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# On a general divisor problem related to a certain Dedekind zeta-function over a specific sequence of positive integers

Anubhav Sharma and Ayyadurai Sankaranarayanan

**Abstract.** We investigate the average behavior of coefficients of the Dirichlet series of positive integral power of the Dedekind zeta-function  $\zeta_{\mathbb{K}_3}(s)$  of a non-normal cubic extension  $\mathbb{K}_3$  of  $\mathbb{Q}$  over a certain sequence of positive integers. More precisely, we prove an asymptotic formula with an error term for the sum

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \leq x \\ (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{Z}^6}} a_{k, \mathbb{K}_3}(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2),$$

where  $(\zeta_{\mathbb{K}_3}(s))^k := \sum_{n=1}^{\infty} \frac{a_{k, \mathbb{K}_3}(n)}{n^s}$ .

**Keywords.** Dedekind zeta-function, Hecke eigenvalues, non-normal cubic extension.

**2010 Mathematics Subject Classification.** 11F30, 11R42.

## 1. Introduction

Let  $\mathbb{K}$  be an algebraic extension of degree  $m$  of rational field  $\mathbb{Q}$ . Define

$$\zeta_{\mathbb{K}}(s) := \sum_{\alpha} \frac{1}{N(\alpha)^s},$$

for  $\Re(s) > 1$ , where the summation is running over all the integral ideals  $\alpha$  of  $\mathbb{K}$  and norm of integral ideal  $\alpha$  is denoted by  $N(\alpha)$ . The function  $\zeta_{\mathbb{K}}(s)$  can also be written as

$$\zeta_{\mathbb{K}}(s) = \sum_{n=1}^{\infty} \frac{a_{\mathbb{K}}(n)}{n^s},$$

where  $a_{\mathbb{K}}(n)$  denotes the number of integral ideals of  $\mathbb{K}$  with norm  $n$ . It is shown (by Chandrasekharan and Good [ChGo83]) that these coefficients are multiplicative and satisfies the upper bound

$$a_{\mathbb{K}}(n) \leq d(n)^m,$$

where  $m$  is the degree of extension, i.e.,  $m = [\mathbb{K} : \mathbb{Q}]$  and  $d(n)$  is the number of divisors of  $n$ .

In 1949, Landau [Lan49] showed that

$$\sum_{n \leq x} a_{\mathbb{K}}(n) = cx + O\left(x^{1-\frac{2}{m+1}+\epsilon}\right),$$

where  $c$  is the residue of  $\zeta_{\mathbb{K}}(s)$  at its simple pole at  $s = 1$ , which is further improved to

$$\sum_{n \leq x} a_{\mathbb{K}}(n) = cx + O\left(x^{\frac{23}{73}} \log^{\frac{315}{146}} x\right),$$

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for quadratic field by Huxley and Watt [HuWa00]. Some further improvement is also available for cubic fields by Müller [Mül88]. In 1993, W.G. Nowak [Now93] established that

$$\sum_{n \leq x} a_{\mathbb{K}}(n) = cx + \begin{cases} O\left(x^{1-\frac{2}{m}+\frac{8}{m(5m+2)}} \log^{\frac{10}{5m+2}} x\right) & \text{for } 3 \leq m \leq 6, \\ O\left(x^{1-\frac{2}{m}+\frac{3}{2m^2}} \log^{\frac{2}{m}} x\right) & \text{for } m \geq 7. \end{cases}$$

We also have some significant results (by Chandrasekharan and Narasimhan [ChNa63] and by Chandrasekharan and Good [ChGo83]) of  $\sum_{n \leq x} a_{\mathbb{K}}(n)^k$  for some higher powers  $k$ , if  $\mathbb{K}$  is the Galois extension of  $\mathbb{Q}$ .

If  $h$  is the class number of  $\mathbb{K}$  and  $[\mathbb{K} : \mathbb{Q}] = r_1 + 2r_2$ , where  $r_1$  is the number of real conjugate fields and  $2r_2$  is the number of complex conjugate fields, then we can write

$$\sum_{n \leq x} a_{\mathbb{K}}(n) = h\lambda x + E(x),$$

where

$$\lambda := \frac{2^{r_1+r_2} \pi^{r_2} R}{w|\Delta|^{\frac{1}{2}}}.$$

Here,  $w$  is the number of roots of unity in  $\mathbb{K}$ ;  $R$  is the regulator of  $\mathbb{K}$  and  $\Delta$  is the discriminant of  $\mathbb{K}$ .

When  $[\mathbb{K} : \mathbb{Q}] = m \geq 10$ , B. Paul and A. Sankaranarayanan proved that

$$E(x) \ll x^{1-\frac{3}{m+6}+\epsilon},$$

where implied constants depend only on  $\mathbb{K}$  and  $\epsilon$  (see [PaSa20]).

Also, if  $\mathbb{K} = \mathbb{Q}(\zeta_l)$ , where  $l$  is some positive integer and  $[\mathbb{K} : \mathbb{Q}] = m \geq 8$ , then,

$$E(x) \ll x^{1-\frac{3}{m+5}+\epsilon},$$

where the implied constants depend only on  $\mathbb{K}$  and  $\epsilon$  (see [PaSa20]).

It is of great interest to study the  $L$ -functions related to primitive holomorphic cusp forms. For many years, it has been a profound area in which many authors have contributed.

Let  $L(s, f)$  be the  $L$ -function connected with the primitive holomorphic cusp form  $f$  of weight  $w$  for the full modular group  $SL(2, \mathbb{Z})$  and  $\lambda_f(n)$  are the normalized  $n^{\text{th}}$  Fourier coefficients of Fourier expansion of  $f(z)$  at the cusp  $\infty$ , i.e.,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{w-1}{2}} e^{2\pi i n z}$$

where  $\Im(z) > 0$ , then the  $L$ -function attached to  $\lambda_f(n)$  is defined as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s},$$

for  $\Re(s) > 1$ , where  $\lambda_f(n)$  are Hecke eigenvalues of Hecke operators  $T_n$ .

Also,

$$L^k(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_{k,f}(n)}{n^s},$$

where

$$\lambda_{k,f}(n) = \sum_{n=n_1 n_2 \dots n_k} \lambda_f(n_1) \lambda_f(n_2) \dots \lambda_f(n_k).$$

In 2012, Kanemitsu, Sankaranarayanan and Tanigawa [KST02] proved that for  $k \geq 2$ ,

$$\sum_{n \leq x} \lambda_{k,f}(n) \ll x^{1 - \frac{3}{2k+2} + \epsilon},$$

where implied constant depends only on  $f$  and  $\epsilon$ , which is further improved by Lü in [Lü12].

For such divisor problems connected to holomorphic cusp forms, see the work of H.F. Liu [Liu18], [LiuZha19] and Lü [Lü12]. Recently, several authors considered the average behavior of  $\lambda_{\text{sym}^j f}(n)$  over certain sequences of positive integers and established some interesting asymptotic formulas (see, for instance [ShSa22a, ShSa22b, ShSa22c, Hua22]).

For  $k \geq 2$ , let  $\Delta_k(x)$  denotes the error term in the asymptotic formula for  $\sum_{n \leq x} d_k(n)$ , where  $\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n) n^{-s}$ . The estimation of  $\Delta_k(x)$  is popularly known as the general Dirichlet divisor problem. From elementary arguments, one can get  $\Delta_k(x) \ll x^{\frac{k-1}{k}} \log^{k-2} x$ . For  $k = 2$ , we have  $\Delta_2(x) \ll x^{\frac{35}{108} + \epsilon}$ , see [Ivi12]. For  $k = 3$ ,  $\Delta_3(x) \ll x^{\frac{43}{96} + \epsilon}$  is the best result available, due to G. Kolesnik [Kol79]. We may define the order  $\alpha_k$  of  $\Delta_k(x)$  as the least number such that  $\Delta_k(x) \ll x^{\alpha_k + \epsilon}$  for every  $\epsilon > 0$ . The following results are known (see [Titch86]):

$$\alpha_k \leq \frac{k-1}{k} \quad \text{for } k = 2, 3, \dots$$

and

$$\alpha_k \leq \frac{k-1}{k+1} \quad \text{for } k = 2, 3, \dots$$

But the exact value of  $\alpha_k$  has not been determined for any value of  $k$ . For an extensive literature and detailed discussion of general divisor problem, see [Titch86, Chapter 12].

Estimating the average behavior of some special functions over polynomial values has been of interest since the early 1950s. In 1952, Erdős [Erd52] proved that

$$c_1 x \log x < \sum_{k=1}^x d(f(k)) < c_2 x \log x$$

and

$$\sum_{p \leq x} d(f(p)) \ll x,$$

where  $f(x) \in \mathbb{Z}[x]$  and  $c_1, c_2$  are some positive constants. For a quadratic polynomial  $f(x)$ , McKee [Mck95, McK99] proved that

$$\sum_{k=1}^x d(f(k)) \sim \lambda(f) x \log x,$$

where  $\lambda(f)$  can be written in terms of Hurwitz class numbers. No similar results have been established for polynomials that have higher degrees. Then, Titchmarsh considered the linear polynomials  $f(x) = a + x$  ( $a \neq 0$ ) and observed the average behavior of divisor function over shifted primes. More precisely, he proved that

$$\sum_{p \leq x} d(p+a) \sim C(a)x,$$

where  $C(a)$  is some constant depending upon  $a$ . Motivated by the previous result of Titchmarsh, many authors studied the problem of finding an optimal error term of  $\sum_{p \leq x} d(n_p)$ , where the  $n_p$ 's are quantities of arithmetic significance, for instance see [Pol16, AkDr12, Chi22].

One topic that has drawn a lot of attention involves figuring out the average order of  $d_k(n)$  over sparse sequences of values taken by polynomials, i.e.,

$$D_k(f(x_1, x_2, \dots, x_n), x) := \sum_{|f(x_1, x_2, \dots, x_n)| \leq x} d_k(|f(x_1, x_2, \dots, x_n)|).$$

For  $f(x_1, x_2)$  a binary irreducible cubic form, Greaves [Gre70] proved that there exist real constants  $c_1 > 0$  and  $c_2$  depending only on  $f(x_1, x_2)$ , such that

$$D_2(f(x_1, x_2, \dots, x_n), x) = c_1 x^{\frac{2}{3}} \log x + c_2 x^{\frac{2}{3}} + O\left(x^{\frac{9}{14} + \epsilon}\right),$$

for any  $\epsilon > 0$  as  $x \rightarrow \infty$ . If  $f(x_1, x_2)$  is an irreducible quartic form, Daniel [Dan99] proved that

$$D_2(f(x_1, x_2, \dots, x_n), x) = c_1 x^{\frac{1}{2}} \log x + O\left(x^{\frac{1}{2}} \log \log x\right),$$

where  $c_1 > 0$  is a constant depending only on  $f$ . More related work can be found in [LaZh21].

In 1998, Friedlander and Iwaniec [FrIw98] established the asymptotic formula for the distribution of prime values of  $a^4 + b^2$ . More precisely, they proved

$$\sum_{\substack{a=1 \\ a^2+b^4 \leq x}}^{\infty} \sum_{b=1}^{\infty} \Lambda(a^2 + b^4) = \frac{4}{\pi} \kappa x^{\frac{3}{4}} \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right),$$

where  $\kappa$  is some constant. Motivated from the above result, they [FrIw06] replaced  $\Lambda$  by  $d_k$  and established the following asymptotic formula for  $d_3(a^6 + b^2)$ :

$$\sum_{\substack{a=1 \\ a^2+b^4 \leq x \\ (a,b)=1}}^{\infty} \sum_{b=1}^{\infty} d_3(a^6 + b^2) = c \kappa x^{\frac{2}{3}} (\log x)^2 + O\left(x^{\frac{2}{3}} (\log x)^{\frac{7}{4}} (\log \log x)^{\frac{1}{2}}\right),$$

where  $c$  and  $\kappa$  are some constants. For irreducible binary definite quadratic forms  $f$ , Daniel [Dan97] proved an asymptotic formula for  $D_k(f(x_1, x_2, \dots, x_n), x)$  for any  $k \geq 2$ . With the help of circle method, Sun and Zhang [SuZh16] proved that

$$\sum_{1 \leq a_1, a_2, a_3 \leq x} d_3(a_1^2 + a_2^2 + a_3^2) = c_1 x^3 (\log x)^2 + c_2 x^3 \log x + c_3 x^3 + O\left(x^{\frac{11}{4} + \epsilon}\right),$$

where  $c_1, c_2, c_3$  are some constants and  $\epsilon$  is any positive number. Finally, Blomer [Blo18] proved an asymptotic formula for  $D_k(f(x_1, x_2, \dots, x_n), x)$  for any  $k \geq 2$  where  $f(x_1, x_2, \dots, x_n)$  is a form of degree  $k$  in  $n = k - 1$  variables, coming from incomplete norm form. Very recently, Lapkova and Zhou [LaZh21] investigated the average sum of the  $k^{\text{th}}$  divisor function over values of quadratic polynomials  $f$ , not necessarily homogenous, in  $n \geq 3$  variables for any  $k \geq 2$ .

Let  $\mathbb{K}_3$  be a non-normal cubic extension of a rational field  $\mathbb{Q}$  given by an irreducible polynomial  $f(x) = x^3 + ax^2 + bx + c$  of discriminant  $D(< 0)$ . It is natural to study the  $k^{\text{th}}$  integral power of Dedekind zeta function, i.e.,

$$(\zeta_{\mathbb{K}_3}(s))^k = \sum_{n=1}^{\infty} \frac{a_{k, \mathbb{K}_3}(n)}{n^s}, \quad (1.1)$$

for  $\Re(s) > 1$ , where  $a_{k,\mathbb{K}_3}(n) = \sum_{n=n_1 n_2 \dots n_k} a_{\mathbb{K}_3}(n_1) a_{\mathbb{K}_3}(n_2) \dots a_{\mathbb{K}_3}(n_k)$ .

In 2012, Lü [Lü13] was able to refine the previously known results (by Fomenko [Fom08]) of mean square and third power of  $a_{\mathbb{K}_3}(n)$  to

$$\sum_{n \leq x} a_{\mathbb{K}_3}(n)^2 = a_1 x \log x + a_2 x + O\left(x^{\frac{23}{31} + \epsilon}\right)$$

where  $a_1$  and  $a_2$  are constants and

$$\sum_{n \leq x} a_{\mathbb{K}_3}(n)^3 = x P_3(\log x) + O\left(x^{\frac{235}{259} + \epsilon}\right),$$

where  $P_3(t)$  is a suitable polynomial in  $t$  of degree 4.

In this paper, we will consider the average of the Dirichlet coefficients  $a_{k,\mathbb{K}_3}(n)$  of the  $k^{\text{th}}$  power  $(\zeta_{\mathbb{K}_3}(s))^k$  of the Dedekind zeta-function of a non-normal cubic extension  $\mathbb{K}_3$  of  $\mathbb{Q}$  over the sequence of values of a binary quadratic form  $F(x_1, x_2, \dots, x_6) = \sum_{k=1}^6 x_k^2$ . More precisely, we are interested in the asymptotic formula for the sum

$$\sum_{\substack{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 \leq x \\ (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{Z}^6}} a_{k,\mathbb{K}_3}(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2),$$

for any integer  $k \geq 1$ , where

$$a_{k,\mathbb{K}_3}(n) = \sum_{n=n_1 n_2 \dots n_k} a_{\mathbb{K}_3}(n_1) a_{\mathbb{K}_3}(n_2) \dots a_{\mathbb{K}_3}(n_k).$$

Note that,  $a_{1,\mathbb{K}_3}(n) = a_{\mathbb{K}_3}(n)$ .

First, we make the following remark.

**Remark 1.** Let  $|t| \geq 1$  and  $\epsilon > 0$  be any small constant. Then we have

$$\zeta\left(\frac{1}{2} + it\right) \ll (|t| + 1)^{\mu + \epsilon},$$

where  $\mu = \mu(\frac{1}{2})$ . Moreover, Phragmén Lindelöf principle leads to

$$\zeta(\sigma + it) \ll (|t| + 1)^{2\mu(1-\sigma) + \epsilon}$$

uniformly for  $\frac{1}{2} \leq \sigma \leq 1$  and  $|t| \geq 1$ .

For any integer  $k \geq 1$ , writing,

$$\begin{aligned} \sum_{\substack{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 \leq x \\ (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{Z}^6}} a_{k,\mathbb{K}_3}(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2) &= \sum_{n \leq x} a_{k,\mathbb{K}_3}(n) \sum_{\substack{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 \leq x \\ (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{Z}^6}} 1 \\ &= \sum_{n \leq x} a_{k,\mathbb{K}_3}(n) r_6(n) \\ &= M_{k,\mathbb{K}_3}(x) + \tilde{E}_{k,\mathbb{K}_3}(x), \end{aligned} \tag{1.2}$$

where  $M_{k,\mathbb{K}_3}(x)$  is the main term which is of the form  $x^3 P_{k-1}(\log x)$ , where  $P_{k-1}(t)$  is a polynomial in  $t$  of degree  $k - 1$ . We prove the following theorem.

**Theorem 1.** *Let  $\epsilon > 0$  (be any small constant) and define (for  $k \geq 1$ )  $\tilde{\lambda}_k = \max(\lambda_k, \lambda'_k)$ , where  $\lambda_k, \lambda'_k$  are defined in Theorems 2, 3 resp. Then we have for any integer  $k \geq 1$ ,*

$$\tilde{E}_{k, \mathbb{K}_3}(x) \ll x^{3 - \frac{1}{2(1+\tilde{\lambda}_k)} + 3k\epsilon}.$$

To prove Theorem 1, first, we demonstrate the following theorems:

**Theorem 2.** *Let  $\epsilon > 0$  (be any small constant) and define  $\lambda_1 = 3\epsilon$ ,  $\lambda_2 = \min(2\mu, \frac{1}{4})$ ,  $\lambda_3 = \min(\mu + \frac{1}{2}, \frac{5}{8})$ ,  $\lambda_4 = \min(2\mu + \frac{3}{4}, 1)$ ,  $\lambda_5 = \min(3\mu + 1, \frac{3}{2})$  and  $\lambda_k = \mu(k - 6) + \frac{k}{3}$  for  $k \geq 6$ . Then we have for any integer  $k \geq 1$ ,*

$$E_{k, \mathbb{K}_3}(x) \ll x^{3 - \frac{1}{2(1+\lambda_k)} + 3k\epsilon},$$

where

$$\sum_{n \leq x} a_{k, \mathbb{K}_3}(n) l_1(n) = M_{k, \mathbb{K}_3}(x) + E_{k, \mathbb{K}_3}(x),$$

and  $l_1(n)$  is defined in Section 2.

**Remark 2.** From [Bou17] of Bourgain, we can very well take  $\mu = \frac{13}{84}$ . Thus, the theorem is unconditional with  $\mu = \frac{13}{84}$ .

**Theorem 3.** *Let  $\epsilon > 0$  (be any small constant) and define  $\lambda'_1 = 3\epsilon$ ,  $\lambda'_2 = \frac{1}{4}$ , and  $\lambda'_k = \frac{3k-5}{6}$  for  $k \geq 3$ . Then we have for any integer  $k \geq 1$ ,*

$$E'_{k, \mathbb{K}_3}(x) \ll x^{3 - \frac{1}{2(1+\lambda'_k)} + 3k\epsilon},$$

where

$$\sum_{n \leq x} a_{k, \mathbb{K}_3}(n) v_1(n) = E'_{k, \mathbb{K}_3}(x),$$

and  $v_1(n)$  is defined in Section 2.

From (1.2), one can easily see that the proof of Theorem 1 follows from the proof of Theorems 2 and 3.

## 2. Preliminaries and some important lemmas

Let  $r_k(n) := \#\{(n_1, n_2, \dots, n_k) \in \mathbb{Z}^k : n_1^2 + n_2^2 + \dots + n_k^2 = n\}$  allowing zeros, distinguishing signs, and order. We will be concerned with the function  $r_6(n)$ .

**Lemma 1.** *For any positive integer  $n$ , we have*

$$r_6(n) = 16 \sum_{d|n} \chi(d') d^2 - 4 \sum_{d|n} \chi(d) d^2, \quad (2.3)$$

where  $dd' = n$ , and  $\chi$  is the non-principal Dirichlet character modulo 4, i.e.,

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{4} \\ -1 & \text{if } n \equiv -1 \pmod{4} \\ 0 & \text{if } n \equiv 0 \pmod{2} \end{cases}.$$

*Proof.* See, for instance, Lemma 1 of [ShSa22c].

We can reframe the equation (2.3) as

$$\begin{aligned} r_6(n) &= 16 \sum_{d|n} \chi(d) \frac{n^2}{d^2} - 4 \sum_{d|n} \chi(d) d^2 \\ &=: 16l(n) - 4v(n). \end{aligned}$$

We write  $l_1(n) = 16l(n)$ , and  $v_1(n) = 4v(n)$ .

The functions  $\chi(d)$  and  $\frac{n^2}{d^2}$  are completely multiplicative functions. This implies that  $\chi(d) \frac{n^2}{d^2}$  is multiplicative. If  $g(d)$  is any multiplicative function, then  $\sum_{d|n} g(d)$  is also multiplicative. Therefore,  $l(n)$  is a multiplicative function. Similarly,  $v(n)$  is also multiplicative.

Note that

$$\begin{aligned} l(p) &= p^2 + \chi(p), \\ l(p^2) &= p^4 + p^2\chi(p) + \chi(p^2), \end{aligned}$$

and

$$\begin{aligned} v(p) &= 1 + p^2\chi(p), \\ v(p^2) &= 1 + p^2\chi(p) + p^4\chi(p^2). \end{aligned}$$

**Lemma 2.** ([Lü13]) For  $\Re(s) > 1$ , we have

$$\zeta_{\mathbb{K}_3}(s) = \zeta(s)L(s, f),$$

where  $f$  is a holomorphic cusp form of weight 1 with respect to the congruence group  $\Gamma_0(|D|)$  and  $D(< 0)$  be the discriminant of  $f(x) = x^3 + ax^2 + bx + c$ .

From Lemma 2, we can write

$$a_{\mathbb{K}_3}(n) = \sum_{d|n} \lambda_f(d).$$

Also, note that

$$a_{\mathbb{K}_3}(p) = 1 + \lambda_f(p).$$

**Lemma 3.** For any  $\epsilon > 0$ , we have

$$\int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \sim \frac{T(\log T)^4}{2\pi} \quad (2.4)$$

and

$$\int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{12} dt \ll T^{2+\epsilon} \quad (2.5)$$

uniformly for  $T \geq 1$ .

*Proof.* For the proof of (2.4) see (Theorem 5.1 of [Ivi12]), and (2.5) result is due to Heath-Brown [Hea78].



**Lemma 4.** ([Go82]) For any  $\epsilon > 0$ , we have

$$\int_1^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 dt \ll T \log T,$$

uniformly for  $T \geq 1$  and

$$L(\sigma + it, \chi) \ll_\epsilon (1 + |t|)^{\frac{1}{3}(1-\sigma)+\epsilon},$$

uniformly for  $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$ , and  $|t| \geq t_0$  (where  $t_0$  is sufficiently large).

**Lemma 5.** ([Rama74]) For any  $\epsilon > 0$ , we have

$$\int_1^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \ll T^{1+\epsilon},$$

uniformly for  $T \geq 1$ .

**Lemma 6.** For any  $\epsilon > 0$  and for any  $T \geq 1$  uniformly, we have

$$\int_1^T \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \sim cT \log T \quad (2.6)$$

and

$$\int_1^T \left| L\left(\frac{1}{2} + it, f\right) \right|^6 dt \ll T^{2+\epsilon}. \quad (2.7)$$

*Proof.* Proofs of (2.6) and (2.7) follow by A. Good [Go82] and Jutila [Jut87], respectively.

**Lemma 7.** For any  $\epsilon > 0$ , we have

$$L(\sigma + it, f) \ll (1 + |t|)^{\max(\frac{2(1-\sigma)}{3}, 0) + \epsilon}$$

uniformly for  $\frac{1}{2} \leq \sigma \leq 2$ ,  $|t| \geq 1$ .

*Proof.* Proof follows from a result of A. Good [Go82] on using maximum-modulus principle to a suitable function.

**Lemma 8.** Let  $f$  be defined as in Lemma 2 and  $a_{k, \mathbb{K}_3}(n)$  be defined as in equation (1.1). If

$$F_k(s) = \sum_{n=1}^{\infty} \frac{a_{k, \mathbb{K}_3}(n) l(n)}{n^s},$$

for  $\Re(s) > 3$ , then

$$F_k(s) = G_k(s) H_k(s),$$

where

$$G_k(s) = \zeta(s-2)^k L(s, \chi)^k L(s-2, f)^k L(s, f \otimes \chi)^k,$$

and  $\chi$  is the non-principal character modulo 4. Here,  $H_k(s)$  is a Dirichlet series which converges uniformly and absolutely in the half plane  $\Re(s) > \frac{5}{2}$ , and  $H_k(s) \neq 0$  on  $\Re(s) = 3$ .

*Proof.* We observe that  $a_{k, \mathbb{K}_3}(n)l(n)$  is multiplicative, and hence

$$F_k(s) = \prod_p \left( 1 + \frac{a_{k, \mathbb{K}_3}(p)l(p)}{p^s} + \dots + \frac{a_{k, \mathbb{K}_3}(p^m)l(p^m)}{p^{ms}} + \dots \right).$$

Note that

$$\begin{aligned} a_{k, \mathbb{K}_3}(n)l(p) &= ka_{\mathbb{K}_3}(p)l(p) \\ &= k(1 + \lambda_f(p))(p^2 + \chi(p)) \\ &= kp^2 + k\chi(p) + kp^2\lambda_f(p) + k\lambda_f(p)\chi(p) \\ &=: b(p). \end{aligned}$$

From the structure of  $b(p)$ , we define the coefficients  $b(n)$  as

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \zeta(s-2)^k L(s, \chi)^k L(s-2, f)^k L(s, f \otimes \chi)^k,$$

which is absolutely convergent in  $\Re(s) > 3$ . We also note that

$$\begin{aligned} \prod_p \left( 1 + \frac{b(p)}{p^s} + \dots + \frac{b(p^m)}{p^{ms}} + \dots \right) \\ = \zeta(s-2)^k L(s, \chi)^k L(s-2, f)^k L(s, f \otimes \chi)^k \\ =: G_k(s), \end{aligned}$$

for  $\Re(s) > 3$ . Observe that  $b(n) \ll_{\epsilon} n^{2+\epsilon}$  for any small positive constant  $\epsilon$ .

Now, we note that in the half plane  $\Re(s) \geq 3 + 2\epsilon$ , we have

$$\begin{aligned} \left| \frac{b(p)}{p^s} + \frac{b(p^2)}{p^{2s}} + \dots + \frac{b(p^m)}{p^{ms}} + \dots \right| &\ll \sum_{m=1}^{\infty} \frac{p^{(2+\epsilon)m}}{p^{ms}} \\ &\leq \sum_{m=1}^{\infty} \frac{p^{(2+\epsilon)m}}{p^{(3+2\epsilon)m}} \\ &= \sum_{m=1}^{\infty} \frac{1}{p^{(1+\epsilon)m}} \\ &= \frac{1}{p^{1+\epsilon}} \\ &= \frac{1}{1 - \frac{1}{p^{1+\epsilon}}} \\ &= \frac{1}{p^{1+\epsilon} - 1} \\ &< 1. \end{aligned}$$

Let us write

$$A = \frac{a_{k, \mathbb{K}_3}(p)l(p)}{p^s} + \dots + \frac{a_{k, \mathbb{K}_3}(p^m)l(p^m)}{p^{ms}} + \dots, \quad \text{and} \quad B = \frac{b(p)}{p^s} + \dots + \frac{b(p^m)}{p^{ms}} + \dots.$$

From the above calculations, we observe that  $|B| < 1$  in  $\Re(s) \geq 3 + 2\epsilon$ .

We note that in the half plane  $\Re(s) \geq 3 + 2\epsilon$ , we have

$$\begin{aligned} \frac{1+A}{1+B} &= (1+A)(1-B+B^2-B^3+\dots) \\ &= 1+A-B-AB+\text{higher terms} \\ &= 1 + \frac{a_{k,\mathbb{K}_3}(p^2)l(p^2) - b(p^2)}{p^{2s}} + \dots + \frac{c_m(p^m)}{p^{ms}} + \dots, \end{aligned}$$

with  $c_m(n) \ll_\epsilon n^{2+\epsilon}$ . So, we have (in the half plane  $\Re(s) > \frac{5}{2}$ )

$$\prod_p \left( \frac{1+A}{1+B} \right) = \prod_p \left( 1 + \frac{a_{k,\mathbb{K}_3}(p^2)l(p^2) - b(p^2)}{p^{2s}} + \dots + \frac{c_m(p^m)}{p^{ms}} + \dots \right) \ll_\epsilon 1.$$

Thus, we have (in the half plane  $\Re(s) > \frac{5}{2}$ )

$$\begin{aligned} H_k(s) &:= \frac{F_k(s)}{G_k(s)} \\ &= \prod_p \left( \frac{1+A}{1+B} \right) \\ &\ll_\epsilon 1, \end{aligned}$$

and also  $H_k(s) \neq 0$  on  $\Re(s) = 3$ .

**Lemma 9.** *Let  $f$  be defined as in Lemma 2 and  $a_{k,\mathbb{K}_3}(n)$  be defined as in equation (1.1). If*

$$\tilde{F}_k(s) = \sum_{n=1}^{\infty} \frac{a_{k,\mathbb{K}_3}(n)v(n)}{n^s},$$

for  $\Re(s) > 3$ , then

$$\tilde{F}_k(s) = \tilde{G}_k(s)\tilde{H}_k(s),$$

where

$$\tilde{G}_k(s) = \zeta(s)^k L(s-2, \chi)^k L(s, f)^k L(s-2, f \otimes \chi)^k,$$

and  $\chi$  is the non-principal character modulo 4. Here,  $\tilde{H}_k(s)$  is a Dirichlet series which converges uniformly and absolutely in the half plane  $\Re(s) > \frac{5}{2}$ , and  $\tilde{H}_k(s) \neq 0$  on  $\Re(s) = 3$ .

*Proof.* The proof of Lemma 9 follows along similar lines as the proof of Lemma 8.

**Lemma 10.** ([JiLü14]) *Let  $\chi$  be a primitive character modulo  $q$  and  $\mathfrak{L}_{m,n}^d(s, \chi)$  be a general  $L$ -function of degree  $2A$ . For any  $\epsilon > 0$ , we have*

$$\int_T^{2T} \left| \mathfrak{L}_{m,n}^d(\sigma + it, \chi) \right|^2 dt \ll (qT)^{2A(1-\sigma)+\epsilon},$$

uniformly for  $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$ , and  $T \geq 1$ . Also,

$$\mathfrak{L}_{m,n}^d(\sigma + it, \chi) \ll (q(1+|t|))^{\max\{A(1-\sigma), 0\}+\epsilon},$$

uniformly for  $-\epsilon \leq \sigma \leq 1 + \epsilon$ .

### 3. Proof of Theorem 2

Let  $k \geq 1$  be an integer. Firstly, we consider the sum  $\sum_{n \leq x} a_{k, \mathbb{K}_3}(n) l_1(n)$ . We begin by applying Perron's formula (see [GrSo14, Chapter 2.4]) to  $F_k(s)$  with  $\eta = 3 + \epsilon$  and  $10 \leq T \leq x$ . Thus, we have

$$\begin{aligned} \sum_{n \leq x} a_{k, \mathbb{K}_3}(n) l_1(n) &= 16 \sum_{n \leq x} a_{k, \mathbb{K}_3}(n) l(n) \\ &= \frac{16}{2\pi i} \int_{\eta-iT}^{\eta+iT} F_k(s) \frac{x^s}{s} ds + O\left(\frac{x^{3+3\epsilon}}{T}\right). \end{aligned}$$

We move the line of integration to  $\Re(s) = \frac{5}{2} + \epsilon$ . By Cauchy's residue theorem there is only one pole at  $s = 3$  of order  $k$ , coming from the factor  $\zeta(s-2)^k$ .

So, we obtain

$$\begin{aligned} \sum_{n \leq x} a_{k, \mathbb{K}_3}(n) l_1(n) &= \operatorname{Res}_{s=3} \left\{ F_k(s) \frac{x^s}{s} \right\} + \frac{16}{2\pi i} \left\{ \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} + \int_{3+\epsilon-iT}^{\frac{5}{2}+\epsilon-iT} + \int_{\frac{5}{2}+\epsilon+iT}^{3+\epsilon+iT} \right\} F_k(s) \frac{x^s}{s} ds \\ &\quad + O\left(\frac{x^{3+3\epsilon}}{T}\right) \\ &=: x^3 P_{k-1}(\log x) + \frac{16}{2\pi i} (J_1(k) + J_2(k) + J_3(k)) + O\left(\frac{x^{3+3\epsilon}}{T}\right), \end{aligned}$$

where  $P_{k-1}(t)$  is a polynomial in  $t$  of degree  $k-1$ .

Note that the horizontal lines ( $J_2(k)$  and  $J_3(k)$ ) contribute (for any fixed integer  $k \geq 1$ ), using Lemma 6, Lemma 7 and Remark 1

$$\begin{aligned} J_2(k) + J_3(k) &\ll (x^2)^{\max_{\frac{1}{2}+\epsilon \leq \sigma \leq 1+\epsilon}} x^\sigma T^{(2k\mu + \frac{2k}{3})(1-\sigma) + \epsilon} T^{-1} \\ &\ll (x^{2+\epsilon})^{\max_{\frac{1}{2}+\epsilon \leq \sigma \leq 1+\epsilon}} \left( \frac{x}{T^{2k\mu + \frac{2k}{3}}} \right)^\sigma T^{2k\mu + \frac{2k}{3} - 1 + \epsilon}. \end{aligned}$$

For any fixed  $k, \mu (> 0)$ ,  $\left( \frac{x}{T^{2k\mu + \frac{2k}{3}}} \right)^\sigma$  is monotonic as a function of  $\sigma$  for  $\frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon$  and hence the maximum is attained at the extremities of the interval  $[\frac{1}{2} + \epsilon, 1 + \epsilon]$ . Thus,

$$J_2(k) + J_3(k) \ll \frac{x^{3+3\epsilon}}{T} + x^{\frac{5}{2}+3\epsilon} T^{\frac{1}{2}(2k\mu + \frac{2k}{3}) - 1}.$$

**Vertical line contributions:**

1. **For  $k=1$ :**

$$J_1(1) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_1(s) \frac{x^s}{s} ds.$$

Using Lemma 5, Lemma 6, Lemma 7 and Cauchy-Schwarz inequality,

$$\begin{aligned}
J_1(1) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta\left(\frac{1}{2} + it\right) \right| \left| L\left(\frac{1}{2} + it, f\right) \right| dt \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left( \int_{\frac{U}{2}}^U \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \right)^{\frac{1}{2}} \left( \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \right)^{\frac{1}{2}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\frac{1}{2}+\epsilon} U^{\frac{1}{2}+\epsilon} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} T^{3\epsilon},
\end{aligned}$$

which dominates over  $J_2(1) + J_3(1)$ .

**2. For  $k=2$ :**

$$J_1(2) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_2(s) \frac{x^s}{s} ds.$$

Using Lemma 7, Remark 1 and Lemma 5,

$$\begin{aligned}
J_1(2) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{2\mu+2\epsilon} U \log U \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} T^{2\mu+4\epsilon}.
\end{aligned}$$

Note that, by Lemma 5

$$\begin{aligned}
\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f\right) \right|^4 dt &\ll \left( \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \right)^{\frac{1}{2}} \left( \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f\right) \right|^6 dt \right)^{\frac{1}{2}} \\
&\ll U^{\frac{3}{2}+\epsilon}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
J_1(2) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left( \int_{\frac{U}{2}}^U \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \right)^{\frac{1}{2}} \left( \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f\right) \right|^4 dt \right)^{\frac{1}{2}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\frac{1}{2}+\epsilon} U^{\frac{1}{2}(\frac{3}{2}+\epsilon)} \right\} \text{ (using Lemma 2 and above observation)} \\
&\ll x^{\frac{5}{2}+\epsilon} T^{\frac{1}{4}+2\epsilon}.
\end{aligned}$$

Thus, we have

$$J_1(2) \ll x^{\frac{5}{2}+4\epsilon} T^{\min(2\mu, \frac{1}{4})},$$

which dominates over  $J_2(2) + J_3(2)$ .

**3. For  $k=3$ :**

$$J_1(3) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_3(s) \frac{x^s}{s} ds.$$

Using Lemma 2, Lemma 5, Cauchy-Schwarz Inequality and Remark 1,

$$\begin{aligned}
J_1(3) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^3 \left| L \left( \frac{1}{2} + it, f \right) \right|^3 dt \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{\mu+\epsilon} \left( \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 dt \right)^{\frac{1}{2}} \right. \\
&\quad \left. \times \left( \int_{\frac{U}{2}}^U \left| L \left( \frac{1}{2} + it, f \right) \right|^6 dt \right)^{\frac{1}{2}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\mu+\epsilon} U^{\frac{1}{2}(1+\epsilon)} U^{\frac{1}{2}(2+\epsilon)} \right\} \\
&\ll x^{\frac{5}{2}+4\epsilon} T^{\mu+\frac{1}{2}}.
\end{aligned}$$

Also, we have (using Lemma 2, Lemma 5 and above observation)

$$\begin{aligned}
J_1(3) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left( \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^{12} dt \right)^{\frac{1}{4}} \left( \int_{\frac{U}{2}}^U \left| L \left( \frac{1}{2} + it, f \right) \right|^4 dt \right)^{\frac{3}{4}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\frac{1}{2}+\epsilon} U^{\frac{3}{4}(\frac{3}{2}+\epsilon)} \right\} \\
&\ll x^{\frac{5}{2}+5\epsilon} T^{\frac{5}{8}}.
\end{aligned}$$

Thus, we have

$$J_1(3) \ll x^{\frac{5}{2}+5\epsilon} T^{\min(\mu+\frac{1}{2}, \frac{5}{8})},$$

which dominates over  $J_2(3) + J_3(3)$ .

4. **For  $k=4$ :** First we observe, (using Lemma 2 and Cauchy-Schwarz inequality)

$$\begin{aligned}
\int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^8 dt &\ll \left( \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 dt \right)^{\frac{1}{2}} \left( \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^{12} dt \right)^{\frac{1}{2}} \\
&\ll U^{\frac{1}{2}(1+\epsilon)} U^{\frac{1}{2}(2+\epsilon)} \\
&\ll U^{\frac{3}{2}+\epsilon}.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^6 dt &\ll \left( \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 dt \right)^{\frac{1}{2}} \left( \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^8 dt \right)^{\frac{1}{2}} \\
&\ll U^{\frac{1}{2}(1+\epsilon)} T^{\frac{1}{2}(\frac{3}{2}+\epsilon)} \\
&\ll U^{\frac{5}{4}+\epsilon}.
\end{aligned}$$

Now,

$$J_1(4) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_4(s) \frac{x^s}{s} ds.$$

Using Lemma 2, Lemma 5, Hölder's inequality and Remark 1,

$$\begin{aligned}
J_1(4) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 \left| L \left( \frac{1}{2} + it, f \right) \right|^4 dt \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{2\mu+2\epsilon} \left( \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^6 dt \right)^{\frac{1}{3}} \right. \\
&\quad \left. \times \left( \int_{\frac{U}{2}}^U \left| L \left( \frac{1}{2} + it, f \right) \right|^6 dt \right)^{\frac{2}{3}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{2\mu+2\epsilon} U^{\frac{1}{3}(\frac{5}{4}+\epsilon)} U^{\frac{2}{3}(2+\epsilon)} \right\} \text{ (using above observation)} \\
&\ll x^{\frac{5}{2}+5\epsilon} T^{2\mu+\frac{3}{4}}.
\end{aligned}$$

Also, we have (using Lemma 7, Lemma 2, Lemma 5, and Hölder's inequality)

$$\begin{aligned}
J_1(4) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left( \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^{12} dt \right)^{\frac{1}{3}} \left( \int_{\frac{U}{2}}^U \left| L \left( \frac{1}{2} + it, f \right) \right|^6 dt \right)^{\frac{2}{3}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\frac{1}{3}(2+\epsilon)} U^{\frac{2}{3}(2+\epsilon)} \right\} \\
&\ll x^{\frac{5}{2}+5\epsilon} T.
\end{aligned}$$

Thus, we have

$$J_1(4) \ll x^{\frac{5}{2}+5\epsilon} T^{\min(2\mu+\frac{3}{4}, 1)},$$

which dominates over  $J_2(4) + J_3(4)$ .

**5. For  $k=5$ :**

$$J_1(5) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_5(s) \frac{x^s}{s} ds.$$

Using Lemma 2, Lemma 5, Hölder's inequality and Remark 1,

$$\begin{aligned}
J_1(5) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^5 \left| L \left( \frac{1}{2} + it, f \right) \right|^5 dt \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{3\mu+3\epsilon} \left( \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^{12} dt \right)^{\frac{1}{6}} \right. \\
&\quad \left. \times \left( \int_{\frac{U}{2}}^U \left| L \left( \frac{1}{2} + it, f \right) \right|^6 dt \right)^{\frac{5}{6}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{3\mu+3\epsilon} U^{\frac{1}{6}(2+\epsilon)} U^{\frac{5}{6}(2+\epsilon)} \right\} \\
&\ll x^{\frac{5}{2}+6\epsilon} T^{3\mu+1}.
\end{aligned}$$

Also, we have

$$\begin{aligned} J_1(5) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left( \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^{12} dt \right)^{\frac{5}{12}} \left( \int_{\frac{U}{2}}^U \left| L \left( \frac{1}{2} + it, f \right) \right|^{5 \cdot \frac{12}{7}} dt \right)^{\frac{7}{12}} \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{\frac{5}{12}(2+\epsilon)} U^{\frac{7}{12}(2+\epsilon)} U^{\frac{18}{7} \cdot \frac{1}{3} \cdot \frac{7}{12} + 2\epsilon} \right\} \\ &\ll x^{\frac{5}{2}+5\epsilon} T^{\frac{3}{2}}. \end{aligned}$$

Thus, we have

$$J_1(5) \ll x^{\frac{5}{2}+5\epsilon} T^{\min(3\mu+1, \frac{3}{2})},$$

which dominates over  $J_2(5) + J_3(5)$ .

**6. For  $k \geq 6$ :**

$$J_1(k) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_k(s) \frac{x^s}{s} ds.$$

Using Lemma 2, Cauchy-Schwarz Inequality, Lemma 5, Lemma 6, and Remark 1 we get

$$\begin{aligned} J_1(k) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^k \left| L \left( \frac{1}{2} + it, f \right) \right|^k dt \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{(k-6)(\mu+\epsilon)} U^{(k-3)(\frac{1}{3}+\epsilon)} \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^6 \left| L \left( \frac{1}{2} + it, f \right) \right|^3 dt \right\} \\ &\ll x^{\frac{5}{2}+2k\epsilon} \left\{ \max_{1 \leq U \leq T} U^{\mu(k-6)+\frac{1}{3}(k-3)-1} \left( \int_{\frac{U}{2}}^U \left| \zeta \left( \frac{1}{2} + it \right) \right|^{12} dt \right)^{\frac{1}{2}} \right. \\ &\quad \left. \times \left( \int_{\frac{U}{2}}^U \left| L \left( \frac{1}{2} + it, f \right) \right|^6 dt \right)^{\frac{1}{2}} \right\} \\ &\ll x^{\frac{5}{2}+2k\epsilon} \left\{ \max_{1 \leq U \leq T} U^{\mu(k-6)+\frac{1}{3}(k-3)-1} U^{\frac{1}{2}(2+\epsilon)} U^{\frac{1}{2}(2+\epsilon)} \right\} \\ &\ll x^{\frac{5}{2}+3k\epsilon} T^{\mu(k-6)+\frac{k}{3}}. \end{aligned}$$

Define

$$\lambda_k := \mu(k-6) + \frac{k}{3},$$

for  $k \geq 6$ , then

$$J_1(k) \ll x^{\frac{5}{2}+3k\epsilon} T^{\lambda_k}.$$

Therefore, we have (for  $k \geq 1$ )

$$\sum_{n \leq x} a_{k, \mathbb{K}_3}(n) l_1(n) = x^3 P_{k-1}(\log x) + E_{k, \mathbb{K}_3}(x),$$

where  $E_{k, \mathbb{K}_3}(x) \ll \frac{x^{3+3\epsilon}}{T} + x^{\frac{5}{2}+3k\epsilon} T^{\lambda_k}$ . We choose  $T$  such that  $\frac{x^3}{T} \sim x^{\frac{5}{2}} T^{\lambda_k}$ , i.e.,  $T^{1+\lambda_k} \sim x^{\frac{1}{2}}$ , i.e.,  $T \sim x^{\frac{1}{2(1+\lambda_k)}}$ .

So finally, we have

$$E_{k, \mathbb{K}_3}(x) \ll x^{3-\frac{1}{2(1+\lambda_k)}+3k\epsilon}.$$

This proves the theorem.



## 4. Proof of Theorem 3

Let  $k \geq 1$  be an integer. Now, we consider the sum  $\sum_{n \leq x} a_{k, \mathbb{K}_3}(n) v_1(n)$ . We begin by applying Perron's formula (see [GrSo14, Chapter 2.4]) to  $\tilde{F}_k(s)$  with  $\eta = 3 + \epsilon$  and  $10 \leq T \leq x$ . Thus, we have

$$\begin{aligned} \sum_{n \leq x} a_{k, \mathbb{K}_3}(n) v_1(n) &= 4 \sum_{n \leq x} a_{k, \mathbb{K}_3}(n) v(n) \\ &= \frac{4}{2\pi i} \int_{\eta - iT}^{\eta + iT} \tilde{F}_k(s) \frac{x^s}{s} ds + O\left(\frac{x^{3+3\epsilon}}{T}\right). \end{aligned}$$

We move the line of integration to  $\Re(s) = \frac{5}{2} + \epsilon$ . There is no singularity in the rectangle obtained and the function  $\tilde{F}_k(s) \frac{x^s}{s}$  is analytic in this region. Thus, using Cauchy's theorem for rectangles pertaining to analytic functions, we get

$$\begin{aligned} \sum_{n \leq x} a_{k, \mathbb{K}_3}(n) v_1(n) &= \operatorname{Res}_{s=3} \left\{ \tilde{F}_k(s) \frac{x^s}{s} \right\} + \frac{4}{2\pi i} \left\{ \int_{\frac{5}{2} + \epsilon - iT}^{\frac{5}{2} + \epsilon + iT} + \int_{3 + \epsilon - iT}^{\frac{5}{2} + \epsilon - iT} + \int_{\frac{5}{2} + \epsilon + iT}^{3 + \epsilon + iT} \right\} \tilde{F}_k(s) \frac{x^s}{s} ds \\ &\quad + O\left(\frac{x^{3+3\epsilon}}{T}\right) \\ &=: \frac{4}{2\pi i} (J'_1(k) + J'_2(k) + J'_3(k)) + O\left(\frac{x^{3+3\epsilon}}{T}\right). \end{aligned}$$

Note that the horizontal lines ( $J'_2(k)$  and  $J'_3(k)$ ) contribute (for any fixed integer  $k \geq 1$ ), using Lemma 9, Lemma 10 and Lemma 6

$$\begin{aligned} J'_2(k) + J'_3(k) &\ll (x^2) \max_{\frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon} x^\sigma T^{(\frac{k}{3} + \frac{2k}{3})(1-\sigma) + \epsilon} T^{-1} \\ &\ll (x^{2+\epsilon}) \max_{\frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon} \left(\frac{x}{T^k}\right)^\sigma T^{k-1+\epsilon}. \end{aligned}$$

For any fixed  $k$ ,  $\mu(> 0)$ ,  $\left(\frac{x}{T^k}\right)^\sigma$  is monotonic as a function of  $\sigma$  for  $\frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon$  and hence the maximum is attained at the extremities of the interval  $[\frac{1}{2} + \epsilon, 1 + \epsilon]$ . Thus,

$$J'_2(k) + J'_3(k) \ll \frac{x^{3+3\epsilon}}{T} + x^{\frac{5}{2}+3\epsilon} T^{\frac{k}{2}-1}.$$

### Vertical line contributions:

#### 1. For $k=1$ :

$$J'_1(1) := \int_{\frac{5}{2} + \epsilon - iT}^{\frac{5}{2} + \epsilon + iT} \tilde{F}_1(s) \frac{x^s}{s} ds.$$

Using Lemma 9, Lemma 3, Lemma 10 and Cauchy-Schwarz inequality,

$$\begin{aligned} J'_1(1) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right| \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right| dt \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left( \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 dt \right)^{\frac{1}{2}} \left( \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^2 dt \right)^{\frac{1}{2}} \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\frac{1}{2}+\epsilon} U^{\frac{1}{2}+\epsilon} \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} T^{3\epsilon}, \end{aligned}$$

which dominates over  $J'_2(1) + J'_3(1)$ .

**2. For  $k=2$**

$$J'_1(2) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} \widetilde{F}_2(s) \frac{x^s}{s} ds.$$

Note that, by Lemma 4

$$\begin{aligned} \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^4 dt &\ll \left( \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^2 dt \right)^{\frac{1}{2}} \left( \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^6 dt \right)^{\frac{1}{2}} \\ &\ll U^{\frac{3}{2}+\epsilon}. \end{aligned}$$

Using Lemma 4 and Lemma 10,

$$\begin{aligned} J'_1(2) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^2 dt \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left( \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \right)^{\frac{1}{2}} \left( \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^4 dt \right)^{\frac{1}{2}} \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\frac{1}{2}+\epsilon} U^{\frac{1}{2}(\frac{3}{2}+\epsilon)} \right\} \text{ (using above observation)} \\ &\ll x^{\frac{5}{2}+\epsilon} T^{\frac{1}{4}+2\epsilon}, \end{aligned}$$

which dominates over  $J'_2(2) + J'_3(2)$ .

**3. For  $k \geq 3$ :**

$$J'_1(k) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} \widetilde{F}_k(s) \frac{x^s}{s} ds.$$

Using Lemma 8, Cauchy-Schwarz Inequality, Lemma 4, Lemma 5, and Lemma 6 we get

$$\begin{aligned} J'_1(k) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right|^k \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^k dt \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{(k-2)(1-\sigma)} U^{\frac{2}{3}(k-3)(1-\sigma)} \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^3 dt \right\} \\ &\ll x^{\frac{5}{2}+2k\epsilon} \left\{ \max_{1 \leq U \leq T} U^{\frac{k-2}{6} + \frac{k-3}{3} - 1} \left( \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \right)^{\frac{1}{2}} \right. \\ &\quad \left. \times \left( \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^6 dt \right)^{\frac{1}{2}} \right\} \\ &\ll x^{\frac{5}{2}+2k\epsilon} \left\{ \max_{1 \leq U \leq T} U^{\frac{k-2}{6} + \frac{k-3}{3} - 1} U^{\frac{1}{2}(1+\epsilon)} U^{\frac{1}{2}(2+\epsilon)} \right\} \\ &\ll x^{\frac{5}{2}+3k\epsilon} T^{\frac{k}{2} - \frac{5}{6}}, \end{aligned}$$

which dominates over  $J'_2(k) + J'_3(k)$ .

Therefore, we have (for  $k \geq 1$ )

$$\sum_{n \leq x} a_{k, \mathbb{K}_3}(n) v_1(n) = E'_{k, \mathbb{K}_3}(x),$$

where  $E'_{k, \mathbb{K}_3}(x) \ll \frac{x^{3+3\epsilon}}{T} + x^{\frac{5}{2}+3k\epsilon} T^{\lambda'_k}$ . We choose  $T$  such that  $\frac{x^3}{T} \sim x^{\frac{5}{2}} T^{\lambda'_k}$ , i.e.,  $T^{1+\lambda'_k} \sim x^{\frac{1}{2}}$ , i.e.,  $T \sim x^{\frac{1}{2(1+\lambda'_k)}}$ .

So finally, we have

$$E'_{k, \mathbb{K}_3}(x) \ll x^{3 - \frac{1}{2(1+\lambda'_k)} + 3k\epsilon}.$$

This proves the theorem.

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## References

- [AkDr12] A. Akbary and G. Dragos, A geometric variant of Titchmarsh divisor problem. *International Journal of Number Theory*, **8(1)** (2012), pp. 53–69.
- [Blo18] V. Blomer, Higher order divisor problems. *Mathematische Zeitschrift*, **290(3–4)** (2018), pp. 937–952.
- [Bou17] J. Bourgain, Decoupling, exponential sums and the Riemann zeta function. *Journal of the American Mathematical Society*, **30(1)** (2017), pp.205–224.
- [CaFr67] J.W.S. Cassels and A. Fröhlich, *Algebraic Number Theory*, Thompson Book Company. Inc., Washington, DC (1967).
- [ChGo83] K. Chandrasekharan and A. Good, On the number of integral ideals in Galois extensions. *Monatshefte für Mathematik*, **95(2)** (1983), pp.99–109.
- [ChNa63] K. Chandrasekharan and R. Narasimhan, The approximate functional equation for a class of zeta-functions. *Mathematische Annalen*, **152(1)** (1963), pp.30–64.
- [Chi22] L. Chiriac, The Average Number of Divisors in Certain Arithmetic Sequences. *Mathematica Pannonica*, **28(2)** (2022), pp. 136–142.
- [Dan97] S. Daniel, Teilerprobleme für homogene Polynome. Dissertation, Stuttgart (1997)
- [Dan99] S. Daniel, On the divisor-sum problem for binary forms, *Journal für die reine und angewandte Mathematik*, **507** (1999), pp. 107–129.
- [DeSe74] P. Deligne, and J.P. Serre, Formes modulaires de poids 1. *Annales scientifiques de l'École Normale Supérieure*, **7(4)** (1974), pp. 507–530.
- [Erd52] P. Erdős, On the sum  $\sum_{k=1}^x d(f(k))$ . *Journal of the London Mathematical Society*, **27** (1952), pp. 7–15.
- [Fom08] O.M. Fomenko, Mean values connected with the Dedekind zeta function. *Journal of Mathematical Sciences*, **150(3)** (2008), pp. 2115–2122.
- [FrIw06] J.B. Friedlander and H. Iwaniec, A polynomial divisor problem. *Journal für die reine und angewandte Mathematik*, **601** (2006), pp. 109–137.
- [FrIw98] J.B. Friedlander and H. Iwaniec, The polynomial  $x^2 + y^4$  captures its primes. *Annals of Mathematics*, **148** (1998), pp. 945–1040.
- [Go82] A. Good, The square mean of Dirichlet series associated with cusp forms. *Mathematika*, **29(2)** (1982), pp.278–295.
- [GrSo14] A. Granville and K. Soundararajan, Multiplicative number theory: The pretentious approach, available at <https://dms.umontreal.ca/~andrew/PDF/BookChaps1n2.pdf>.
- [Gre70] G. Greaves, On the divisor-sum problem for binary cubic forms. *Acta arithmetica*, **17** (1970), pp. 1–28.
- [Hea78] D.R. Heath-Brown, The twelfth power moment of the Riemann-function. *The Quarterly Journal of Mathematics*, **29(4)** (1978), pp.443–462.
- [Hua22] G. Hua, The average behaviour of Hecke eigenvalues over certain sparse sequence of positive integers. *Research in Number Theory* **8(4)** (2022), pp. 1–20.

- [HuWa00] M.N. Huxley and N. Watt, The number of ideals in a quadratic field, II. *Israel Journal of Mathematics*, **120(1)** (2000), pp.125–153.
- [Ivi12] A. Ivic, *The Riemann zeta-function: theory and applications*. Courier Corporation (2012).
- [JiLü14] Y. Jiang and G. Lü, On the higher mean over arithmetic progressions of Fourier coefficients of cusp forms. *Acta Arith.* **3(166)** (2014), pp. 231–252.
- [Jut87] M. Jutila, Tata Institute of Fundamental Research (Bombay), *Lectures on a Method in the Theory of Exponential Sums* **80**. Berlin: Springer (1987).
- [KST02] S. Kanemitsu, A. Sankaranarayanan, and Y. Tanigawa, A mean value theorem for Dirichlet series and a general divisor problem. *Monatshefte für Mathematik*, **136(1)** (2002), pp. 17–34.
- [Kol79] G. Kolesnik, On the estimation of multiple exponential sums, in Recent progress in analytic number theory, Symposium Durham (1979), pp. 231–246.
- [LaZh21] K. Lapkova and N.H. Zhou, On the average sum of the  $k$ -th divisor function over values of quadratic polynomials. *Ramanujan Journal*, **55** (2021), pp. 849–872.
- [Lan49] E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, Leipzig, (1918); also Chelsea Pub, Co, (1949).
- [Liu18] H.F. Liu, Mean value estimates of the coefficients of product  $L$ -functions. *Acta Mathematica Hungarica*, **156(1)** (2018), pp. 102–111.
- [LiuZha19] H.F. Liu, and R. Zhang, Some problems involving Hecke eigenvalues. *Acta Mathematica Hungarica*, **159(1)** (2019), pp. 287–298.
- [Lü12] G. Lü, On general divisor problems involving Hecke eigenvalues. *Acta Mathematica Hungarica*, **135(1-2)** (2012), pp. 148–159.
- [Lü13] G. Lü, Mean values connected with the Dedekind zeta-function of a non-normal cubic field. *Central European Journal of Mathematics*, **11(2)** (2013), pp. 274–282.
- [Mck95] J. McKee, On the average number of divisors of quadratic polynomials. *Mathematical Proceedings of the Cambridge Philosophical Society*, **117(3)** (1995), pp. 389–392.
- [McK99] J. McKee, The average number of divisors of an irreducible quadratic polynomial. *Mathematical Proceedings of the Cambridge Philosophical Society*, **126(1)** (1999), pp. 17–22.
- [Mül88] W. Müller, On the distribution of ideals in cubic number fields. *Monatshefte für Mathematik*, **106(3)** (1988), pp. 211–219.
- [Now93] W.G. Nowak, On the distribution of integer ideals in algebraic number-fields. *Mathematische Nachrichten*, **161** (1993), pp. 59–74.
- [PaSa20] B. Paul, and A. Sankaranarayanan, On the error term and zeros of the Dedekind zeta function. *Journal of Number Theory*, **215** (2020), pp. 98–119.
- [Pol16] P. Pollack, A Titchmarsh divisor problem for elliptic curves. *Mathematical Proceedings of the Cambridge Philosophical Society*, **160(1)** (2016), pp. 167–189.
- [Rama74] K. Ramachandra, A simple proof of the mean fourth power estimate for  $\zeta(\frac{1}{2} + it)$  and  $L(\frac{1}{2} + it, \chi)$ . *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze* **1(1-2)** (1974), pp. 81–97.
- [ShSa22a] A. Sharma and A. Sankaranarayanan, Discrete mean square of the coefficients of symmetric square  $L$ -functions on certain sequence of positive numbers. *Res. Number Theory*, **8(1)** (2022), pp. 1–13.
- [ShSa22b] A. Sharma and A. Sankaranarayanan, Higher moments of the Fourier coefficients of symmetric square  $L$ -functions on certain sequence. *Rend. Circ. Mat. Palermo (2)* (2022), pp. 1–18.
- [ShSa22c] A. Sharma and A. Sankaranarayanan, Average behavior of the Fourier coefficients of symmetric square  $L$ -function over some sequence of integers. *Integers* **22** (2022).
- [SuZh16] Q. Sun and D. Zhang, Sums of the triple divisor function over values of a ternary quadratic form. *Journal of Number Theory*, **168** (2016), pp. 215–246.
- [Titch86] E.C. Titchmarsh and D.R. Heath-Brown, *The theory of the Riemann zeta-function*. Oxford university press, 1986.

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