed-loop matrix after change of coordinates

$$\widehat{A}_{\mathrm{cl}} = (T_J\otimes\mathrm{I}_{N-1})A_{\mathrm{cl}}(T_J^{-1}\otimes\mathrm{I}_{N-1}),$$

Schur-Cohn stability of $\widehat{A}_{\mathrm{cl}}$ implies Schur-Cohn stability of A_{cl}. By the properties of the Kronecker product and (14b), we have

$$\widehat{A}_{\mathrm{cl}} = (\mathrm{I}_{N-1}\otimes A) + (\mathcal{L}\otimes BK). \tag{17}$$

Since $\mathcal{L}$ is in its Jordan form, the former matrix is block triangular with diagonal block equal to $(A+\lambda BK)$ with λ in $\mathrm{spec}(\mathcal{L})$.

Hence, Schur-Cohn stability of $\widehat{A}_{\mathrm{cl}}$ holds if and only if the complex matrix $(A+\lambda BK)$ is Schur-Cohn stable for all $\lambda\in\mathrm{spec}(\mathcal{L})$. Due to the similarity transformations,

$$\mathrm{spec}(\mathcal{L}) = \mathrm{spec}(L_{22}-\mathbf{1}\,L_{12}) = \mathrm{spec}(L)\setminus\{\lambda_1\}\,,$$

where $\lambda_1=0$ is the eigenvalue associated to the eigenvector $\mathbf{1}$. Consider now the case in item *(i)* where the graph is connected. Then, by Lemma 1, L has only one zero eigenvalue and the gain K is such that $(A+\lambda BK)$ is Schur-Cohn stable for all $\lambda\in\mathrm{spec}(\mathcal{L})$, which implies Schur-Cohn stability of A_{cl} in (14b). If instead A is Schur-Cohn in item *(i)*, then $(A+\lambda BK)$ is Schur-Cohn stable for all eigenvalues λ of L (including the zero ones) and A_{cl} in (14b) is exponentially stable.

Necessity. Consider a Laplacian matrix of the form (1). Following the lines of the sufficiency proof, the error dynamics between agents and a virtual leader are described by (14a). We first study the connectivity requirement. Suppose that synchronization is achieved, A is unstable and at least one agent is not connected. Without loss of generality, assume x_1 to be such a node. Since it is not connected, the Laplacian takes the form

$$L = \begin{pmatrix} 0 & \mathbf{0}^\top \\ \mathbf{0} & L'\end{pmatrix},$$

where L' is the Laplacian matrix of the connected portion of the graph. Then, by (14a) with $\widetilde{\mathbf{w}}=0$, we have

$$\mathbf{e}^+ = ((\mathrm{I}_N\otimes A) + (L'\otimes BK))\mathbf{e}\,.$$

Notice that L' describes a connected graph. Then, by Lemma 1, it has one zero eigenvalue. By performing similar steps to the ones in the sufficiency proof, we define the transformed closed-loop matrix

$$\widehat{A}_{\mathrm{cl}} = (\mathrm{I}_{N-1}\otimes A) + (\mathcal{L}'\otimes BK),$$

where $\mathcal{L}'$ is in Jordan form. Note that $\widehat{A}_{\mathrm{cl}}$ is Schur-Cohn stable if and only if the complex matrix $(A+\lambda BK)$ is Schur-Cohn stable for all $\lambda\in\mathrm{spec}(\mathcal{L}')$. However, $\mathrm{spec}(\mathcal{L}')$ includes a zero eigenvalue. Hence, $\widehat{A}_{\mathrm{cl}}$ is stable if and only if A is Schur-Cohn stable, showing the first item by establishing contradiction.

We now prove the necessity of item *(ii)*. If the agents are synchronized, the $\mathbf{e}$ subsystem in (14a) is asymptotically stable and the matrix $\widehat{A}_{\mathrm{cl}}$ in (17) is Schur-Cohn. Since $\mathcal{L}$ contains all the nonzero eigenvalues of L and $\widehat{A}_{\mathrm{cl}}$ is block-upper triangular, item *(ii)* must hold, and this concludes the proof. ■

B. Gain margin computation via Riccati design

The results presented in Section III-A are not constructive. Following a similar approach to the one in [4], in this section we provide a design procedure for pencil matrices stabilization, namely we consider a single discrete-time agent. Then, in Section III-C, we will apply this result to the case of networks.

Consider a discrete-time linear system described by

$$x^+ = Ax + Bu\,, \tag{18}$$

with $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$ and, without loss of generality, B is assumed to be full-column rank. The goal is to find a gain matrix $K\in\mathbb{R}^{m\times n}$ such that a state feedback control law $u=Kx$ makes the complex closed-loop matrix $(A+\lambda BK)$ Schur-Cohn for some complex numbers λ. Inspired by [33, Definition 3.13], we formally define this notion as follows.

Definition 2 (Complex gain margin for linear systems) *The matrix K is said to have a complex gain margin with radius $r>0$ if $A+\lambda BK$ is Schur-Cohn for any λ in $\{\lambda\in\mathbb{C}:|\lambda-1|\le r\}$.*

To find the complex gain margin of a matrix K, we propose a solution based on the discrete-time Modified Algebraic Riccati Inequality (MARI) [9], [21], [22] defined as

$$A^\top PA - \sigma A^\top PB(R+B^\top PB)^{-1}B^\top PA \preceq \rho P\,, \tag{19}$$

where $R\in\mathbb{S}^m_{\succeq 0}$, $P\in\mathbb{S}^n_{\succ 0}$ and generally $\sigma\in(0,1]$, $\rho\in(0,1)$. Note that, since B is assumed to be full column rank, the matrix $R+B^\top PB$ is positive definite and, consequently, invertible. The main

difference between the MARI (19) and the more common discrete-time Algebraic Riccati Inequality (DARI)

$$A^\top PA - A^\top PB(R + B^\top PB)^{-1}B^\top PA + Q \prec P \qquad (20)$$

lies in the presence of the scalar σ. First, note that if $R \succ 0$, then the positive semi-definite matrix Q can be embedded in the right-hand side of (19) by exploiting $\rho P \prec P - Q$, which holds for a suitable $\rho \in (0, 1)$ as long as $P - Q \succ 0$. Inequality $P - Q \succ 0$ holds when $R \in \mathbb{S}^m_{\succ 0}$ because one can rearrange (20) as

$$A^\top [P - PB(R + B^\top PB)^{-1}B^\top P]A \prec P - Q$$

and applying the block matrix inversion identity

$$(\mathcal{A} + \mathcal{B}\mathcal{D}^{-1}\mathcal{C})^{-1} = \mathcal{A}^{-1} - \mathcal{A}^{-1}\mathcal{B}(\mathcal{D} + \mathcal{C}\mathcal{A}^{-1}\mathcal{B})^{-1}\mathcal{C}\mathcal{A}^{-1}\,, \quad (21)$$

with $\mathcal{A}^{-1} = P$, $\mathcal{D} = R$, $\mathcal{C} = B^\top$, $\mathcal{B} = B$, yields to

$$0 \prec A^\top (P^{-1} + BR^{-1}B^\top)^{-1}A \prec P - Q\,. \qquad (22)$$

Due to the discussion above, the DARI (20) is a special case of the MARI (19) when $\sigma = 1$ and R is positive definite. As such, the MARI allows for an extra degree of freedom. Its role is to weigh the impact of the input on the solution to the inequality. In other words, the smaller the σ, the less we can rely on the input to stabilize the system. This is evident for the special case $\sigma = 0$, where the MARI boils down to the Lyapunov inequality for autonomous discrete-time systems.

We show next that the degree of freedom offered by the MARI (19) allows stating sufficient conditions for the existence of a state feedback gain K solving the simultaneous stabilization problem. This result reinterprets the findings of [9, Theorem 1].

Proposition 1 *Let the pair (A, B) be stabilizable and $R \in \mathbb{S}^m_{\succ 0}$. Let $P \in \mathbb{S}^n_{\succ 0}$ be a solution to the MARI (19) for some $\sigma \in (0, 1]$ and for some $\rho \in (0, 1)$. Then the matrix*

$$K = -(R + B^\top PB)^{-1}B^\top PA\,, \qquad (23)$$

has a complex gain margin with radius $r = \sqrt{1 - \sigma}$.

Proof: Consider the closed-loop matrix $(A + \lambda BK)$ for some arbitrary $\lambda \in \mathbb{C}$ and define the following matrix

$$\Gamma := (A^\top + \lambda^* K^\top B^\top)P(A + \lambda BK), \qquad (24)$$

with P solution to (19) and superscript $*$ denoting the complex conjugate. Substituting K in (23) into (24) and by defining $Y := Y^\top = (R + B^\top PB)^{-1}$ for the sake of compactness, we obtain

$$\Gamma = (A^\top - \lambda^* A^\top PBYB^\top)P(A - \lambda BYB^\top PA).$$

Expanding the product and adding and subtracting $\sigma A^\top PBYB^\top PA$, by virtue of (19) and using $\lambda\lambda^* = |\lambda|^2$, $\lambda + \lambda^* = 2\mathfrak{Re}(\lambda)$, we obtain

$$\Gamma \preceq \rho P + A^\top PBY((\sigma - 2\mathfrak{Re}(\lambda))\,\mathrm{I}_n + |\lambda|^2 B^\top PBY)B^\top PA\,.$$

Since $R \succeq 0$, by the definition of Y we obtain $YB^\top PBY \preceq YY^{-1}Y \preceq Y$ which implies

$$\Gamma \preceq \rho P + (\sigma - 2\mathfrak{Re}(\lambda) + |\lambda|^2)A^\top PBYB^\top PA.$$

Since $Y \succ 0$, the second term at the right-hand side is negative semidefinite if $\sigma - 2\mathfrak{Re}(\lambda) + |\lambda|^2 \leq 0$. Recall that $|\lambda|^2 = \mathfrak{Re}(\lambda)^2 + \mathfrak{Im}(\lambda)^2$ and define the real scalars $\lambda_R := \mathfrak{Re}(\lambda) - 1$, $\lambda_I := \mathfrak{Im}(\lambda)$. From the previous inequality, $\Gamma \preceq \rho P$ if

$$\sigma - 2(\lambda_R + 1) + (\lambda_R + 1)^2 + \lambda_I^2 \leq 0 \iff \lambda_R^2 + \lambda_I^2 \leq 1 - \sigma\,.$$

By the definition of λ_R and λ_I, these inequalities characterize the circle of radius $r = \sqrt{1 - \sigma}$ centered at the point $c = (1, 0)$ of the complex plane. Therefore, if $|\lambda - 1| \leq r$ we have $\Gamma \preceq \rho P$ and consequently

$$(A^\top + \lambda^* K^\top B^\top)P(A + \lambda BK) - P \preceq -(1 - \rho)P. \qquad (25)$$

By [37, Theorem 3.2], since $P \in \mathbb{S}^n_{\succ 0}$, the closed-loop matrix $(A + \lambda BK)$ is Schur-Cohn stable, which concludes the proof. ■

Remark 1 *Since the DARI can be seen as a special case of the MARI with $\sigma = 1$, the gain margin radius r degenerates to zero. This drastically reduces the set of simultaneously stabilizable matrices.*

Remark 2 *We remark that in [4] the use of the MARI is discouraged, as it is stated that no standard algorithm exists to provide a solution. Also, the authors state that it is not clear when such a solution exists. However, we highlight that recent results showed that LMI approaches provide useful tools for finding such a solution, see e.g., [21], [22]. Moreover, concerning the existence of positive definite stabilizing solutions of the MARI (19), we refer to [9, Proposition 3]. In simple words, the authors of [9] prove the existence of at least one stabilizing positive definite solution to the MARI for $\sigma \in (\underline{\sigma}, 1]$, where $\underline{\sigma} > 0$ depends on R, ρ and the most unstable eigenvalue of A. In particular, stabilizing solutions were shown to exist when the parameter is sufficiently close to 1 and $R \succ 0$ (i.e., when we are sufficiently close to the standard DARI), see [9], [21].*

C. Main result on robust linear synchronization

In this section, we exploit Proposition 1 for discrete-time linear network synchronization. We consider a network of systems (11) and combine the results of Theorem 1 and Proposition 1 to design the state-feedback gain K inducing synchronization over general time-invariant graphs.

To this end, define the following quantities:

$$\eta_i := \left(\frac{|\lambda_i|}{\mathfrak{Re}(\lambda_i)}\right)^2 = 1 + \left(\frac{\mathfrak{Im}(\lambda_i)}{\mathfrak{Re}(\lambda_i)}\right)^2,\ i = 2, \dots, N, \qquad (26a)$$

$$\bar{\eta} := \max\{\eta_2, \dots, \eta_N\}, \quad \underline{\eta} := \min\{\eta_2, \dots, \eta_N\}, \qquad (26b)$$

$$\bar{\lambda} := \max_{i \in \{2,\dots,N\}} \mathfrak{Re}(\lambda_i), \quad \underline{\lambda} := \min_{i \in \{2,\dots,N\}} \mathfrak{Re}(\lambda_i), \qquad (26c)$$

for the non-zero eigenvalues λ_i, $i = 2, \dots, N$, of a connected Laplacian matrix L. Our MARI-based design is effective whenever the following inclusion holds for the graph-induced quantities (26) and the MARI parameter $\sigma \in (0, 1]$:

$$\bar{\eta}\sigma \in \left(0,\ 1 - \frac{(\bar{\eta}\,\bar{\lambda} - \underline{\eta}\,\underline{\lambda})^2}{(\bar{\eta}\,\bar{\lambda} + \underline{\eta}\,\underline{\lambda})^2}\right]. \qquad (27)$$

The following lemma establishes a useful implication of (27).

Lemma 2 *Consider the quantities $\underline{\eta}$, $\bar{\eta}$, $\underline{\lambda}, \bar{\lambda}$ in (26) and let $\sigma \in (0, 1]$. The following interval of the real axis*

$$\mathcal{K} := \left[\frac{1 - \sqrt{1 - \bar{\eta}\,\sigma}}{\underline{\eta}\,\underline{\lambda}},\ \frac{1 + \sqrt{1 - \bar{\eta}\,\sigma}}{\bar{\eta}\,\bar{\lambda}}\right] \qquad (28)$$

is nonempty if and only if (27) holds.

The proof of Lemma 2 is given in Appendix A. We are ready to present the main result on robust synchronization of linear systems.

Theorem 2 *Consider the network (11) and suppose that L is a Laplacian matrix describing a connected, directed and weighted*

communication graph. Let $R \in \mathbb{S}^m_{\succeq 0}$ and suppose that there exists $P \in \mathbb{S}^n_{\succ 0}$ such that (19) *holds for a selection of σ satisfying* (27) *with $\bar{\eta}, \underline{\eta}, \bar{\lambda}, \underline{\lambda}$ defined in* (26)*. Then, the distributed control law u_i in* (4)*, with $\alpha(x) = \kappa K x$ and K selected as in* (23)*, solves Problem 1 for any scalar gain $\kappa \in \mathcal{K}$ as defined in* (28)*.*

Before proving Theorem 2, we highlight the importance of bounds (27) on σ in Proposition 2. First, differently from the continuous-time scenario [30], the bounds on the scalar gain κ depend on the imaginary part of the Laplacian eigenvalues via $\underline{\eta}$ and $\bar{\eta}$. This is expected, as discrete-time stability requires the eigenvalues to lay inside the unit disc, which unlike the negative half-plane imposes bounds on the imaginary part. Hence, in the case where there is at least one complex eigenvalue, definitions (26) imply $\bar{\eta} > 1$. As a consequence, it is necessary that $\sigma < 1$ for a real solution to the square roots in (28) to exist. In the case of real eigenvalues, $\eta_i = 1$ for all $i = 1, \dots, N$. Then, (28) simplifies to

$$\kappa \in \left[\frac{1 - \sqrt{1-\sigma}}{\underline{\lambda}} \, , \, \frac{1 + \sqrt{1-\sigma}}{\bar{\lambda}} \right],$$

with condition (27) on σ simplified to

$$\sigma \in \left(0, \, 1 - \frac{(\bar{\lambda} - \underline{\lambda})^2}{(\bar{\lambda} + \underline{\lambda})^2} \right].$$

This last bound recovers the results in [6], where all eigenvalues of L are supposed to be real. We emphasize that $\sigma = 1$ is a worst-case value not leading to synchronization unless $\underline{\lambda} = \bar{\lambda}$, as in Remark 1. Moreover, smaller values of σ lead to robust synchronization over broader range of graphs. We now show the proof for Theorem 2.

Proof of Theorem 2: By Theorem 1, Problem 1 is solved (equivalently, (6) holds) if the matrices $(A + \lambda_i \kappa B K)$ are Schur-Cohn for all $\lambda_i \in \text{spec}(L) \setminus 0$. By Proposition 1, each one of these matrices is Schur-Cohn if $|\kappa \lambda_i - 1|^2 \leq 1 - \sigma$. By expanding the norm, we conclude that the closed-loop matrix associated to λ_i is Schur-Cohn if $\sigma - 2\kappa \, \mathfrak{Re}(\lambda_i) + \kappa^2 \, |\lambda_i|^2 \leq 0$. Solving for κ and recalling the definition of η_i in (26), we obtain robust synchronization if

$$\kappa \in \left[\frac{1 - \sqrt{1 - \eta_i \, \sigma}}{\eta_i \, \mathfrak{Re}(\lambda_i)}, \frac{1 + \sqrt{1 - \eta_i \, \sigma}}{\eta_i \, \mathfrak{Re}(\lambda_i)} \right] \; \forall i = 2, \dots, N, \qquad (29)$$

because we simultaneously stabilize all the closed-loop matrices. First, note that from (26c) we have $\eta_i \geq 1$. Moreover, $\eta_i < \infty$, because all eigenvalues λ_i have positive real part. Then, since $\sigma > 0$, for any $i = 2, \dots, N$ it holds that

$$\frac{1 + \sqrt{1 - \bar{\eta}\sigma}}{\bar{\eta}\bar{\lambda}} \leq \frac{1 - \sqrt{1 - \eta_i \sigma}}{\eta_i \mathfrak{Re}(\lambda_i)} \leq \frac{1 - \sqrt{1 - \underline{\eta}\sigma}}{\underline{\eta}\underline{\lambda}} \, .$$

Consequently, for any $\kappa \in \mathcal{K}$ as per (28), condition (29) holds and (6) holds, as to be proven. □

Remark 3 *We highlight that, in combination with the results in [9], condition* (27) *implies that there exist $(\underline{\sigma}, \bar{\sigma}) \in (0, 1]^2$ such that synchronizing solutions to the multi-agent problem based on MARI-design exist for some σ satisfying $\sigma \geq \underline{\sigma}$ and $\sigma \leq \bar{\sigma}$. However, this set is not guaranteed to be nonempty. In other words, it may happen that $\underline{\sigma} > \bar{\sigma}$. In particular, for unstable linear systems, the lower bound $\underline{\sigma}$ depends on the choice of R, ρ in* (19) *and the most unstable eigenvalue of A [9]. The upper bound $\bar{\sigma}$ depends on the Laplacian eigenvalues. The first implication is that it may not be possible to synchronize a network of arbitrarily unstable systems under an arbitrary communication graph. Similarly, some choices of convergence rate-control penalty pair (ρ, R) may not be suitable for a given system-graph pair.*

IV. Synchronization of Nonlinear Systems

As shown in Section III, the problem of synchronization can be interpreted as a simultaneous stabilization problem by means of an opportune change of coordinate. Hence, in this section, we first focus on the general properties a single nonlinear system has to fulfill in order to extend the results of Section III-B to the nonlinear case. Successively, we exploit these properties to extend the results of Section III-C to the nonlinear scenario.

A. The case of single nonlinear discrete agents

We consider time-varying discrete-time nonlinear system of the form

$$x^+ = \varphi(x, \mathfrak{t}) + w \qquad (30)$$

where the function $\varphi : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^n$ is such that the following mild property holds.

Property 1 *Function $\varphi : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^n$ is continuous in its first argument and there exists a (possibly unbounded) set of matrices $\mathcal{D}\varphi \subset \mathbb{R}^{n \times n}$ such that, for each $x_a, x_b \in \mathbb{R}^n$ and all $\mathfrak{t} \in \mathbb{N}$, there exists an integrable function $\psi : [0, 1] \to \mathbb{R}^{n \times n}$ satisfying*

$$\varphi(x_a, \mathfrak{t}) - \varphi(x_b, \mathfrak{t}) = \int_0^1 \psi(s) \, ds \, (x_a - x_b) \qquad (31a)$$

$$\psi(s) \in \mathcal{D}\varphi, \quad \forall s \in [0, 1]. \qquad (31b)$$

The above definition allows considering a wide class of dynamical systems. First, note that when $n = 1$, Property 1 boils down to the requirement of φ being absolutely continuous. Trivially, such a class of systems includes continuously differentiable ones with $\mathcal{D}\varphi$ containing all of their Jacobians. Moreover, Property 1 includes functions that are differentiable almost everywhere (i.e. everywhere but on a set of measure zero), such as piecewise smooth and Lipschitz functions. In this case, $\mathcal{D}\varphi$ contains all the possible Clarke generalized gradients [38]. As a particular case, for linear systems of the form (18), $\mathcal{D}\varphi = \{A\}$. When moving to nonlinear systems, this allows the inclusion of some useful nonlinearities, such as saturations and arctangents, by selecting $\mathcal{D}\varphi$ as the vertices of the convex hull of all possible Jacobians.

We now explore the design of stabilizers showing gain margin properties in the nonlinear framework by means of incremental input-to-state stability (δISS) arguments. Typically, δISS is obtained via incremental Lyapunov functions [17]–[19], [25]. However, in [13, Theorem 15], the equivalence between uniform global exponentially δISS and global contractivity is shown for continuously differentiable discrete-time dynamics. Hence, we aim at exploiting contraction to obtain δISS.

A-1. Sufficient conditions for exponential δISS

While the first in-depth analysis of the relation between contraction, incremental stability and convergence in discrete-time appeared in [13], to the authors' knowledge the first results date back to [17]. We now generalize these existing results to the framework of non-smooth dynamics whose vector fields satisfy Property 1. Moreover, we extend recent advances of [25] to time-varying dynamics. We start by recalling the definition of δISS for discrete-time systems, see e.g. [18], [19], [25].

Definition 3 (Exponential δISS) *System* (30) *is globally uniformly Incrementally Input-to-State Stable with exponential convergence rate (exponentially δISS) if there exist $c, \gamma > 0$ and $\rho \in [0, 1)$ such that, for all $\mathfrak{t} \geq \mathfrak{t}_0$ with $\mathfrak{t}_0 \in \mathbb{N}$ and for any initial states $x_1(\mathfrak{t}_0), x_2(\mathfrak{t}_0)$*

and any pair of disturbance sequences $\mathfrak{t} \mapsto w_1(\mathfrak{t}), \mathfrak{t} \mapsto w_2(\mathfrak{t})$*, the resulting solutions* $x_1(\mathfrak{t}), x_2(\mathfrak{t})$ *of* (30) *satisfy*

$$|x_1(\mathfrak{t}) - x_2(\mathfrak{t})| \leq c\rho^{\mathfrak{t}-\mathfrak{t}_0}|x_1(\mathfrak{t}_0) - x_2(\mathfrak{t}_0)| + \sup_{t\in[\mathfrak{t}_0,\mathfrak{t}]} \gamma|w_1(t) - w_2(t)|\,. \quad (32)$$

We prove below a sufficient δISS condition extending the results in [25, Theorem 2], [13, Theorem 14], [17, Theorem 6.1] to the case of time-varying non-smooth vector fields satisfying Property 1.

Lemma 3 *Consider system* (30) *and suppose that* φ *satisfies Property 1 with a specific set-valued map* $\mathcal{D}\varphi$*. Moreover, suppose that there exists* $P \in \mathbb{S}^n_{\succ 0}$ *and* $\rho \in (0,1)$ *satisfying*

$$J^\top P J \preceq \rho^2 P\,, \quad \forall J \in \mathcal{D}\varphi. \quad (33)$$

Then, system (30) *is exponentially* δ*ISS according to Definition 3.*

The proof of Lemma 3 is given in Appendix B. Equipped with sufficient conditions for contraction of non-smooth dynamics, we conclude the subsection with some pedagogical examples, providing a useful insight into the applicability of the result.

Example 1 *Consider a system of the form*

$$x^+ = \varphi(x) = \mathrm{sat}_r(Ax)\,, \quad (34)$$

where $r \in \mathbb{R}^n_{>0}$ *and the vector saturation function* $\mathrm{sat}(\cdot)$ *has components* $\mathrm{sat}_i(\cdot) := \max(\min(\cdot, r_i), -r_i)$*. It can be easily verified that, for all* $x \in \mathbb{R}^n$*, the generalized Jacobian of* φ *[38] satisfies*

$$\partial\varphi(x) \subset \mathbf{co}\{\Delta A,\ \Delta \in \mathbf{\Delta}\},$$

where $\mathbf{\Delta} := \{\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n) : \delta_i \in \{0,1\},\ \forall i = 1, \ldots, n\}$ *is a finite set of matrices representing the vertices of a polytope. More generally, let* $\mathcal{V}$ *be a set of matrices* $\mathcal{V} := \{A_1, \ldots, A_v\}$ *with* $v \in \mathbb{N}$*, such that* $\partial\varphi(x) \in \mathbf{co}\{\mathcal{V}\}$ *for all* $x \in \mathbb{R}^n$*. Then, it suffices to verify* (33) *on* $\mathcal{V}$ *and convexity of the equivalent formulation*

$$\begin{pmatrix} \rho^2 P & J^\top \\ J & P^{-1} \end{pmatrix} \succeq 0$$

(obtained via a Schur complement) ensures that (33) *holds for* $\mathcal{D}\varphi = \mathbf{co}\{\mathcal{V}\}$*. Similar reasonings can be followed for smooth monotone saturation-like functions, such as arctangents or hyperbolic tangents.*

Example 2 *The study of incremental stability properties of neural networks is gaining attention in the research community, e.g. [25], [39], [40]. The presented contraction analysis tools can be valuable to derive such properties. For example, a multilayer perceptron with* $L \in \mathbb{N}$ *layers and ReLU activation functions can be described by the following dynamics*

$$\begin{cases} x^+ = y_L \\ y_\ell = W_\ell\, \nu(y_{\ell-1}) + b_\ell, \quad \ell = 1, \ldots, L \\ y_0 = x \end{cases} \quad (35)$$

with the ReLU function $\nu(\cdot)$ *applied component-wise, i.e.,* $\nu(x)$ *has components* $\nu_i(x_i) := \max(0, x_i)$*,* $y_\ell, b_\ell \in \mathbb{R}^{n_\ell}$ *and* $W_\ell \in \mathbb{R}^{n_\ell \times n_{\ell-1}}$*. Denoting by* $x \mapsto \varphi(x)$ *the function satisfying* $y_L = \varphi(x)$ *recursively defined in* (35)*, by the chain rule [38, Theorem 2.6.6], for all* $x \in \mathbb{R}^n$*, we have*

$$\partial\varphi(x) \subset \mathbf{co}\{W_L\Delta_{L-1}W_{L-1}\ldots\Delta_1 W_1, \ \Delta_i \in \mathbf{\Delta},\ \forall i = 1, \ldots, L-1\},$$

with $\mathbf{\Delta}$ *defined as in Example 1. Proceeding as in Example 1, we can conclude exponential* δ*ISS properties of* (35) *by checking* (37) *on a set* $\mathcal{V}$ *satisfying* $\partial\varphi(x) \in \mathbf{co}\{\mathcal{V}\}$ *for all* $x \in \mathbb{R}^n$*. Similar results extend to more complex recurrent neural networks, as shown in [25].*

A-2. Nonlinear robust feedback with gain margin design via contraction

Paralleling the linear derivation in Section III-B, we now exploit the results of Lemma 3 to design a feedback stabilizers $u = \alpha(x, \mathfrak{t})$ inducing exponential δISS (as defined in Definition 3) with respect to w and for a nonlinear system of the form

$$x^+ = f(x, \mathfrak{t}) + Bu + w, \quad (36)$$

where $f : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^n$ satisfies Property 1 and B is full column rank. Again, the result we obtain establishes a gain margin property as defined below. We emphasize the similarity with (2), promising that this construction will be used in Section IV-B to solve the robust nonlinear synchronization problem.

Definition 4 (Gain margin for nonlinear systems) *A function* $\alpha : \mathbb{R}^n \times \mathbb{N} \mapsto \mathbb{R}^m$ *is a* δ*ISS feedback with gain margin of radius* $r > 0$ *if, for any real number* $\kappa \in [1-r, 1+r]$*, system* (36) *with* $u = \kappa\alpha(x,t)$ *is exponentially* δ*ISS with respect to* w*.*

The following proposition can be seen as a non-smooth nonlinear counterpart of Proposition 1. We now state our first result for the nonlinear framework.

Proposition 2 *Let* $R \in \mathbb{S}^m_{\succeq 0}$ *and assume that* f *in* (36) *satisfies Property 1 for some* $\mathcal{D}f \subset \mathbb{R}^{n\times n}$*. Moreover, suppose that there exists* $P \in \mathbb{S}^n_{\succ 0}$ *satisfying*

$$J^\top Q J \preceq \rho^2 P, \quad \forall J \in \mathcal{D}f, \quad (37a)$$

$$Q := P - \sigma PB\left(R + B^\top PB\right)^{-1} B^\top P, \quad (37b)$$

for some $\rho \in (0,1)$ *and* $\sigma \in (0,1]$*. Then for the system* (36) *the function*

$$u = \alpha(x, \mathfrak{t}) = -\kappa\left(R + B^\top PB\right)^{-1} B^\top P f(x, \mathfrak{t}) \quad (38)$$

is a δ*ISS feedback with gain margin of radius* $r = \sqrt{1-\sigma}$*.*

Proof: For the sake of compactness, let us start by defining

$$Y = Y^\top := \left(R + B^\top PB\right)^{-1},$$
$$\Omega = \mathrm{I}_n - \kappa BYB^\top P.$$

Since B is assumed to be full column rank, the matrix $R + B^\top PB$ is invertible, and Y exists. Then, Lemma 3 states that the closed-loop (36), (38), which can be written as (30) with

$$\varphi(x, \mathfrak{t}) = f(x, \mathfrak{t}) + B\alpha(x, \mathfrak{t}) = \Omega f(x, \mathfrak{t}), \quad (39)$$

is exponentially δISS if

$$J^\top \Omega^\top P \Omega J \preceq \rho^2 P, \qquad \forall J \in \mathcal{D}f. \quad (40)$$

By expanding the left-hand side in (40) and by adding and subtracting $\sigma J^\top PBYB^\top PJ$, due to (40) we obtain the equality

$$J^\top\Omega^\top P\Omega J = J^\top PJ - \sigma J^\top PBYB^\top PJ + (\sigma - 2\kappa)J^\top PBYB^\top PJ + \kappa^2 J^\top PBYB^\top PBYB^\top PJ,$$

where we note that the first two terms at the right-hand side coincide with $J^\top QJ$. Then inequality (37) implies, for all $J \in \mathcal{D}f$

$$J^\top\Omega^\top P\Omega J \preceq \rho^2 P + J^\top PBY\left((\sigma - 2\kappa)Y^{-1} + \kappa^2 B^\top PB\right)YB^\top PJ \preceq \rho^2 P + (\kappa^2 - 2\kappa + \sigma)J^\top PBYB^\top PJ,$$

where we expanded Y^{-1} and added $\kappa^2 R \succeq 0$ inside the brackets. Since Y is positive definite, (40) holds and Lemma 3 applies if $\kappa^2 - 2\kappa + \sigma \leq 0$, which holds if and only if

$$1 - \sqrt{1-\sigma} \leq \kappa \leq 1 + \sqrt{1-\sigma}\,,$$

concluding the proof. ■

Remark 4 *Consider the role of the parameter σ in* (37b). *On the one hand, a strictly positive σ implies that the system can be made exponentially δISS with the addition of an input acting in the correct directions. On the other hand, $\sigma = 0$ implies that the autonomous system $x^+ = f(x, \mathfrak{t})$ is already contracting, while a negative σ would mean that the autonomous system is sufficiently robust to withstand inputs in the wrong directions. Then, the parameter σ in* (37b) *plays a similar role to the one in the MARI* (19). *As a consequence, aside from motivating our choice of bounding $\sigma \in (0, 1]$, these facts show that inequality* (37) *can be seen as a nonlinear version of the MARI* (19).

Remark 5 *Our nonlinear δISS results parallel the linear ones on robust stabilization via MARI-based design. Indeed, mimicking the linear scenario, nonlinear feedbacks designed via Proposition 2 present a gain margin property, since any gain $\kappa \in [1-\sqrt{1-\sigma}, 1+\sqrt{1-\sigma}]$ is a good controller for system* (36). *This parallels the continuous-time scenario in [41], where contractive laws based on Riccati-like design show infinite-gain margin properties. Finally, similarly to the MARI-based approach, the allowable range for κ is centered at 1 for any σ. Then, if $\sigma = 1$, the range degenerates to a single point $\kappa = 1$, and the gain margin property is lost.*

Remark 6 *Note that* (37b) *implies $Q \in \mathbb{S}^n_{\succ 0}$ if $R \in \mathbb{S}^m_{\succ 0}$. Indeed, by* (37b) *and* (21) *with $\mathcal{A} = P$, $\mathcal{B} = B$, $\mathcal{C} = B^\top$ and $\mathcal{D} = R$, the invertibility of R yields the equivalent formulation*

$$Q = (1-\sigma)P + \sigma(P^{-1} + BR^{-1}B^\top)^{-1}.$$

Since $\sigma \in (0, 1]$ and $P \succ 0$, matrix Q is a σ-governed linear interpolation between positive definite matrices and, thus, it is positive.

B. Main result on nonlinear synchronization

We now exploit the δISS results of Proposition 2 to derive a solution to the nonlinear multi-agent robust synchronization problem. The idea is to exploit contraction properties to show convergence to the synchronization manifold. The link between Proposition 1 and Proposition 2 is evident. However, the Jordan transformation which is the basis of the proof of Theorem 1 and, consequently, of Theorem 2 cannot be easily applied in the nonlinear scenario. Thus, in order to exploit Proposition 2 for synchronization, we introduce the following technical lemma, which we will be used in the proof to design an appropriate transformation.

Lemma 4 *Let the weighted graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ be directed and connected, with Laplacian L and L_{11}, L_{12} defined as in* (1). *Then, there exist $M \in \mathbb{S}^{N-1}_{\succ 0}$ and constants $\underline{m}, \overline{m}, \underline{\mu}, \bar{\mu} > 0$, $\rho_M \in (0, 1]$ such that*

$$\underline{m}\,\mathrm{I}_{N-1} \preceq M \preceq \overline{m}\,\mathrm{I}_{N-1}, \quad \rho_M \leq \frac{\underline{m}}{\overline{m}} \tag{41a}$$

$$M(L_{22} - \mathbf{1}\,L_{12}) + (L_{22} - \mathbf{1}\,L_{12})^\top M \succeq 2\underline{\mu} M \tag{41b}$$

$$(L_{22} - \mathbf{1}\,L_{12})^\top M (L_{22} - \mathbf{1}\,L_{12}) \preceq \bar{\mu}^2 M. \tag{41c}$$

The proof of Lemma 4 is given in Appendix C. We now present the following main result on network synchronization of nonlinear systems.

Theorem 3 *Consider the network* (2) *and suppose that f satisfies Property 1 for some $\mathcal{D}f \subset \mathbb{R}^{n\times n}$ and L is a Laplacian matrix describing a connected, directed and weighted communication graph. Let $\rho \in (0, 1)$ and $\sigma \in (0, 1]$ satisfy*

$$\rho \leq \rho_M, \qquad \sigma \leq \frac{1}{\varsigma}, \qquad \varsigma := \left(\frac{\bar{\mu}}{\underline{\mu}}\right)^2, \tag{42}$$

with $\rho_M, \underline{\mu}, \bar{\mu}$ as in Lemma 4. If, for some $R \in \mathbb{S}^m_{\succ 0}$, there exists $P \in \mathbb{S}^n_{\succ 0}$ satisfying (37a), (37b), *then, the distributed control law $u_i = \sum_{j=1}^N \ell_{ij}\alpha(x_j, \mathfrak{t})$ in* (4) *with α defined as in* (38) *and κ satisfying*

$$\kappa \in \left[\frac{\left(1-\sqrt{1-\varsigma\sigma}\right)}{\varsigma\underline{\mu}}\,,\ \frac{\left(1+\sqrt{1-\varsigma\sigma}\right)}{\varsigma\underline{\mu}}\right], \tag{43}$$

solves Problem 1 for the network (2), *namely,* (6) *holds.*

Proof: Mimicking the linear framework, we show convergence to the synchronization manifold $\mathcal{M}$ by focusing our analysis on the error between agents. If these error dynamics are robustly stable (ISS) with respect to an incremental version of $\mathbf{w}$, then Problem 1 is solved. Bearing in mind the steps of the proof of Theorem 1, without loss of generality, we define a virtual leader $z = x_1$ and define $N-1$ error coordinates with respect to such a leader node, $\mathbf{e} := \mathrm{col}(e_2, \ldots, e_N) \in \mathbb{R}^{Nn}$ with $e_i := x_i - z$ for all $i = 1, \ldots, N$. Similarly, we define the incremental disturbance $\widetilde{\mathbf{w}} := \mathrm{col}(\widetilde{w}_2, \ldots, \widetilde{w}_N)$ with $\widetilde{w}_i := w_i - w_1$. The error dynamics are described, for all $i = 2, \ldots, N$, by

$$e_i^+ = f(z + e_i, \mathfrak{t}) - f(z, \mathfrak{t}) + B\sum_{j=1}^N \left(\ell_{ij} - \ell_{1j}\right)\alpha(z + e_j, \mathfrak{t}) + \widetilde{w}_i. \tag{44}$$

Since by definition of the Laplacian entries $\sum_{j=1}^N \ell_{ij} = 0$ for any agent i, we can subtract $B\sum_{j=1}^N \left(\ell_{ij} - \ell_{1j}\right)\alpha(z, \mathfrak{t}) = 0$ from the right-hand side so that (44) becomes

$$\begin{aligned} e_i^+ &= \tilde{f}(e_i, \mathfrak{t}) + B\sum_{j=2}^N \tilde{\ell}_{ij}\tilde{\alpha}(e_j, \mathfrak{t}) + \widetilde{w}_i \\ \tilde{f}(e_i, \mathfrak{t}) &:= f(z + e_i, \mathfrak{t}) - f(z, \mathfrak{t}) \\ \tilde{\alpha}(e_j, \mathfrak{t}) &:= \alpha(z + e_j, \mathfrak{t}) - \alpha(z, \mathfrak{t}) \\ \tilde{\ell}_{ij} &:= \ell_{ij} - \ell_{1j}. \end{aligned} \tag{45}$$

Overall, the closed-loop system can be written in compact form as

$$\mathbf{e}^+ = \boldsymbol{\varphi}(\mathbf{e}, \mathfrak{t}) + \widetilde{\mathbf{w}}, \tag{46}$$

where we defined

$$\boldsymbol{\varphi}(\mathbf{e}, \mathfrak{t}) := \begin{pmatrix} \tilde{f}(e_2, \mathfrak{t}) + B\sum_{j=2}^N \tilde{\ell}_{2j}\tilde{\alpha}(e_j, \mathfrak{t}) \\ \vdots \\ \tilde{f}(e_N, \mathfrak{t}) + B\sum_{j=2}^N \tilde{\ell}_{Nj}\tilde{\alpha}(e_j, \mathfrak{t}) \end{pmatrix}. \tag{47}$$

Now, select the following candidate Lyapunov function

$$V(\mathbf{e}) = \mathbf{e}^\top (M \otimes P)\mathbf{e}, \tag{48}$$

with M defined in Lemma 4. Note that, due to the properties of the Kronecker product, since $M \succ 0$ and $P \succ 0$, $M \otimes P$

is symmetric and positive-definite. Now, for each value of z and $\mathbf{e} = \mathrm{col}(e_2, \dots, e_N)$, define the function $F_{\mathfrak{t}} : \mathbb{R} \to \mathbb{R}^{Nn}$ as

$$F_{\mathfrak{t}}(s) := \begin{pmatrix} \tilde{f}_s(s, e_2, \mathfrak{t}) + B \sum_{j=2}^{N} \tilde{\ell}_{2j} \tilde{\alpha}_s(s, e_j, \mathfrak{t}) \\ \vdots \\ \tilde{f}_s(s, e_N, \mathfrak{t}) + B \sum_{j=2}^{N} \tilde{\ell}_{Nj} \tilde{\alpha}_s(s, e_j, \mathfrak{t}) \end{pmatrix}, \tag{49}$$

parametrized by $\mathfrak{t} \in \mathbb{N}$, with the definitions

$$\tilde{f}_s(s, e_i, \mathfrak{t}) := f(z + s e_i, \mathfrak{t}) - f(z, \mathfrak{t}), \tag{50a}$$

$$\begin{aligned} \tilde{\alpha}_s(s, e_j, \mathfrak{t}) &:= \alpha(z + s e_j, \mathfrak{t}) - \alpha(z, \mathfrak{t}), \\ &= -\kappa Y B^\top P \tilde{f}_s(s, e_i, \mathfrak{t}), \end{aligned} \tag{50b}$$

where we used (38) and $Y := \left(R + B^\top P B\right)^{-1}$. From (49)-(50) we have $F_{\mathfrak{t}}(0) = 0$ and from (31) we get

$$\boldsymbol{\varphi}(\mathbf{e}, \mathfrak{t}) = F_{\mathfrak{t}}(1) = F_{\mathfrak{t}}(1) - F_{\mathfrak{t}}(0) = \int_0^1 \partial F(s) ds \, \mathbf{e}, \tag{51}$$

where

$$\partial F(s) := [\mathrm{I}_{(N-1)n} - \kappa((L_{22} - \mathbf{1}L_{12}) \otimes B Y B^\top P)] \Psi(s), \tag{52}$$

$$\Psi(s) := \mathrm{diag}\left(\tilde{\psi}_2(s) \, \dots, \, \tilde{\psi}_N(s)\right), \tag{53}$$

$$\tilde{\psi}_i(s) \in \mathcal{D} f, \qquad \forall i = 2, \dots, N \tag{54}$$

are obtained from (49) by proceeding as in (14). Since $V(\mathbf{e}^+) = 2(\mathbf{e}^+)^\top (M \otimes P)\mathbf{e}^+ - V(\mathbf{e}^+)$, subtracting $\rho V(\mathbf{e})$ on both sides, by combining (46) and (51), we obtain

$$\begin{aligned} V(\mathbf{e}^+) - \rho V(\mathbf{e}) = {} & 2\,(\mathbf{e}^+)^\top (M \otimes P) \int_0^1 \partial F(s)\, ds\, \mathbf{e} \\ & - \left[(\mathbf{e}^+)^\top (M \otimes P)\mathbf{e}^+ + \rho\, \mathbf{e}^\top (M \otimes P)\mathbf{e}\right] \int_0^1 ds \\ & + 2(\mathbf{e}^+)^\top (M \otimes P)\widetilde{\mathbf{w}}. \end{aligned}$$

Then, by collecting everything under the integral and defining the extended error vector $\boldsymbol{\xi} = \mathrm{col}(\mathbf{e}, \mathbf{e}^+)$ we obtain

$$V(\mathbf{e}^+) - \rho V(\mathbf{e}) = -\int_0^1 \boldsymbol{\xi}^\top \Upsilon(s) \boldsymbol{\xi}\, ds + 2(\mathbf{e}^+)^\top (M \otimes PB)\widetilde{\mathbf{w}}, \tag{55a}$$

$$\Upsilon(s) := \begin{pmatrix} \rho(M \otimes P) & -\partial F^\top(s)(M \otimes P) \\ -(M \otimes P)\partial F(s) & (M \otimes P) \end{pmatrix}. \tag{55b}$$

Since $P \succ 0$, $M \succ 0$, and $M \otimes P$ is invertible, we can study the positive definiteness of $\Upsilon(s)$ via its Schur complement

$$\hat{\Upsilon}(s) = \rho(M \otimes P) - \partial F^\top(s)(M \otimes P)\partial F(s)\,.$$

By using the definition of $\partial F(s)$ in (52) and the properties of Kronecker products, we obtain

$$\hat{\Upsilon}(s) = \rho(M \otimes P) - \Psi(s)^\top (T_a + \mathrm{He}\,\{T_b\} + T_c)\Psi(s) \tag{56}$$

where we defined

$$\begin{aligned} T_a &:= M \otimes P, \\ T_b &:= -\kappa(M(L_{22} - \mathbf{1}L_{12}) \otimes PBYB^\top P), \\ T_c &:= \kappa^2[(L_{22} - \mathbf{1}L_{12})^\top M(L_{22} - \mathbf{1}L_{12}) \otimes PBYB^\top PBYB^\top P]. \end{aligned}$$

For T_b, by the properties of the Kronecker product, since $PBYB^\top P$ is symmetric we obtain

$$\mathrm{He}\,\{T_b\} = -\kappa(\mathrm{He}\,\{M(L_{22} - \mathbf{1}L_{12})\} \otimes PBYB^\top P).$$

Consequently, by Lemma 4 and using again the properties of the Kronecker product, the following holds

$$\mathrm{He}\,\{T_b\} \preceq -2\kappa\underline{\mu}(M \otimes PBYB^\top P). \tag{57}$$

Similarly, since $R \succeq 0$, by exploiting the Kronecker product and by using again Lemma 4, we get

$$\begin{aligned} T_c &\preceq \kappa^2\bar{\mu}^2[M \otimes PBY(R + B^\top PB)YB^\top P] \\ &\preceq \kappa^2\bar{\mu}^2(M \otimes PBYB^\top P). \end{aligned} \tag{58}$$

Using (57) and (58), matrix $\hat{\Upsilon}$ in (56) can be bounded as

$$\begin{aligned} \hat{\Upsilon}(s) &\succeq \rho(M \otimes P) - \Psi(s)^\top (M \otimes \overline{P})\Psi(s) \\ \overline{P} &= P + (\kappa^2\bar{\mu}^2 - 2\kappa\underline{\mu})PBYB^\top P \end{aligned} \tag{59}$$

Now, consider $\overline{P}$. By addition and subtraction, it can be rewritten as

$$\overline{P} = P - \sigma PBYB^\top P + (\kappa^2\bar{\mu}^2 - 2\kappa\underline{\mu} + \sigma)PBYB^\top P.$$

Then, if $\kappa^2\bar{\mu}^2 - 2\kappa\underline{\mu} + \sigma \leq 0$, namely if

$$\frac{\underline{\mu}}{\bar{\mu}^2}\left(1 - \sqrt{1 - \left(\frac{\bar{\mu}}{\underline{\mu}}\right)^2 \sigma}\right) \leq \kappa \leq \frac{\underline{\mu}}{\bar{\mu}^2}\left(1 + \sqrt{1 - \left(\frac{\bar{\mu}}{\underline{\mu}}\right)^2 \sigma}\right),$$

which holds due to the selection in (43), we obtain

$$\overline{P} \preceq P - \sigma PBYB^\top P = Q, \tag{60}$$

with Q defined in (37b). Using (41a) from Lemma 4 and (60), $\hat{\Upsilon}(s)$ in (59) satisfies

$$\hat{\Upsilon}(s) \succeq \underline{m}\rho(\mathrm{I}_{N-1} \otimes P) - \overline{m}\Psi(s)^\top (\mathrm{I}_{N-1} \otimes Q)\Psi(s).$$

Recalling from (53) the block-diagonal structure of $\Psi(s)$ and exploiting (37), we obtain

$$\begin{aligned} \hat{\Upsilon}(s) &\succeq \underline{m}\rho(\mathrm{I}_{N-1} \otimes P) - \overline{m}\,\mathrm{diag}(\{\tilde{\psi}_i(s)^\top Q \tilde{\psi}_i(s)\}_{i=2}^N) \\ &\succeq \underline{m}\rho(\mathrm{I}_{N-1} \otimes P) - \overline{m}\,\mathrm{diag}(\{\rho^2 P\}_{i=2}^N) \\ &= \underline{m}\rho(\mathrm{I}_{N-1} \otimes P) - \overline{m}\rho^2(\mathrm{I}_{N-1} \otimes P) \\ &\succeq \rho(\underline{m} - \overline{m}\,\rho)(\mathrm{I}_{N-1} \otimes P) \\ &\succeq \rho(\underline{m} - \overline{m}\,\rho_M)(\mathrm{I}_{N-1} \otimes P) \succeq 0, \end{aligned}$$

where we used $0 < \rho \leq \rho_M = \underline{m}\overline{m}^{-1}$. Since $\hat{\Upsilon}(s) \succeq 0$ for each $s \in [0, 1]$, we conclude that also $\Upsilon(s)$ defined in (55b) satisfies $\Upsilon(s) \succeq 0$ for all $s \in [0, 1]$, and (55a) implies

$$V(\mathbf{e}^+) - \rho V(\mathbf{e}) \leq 2(\mathbf{e}^+)^\top (M \otimes P)\widetilde{\mathbf{w}} \tag{61}$$

By the generalized Young's inequality and by considering the factorization $M \otimes P = \sqrt{M \otimes P}^\top \sqrt{M \otimes P} = (\sqrt{M \otimes P})^2$ (with $\sqrt{M \otimes P}$ denoting the unique positive square root of $M \otimes P \succ 0$), we have

$$\begin{aligned} 2(\mathbf{e}^+)^\top (M \otimes P)\widetilde{\mathbf{w}} &= 2(\mathbf{e}^+)^\top (\sqrt{M \otimes P})^2 \widetilde{\mathbf{w}} \\ &\leq (1 - \sqrt{\rho})V(\mathbf{e}^+) + \frac{1}{1 - \sqrt{\rho}} \widetilde{\mathbf{w}}^\top (M \otimes P)\widetilde{\mathbf{w}}. \end{aligned}$$

Then, since $\rho \in (0, 1)$, inequality (61) implies

$$V(\mathbf{e}^+) - \sqrt{\rho} V(\mathbf{e}) \leq \frac{1}{\sqrt{\rho}(1 - \sqrt{\rho})} \widetilde{\mathbf{w}}^\top (M \otimes P)\widetilde{\mathbf{w}},$$

thus proving exponential ISS properties of the $\mathbf{e}$ dynamics due to the quaratic form of (48). Finally, similarly to the linear scenario of Theorem 1, relations (16) hold and robust synchronization as in Problem 1 is obtained, thus concluding the proof. ■

Remark 7 *Note that the contraction inequality* (37a) *in the context of Theorem 3 is tightly related to the structure of the Laplacian matrix and its eigenvalues. To appreciate this link, consider the network graph to be undirected and leader-connected. Under these conditions, the Laplacian L in* (1) *satisfies $L_{11} = 0$, $L_{12} = \mathbf{0}$. Then, $L_{22} = L_{22}^\top \succ 0$ and we can select $M = \mathrm{I}_{N-1}$ and $\underline{\mu}$ (resp. $\bar{\mu}$) in Lemma 4*

as the smallest (resp. largest) eigenvalue of L_{22}. Consequently, the admissible values of σ are related to the condition number of L_{22}. Moreover, by picking $M = \mathrm{I}_{N-1}$, the contraction rate ρ disentangles from the network structure, as $\underline{m} = \overline{m} = 1$. Indeed, from (41a) *we can select $\rho_M = 1$ and, consequently, condition* (42) *in Theorem 3 imposes no constraints on $\rho \in (0,1)$, as in Lemma 3.*

To the best of the authors' knowledge, while a general contraction-based approach appeared in [42], there is no result paralleling Theorem 3 in the continuous-time framework, nor there is a Lyapunov-based result addressing robust exponential synchronization of nonlinear agents under general weighted, directed graphs. The main issue in continuous-time arises when considering a Lyapunov function of the form (48). Indeed, it is not trivial to perform continuous-time parallel steps similar to those at the end of the proof of Theorem 3, which exploit Demidovich-like conditions [43, Theorem 1] to derive upper bounds on the Lyapunov decrease.

V. GEVPs FOR EXPONENTIAL δISS

We now discuss numerically efficient formulations of the results of Section IV, and apply them to specific classes of systems. LMI-based conditions for robust stabilization are a valuable tool for control design for discrete-time nonlinear systems, see e.g. [8], [16], [20], [44]. Hence, inspired by these works and recent LMI approaches for solving MARI inequalities [21], [22], we propose LMI-based conditions to obtain the solution of the MARI-like inequality (37). Our numerically efficient formulation provides a viable solution to the design problem of robustly synchronizing controllers.

First, we introduce an equivalent formulation for (37). The parameters of the proposed reformulation can be obtained by solving a generalized eigenvalue problem (GEVP). Then, we focus our attention on the case where the set of possible open-loop Jacobians of the system dynamics (36) is polytopic. Finally, we target the specific case of Lur'e systems.

A. Formulation as a GEVP

We start by reformulating Proposition 2 as an LMI problem. This provides convex analysis conditions for constructing matrix P. Given this new formulation, the convergence rate ρ and the parameter σ in (37) can be estimated as part of a GEVP.

Proposition 3 *Let $R \in \mathbb{S}^m_{\succ 0}, \sigma \in (0,1], \rho \in (0,1)$. The following LMIs in the decision variables W and Σ*

$$W \succ 0, \quad \Sigma \succ 0, \quad \begin{pmatrix} \rho W & WJ^\top \\ JW & \rho\Sigma \end{pmatrix} \succeq 0, \tag{62a}$$

$$\begin{pmatrix} W + \sigma BR^{-1}B^\top - \Sigma & BR^{-1}B^\top \\ BR^{-1}B^\top & \frac{1}{\sigma}\left(\frac{W}{(1-\sigma)} + BR^{-1}B^\top\right) \end{pmatrix} \succeq 0, \tag{62b}$$

hold if and only if conditions (37) *hold with $P = W^{-1} \succ 0$. Moreover,* (62) *is a generalized eigenvalue problem in (σ, ρ, R), namely, if it is feasible for $(\bar\sigma, \bar\rho, \bar R)$, then it is feasible for any (σ, ρ, R) such that $\sigma \geq \bar\sigma$, $\rho \geq \bar\rho$, $R \preceq \bar R$ and, conversely, if it is infeasible for $(\underline{\sigma}, \underline{\rho}, \underline{R})$, then it is infeasible for any (σ, ρ, R) such that $\sigma \leq \underline{\sigma}$, $\rho \leq \underline{\rho}$, $R \succeq \underline{R}$.*

Proof: Consider the last LMI of (62). Since $W \succ 0$ and $R^{-1} \succ 0$, then its $(2,2)$ entry is positive definite. Then, by the Schur complement, (62b) holds if and only if

$$W + \sigma BR^{-1}B^\top - \Sigma \succeq \sigma(1-\sigma)BR^{-1}B^\top(W + (1-\sigma)BR^{-1}B^\top)^{-1}BR^{-1}B^\top,$$

which can be rearranged as

$$\Sigma \preceq W + \sigma B\Big(R^{-1} - (1-\sigma)R^{-1}B^\top \times (W + (1-\sigma)BR^{-1}B^\top)^{-1}BR^{-1}\Big)B^\top. \tag{63}$$

By the matrix inversion lemma (21) with $\mathcal{A} = R$, $\mathcal{B} = \sqrt{1-\sigma}B^\top$, $\mathcal{C} = \sqrt{1-\sigma}B$, $\mathcal{D} = W$, inequality (63) is equivalent to

$$\Sigma \preceq W + \sigma B(R + (1-\sigma)B^\top W^{-1}B)^{-1}B^\top.$$

Left and right multiplication of both sides by $W^{-1} \succ 0$ yields the equivalent condition

$$W^{-1}\Sigma W^{-1} \preceq W^{-1} - \sigma W^{-1}B(-(R + B^\top W^{-1}B) + \sigma B^\top W^{-1}B)^{-1}B^\top W^{-1}. \tag{64}$$

Once again, by the matrix inversion lemma (21) applied with $\mathcal{A} = W$, $\mathcal{B} = \sqrt{\sigma}B$, $\mathcal{C} = \sqrt{\sigma}B^\top$, $\mathcal{D} = -(R + B^\top W^{-1}B)$, inequality (64) is equivalent to

$$W^{-1}\Sigma W^{-1} \preceq (W - \sigma B(R + B^\top W^{-1}B)^{-1}B^\top)^{-1}.$$

By left and right multiplying both sides by W, we obtain the equivalent inequality

$$\Sigma \preceq W(W - \sigma B(R + B^\top W^{-1}B)^{-1}B^\top)^{-1}W. \tag{65}$$

Since $(\mathcal{ABC})^{-1} = \mathcal{C}^{-1}\mathcal{B}^{-1}\mathcal{A}^{-1}$ for any invertible matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, inequality (65) is equivalent to

$$\begin{aligned}\Sigma &\preceq (W^{-1}(W - \sigma B(R + B^\top W^{-1}B)^{-1}B^\top)W^{-1})^{-1} \\ &\preceq (P - \sigma PB(R + B^\top PB)^{-1}B^\top P)^{-1} = Q^{-1},\end{aligned} \tag{66}$$

where we used $P = W^{-1}$ and the definition of Q in (37b). Consider now the right LMI in (62a). By left and right multiplication by the matrix

$$T = \begin{pmatrix} W^{-1} & 0 \\ 0 & \mathrm{I}_n \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & \mathrm{I}_n \end{pmatrix},$$

we have

$$\begin{pmatrix} \rho P & J^\top \\ J & \rho\Sigma \end{pmatrix} = \begin{pmatrix} \rho W^{-1} & J^\top \\ J & \rho\Sigma \end{pmatrix} \succeq 0.$$

Then, by the Schur complement, we obtain the equivalent condition $\rho W^{-1} - \rho^{-1}J^\top\Sigma^{-1}J \succeq 0$, which, with the selection $P = W^{-1}$, can be written as

$$J^\top\Sigma^{-1}J \preceq \rho^2 P. \tag{67}$$

Summarizing, we proved the equivalence of (62) with the four inequalities $P \succ 0$, $\Sigma \succ 0$, (66) and (67), where we emphasize that, under (62), Q^{-1} exists due to the positive definiteness of Q implied by $R \succ 0$, as established in Remark 6. More specifically, (62) is equivalent to

$$P \succ 0, \quad \Sigma \succ 0, \quad \Sigma^{-1} \succeq Q, \quad J^\top\Sigma^{-1}J \preceq \rho^2 P. \tag{68}$$

To complete the first part of the proof, we show that (68) is equivalent to (37). If (37) holds, then $Q \succ 0$ and (68) holds with $\Sigma = Q^{-1}$. If (68) holds, then

$$J^\top QJ \preceq J^\top\Sigma^{-1}J \preceq \rho^2 P,$$

thus completing the first part of the proof. To prove that (62) is a GEVP in (σ, ρ, R), let us denote by $\bar W, \bar\Sigma$ the solution of (62) with $(\bar\sigma, \bar\rho, \bar R)$. If $\sigma \geq \bar\sigma > 0$, $0 \prec R \preceq \bar R$, we obtain

$$\begin{aligned}\bar Q &= \bar P - \bar\sigma\bar P B(\bar R + B^\top\bar P B)^{-1}B^\top\bar P \\ &\succeq \bar P - \sigma\bar P B(R + B^\top\bar P B)^{-1}B^\top\bar P = Q,\end{aligned}$$

where $\bar{P} = \bar{W}^{-1} \succ 0$. Then, by (68) and since $\bar{\Sigma}^{-1} \succeq \bar{Q}$ and $\rho \geq \bar{\rho}$, the following inequalities hold

$$\bar{P} \succ 0, \quad \bar{\Sigma} \succ 0, \quad \bar{\Sigma}^{-1} \succeq Q, \quad J^\top \bar{\Sigma}^{-1} J \preceq \bar{\rho}^2 \bar{P}.$$

Due to the equivalence between inequalities (68) and (62), we conclude that $P = \bar{P}$ is solution to (62) with (σ, ρ, R). Similar reasonings prove infeasibility of (68) for any (σ, ρ, R) such that $\sigma \leq \underline{\sigma}$, $\rho \leq \underline{\rho}$, $R \succeq \underline{R}$ if (68) is infeasible for (σ, ρ, R). ■

Combined with Proposition 2, Proposition 3 requires the satisfaction of (62) for all $J \in \mathcal{D}f$. This may turn out to be impracticable, as $\mathcal{D}f$ could be infinite dimensional. However, under some additional assumptions on system (36), we can follow a polytopic approach similar to the one in Examples 1 and 2. Hence, we propose the following result addressing the case where the open-loop system Jacobian belongs to a polytopic set defined by a finite number of vertices.

Corollary 1 *Let $R \in \mathbb{S}^m_{\succ 0}$ and assume that f in (36) satisfies Property 1 for some $\mathcal{D}f \subset \mathbb{R}^{n\times n}$. Moreover, suppose there exists a finite set of matrices $\mathcal{V} := \{A_1, \dots, A_v\} \subset \mathbb{R}^{n\times n}$ such that $\mathcal{D}f \subseteq \mathbf{co}\{\mathcal{V}\}$. If there exist matrices $W, \Sigma \in \mathbb{S}^n_{\succ 0}$ and scalars $\rho \in (0,1)$, $\sigma \in (0,1]$ satisfying (62) for all $J \in \mathcal{V}$, the control law $u = \alpha(x, \mathfrak{t})$ with α defined in (38) and $P = W^{-1}$ makes the closed-loop exponentially δISS with respect to w with gain margin of radius $r = \sqrt{1-\sigma}$.*

B. Lur'e systems

We further specialize our result to the case of Lur'e systems. Namely, we now consider nonlinear discrete-time systems of the form

$$x^+ = f(x) + Bu = Ax + F\phi(Cx) + Bu, \tag{69}$$

where $C \in \mathbb{R}^{p\times n}$, $F \in \mathbb{R}^{n\times p}$ and the square nonlinearity $\phi : \mathbb{R}^p \to \mathbb{R}^p$ is a pool of p, possibly different, feedback nonlinear elements $\phi(y) := \mathrm{diag}(\phi_1(y_1), \dots, \phi_p(y_p))$ whose components $\phi_i, i = 1, \dots, p$, satisfy Property 1 for some intervals $\mathcal{D}\phi_i \subset \mathbb{R}$, $i = 1, \dots, p$. We assume that each function ψ_i belongs to an incremental sector $[0, \omega_i]$, with $\omega_i \geq 0$, in the following classical sense:

$$(\phi_i(s_1) - \phi_i(s_2))(\phi_i(s_1) - \phi_i(s_2) - \omega_i(s_1 - s_2)) \leq 0, \ \forall s_1, s_2 \in \mathbb{R}. \tag{70}$$

By the non-smooth mean value theorem [38, Theorem 2.3.7], we may combine bounds (70) into

$$\mathrm{He}\left\{J_\phi S(J_\phi - \Omega)\right\} \preceq 0 \tag{71}$$

which holds for all diagonal $J_\phi \in \mathcal{D}\phi = \mathrm{diag}(\mathcal{D}\phi_1, \dots, \mathcal{D}\phi_p)$, for any diagonal $S \in \mathbb{S}^p_{\succeq 0}$ and for some diagonal $\Omega = \mathrm{diag}(\omega_1, \dots, \omega_p) \in \mathbb{S}^p_{\succeq 0}$. We then have the following result.

Proposition 4 *Let $R \in \mathbb{S}^m_{\succ 0}$ and suppose that ϕ in (69) satisfies (70) for some $\Omega = \mathrm{diag}(\omega_1, \dots, \omega_p)$. If there exist symmetric matrices W, Σ, a diagonal matrix $S \in \mathbb{S}^p_{\succeq 0}$ and scalars $\rho \in (0,1)$, $\sigma \in (0,1]$ satisfying (62b) and*

$$W \succ 0, \quad \Sigma \succ 0, \quad \begin{pmatrix} \rho W & WA^\top & WC^\top \\ AW & \rho\Sigma & -F\Omega^\top S \\ CW & -S\Omega F^\top & 2S \end{pmatrix} \succeq 0, \tag{72}$$

then the control law $u = \alpha(x, \mathfrak{t})$ with α defined in (38) and $P = W^{-1}$ makes the closed-loop (69)-(38) exponentially δISS with respect to w with gain margin of radius $r = \sqrt{1-\sigma}$.

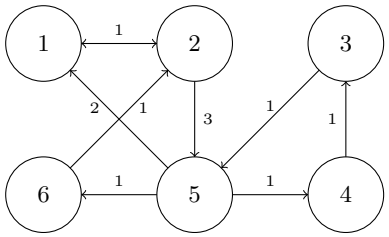

Fig. 1: Communication graph considered in Section V-C

Proof: Due to [38, Theorem 2.3.7], (71) holds for any diagonal $S \in \mathbb{S}^p_{\succeq 0}$ and any $J_\phi \in \mathcal{D}\phi$. Define the matrix $\Lambda^\top := \left(\begin{smallmatrix} \mathrm{I}_n & 0 & 0 \\ 0 & \mathrm{I}_n & FJ_\phi \end{smallmatrix}\right)$ with any diagonal $J_\phi \in \mathcal{D}\phi$. It is easy to verify that (71) implies

$$\Lambda^\top \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -F\Omega^\top S \\ 0 & -S\Omega F^\top & 2S \end{pmatrix} \Lambda = \Lambda^\top \Pi \Lambda \preceq 0. \tag{73}$$

Consider now (72), which implies

$$\Lambda^\top \begin{pmatrix} \rho W & WA^\top & WC^\top \\ AW & \rho\Sigma & -F\Omega^\top S \\ CW & -S\Omega F^\top & 2S \end{pmatrix} \Lambda = \Lambda^\top (\Xi + \Pi) \Lambda \succeq 0,$$

where we defined

$$\Xi := \begin{pmatrix} \rho W & WA^\top & WC^\top \\ AW & \rho\Sigma & 0 \\ CW & 0 & 0 \end{pmatrix}. \tag{74}$$

By (73), we then have

$$\Lambda^\top \Xi \Lambda \succeq \Lambda^\top (\Xi + \Pi) \Lambda \succeq 0,$$

thus showing $\Lambda^\top \Xi \Lambda \succeq 0$. Then, the expansion of the product leads to

$$\Lambda^\top \Xi \Lambda = \begin{pmatrix} \rho W & W(A + FJ_\phi C)^\top \\ (A + FJ_\phi C)W & \rho\Sigma \end{pmatrix} \succeq 0, \forall J_\phi \in \mathcal{D}\phi.$$

By Proposition 3, the assumption (62b) implies that conditions (37) hold with $P = W^{-1} \succ 0$ for any $J \in \mathcal{D}f = A + F\mathcal{D}\phi C$. The proof is concluded by Proposition 2. ■

C. Numerical example

In what follows, we propose a simple numerical example to validate the results of Theorem 3 with the construction in Proposition 4. Consider a network of $N = 6$ agents connected according to the weighted, directed graph in Figure 1 and evolving according to the planar Lur'e dynamics (69) as follows

$$x_i^+ = Ax_i + F\phi(Cx_i) + Bu_i, \quad i = 1, \dots, 6,$$

where

$$A = \begin{pmatrix} 1.1 & 0.1 \\ -0.3 & 0.5 \end{pmatrix}, \ B = \begin{pmatrix} 2 \\ 0.3 \end{pmatrix}, \ C = \begin{pmatrix} 1 & -1 \end{pmatrix}, \ F = \begin{pmatrix} -0.1 \\ 0.7 \end{pmatrix},$$

and $\phi(\cdot) = \mathrm{sat}_{10}(\cdot) = \max(\min(\cdot, 10), -10)$. It is simple to verify that $J_\phi \in \{0, 1\}$ and the Laplacian matrix is

$$L = \begin{pmatrix} 3 & -1 & 0 & 0 & -2 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -3 & -1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Conditions (71), (72) and (62b) are easily solved with $\rho = 0.9$, $\sigma = 0.285$, $S = \Omega = 1$ and provide

$$W = \begin{pmatrix} 0.0408 & -0.1747 \\ -0.1747 & 1.1273 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 0.0388 & -0.1294 \\ -0.1294 & 0.8495 \end{pmatrix}.$$

Then, we select a control law of the form (4) with α as in (38), $R = 1$ and $\kappa = 0.2$. It is interesting to notice that, by solving (41) via semi-definite programming, ρ, σ and κ fall outside the required bounds. This shows the conservativeness of our Lyapunov analysis, that is aimed at obtaining a very general result. Finally, we simulate the proposed closed-loop under the action of a random Gaussian noise $w \in \mathcal{N}(0, 0.5)$. Robust exponential convergence to a non-trivial trajectory with initial conditions sampled from a normal distribution $\mathcal{N}(0, 100)$ are presented in Figure 2 and Figure 3. As expected, Figure 3 shows an exponential decrease of the average error between the agents, which converges to a bounded value in the presence of additive noise.

VI. Conclusions

In this paper, we explored the design of incrementally stabilizing feedbacks for discrete-time nonlinear systems and their application to multiagent synchronization under generic connected communication graphs. Starting from the linear scenario, we provided constructive designs for robust stabilization and sufficient convex conditions for network synchronization. We exploited new contraction analysis results and focused on Euclidean metrics. The analysis is focused on input-linear systems. An interesting direction that will be the subject of future research is the generalization to different input-vector fields, by means of non-quadratic Lyapunov functions or Riemannian metrics.

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Appendix

A. Proof of Lemma 2

It is trivial that (27) implies $\bar\eta\sigma \le 1$ and then the square root in (2) is well defined. To complete the proof, we show that the bound

$$\frac{1-\sqrt{1-\bar\eta\sigma}}{\underline\eta\underline\lambda} \le \frac{1+\sqrt{1-\bar\eta\sigma}}{\bar\eta\bar\lambda} \tag{75}$$

holds if and only if (27) is satisfied, namely if and only if

$$\bar\eta\,\sigma \le 1 - \frac{(\bar\eta\,\bar\lambda - \underline\eta\,\underline\lambda)^2}{(\bar\eta\,\bar\lambda + \underline\eta\,\underline\lambda)^2} = \frac{4\bar\eta\,\underline\eta\,\bar\lambda\,\underline\lambda}{(\bar\eta\,\bar\lambda + \underline\eta\,\underline\lambda)^2} = \left(\frac{2\bar\eta\,\bar\lambda}{\bar\eta\,\bar\lambda + \underline\eta\,\underline\lambda}\right)^2 \frac{\underline\eta\,\underline\lambda}{\bar\eta\,\bar\lambda},$$

which, due to the positivity of the squared term, is equivalent to

$$\left(\frac{\bar\eta\,\bar\lambda + \underline\eta\,\underline\lambda}{2\bar\eta\,\bar\lambda}\right)^2 \bar\eta\,\sigma - \frac{\underline\eta\,\underline\lambda}{\bar\eta\,\bar\lambda} \le 0\,. \tag{76}$$

Thus, we must show that (75) $\iff$ (76). By the lower bound of (27), $\bar\eta\,\sigma > 0$. Then, multiplying (76) by $\bar\eta\,\sigma$ paired with addition and subtraction of $1-\bar\eta\,\sigma$ at the right-hand side yields the equivalent inequality

$$\begin{aligned} 1 - \bar\eta\,\sigma &\ge \left(\frac{\bar\eta\,\bar\lambda + \underline\eta\,\underline\lambda}{2\bar\eta\,\bar\lambda}\right)^2 (\bar\eta\,\sigma)^2 + 1 - \left(1 + \frac{\underline\eta\,\underline\lambda}{\bar\eta\,\bar\lambda}\right)\bar\eta\,\sigma \\ &\ge \left(1 - \frac{\bar\eta\,\bar\lambda + \underline\eta\,\underline\lambda}{2\bar\eta\,\bar\lambda}\,\bar\eta\,\sigma\right)^2. \end{aligned} \tag{77}$$

By taking the square root, (77) is equivalent to

$$\sqrt{1-\bar\eta\,\sigma} \ge 1 - \frac{\bar\eta\,\bar\lambda + \underline\eta\,\underline\lambda}{2\bar\eta\,\bar\lambda}\bar\eta\,\sigma, \tag{78}$$

where the right-hand side is non-negative because $\underline\eta\,\underline\lambda \le \bar\eta\,\bar\lambda$ and $\bar\eta\,\sigma \le 1$. Exploiting the expansion $\bar\eta\,\sigma = (1-\sqrt{1-\bar\eta\,\sigma})(1+\sqrt{1-\bar\eta\,\sigma}) > 0$, inequality (78) is equivalent to

$$\begin{aligned} \frac{\underline\eta\,\underline\lambda}{\bar\eta\,\bar\lambda} &\ge \left(1-\sqrt{1-\bar\eta\,\sigma}\right)\frac{2}{\bar\eta\,\sigma} - 1 \ge \frac{1+(1-\bar\eta\,\sigma) - 2\sqrt{1-\bar\eta\,\sigma}}{\bar\eta\,\sigma} \\ &\ge \frac{\left(1-\sqrt{1-\bar\eta\,\sigma}\right)^2}{(1-\sqrt{1-\bar\eta\,\sigma})(1+\sqrt{1-\bar\eta\,\sigma})} \ge \frac{1-\sqrt{1-\bar\eta\,\sigma}}{1+\sqrt{1-\bar\eta\,\sigma}}\,, \end{aligned}$$

which coincides with (75), thus completing the proof.

B. Proof of Lemma 3

Consider the candidate Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ defined as

$$V(x_1, x_2) := (x_1 - x_2)^\top P (x_1 - x_2),$$

for any two states $x_1, x_2 \in \mathbb{R}^n$. Given any selection of x_1, x_2 and $w_1, w_2 \in \mathbb{R}^n$, define function $\tilde\Phi : \mathbb{R} \to \mathbb{R}^n$ as

$$\tilde\Phi(s) = \varphi(sx_1 + (1-s)x_2, \mathfrak{t}) + sw_1 + (1-s)w_2\,.$$

We have

$$x_1^+ - x_2^+ = \tilde\Phi(1) - \tilde\Phi(0) = \varphi(x_1, \mathfrak{t}) - \varphi(x_2, \mathfrak{t}) + w_1 - w_2\,.$$

In view of (31), we obtain

$$\begin{aligned} V^+ &= V(x_1^+, x_2^+) \\ &= (x_1^+ - x_2^+)^\top P \left[\int_0^1 \psi(s)\,ds\,(x_1 - x_2) + w_1 - w_2\right] \end{aligned} \tag{79}$$

for some $\psi(s) \in \mathcal{D}\varphi$, for all $s \in [0,1]$. Then, adding and subtracting $\rho^2 V(x_1, x_2)$ and $V(x_1^+, x_2^+)$ to the right-hand side of (79) yields

$$\begin{aligned} &V(x_1^+, x_2^+) - \rho^2 V(x_1, x_2) \\ &= 2(x_1^+ - x_2^+)^\top P \left[\int_0^1 \psi(s)\,ds\,(x_1 - x_2) + w_1 - w_2\right] \\ &\quad - (x_1^+ - x_2^+)^\top P (x_1^+ - x_2^+) \int_0^1 ds \\ &\quad - \rho^2 (x_1 - x_2)^\top P (x_1 - x_2) \int_0^1 ds \\ &= \int_0^1 \xi^\top \Upsilon(s) \xi\, ds + 2(x_1^+ - x_2^+)^\top P (w_1 - w_2) \end{aligned}$$

where we defined $\xi := \mathrm{col}(x_1 - x_2, x_1^+ - x_2^+)$ and

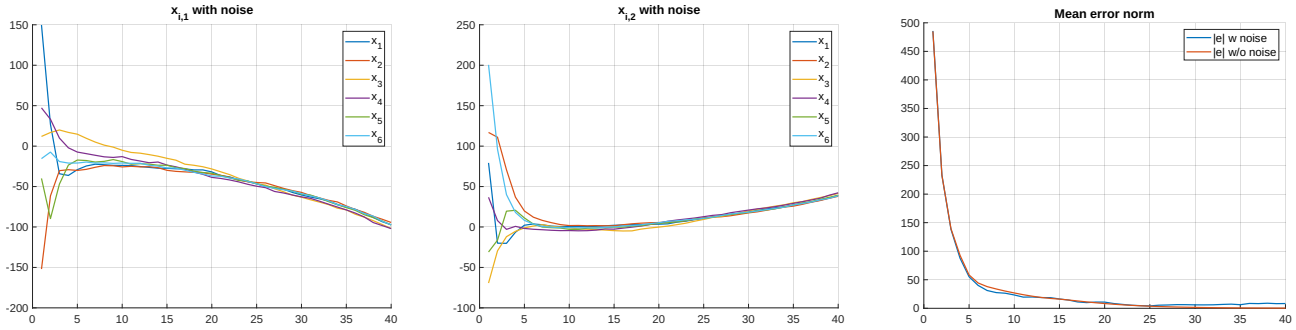
$$\Upsilon(s) := \begin{pmatrix} -\rho^2 P & \psi^\top(s) P \\ P\psi(s) & -P \end{pmatrix}.$$

By performing steps similar to the ones in [45, Theorem 1], due to (33) and a Schur complement, $\Upsilon(s) \preceq 0$ for all $s \in [0,1]$. As a consequence, since $\rho \in (0,1)$, we obtain

$$V(x_1^+, x_2^+) - \rho^2 V(x_1, x_2) \le 2(x_1^+ - x_2^+)^\top P (w_1 - w_2)\,.$$

By the generalized Young's inequality and by considering the decomposition $P = \sqrt{P}^\top \sqrt{P}$, we have

$$\begin{aligned} &2(x_1^+ - x_2^+)^\top P (w_1 - w_2) \le \\ &\quad (1-\rho) V(x_1^+, x_2^+) + \frac{1}{1-\rho}(w_1 - w_2)^\top P (w_1 - w_2)\,. \end{aligned}$$



(a) (b) (c)

Fig. 2: Trajectories during transient. a-b) state components with noise. c) mean error wrt agent 1 with and without noise.

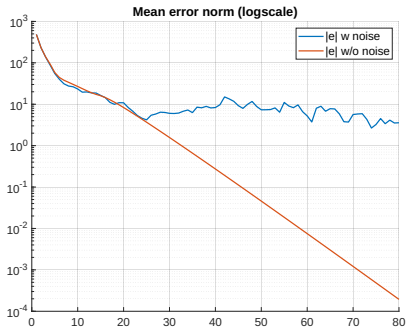


Fig. 3: Long-term mean error wrt agent 1 in logarithmic scale.

Then, by combining the previous inequalities we obtain

$$V(x_1^+, x_2^+) - \rho V(x_1, x_2) \leq \frac{|P|}{\rho(1-\rho)} |w_1 - w_2|^2 .$$

As $\rho \in (0,1)$ and $P \succ 0$, the function V is a dissipation-form incremental Lyapunov function [19, Definition 7]. Then, the result follows by [19, Theorem 8]. Finally, by using standard arguments (i.e. [13, Theorem 14]) one can conclude the exponential behavior of solutions.

C. Proof of Lemma 4

Since the graph is connected, Lemma 1 ensures that the Laplacian L, as in (1), has one zero eigenvalue and $N-1$ eigenvalues with positive real part. Consider the transformation

$$T = T^{-1} := \begin{pmatrix} 1 & 0 \\ \mathbf{1} & -\mathrm{I}_{N-1} \end{pmatrix}$$

and consider the change of coordinates on the Laplacian defined by

$$\mathcal{L} := TLT^{-1} = T \begin{pmatrix} 0 & -L_{12} \\ \mathbf{0} & -L_{22} \end{pmatrix} = \begin{pmatrix} 0 & -L_{12} \\ \mathbf{0} & L_{22} - \mathbf{1}\, L_{12} \end{pmatrix},$$

where we exploited $L\,\mathbf{1} = \mathbf{0}$ (see Section II-A). Since T is full rank, by similarity transformation $\mathrm{spec}(\mathcal{L}) = \mathrm{spec}(L)$, namely, it has one zero eigenvalue and $N-1$ eigenvalues with positive real part. Then, due to the block-triangular structure of $\mathcal{L}$, all the eigenvalues of $L_{22} - \mathbf{1}\, L_{12}$ have positive real part. Define $\tilde{L} := \mathbf{1}\, L_{12} - L_{22}$. Since all eigenvalues of $\tilde{L}$ have negative real part, by the Lyapunov equation there exists $M \in \mathbb{S}^{N-1}_{\succ 0}$ satisfying

$$M\tilde{L} + \tilde{L}^\top M = -\mathrm{I}_{N-1} \prec 0.$$

In turn, this implies $M(-\tilde{L}) + (-\tilde{L})^\top M \succ 0$. As a consequence, there exists a sufficiently small scalar $\underline{\mu} > 0$ such that (41b) holds. We now move to the other inequalities in Lemma 4. Since $M \in \mathbb{S}^{N-1}_{\succ 0}$, (41a) and (41c) trivially hold with of $\underline{m}$ and $\overline{m}$ being the smallest and largest eigenvalues of M respectively, $\rho_M = \underline{m}\overline{m}^{-1}$ and a sufficiently large $\bar{\mu}$.

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