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Rates in the central limit theorem for random projections of Martingales

J. Dedecker*, F. Merlevède †and M. Peligrad ‡

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Abstract

In this paper, we consider partial sums of martingale differences weighted by random variables drawn uniformly on the sphere, and globally independent of the martingale differences. Combining Lindeberg’s method and a series of arguments due to Bobkov, Chistyakov and Götze, we show that the Kolmogorov distance between the distribution of these weighted sums and the limiting Gaussian is “super-fast” of order $(\log n)^2/n$, under conditions allowing us to control the higher-order conditional moments of the martingale differences. We give an application of this result to the least squares estimator of the slope in the linear model with Gaussian design.

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Key words and phrases. Random projections; Berry-Esseen theorem; Martingales

1 Introduction

Let $(X_k)_{k \in \mathbb{Z}}$ be a strictly stationary process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}(X_0) = 0$ and $\mathbb{E}(X_0^2) = 1$. Consider the weighted sums

$$\langle X, \theta \rangle := \theta_1 X_1 + \cdots + \theta_n X_n := S_n(\theta),$$

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where $X = (X_1, \dots, X_n)$, $\theta = (\theta_1, \dots, \theta_n)$ is defined on $(\Omega, \mathcal{F}, \mathbb{P})$, is independent of $(X_k)_{k \in \mathbb{Z}}$ and has uniform distribution μ_{n-1} on the unit sphere S^{n-1} of \mathbb{R}^n ($n \geq 2$). Let $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ be a non decreasing stationary filtration in \mathcal{F} such that X_0 is \mathcal{F}_0 -adapted. In this paper, we shall often use the notation $\mathbb{E}_\ell(X)$ to mean $\mathbb{E}(X|\mathcal{F}_\ell)$. Assume that $\mathbb{E}_{i-1}(X_i) = 0$ \mathbb{P} -almost surely: in other words $(X_k)_{k \in \mathbb{Z}}$ is a sequence of martingale differences adapted to $(\mathcal{F}_i)_{i \in \mathbb{Z}}$. By the properties of the uniform distribution on the sphere, we have that

$$\max_{1 \leq i \leq n} |\theta_i| \rightarrow 0, \quad \mathbb{P}\text{-a.s.}$$

Therefore, if we assume in addition that $\mathbb{E}(X_0^2|\mathcal{F}_{-\infty}) = 0$, according to Hannan [12, p. 284], $S_n(\theta)$ converges in distribution to a standard Gaussian random variable. Actually [12, Theorem 1] implies the following conditional version of the central limit theorem (CLT):

$$\kappa_\theta(P_{S_n(\theta)}, P_G) \rightarrow 0 \quad \mathbb{P}\text{-a.s., as } n \rightarrow \infty,$$

where $G \sim N(0, 1)$ and

$$\kappa_\theta(P_{S_n(\theta)}, P_G) = \sup_{t \in \mathbb{R}} |\mathbb{P}_{|\theta}(S_n(\theta) \leq t) - \mathbb{P}(G \leq t)|.$$

Above the notation $\mathbb{P}_{|\theta}$ (resp. $\mathbb{E}_{|\theta}$) means the conditional probability (resp. the conditional expectation) with respect to θ .

In this paper, we are interested in rates in this CLT in terms of the Kolmogorov distance.

When $(X_k)_{k \in \mathbb{Z}}$ forms a sequence of independent centered random variables in \mathbb{L}^4 with variance one, from Corollary 3.8 in Klartag and Sodin [13], it follows that

$$\mathbb{E}\left(\kappa_\theta(P_{S_n(\theta)}, P_G)\right) \leq \frac{cM_{4,n}}{n} \quad \text{where } M_{4,n} = \frac{1}{n} \sum_{k=1}^n \mathbb{E}(X_k^4). \quad (1.1)$$

This proves that, when $M_{4,n}$ is uniformly bounded (which is the case in the setting of independent and identically distributed (iid) r.v.'s), projecting the variables on the sphere allows to derive a much faster rate than in the usual Berry-Esseen theorem, where the rate is $1/\sqrt{n}$. In a recent paper, Bobkov et al. [2] have extended this interesting phenomenon to isotropic random vectors (meaning that the coordinates are uncorrelated with variance one) having a symmetric distribution, and under a suitable second order correlation condition. They obtained a similar $1/n$ -rate modulo a logarithmic factor. More precisely, their second order correlation condition reads as: there exists a constant Λ such that, for any collection $a_{ij} \in \mathbb{R}$,

$$\text{Var}\left(\sum_{i,j=1}^n a_{ij} X_i X_j\right) \leq \Lambda \sum_{i,j=1}^n a_{ij}^2. \quad (1.2)$$

Theorem 1.1 in [2] asserts that if $(X_k)_{k \in \mathbb{Z}}$ is a sequence of uncorrelated centered random variables with variance one, satisfying (1.2) and such that (X_1, \dots, X_n) has a symmetric distribution then

$$\mathbb{E}\left(\kappa_\theta(P_{S_n(\theta)}, P_G)\right) \leq \frac{c \log n}{n} \Lambda. \quad (1.3)$$

As shown in [2], condition (1.2) can be verified for random vectors which satisfy a Poincaré-type inequality with positive constant (see [2, Proposition 3.4]). Moreover, if we assume that $(X_k)_{k \in \mathbb{Z}}$ is a sequence of martingale differences such that $\sup_{i \geq 1} \mathbb{E}(X_i^4) < \infty$, one can check that condition (1.2) is satisfied provided that

$$\sum_{k \geq 1} \tilde{\gamma}(k) < \infty, \quad (1.4)$$

with $\tilde{\gamma}(k) = \max(\gamma_{2,2}(k), \gamma_{1,3}(k))$ where

$$\gamma_{2,2}(k) = \sup_{\ell \geq u \geq 0} \|X_u X_\ell (\mathbb{E}_\ell(X_{k+\ell}^2) - \mathbb{E}(X_{k+\ell}^2))\|_1, \quad (1.5)$$

and

$$\gamma_{1,3}(k) = \sup_{\ell, v \geq 0} \|X_\ell (\mathbb{E}_\ell(X_{k+\ell} X_{k+v+\ell}^2) - \mathbb{E}(X_{k+\ell} X_{k+v+\ell}^2))\|_1. \quad (1.6)$$

For instance, if $(X_k)_{k \in \mathbb{Z}}$ is additionally strictly stationary and strongly mixing, condition (1.4) is satisfied provided that $\sum_{k \geq 0} \int_0^{\alpha_2(k)} Q^4(u) du < \infty$ (see Section 2.2.1 for a definition of the coefficients $\alpha_2(k)$ and of the quantile function Q). The most stringent condition in the assumptions made in [2, Theorem 1.1] is probably the fact that the distribution of (X_1, \dots, X_n) is assumed to be symmetric. In [3, Chapter 17.4] the authors consider the case of non-symmetric distributions. Their Proposition 17.4.1 states that if $(X_k)_{k \in \mathbb{Z}}$ is a sequence of uncorrelated centered random variables with variance one, satisfying (1.2) then we can still provide an explicit bound for $\mathbb{E}(\kappa_\theta(P_{S_n(\theta)}, P_G))$ and that a certain term has to be added to the right-hand side of (1.3). This additional term is

$$\left(\frac{\log n}{n}\right)^{1/4} \left(\mathbb{E} \frac{\langle X, Y \rangle}{\sqrt{\|X\|_e^2 + \|Y\|_e^2}}\right)^{1/2}, \quad (1.7)$$

where Y is an independent copy of X and $\|X\|_e^2 = \langle X, X \rangle$ denotes the euclidian norm of X . As proved in [3, Chapter 17.5], the term (1.7) can be upper-bounded by $C(\log n)^{1/4}/n$ provided (X_1, \dots, X_n) satisfies a Poincaré-type inequality. For instance, when $n = 1$, for this Poincaré-type inequality to be satisfied it is necessary that \mathbb{P}_{X_1} has an absolutely continuous component. Now in case of random vectors ($n \geq 2$), the required Poincaré-type inequality is quite complicated to obtain except in the case where the random variables are independent with marginal distributions satisfying a Poincaré-type inequality (see [3, Pages 108-109]).

The aim of this paper is to provide a new method allowing us to show that sequences $(X_k)_{k \in \mathbb{Z}}$ of martingale differences satisfy an upper bound of the type (1.1) (up to some logarithmic term)

without requiring that the law of the vector (X_1, \dots, X_n) is symmetric, nor satisfies a Poincaré-type inequality. More precisely, as stated and proved in Section 3, an upper bound of the type (1.1) will be achieved with the help of the Lindeberg method, where the random variables $\theta_i X_i$ will be replaced one by one by random variables $Y_i(\theta)$ taking only two values with some desired characteristics (in particular they are independent conditionally to θ). Then, following the approach of Klartag and Sodin [13], we shall compare the distribution of $\sum_{i=1}^n Y_i(\theta)$ with a normal distribution. We select this method because Stein’s method, which is also successfully used in some situations to get sharp Berry-Esseen bounds, would require strong assumptions on the conditional moments of the martingale differences (see for instance [15]). The rate we achieve is “super fast” of order $(\log n)^2/n$. It should be mentioned that, in the absence of the randomization considered in this section and in the presence of dependence, it is very rare to achieve a Berry-Esseen upper bound of order $n^{-1/2}$ for stationary sequences. Especially, for a martingale with stationary and ergodic differences, constant conditional variance and finite third moments, the upper bound in the classical Berry-Esseen inequality cannot be better than $n^{-1/4}$, (see [11] and [4]).

In Section 4, we shall see that, with additional computations, our result applies to the ordinary least square estimator of the slope in the linear regression model with Gaussian design and martingale differences errors.

In a Berry-Esseen bound the constants are important. However, in this paper, due to the difficulty and complexity of the problem, we shall not compute the constants exactly. For any two positive sequences of random variables a_n and b_n we denote by $a_n \ll b_n$ the fact that there is a universal positive constant c such that $a_n \leq cb_n$ for all n .

The paper is organized as follows. In Section 2.1 we present the result for martingales and comment on the method of proof. Section 2.2 is devoted to applications to martingales differences satisfying a mixing-type condition, functions of Markov chains and ARCH(∞) models. The proof on the general martingale result is given in Section 3. Finally, in Section 4, we present the application to linear regression with Gaussian design and give its proof.

2 Normal approximation for weighted sums of martingale differences

2.1 Main Result

In what follows we assume that $(X_i)_{i \in \mathbb{Z}}$ is a strictly stationary sequence of martingale differences. Assume moreover that $\|X_0\|_4 < \infty$ and $\|X_0\|_2 = 1$. Let us introduce the weak dependence

coefficients we will use in this paper. For any positive integer v , let

$$\gamma_{0,2}(v) = \|\mathbb{E}_0(X_v^2) - \mathbb{E}(X_v^2)\|_1, \quad \gamma_{1,2}(v) = \|X_0(\mathbb{E}_0(X_v^2) - \mathbb{E}(X_v^2))\|_1.$$

Recall also the coefficients $\gamma_{2,2}(v)$ and $\gamma_{1,3}(v)$ as defined in the introduction that can be rewritten as follows in the stationary setting:

$$\gamma_{2,2}(v) = \sup_{\ell \geq 0} \|X_0 X_\ell (\mathbb{E}_\ell(X_{v+\ell}^2) - \mathbb{E}(X_{v+\ell}^2))\|_1, \quad \gamma_{1,3}(v) = \sup_{\ell \geq 0} \|X_0 (\mathbb{E}_0(X_v X_{v+\ell}^2) - \mathbb{E}(X_v X_{v+\ell}^2))\|_1.$$

Define then

$$\gamma(v) = \max(\gamma_{0,2}(v), \gamma_{1,2}(v), \gamma_{2,2}(v), \gamma_{1,3}(v)). \quad (2.1)$$

Our general result for martingales is the following theorem:

Theorem 2.1. *Let $(X_n)_{n \in \mathbb{Z}}$ be a stationary sequence of martingale differences in \mathbb{L}^4 such that $\mathbb{E}(X_0^2) = 1$. Let $(\gamma(k))_{k \geq 0}$ be the sequence of dependent coefficients defined in (2.1). Assume that $\sum_{k \geq 1} k\gamma(k) < \infty$. Then*

$$\mathbb{E}(\kappa_\theta(P_{S_n(\theta)}, P_G)) \ll \frac{(\log n)^2}{n}. \quad (2.2)$$

As a proof technique, we shall use as a starting point (following [1, 2]) a new variant of the smoothing inequality, which is custom built for the type of randomization we used and sequences of martingale differences. Let us give some details.

Let $(X_k)_{k \in \mathbb{Z}}$ be a stationary sequence of martingale differences. We start from the so-called Berry-Esseen smoothing inequality (see e.g. [9, Ineq. (3.13) p. 538]) together with Lemma 5.2 in [1] which takes advantage of the fact that the random variables are projected on the sphere. More precisely, note first that for any fixed point θ on the sphere S^{n-1} , in case of stationary sequences of martingale differences, Theorem 3.10 in [14] implies

$$(\mathbb{E}(|\langle X, \theta \rangle|^4))^{1/4} \leq c_2 \|X_0\|_4.$$

In addition, if

$$\sum_{k \geq 1} \text{Cov}(X_0^2, X_k^2) < \infty, \quad (2.3)$$

then

$$\sigma_4^2 := n^{-1} \mathbb{E} \left(\left| \sum_{i=1}^n (X_i^2 - 1) \right|^2 \right) < \infty.$$

Therefore, if the moment of order 4 is finite and (2.3) holds, [1, Lemma 5.2] together with [1, Corollary 2.3] then imply that for all $T \geq T_0 \geq 1$,

$$\int_{T_0}^T t^{-1} \mathbb{E}(|f_\theta(t)|) dt \ll \frac{\log(T/T_0)}{n} + e^{-T_0^2/16}, \quad (2.4)$$

where $f_\theta(t) = \mathbb{E}(e^{it\langle X, \theta \rangle} | \theta)$. The upper bound (2.4) combined with the Berry-Esseen's smoothing inequality entails the following Proposition:

Proposition 2.1. *Let $(X_k)_{k \in \mathbb{Z}}$ be a sequence of stationary martingale differences with finite fourth moment and G a standard normal variable. If (2.3) is satisfied then, for all $T \geq T_0 \geq 1$,*

$$\mathbb{E}(\kappa_\theta(P_{\langle X, \theta \rangle}, P_G)) \ll \int_0^{T_0} \mathbb{E}(|f_\theta(t) - e^{-t^2/2}|) \frac{dt}{t} + \frac{\log(T/T_0)}{n} + e^{-T_0^2/16} + \frac{1}{T}. \quad (2.5)$$

In the rest of the paper we shall chose $T = n$ and $T_0 = 4\sqrt{\log n}$. To derive an upper bound for $\mathbb{E}(\kappa_\theta(P_{\langle X, \theta \rangle}, P_G))$ of order $1/n$ modulo an extra-logarithmic term $(\log n)^\alpha$ for $\alpha \geq 1$, one needs to prove that

$$\int_0^{T_0} \mathbb{E}(|f_\theta(t) - e^{-t^2/2}|) \frac{dt}{t} \ll \frac{(\log n)^\alpha}{n}. \quad (2.6)$$

This will be achieved by a two steps procedure. Using Lindeberg's method we shall replace one by one the variables $\theta_i X_i$ by independent random variables $Y_i(\theta)$ taking only two values with some desired characteristics. After that we shall compare the characteristic function of $\sum_{i=1}^n Y_i(\theta)$ with $e^{-t^2/2}$.

2.2 Applications

2.2.1 Martingale differences sequences and functions of Markov chains

Recall that the strong mixing coefficient of Rosenblatt [16] between two σ -algebras \mathcal{A} and \mathcal{B} is defined by $\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : (A, B) \in \mathcal{A} \times \mathcal{B}\}$. For a strictly stationary sequence $(X_i)_{i \in \mathbb{Z}}$, let $\mathcal{F}_i = \sigma(X_k, k \leq i)$. Define the mixing coefficients $\alpha_2(n)$ of the sequence $(X_i)_{i \in \mathbb{Z}}$ by

$$\alpha_2(n) = \sup_{\ell \geq 0} \alpha(\mathcal{F}_0, \sigma(X_n, X_{n+\ell})).$$

For the sake of brevity, let $Q = Q_{X_0}$ where Q_{X_0} is the quantile function of X_0 , that is the generalized inverse of $t \mapsto \mathbb{P}(|X_0| > t)$. Applying Theorem 2.1 and proceeding as in (6.86) and (6.87) of [14], the following result holds:

Corollary 2.1. *Let $(X_n)_{n \in \mathbb{Z}}$ be a stationary sequence of martingale differences in \mathbb{L}^4 such that $\mathbb{E}(X_0^2) = 1$. Assume that*

$$\sum_{k \geq 1} k \int_0^{\alpha_2(k)} Q^4(u) du < \infty \quad (2.7)$$

Then the conclusion of Theorem 2.1 holds.

Examples of martingale differences that are additionally strongly mixing are harmonic function of a Harris recurrent Markov chain. Let us for instance consider the following example as described in [7, Section 4.1]: Let $(Y_i)_{i \in \mathbb{Z}}$ be the homogeneous Markov chain with state space \mathbb{Z} described at page 320 in [6]. The transition probabilities are given by $p_{n,n+1} = p_{-n,-n-1} = a_n$ for $n \geq 0$, $p_{n,0} = p_{-n,0} = 1 - a_n$ for $n > 0$, $p_{0,0} = 0$, $a_0 = 1/2$ and $1/2 \leq a_n < 1$ for $n \geq 1$. This chain is Harris positively recurrent as soon as $\sum_{n \geq 2} \prod_{k=1}^{n-1} a_k < \infty$ and in that case the stationary chain is strongly mixing in the sense of Rosenblatt [16].

Denote by K the Markov kernel of the chain $(Y_i)_{i \in \mathbb{Z}}$. The functions f such that $K(f) = 0$ almost everywhere are obtained by linear combinations of the two functions f_1 and f_2 given by $f_1(1) = 1$, $f_1(-1) = -1$ and $f_1(n) = f_1(-n) = 0$ if $n \neq 1$, and $f_2(0) = 1$, $f_2(1) = f_2(-1) = 0$ and $f_2(n+1) = f_2(-n-1) = 1 - a_n^{-1}$ if $n > 0$. Hence the functions f such that $K(f) = 0$ are bounded.

If $(X_i)_{i \in \mathbb{Z}}$ is defined by $X_i = f(Y_i)$, with $K(f) = 0$, then Corollary 2.1 applies if $\sum_{k \geq 1} k \alpha_2(k) < \infty$ that holds as soon as $\mathbb{E}_0(\tau^3) < \infty$ where $\tau = \inf\{n > 0, X_n = 0\}$ and \mathbb{E}_0 is the expectation under P_0 , which is the probability of the chain starting from 0. Now $P_0(\tau = n) = (1 - a_n) \prod_{i=1}^{n-1} a_i$ for $n \geq 2$. Consequently, if for some $\epsilon > 0$, $a_i = 1 - \frac{1}{i} \left(3 + \frac{1+\epsilon}{\log i}\right)$ for i large enough, Corollary 2.1 applies.

2.2.2 ARCH models

Theorem 2.1 applies to the case where $(X_i)_{i \in \mathbb{Z}}$ has an ARCH(∞) structure as described by Giraitis *et al.* [10], that is

$$X_n = \sigma_n \eta_n, \text{ with } \sigma_n \in \mathbb{R}^+ \text{ that satisfies } \sigma_n^2 = c + \sum_{j=1}^{\infty} c_j X_{n-j}^2, \quad (2.8)$$

where $(\eta_n)_{n \in \mathbb{Z}}$ is a sequence of iid centered random variables such that $\mathbb{E}(\eta_0^2) = 1$ and independent of \mathcal{F}_{n-1} , and where $c \geq 0$, $c_j \geq 0$, and $\sum_{j \geq 1} c_j < 1$. Since $\mathbb{E}(\eta_0^2) = 1$ and $\sum_{j \geq 1} c_j < 1$, the unique stationary solution to (2.8) is given by Giraitis *et al.* [10]:

$$\sigma_n^2 = c + c \sum_{\ell=1}^{\infty} \sum_{j_1, \dots, j_\ell=1}^{\infty} c_{j_1} \dots c_{j_\ell} \eta_{n-j_1}^2 \dots \eta_{n-(j_1+\dots+j_\ell)}^2. \quad (2.9)$$

Let $v^2 = \mathbb{E}(X_0^2)$ and note that $v^2 = c(1 - \sum_{j \geq 1} c_j)^{-1}$. Applying Theorem 2.1, we get the following result:

Corollary 2.2. *Let $p \in]4, 6]$. Assume that $\|\eta_0\|_p < \infty$ and*

$$c_j = O(j^{-b}) \text{ for } b > 1 + 2(p-2)/(p-4). \quad (2.10)$$

Then

$$\mathbb{E}(\kappa_\theta(P_{S_n(\theta)}, P_{G_{v^2}})) \ll \frac{(\log n)^2}{n},$$

where $G_{v^2} \sim \mathcal{N}(0, v^2)$.

Proof of Corollary 2.2. Let $k \geq 2$ and f be such that $\sigma_k = f(\eta_{k-1}, \dots, \eta_1, \eta_0, \eta_{-1}, \dots)$. Let $(\eta'_n)_{n \in \mathbb{Z}}$ be an independent copy of $(\eta_n)_{n \in \mathbb{Z}}$ and set

$$\sigma_k^* = f(\eta_{k-1}, \dots, \eta_1, \eta'_0, \eta'_{-1}, \dots).$$

Let also $X_k^* = \sigma_k^* \eta_k$, $k \geq 2$. Let us estimate the coefficients $(\gamma(k))_{k \geq 2}$. With this aim, we start by noticing that, for any $k \geq 2$,

$$\gamma_{0,2}(k) \leq \mathbb{E}(|X_k^2 - X_k^{*2}|) = \mathbb{E}(\eta_k^2) \mathbb{E}(|\sigma_k^2 - \sigma_k^{*2}|) = \mathbb{E}(|\sigma_k^2 - \sigma_k^{*2}|) := \delta_k.$$

Now, according to [5, Prop. 5.1],

$$\delta_k \ll \inf_{1 \leq \ell \leq k} \left\{ \kappa^{k/\ell} + \sum_{i \geq \ell+1} c_j \right\} \text{ where } \kappa = \sum_{i \geq 1} c_j. \quad (2.11)$$

Next, for any $k \geq 2$,

$$\begin{aligned} \gamma_{1,2}(k) &\leq \mathbb{E}(|X_0| |X_k^2 - X_k^{*2}|) = \mathbb{E}(|X_0| |\sigma_k^2 - \sigma_k^{*2}|), \\ \gamma_{2,2}(k) &\leq \sup_{\ell \geq 0} \mathbb{E}(|X_0 X_\ell| |X_{k+\ell}^2 - X_{k+\ell}^{*2}|) = \sup_{\ell \geq 0} \mathbb{E}(|X_0 X_\ell| |\sigma_{k+\ell}^2 - \sigma_{k+\ell}^{*2}|), \end{aligned}$$

and, there exists a numerical constant C such that, for any $k \geq 2$,

$$\gamma_{1,3}(k) \leq C \sup_{\ell \geq 0} \mathbb{E}(\{|X_\ell|^3 + |X_\ell^*|^3\} |X_k - X_k^*|).$$

Let us give an upper bound for the coefficients $\gamma_{1,3}(k)$, the other coefficients being bounded by similar arguments.

For any $k \geq 2$, let $u(k) := \sup_\ell \| |X_\ell|^3 (X_k - X_k^*) \|_1$. Let M be a positive real. By stationarity and Cauchy-Schwarz's inequality, note that

$$\begin{aligned} u(k) &\leq M^{3-p/2} \sqrt{\mathbb{E}(|X_0|^p)} \sqrt{\mathbb{E}(|X_k - X_k^*|^2)} + M^{4-p} \sup_\ell \| |X_\ell|^{p-1} (X_k - X_k^*) \|_1 \\ &\leq M^{3-p/2} \sqrt{\mathbb{E}(|X_0|^p)} \sqrt{\mathbb{E}(|X_k - X_k^*|^2)} + 2M^{4-p} \mathbb{E}(|X_0|^p). \end{aligned}$$

Note that

$$\mathbb{E}(|X_k - X_k^*|^2) = \mathbb{E}(|\sigma_k - \sigma_k^*|^2) \leq \mathbb{E}(|\sigma_k^2 - \sigma_k^{*2}|).$$

Therefore

$$u(k) \ll M^{3-p/2} \sqrt{\delta_k} + M^{4-p}.$$

The quantity $\sup_\ell \| |X_\ell^*|^3 (X_k - X_k^*) \|_1$ can be bounded similarly. Selecting $M = \delta_k^{-1/(p-2)}$, it follows that

$$\gamma_{1,3}(k) \ll \delta_k^{(p-4)/(p-2)}.$$

Using similar arguments we infer that $\gamma_{1,2}(k) \ll \delta_k^{(p-3)/(p-2)}$ and $\gamma_{2,2}(k) \ll \delta_k^{(p-4)/(p-2)}$. So, overall, Theorem 2.1 applies if $\sum_{k \geq 1} k \delta_k^{(p-4)/(p-2)} < \infty$ that clearly holds under (2.10) by taking into account (2.11). \square

3 Proof of Theorem 2.1

If (1.4) is satisfied then (2.3) holds and we apply the smoothing Proposition 2.1. Hence we want to prove that (2.6) holds with $\alpha = 2$, namely:

$$\int_0^{T_0} \mathbb{E} \left(\left| \mathbb{E}_{|\theta} \left(e^{it \sum_{k=1}^n \theta_k X_k} \right) - e^{-t^2/2} \right| \right) \frac{dt}{t} \ll \frac{(\log n)^2}{n}, \quad (3.12)$$

with $T_0 = 4\sqrt{\log n}$. With this aim, for any fixed θ , we shall use the Lindeberg method and consider a sequence $(Y_i(\theta))_{i \geq 1}$ of independent random variables with specific distribution, which is also independent of $(X_i)_{i \geq 1}$. To specify the distribution, let us mention the following fact.

Fact 3.1. Let $\sigma^2 > 0$ and $\beta_3 \in \mathbb{R}$. There exists a random variable Y taking only 2 values m and m' (depending only on σ^2 and β_3) and such that $\mathbb{E}(Y) = 0$, $\mathbb{E}(Y^2) = \sigma^2$, $\mathbb{E}(Y^3) = \beta_3$ and $\mathbb{E}(Y^4) = \sigma^4 + \frac{\beta_3^2}{\sigma^2}$.

Proof. According to Lemma 5.1 in [8], we can select m and m' as follows:

$$m = \frac{\beta_3 + \sqrt{\beta_3^2 + 4\sigma^6}}{2\sigma^2}, \quad m' = -\frac{\sigma^2}{m},$$

and consider a r.v. Y with values in $\{m, m'\}$ such that

$$\mathbb{P}(Y = m) = t \text{ and } \mathbb{P}(Y = m') = 1 - t,$$

where

$$t = \frac{2\sigma^6}{4\sigma^6 + \beta_3(\beta_3 + \sqrt{\beta_3^2 + 4\sigma^6})}.$$

Indeed, in this case, by straightforward computations, $\mathbb{E}(Y) = 0$, $\mathbb{E}(Y^2) = \sigma^2$ and $\mathbb{E}(Y^3) = \beta_3$. Let us now compute $\|Y\|_4^4$. Note that

$$m - \sigma^2/m = m + m' = \frac{\beta_3}{\sigma^2}.$$

Setting $\kappa_3 = \beta_3/\sigma^2$, we have

$$\begin{aligned}\|Y\|_4^4 &= m^2(m^2t + m'^2(1-t)) + m'^2(m'^2 - m^2)(1-t) \\ &= m^2\sigma^2 - m'^2(m - m')\kappa_3(1-t).\end{aligned}$$

But $-m'^2(1-t) = m^2t - \sigma^2$ and $m - m' = \sqrt{\kappa_3^2 + 4\sigma^2}$. Then, simple computations lead to

$$\|Y\|_4^4 = \sigma^4 + \sigma^2\kappa_3^2 = \sigma^4 + \frac{\beta_3^2}{\sigma^2},$$

which ends the proof of the lemma. \square

Let $(Y_i(\theta))_{i \geq 1}$ be a sequence of random variables that are independent for any fixed θ , independent of $(X_i)_{i \geq 1}$, and such that, for each $i \geq 1$, the conditional law of $Y_i(\theta)$ given θ takes 2 values and is such that $\mathbb{E}_{|\theta}(Y_i(\theta)) = 0$, $\mathbb{E}_{|\theta}(Y_i^2(\theta)) = \theta_i^2\mathbb{E}(X_0^2)$, $\mathbb{E}_{|\theta}(Y_i^3(\theta)) = \beta_{i,3}(\theta)$ and $\mathbb{E}_{|\theta}(Y_i^4(\theta)) = \beta_{i,4}(\theta)$ where

$$\beta_{k,3}(\theta) = \theta_k^3\mathbb{E}(X_k^3) + 3 \sum_{\ell=1}^{k-1} \theta_\ell \theta_k^2 \mathbb{E}(X_\ell X_k^2) := \theta_k^3\mathbb{E}(X_k^3) + 3\tilde{\beta}_{k,3}(\theta). \quad (3.13)$$

and

$$\beta_{k,4}(\theta) = \theta_k^4(\mathbb{E}(X_k^2))^2 + \frac{\beta_{k,3}^2(\theta)}{\theta_k^2\mathbb{E}(X_k^2)}. \quad (3.14)$$

Note that this is always possible according to Fact 3.1.

Setting $f_t(x) = e^{itx}$, $M_k(\theta) = \sum_{i=1}^k \theta_i X_i$ and $T_{k,n}(\theta) = \sum_{i=k+1}^n Y_i(\theta)$, we have

$$\begin{aligned}\mathbb{E}_{|\theta}(f_t(\sum_{i=1}^n \theta_i X_i)) - \mathbb{E}_{|\theta}(f_t(\sum_{i=1}^n Y_i(\theta))) \\ = \sum_{k=1}^n \left\{ \mathbb{E}_{|\theta}(f_t(M_{k-1}(\theta) + \theta_k X_k + T_{k,n}(\theta))) - \mathbb{E}_{|\theta}(f_t(M_{k-1}(\theta) + Y_k(\theta) + T_{k,n}(\theta))) \right\}.\end{aligned}$$

Let

$$f_{t,k,n}(x) = \mathbb{E}_{|\theta}(f_t(x + T_{k,n}(\theta))).$$

This function is in \mathcal{C}^∞ and all its successive derivatives are bounded and satisfy: for any $i \geq 0$, $\|f_{t,k,n}^{(i)}(x)\|_\infty \leq t^i$. By Taylor's expansion and independence between sequences, it follows that

$$\mathbb{E}_{|\theta}(f_t(\sum_{i=1}^n \theta_i X_i)) - \mathbb{E}_{|\theta}(f_t(\sum_{i=1}^n Y_i(\theta))) = \sum_{i=1}^3 \sum_{k=1}^n I_{i,k} + \sum_{k=1}^n (R_{1,k}(f_t) + R_{2,k}(f_t)), \quad (3.15)$$

where the following notations have been used: for any integer $i \geq 1$,

$$I_{i,k} = \frac{1}{i!} \mathbb{E}_{|\theta} \left(f_{t,k,n}^{(i)}(M_{k-1}(\theta)) (\theta_k^i \mathbb{E}_{|\theta,k-1}(X_k^i) - \mathbb{E}_{|\theta}(Y_k^i(\theta))) \right),$$

$$R_{1,k}(f_t) = \frac{1}{6} \int_0^1 (1-s)^3 \mathbb{E}_\theta \{ \theta_k^4 X_k^4 f_{t,k,n}^{(4)}(M_{k-1}(\theta) + s\theta_k X_k) \} ds$$

and

$$R_{2,k}(f_t) = \frac{1}{6} \int_0^1 (1-s)^3 \mathbb{E}_\theta \{ Y_k^4(\theta) f_{t,k,n}^{(4)}(M_{k-1}(\theta) + sY_k(\theta)) \} ds.$$

Above the notation $\mathbb{E}_{|\theta, k-1}(X_k^i)$ means $\mathbb{E}(X_k^i | \sigma(\theta) \vee \mathcal{F}_{k-1})$ (note that by independence between θ and X , $\mathbb{E}_{|\theta, k-1}(X_k^i) = \mathbb{E}_{k-1}(X_k^i)$). Clearly $\sum_{k=1}^n I_{1,k} = 0$ (by the martingale property and the fact that $\mathbb{E}_{|\theta}(Y_i(\theta)) = 0$), and

$$\left| \sum_{k=1}^n (R_{1,k}(f_t) + R_{2,k}(f_t)) \right| \leq \frac{t^4}{24} \sum_{k=1}^n \{ \beta_{k,4}(\theta) + \theta_k^4 \mathbb{E}(X_k^4) \}. \quad (3.16)$$

On another hand, since $\mathbb{E}_{|\theta}(Y_k^2(\theta)) = \theta_k^2 \mathbb{E}(X_0^2)$,

$$\mathbb{E}_\theta \left(f_{t,k,n}^{(2)}(0) (\theta_k^2 \mathbb{E}_{|\theta, k-1}(X_k^2) - \mathbb{E}_{|\theta}(Y_k^2(\theta))) \right) = \theta_k^2 f_{t,k,n}^{(2)}(0) \mathbb{E}(\mathbb{E}_{k-1}(X_k^2) - \mathbb{E}(X_k^2)) = 0.$$

Hence, since $M_0(\theta) = 0$,

$$\begin{aligned} \sum_{k=1}^n I_{2,k} &= \frac{1}{2} \sum_{k=1}^n \theta_k^2 \sum_{\ell=1}^{k-1} \mathbb{E}_\theta \left\{ \left(f_{t,k,n}^{(2)}(M_\ell(\theta)) - f_{t,k,n}^{(2)}(M_{\ell-1}(\theta)) \right) \left(\mathbb{E}_\ell(X_k^2) - \mathbb{E}(X_k^2) \right) \right\} \\ &= \frac{1}{2} \sum_{k=1}^n \theta_k^2 \sum_{\ell=1}^{k-1} \theta_\ell \mathbb{E}_\theta \left\{ f_{t,k,n}^{(3)}(M_{\ell-1}(\theta)) X_\ell \left(\mathbb{E}_\ell(X_k^2) - \mathbb{E}(X_k^2) \right) \right\} \\ &\quad + \frac{1}{2} \sum_{k=1}^n \theta_k^2 \sum_{\ell=1}^{k-1} \theta_\ell^2 \int_0^1 (1-s) \mathbb{E}_\theta \left\{ f_{t,k,n}^{(4)}(M_{\ell-1}(\theta) + s\theta_\ell X_\ell) X_\ell^2 \left(\mathbb{E}_\ell(X_k^2) - \mathbb{E}(X_k^2) \right) \right\} \\ &:= \sum_{k=1}^n I_{2,3,k} + \sum_{k=1}^n I_{2,4,k}. \end{aligned} \quad (3.17)$$

We have

$$\left| \sum_{k=1}^n I_{2,4,k} \right| \leq \frac{t^4}{4} \sum_{k=1}^n \theta_k^2 \sum_{\ell=1}^{k-1} \theta_\ell^2 \gamma_{2,2}(k-\ell) \leq \frac{t^4}{4} \sum_{k=1}^n \theta_k^4 \sum_{v=1}^n \gamma_{2,2}(v). \quad (3.18)$$

Next we deal with the quantity $\sum_{k=1}^n (I_{3,k} + I_{2,3,k})$. We have

$$\begin{aligned} \sum_{k=1}^n (I_{3,k} + I_{2,3,k}) &= \frac{1}{6} \sum_{k=1}^n \theta_k^3 \mathbb{E}_\theta \left(f_{t,k,n}^{(3)}(M_{k-1}(\theta)) (\mathbb{E}_{k-1}(X_k^3) - \mathbb{E}(X_k^3)) \right) \\ &\quad + \frac{1}{2} \sum_{k=1}^n \theta_k^2 \sum_{\ell=1}^{k-1} \theta_\ell \mathbb{E}_\theta \left\{ f_{t,k,n}^{(3)}(M_{\ell-1}(\theta)) X_\ell \left(\mathbb{E}_\ell(X_k^2) - \mathbb{E}(X_k^2) \right) \right\} - \frac{1}{2} \sum_{k=1}^n \tilde{\beta}_{k,3}(\theta) \mathbb{E}_\theta \left(f_{t,k,n}^{(3)}(M_{k-1}(\theta)) \right). \end{aligned}$$

Next define

$$\begin{aligned} J_n &:= \frac{1}{2} \sum_{k=1}^n \left\{ \tilde{\beta}_{k,3}(\theta) \mathbb{E}_{|\theta} \left(f_{t,k,n}^{(3)}(M_{k-1}(\theta)) \right) - \sum_{\ell=1}^{k-1} \theta_k^2 \theta_\ell \mathbb{E}_{|\theta} \left(f_{t,k,n}^{(3)}(M_{\ell-1}(\theta)) \right) \mathbb{E} \left(X_\ell (\mathbb{E}_\ell(X_k^2) - \mathbb{E}(X_k^2)) \right) \right\} \\ &= \frac{1}{2} \sum_{k=1}^n \sum_{\ell=1}^{k-1} \theta_k^2 \theta_\ell \left\{ \mathbb{E}_{|\theta} \left(f_{t,k,n}^{(3)}(M_{k-1}(\theta)) \right) - \mathbb{E}_{|\theta} \left(f_{t,k,n}^{(3)}(M_{\ell-1}(\theta)) \right) \right\} \mathbb{E}(X_\ell X_k^2). \end{aligned}$$

For $\ell \leq k$,

$$\begin{aligned} \mathbb{E}_{|\theta} \left(f_{t,k,n}^{(3)}(M_{k-1}(\theta)) \right) - \mathbb{E}_{|\theta} \left(f_{t,k,n}^{(3)}(M_{\ell-1}(\theta)) \right) &= \sum_{u=\ell}^{k-1} \left\{ \mathbb{E}_{|\theta} \left(f_{t,k,n}^{(3)}(M_u(\theta)) \right) - \mathbb{E}_{|\theta} \left(f_{t,k,n}^{(3)}(M_{u-1}(\theta)) \right) \right\} \\ &= \sum_{u=\ell}^{k-1} \theta_u \mathbb{E}_{|\theta} \left(f_{t,k,n}^{(4)}(M_{u-1}(\theta)) \right) \mathbb{E}_{u-1}(X_u) + \int_0^1 (1-s) \sum_{u=\ell}^{k-1} \theta_u^2 \mathbb{E}_{|\theta} \left(f_{t,k,n}^{(5)}(M_{u-1}(\theta) + sX_u) X_u^2 \right) ds. \end{aligned}$$

By the Martingale property, $\mathbb{E}_{u-1}(X_u) = 0$. Therefore

$$|J_n| \leq \frac{t^5}{4} \sum_{k=1}^n \sum_{\ell=1}^{k-1} \sum_{u=\ell}^{k-1} \theta_k^2 |\theta_\ell| \theta_u^2 \gamma_{1,2}(k-\ell). \quad (3.19)$$

Next, since $\mathbb{E}_{|\theta}(Y_i^3(\theta)) = \beta_{i,3}(\theta)$, note that

$$\sum_{k=1}^n (I_{3,k} + I_{2,3,k}) - J_n = \frac{1}{6} \sum_{k=1}^n \theta_k^2 \sum_{\ell=1}^k (1 + 3\mathbf{1}_{k \neq \ell}) \mathbb{E}_{|\theta} \left\{ f_{t,k,n}^{(3)}(M_{\ell-1}(\theta)) (Z_{\ell,k} - \mathbb{E}(Z_{\ell,k})) \right\},$$

where

$$Z_{\ell,k} = X_\ell (\mathbb{E}_\ell(X_k^2) - \mathbb{E}(X_k^2)).$$

Hence, since $\mathbb{E}_{|\theta} \left\{ f_{t,k,n}^{(3)}(0) (Z_{\ell,k} - \mathbb{E}(Z_{\ell,k})) \right\} = 0$,

$$\begin{aligned} &\sum_{k=1}^n (I_{3,k} + I_{2,3,k}) - J_n \\ &= \frac{1}{6} \sum_{k=1}^n \theta_k^2 \sum_{\ell=1}^k (1 + 3\mathbf{1}_{k \neq \ell}) \sum_{u=1}^{\ell-1} \theta_\ell \mathbb{E}_{|\theta} \left\{ \left(f_{t,k,n}^{(3)}(M_{\ell-u}(\theta)) - f_{t,k,n}^{(3)}(M_{\ell-u-1}(\theta)) \right) (Z_{\ell,k} - \mathbb{E}(Z_{\ell,k})) \right\} \\ &= \frac{1}{6} \sum_{k=1}^n \theta_k^2 \sum_{\ell=1}^k (1 + 3\mathbf{1}_{k \neq \ell}) \sum_{u=1}^{\ell-1} \theta_\ell \theta_{\ell-u} \int_0^1 \mathbb{E}_{|\theta} \left\{ f_{t,k,n}^{(4)}(M_{\ell-u-1}(\theta) + s\theta_{\ell-u} X_{\ell-u}) X_{\ell-u} (Z_{\ell,k} - \mathbb{E}(Z_{\ell,k})) \right\} ds. \end{aligned}$$

We have

$$\begin{aligned} &\left| \mathbb{E}_{|\theta} \left\{ f_{t,k,n}^{(4)}(M_{\ell-u-1}(\theta) + s\theta_{\ell-u} X_{\ell-u}) X_{\ell-u} (Z_{\ell,k} - \mathbb{E}(Z_{\ell,k})) \right\} \right| \\ &= \left| \mathbb{E}_{|\theta} \left\{ f_{t,k,n}^{(4)}(M_{\ell-u-1}(\theta) + s\theta_{\ell-u} X_{\ell-u}) X_{\ell-u} \mathbb{E}_{\ell-u}(Z_{\ell,k} - \mathbb{E}(Z_{\ell,k})) \right\} \right| \\ &\leq t^4 \|X_{\ell-u} (\mathbb{E}_{\ell-u}(X_\ell X_k^2) - \mathbb{E}(X_\ell X_k^2))\|_1 \leq t^4 \gamma_{1,3}(u). \end{aligned}$$

On another hand

$$\begin{aligned}
& \left| \mathbb{E}_{|\theta} \left\{ f_{t,k,n}^{(4)}(M_{\ell-u-1}(\theta) + s\theta_{\ell-u}X_{\ell-u})X_{\ell-u}(Z_{\ell,k} - \mathbb{E}(Z_{\ell,k})) \right\} \right| \\
& \leq \left| \mathbb{E}_{|\theta} \left\{ f_{t,k,n}^{(4)}(M_{\ell-u-1}(\theta) + s\theta_{\ell-u}X_{\ell-u})X_{\ell-u}X_{\ell}(\mathbb{E}_{\ell}(X_k^2) - \mathbb{E}(X_k^2)) \right\} \right| \\
& \quad + \left| \mathbb{E}_{|\theta} \left\{ f_{t,k,n}^{(4)}(M_{\ell-u-1}(\theta) + s\theta_{\ell-u}X_{\ell-u})X_{\ell-u} \right\} \right| \left\| X_{\ell}(\mathbb{E}_{\ell}(X_k^2) - \mathbb{E}(X_k^2)) \right\|_1 \\
& \leq t^4 \|X_0 X_u (\mathbb{E}_u(X_{k-\ell+u}^2 - \mathbb{E}(X_0^2)))\|_1 + t^4 \|X_0\|_1 \|X_0 (\mathbb{E}_0(X_{k-\ell}^2) - \mathbb{E}(X_0^2))\|_1 \leq t^4 (\gamma_{2,2}(k-\ell) + \gamma_{1,2}(k-\ell)).
\end{aligned}$$

Bearing in mind Definition (2.1), we get

$$\left| \sum_{k=1}^n (I_{3,k} + I_{2,3,k}) - J_n \right| \leq t^4 \sum_{k=1}^n \theta_k^2 \sum_{\ell=1}^{k-1} \sum_{u=1}^{\ell-1} |\theta_{\ell} \theta_{\ell-u}| (\gamma(u) \wedge \gamma(k-\ell)). \quad (3.20)$$

Starting from (3.15) and taking into account (3.16), (3.17), (3.18), (3.19) and (3.20), we derive that

$$\begin{aligned}
& \left| \mathbb{E}_{|\theta} (f_t(\sum_{i=1}^n \theta_i X_i)) - \mathbb{E}_{|\theta} (f_t(\sum_{i=1}^n Y_i(\theta))) \right| \leq \frac{t^4}{12} \sum_{k=1}^n \{ \beta_{k,4}(\theta) + \theta_k^4 \mathbb{E}(X_0^4) \} + \frac{t^4}{4} \sum_{k=1}^n \theta_k^4 \sum_{v=1}^n \gamma(v) \\
& \quad + t^4 \sum_{k=1}^n \theta_k^2 \sum_{\ell=1}^{k-1} \sum_{u=1}^{\ell-1} |\theta_{\ell} \theta_{\ell-u}| (\gamma(u) \wedge \gamma(k-\ell)) + \frac{t^5}{4} \sum_{k=1}^n \sum_{\ell=1}^{k-1} \sum_{u=\ell}^{k-1} \theta_k^2 |\theta_{\ell}| \theta_u^2 \gamma(k-\ell). \quad (3.21)
\end{aligned}$$

By Young's inequality,

$$\theta_k^2 |\theta_{\ell} \theta_{\ell-u}| \leq \frac{1}{2\sqrt{2}} (\theta_k^4 + \theta_{\ell}^4 + \theta_{\ell-u}^4) \quad \text{and} \quad \theta_k^2 |\theta_{\ell}| \theta_u^2 \leq \frac{2}{5} (|\theta_k|^5 + |\theta_u|^5 + |\theta_{\ell}|^5).$$

Now, for any $m \geq 1$, $\mathbb{E}(|\theta_v|^m) \leq c_m n^{-m/2}$. Hence

$$\mathbb{E}(\theta_k^2 |\theta_{\ell} \theta_{\ell-u}|) \ll n^{-2} \quad \text{and} \quad \mathbb{E}(\theta_k^2 |\theta_{\ell}| \theta_u^2) \ll n^{-5/2}.$$

In addition,

$$\mathbb{E}(\beta_{k,4}(\theta)) \ll n^{-2} \left(1 + \sum_{\ell=1}^{k-1} \gamma(\ell) \right)^2. \quad (3.22)$$

Indeed,

$$\beta_{k,4}(\theta) \leq \theta_k^4 (1 + 2\|X_0\|_3^2) + 18\theta_k^2 \left(\sum_{\ell=1}^{k-1} \theta_{\ell} \mathbb{E}(X_{\ell} X_k^2) \right)^2.$$

Then we use the fact that $\sup_{1 \leq v \leq n} \mathbb{E}(\theta_v^4) \leq Cn^{-2}$ and that

$$\mathbb{E} \left(\theta_k^2 \left(\sum_{\ell=1}^{k-1} \theta_{\ell} \mathbb{E}(X_{\ell} X_k^2) \right)^2 \right) \leq \sum_{\ell, \ell'=1}^{k-1} \gamma(k-\ell) \gamma(k-\ell') \mathbb{E}(|\theta_k^2 \theta_{\ell} \theta_{\ell'}|) \leq Cn^{-2} \left(\sum_{\ell=1}^{k-1} \gamma(\ell) \right)^2.$$

So, overall, starting from (3.21), we derive that

$$\begin{aligned} \mathbb{E} \left| \mathbb{E}_{|\theta} \left(f_t \left(\sum_{i=1}^n \theta_i X_i \right) \right) - \mathbb{E}_{|\theta} \left(f_t \left(\sum_{i=1}^n Y_i(\theta) \right) \right) \right| &\ll t^4 n^{-1} \left(1 + \sum_{\ell=1}^n \gamma(\ell) \right)^2 \\ &+ t^4 n^{-1} \sum_{v=1}^n \sum_{u=1}^n \gamma(u) \wedge \gamma(v) + t^5 n^{-3/2} \sum_{v=1}^n v \gamma(v). \end{aligned}$$

Hence, the following bound is valid:

$$\mathbb{E} \left| \mathbb{E}_{|\theta} \left(f_t \left(\sum_{i=1}^n \theta_i X_i \right) \right) - \mathbb{E}_{|\theta} \left(f_t \left(\sum_{i=1}^n Y_i(\theta) \right) \right) \right| \ll \frac{t^4}{n} \left\{ 1 + \left(\sum_{\ell=1}^n \gamma(\ell) \right)^2 + \sum_{v=1}^n v \gamma(v) + \frac{t}{\sqrt{n}} \sum_{v=1}^n v \gamma(v) \right\},$$

implying that

$$\int_0^{T_0} t^{-1} \mathbb{E} \left| \mathbb{E}_{|\theta} \left(f_t \left(\sum_{i=1}^n \theta_i X_i \right) \right) - \mathbb{E}_{|\theta} \left(f_t \left(\sum_{i=1}^n Y_i(\theta) \right) \right) \right| dt \ll \frac{T_0^4}{n} \left\{ 1 + \left(\sum_{\ell=1}^n \gamma(\ell) \right)^2 + \sum_{v=1}^n v \gamma(v) \right\}.$$

Since $\sum_{v \geq 1} v \gamma(v) < \infty$, it follows that

$$\int_0^{T_0} \mathbb{E} \left(\left| \mathbb{E}_{|\theta} \left(e^{it \sum_{k=1}^n X_k(\theta)} \right) - \mathbb{E}_{|\theta} \left(e^{it \sum_{k=1}^n Y_k(\theta)} \right) \right| \right) \frac{dt}{t} \ll \frac{(\log n)^2}{n}. \quad (3.23)$$

Therefore the upper bound (3.12) will follow from (3.23) provided one can prove that

$$\int_0^{T_0} \mathbb{E} \left(\left| \mathbb{E}_{|\theta} \left(e^{it \sum_{k=1}^n Y_k(\theta)} \right) - e^{-t^2/2} \right| \right) \frac{dt}{t} \ll \frac{(\log n)^2}{n}. \quad (3.24)$$

With this aim, we shall adapt the proof of [13, Lemma 2.1]. Their result cannot be applied directly since $Y_k(\theta)$ is not of the form $\theta_k \eta_k$ where $(\eta_k)_{k \in \mathbb{Z}}$ is a sequence of independent r.v.'s independent of θ . Let

$$\Gamma_n(\theta) = \left\{ \max_{1 \leq k \leq n} |\beta_{k,3}(\theta)| T_0 \leq 1 \right\} \cap \left\{ T_0^3 \left| \sum_{k=1}^n \beta_{k,3}(\theta) \right| \leq 1 \right\} \cap \left\{ T_0^4 \sum_{k=1}^n \beta_{k,4}(\theta) \leq 1 \right\}.$$

Since, when θ is fixed, $(Y_k(\theta))_{1 \leq k \leq n}$ are independent random variables that are centered, in \mathbb{L}^4 and such that $\sum_{k=1}^n \mathbb{E}(Y_k^2(\theta)) = 1$, we infer that, by standard arguments (see the proof of [13, Lemma 2.1]), the following estimate holds : for any positive t such that $t \leq T_0$,

$$\left| \mathbb{E}_{|\theta} \left(e^{it \sum_{k=1}^n Y_k(\theta)} \right) - e^{-t^2/2} \right| \mathbf{1}_{\Gamma_n(\theta)} \ll e^{-t^2/2} \left(t^3 \left| \sum_{k=1}^n \beta_{k,3}(\theta) \right| + t^4 \sum_{k=1}^n \beta_{k,4}(\theta) \right). \quad (3.25)$$

Let $a(u) = \mathbb{E}(X_0 X_u^2)$. Note that

$$\mathbb{E} \left| \sum_{k=1}^n \beta_{k,3}(\theta) \right| \leq \|X_0\|_3^3 \mathbb{E} \left| \sum_{k=1}^n \theta_k^3 \right| + 3 \mathbb{E} \left| \sum_{k=1}^n \theta_k^2 \sum_{\ell=1}^{k-1} a(k-\ell) \theta_\ell \right|.$$

We shall use the fact that $(\theta_1, \dots, \theta_n) \stackrel{\mathcal{D}}{=} (\xi_1, \dots, \xi_n) / \|\xi\|_e$ where $(\xi_i)_{i \geq 1}$ is a sequence of iid $\mathcal{N}(0, 1)$ -distributed r.v.'s and $\|\xi\|_e^2 = \sum_{i=1}^n \xi_i^2$. Note first that, because $\mathbb{E}\left(n^3 / \|\xi\|_e^6\right) \leq K$ (by the properties of the χ^2 -distribution),

$$\mathbb{E}\left|\sum_{k=1}^n \theta_k^3\right| = \mathbb{E}\left|\frac{1}{\|\xi\|_e^3} \sum_{k=1}^n \xi_k^3\right| \leq \left(\mathbb{E}\left(\frac{n^3}{\|\xi\|_e^6}\right)\right)^{1/2} \left(\frac{1}{n^3} \mathbb{E}\left(\sum_{k=1}^n \xi_k^3\right)^2\right)^{1/2} \ll n^{-1}.$$

Next

$$\begin{aligned} \mathbb{E}\left|\sum_{k=1}^n \theta_k^2 \sum_{\ell=1}^{k-1} a(k-\ell)\theta_\ell\right| &= \mathbb{E}\left|\frac{1}{\|\xi\|_e^3} \sum_{k=1}^n \xi_k^2 \sum_{\ell=1}^{k-1} a(k-\ell)\xi_\ell\right| \\ &\leq \left(\mathbb{E}\left(\frac{n^3}{\|\xi\|_e^6}\right)\right)^{1/2} \left(\frac{1}{n^3} \mathbb{E}\left(\sum_{k=1}^n \xi_k^2 \sum_{\ell=1}^{k-1} a(k-\ell)\xi_\ell\right)^2\right)^{1/2}. \end{aligned}$$

Now,

$$\begin{aligned} \mathbb{E}\left(\sum_{k=1}^n \xi_k^2 \sum_{\ell=1}^{k-1} a(k-\ell)\xi_\ell\right)^2 &= \sum_{k=1}^n \mathbb{E}(\xi_k^4) \sum_{\ell=1}^{k-1} a^2(k-\ell)\mathbb{E}(\xi_\ell^2) \\ &\quad + 2 \sum_{k=1}^{n-1} \sum_{k'=k+1}^n \mathbb{E}(\xi_{k'}^2) \sum_{\ell=1}^{k-1} \sum_{\ell'=1}^{k'-1} a(k-\ell)a(k'-\ell')\mathbb{E}(\xi_\ell \xi_k^2 \xi_{\ell'}). \end{aligned}$$

By independence,

$$\sum_{k=1}^{n-1} \sum_{k'=k+1}^n \sum_{\ell=1}^{k-1} \sum_{\ell'=1}^{k'-1} a(k-\ell)a(k'-\ell')\mathbb{E}(\xi_\ell \xi_k^2 \xi_{\ell'}) = \sum_{k=1}^{n-1} \sum_{k'=k+1}^n \sum_{\ell=1}^{k-1} a(k-\ell)a(k'-\ell).$$

Hence

$$\mathbb{E}\left(\sum_{k=1}^n \xi_k^2 \sum_{\ell=1}^{k-1} a(k-\ell)\xi_\ell\right)^2 \ll n \left(\sum_{\ell=1}^n \gamma(\ell)\right)^2.$$

It follows that

$$\mathbb{E}\left|\sum_{k=1}^n \beta_{k,3}(\theta)\right| \ll n^{-1} \left(1 + \sum_{\ell=1}^n \gamma(\ell)\right). \quad (3.26)$$

Starting from (3.25) and taking into account (3.26) and (3.22), we derive that

$$\int_0^{T_0} \mathbb{E}\left(\left|\mathbb{E}_\theta\left(e^{it \sum_{k=1}^n Y_k(\theta)}\right) - e^{-t^2/2}\right| \mathbf{1}_{\Gamma_n(\theta)}\right) \frac{dt}{t} \ll n^{-1} \left(1 + \sum_{\ell=1}^n \gamma(\ell)\right)^2. \quad (3.27)$$

Next, note that

$$\mathbb{P}(\Gamma_n^c(\theta)) \leq \sum_{k=1}^n \mathbb{P}(T_0 |\beta_{k,3}(\theta)| > 1) + \mathbb{P}\left(T_0^3 \left|\sum_{k=1}^n \beta_{k,3}(\theta)\right| > 1\right) + \mathbb{P}\left(T_0^4 \left|\sum_{k=1}^n \beta_{k,4}(\theta)\right| > 1\right). \quad (3.28)$$

We first deal with the first term in the right-hand side of (3.28). By Markov's inequality,

$$\sum_{k=1}^n \mathbb{P}(T_0|\beta_{k,3}(\theta)| > 1) \ll T_0^2 \sum_{k=1}^n \left(\mathbb{E}(\theta_k^6) + \mathbb{E}\left\{ \theta_k^4 \left(\sum_{\ell=1}^{k-1} \theta_\ell a(k-\ell) \right)^2 \right\} \right).$$

Now $\mathbb{E}(\theta_k^6) \leq Cn^{-3}$, and

$$\mathbb{E}\left\{ \theta_k^4 \left(\sum_{\ell=1}^{k-1} \theta_\ell a(k-\ell) \right)^2 \right\} \leq \sum_{\ell, \ell'=1}^{k-1} |a(k-\ell)| |a(k-\ell')| \mathbb{E}(\theta_k^4 |\theta_\ell \theta_{\ell'}|) \leq Cn^{-3} \left(\sum_{\ell=1}^{k-1} \gamma(\ell) \right)^2.$$

Hence

$$\sum_{k=1}^n \mathbb{P}(T_0|\beta_{k,3}(\theta)| > 1) \ll T_0^2 n^{-2} \left(1 + \sum_{\ell=1}^n \gamma(\ell) \right)^2. \quad (3.29)$$

Starting from (3.28), applying Markov's inequality and taking into account the upper bounds (3.29), (3.26) and (3.22), we derive that

$$\mathbb{P}(\Gamma_n^c(\theta)) \ll T_0^4 n^{-1} \left(1 + \sum_{\ell=1}^n \gamma(\ell) \right)^2. \quad (3.30)$$

On another hand, note that

$$\begin{aligned} |\mathbb{E}_\theta \left(e^{it \sum_{k=1}^n Y_k(\theta)} \right) - e^{-t^2/2}| &= \left| \prod_{k=1}^n \mathbb{E}_\theta \left(e^{it Y_k(\theta)} \right) - \prod_{k=1}^n e^{-t^2 \theta_k^2 / 2} \right| \\ &\leq \sum_{k=1}^n |\mathbb{E}_\theta \left(e^{it Y_k(\theta)} \right) - e^{-t^2 \theta_k^2 / 2}| \leq \frac{|t|^3}{6} \sum_{k=1}^n \mathbb{E}_\theta(|Y_k^3(\theta)|) + \frac{t^4}{8} \sum_{k=1}^n \theta_k^4. \end{aligned}$$

Now $\mathbb{E}_\theta(|Y_k^3(\theta)|) \leq \beta_{k,4}^{3/4}(\theta)$. Since $|\theta_k| \leq 1$ it follows $\theta_k^4 \leq \beta_{k,4}^{3/4}(\theta)$. Therefore

$$|\mathbb{E}_\theta \left(e^{it \sum_{k=1}^n Y_k(\theta)} \right) - e^{-t^2/2}| \leq \frac{|t|^3}{6} (1 + |t|) \sum_{k=1}^n \beta_{k,4}^{3/4}(\theta). \quad (3.31)$$

Taking into account (3.30), (3.31) and the fact that $\sum_{\ell \geq 1} \gamma(\ell) < \infty$, we infer that for any $r \geq 1$,

$$\int_0^{T_0} \mathbb{E} \left(\left| \mathbb{E}_\theta \left(e^{it \sum_{k=1}^n Y_k(\theta)} \right) - e^{-t^2/2} \right| \mathbf{1}_{\Gamma_n^c(\theta)} \right) \frac{dt}{t} \ll T_0^4 \sum_{k=1}^n \left(\mathbb{E}(\beta_{k,4}^{3r/4}(\theta)) \right)^{1/r} \left(T_0^4 n^{-1} \right)^{(r-1)/r}. \quad (3.32)$$

Proceeding as to prove (3.22), we infer that if $\sum_{\ell \geq 1} \gamma(\ell) < \infty$, for any $m \geq 1$,

$$\max_{1 \leq k \leq n} \mathbb{E}(\beta_{k,4}^m(\theta)) \ll n^{-2m}. \quad (3.33)$$

So, for any $r \geq 4/3$,

$$\int_0^{T_0} \mathbb{E} \left(\left| \mathbb{E}_{|\theta} \left(e^{it \sum_{k=1}^n Y_k(\theta)} \right) - e^{-t^2/2} \right| \mathbf{1}_{\Gamma_n^c(\theta)} \right) \frac{dt}{t} \ll T_0^{8-4/r} \frac{n^{1/r}}{n^{3/2}}.$$

Taking $r > 2$ in the inequality above, we derive that

$$\int_0^{T_0} \mathbb{E} \left(\left| \mathbb{E}_{|\theta} \left(e^{it \sum_{k=1}^n Y_k(\theta)} \right) - e^{-t^2/2} \right| \mathbf{1}_{\Gamma_n^c(\theta)} \right) \frac{dt}{t} \ll \frac{1}{n},$$

which combined with (3.27) implies (3.24) in case $\sum_{\ell \geq 1} \gamma(\ell) < \infty$. This ends the proof of the theorem. \square

4 Application to linear regression with Gaussian design

Let us consider the following linear model

$$Y_i = \alpha + \beta Z_i + X_i, \quad 1 \leq i \leq n,$$

where $(X_i)_{i \in \mathbb{Z}}$ is a strictly stationary sequence of martingale differences, and $(Z_i)_{i \in \mathbb{Z}}$ is a sequence of iid $\mathcal{N}(\mu, \sigma^2)$ -distributed random variables, which is independent of $(X_i)_{i \in \mathbb{Z}}$. As usual, the observations are $(Y_i, Z_i)_{1 \leq i \leq n}$ and the aim is to estimate the unknown parameter β . The ordinary least squares estimator $\hat{\beta}$ of β is then given by

$$\hat{\beta} = \sum_{i=1}^n \frac{(Y_i - \bar{Y}_n)(Z_i - \bar{Z}_n)}{\|Z - \bar{Z}_n\|_e^2},$$

and satisfies

$$\hat{\beta} - \beta = \sum_{i=1}^n \frac{(Z_i - \bar{Z}_n)}{\|Z - \bar{Z}_n\|_e^2} X_i = \sum_{i=1}^n \frac{(Z_i - \bar{Z}_n)}{\|Z - \bar{Z}_n\|_e^2} (X_i - \bar{X}_n),$$

where $Z = (Z_1, \dots, Z_n)$ and $\|Z - \bar{Z}_n\|_e^2 = \sum_{i=1}^n (Z_i - \bar{Z}_n)^2$. Let

$$T_n := \|Z - \bar{Z}_n\|_e (\hat{\beta} - \beta) = \sum_{i=1}^n \frac{(Z_i - \bar{Z}_n)}{\|Z - \bar{Z}_n\|_e} (X_i - \bar{X}_n).$$

Note that

$$T_n = \|\tilde{\xi}\|_e^{-1} \langle \tilde{\xi}, \tilde{X} \rangle_{\mathbb{R}^n}.$$

where $\tilde{X} = (X_1 - \bar{X}_n, \dots, X_n - \bar{X}_n)$, and $\tilde{\xi} = (\xi_1 - \bar{\xi}_n, \dots, \xi_n - \bar{\xi}_n)$ with $\xi_i = (Z_i - \mu)/\sigma$.

Note that $\sum_{i=1}^n \tilde{\xi}_i = \sum_{i=1}^n \tilde{X}_i = 0$. Let $(u_i)_{1 \leq i \leq n}$ be the vectors of \mathbb{R}^n defined as follows:

$$u_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \dots, 0 \right)^t, \quad u_n = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}} \right)^t$$

and, for $2 \leq k \leq n-1$,

$$u_k = \left(\underbrace{\frac{1}{\sqrt{k(k+1)}}, \dots, \frac{1}{\sqrt{k(k+1)}}}_{k \text{ times}}, -\frac{k}{\sqrt{k(k+1)}}, \underbrace{0, \dots, 0}_{n-k-1 \text{ times}} \right)^t.$$

Note that $(u_i)_{1 \leq i \leq n}$ is an orthonormal basis of \mathbb{R}^n . Let A be the change-of-basis matrix from the basis $(u_i)_{1 \leq i \leq n}$ to the canonical basis $(e_i)_{1 \leq i \leq n}$ of \mathbb{R}^n . Hence $A = (u_1, u_2, \dots, u_n)$. Since $(e_i)_{1 \leq i \leq n}$ and $(u_i)_{1 \leq i \leq n}$ are both orthonormal bases of \mathbb{R}^n , A is an orthonormal matrix and the change-of-basis matrix B from $(e_i)_{1 \leq i \leq n}$ to $(u_i)_{1 \leq i \leq n}$ satisfies $B = A^t$. Hence $\xi = \sum_{i=1}^n (B\xi)_i u_i$ and $X = \sum_{i=1}^n (BX)_i u_i$. Since $\tilde{\xi}$ (resp. \tilde{X}) is the orthogonal projection of ξ (resp. X) on the space generated by (u_1, \dots, u_{n-1}) , $\tilde{\xi} = \sum_{i=1}^{n-1} (B\xi)_i u_i$ and $\tilde{X} = \sum_{i=1}^{n-1} (BX)_i u_i$. It follows that

$$\langle \tilde{\xi}, \tilde{X} \rangle_{\mathbb{R}^n} = \sum_{i=1}^{n-1} (B\xi)_i (BX)_i.$$

In addition,

$$\|\tilde{\xi}\|_e^2 = \sum_{i=1}^{n-1} (B\xi)_i^2 \quad \text{and} \quad \|\tilde{X}\|_e^2 = \sum_{i=1}^{n-1} (BX)_i^2. \quad (4.1)$$

So, overall,

$$T_n = \|\tilde{\xi}\|_e^{-1} \langle \tilde{\xi}, \tilde{X} \rangle_{\mathbb{R}^n} := \langle \theta, Y \rangle_{\mathbb{R}^{n-1}}$$

where $\theta = ((B\xi)_1, \dots, (B\xi)_{n-1}) / \|\tilde{\xi}\|_e$ and $Y = ((BX)_1, \dots, (BX)_{n-1})$. Note that since ξ is a standard Gaussian vector (i.e. with $\mathcal{N}(0, I_n)$ distribution) and B is an orthonormal matrix, θ is uniformly distributed on the sphere S^{n-2} .

Next, note that

$$\langle \theta, Y \rangle_{\mathbb{R}^{n-1}} = \sum_{k=1}^{n-1} \theta_k \left\{ \left(\sum_{\ell=1}^k \frac{X_\ell}{\sqrt{k(k+1)}} \right) - \frac{k}{\sqrt{k(k+1)}} X_{k+1} \right\} := \sum_{k=1}^{n-1} \theta_k X_k^*.$$

Hence by interchanging the sums, it follows that

$$T_n = \langle \theta, Y \rangle_{\mathbb{R}^{n-1}} = \sum_{\ell=1}^n X_\ell \theta_\ell^*.$$

where

$$\theta_n^* = -\frac{\sqrt{n-1}}{\sqrt{n}} \theta_{n-1} \quad \text{and} \quad \theta_\ell^* = \tilde{\theta}_\ell - \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \theta_{\ell-1}, \quad 1 \leq \ell \leq n-1, \quad (4.2)$$

with $\tilde{\theta}_\ell = \sum_{v=\ell}^{n-1} \theta_v / \sqrt{v(v+1)}$ and the convention that $\theta_0 = 0$.

To summarize T_n can be viewed either as the projection of (X_1^*, \dots, X_k^*) on the sphere S_{n-2} (however $(X_k^*)_{k \in \mathbb{Z}}$ is not anymore a sequence of martingale differences) or as the weighted sums $\sum_{\ell=1}^n X_\ell \theta_\ell^*$. Even if $(\theta_1^*, \dots, \theta_n^*)$ is not uniformly distributed on the sphere, we can use both ways of writings T_n and adapt the proof of Theorem 2.1 to derive the following:

Theorem 4.1. Let $(X_n)_{n \in \mathbb{Z}}$ be a stationary sequence of martingale differences in \mathbb{L}^4 such that $\mathbb{E}(X_0^2) = 1$. Assume that $\sum_{v \geq 1} v\gamma(v) < \infty$ (where $\gamma(k)$ is defined in (2.1)). Then

$$\mathbb{E}\left(\sup_{t \in \mathbb{R}} \left| \mathbb{P}_{|Z}(T_n \leq t) - \mathbb{P}(G \leq t) \right| \right) \ll \frac{(\log n)^2}{n}.$$

Proof of Theorem 4.1. Recall that $T_n = \sum_{\ell=1}^{n-1} X_\ell^* \theta_\ell$ where $(\theta_1, \dots, \theta_{n-1})$ is uniformly distributed on the sphere S_{n-2} and

$$X_k^* = \sqrt{\frac{k}{k+1}} (\bar{X}_k - X_{k+1}),$$

with $\bar{X}_k = k^{-1} \sum_{v=1}^k X_v$. We first notice that

$$\sum_{k=1}^{n-1} \mathbb{E}(X_k^*)^2 = n + r_n \text{ where } |r_n| \leq c \log n.$$

We then infer that the conclusion of Lemma 5.2 in [1] is still valid. Therefore, if one can prove that

$$\left\| \sum_{k=1}^{n-1} ((X_k^*)^2 - \mathbb{E}(X_k^*)^2) \right\|_2^2 \ll n \text{ and } \sup_{\theta \in S_{n-2}} \mathbb{E}_{|\theta} \left(\sum_{k=1}^{n-1} \theta_k X_k^* \right)^4 \ll (\log n)^2, \quad (4.3)$$

then, with $T_0 = 4\sqrt{\log n}$, we will have

$$\mathbb{E}\left(\sup_{t \in \mathbb{R}} \left| \mathbb{P}_{|Z}(T_n \leq t) - \mathbb{P}(G \leq t) \right| \right) \ll \int_0^{T_0} \mathbb{E} \left| \mathbb{E}_{|\theta} \left(e^{i \sum_{k=1}^{n-1} \theta_k X_k^*} \right) - e^{-t^2/2} \right| \frac{dt}{t} + \frac{(\log n)^2}{n}, \quad (4.4)$$

in place of (2.5) of Proposition 2.1. Therefore, using that $\sum_{k=1}^{n-1} \theta_k X_k^* = \sum_{k=1}^n \theta_k^* X_k$, where the θ_k^* 's are defined in (4.2), it will suffice to prove that

$$\int_0^{T_0} \mathbb{E} \left| \mathbb{E}_{|\theta} \left(e^{i \sum_{k=1}^n \theta_k^* X_k} \right) - e^{-t^2/2} \right| \frac{dt}{t} \ll \frac{(\log n)^2}{n}. \quad (4.5)$$

Let us start by proving (4.3). Note first that

$$\begin{aligned} \left\| \sum_{k=1}^{n-1} ((X_k^*)^2 - \mathbb{E}(X_k^*)^2) \right\|_2 &\leq \left\| \sum_{k=1}^{n-1} \frac{k}{k+1} (X_{k+1}^2 - \mathbb{E}(X_{k+1}^2)) \right\|_2 \\ &\quad + 4 \left\| \sum_{k=1}^{n-1} \frac{k}{k+1} \bar{X}_k X_{k+1} \right\|_2 + \sum_{k=1}^{n-1} \|\bar{X}_k\|_4^2. \end{aligned}$$

Since $(X_k)_{k \geq 0}$ is a sequence of martingale differences in \mathbb{L}^4 , Burkholder's inequality implies that

$$\sum_{k=1}^{n-1} \|\bar{X}_k\|_4^2 \ll \|X_0\|_4^2 \sum_{k=1}^{n-1} k^{-1} \ll \|X_0\|_4^2 (\log n)$$

and

$$\left\| \sum_{k=1}^{n-1} \frac{k}{k+1} \bar{X}_k X_{k+1} \right\|_2^2 = \sum_{k=1}^{n-1} \frac{k^2}{(k+1)^2} \|\bar{X}_k X_{k+1}\|_2^2 \leq \|X_0\|_4^2 \sum_{k=1}^{n-1} \|\bar{X}_k\|_4^2 \ll \|X_0\|_4^4 (\log n).$$

On another hand,

$$\left\| \sum_{k=1}^{n-1} \frac{k}{k+1} (X_{k+1}^2 - \mathbb{E}(X_{k+1}^2)) \right\|_2^2 \leq 2n \sum_{v=0}^{n-1} \gamma_{2,2}(v).$$

All the above considerations prove that the first part of condition (4.3) is satisfied as soon as $\sum_{v \geq 0} \gamma(v) < \infty$. To prove its second part, we first write that

$$\mathbb{E}_{|\theta} \left(\sum_{k=1}^{n-1} \theta_k X_k^* \right)^4 \leq 2^3 \mathbb{E}_{|\theta} \left(\sum_{k=1}^{n-1} \frac{\sqrt{k}}{\sqrt{k+1}} \theta_k X_{k+1} \right)^4 + 2^3 \mathbb{E}_{|\theta} \left(\sum_{k=1}^{n-1} \frac{\sqrt{k}}{\sqrt{k+1}} \theta_k \bar{X}_k \right)^4.$$

Burkholder's inequality implies that

$$\mathbb{E}_{|\theta} \left(\sum_{k=1}^{n-1} \frac{\sqrt{k}}{\sqrt{k+1}} \theta_k X_{k+1} \right)^4 \ll \|X_0\|_4^4 \left(\sum_{k=1}^{n-1} \frac{k}{k+1} \theta_k^2 \right)^2 \ll \|X_0\|_4^4.$$

Next, again by Burkholder's inequality,

$$\mathbb{E}_{|\theta} \left(\sum_{k=1}^{n-1} \frac{\sqrt{k}}{\sqrt{k+1}} \theta_k \bar{X}_k \right)^4 \leq \mathbb{E} \left(\sum_{k=1}^{n-1} \bar{X}_k^2 \right)^2 \leq \left(\sum_{k=1}^{n-1} \|\bar{X}_k\|_4^2 \right)^2 \ll \|X_0\|_4^4 (\log n)^2.$$

So, overall, the second part of (4.3) is proved.

It remains to show that (4.5) is satisfied. With this aim, we consider $(Y_i(\theta^*))_{1 \leq i \leq n}$ a sequence of random variables that are independent for any fixed θ , independent of $(X_i)_{i \geq 1}$, and such that, for each $i \geq 1$, the conditional law of $Y_i(\theta^*)$ given θ takes 2 values and is such that $\mathbb{E}_{|\theta}(Y_i(\theta^*)) = 0$, $\mathbb{E}_{|\theta}(Y_i^2(\theta^*)) = (\theta_i^*)^2 \mathbb{E}(X_0^2)$, $\mathbb{E}_{|\theta}(Y_i^3(\theta^*)) = \beta_{i,3}(\theta^*)$ and $\mathbb{E}_{|\theta}(Y_i^4(\theta^*)) = \beta_{i,4}(\theta^*)$ where $\beta_{i,3}(\theta^*)$ (resp. $\beta_{i,4}(\theta^*)$) is defined by (3.13) (resp. (3.14)) with θ^* replacing θ . Recall that this is always possible according to Fact 3.1.

To show that (4.5) is satisfied, we shall prove that

$$\int_0^{T_0} \mathbb{E} \left(\left| \mathbb{E}_{|\theta} \left(e^{i \sum_{k=1}^n \theta_k^* X_k} \right) - \mathbb{E}_{|\theta} \left(e^{i \sum_{k=1}^n Y_k(\theta^*)} \right) \right| \right) \frac{dt}{t} \ll \frac{(\log n)^2}{n}, \quad (4.6)$$

$$\int_0^{T_0} \mathbb{E} \left(\left| \mathbb{E}_{|\theta} \left(e^{i \sum_{k=1}^n Y_k(\theta^*)} \right) - e^{-t^2 \sum_{k=1}^n (\theta_k^*)^2 / 2} \right| \right) \frac{dt}{t} \ll \frac{(\log n)^2}{n}, \quad (4.7)$$

and

$$\int_0^{T_0} \mathbb{E} \left| e^{-t^2 \sum_{k=1}^n (\theta_k^*)^2 / 2} - e^{-t^2 / 2} \right| \frac{dt}{t} \ll \frac{(\log n)^2}{n}, \quad (4.8)$$

To prove (4.6) and (4.7), we proceed as in the proof of Theorem 2.1. Then starting from (3.21) with θ^* instead of θ , the estimate (4.6) will follow if one can prove that, for any $m \geq 1$, there exists a positive constant c_m such that, any $1 \leq k \leq n$,

$$\mathbb{E}(|\theta_k^*|^m) \ll c_m n^{-m/2}, \quad (4.9)$$

and, for any $\eta > 0$,

$$\sum_{k=1}^n \mathbb{E}(\beta_{k,4}(\theta^*)) \ll n^{-1}. \quad (4.10)$$

On another hand, analyzing the proof of (3.24), we infer that (4.7) holds provided (4.10) is satisfied and

$$\mathbb{E} \left| \sum_{k=1}^n \beta_{k,3}(\theta^*) \right| \ll n^{-1}, \quad \sum_{k=1}^n \mathbb{E}(\beta_{k,3}^2(\theta^*)) \ll n^{-1} \quad \text{and} \quad \max_{1 \leq k \leq n} \mathbb{E}(\beta_{k,4}^m(\theta^*)) \ll n^{-2m}. \quad (4.11)$$

Finally, to prove (4.8), we note that

$$\mathbb{E} \left| e^{-t^2 \sum_{k=1}^n (\theta_k^*)^2 / 2} - e^{-t^2 / 2} \right| \leq \frac{t^2}{2} \mathbb{E} \left| \sum_{k=1}^n (\theta_k^*)^2 - 1 \right|.$$

Hence to prove (4.8), it suffices to show that

$$\mathbb{E} \left| \sum_{k=1}^n (\theta_k^*)^2 - 1 \right| \ll n^{-1} \log n. \quad (4.12)$$

We start by proving (4.9). Note first that

$$\mathbb{E}(|\theta_k^*|^m) \leq 2^{m-1} \mathbb{E}(|\theta_{k-1}|^m) + 2^{m-1} \mathbb{E} \left(\left| \sum_{\ell=k}^{n-1} \frac{\theta_\ell}{\sqrt{\ell(\ell+1)}} \right|^m \right)$$

Since θ is uniformly distributed on the sphere S^{n-2} , $\mathbb{E}(|\theta_{k-1}|^m) \ll c_m n^{-m/2}$. In addition $(\theta_1, \dots, \theta_{n-1}) =^D (\xi_1, \dots, \xi_{n-1}) / \|\xi\|_e$ where $(\xi_i)_{i \geq 1}$ is a sequence of iid $\mathcal{N}(0, 1)$ -distributed r.v.'s and $\|\xi\|_e^2 = \sum_{i=1}^{n-1} \xi_i^2$. Hence

$$\begin{aligned} \mathbb{E} \left(\left| \sum_{\ell=k}^{n-1} \frac{\theta_\ell}{\sqrt{\ell(\ell+1)}} \right|^m \right) &\ll n^{-m/2} \mathbb{E} \left(\left| \sum_{\ell=k}^{n-1} \frac{\xi_\ell}{\sqrt{\ell(\ell+1)}} \right|^m \right) \\ &\quad + \mathbb{E}^{1/2} \left(\left| \sum_{\ell=k}^{n-1} \frac{\theta_\ell}{\sqrt{\ell(\ell+1)}} \right|^{2m} \right) \left(\mathbb{P}(\|\xi\|_e^2 < n/2) \right)^{1/2}. \end{aligned}$$

By Burkholder's inequality

$$\mathbb{E} \left(\left| \sum_{\ell=k}^{n-1} \frac{\xi_\ell}{\sqrt{\ell(\ell+1)}} \right|^m \right) \ll \left| \sum_{\ell=k}^{n-1} \frac{1}{\ell(\ell+1)} \right|^{m/2} \ll k^{-m/2}.$$

On another hand, using that $\mathbb{P}(\|\xi\|_e^2 < n/2) \leq e^{-cn}$, for some $c > 0$, and the fact that $\theta_k \leq 1$, we get

$$\mathbb{E}^{1/2} \left(\left| \sum_{\ell=k}^{n-1} \frac{\theta_\ell}{\sqrt{\ell(\ell+1)}} \right|^{2m} \right) \left(\mathbb{P}(\|\xi\|_e^2 < n/2) \right)^{1/2} \ll n^{-m/2} k^{-m/2}.$$

So, overall,

$$\mathbb{E} \left(\left| \sum_{\ell=k}^{n-1} \frac{\theta_\ell}{\sqrt{\ell(\ell+1)}} \right|^m \right) \ll n^{-m/2} k^{-m/2} \text{ and then } \mathbb{E}(|\theta_k^*|^m) \ll n^{-m/2}, \quad (4.13)$$

which proves (4.9). We turn now to the proof of the last part of (4.11) that will also imply (4.10). Setting $a(k) = \mathbb{E}(X_0 X_k^2)$, note first that

$$\beta_{k,4}(\theta^*) \ll \theta_{k-1}^4 + \left(\sum_{\ell=k}^{n-1} \frac{\theta_\ell}{\sqrt{\ell(\ell+1)}} \right)^4 + \left(\sum_{\ell=1}^{k-1} \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \theta_{\ell-1} a(k-\ell) \right)^4 + \left(\sum_{\ell=1}^{k-1} \tilde{\theta}_\ell a(k-\ell) \right)^4.$$

Proceeding as in the proof of (4.13), since $\sum_{k \geq 1} a^2(k) < \infty$, we infer that

$$\max_{1 \leq k \leq n} \mathbb{E} \left(\sum_{\ell=1}^{k-1} \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \theta_{\ell-1} a(k-\ell) \right)^{4m} \ll n^{-2m},$$

which combined with (4.13) and the fact that $\max_{1 \leq k \leq n} \mathbb{E}(\theta_{k-1}^{4m}) \ll n^{-2m}$, implies that, for any $m \geq 1$,

$$\max_{1 \leq k \leq n} \mathbb{E}(\beta_{k,4}^m(\theta^*)) \ll n^{-2m} + \max_{1 \leq k \leq n} \mathbb{E} \left(\sum_{\ell=1}^{k-1} \tilde{\theta}_\ell a(k-\ell) \right)^{4m}.$$

Next, proceeding again as in the proof of (4.13), for any $k \leq n$, we get

$$\begin{aligned} \mathbb{E} \left(\sum_{\ell=1}^{k-1} \tilde{\theta}_\ell a(k-\ell) \right)^{4m} &= \mathbb{E} \left(\sum_{v=1}^{n-1} \frac{\theta_v}{\sqrt{v(v+1)}} \sum_{\ell=1}^{(k-1) \wedge v} a(k-\ell) \right)^{4m} \\ &\ll n^{-2m} + n^{-2m} \left[\sum_{v=1}^{n-1} \frac{1}{v(v+1)} \left(\sum_{\ell=1}^{(k-1) \wedge v} a(k-\ell) \right)^2 \right]^{2m} \ll n^{-2m} \left(1 + \sum_{\ell=1}^{n-1} \gamma(v) \right)^{4m}. \end{aligned} \quad (4.14)$$

So, overall, for any $m \geq 1$,

$$\max_{1 \leq k \leq n} \mathbb{E}(\beta_{k,4}^m(\theta^*)) \ll n^{-2m} \left(1 + \sum_{\ell=1}^{n-1} \gamma(v) \right)^{4m},$$

which proves the last part of (4.11) (and then (4.10)) since $\sum_{v \geq 1} \gamma(v) < \infty$. On another hand, by Young's inequality,

$$\sum_{k=1}^n \mathbb{E}(\beta_{k,3}^2(\theta^*)) \ll \sum_{k=1}^{n-1} \mathbb{E}(\theta_k^{*6}) + \sum_{k=1}^n \mathbb{E} \left(\sum_{\ell=1}^{k-1} \frac{\sqrt{\ell-1}}{\sqrt{\ell}} a(k-\ell) \theta_{\ell-1} \right)^6 + \sum_{k=1}^n \mathbb{E} \left(\sum_{\ell=1}^{k-1} \tilde{\theta}_\ell a(k-\ell) \right)^6.$$

By (4.9), the first term in the right-hand side is bounded by a constant times n^{-2} . In addition, by the same arguments used to prove (4.13), we infer that

$$\sum_{k=1}^n \mathbb{E} \left(\sum_{\ell=1}^{k-1} \frac{\sqrt{\ell-1}}{\sqrt{\ell}} a(k-\ell) \theta_{\ell-1} \right)^6 \ll n^{-3} \sum_{k=1}^n \left(\sum_{\ell=1}^{k-1} a^2(k-\ell) \right)^3 + n^{-2} \ll n^{-2}, \quad (4.15)$$

and

$$\sum_{k=1}^n \mathbb{E} \left(\sum_{\ell=1}^{k-1} \tilde{\theta}_\ell a(k-\ell) \right)^6 \ll n^{-3} \sum_{k=1}^n \left(1 + \sum_{\ell=1}^{k-1} \gamma(v) \right)^6 \ll n^{-2}.$$

So, overall, the second part of (4.11) is satisfied. We turn to the proof of (4.12). With this aim, we note that $\sum_{k=1}^{n-1} \theta_k^2 = 1$. Hence

$$\sum_{k=1}^n \theta_k^{*2} - 1 = \sum_{\ell=1}^{n-1} (\tilde{\theta}_\ell)^2 - 2 \sum_{\ell=2}^{n-1} \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \tilde{\theta}_\ell \theta_{\ell-1} - \sum_{\ell=1}^{n-1} \frac{1}{\ell+1} \theta_\ell^2.$$

Taking the expectation of the absolute values of the above quantity, and considering the first part of (4.13), we then infer that (4.12) holds. Indeed we can use the same arguments used to prove (4.13) and the fact that

$$\begin{aligned} \mathbb{E} \left| \sum_{\ell=2}^{n-1} \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \xi_{\ell-1} \sum_{v=\ell}^{n-1} \frac{\xi_v}{\sqrt{v(v+1)}} \right|^2 &= \mathbb{E} \left| \sum_{v=2}^{n-1} \frac{\xi_v}{\sqrt{v(v+1)}} \sum_{\ell=2}^v \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \xi_{\ell-1} \right|^2 \\ &= \sum_{v=2}^{n-1} \frac{1}{v(v+1)} \mathbb{E} \left(\sum_{\ell=2}^v \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \xi_{\ell-1} \right)^2 \leq \sum_{v=2}^{n-1} \frac{1}{v+1} \leq \log n. \end{aligned}$$

to derive that

$$\mathbb{E} \left| \sum_{\ell=2}^{n-1} \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \theta_{\ell-1} \sum_{v=\ell}^{n-1} \frac{\theta_v}{\sqrt{v(v+1)}} \right|^2 \ll n^{-2} \log n. \quad (4.16)$$

To end the proof of the theorem, it remains to prove the first part of (4.11). Recall that

$$\sum_{k=1}^n \beta_{k,3}(\theta^*) = \mathbb{E}(X_0^3) \sum_{k=1}^n \theta_k^{*3} + 3 \sum_{k=1}^n \theta_k^{*2} \sum_{\ell=1}^{k-1} \theta_\ell^* a(k-\ell).$$

Note that

$$\left| \sum_{k=1}^n \theta_k^{*3} \right| \ll \left| \sum_{k=1}^n (\tilde{\theta}_k)^3 \right| + \left| \sum_{k=1}^n \left(\frac{k-1}{k} \right)^{3/2} \theta_{k-1}^3 \right| + \left| \sum_{k=1}^n (\tilde{\theta}_k)^2 \left(\frac{k-1}{k} \right)^{1/2} \theta_{k-1} \right| + \left| \sum_{k=1}^n \tilde{\theta}_k \frac{k-1}{k} \theta_{k-1}^2 \right|.$$

By the first part of (4.13),

$$\mathbb{E} \left| \sum_{k=1}^n (\tilde{\theta}_k)^3 \right| \leq \sum_{k=1}^n \mathbb{E}(|(\tilde{\theta}_k)^3|) \ll n^{-3/2} \sum_{k=1}^n k^{-3/2} \ll n^{-3/2}$$

and

$$\mathbb{E} \left| \sum_{k=1}^n (\tilde{\theta}_k)^2 \left(\frac{k-1}{k} \right)^{1/2} \theta_{k-1} \right| \leq \sum_{k=1}^n \mathbb{E}^{1/2}(\tilde{\theta}_k^4) \mathbb{E}^{1/2}(\theta_{k-1}^2) \ll n^{-3/2} \sum_{k=1}^n k^{-1} \ll n^{-3/2} \log n.$$

Next, using the same arguments used for proving (4.13), we infer that

$$\mathbb{E} \left| \sum_{k=1}^n \left(\frac{k-1}{k} \right)^{3/2} \theta_{k-1}^3 \right| \ll n^{-1} + n^{-3/2} \sqrt{\mathbb{E} \left| \sum_{k=1}^n \left(\frac{k-1}{k} \right)^{3/2} \xi_{k-1}^3 \right|^2} \ll n^{-1}.$$

On another hand, using again the same arguments used to prove (4.13), we get

$$\begin{aligned} \mathbb{E} \left(\sum_{k=1}^n \tilde{\theta}_k \frac{k-1}{k} \theta_{k-1}^2 \right)^2 &= \mathbb{E} \left(\sum_{\ell=1}^{n-1} \frac{\theta_\ell}{\sqrt{\ell(\ell+1)}} \sum_{k=1}^{\ell} \frac{k-1}{k} \theta_{k-1}^2 \right)^2 \\ &\ll n^{-3} + n^{-3} \mathbb{E} \left(\sum_{\ell=1}^{n-1} \frac{\xi_\ell}{\sqrt{\ell(\ell+1)}} \sum_{k=1}^{\ell} \frac{k-1}{k} \xi_{k-1}^2 \right)^2 \\ &\ll n^{-3} + n^{-3} \sum_{\ell=1}^n \frac{1}{\ell(\ell+1)} \text{Var} \left(\sum_{k=1}^{\ell} \frac{k-1}{k} \xi_{k-1}^2 \right)^2 \ll n^{-3} (\log n). \end{aligned}$$

So, overall,

$$\mathbb{E} \left| \sum_{k=1}^n \theta_k^{*3} \right| \ll n^{-1}. \quad (4.17)$$

We now give an upper bound for $\mathbb{E} \left| \sum_{k=1}^n \theta_k^{*2} \sum_{\ell=1}^{k-1} \theta_\ell^* a(k-\ell) \right|$. By using (4.9), (4.14) and the fact that $\sum_{v \geq 1} \gamma(v) < \infty$, we first notice that

$$\mathbb{E} \left| \sum_{k=1}^n (\tilde{\theta}_k)^2 \sum_{\ell=1}^{k-1} \tilde{\theta}_\ell a(k-\ell) \right| \leq \sum_{k=1}^n \mathbb{E}^{1/2}(\tilde{\theta}_k^4) \mathbb{E}^{1/2} \left(\sum_{\ell=1}^{k-1} \tilde{\theta}_\ell a(k-\ell) \right)^2 \ll n^{-3/2} \log n. \quad (4.18)$$

On another hand, proceeding as to prove (3.26) and since $\sum_{v \geq 1} \gamma(v) < \infty$, we get that

$$\mathbb{E} \left| \sum_{k=1}^n \frac{k-1}{k} \theta_{k-1}^2 \sum_{\ell=1}^{k-1} \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \theta_{\ell-1} a(k-\ell) \right| \ll n^{-1}. \quad (4.19)$$

Next

$$\mathbb{E} \left| \sum_{k=1}^n (\tilde{\theta}_k)^2 \sum_{\ell=1}^{k-1} \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \theta_{\ell-1} a(k-\ell) \right| \leq \sum_{k=1}^n \mathbb{E}^{1/2}(\tilde{\theta}_k^4) \mathbb{E}^{1/2} \left(\sum_{\ell=1}^{k-1} \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \theta_{\ell-1} a(k-\ell) \right)^2.$$

Using (4.9) and proceeding as in the proof of (4.15), we derive that

$$\mathbb{E} \left| \sum_{k=1}^n (\tilde{\theta}_k)^2 \sum_{\ell=1}^{k-1} \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \theta_{\ell-1} a(k-\ell) \right| \ll n^{-3/2} \log n. \quad (4.20)$$

Next

$$\begin{aligned} & \left| \sum_{k=1}^n \frac{k-1}{k} \theta_{k-1}^2 \sum_{\ell=1}^{k-1} \tilde{\theta}_\ell a(k-\ell) \right| = \left| \sum_{k=1}^n \frac{k-1}{k} \theta_{k-1}^2 \sum_{v=1}^{n-1} \frac{\theta_v}{\sqrt{v(v+1)}} \sum_{\ell=1}^{(k-1) \wedge v} a(k-\ell) \right| \\ & \leq \left| \sum_{k=1}^n \frac{k-1}{k} \theta_{k-1}^2 \sum_{v=1}^{k-1} \frac{\theta_v}{\sqrt{v(v+1)}} \sum_{\ell=1}^v a(k-\ell) \right| + \left| \sum_{k=1}^{n-1} \frac{k-1}{k} \theta_{k-1}^2 \sum_{v=k}^{n-1} \frac{\theta_v}{\sqrt{v(v+1)}} \sum_{\ell=1}^{k-1} a(k-\ell) \right|. \end{aligned}$$

Using the same arguments used to prove (4.13), we infer that

$$\begin{aligned} & \mathbb{E} \left| \sum_{k=1}^n \frac{k-1}{k} \theta_{k-1}^2 \sum_{v=[k/2]+1}^{k-1} \frac{\theta_v}{\sqrt{v(v+1)}} \sum_{\ell=1}^v a(k-\ell) \right| \\ & \ll n^{-1} + n^{-3/2} \mathbb{E} \left| \sum_{k=1}^n \frac{k-1}{k} \xi_{k-1}^2 \sum_{v=[k/2]+1}^{k-1} \frac{\xi_v}{\sqrt{v(v+1)}} \sum_{\ell=1}^v a(k-\ell) \right| \\ & \ll n^{-1} + n^{-3/2} \sum_{k=1}^n \mathbb{E}^{1/2} \left(\sum_{v=[k/2]+1}^{k-1} \frac{\xi_v}{\sqrt{v(v+1)}} \sum_{\ell=k-v}^{k-1} a(u) \right)^2 \\ & \ll n^{-1} + n^{-3/2} \sum_{k=1}^n k^{-1} \sqrt{\sum_{v=[k/2]+1}^{k-1} \left(\sum_{\ell=k-v}^{k-1} a(u) \right)^2} \ll n^{-1} + n^{-3/2} \sum_{k=1}^n k^{-1/2} \ll n^{-1}. \end{aligned}$$

In addition

$$\begin{aligned} & \mathbb{E} \left| \sum_{k=1}^n \frac{k-1}{k} \theta_{k-1}^2 \sum_{v=1}^{[k/2]} \frac{\theta_v}{\sqrt{v(v+1)}} \sum_{\ell=1}^v a(k-\ell) \right| \\ & \ll n^{-3/2} + n^{-3/2} \mathbb{E} \left| \sum_{k=1}^n \frac{k-1}{k} \xi_{k-1}^2 \sum_{v=1}^{[k/2]} \frac{\xi_v}{\sqrt{v(v+1)}} \sum_{\ell=1}^v a(k-\ell) \right| \\ & \ll n^{-3/2} + n^{-3/2} \sum_{k=1}^n k \gamma([k/2]) \ll n^{-3/2}. \end{aligned}$$

On another hand

$$\mathbb{E} \left| \sum_{k=1}^{n-1} \frac{k-1}{k} \theta_{k-1}^2 \sum_{v=k}^{n-1} \frac{\theta_v}{\sqrt{v(v+1)}} \sum_{\ell=1}^{k-1} a(k-\ell) \right| = \mathbb{E} \left| \sum_{v=1}^{n-1} \frac{\theta_v}{\sqrt{v(v+1)}} \sum_{k=1}^v \sum_{\ell=1}^{k-1} a(k-\ell) \frac{k-1}{k} \theta_{k-1}^2 \right|.$$

By the same arguments as to prove (4.13), note that

$$\begin{aligned} & \mathbb{E} \left| \sum_{v=1}^{n-1} \frac{\theta_v}{\sqrt{v(v+1)}} \sum_{k=1}^v \sum_{\ell=1}^{k-1} a(k-\ell) \frac{k-1}{k} \theta_{k-1}^2 \right| \\ & \ll n^{-1} + n^{-3/2} \mathbb{E} \left| \sum_{v=1}^{n-1} \frac{\xi_v}{\sqrt{v(v+1)}} \sum_{k=1}^v \sum_{\ell=1}^{k-1} a(k-\ell) \frac{k-1}{k} \xi_{k-1}^2 \right|. \end{aligned}$$

In addition

$$\begin{aligned}
& n^{-3/2} \mathbb{E}^{1/2} \left| \sum_{v=1}^{n-1} \frac{\xi_v}{\sqrt{v(v+1)}} \sum_{k=1}^v \sum_{\ell=1}^{k-1} a(k-\ell) \frac{k-1}{k} \xi_{k-1}^2 \right|^2 \\
& \leq n^{-3/2} \left(\sum_{v=1}^{n-1} \frac{1}{v(v+1)} \mathbb{E} \left(\sum_{k=1}^v \sum_{\ell=1}^{k-1} a(k-\ell) \frac{k-1}{k} \xi_{k-1}^2 \right)^2 \right)^{1/2} \\
& \ll n^{-3/2} \left(\sum_{v=1}^{n-1} \left(\sum_{u=1}^{v-1} \gamma(u) \right)^2 \right)^{1/2} \ll n^{-1}.
\end{aligned}$$

So, overall,

$$\mathbb{E} \left| \sum_{k=1}^n \frac{k-1}{k} \theta_{k-1}^2 \sum_{\ell=1}^{k-1} \tilde{\theta}_\ell a(k-\ell) \right| \ll n^{-1}. \quad (4.21)$$

With similar but easier arguments, one can prove that

$$\mathbb{E} \left| \sum_{k=1}^n \frac{\sqrt{k-1}}{\sqrt{k}} \theta_{k-1} \tilde{\theta}_k \sum_{\ell=1}^{k-1} \tilde{\theta}_\ell a(k-\ell) \right| \ll n^{-1}, \quad (4.22)$$

and

$$\mathbb{E} \left| \sum_{k=1}^n \frac{\sqrt{k-1}}{\sqrt{k}} \theta_{k-1} \tilde{\theta}_k \sum_{\ell=1}^{k-1} \frac{\sqrt{\ell-1}}{\sqrt{\ell}} \theta_{\ell-1} a(k-\ell) \right| \ll n^{-1}. \quad (4.23)$$

Hence considering the upper bounds (4.18)-(4.23), we get that

$$\mathbb{E} \left| \sum_{k=1}^n \theta_k^{*2} \sum_{\ell=1}^{k-1} \theta_\ell^* a(k-\ell) \right| \ll n^{-1}. \quad (4.24)$$

The first part of (4.11) then follows by considering the upper bounds (4.17) and (4.24). This ends the proof of the theorem. \square

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References

- [1] Bobkov, S. G., Chistyakov, G. P. and Götze, F. Berry-Esseen bounds for typical weighted sums. Electron. J. Probab. 23 (2018), Paper No. 92, 22 pp.
- [2] Bobkov, S. G., Chistyakov, G. P. and Götze, F. Normal approximation for weighted sums under a second-order correlation condition. Ann. Probab. 48 (2020), no. 3, 1202–1219.

- [3] Bobkov, S. G., Chistyakov, G. P. and Götze, F. Concentration and Gaussian Approximation for Randomized Sums. *Probability Theory and Stochastic Modelling*, 104. Springer, Cham, 2023.
- [4] Bolthausen, E. Exact Convergence Rates in Some Martingale Central Limit Theorems. *Ann. Probab.* 10 (1982), 3 672–688.
- [5] Comte, F., Dedecker, J. and Taupin, M. L. Adaptive density estimation for general ARCH models. *Econometric Theory* 24 (2008), no. 6, 1628–1662.
- [6] Davydov, Yu. A. Mixing conditions for Markov chains. *Theor. Probab. Appl.* 18 (1973), 312–328.
- [7] Dedecker, J., Merlevède, F. and Rio, E. Rates of convergence for minimal distances in the central limit theorem under projective criteria. *Electron. J. Probab.* 14 (2009), 35, 978–1011.
- [8] Dedecker, J. and Rio, E. On mean central limit theorems for stationary sequences. *Ann. Inst. Henri Poincaré Probab. Stat.* 44 (2008), 4, 693–726.
- [9] Feller, W. An introduction to probability theory and its applications. Vol. II. Second edition John Wiley & Sons, Inc., New York-London-Sydney 1971 xxiv+669 pp.
- [10] Giraitis, L., Kokoszka, P. and Leipus, R. Stationary ARCH models: dependence structure and central limit theorem. *Econometric Theory*. 16 (2000) 3–22.
- [11] Grams, W. F. Rates of Convergence in the Central Limit Theorem for Dependent Variables, Thesis, Florida State University, 1972.
- [12] Hannan, E. J. Central limit theorems for time series regression. *Stochastic Process. Appl.* 9 (1979), no. 3, 281–289.
- [13] Klartag, B. and Sodin, S. Variations on the Berry-Esseen theorem. *Theory Probab. Appl.* 56 (2012), 3, 403–419.
- [14] Merlevède, F., Peligrad, M. and Utev, S. Functional Gaussian approximation for dependent structures. *Oxford Studies in Probability*, 6. Oxford University Press, Oxford, 2019. xv+478 pp.
- [15] Röllin, A. On quantitative bounds in the mean martingale central limit theorem. *Statist. Probab. Lett.* 138 (2018) 171–176.
- [16] Rosenblatt, M., A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci. U. S. A.* 42 (1956) 43–47.