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# Thin Ferromagnetic plates with Magnetostriction

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**Abstract.** In this paper, we establish the existence of global-in-time weak solutions for the Landau-Lifschitz-Gilbert equation with magnetostriction in the case of mixed boundary conditions. From this model, we derive by asymptotic method a two-dimensional model for thin ferromagnetic plates taking into account magnetostrictive effects.

**Keywords:** ferromagnetism, magnetostriction, weak solutions, thin plates.

## 1 Introduction

Ferromagnetic materials are characterized by a spontaneous magnetization, even in the absence of an external magnetic field. This magnetization influences the shape of the materials, and conversely, a deformation of the material leads to a change in magnetization. This phenomenon, called magnetostriction, is used for industrial applications, for example in magnetostrictive motion or pressure sensors or ultrasonic transducers. For these applications, the devices are often thin plates of ferromagnetic material (see [22, 25]).

In this paper, starting from the 3D model describing the time behavior of the magnetization of a material taking into account the magnetostriction, our goal is to obtain and justify by asymptotic method a two-dimensional model of thin ferromagnetic plate. Note that asymptotic studies of this type are carried out in [3, 15, 16, 17, 19] for ferromagnetic films without taking magnetostriction into account.

We first describe the three-dimensional model coupling Landau-Lifschitz-Gilbert with elasticity equations (see [21]).

### 1.1 Three dimensional model

We denote by  $\Omega$ , the domain occupied by the ferromagnetic material, and by  $\mathbf{m}(\mathbf{t}, \mathbf{x}) \in \mathbb{R}^3$  the magnetic moment at time  $\mathbf{t}$  and at point  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \Omega$ . We assume that the material is saturated, so that  $\mathbf{m}$  satisfies the saturation constraint  $|\mathbf{m}(\mathbf{t}, \mathbf{x})| = m_s$  a.e., where  $m_s$  is a constant expressed in A.m<sup>-1</sup>. The dynamics of  $\mathbf{m}$  is described by the Landau-Lifschitz-Gilbert equation (see [1], [7], [10] and [20]):

$$\begin{cases} \frac{\partial \mathbf{m}}{\partial \mathbf{t}} - \frac{\alpha}{m_s} \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial \mathbf{t}} = -(1 + \alpha^2) \gamma \mathbf{m} \times \mathbf{H}_{\text{eff}} & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_{\mathbf{n}} \mathbf{m} = 0 & \text{on } \mathbb{R}^+ \times \partial \Omega, \end{cases} \quad (1.1)$$

where  $\times$  is the cross product in  $\mathbb{R}^3$ ,  $\gamma$  is the gyromagnetic ratio (expressed in A.s.kg<sup>-1</sup>),  $\alpha$  is the dimensionless damping coefficient and  $\mathbf{n}$  the outward unit normal vector on  $\partial \Omega$ . The effective field  $\mathbf{H}_{\text{eff}}$  (expressed in T = kg.s<sup>-2</sup>.A<sup>-1</sup>) is given by:

$$\mathbf{H}_{\text{eff}} = \frac{\mathbf{A}}{m_s^2} \Delta \mathbf{m} + \mu_0 \mathbf{h}_d(\mathbf{m}) + \mu_0 \Psi(\mathbf{m}) + (\mathbf{\Lambda}^{\text{m}} : \boldsymbol{\sigma}) \mathbf{m}, \quad (1.2)$$

where

- $\mathbf{A}$  is the exchange constant (expressed in  $\text{J}\cdot\text{m}^{-1}$ ),
- $\mu_0 = 4\pi \cdot 10^{-7} \text{kg}\cdot\text{m}\cdot\text{s}^{-2}\cdot\text{A}^{-2}$  is the vacuum permeability,
- $\Psi$  is an anisotropic linear term satisfying:

$$\Psi(m) = -\nabla\Phi(m), \quad (1.3)$$

where  $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a non negative quadratic form,

- the demagnetizing field  $\mathbf{h}_d(\mathbf{m})$  is calculated from  $\mathbf{m}$  solving the static Maxwell equation coupled with the law of Faraday:

$$\text{curl } \mathbf{h}_d(\mathbf{m}) = 0 \quad \text{and} \quad \text{div}(\mathbf{h}_d(\mathbf{m}) + \overline{\mathbf{m}}) = 0, \quad (1.4)$$

where  $\overline{\mathbf{m}}$  is the extension of  $\mathbf{m}$  by zero outside  $\Omega$ ,

- in the magnetostrictive term,  $\boldsymbol{\sigma}$  is the stress tensor (2-tensor expressed in  $\text{kg}\cdot\text{m}^{-1}\cdot\text{s}^{-2}$ ),  $\mathbf{\Lambda}^{\text{m}}$  is a 4-tensor expressed in  $\text{m}^2\cdot\text{A}^{-2}$ , and  $:$  is the contraction operator (see below).

**Notation 1.1.** Let  $\Lambda$  be a 4-tensor in  $\mathbb{R}^3$ . Let  $\xi$  be a  $3 \times 3$  matrix. We denote  $\Lambda : \xi$  the  $3 \times 3$  matrix which entries are given by:

$$(\Lambda : \xi)_{ij} = \sum_{k=1}^3 \sum_{l=1}^3 \Lambda_{ijkl} \xi_{kl}.$$

Let  $\xi$  and  $\zeta$  be two  $3 \times 3$  matrices. We denote  $\xi : \zeta$  the scalar given by:

$$\xi : \zeta = \sum_{i=1}^3 \sum_{j=1}^3 \xi_{ij} \zeta_{ij}.$$

The Landau-Lifschitz-Gilbert Equation (1.1) is coupled with the wave elasticity equation:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \text{div } \boldsymbol{\sigma} = 0. \quad (1.5)$$

In (1.5),  $\rho$  is the mass density (expressed in  $\text{kg}\cdot\text{m}^{-3}$ ),  $\mathbf{u}$  is the displacement field and the stress tensor  $\boldsymbol{\sigma}$  is given by  $\boldsymbol{\sigma} = \mathbf{\Lambda}^{\text{e}} : \boldsymbol{\varepsilon}^{\text{e}}$ , where

- $\mathbf{\Lambda}^{\text{e}}$  is a 4-tensor (expressed in  $\text{kg}\cdot\text{m}^{-1}\text{s}^{-2}$ ),
- the total strain  $\boldsymbol{\varepsilon}^{\text{e}}$  is a  $3 \times 3$  matrix obtained from the linearized strain tensor  $\boldsymbol{\varepsilon}(\mathbf{u})$  and the magnetostrictive strain tensor  $\boldsymbol{\varepsilon}^{\text{m}}$  by the relation  $\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varepsilon}^{\text{e}} + \boldsymbol{\varepsilon}^{\text{m}}$ , with

$$\boldsymbol{\varepsilon}_{ij}(\mathbf{u}) = \frac{1}{2} \left( \frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} + \frac{\partial \mathbf{u}_j}{\partial \mathbf{x}_i} \right) \quad \text{and} \quad \boldsymbol{\varepsilon}^{\text{m}} = \mathbf{\Lambda}^{\text{m}} : \mathbf{m} \otimes \mathbf{m}, \quad (1.6)$$

where  $\mathbf{m} \otimes \mathbf{m} = \mathbf{m} \cdot {}^t \mathbf{m}$  is the  $3 \times 3$  matrix which entries are given by  $(\mathbf{m} \otimes \mathbf{m})_{ij} = \mathbf{m}_i \mathbf{m}_j$ .

The 4-tensors  $\mathbf{\Lambda}^{\text{e}}$  and  $\mathbf{\Lambda}^{\text{m}}$  are supposed to be symmetric and positive, as defined below:

**Definition 1.1.** Let  $\Lambda$  be a 4-tensor in  $\mathbb{R}^3$ . This tensor is said to be symmetric if:

$$\forall (i, j, k, l) \in \{1, 2, 3\}^4, \quad \Lambda_{ijkl} = \Lambda_{jikl} = \Lambda_{ijlk} = \Lambda_{klij}.$$

This tensor is said to be positive if there exists  $\lambda^* \in \mathbb{R}^{*+}$  such that for all symmetric matrix  $\xi$  of entries  $\xi_{ij}$ , we have:

$$\sum_{ijkl} \Lambda_{ijkl} \xi_{ij} \xi_{kl} \geq \lambda^* \sum_{ij} |\xi_{ij}|^2.$$

We assume that the material is clamped on  $\Gamma_1 \subset \partial\Omega$  and that a surface force  $\mathbf{f}$  (expressed in  $\text{N.m}^{-2}$ ) is applied on  $\Gamma_2 = \partial\Omega \setminus \Gamma_1$ , so that the following boundary conditions hold:

$$\begin{cases} \mathbf{u} = 0 & \text{on } \Gamma_1, \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{f} & \text{on } \Gamma_2, \end{cases} \quad (1.7)$$

In order to obtain a dimensionless model, we write

$$\mathbf{m}(\mathbf{t}, \mathbf{x}) = \mathbf{m}_s m\left(\frac{\mathbf{t}}{\tau}, \frac{\mathbf{x}}{\ell}\right) \quad \text{and} \quad \mathbf{u}(\mathbf{t}, \mathbf{x}) = \ell u\left(\frac{\mathbf{t}}{\tau}, \frac{\mathbf{x}}{\ell}\right), \quad (1.8)$$

where the characteristic time  $\tau$  and the exchange length  $\ell$  are respectively given by:

$$\tau = \frac{1}{\gamma \mu_0 \mathbf{m}_s} \quad \text{and} \quad \ell^2 = \frac{\mathbf{A}}{\mu_0 \mathbf{m}_s^2}.$$

We denote by  $t = \mathbf{t}/\tau$  the dimensionless time and by  $x = (x_1, x_2, x_3) = \mathbf{x}/\ell$  the dimensionless position. We define:

$$\Omega = \Omega/\ell, \quad \Gamma_1 = \Gamma_1/\ell \quad \text{and} \quad \Gamma_2 = \Gamma_2/\ell. \quad (1.9)$$

The dimensionless demagnetizing field is given by:

$$\text{curl } h_d(m) = 0 \quad \text{and} \quad \text{div}(h_d(m) + \bar{m}) = 0, \quad (1.10)$$

where  $\bar{m}$  is the extension of  $m$  by zero outside  $\Omega$ .

We introduce the dimensionless tensors  $\lambda^m$  and  $\lambda^e$  such that:

$$\Lambda^m = \frac{1}{\mathbf{m}_s^2} \lambda^m \quad \text{and} \quad \Lambda^e = \mu_0 \mathbf{m}_s^2 \lambda^e, \quad (1.11)$$

and we define the dimensionless surface force  $f \in L^2(\Gamma_2)$  and density  $\rho$  by

$$\mathbf{f} = \mu_0 \mathbf{m}_s^2 f \quad \text{and} \quad \rho = \frac{\rho \gamma^2 \mathbf{A}}{\mathbf{m}_s^2}. \quad (1.12)$$

We denote then by  $\sigma$  the dimensionless stress tensor given by:

$$\sigma(t, x) = \frac{1}{\mu_0 \mathbf{m}_s^2} \boldsymbol{\sigma}(\tau t, \ell x) = (\lambda^e : \varepsilon(u) - \lambda^e : (\lambda^m : m \otimes m))(t, x), \quad (1.13)$$

with

$$\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (1.14)$$

We obtain then the following dimensionless model:  $m(t, x)$  and  $u(t, x)$  are defined for  $t \in \mathbb{R}^+$  and  $x \in \Omega$  and satisfy:

$$\left\{ \begin{array}{l} \frac{\partial m}{\partial t} - \alpha m \times \frac{\partial m}{\partial t} = -(1 + \alpha^2)m \times H_{\text{eff}} \text{ in } \mathbb{R}^+ \times \Omega, \\ H_{\text{eff}} = \Delta m + h_d(m) + \Psi(m) + (\lambda^m : \sigma)m, \\ \sigma = \lambda^e : \varepsilon^e \quad \text{with} \quad \varepsilon^e = \varepsilon(u) - \lambda^m : m \otimes m, \\ \rho \frac{\partial^2 u}{\partial t^2} - \text{div } \sigma = 0 \text{ in } \mathbb{R}^+ \times \Omega, \end{array} \right. \quad (1.15)$$

with the following boundary conditions

$$\left\{ \begin{array}{l} \partial_n m = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega, \\ u(t, x) = 0 \text{ on } \mathbb{R}^+ \times \Gamma_1, \\ \sigma \cdot n = f \text{ on } \mathbb{R}^+ \times \Gamma_2, \end{array} \right. \quad (1.16)$$

where  $n$  is the unit outward normal vector on  $\partial\Omega$ . We remark that from the assumptions on  $\mathbf{\Lambda}^e$ , there exists a constant  $\lambda^{e*} > 0$  such that for all symmetric 2-tensor  $\xi$ :

$$\sum_{ijkl} \lambda_{ijkl}^e \xi_{ij} \xi_{kl} \geq \lambda^{e*} \sum_{ij} |\xi_{ij}|^2. \quad (1.17)$$

Note that  $\alpha$  and  $\rho$  are dimensionless constants whose values do not affect the mathematical analysis of the equations. We therefore take them equal to 1. We define the space

$$V(\Omega) = \{v \in H^1(\Omega; \mathbb{R}^3); v = 0 \text{ on } \Gamma_1\}. \quad (1.18)$$

For  $m \in H^1(\Omega; S^2)$ ,  $v \in V(\Omega)$  and  $w \in L^2(\Omega; \mathbb{R}^3)$ , we define the energy  $\mathcal{E}(m, v, w)$  by:

$$\begin{aligned} \mathcal{E}(m, v, w) &= \int_{\Omega} |\nabla m(x)|^2 dx + 2 \int_{\Omega} \Phi(m(x)) dx + \int_{\mathbb{R}^3} |h_d(m)(x)|^2 dx \\ &+ \frac{1}{2} \int_{\Omega} \left[ \lambda^e : \left( \varepsilon(v(x)) - \lambda^m : m(x) \otimes m(x) \right) \right] : \left( \varepsilon(v(x)) - \lambda^m : m(x) \otimes m(x) \right) dx \\ &+ \frac{1}{2} \int_{\Omega} |w(x)|^2 dx - \int_{\Gamma_2} f(s) \cdot v(s) d\Gamma_s. \end{aligned} \quad (1.19)$$

We aim to solve the Cauchy problem coupling (1.15)-(1.16) with the following initial conditions:

$$\left\{ \begin{array}{l} m(t=0) = m_0 \text{ in } \Omega, \\ u(t=0) = u_0 \text{ in } \Omega, \\ \frac{\partial u}{\partial t}(t=0) = u_1 \text{ in } \Omega, \end{array} \right. \quad (1.20)$$

where  $m_0 \in H^1(\Omega; S^2)$ ,  $u_0 \in V(\Omega)$  and  $u_1 \in L^2(\Omega; \mathbb{R}^3)$ . We define the notion of weak solutions for the previous Cauchy problem:

**Definition 1.2.** We say that  $(m, u)$  is a weak solution for (1.15)-(1.16)-(1.20) if

1.  $m \in L^\infty(\mathbb{R}^+; H^1(\Omega; \mathbb{R}^3)), \frac{\partial m}{\partial t} \in L^2(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^3))$  and  $m(0, \cdot) = m_0$ ,

2.  $m$  satisfies the saturation constraint

$$|m(t, x)| = 1 \text{ for almost every } (t, x) \in \mathbb{R}^+ \times \Omega, \quad (1.21)$$

3.  $u \in L^\infty(\mathbb{R}^+; V(\Omega)), \frac{\partial u}{\partial t} \in L^\infty(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^3))$  and  $u(0, \cdot) = u_0$ ,

4. for all  $\chi \in C_c^\infty(\mathbb{R}^+; H^1(\Omega; \mathbb{R}^3))$ ,

$$\begin{aligned} \int_{\mathbb{R}^+ \times \Omega} \left( \frac{\partial m}{\partial t} - m \times \frac{\partial m}{\partial t} \right) \cdot \chi &= 2 \int_{\mathbb{R}^+ \times \Omega} \sum_{i=1}^3 m \times \frac{\partial m}{\partial x_i} \cdot \frac{\partial \chi}{\partial x_i} \\ &- 2 \int_{\mathbb{R}^+ \times \Omega} m \times (h_d(m) + \Psi(m) + (\lambda^m : \sigma)m) \cdot \chi, \end{aligned} \quad (1.22)$$

$$\text{with } \sigma = \lambda^e : \varepsilon(u) - \lambda^e : (\lambda^m : m \otimes m),$$

5. for all  $\chi \in C_c^\infty(\mathbb{R}^+; V(\Omega))$ ,

$$\int_{\mathbb{R}^+ \times \Omega} \frac{\partial u}{\partial t} \cdot \frac{\partial \chi}{\partial t} - \int_{\mathbb{R}^+ \times \Omega} \sigma : \varepsilon(\chi) + \int_{\mathbb{R}^+ \times \Gamma_2} f \cdot \chi + \int_{\Omega} u_1 \chi(0, x) = 0, \quad (1.23)$$

6. for all  $t \geq 0$ ,

$$\mathcal{E}(m(t, \cdot), u(t, \cdot), \frac{\partial u}{\partial t}(t, \cdot)) + \int_0^t \int_{\Omega} \left| \frac{\partial m}{\partial t}(\tau, x) \right|^2 d\tau dx \leq \mathcal{E}(m_0, u_0, u_1) \quad (1.24)$$

where  $\mathcal{E}$  is defined by (1.19) (energy inequality).

## 1.2 Statement of the results

First, we establish the existence of global-in-time weak solutions for the Cauchy problem (1.15)-(1.16)-(1.20). Such a result is proved in [10] for homogeneous Dirichlet boundary conditions for the deformation. With the same method, we address mixed boundary conditions and we obtain the following theorem:

**Theorem 1.1.** Let  $m_0 \in H^1(\Omega; S^2)$ ,  $u_0 \in V(\Omega)$ ,  $u_1 \in L^2(\Omega)$  and  $f \in L^2(\Gamma_2)$ . Then, there exists a weak solution  $(m, u)$  for (1.15)-(1.16)-(1.20).

In a second time, we aim to obtain a reduced two-dimensional model for thin ferromagnetic plate. Let  $\omega \subset \mathbb{R}^2$  be a smooth open domain. We assume that

$$\partial\omega = \mathcal{C}_1 \cup \mathcal{C}_2, \quad \text{with } \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset \quad (1.25)$$

such that the one-dimensional measure of  $\mathcal{C}_1$  is non vanishing. We consider the thin plate  $\Omega_\eta$  given by:

$$\Omega_\eta = \omega \times ]-\eta, \eta[, \quad (1.26)$$

whose boundary splits as  $\partial\Omega = \Gamma_1^\eta \cap \Gamma_2^\eta$  with

$$\begin{aligned} \Gamma_1^\eta &= \mathcal{C}_1 \times ]-\eta, \eta[, \quad \text{and} \quad \Gamma_2^\eta = \Gamma_b^\eta \cup \Gamma_+^\eta \cup \Gamma_-^\eta, \\ \text{with } \Gamma_b^\eta &= \mathcal{C}_2 \times ]-\eta, \eta[, \quad \Gamma_+^\eta = \bar{\omega} \times \{+\eta\}, \quad \Gamma_-^\eta = \bar{\omega} \times \{-\eta\}. \end{aligned} \tag{1.27}$$

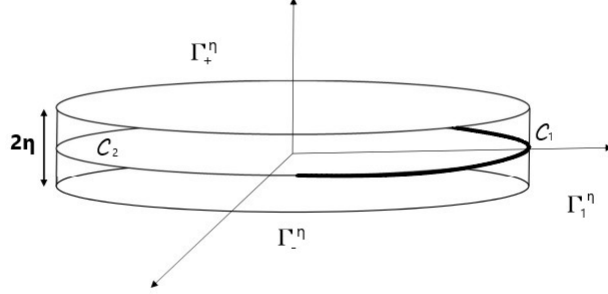


Figure 1:  $\Omega_\eta = \omega \times ]-\eta, \eta[$

We assume that this plate is clamped on  $\Gamma_1^\eta$ .

We define the spaces:

$$V(\Omega_\eta) = \{v \in H^1(\Omega_\eta; \mathbb{R}^3) ; v = 0 \text{ on } \Gamma_1^\eta\},$$

$$\mathcal{W}(\omega) = \{v = (v_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) ; v_i = 0 \text{ on } \mathcal{C}_1 \text{ and } \partial_1 v_3 = \partial_2 v_3 = 0 \text{ on } \mathcal{C}_1\}. \tag{1.28}$$

**Notation:** for  $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ , we denote  $y_T = (y_1, y_2)$ .

Let  $h \in L^2(\mathcal{C}_2; \mathbb{R}^3)$  and  $g^+$  and  $g^-$  in  $L^2(\omega; \mathbb{R}^3)$ . We define the  $\eta$ -depending boundary data on  $\Gamma_2^\eta$  by:

$$\begin{cases} \forall x_T \in \omega, \forall \alpha \in \{1, 2\}, f_\alpha^\eta(x_T, \pm\eta) = g_\alpha^\pm(x_T) \text{ and } f_3^\eta(x_T, \pm\eta) = \eta^2 g_3^\pm(x_T), \\ \forall (x_T, x_3) \in \mathcal{C}_2 \times ]-\eta, \eta[, \forall \alpha \in \{1, 2\}, f_\alpha^\eta(x_T, x_3) = h_\alpha(x_T) \text{ and } f_3^\eta(x_T, x_3) = \eta h_3(x_T). \end{cases} \tag{1.29}$$

Let  $m_0 \in H^1(\omega; S^2(0, 1))$ ,  $\tilde{u}_0 \in \mathcal{W}(\omega)$  and  $u_1 \in L^2(\omega; \mathbb{R}^3)$ . We assume that the third component of  $u_1$  vanishes:  $u_{1,3} = 0$  on  $\omega$ . We define the  $\eta$ -depending initial data by: for

$x = (x_T, x_3) \in \Omega_\eta,$

$$\left\{ \begin{array}{l} m_0^\eta(x_T, x_3) = m_0(x_T), \\ \forall \alpha \in \{1, 2\}, u_{0,\alpha}^\eta(x_T, x_3) = \tilde{u}_{0,\alpha}(x_T) - \frac{x_3}{\eta} \partial_\alpha \tilde{u}_{0,3}(x_T), \\ u_{0,3}^\eta(x_T, x_3) = \frac{1}{\eta} \tilde{u}_{0,3}(x_T), \\ u_1^\eta(x_T, x_3) = u_1(x_T). \end{array} \right. \quad (1.30)$$

We consider  $(m^\eta, u^\eta)$  the weak solution of (1.15)-(1.16)-(1.20), given by Theorem 1.1, with initial data  $m_0^\eta, u_0^\eta, u_1^\eta$  and external force  $f^\eta$  previously defined. We rescale this solution in order to work on the fixed domain  $\Omega_1 = \omega \times ]-1, 1[$ : for  $(x_T, x_3) \in \Omega_1$ , we set:

$$\begin{aligned} M^\eta(t, x_T, x_3) &= m^\eta(t, x_T, \eta x_3), & H^\eta(t, x_T, x_3) &= (h_d(m^\eta))(t, x_T, \eta x_3), \\ \forall \alpha \in \{1, 2\}, U_\alpha^\eta(t, x_T, x_3) &= u_\alpha^\eta(t, x_T, \eta x_3), & U_3^\eta(t, x_T, x_3) &= \eta u_3^\eta(t, x_T, \eta x_3). \end{aligned} \quad (1.31)$$

We obtain the following convergence result, announced in [18] in the isotropic case:

**Theorem 1.2.** *Using the notations above, when  $\eta$  tends to zero, there exists a subsequence still denoted  $(M^\eta, H^\eta, U^\eta)_\eta$  such that  $M^\eta$  tends to  $M$  in  $L^\infty(0, T; H^1(\Omega_1))$  weak \*,  $H^\eta$  tends to  $H$  in  $L^\infty(0, T; L^2(\Omega_1))$  strongly,  $U^\eta$  tends to  $U$  in  $L^\infty(0, T; V(\Omega_1))$  weak \*. In addition,  $M$  does not depend on its third variable and there exists  $\tilde{u} \in \mathcal{W}(\omega)$  such that:*

$$U_\alpha(x_T, x_3) = \tilde{u}_\alpha(x_T) - x_3 \partial_\alpha \tilde{u}_3 \text{ for } \alpha \in \{1, 2\}, \text{ and } U_3(x_T, x_3) = \tilde{u}_3(x_T).$$

The limit  $(M, \tilde{u})$  is a weak solution for the following initial and boundary value problem:

$$\left\{ \begin{array}{l} \frac{\partial M}{\partial t} - M \times \frac{\partial M}{\partial t} = -2M \times H^{\text{eq}} \quad \text{in } \mathbb{R}^+ \times \omega \\ \text{with } H^{\text{eq}} = \Delta M - (M, \vec{e}_3) \vec{e}_3 + \Psi(M) + (\lambda^{\text{m}} : \tilde{\sigma}) M, \\ \frac{\partial^2 \tilde{u}_T}{\partial t^2} - \widetilde{\text{div}} \tilde{\sigma} = -\frac{1}{2}(g_T^+ + g_T^-) \quad \text{in } \mathbb{R}^+ \times \omega, \\ \tilde{\sigma} = \lambda^{\text{eq}} : (\varepsilon(\tilde{u}_T) - \lambda^{\text{m}} : M \otimes M), \\ M(0, \cdot) = m_0, \tilde{u}_T(0, \cdot) = \tilde{u}_{0,T} \quad \text{and} \quad \frac{\partial \tilde{u}_T}{\partial t}(0, \cdot) = u_{1,T}, \\ \partial_n M = 0 \text{ on } \partial\omega, \quad \tilde{u}_T = 0 \text{ on } \mathcal{C}_1, \quad \text{and} \quad \tilde{\sigma} \cdot n = h_T \text{ on } \mathcal{C}_2, \end{array} \right. \quad (1.32)$$

where  $\lambda^{\text{eq}}$  is a 4-tensor only depending on the coefficients of  $\lambda^e$  (see below), and where  $\widetilde{\text{div}} S$  is defined by

$$\left( \widetilde{\text{div}} S \right)_\alpha = \sum_{\beta=1}^2 \partial_\beta S_{\alpha\beta}. \quad (1.33)$$



The new tensor  $\lambda^{\text{eq}}$  arising in the two-dimensional model (1.32) is defined as follows: we denote by  $\mathcal{G}$  the set of symmetric real  $3 \times 3$  matrices and by  $\mathcal{G}_n$  the set of the matrices  $D = (d_{ij}) \in \mathcal{G}$  such that  $d_{\alpha\beta} = 0$  for  $(\alpha, \beta) \in \{1, 2\}^2$ . We define the linear operator  $\mathcal{K} : \mathcal{G} \rightarrow \mathcal{G}_n$  by:

$$\forall D \in \mathcal{G}, (\mathcal{K}(D))_{ij} = \begin{cases} 0 & \text{if } (i, j) \in \{1, 2\}^2, \\ (\lambda^e : D)_{ij} = \sum_{(k,l) \in \{1,2,3\}^2} \lambda_{ijkl}^e d_{kl} & \text{if } i = 3 \text{ or } j = 3. \end{cases} \quad (1.34)$$

We claim that the restriction of  $\mathcal{K}$  to  $\mathcal{G}_n$  is a bijection from  $\mathcal{G}_n$  to itself (see the proof in subsection 3.4). We denote by  $\mathcal{T} : \mathcal{G}_n \rightarrow \mathcal{G}_n$  the inverse of  $\mathcal{K}|_{\mathcal{G}_n}$ . Then  $\lambda^{\text{eq}}$  is given by:

$$\forall D \in \mathcal{G}, \lambda^{\text{eq}} : D = \lambda^e : (D - \mathcal{T}(\mathcal{K}(D))). \quad (1.35)$$

We remark that the resulting model (1.32) is indeed two-dimensional, but it does not describe the behavior of the normal deformations. We remark also that in this regime, the magnetization is not influenced by the normal deformations, so that our model is closed.

This paper is organized as follows: Theorem 1.1 is proved in section 2. We follow the method due to Alouges and Soyeur [2] and generalized in [11]. First, we prove the existence of solution for a penalized system, in which the saturation constraint is relaxed. Then, we take the limit when the penalization constant tends toward zero. In [10], global existence for (1.15)-(1.20) is obtained in the case of a clamped sample, that is with  $u = 0$  on  $\partial\Omega$ . Our proof is very close to the proof in [10]. We reproduce it for the convenience of the reader in order to present a self-contained paper.

Theorem 1.2 is established in section 3. In order to avoid working on a domain depending on  $\eta$ , we perform a rescaling inspired both by [9] for the magnetization and by [14] for the deformation. The thickness parameter  $\eta$  appears then as a stiff term in the equations. Nevertheless, we are able to obtain a limit model thanks to the energy inequality and by choosing convenient test functions in the rescaled weak formulation.

### 1.3 Index of symbols and notations.

In the table below, for each notation used hereafter, we mention the equation number (or the page number) on which it is defined.

:	p. 2	$m^\zeta$	Eq. (2.1)	$\Gamma_2$	Eq. (1.9)
$\times$	p. 1	$m_N$	Eq. (2.5)	$\Gamma_1^\eta$	Eq. (1.27)
$\otimes$	p. 2	$M$	p. 7	$\Gamma_2^\eta$	Eq. (1.27)
$A$	p. 22	$M^\eta$	Eq. (1.31)	$\Gamma_b^\eta$	Eq. (1.27)
$\mathcal{C}_1$	Eq. (1.25)	$n$	p. 4	$\Gamma_+^\eta$	Eq. (1.27)
$\mathcal{C}_2$	Eq. (1.25)	$P_N$	p. 10	$\Gamma_-^\eta$	Eq. (1.27)
$\widetilde{\text{div}}$	Eq. (1.33)	$\mathcal{T}$	p. 8	$\varepsilon(u)$	Eq. (1.14)
$\mathcal{E}$	Eq. (1.19)	$u$	Eq. (1.8)	$\varepsilon^e$	p. 4
$\mathcal{E}_N$	Eq. (2.6)	$u_0$	Eq. (1.20)	$\varepsilon_N^e$	Eq. (2.5)
$\mathcal{E}^\eta$	Eq. (3.9)	$u_1$	Eq. (1.20)	$\varepsilon(\eta, \xi)$	Eq. (3.2)
$f$	Eq. (1.12)	$u_N$	Eq. (2.5)	$\varepsilon^e$	Eq. (3.17)
$f^\eta$	Eq. (1.29)	$u_0^\eta$	Eq. (1.30)	$\zeta$	p. 9
$g^\pm$	Eq. (1.29)	$u_1^\eta$	Eq. (1.30)	$\eta$	p. 6
$\mathcal{G}$	p. 8	$u^\zeta$	Eq. (2.1)	$\lambda^e$	Eq. (1.11)
$\mathcal{G}_n$	p. 8	$\tilde{u}$	Eq. (1.32)	$\lambda^m$	Eq. (1.11)
$\mathcal{G}_T$	p. 25	$\tilde{u}_0$	p. 7	$\lambda^{\text{eq}}$	Eq. (1.35)
$h$	p. 6	$U^\eta$	Eq. (1.31)	$\Pi_N$	p. 11
$h_d$	Eq. (1.10)	$V(\Omega)$	Eq. (1.18)	$\rho$	Eq. (1.12)
$H^\eta$	Eq. (1.31)	$V(\Omega_\eta)$	Eq. (1.28)	$\sigma$	Eq. (1.13)
$H_{\text{eff}}$	Eq. (1.15)	$V_{KL}(\Omega_1)$	Eq. (3.1)	$\sigma^\zeta$	Eq. (2.1)
$H_{\text{eff}}^\zeta$	Eq. (2.1)	$\mathbf{V}_N$	p. 10	$\sigma_N$	Eq. (2.5)
$H_{\text{eff}}^N$	Eq. (2.5)	$\mathcal{V}(\omega)$	Eq. (3.1)	$\tilde{\sigma}$	Eq. (1.32)
$H^{\text{eq}}$	Eq. (1.32)	$\mathbf{W}_N$	p. 11	$\Phi$	Eq. (1.3)
$\mathcal{K}$	Eq. (1.34)	$\mathcal{W}(\omega)$	Eq. (1.28)	$\Psi$	Eq. (1.3)
$m$	Eq. (1.8)	$x$	p. 3	$\omega$	p. 6
$m_0$	Eq. (1.20)	$x_i$	p. 3	$\Omega$	; Eq. (1.9)
$m_0^\eta$	Eq. (1.30)	$y_T$	p. 6	$\Omega_\eta$	p. 6
$m^\eta$	p. 7	$\Gamma_1$	Eq. (1.9)		

## 2 Weak solutions for the Landau-Lifschitz-Gilbert equation with magnetostriction

Let  $m_0 \in H^1(\Omega)$  with values in the unit sphere  $S^2$ ,  $u_0 \in V(\Omega)$ ,  $u_1 \in L^2(\Omega)$  and  $f \in L^2(\Gamma_2)$ .

### 2.1 Penalized system

As in [2] and [10], we relax the saturation constraint and for  $\zeta > 0$ , we consider the following penalized system:

$$\left\{ \begin{array}{l}
\frac{\partial m^\zeta}{\partial t} + m^\zeta \times \frac{\partial m^\zeta}{\partial t} - 2H_{\text{eff}}^\zeta + \frac{1}{\zeta} (|m^\zeta|^2 - 1)m^\zeta = 0 \text{ in } \mathbb{R}^+ \times \Omega, \\
H_{\text{eff}}^\zeta = \Delta m^\zeta + h_d(m^\zeta) + \Psi(m^\zeta) + (\lambda^m : \sigma^\zeta)m^\zeta, \\
\sigma^\zeta = \lambda^e : \varepsilon(u^\zeta) - \lambda^e : (\lambda^m : m^\zeta \otimes m^\zeta), \\
\frac{\partial^2 u^\zeta}{\partial t^2} - \text{div } \sigma^\zeta = 0 \text{ in } \mathbb{R}^+ \times \Omega, \\
m^\zeta(t=0) = m_0, \quad u^\zeta(t=0) = u_0, \quad \frac{\partial u^\zeta}{\partial t}(t=0) = u_1, \\
\partial_n m^\zeta = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega, \quad u^\zeta(t, x) = 0 \text{ on } \mathbb{R}^+ \times \Gamma_1, \quad \sigma^\zeta \cdot n = f \text{ on } \mathbb{R}^+ \times \Gamma_2.
\end{array} \right. \quad (2.1)$$

For a fixed penalization parameter  $\zeta > 0$ , we construct by Galerkin method a weak solution for (2.1) such that

1.  $m^\zeta \in L^\infty(\mathbb{R}^+; H^1(\Omega; \mathbb{R}^3))$  and  $\frac{\partial m^\zeta}{\partial t} \in L^2(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^3))$ ,
2.  $u^\zeta \in L^\infty(\mathbb{R}^+; V(\Omega))$  and  $\frac{\partial u^\zeta}{\partial t} \in L^\infty(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^3))$ ,
3. for all  $t \geq 0$ , we have the following energy inequality:

$$\begin{aligned}
& \mathcal{E}(m^\zeta(t), u^\zeta(t), \frac{\partial u^\zeta}{\partial t}(t)) + \frac{1}{4\zeta} \int_\Omega (|m^\zeta(t, x)|^2 - 1)^2 dx \\
& + \int_0^t \int_\Omega \left| \frac{\partial m^\zeta(s, x)}{\partial t} \right|^2 ds dx \leq \mathcal{E}(m_0, u_0, u_1).
\end{aligned}$$

We recall that since the surface measure of  $\Gamma_1$  is positive, the Korn inequality below is valid (see [13], Theorem 6.3-4 page 292): there exists a constant  $c(\Omega)$  such that for all  $v \in V(\Omega)$ ,

$$\int_\Omega \varepsilon(v) : \varepsilon(v) \geq c(\Omega) \int_\Omega |v|^2. \quad (2.2)$$

This inequality yields that  $\|\cdot\|_{V(\Omega)} = \left( \int_\Omega \varepsilon(\cdot) : \varepsilon(\cdot) \right)^{\frac{1}{2}}$  is a norm on  $V(\Omega)$  equivalent to the norm  $\|\cdot\|_{H^1(\Omega)} = \left( \int_\Omega |\cdot|^2 + |\nabla \cdot|^2 \right)^{\frac{1}{2}}$ , and then it is also equivalent to the norm  $|\cdot|_{1, \Omega} = \left( \int_\Omega |\nabla \cdot|^2 \right)^{\frac{1}{2}}$ .

### 2.1.1 First step: Galerkin approximation

For  $m$ , we use a Galerkin basis  $(e_1, e_2, \dots)$  of eigenvectors of  $-\Delta$  with homogeneous Neumann conditions at the boundary.

$$\left\{ \begin{array}{l}
-\Delta e_i = \alpha_i e_i \text{ in } \Omega, \\
\partial_n e_i = 0 \text{ on } \partial\Omega.
\end{array} \right. \quad (2.3)$$

We denote by  $\mathbf{V}_N = \text{span}(e_1, \dots, e_N)$  and by  $P_N$  the orthogonal projection map onto  $\mathbf{V}_N$ .

For  $u$ , we use a the Galerkin basis  $(\phi_1, \phi_2, \dots)$  of eigenvectors of  $-\text{div}(\lambda^e : \varepsilon)$  with homogeneous mixed conditions at the boundary:

$$\begin{cases} -\text{div}(\lambda^e : \varepsilon(\phi_j)) = b_j \phi_j \text{ in } \Omega, \\ \phi_j = 0 \text{ on } \Gamma_1, \\ (\lambda^e : \varepsilon(\phi_j)) \cdot n = 0 \text{ on } \Gamma_2. \end{cases} \quad (2.4)$$

We denote by  $\mathbf{W}_N = \text{span}(\phi_1, \dots, \phi_N)$  and by  $\Pi_N$  the orthogonal projection map into  $\mathbf{W}_N$ .

For a fixed  $N$ , we consider the following Cauchy problem: Find  $m_N : [0, T_N[ \rightarrow \mathbf{V}_N$  and  $u_N : [0, T_N[ \rightarrow \mathbf{W}_N$ , such that  $\forall g_1 \in \mathbf{V}_N$  and  $g_2 \in \mathbf{W}_N$ , we have

$$\left\{ \begin{array}{l} \int_{\Omega} \frac{\partial m_N}{\partial t} \cdot g_1 + \int_{\Omega} m_N \times \frac{\partial m_N}{\partial t} \cdot g_1 - 2 \int_{\Omega} H_{\text{eff}}^N \cdot g_1 + \frac{1}{\zeta} \int_{\Omega} (|m_N|^2 - 1) m_N \cdot g_1 = 0 \text{ in } \mathbb{R}^+, \\ H_{\text{eff}}^N = \Delta m_N + h_d(m_N) + \Psi(m_N) + (\lambda^m : \sigma_N) m_N, \\ \sigma_N = \lambda^e : \varepsilon_N^e \text{ with } \varepsilon_N^e = \varepsilon(u_N) - \lambda^m : m_N \otimes m_N, \\ \int_{\Omega} \frac{d^2 u_N}{dt^2} \cdot g_2 + \int_{\Omega} (\lambda^e : \sigma_N) : \varepsilon(g_2) - \int_{\Gamma_2} f \cdot g_2 = 0 \text{ in } \mathbb{R}^+, \\ m_N(t=0) = P_N(m_0), \quad u_N(t=0) = \Pi_N(u_0), \quad \frac{\partial u_N}{\partial t}(t=0) = \Pi_N(u_1). \end{array} \right. \quad (2.5)$$

The quantities  $m_N$ ,  $u_N$ ,  $\sigma_N$  and  $\varepsilon_N^e$  depend on  $\zeta$ . We don't mention this dependence to lighten the notations. By the Cauchy-Lipschitz theorem, as in subsection 2.1 in [10], there exists a unique solution  $(m_N, u_N)$  for (2.5) whose maximal existence time is denoted by  $T_N$ .

### 2.1.2 Energy estimate on the Galerkin approximation

We denote by  $\mathcal{E}_N(t)$  the following quantity:

$$\begin{aligned} \mathcal{E}_N(t) &:= \mathcal{E}(m_N(t, \cdot), u_N(t, \cdot), \frac{\partial u_N}{\partial t}(t, \cdot)), \\ &= \int_{\Omega} |\nabla m_N(t, x)|^2 dx + 2 \int_{\Omega} \Phi(m_N(t, x)) dx + \int_{\mathbb{R}^3} |h_d(m_N(t, \cdot))(x)|^2 dx \\ &\quad + \frac{1}{2} \int_{\Omega} [\lambda^e : \varepsilon_N^e(t, x)] : \varepsilon_N^e(t, x) dx + \frac{1}{2} \int_{\Omega} \left| \frac{\partial u_N}{\partial t}(t, x) \right|^2 dx \\ &\quad - \int_{\Gamma_2} f(s) \cdot u_N(t, s) d\Gamma_s. \end{aligned} \quad (2.6)$$

Using the symmetry of  $\lambda^e$ , we have:

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_N &= 2 \int_{\Omega} \nabla m_N : \nabla \frac{\partial m_N}{\partial t} + 2 \int_{\Omega} \nabla \Phi(m_N) \cdot \frac{\partial m_N}{\partial t} + 2 \int_{\mathbb{R}^3} h_d(m_N) \cdot h_d\left(\frac{\partial m_N}{\partial t}\right) \\ &\quad + \int_{\Omega} (\lambda^e : \varepsilon_N^e) : \frac{\partial \varepsilon_N^e}{\partial t} + \int_{\Omega} \frac{\partial^2 u_N}{\partial t^2} \cdot \frac{\partial u_N}{\partial t} - \int_{\Gamma_2} f \cdot \frac{\partial u_N}{\partial t}. \end{aligned}$$

Since  $-h_d$  is an orthogonal projection for the  $L^2(\mathbb{R}^3)$ -inner product, we have:

$$\int_{\mathbb{R}^3} h_d(m_N) \cdot h_d\left(\frac{\partial m_N}{\partial t}\right) = - \int_{\Omega} h_d(m_N) \cdot \frac{\partial m_N}{\partial t}.$$

In addition, using the symmetry of  $\lambda^m$ , we have:

$$\frac{\partial \varepsilon_N^e}{\partial t} = \varepsilon\left(\frac{\partial u_N}{\partial t}\right) - 2\lambda^m : m_N \otimes \frac{\partial m_N}{\partial t}.$$

Using that  $\sigma_N = \lambda^e : \varepsilon_N^e$ , by the symmetry of  $\lambda^e$  and  $\lambda^m$ , we have:

$$\sigma_N : \left(\lambda^m : m_N \otimes \frac{\partial m_N}{\partial t}\right) = [(\lambda^m : \sigma_N)m_N] \cdot \frac{\partial m_N}{\partial t}.$$

Therefore, we have:

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_N = & 2 \int_{\Omega} \nabla m_N : \nabla \frac{\partial m_N}{\partial t} - 2 \int_{\Omega} \Psi(m_N) \cdot \frac{\partial m_N}{\partial t} - 2 \int_{\Omega} h_d(m_N) \cdot \frac{\partial m_N}{\partial t} \\ & - 2 \int_{\Omega} [(\lambda^m : \sigma_N)m_N] \cdot \frac{\partial m_N}{\partial t} \int_{\Omega} \frac{\partial^2 u_N}{\partial t^2} \cdot \frac{\partial u_N}{\partial t} + \int_{\Omega} \sigma_N : \varepsilon\left(\frac{\partial u_N}{\partial t}\right) - \int_{\Gamma_2} f \cdot \frac{\partial u_N}{\partial t}. \end{aligned}$$

Taking  $g_1 = \frac{\partial m_N}{\partial t}$  and  $g_2 = \frac{\partial u_N}{\partial t}$  in (2.5), we obtain that

$$\frac{d}{dt} \left( \mathcal{E}_N + \frac{1}{4\zeta} \int_{\Omega} (|m_N|^2 - 1)^2 \right) + \int_{\Omega} \left| \frac{\partial m_N}{\partial t} \right|^2 = 0,$$

and integrating from  $t = 0$  to  $t = T$ , we get that for all  $T < T_N$

$$\mathcal{E}_N(T) + \frac{1}{4\zeta} \int_{\Omega} (|m_N|^2 - 1)^2 + \int_0^T \int_{\Omega} \left| \frac{\partial m_N}{\partial t} \right|^2 = \mathcal{E}_N(0) + \frac{1}{4\zeta} \int_{\Omega} (|P_N(m_0)|^2 - 1)^2. \quad (2.7)$$

### 2.1.3 Limit in the Galerkin Approximation

We claim the following proposition:

**Proposition 2.1.** *For all fixed  $\zeta > 0$ , the right-hand-side of (2.7) tends to  $\mathcal{E}(m_0, u_0, u_1)$  when  $N$  tends to  $+\infty$ .*

*Proof.* (see also [10]) Since the  $e'_i$ 's form an hilbertian basis in  $L^2(\Omega)$ ,  $P_N(m_0)$  tends to  $m_0$  in  $L^2(\Omega)$ . Writing that:

$$\begin{aligned} \int_{\Omega} |\nabla P_N(m_0)|^2 &= - \int_{\Omega} \Delta P_N(m_0) \cdot P_N(m_0) \quad \text{since } \partial_n P_N(m_0) = 0 \text{ on } \partial\Omega, \\ &= - \int_{\Omega} \Delta P_N(m_0) \cdot m_0 \quad \text{since } P_N \text{ is self-adjoint and } \mathbf{V}_N \text{ is stable by } \Delta, \\ &= \int_{\Omega} \nabla P_N(m_0) \cdot \nabla m_0, \end{aligned} \quad (2.8)$$

we obtain that the sequence  $(\nabla P_N(m_0))_N$  is bounded in  $L^2(\Omega)$  by  $\|\nabla m_0\|_{L^2(\Omega)}$ , so this sequence tends to  $\nabla m_0$  in  $L^2(\Omega)$  weak, and since (2.8) yields that  $\|\nabla P_N(m_0)\|_{L^2(\Omega)}$  tends to  $\|\nabla m_0\|_{L^2(\Omega)}$ , we obtain that:

$$P_n(m_0) \longrightarrow m_0 \text{ strongly in } H^1(\Omega) \text{ and in } L^4(\Omega) \text{ by Sobolev embedding.} \quad (2.9)$$

In the same way and using the boundary conditions in (2.4), we have:

$$\int_{\Omega} (\lambda^e : \varepsilon(\Pi_N(u_0))) : \varepsilon(\Pi_N(u_0)) = \int_{\Omega} (\lambda^e : \varepsilon(\Pi_N(u_0))) : \varepsilon(u_0). \quad (2.10)$$

Since  $\lambda^e$  is positive, by Cauchy–Schwartz type inequality, we have:

$$\begin{aligned} & \int_{\Omega} (\lambda^e : \varepsilon(\Pi_N(u_0))) : \varepsilon(\Pi_N(u_0)) \\ & \leq \left( \int_{\Omega} (\lambda^e : \varepsilon(\Pi_N(u_0))) : \varepsilon(\Pi_N(u_0)) \right)^{\frac{1}{2}} \left( \int_{\Omega} (\lambda^e : \varepsilon(u_0)) : \varepsilon(u_0) \right)^{\frac{1}{2}}, \end{aligned}$$

so that, the sequence  $(\varepsilon(\Pi_N(u_0)))_N$  is bounded in  $L^2(\Omega)$ , so converges weakly to  $\varepsilon(u_0)$  in  $L^2(\Omega)$ . Using (2.10) and the positivity of  $\lambda^e$ , we obtain that:

$$\varepsilon(\Pi_N(u_0)) \longrightarrow \varepsilon(u_0) \quad \text{strongly in } L^2(\Omega), \quad (2.11)$$

and by Korn inequality (2.2),

$$\Pi_N(u_0) \longrightarrow u_0 \quad \text{strongly in } H^1(\Omega). \quad (2.12)$$

Therefore, using that  $\Phi$  is quadratic,  $h_d$  is continuous in  $L^2$ ,  $\Pi_N(u_1)$  tends to  $u_1$  in  $L^2(\Omega)$ , (2.9), (2.11) and (2.12), we obtain that

$$\mathcal{E}_N(0) + \frac{1}{4\zeta} \int_{\Omega} (|P_N(m_0)|^2 - 1)^2 \longrightarrow \mathcal{E}(m_0, u_0, u_1) + \frac{1}{4\zeta} \int_{\Omega} (|m_0|^2 - 1)^2 \quad \text{when } N \rightarrow +\infty,$$

and since  $m_0$  satisfies the saturation constraint  $|m_0| = 1$  *a.e.*, we have:

$$\mathcal{E}_N(0) + \frac{1}{4\zeta} \int_{\Omega} (|P_N(m_0)|^2 - 1)^2 \longrightarrow \mathcal{E}(m_0, u_0, u_1) \quad \text{when } N \rightarrow +\infty. \quad (2.13)$$

□

We establish the following lemma:

**Lemma 2.1.** *There exists a constant  $C$  such that for all  $u \in V(\Omega)$  and for all  $m \in H^1(\Omega; \mathbb{R}^3)$ ,*

$$\frac{1}{2} \int_{\Omega} [\lambda^e : (\varepsilon(u) - \lambda^m : m \otimes m)] : (\varepsilon(u) - \lambda^m : m \otimes m) \geq \frac{\lambda^{e*}}{4} \|\varepsilon(u)\|_{L^2(\Omega)}^2 - C \|m\|_{L^4(\Omega)}^4.$$

where  $\lambda^{e*}$  is the coercivity constant appearing in (1.17).

*Proof.* From (1.17), we have:

$$\begin{aligned}
[\lambda^e : (\varepsilon(u) - \lambda^m : m \otimes m)] : (\varepsilon(u) - \lambda^m : m \otimes m) &\geq \lambda^{e*} |\varepsilon(u) - \lambda^m : m \otimes m|^2, \\
&\geq \lambda^{e*} \left| |\varepsilon(u)| - |\lambda^m : m \otimes m| \right|^2, \\
&\geq \frac{\lambda^{e*}}{2} |\varepsilon(u)|^2 - \lambda^{e*} |\lambda^m : m \otimes m|^2
\end{aligned}$$

using that  $(a - b)^2 \geq \frac{1}{2}a^2 - b^2$ . By integrating on  $\Omega$  and using that there exists a constant  $K$  such that  $|\lambda^m : m \otimes m|^2 \leq K|m|^4$ , we conclude the proof of Lemma 2.1.  $\square$

By the previous lemma, we obtain that

$$\begin{aligned}
\frac{1}{2} \int_{\Omega} [\lambda^e : (\varepsilon(u_N) - \lambda^m : m_N \otimes m_N)] : (\varepsilon(u_N) - \lambda^m : m_N \otimes m_N) &\geq \\
&\frac{1}{4} \lambda^{e*} \|\varepsilon(u_N)\|_{L^2(\Omega)}^2 - C \|m_N\|_{L^4(\Omega)}^4.
\end{aligned}$$

Moreover, since the trace application  $H^1(\Omega) \rightarrow H^{\frac{1}{2}}(\Gamma)$  is linear continuous, using also that the  $H^1(\Omega)$  is equivalent to  $\|\varepsilon(u_N)\|_{L^2(\Omega)}$  by the Korn inequality (2.2)). there exist a constant  $c$  such that

$$\left| \int_{\Gamma_2} f \cdot u_N d\Gamma \right| \leq c \|f\|_{L^2(\Gamma_2)} \|\varepsilon(u_N)\|_{L^2(\Omega)}.$$

So we obtain that:

$$\begin{aligned}
\mathcal{E}_N + \frac{1}{4\zeta} \int_{\Omega} (|m_N|^2 - 1)^2 &\geq \|\nabla m_N\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{\partial u_N}{\partial t} \right\|_{L^2(\Omega)}^2 + \frac{1}{4} \lambda^{e*} \|\varepsilon(u_N)\|_{L^2(\Omega)}^2 \\
&\quad - C \|m_N\|_{L^4(\Omega)}^4 - c \|f\|_{L^2(\Gamma_2)} \|\varepsilon(u_N)\|_{L^2(\Omega)} + \frac{1}{4\zeta} \int_{\Omega} (|m_N|^2 - 1)^2.
\end{aligned}$$

Using that  $(|\xi|^2 - 1)^2 \geq \frac{1}{2}|\xi|^4 - 1$ , we have:

$$\frac{1}{8\zeta} \int_{\Omega} (|m_N|^2 - 1)^2 \geq \frac{1}{16\zeta} \|m_N\|_{L^4(\Omega)}^4 - \frac{1}{8\zeta} \text{meas}(\Omega).$$

Thus, if  $\zeta$  is small enough,

$$\frac{1}{8\zeta} \int_{\Omega} (|m_N|^2 - 1)^2 \geq C \|m_N\|_{L^4(\Omega)}^4 - \frac{1}{8\zeta} \text{meas} \Omega.$$

so that:

$$\begin{aligned}
\mathcal{E}_N + \frac{1}{4\zeta} \int_{\Omega} (|m_N|^2 - 1)^2 &\geq \|\nabla m_N\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{\partial u_N}{\partial t} \right\|_{L^2(\Omega)}^2 + \frac{1}{4} \lambda^{e*} \|\varepsilon(u_N)\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{8\zeta} \int_{\Omega} (|m_N|^2 - 1)^2 - \frac{1}{8\zeta} \text{meas}(\Omega) - c \|f\|_{L^2(\Gamma_2)} \|\varepsilon(u_N)\|_{L^2(\Omega)}.
\end{aligned} \tag{2.14}$$

By (2.7) and (2.13), for a fixed  $\zeta$  small enough, the left-hand-side of (2.14) is uniformly bounded with respect to  $N$ , so there exists  $K$  such that for all  $N$ ,

$$\|\nabla m_N\|_{L^\infty(0,T_N;L^2(\Omega))} + \|m_N\|_{L^\infty(0,T_N;L^4(\Omega))} + \left\|\frac{\partial m_N}{\partial t}\right\|_{L^2(0,T_N;L^2(\Omega))} \leq K,$$

$$\left\|\frac{\partial u_N}{\partial t}\right\|_{L^\infty(0,T_N;L^2(\Omega))} + \|\varepsilon(u_N)\|_{L^\infty(0,T_N;L^2(\Omega))} \leq K.$$

This implies directly that  $T_N = +\infty$ . In addition, since the bounds do not depend on  $N$ , we can assume that there exists a sub-sequence still denoted by  $(m_N, u_N)_N$ , and there exists  $(m^\zeta, u^\zeta)$  such that for all  $T > 0$ :

- (i)  $m_N \rightharpoonup m^\zeta$  in  $L^\infty(0, T; H^1(\Omega))$  weak\*,
- (ii)  $m_N \rightarrow m^\zeta$  in  $L^\infty(0, T; L^4(\Omega))$  strong (by applying the Aubin-Simon Lemma, see [4] and [24], or [5] Theorem II.5.16),
- (iii)  $\frac{\partial m_N}{\partial t} \rightharpoonup \frac{\partial m^\zeta}{\partial t}$  in  $L^2(0, T; L^2(\Omega))$  weak,
- (iv)  $u_N \rightharpoonup u^\zeta$  in  $L^\infty(0, T; V(\Omega))$  weak \*,
- (v)  $\frac{\partial u_N}{\partial t} \rightharpoonup \frac{\partial u^\zeta}{\partial t}$  in  $L^\infty(0, T; L^2(\Omega))$  weak \*.

(2.15)

From (ii) and (iv) in (2.15), since  $\sigma_N = \lambda^e : (\varepsilon(u_N) - \lambda^m : m_N \otimes m_N)$ ,

$$\sigma_N \rightharpoonup \sigma^\zeta = \lambda^e : (\varepsilon(u^\zeta) - \lambda^m : m^\zeta \otimes m^\zeta) \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak *}. \quad (2.16)$$

Using Aubin-Simon Lemma,  $m_N$  (resp.  $u_N$ ) tends to  $m^\zeta$  (resp.  $u^\zeta$ ) in  $\mathcal{C}^0([0, T]; L^2(\Omega))$ . Since  $P_N(m_0) = m_N(0)$  (resp.  $\Pi_N(u_0) = u_N(0)$ ) tends to  $m_0$  (resp.  $u_0$ ) in  $L^2(\Omega)$ , by the uniqueness of the limit, we obtain:

$$m^\zeta(t=0) = m_0 \text{ and } u^\zeta(t=0) = u_0.$$

In order to obtain the equations satisfied by  $m^\zeta$  and  $u^\zeta$ , we fix  $N_0$  and we consider  $g_1 \in \mathbf{V}_{N_0}$ ,  $g_2 \in \mathbf{W}_{N_0}$  and  $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}$ , a smooth function with compact support included in  $[0, T]$ . For all  $N \geq N_0$ ,  $g_1 \in \mathbf{V}_N$  and  $g_2 \in \mathbf{W}_N$ , so we can take these test functions in (2.5). Multiplying by  $\tau(t)$  and integrating on the time interval  $[0, T]$ , we obtain using an integration by part in the space variable in the first equation, we obtain that for all  $N \geq N_0$ :

$$\begin{aligned} & \int_{[0,T] \times \Omega} \left( \frac{\partial m_N}{\partial t} + m_N \times \frac{\partial m_N}{\partial t} \right) \cdot g_1(x) \tau(t) + 2 \int_{[0,T] \times \Omega} \nabla m_N \cdot \nabla g_1(x) \tau(t) \\ & - 2 \int_{[0,T] \times \Omega} (h_d(m_N) + \Psi(m_N) + (\lambda^m : \sigma_N) m_N) \cdot g_1(x) \tau(t) \\ & + \frac{1}{\zeta} \int_{[0,T] \times \Omega} (|m_N|^2 - 1) m_N \cdot g_1(x) \tau(t) = 0 \text{ in } \mathbb{R}^+. \end{aligned} \quad (2.17)$$



Using (i)-(iii) in (2.15) and (2.16), using also the strong convergence in  $L^\infty(0, T; L^4(\Omega))$  for the non linear terms (ii), we obtain when  $N$  tends to  $+\infty$  that:

$$\begin{aligned} & \int_{[0, T] \times \Omega} \left( \frac{\partial m^\zeta}{\partial t} + m^\zeta \times \frac{\partial m^\zeta}{\partial t} \right) \cdot g_1(x) \tau(t) + 2 \int_{[0, T] \times \Omega} \nabla m^\zeta \cdot \nabla g_1(x) \tau(t) \\ & - 2 \int_{[0, T] \times \Omega} \left( h_d(m^\zeta) + \Psi(m^\zeta) + (\lambda^m : \sigma^\zeta) m^\zeta \right) \cdot g_1(x) \tau(t) \\ & + \frac{1}{\zeta} \int_{[0, T] \times \Omega} (|m^\zeta|^2 - 1) m^\zeta \cdot g_1(x) \tau(t) = 0 \text{ in } \mathbb{R}^+. \end{aligned} \quad (2.18)$$

Concerning the wave equation in (2.5), using an integration by part both in the variables  $t$  and  $x$ , using that  $\frac{\partial u_N}{\partial t}(0) = \Pi_N(u_1)$  and that  $\tau(T) = 0$ , we obtain that for all  $N \geq N_0$ :

$$\begin{aligned} & - \int_{[0, T] \times \Omega} \frac{\partial u_N}{\partial t} \cdot g_2(x) \frac{\partial \tau}{\partial t} - \int_{\Omega} \Pi_N(u_1) g_2(x) \tau(0) + \int_{[0, T] \times \Omega} (\lambda^e : \sigma_N) : \varepsilon(g_2)(x) \tau(t) \\ & - \int_{[0, T] \times \Gamma_2} f \cdot g_2(x) \tau(t) = 0. \end{aligned} \quad (2.19)$$

Using (iv)-(v) in (2.15), (2.16), and that  $\Pi_N(u_1) \rightarrow u_1$  in  $L^2(\Omega)$ , we obtain that:

$$\begin{aligned} & - \int_{[0, T] \times \Omega} \frac{\partial u^\zeta}{\partial t} \cdot g_2(x) \frac{\partial \tau}{\partial t} - \int_{\Omega} u_1 g_2(x) \tau(0) + \int_{[0, T] \times \Omega} (\lambda^e : \sigma^\zeta) : \varepsilon(g_2)(x) \tau(t) \\ & - \int_{[0, T] \times \Gamma_2} f \cdot g_2(x) \tau(t) = 0. \end{aligned} \quad (2.20)$$

The obtained limit equations (2.18)-(2.20) are true for all  $g_1 \in \bigcup_{N_0} \mathbf{V}_{N_0}$ ,  $g_2 \in \bigcup_{N_0} \mathbf{W}_{N_0}$  and  $\tau \in \mathcal{C}^\infty(\mathbb{R}^+; \mathbb{R})$  of compact support, and by density arguments, we obtain that: for all  $\chi \in \mathcal{C}^1(\mathbb{R}^+; H^1(\Omega))$  with  $\text{supp } \chi \subset [0, T]$ ,

$$\begin{aligned} & \int_{[0, T] \times \Omega} \left( \frac{\partial m^\zeta}{\partial t} + m^\zeta \times \frac{\partial m^\zeta}{\partial t} \right) \cdot \chi(t, x) + 2 \int_{[0, T] \times \Omega} \nabla m^\zeta \cdot \nabla \chi(t, x) \\ & - 2 \int_{[0, T] \times \Omega} \left( h_d(m^\zeta) + \Psi(m^\zeta) + (\lambda^m : \sigma^\zeta) m^\zeta \right) \cdot \chi(t, x) \\ & + \frac{1}{\zeta} \int_{[0, T] \times \Omega} (|m^\zeta|^2 - 1) m^\zeta \cdot \chi(t, x) = 0 \text{ in } \mathbb{R}^+, \end{aligned} \quad (2.21)$$

and for all  $\chi \in \mathcal{C}^1(\mathbb{R}^+; V(\Omega))$  with  $\text{supp } \chi \subset [0, T]$ ,

$$\begin{aligned} & - \int_{[0, T] \times \Omega} \frac{\partial u^\zeta}{\partial t} \cdot \frac{\partial \chi}{\partial t}(t, x) - \int_{\Omega} u_1 \chi(0, x) + \int_{[0, T] \times \Omega} (\lambda^e : \sigma^\zeta) : \varepsilon(\chi)(t, x) \\ & - \int_{[0, T] \times \Gamma_2} f \cdot \chi(t, x) = 0. \end{aligned} \quad (2.22)$$

Finally, using weak convergence and convexity arguments for the quadratic terms in  $\mathcal{E}$ , using strong convergence in  $L^\infty(0, T; L^4(\Omega))$  for the penalisation term in (2.7), by the convergence at initial time (2.13), we obtain that  $(m^\zeta, u^\zeta)$  satisfies the following energy formula

$$\mathcal{E}(m^\zeta(t, \cdot), u^\zeta(t, \cdot), \frac{\partial u^\zeta}{\partial t}(t, \cdot)) + \frac{1}{4\zeta} \int_{\Omega} (|m^\zeta| - 1)^2 + \int_0^t \int_{\Omega} \left| \frac{\partial m^\zeta}{\partial t} \right|^2 \leq \mathcal{E}(m_0, u_0, u_1). \quad (2.23)$$

## 2.2 Limit when $\zeta$ tends to zero

We remark that the right side of the estimate of energy (2.23) does not depend on  $\zeta$ , then, by using the same arguments as in the previous section, we obtain uniform bounds with respect to  $\zeta$  for the following quantities:

- $\frac{\partial m^\zeta}{\partial t}$  in  $L^2(\mathbb{R}^+; L^2(\Omega))$ ,
- $\nabla m^\zeta$  in  $L^\infty(\mathbb{R}^+; L^2(\Omega))$ ,
- $m^\zeta$  in  $L^\infty(\mathbb{R}^+; L^4(\Omega))$ ,
- $\varepsilon(u^\zeta)$  in  $L^\infty(\mathbb{R}^+; L^2(\Omega))$ ,
- $\frac{\partial u^\zeta}{\partial t}$  in  $L^\infty(\mathbb{R}^+; L^2(\Omega))$ .

Therefore, there exists a subsequence still noted  $(m^\zeta, u^\zeta)_\zeta$ , such that for all  $T > 0$ ,

- $m^\zeta \rightharpoonup m$  in  $L^\infty(0, T; H^1(\Omega))$  weak  $*$ ,
- $m^\zeta \rightarrow m$  in  $L^\infty(0, T; L^4(\Omega))$  strong (by applying the Aubin-Simon lemma [5] Theorem II.5.16),
- $\frac{\partial m^\zeta}{\partial t} \rightharpoonup \frac{\partial m}{\partial t}$  in  $L^2(0, T; L^2(\Omega))$  weak,
- $u^\zeta \rightharpoonup u$  in  $L^\infty(0, T; V(\Omega))$  weak  $*$ ,
- $\frac{\partial u^\zeta}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$  in  $L^\infty(0, T; L^2(\Omega))$  weak  $*$ .

Since

$$\mathcal{E}(m^\zeta(t, \cdot), u^\zeta(t, \cdot), \frac{\partial u^\zeta}{\partial t}(t, \cdot)) \geq -\|f\|_{L^2(\Gamma_2)} \|u^\zeta\|_{L^\infty(0, T; L^2(\Gamma_2))},$$

since  $u^\zeta$  is uniformly bounded in  $L^\infty(0, T; H^1(\Omega))$ , we obtain by (2.23) that  $\frac{1}{4\zeta} \int_{\Omega} (|m^\zeta|^2 - 1)^2$  is bounded uniformly with respect to  $\zeta$ . So  $\int_{\Omega} (|m^\zeta|^2 - 1)^2 \rightarrow 0$  as  $\zeta$  tends to zero, and since  $m^\zeta \rightarrow m$  in  $L^\infty(0, T; L^4(\Omega))$  strong, then we obtain that  $m$  satisfies the saturation constraint (1.21).

Now, using the strong and weak convergence and by taking the limit of (2.23) as  $\zeta \rightarrow 0$ , we obtain that for all  $t > 0$

$$\mathcal{E}(m(t), u(t), \frac{\partial u}{\partial t}(t)) + \int_0^t \int_{\Omega} \left| \frac{\partial m}{\partial t}(s, x) \right|^2 dx \leq \mathcal{E}(m_0, u_0, u_1). \quad (2.24)$$

In order to show that  $(m, u)$  satisfies the Landau-Lifschitz-Gilbert equation, we take the test function  $(t, x) \mapsto m^\zeta(t, x) \times \chi(t, x)$  in (2.21), where  $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^+; H^1(\Omega))$  compactly supported in  $[0, T[$  (as in [10], [2] and [11]), then by making the limit as  $\zeta$  goes to zero, we obtain

$$\begin{aligned} \int_{\mathbb{R}^+ \times \Omega} \left( \frac{\partial m}{\partial t} - m \times \frac{\partial m}{\partial t} \right) \chi(t, x) dt dx &= 2 \int_{\mathbb{R}^+ \times \Omega} \sum_{i=1}^3 m \times \frac{\partial m}{\partial x_i} \cdot \frac{\partial \chi}{\partial x_i} \\ &\quad - 2 \int_{\mathbb{R}^+ \times \Omega} m \times (h_d(m) + \Psi(m)) \cdot \chi \\ &\quad - 2 \int_{\mathbb{R}^+ \times \Omega} m \times \left( (\lambda^m : (\lambda^e : (\varepsilon(u) - \lambda^m : m \otimes m))) m \right) \cdot \chi. \end{aligned}$$

Furthermore, taking the limit when  $\zeta$  tends to zero in (2.22) with a test function  $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^+; V(\Omega))$ , we obtain

$$\begin{aligned} \int_{\mathbb{R}^+ \times \Omega} \frac{\partial u}{\partial t} \cdot \frac{\partial \chi}{\partial t} - \int_{\mathbb{R}^+ \times \Omega} (\lambda^e : \varepsilon(u)) : \varepsilon(\chi) + \int_{\Omega} u_1 \chi(0, x) dx + \int_{\mathbb{R}^+ \times \Gamma_2} f \cdot \chi \\ = - \int_{\mathbb{R}^+ \times \Omega} (\lambda^e : (\lambda^m : m \otimes m)) : \varepsilon(\chi). \end{aligned}$$

Consequently,  $(m, u)$  is a global in time weak solution of (1.15)-(1.16)-(1.20), which concludes the proof of Theorem 1.1 .

### 3 Proof of Theorem 1.2

We define the following spaces:

$$\begin{aligned} \mathcal{V}(\omega) &= \{v \in H^1(\omega) ; v = 0 \text{ on } \mathcal{C}_1\}, \\ \mathcal{W}(\omega) &= \{v = (v_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega), v_i = 0 \text{ on } \mathcal{C}_1 \text{ and } \partial_1 v_3 = \partial_2 v_3 = 0 \text{ on } \mathcal{C}_1\}, \\ V_{KL}(\Omega_1) &= \{\xi \in H^1(\Omega_1) ; \xi = 0 \text{ on } \Gamma_1^1 \text{ and } \varepsilon_{i3}(\xi) = 0 \text{ in } \Omega_1 \text{ for } i \in \{1, 2, 3\}\}. \end{aligned} \quad (3.1)$$

We recall without proof the following result, proved in details in [12] (of Th. 1.4.1.).

**Lemma 3.1.** *The application  $I$  defined as follows:*

$$\begin{aligned} I : \mathcal{W}(\omega) &\longrightarrow V_{KL}(\Omega_1) \\ v &\longmapsto ((v_\alpha - x_3 \partial_\alpha v_3), v_3), \end{aligned}$$

*is an isomorphism.*

Let  $m_0 \in H^1(\omega; S^2(0, 1))$ ,  $\tilde{u}_0 \in \mathcal{W}(\omega)$  and  $u_1 \in L^2(\omega; \mathbb{R}^3)$  such that  $(u_1)_3 = 0$ . We define  $m_0^\eta \in H^1(\Omega_\eta; S^2)$ ,  $u_0^\eta \in V(\Omega_\eta)$  and  $u_1^\eta \in L^2(\Omega_\eta; \mathbb{R}^3)$  by (1.30).

Let  $g^+$  and  $g^- \in L^2(\omega)$  and  $h \in L^2(\mathcal{C}_2)$ . From these data, we define on  $\Gamma_2^\eta$  the  $\eta$ -depending boundary data  $f^\eta$  by (1.29).

We consider then the solution for the initial and boundary value problem (1.15)-(1.16)-(1.20) with these initial and boundary data, and we define the rescaled variables  $M^\eta$ ,  $U^\eta$  and  $H^\eta$  by (1.31).

For  $\xi \in \mathcal{C}^\infty(\mathbb{R}^+; V(\Omega_1))$ , we denote by  $\varepsilon(\eta, \xi)$  the 2-tensor given by:

$$\begin{aligned} \varepsilon_{\alpha\beta}(\eta, \xi) &= \varepsilon_{\alpha\beta}(\xi) \text{ for } \alpha, \beta \in \{1, 2\}, \\ \varepsilon_{\alpha 3}(\eta, \xi) &= \frac{1}{\eta} \varepsilon_{\alpha 3}(\xi) \text{ for } \alpha \in \{1, 2\}, \\ \varepsilon_{33}(\eta, \xi) &= \frac{1}{\eta^2} \varepsilon_{33}(\xi), \end{aligned} \quad (3.2)$$

So that we have:

$$\varepsilon(u^\eta) = \varepsilon(\eta, U^\eta). \quad (3.3)$$

Now we rewrite the properties satisfied by  $m^\eta$  and  $u^\eta$  (see Definition 1.2) for the rescaled variables. We obtain that:

- $M^\eta \in L^\infty(\mathbb{R}^+; H^1(\Omega_1; \mathbb{R}^3))$ ,  $\frac{\partial M^\eta}{\partial t} \in L^2(\mathbb{R}^+; L^2(\Omega_1; \mathbb{R}^3))$ , and  $|M^\eta(t, x)| = 1$  for a.e.  $(t, x) \in \mathbb{R}^+ \times \Omega_1$ ,
- $M^\eta(0, x_1, x_2, x_3) = m_0^\eta(0, x_1, x_2, \eta x_3) = m_0(x_1, x_2)$  in the trace sense,
- $U^\eta \in L^\infty(\mathbb{R}^+; V(\Omega_1))$  and  $\frac{\partial U^\eta}{\partial t} \in L^\infty(\mathbb{R}^+; L^2(\Omega_1))$ ,
- $U^\eta(0, x_1, x_2, x_3) = u_0(x_1, x_2)$  in the trace sense,
- for all  $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^+; H^1(\Omega_1; \mathbb{R}^3))$ , we define  $\chi^\eta \in \mathcal{C}_c^\infty(\mathbb{R}^+; H^1(\Omega_\eta; \mathbb{R}^3))$  by:

$$\chi^\eta(t, x_1^\eta, x_2^\eta, x_3^\eta) = \chi(t, x_1, x_2, \frac{x_3}{\eta})$$

Taking  $\chi^\eta$  as a test function in (1.22), we obtain that:

$$\begin{aligned} \int_{\mathbb{R}^+ \times \Omega_1} \left( \frac{\partial M^\eta}{\partial t} - M^\eta \times \frac{\partial M^\eta}{\partial t} \right) \chi &= 2 \int_{\mathbb{R}^+ \times \Omega_1} \sum_{\alpha=1}^2 M^\eta \times \partial_\alpha M^\eta \cdot \partial_\alpha \chi \\ &+ \frac{2}{\eta^2} \int_{\mathbb{R}^+ \times \Omega_1} M^\eta \times \partial_3 M^\eta \cdot \partial_3 \chi - 2 \int_{\mathbb{R}^+ \times \Omega_1} M^\eta \times (H^\eta + \Psi(M^\eta)) \cdot \chi \\ &- 2 \int_{\mathbb{R}^+ \times \Omega_1} M^\eta \times (\lambda^m : (\lambda^e : (\varepsilon(\eta, U^\eta) - \lambda^m : M^\eta \otimes M^\eta))) M^\eta \cdot \chi, \end{aligned} \quad (3.4)$$

- for all  $\xi \in \mathcal{C}^\infty(\mathbb{R}^+; V(\Omega_1))$ , we define  $\xi^\eta \in \mathcal{C}^\infty(\mathbb{R}^+; V(\Omega_\eta))$  by:

$$\begin{aligned} \xi_\alpha^\eta(t, x_1, x_2, x_3) &= \xi_\alpha(t, x_1, x_2, \frac{x_3}{\eta}) \text{ for } \alpha = 1, 2, \\ \xi_3^\eta(t, x_1, x_2, x_3) &= \frac{1}{\eta} \xi_3(t, x_1, x_2, \frac{x_3}{\eta}). \end{aligned} \quad (3.5)$$

We remark then that

$$\varepsilon(\xi^\eta) = \varepsilon(\eta, \xi). \quad (3.6)$$

Taking  $\xi^\eta$  as a test function in (1.23), using that the third component of  $u_1 = 0$ , we obtain that:

$$\begin{aligned} & \int_{\mathbb{R}^+ \times \Omega_1} \sum_{\alpha=1}^2 \frac{\partial U_\alpha^\eta}{\partial t} \frac{\partial \xi_\alpha}{\partial t} + \frac{1}{\eta^2} \int_{\mathbb{R}^+ \times \Omega_1} \frac{\partial U_3^\eta}{\partial t} \frac{\partial \xi_3}{\partial t} + \int_{\Omega_1} u_1 \cdot \xi(0, x) \\ & - \int_{\mathbb{R}^+ \times \Omega_1} (\lambda^e : (\varepsilon(\eta, U^\eta) - \lambda^m : M^\eta \otimes M^\eta)) : \varepsilon(\eta, \xi) \\ & + \int_{\mathbb{R}^+ \times \Gamma_-^1} g^- \cdot \xi + \int_{\mathbb{R}^+ \times \Gamma_+^1} g^+ \cdot \xi + \int_{\mathbb{R}^+ \times \Gamma_2^1} h \cdot \xi = 0. \end{aligned} \quad (3.7)$$

- for all  $t > 0$ , from (1.24), we have

$$\mathcal{E}^\eta(t) + \int_0^t \int_{\Omega_1} \left| \frac{\partial M^\eta}{\partial t} \right|^2 \leq \mathcal{E}^\eta(0), \quad (3.8)$$

where

$$\begin{aligned} \mathcal{E}^\eta(t) & := \frac{1}{\eta} \mathcal{E}(m^\eta, u^\eta, \frac{\partial u^\eta}{\partial t}), \\ & = \int_{\Omega_1} \sum_{\alpha=1}^2 |\partial_\alpha M^\eta|^2 + \frac{1}{\eta^2} \int_{\Omega_1} |\partial_3 M^\eta|^2 + 2 \int_{\Omega_1} \Phi(M^\eta) + \int_{\mathbb{R}^3} |H^\eta|^2 \\ & + \frac{1}{2} \int_{\Omega_1} [\lambda^e : (\varepsilon(\eta, U^\eta) - \lambda^m : M^\eta \otimes M^\eta)] : (\varepsilon(\eta, U^\eta) - \lambda^m : M^\eta \otimes M^\eta) \\ & + \frac{1}{2} \int_{\Omega_1} \left( \sum_{\alpha=1}^2 \left| \frac{\partial U_\alpha^\eta}{\partial t} \right| + \frac{1}{2\eta^2} \left| \frac{\partial U_3^\eta}{\partial t} \right|^2 \right) - \int_{\Gamma_-^1} g^- \cdot U^\eta - \int_{\Gamma_+^1} g^+ \cdot U^\eta - \int_{\Gamma_2^1} h \cdot U^\eta. \end{aligned} \quad (3.9)$$

### 3.1 Uniform bound and limit when $\eta$ tends to zero

Since  $M^\eta(0)$  does not depend on  $x_3$ , we have:

$$\begin{aligned} \mathcal{E}^\eta(0) & := \frac{1}{\eta} \mathcal{E}(m_0^\eta, u_0^\eta, u_1^\eta) = \int_\omega \left[ \sum_{\alpha=1}^2 |\partial_\alpha m_0|^2 + 2\Phi(m_0) \right] dx_T + \int_{\mathbb{R}^3} |H^\eta(0)|^2 dx \\ & + \frac{1}{2} \int_{\Omega_1} [\lambda^e : (\varepsilon(\eta, U^\eta(0)) - \lambda^m : m_0 \otimes m_0)] : (\varepsilon(\eta, U^\eta(0)) - \lambda^m : m_0 \otimes m_0) \\ & + \frac{1}{2} \int_\omega |u_{1,T}|^2 dx_T - \int_\omega (g^- + g^+) dx_T \cdot u_0 - \int_{\Gamma_2^1} h \cdot u_0. \end{aligned}$$

Since  $\varepsilon_{i3}(U^\eta(0)) = 0$  for  $i \in \{1, 2, 3\}$ ,  $\varepsilon(\eta, U^\eta(0))$  does not depend on  $\eta$ . In addition,

$$\int_{\mathbb{R}^3} |H^\eta(0)|^2 dx = \frac{1}{\eta} \int_{\mathbb{R}^3} |h_d(m^\eta(0))|^2 \leq \frac{1}{\eta} \int_{\Omega_\eta} |m^\eta(0)|^2 \leq 2\text{meas}(\omega).$$

Therefore, there exists a constant  $C_1$ , independent of  $\eta$ , such that:

$$\forall \eta > 0, \mathcal{E}^\eta(0) \leq C_1. \quad (3.10)$$

Using (3.10) and the same arguments as in the proof of Lemma 2.1, using also the saturation constraint satisfied by  $M^\eta$ , we obtain that there exists a constant  $C$  such that for all  $\eta$ ,

$$\frac{1}{2} \int_{\Omega_1} [\lambda^e : (\varepsilon(\eta, U^\eta) - \lambda^m : M^\eta \otimes M^\eta)] : (\varepsilon(\eta, U^\eta) - \lambda^m : M^\eta \otimes M^\eta) \geq \frac{\lambda^{e*}}{4} \|\varepsilon(\eta, U^\eta)\|_{L^2(\Omega_1)}^2 - C. \quad (3.11)$$

By continuity of the trace from  $H^1(\Omega)$  into  $H^{\frac{1}{2}}(\partial\Omega)$ , we have

$$\int_{\Gamma_-^1} g^- \cdot U^\eta + \int_{\Gamma_+^1} g^+ \cdot U^\eta + \int_{\Gamma_+^{\frac{1}{2}}} h \cdot U^\eta \leq C_2 \|U^\eta\|_{H^1(\Omega_1)}, \quad (3.12)$$

where  $C_2$  is a constant depending only on  $\Omega_1$ .

Now, using the energy inequality (3.8), and inequalities (3.11), (3.12) and the fact that  $\Phi(M^\eta)$  is non negative, for all  $t > 0$ , we have

$$\begin{aligned} \sum_{\alpha=1}^2 \|\partial_\alpha M^\eta\|_{L^2(\Omega_1)}^2 + \frac{1}{\eta^2} \|\partial_3 M^\eta\|_{L^2(\Omega_1)}^2 + \|H^\eta\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \int_{\Omega_1} \left| \frac{\partial M^\eta}{\partial t} \right|^2 \\ + \frac{\lambda^{e*}}{4} \|\varepsilon(\eta, U^\eta)\|_{L^2(\Omega_1)}^2 - C_1 - C_2 \|U^\eta\|_{H^1(\Omega_1)} \\ + \sum_{\alpha=1}^2 \left\| \frac{\partial U_\alpha^\eta}{\partial t} \right\|_{L^2(\Omega_1)}^2 + \frac{1}{\eta^2} \left\| \frac{\partial U_3^\eta}{\partial t} \right\|_{L^2(\Omega_1)}^2 \leq \mathcal{E}^\eta(0), \end{aligned} \quad (3.13)$$

Using Definition (3.2) and Korn inequality 2.2 we can prove that

$$\|\varepsilon(\eta, \xi)\|_{L^2(\Omega_1)} \geq \|\varepsilon(\xi)\|_{L^2(\Omega_1)} \geq c \|\xi\|_{H^1(\Omega)} \text{ for all } \xi \in V(\Omega_1). \quad (3.14)$$

Hence, applying inequality (3.14) on (3.13), we obtain that, for all  $T > 0$  and  $\eta$  in a neighborhood of zero, there exists a constant  $C$  independent of  $\eta$  such that

- $\left\| \frac{\partial M^\eta}{\partial t} \right\|_{L^2(0,T;L^2(\Omega_1))} \leq C,$
- $\|\partial_1 M^\eta\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\partial_2 M^\eta\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C,$
- $\frac{1}{\eta} \|\partial_3 M^\eta\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C,$
- $\|U^\eta\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C,$
- $\left\| \frac{\partial U_1^\eta}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega_1))} + \left\| \frac{\partial U_2^\eta}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C,$
- $\frac{1}{\eta} \left\| \frac{\partial U_3^\eta}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C,$
- $\|\varepsilon(\eta, U^\eta)\|_{L^2(\Omega_1)} \leq C,$
- $\|H^\eta\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq C.$

So, we can extract subsequences, still denoted  $(M^\eta, U^\eta)$  and  $H^\eta$ , such that when  $\eta$  tends to zero:

$$\begin{aligned}
M^\eta &\rightharpoonup M \text{ in } L^\infty(0, T; H^1(\Omega_1)) \text{ weak } *, & \frac{\partial M^\eta}{\partial t} &\rightharpoonup \frac{\partial M}{\partial t} \text{ in } L^2(0, T; L^2(\Omega_1)) \text{ weak } *, \\
U^\eta &\rightharpoonup U \text{ in } L^\infty(0, T; V(\Omega_1)) \text{ weak } *, & \frac{\partial U^\eta}{\partial t} &\rightharpoonup \frac{\partial U}{\partial t} \text{ in } L^\infty(0, T; L^2(\Omega_1)) \text{ weak } *, \\
\varepsilon(\eta, U^\eta) &\rightharpoonup A \text{ in } L^\infty(0, T; L^2(\Omega_1)) \text{ weak } *, & \frac{\partial U_3^\eta}{\partial t} &\rightarrow 0 \text{ in } L^\infty(0, T; L^2(\Omega_1)), \\
H^\eta &\rightharpoonup H \text{ in } L^\infty(0, T; L^2(\mathbb{R}^3)) \text{ weak } *. & &
\end{aligned} \tag{3.15}$$

Using the Aubin-Simon lemma (see [5], Theorem II.5.16, or the original papers [4] and [24]), we can prove that  $M^\eta \rightarrow M$  in  $L^\infty(0, T; L^r(\Omega_1)) \cap C^0([0, T]; L^2(\Omega_1))$  strong for  $r < 6$ . In addition, we can extract a subsequence, still denoted by  $(M^\eta)_\eta$ , such that  $M^\eta \rightarrow M$  almost everywhere in  $[0, T] \times \Omega_1$ . Hence  $M$  verifies the saturation constraint

$$|M| = 1 \text{ a.e. in } \mathbb{R}^+ \times \Omega_1,$$

and by continuity in time with values in  $L^2(\Omega_1)$ ,  $M(0, x) = m_0(x)$  in the trace sense.

Furthermore, we have  $\partial_3 M^\eta \rightarrow 0$  in  $L^\infty(0, T; L^2(\Omega_1))$  strong, so  $M$  only depends on  $(t, x_1, x_2) \in \mathbb{R}^+ \times \omega$  and does not depend on  $x_3$ . In addition,  $\frac{\partial U_3}{\partial t} = 0$ .

### 3.2 Limit in the Landau-Lifschitz Equation

In order to characterize  $H$ , the weak limit of  $H^\eta$ , we apply Lemma 2.A in [8] to obtain:

$$H(t, x, y, z) = \begin{cases} - \begin{pmatrix} 0 \\ 0 \\ M_3(t, x, y) \end{pmatrix} & \text{for } (x, y, z) \in \Omega_1, \\ 0 & \text{for } x \notin \mathbb{R}^3 \setminus \Omega_1. \end{cases} \tag{3.16}$$

Let  $\tilde{\chi} \in C_c^\infty(\mathbb{R}^+; \mathcal{D}(\bar{\omega}))$ , and define  $\chi$  in  $\mathbb{R}^+ \times \Omega_1$  by  $\chi(t, x_1, x_2, x_3) = \tilde{\chi}(t, x_1, x_2)$ . Then,  $\chi \in C_c^\infty(\mathbb{R}^+; H^1(\Omega))$  and  $\partial_3 \chi = 0$ , so we can take  $\chi$  as a test function in (3.4). Using the strong convergence of  $M^\eta$  in  $L^\infty(0, T; L^4(\Omega_1))$  and the weak convergence of  $\frac{\partial M^\eta}{\partial t}$  and  $\nabla M^\eta$  in  $L^2(0, T \times \Omega_1)$ , by weak \* convergence of  $\mathcal{E}(\eta; U^\eta)$  in  $L^\infty(0, T; L^2(\Omega_1))$ , we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^+ \times \Omega_1} \left( \frac{\partial M}{\partial t} - M \times \frac{\partial M}{\partial t} \right) \tilde{\chi} &= -2 \int_{\mathbb{R}^+ \times \Omega_1} \sum_{\alpha=1}^2 M \times \partial_\alpha M \cdot \partial_\alpha \tilde{\chi} \\
&- 2 \int_{\mathbb{R}^+ \times \Omega_1} M \times (-M_3 e_3 + \Psi(M)) \cdot \tilde{\chi} - 2 \int_{\mathbb{R}^+ \times \Omega_1} M \times (\lambda^m : (\lambda^e : A)) M \cdot \tilde{\chi} \\
&+ 2 \int_{\mathbb{R}^+ \times \Omega_1} M \times ((\lambda^m : (\lambda^e : (\lambda^m : M \otimes M))) M) \cdot \tilde{\chi},
\end{aligned}$$

We denote  $\bar{A} := \frac{1}{2} \int_{-1}^1 A dx_3$ , so that  $\bar{A}$  is a symmetric 2-tensor. We define  $\tilde{\sigma}$  by:

$$\tilde{\sigma} = \lambda^e : \epsilon^e, \text{ with } \epsilon^e = \bar{A} - \lambda^m : M \otimes M. \tag{3.17}$$

Since  $\tilde{\chi}$  and  $M$  do not depend on  $x_3$ , we obtain that:

$$\begin{aligned} \int_{\mathbb{R}^+ \times \omega} \left( \frac{\partial M}{\partial t} - M \times \frac{\partial M}{\partial t} \right) \cdot \tilde{\chi} &= -2 \int_{\mathbb{R}^+ \times \omega} \sum_{\alpha=1}^2 M \times \partial_\alpha M \cdot \partial_\alpha \tilde{\chi} \\ &\quad -2 \int_{\mathbb{R}^+ \times \omega} M \times (-M_3 e_3 + \Psi(M)) \cdot \tilde{\chi} \\ &\quad -2 \int_{\mathbb{R}^+ \times \omega} M \times (\lambda^m : (\lambda^e : \tilde{\sigma})) M \cdot \tilde{\chi}. \end{aligned} \quad (3.18)$$

### 3.3 Limit in the elasticity wave Equation

We remark that  $(\frac{1}{\eta} \varepsilon_{\alpha 3}(U^\eta))_\eta$  for  $\alpha \in \{1, 2\}$  and  $(\frac{1}{\eta^2} \varepsilon_{33}(U^\eta))_\eta$  are bounded in  $L^\infty(0, T; L^2(\Omega))$ , because  $(\varepsilon(\eta, U^\eta))_\eta$  is bounded, this gives that  $\varepsilon_{i3}(U^\eta) \rightarrow 0$  strongly in  $L^\infty(0, T; L^2(\Omega))$ , for all  $T > 0$ , i.e.  $\varepsilon_{i3}(U) = 0$  for  $i \in \{1, 2, 3\}$ . Therefore,  $U \in L^\infty(\mathbb{R}^+; V_{KL}(\Omega_1))$ , then by Lemma 3.1, there exist  $\tilde{u} \in L^\infty(\mathbb{R}^+; V(\omega))$  such that

$$\begin{aligned} U_\alpha(t, x_1, x_2, x_3) &= \tilde{u}_\alpha(t, x_1, x_2) - x_3 \partial_\alpha \tilde{u}_3 \text{ for } \alpha \in \{1, 2\}, \\ U_3(t, x_1, x_2, x_3) &= \tilde{u}_3(t, x_1, x_2). \end{aligned} \quad (3.19)$$

For abbreviation, we write  $\tilde{u}_T = (\tilde{u}_1, \tilde{u}_2)$ . Using that  $U^\eta \rightharpoonup U$  in  $L^\infty(0, T; V(\Omega_1))$  weak \*, since for all  $(\alpha, \beta) \in \{1, 2\}^2$ ,  $(\varepsilon(\eta, U^\eta))_{\alpha\beta} = (\varepsilon(U^\eta))_{\alpha\beta}$ , we obtain at the weak limit that:

$$A_{\alpha\beta} = \varepsilon_{\alpha\beta}(U) \quad \text{for } (\alpha, \beta) \in \{1, 2\}^2. \quad (3.20)$$

Taking the average in the variable  $x_3 \in [-1, 1]$  and using (3.19), we obtain that:

$$\bar{A}_{\alpha\beta} = \varepsilon_{\alpha\beta}(\tilde{u}_T) \quad \text{for } (\alpha, \beta) \in \{1, 2\}^2. \quad (3.21)$$

For  $v \in \mathcal{C}^\infty(\mathbb{R}^+; \mathcal{W}(\omega))$ , we define  $\xi \in \mathcal{C}^\infty(\mathbb{R}; V_{KL}(\Omega_1))$  by

$$\begin{aligned} \xi_\alpha(x_1, x_2, x_3) &= v_\alpha(x_1, x_2) - x_3 \partial_\alpha v_3(x_1, x_2) \text{ for } \alpha = 1, 2, \\ \xi_3(x_1, x_2, x_3) &= v_3(x_1, x_2). \end{aligned} \quad (3.22)$$

Then,  $\varepsilon_{i3}(\xi) = 0$  for  $i \in \{1, 2, 3\}$ , so that  $\varepsilon(\eta, \xi) = \varepsilon(\xi)$ . With  $\xi$  as a test function in the weak formulation (3.7), we obtain

$$\begin{aligned} \int_{\mathbb{R}^+ \times \Omega_1} \sum_{\alpha=1}^2 \frac{\partial U_\alpha^\eta}{\partial t} \frac{\partial \xi_\alpha}{\partial t} + \frac{1}{\eta^2} \int_{\mathbb{R}^+ \times \Omega_1} \frac{\partial U_3^\eta}{\partial t} \frac{\partial \xi_3}{\partial t} - \int_{\mathbb{R}^+ \times \Omega_1} (\lambda^e : \varepsilon(\eta, U^\eta)) : \varepsilon(\xi) \\ + \int_{\mathbb{R}^+ \times \Gamma_-^1} g^- \cdot \xi + \int_{\mathbb{R}^+ \times \Gamma_+^1} g^+ \cdot \xi + \int_{\mathbb{R}^+ \times \Gamma_2^1} h \cdot \xi + \int_{\Omega_1} u_1 \cdot \xi(0, x) \\ = \int_{\mathbb{R}^+ \times \Omega_1} (\lambda^e : (\lambda^m : M^\eta \otimes M^\eta)) : \varepsilon(\xi), \end{aligned} \quad (3.23)$$

Taking  $v = (v_1, v_2, 0)$  in (3.23), where  $v_1$  and  $v_2$  are in  $\mathcal{C}^\infty(\mathbb{R}^+; \mathcal{V}(\omega))$ , we obtain:

$$\begin{aligned} \int_{\mathbb{R}^+ \times \Omega_1} \sum_{\alpha=1}^2 \frac{\partial U_\alpha^\eta}{\partial t} \frac{\partial v_\alpha}{\partial t} - \int_{\mathbb{R}^+ \times \Omega_1} (\lambda^e : \varepsilon(\eta, U^\eta)) : \varepsilon(\xi) \\ + \int_{\mathbb{R}^+ \times \omega} (g^+ + g^-)_T \cdot v_T + \int_{\mathbb{R}^+ \times \Gamma_2^1} h_T \cdot v_T + \int_{\Omega_1} u_{1,T} \cdot v_T(0, x) \\ = \int_{\mathbb{R}^+ \times \Omega_1} \sum_{(\alpha, \beta) \in \{1, 2\}^2} (\lambda^e : (\lambda^m : M^\eta \otimes M^\eta))_{\alpha\beta} : \varepsilon_{\alpha\beta}(v). \end{aligned}$$



By taking the limit when  $\eta$  tends to zero, we obtain that, for all  $v_T \in \mathcal{C}^\infty(\mathbb{R}^+; \mathcal{V}(\omega))$ :

$$\begin{aligned} \int_{\mathbb{R}^+ \times \omega} \frac{\partial \tilde{u}_T}{\partial t} \frac{\partial v_T}{\partial t} - \int_{\mathbb{R}^+ \times \omega} \sum_{\alpha\beta=1}^2 (\lambda^e : \bar{A})_{\alpha\beta} \varepsilon_{\alpha\beta}(v_T) + \frac{1}{2} \int_{\mathbb{R}^+ \times \omega} (g_T^+ + g_T^-) \cdot v_T + \int_{\mathbb{R}^+ \times \mathcal{C}_2} h_T \cdot v_T \\ + \int_{\mathbb{R}^+ \times \omega} u_{1T} v_T(0, x) = - \int_{\mathbb{R}^+ \times \omega} \sum_{\alpha\beta=1}^2 (\lambda^e : (\lambda^m : M \otimes M))_{\alpha\beta} \varepsilon_{\alpha\beta}(v), \end{aligned}$$

and therefore, using notation (3.17), we have:

$$\begin{aligned} \int_{\mathbb{R}^+ \times \omega} \frac{\partial \tilde{u}_T}{\partial t} \frac{\partial v_T}{\partial t} - \int_{\mathbb{R}^+ \times \omega} \sum_{\alpha\beta=1}^2 \tilde{\sigma}_{\alpha\beta} \varepsilon_{\alpha\beta}(v) + \frac{1}{2} \int_{\mathbb{R}^+ \times \omega} (g_T^+ + g_T^-) \cdot v_T + \int_{\mathbb{R}^+ \times \mathcal{C}_2} h_T \cdot v_T \\ + \int_{\mathbb{R}^+ \times \omega} u_{1T} v_T(0, x) = 0. \end{aligned} \quad (3.24)$$

### 3.4 Characterization of $\tilde{\sigma}$

For all  $\xi \in \mathcal{C}^\infty(\mathbb{R}^+; V(\Omega_1))$ , multiplying (3.7) by  $\eta^2$  and taking the limit when  $\eta$  tends to zero, we obtain that

$$\int_{\mathbb{R}^+ \times \Omega_1} \frac{\partial U_3}{\partial t} \cdot \frac{\partial \xi}{\partial t} - \int_{\mathbb{R}^+ \times \Omega_1} (\lambda^e : A)_{33} \varepsilon_{33}(\xi) = - \int_{\mathbb{R}^+ \times \Omega_1} (\lambda^e : (\lambda^m : M \otimes M))_{33} \varepsilon_{33}(\xi). \quad (3.25)$$

We remark that  $\frac{\partial U_3}{\partial t} = 0$ , so the first integral in (3.25) vanishes. For  $\varphi \in \mathbb{D}(\mathbb{R}^+ \times \omega)$ , we choose  $\xi$  such that:

$$\begin{aligned} \xi_\alpha &= 0 \text{ for } \alpha = 1, 2 \\ \xi_3 &= x_3 \varphi, \end{aligned}$$

then  $e_{33}(\xi) = \varphi$ , and (3.25) implies

$$\int_{\mathbb{R}^+ \times \omega} (\lambda^e : \bar{A})_{33} \varphi = \int_{\mathbb{R}^+ \times \omega} (\lambda^e : (\lambda^m : M \otimes M))_{33} \varphi,$$

since  $M$  and  $\varphi$  are independent of  $x_3$ . Then

$$(\lambda^e : \bar{A})_{33} = (\lambda^e : (\lambda^m : M \otimes M))_{33} \text{ in } L^2(\mathbb{R}^+ \times \omega).$$

Fix  $\alpha \in \{1, 2\}$  and choose  $\xi$  such that  $\xi_\alpha = x_3 \varphi$ , and  $\xi_i = 0$  for  $i \neq \alpha$ , where  $\varphi \in \mathbb{D}(\mathbb{R}^+ \times \omega)$ , then  $e_{\alpha 3}(\xi) = \frac{1}{2} \varphi$  and  $\varepsilon_{33}(\xi) = 0$ . Replacing  $\xi$  by its value in (3.7), multiplying the obtained equation by  $\eta$  and  $\eta \rightarrow 0$ , we get:

$$\int_{\mathbb{R}^+ \times \Omega_1} (\lambda^e : A)_{\alpha 3} \varphi = \int_{\mathbb{R}^+ \times \Omega_1} (\lambda^e : (\lambda^m : M \otimes M))_{\alpha 3} \varphi.$$

This is true for all  $\varphi \in \mathbb{D}(\mathbb{R}^+ \times \omega)$ , then by using the fact that  $\varphi$  and  $M$  are independent of  $x_3$ , we obtain

$$(\lambda^e : \bar{A})_{\alpha 3} = (\lambda^e : (\lambda^m : M \otimes M))_{\alpha 3} \text{ in } L^2(\mathbb{R}^+ \times \omega). \quad (3.26)$$

Therefore, we obtain that:

$$\forall i \in \{1, 2, 3\}, \quad \tilde{\sigma}_{i3} = \tilde{\sigma}_{3i} = 0. \quad (3.27)$$

We define  $I$  and  $J$  by:

$$I = \{(i, j) \in \{1, 2\}^2\} \quad \text{an} \quad J = \{(i, j) \in \{1, 2, 3\}^2, i = 3 \text{ or } j = 3\}.$$

We denote by  $\mathcal{G}$  the set of symmetric real  $3 \times 3$  matrices endowed with the scalar product  $\cdot$ . We denote by  $\mathcal{G}_n$  the set of the matrices  $D = (d_{ij}) \in \mathcal{G}$  such that  $d_{\alpha\beta} = 0$  for  $(\alpha, \beta) \in I$ , and by  $\mathcal{G}_T$  the set of the matrices  $D = (d_{ij}) \in \mathcal{G}$  such that  $d_{ij} = 0$  for  $(i, j) \in J$ , so that

$$\mathcal{G} = \mathcal{G}_T \oplus^\perp \mathcal{G}_n. \quad (3.28)$$

For  $D \in \mathcal{G}$ , we define  $\mathcal{K}(D) \in \mathcal{G}_n$  by:

$$(\mathcal{K}(D))_{ij} = \begin{cases} 0 & \text{if } (i, j) \in I, \\ (\lambda^e : D)_{ij} = \sum_{(k,l) \in \{1,2,3\}^2} \lambda_{ijkl}^e d_{kl} & \text{if } (i, j) \in J. \end{cases}$$

We remark that  $\mathcal{K}$  is linear and that its restriction to  $\mathcal{G}_n$  is a bijection from  $\mathcal{G}_n$  to itself. Indeed, if  $D \in \mathcal{G}_n$  with  $\mathcal{K}(D) = 0$ , we have:

$$\begin{aligned} 0 &= \sum_{(i,j) \in J} (\mathcal{K}(D))_{ij} D_{ij} \text{ since } \mathcal{K}(D) = 0, \\ &= \sum_{(i,j) \in J} (\lambda^e : D)_{ij} D_{ij}, \\ &= \sum_{(i,j) \in \{1,2,3\}^2} (\lambda^e : D)_{ij} D_{ij} \text{ since } D_{ij} = 0 \text{ if } (i, j) \in I, \\ &= (\lambda^e : D) : D. \end{aligned}$$

So by positivity of  $\lambda^e$ , if  $\mathcal{K}(D) = 0$  with  $D \in \mathcal{G}_n$ , then  $D = 0$ . So  $\mathcal{K}$  is bijective from  $\mathcal{G}_n$  into  $\mathcal{G}_n$ . We denote by  $\mathcal{T} : \mathcal{G}_n \rightarrow \mathcal{G}_n$  the inverse of  $\mathcal{K}|_{\mathcal{G}_n}$ . We split  $\epsilon^e$  as  $\epsilon^e = \epsilon_T^e + \epsilon_n^e$  where:

$$(\epsilon_T^e)_{ij} = \begin{cases} 0 & \text{if } (i, j) \in J, \\ \varepsilon(\tilde{u}_T)_{ij} - (\lambda^m : M \otimes M)_{ij} & \text{if } (i, j) \in I, \end{cases}$$

and

$$(\epsilon_n^e)_{ij} = \begin{cases} 0 & \text{if } (i, j) \in I, \\ \bar{A}_{ij} - (\lambda^m : M \otimes M)_{ij} & \text{if } (i, j) \in J. \end{cases}$$

By (3.27),  $\mathcal{K}(\epsilon_T^e + \epsilon_n^e) = 0$ , so  $\mathcal{K}(\epsilon_n^e) = -\mathcal{K}(\epsilon_T^e)$ . Since  $\epsilon_n^e \in \mathcal{G}$ , we obtain that:

$$\epsilon_n^e = -\mathcal{T}(\mathcal{K}(\epsilon_T^e)).$$

Therefore,

$$\tilde{\sigma} = \lambda^e : (\epsilon_T^e - \mathcal{T}(\mathcal{K}(\epsilon_T^e))).$$

We denote by  $\lambda^{\text{eq}}$  the 4-tensor defined by:

$$\text{for } D \in \mathcal{G}, \lambda^{\text{eq}} : D = \lambda^e : (D - \mathcal{T}(\mathcal{K}(D))),$$

so that we have:

$$\tilde{\sigma} = \lambda^{\text{eq}} : (\varepsilon(\tilde{u}_T) - \lambda^m : M \otimes M).$$

**Remark 3.1.** *In the isotropic case, that is for  $\lambda^e$  given by*

$$\forall S \in \mathcal{G}, (\lambda^e : S)_{ij} = \frac{E}{1+\nu} \left( S_{ij} + \frac{\nu}{1-2\nu} (\text{tr } S) \delta_{ij} \right),$$

where  $E$  is the young modulus and  $\nu \in [0, \frac{1}{2}[$  is the Poisson coefficient, we obtain that  $\tilde{\sigma}$  is deduced from  $\varepsilon_T^e$  by:

$$\tilde{\sigma}_{ij} = \begin{cases} 0 & \text{if } i = 3 \text{ or } j = 3, \\ \frac{E}{1+\nu} \left( (\varepsilon_T^e)_{ij} + \frac{\nu}{1-2\nu} (\tilde{\text{tr}} \varepsilon_T^e) \delta_{ij} \right) & \text{if } (i, j) \in \{1, 2\}^2, \end{cases}$$

where  $\tilde{\text{tr}} \varepsilon_T^e = (\varepsilon_T^e)_{11} + (\varepsilon_T^e)_{22}$ . This particular case was studied in [18].

## 4 Conclusion

We have obtained the following 2d-model for thin ferromagnetic plates with magnetostriction:

- $M \in L^\infty(\mathbb{R}^+; H^1(\omega; S^2))$ ,  $\partial_t M \in L^2(\mathbb{R}^+ \times \omega)$ ,
- $\tilde{u}_T \in L^\infty(\mathbb{R}^+; H^1(\omega; \mathbb{R}^2))$  with  $\tilde{u}_T = 0$  on  $\mathbb{R}^+ \times \mathcal{C}_1$  and  $\partial_t \tilde{u}_T \in L^\infty(\mathbb{R}^+; L^2(\omega; \mathbb{R}^2))$
- For all  $\tilde{\chi} \in C^\infty(\mathbb{R}^+; H^1(\omega; \mathbb{R}^3))$ ,

$$\begin{aligned} \int_{\mathbb{R}^+ \times \omega} \left( \frac{\partial M}{\partial t} - M \times \frac{\partial M}{\partial t} \right) \tilde{\chi} &= -2 \int_{\mathbb{R}^+ \times \omega} \sum_{\alpha=1}^2 M \times \partial_\alpha M \cdot \partial_\alpha \tilde{\chi} \\ &\quad - 2 \int_{\mathbb{R}^+ \times \omega} M \times (-M_3 e_3 + \Psi(M) + (\lambda^m : \tilde{\sigma})M) \cdot \tilde{\chi}. \end{aligned}$$

- For all  $v_T \in C^\infty(\mathbb{R}^+; \mathcal{V}(\Omega; \mathbb{R}^2))$ ,

$$\begin{aligned} \int_{\mathbb{R}^+ \times \omega} \frac{\partial \tilde{u}_T}{\partial t} \frac{\partial v_T}{\partial t} - \int_{\mathbb{R}^+ \times \omega} \sum_{\alpha\beta=1}^2 \tilde{\sigma}_{\alpha\beta} \varepsilon_{\alpha\beta}(v_T) + \frac{1}{2} \int_{\mathbb{R}^+ \times \omega} (g_T^+ + g_T^-) \cdot v_T \\ + 2 \int_{\mathbb{R}^+ \times \mathcal{C}_2} h_T \cdot v_T + \int_{\mathbb{R}^+ \times \omega} u_{1T} v_T(0, x) = 0. \end{aligned}$$

- $\tilde{\sigma} = \lambda^e : (\varepsilon_T^e - \mathcal{T}(\mathcal{K}(\varepsilon_T^e)))$  with  $\varepsilon_T^e = \varepsilon(\tilde{u}_T) - \lambda^m : M \otimes M$ .

We remark that the equation in  $v_T$  is hyperbolic since the equivalent tensor  $\lambda^{\text{eq}}$  is positive on  $\mathcal{G}_T$ . Indeed, if  $D \in \mathcal{G}_T$ , then:

$$(\lambda^{\text{eq}} : D) : D = (\lambda^e : (D - \mathcal{T}(\mathcal{K}(D)))) : D.$$

We remark that  $\lambda^e : (D - \mathcal{T}(\mathcal{K}(D))) \in \mathcal{G}_T$  and  $\mathcal{T}(\mathcal{K}(D)) \in \mathcal{G}_n$ . Thus,

$$\begin{aligned} (\lambda^{\text{eq}} : D) : D &= (\lambda^e : (D - \mathcal{T}(\mathcal{K}(D)))) : (D - \mathcal{T}(\mathcal{K}(D))) \\ &\geq \lambda^{e*} \|D - \mathcal{T}(\mathcal{K}(D))\|^2 \\ &\geq \lambda^{e*} \|D\|^2 \text{ since } \mathcal{G}_T \perp \mathcal{G}_n. \end{aligned}$$

It is well known that in 3d, the weak solutions for the Landau Lifschitz equation are not unique (see [2]). The uniqueness is only proved for regular solutions (at least with  $H^2$  regularity in the space variable). For the 2D-model coupling, the uniqueness of weak solutions remains an open problem. Although, using J L Lions' theorem, we can show that if  $M$  is fixed, then the solution of the initial and boundary value hyperbolic problem satisfied by  $\tilde{u}_T$  is unique (see Theorem 10.14 in [6] and [23]).

In conclusion, the model we obtain will be easier to study and to simulate since it is bi-dimensional and since the 2d demagnetizing field is local. It would be interesting to characterize the normal deformations in the 2d model, even though they have no influence on magnetization in the regime we studied.

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