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► **To cite this version:**

Anubhav Sharma, Ayyadurai Sankaranarayanan. On a general divisor problem related to a certain Dedekind zeta-function over a specific sequence of positive integers. Hardy-Ramanujan Journal, 2024, 46 – 2023, pp.23 – 41. 10.46298/hrj.2024.13033 . hal-04443008v2

HAL Id: hal-04443008

<https://hal.science/hal-04443008v2>

Submitted on 1 Mar 2024

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On a general divisor problem related to a certain Dedekind zeta-function over a specific sequence of positive integers

Anubhav Sharma and Ayyadurai Sankaranarayanan

Abstract. We investigate the average behavior of coefficients of the Dirichlet series of positive integral power of the Dedekind zeta-function $\zeta_{\mathbb{K}_3}(s)$ of a non-normal cubic extension \mathbb{K}_3 of \mathbb{Q} over a certain sequence of positive integers. More precisely, we prove an asymptotic formula with an error term for the sum

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \leq x \\ (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{Z}^6}} a_{k, \mathbb{K}_3}(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2),$$

where $(\zeta_{\mathbb{K}_3}(s))^k := \sum_{n=1}^{\infty} \frac{a_{k, \mathbb{K}_3}(n)}{n^s}$.

Keywords. Dedekind zeta-function, Hecke eigenvalues, non-normal cubic extension.

2010 Mathematics Subject Classification. 11F30, 11R42.

1. Introduction

Let \mathbb{K} be an algebraic extension of degree m of rational field \mathbb{Q} . Define

$$\zeta_{\mathbb{K}}(s) := \sum_{\alpha} \frac{1}{N(\alpha)^s},$$

for $\Re(s) > 1$, where the summation is running over all the integral ideals α of \mathbb{K} and norm of integral ideal α is denoted by $N(\alpha)$. The function $\zeta_{\mathbb{K}}(s)$ can also be written as

$$\zeta_{\mathbb{K}}(s) = \sum_{n=1}^{\infty} \frac{a_{\mathbb{K}}(n)}{n^s},$$

where $a_{\mathbb{K}}(n)$ denotes the number of integral ideals of \mathbb{K} with norm n . It is shown (by Chandrasekharan and Good [ChGo83]) that these coefficients are multiplicative and satisfies the upper bound

$$a_{\mathbb{K}}(n) \leq d(n)^m,$$

where m is the degree of extension, i.e., $m = [\mathbb{K} : \mathbb{Q}]$ and $d(n)$ is the number of divisors of n .

In 1949, Landau [Lan49] showed that

$$\sum_{n \leq x} a_{\mathbb{K}}(n) = cx + O\left(x^{1-\frac{2}{m+1}+\epsilon}\right),$$

where c is the residue of $\zeta_{\mathbb{K}}(s)$ at its simple pole at $s = 1$, which is further improved to

$$\sum_{n \leq x} a_{\mathbb{K}}(n) = cx + O\left(x^{\frac{23}{73}} \log^{\frac{315}{146}} x\right),$$

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for quadratic field by Huxley and Watt [HuWa00]. Some further improvement is also available for cubic fields by Müller [Mül88]. In 1993, W.G. Nowak [Now93] established that

$$\sum_{n \leq x} a_{\mathbb{K}}(n) = cx + \begin{cases} O\left(x^{1-\frac{2}{m}+\frac{8}{m(5m+2)}} \log^{\frac{10}{5m+2}} x\right) & \text{for } 3 \leq m \leq 6, \\ O\left(x^{1-\frac{2}{m}+\frac{3}{2m^2}} \log^{\frac{2}{m}} x\right) & \text{for } m \geq 7. \end{cases}$$

We also have some significant results (by Chandrasekharan and Narasimhan [ChNa63] and by Chandrasekharan and Good [ChGo83]) of $\sum_{n \leq x} a_{\mathbb{K}}(n)^k$ for some higher powers k , if \mathbb{K} is the Galois extension of \mathbb{Q} .

If h is the class number of \mathbb{K} and $[\mathbb{K} : \mathbb{Q}] = r_1 + 2r_2$, where r_1 is the number of real conjugate fields and $2r_2$ is the number of complex conjugate fields, then we can write

$$\sum_{n \leq x} a_{\mathbb{K}}(n) = h\lambda x + E(x),$$

where

$$\lambda := \frac{2^{r_1+r_2} \pi^{r_2} R}{w|\Delta|^{\frac{1}{2}}}.$$

Here, w is the number of roots of unity in \mathbb{K} ; R is the regulator of \mathbb{K} and Δ is the discriminant of \mathbb{K} .

When $[\mathbb{K} : \mathbb{Q}] = m \geq 10$, B. Paul and A. Sankaranarayanan proved that

$$E(x) \ll x^{1-\frac{3}{m+6}+\epsilon},$$

where implied constants depend only on \mathbb{K} and ϵ (see [PaSa20]).

Also, if $\mathbb{K} = \mathbb{Q}(\zeta_l)$, where l is some positive integer and $[\mathbb{K} : \mathbb{Q}] = m \geq 8$, then,

$$E(x) \ll x^{1-\frac{3}{m+5}+\epsilon},$$

where the implied constants depend only on \mathbb{K} and ϵ (see [PaSa20]).

It is of great interest to study the L -functions related to primitive holomorphic cusp forms. For many years, it has been a profound area in which many authors have contributed.

Let $L(s, f)$ be the L -function connected with the primitive holomorphic cusp form f of weight w for the full modular group $SL(2, \mathbb{Z})$ and $\lambda_f(n)$ are the normalized n^{th} Fourier coefficients of Fourier expansion of $f(z)$ at the cusp ∞ , i.e.,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{w-1}{2}} e^{2\pi i n z}$$

where $\Im(z) > 0$, then the L -function attached to $\lambda_f(n)$ is defined as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s},$$

for $\Re(s) > 1$, where $\lambda_f(n)$ are Hecke eigenvalues of Hecke operators T_n .

Also,

$$L^k(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_{k,f}(n)}{n^s},$$

where

$$\lambda_{k,f}(n) = \sum_{n=n_1 n_2 \dots n_k} \lambda_f(n_1) \lambda_f(n_2) \dots \lambda_f(n_k).$$

In 2012, Kanemitsu, Sankaranarayanan and Tanigawa [KST02] proved that for $k \geq 2$,

$$\sum_{n \leq x} \lambda_{k,f}(n) \ll x^{1 - \frac{3}{2k+2} + \epsilon},$$

where implied constant depends only on f and ϵ , which is further improved by Lü in [Lü12].

For such divisor problems connected to holomorphic cusp forms, see the work of H.F. Liu [Liu18], [LiuZha19] and Lü [Lü12]. Recently, several authors considered the average behavior of $\lambda_{\text{sym}^j f}(n)$ over certain sequences of positive integers and established some interesting asymptotic formulas (see, for instance [ShSa22a, ShSa22b, ShSa22c, Hua22]).

For $k \geq 2$, let $\Delta_k(x)$ denotes the error term in the asymptotic formula for $\sum_{n \leq x} d_k(n)$, where $\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n) n^{-s}$. The estimation of $\Delta_k(x)$ is popularly known as the general Dirichlet divisor problem. From elementary arguments, one can get $\Delta_k(x) \ll x^{\frac{k-1}{k}} \log^{k-2} x$. For $k = 2$, we have $\Delta_2(x) \ll x^{\frac{35}{108} + \epsilon}$, see [Ivi12]. For $k = 3$, $\Delta_3(x) \ll x^{\frac{43}{96} + \epsilon}$ is the best result available, due to G. Kolesnik [Kol79]. We may define the order α_k of $\Delta_k(x)$ as the least number such that $\Delta_k(x) \ll x^{\alpha_k + \epsilon}$ for every $\epsilon > 0$. The following results are known (see [Titch86]):

$$\alpha_k \leq \frac{k-1}{k} \quad \text{for } k = 2, 3, \dots$$

and

$$\alpha_k \leq \frac{k-1}{k+1} \quad \text{for } k = 2, 3, \dots$$

But the exact value of α_k has not been determined for any value of k . For an extensive literature and detailed discussion of general divisor problem, see [Titch86, Chapter 12].

Estimating the average behavior of some special functions over polynomial values has been of interest since the early 1950s. In 1952, Erdős [Erd52] proved that

$$c_1 x \log x < \sum_{k=1}^x d(f(k)) < c_2 x \log x$$

and

$$\sum_{p \leq x} d(f(p)) \ll x,$$

where $f(x) \in \mathbb{Z}[x]$ and c_1, c_2 are some positive constants. For a quadratic polynomial $f(x)$, McKee [Mck95, McK99] proved that

$$\sum_{k=1}^x d(f(k)) \sim \lambda(f) x \log x,$$

where $\lambda(f)$ can be written in terms of Hurwitz class numbers. No similar results have been established for polynomials that have higher degrees. Then, Titchmarsh considered the linear polynomials $f(x) = a + x$ ($a \neq 0$) and observed the average behavior of divisor function over shifted primes. More precisely, he proved that

$$\sum_{p \leq x} d(p+a) \sim C(a)x,$$

where $C(a)$ is some constant depending upon a . Motivated by the previous result of Titchmarsh, many authors studied the problem of finding an optimal error term of $\sum_{p \leq x} d(n_p)$, where the n_p 's are quantities of arithmetic significance, for instance see [Pol16, AkDr12, Chi22].

One topic that has drawn a lot of attention involves figuring out the average order of $d_k(n)$ over sparse sequences of values taken by polynomials, i.e.,

$$D_k(f(x_1, x_2, \dots, x_n), x) := \sum_{|f(x_1, x_2, \dots, x_n)| \leq x} d_k(|f(x_1, x_2, \dots, x_n)|).$$

For $f(x_1, x_2)$ a binary irreducible cubic form, Greaves [Gre70] proved that there exist real constants $c_1 > 0$ and c_2 depending only on $f(x_1, x_2)$, such that

$$D_2(f(x_1, x_2, \dots, x_n), x) = c_1 x^{\frac{2}{3}} \log x + c_2 x^{\frac{2}{3}} + O\left(x^{\frac{9}{14} + \epsilon}\right),$$

for any $\epsilon > 0$ as $x \rightarrow \infty$. If $f(x_1, x_2)$ is an irreducible quartic form, Daniel [Dan99] proved that

$$D_2(f(x_1, x_2, \dots, x_n), x) = c_1 x^{\frac{1}{2}} \log x + O\left(x^{\frac{1}{2}} \log \log x\right),$$

where $c_1 > 0$ is a constant depending only on f . More related work can be found in [LaZh21].

In 1998, Friedlander and Iwaniec [FrIw98] established the asymptotic formula for the distribution of prime values of $a^4 + b^2$. More precisely, they proved

$$\sum_{\substack{a=1 \\ a^2+b^4 \leq x}}^{\infty} \sum_{b=1}^{\infty} \Lambda(a^2 + b^4) = \frac{4}{\pi} \kappa x^{\frac{3}{4}} \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right),$$

where κ is some constant. Motivated from the above result, they [FrIw06] replaced Λ by d_k and established the following asymptotic formula for $d_3(a^6 + b^2)$:

$$\sum_{\substack{a=1 \\ a^2+b^4 \leq x \\ (a,b)=1}}^{\infty} \sum_{b=1}^{\infty} d_3(a^6 + b^2) = c \kappa x^{\frac{2}{3}} (\log x)^2 + O\left(x^{\frac{2}{3}} (\log x)^{\frac{7}{4}} (\log \log x)^{\frac{1}{2}}\right),$$

where c and κ are some constants. For irreducible binary definite quadratic forms f , Daniel [Dan97] proved an asymptotic formula for $D_k(f(x_1, x_2, \dots, x_n), x)$ for any $k \geq 2$. With the help of circle method, Sun and Zhang [SuZh16] proved that

$$\sum_{1 \leq a_1, a_2, a_3 \leq x} d_3(a_1^2 + a_2^2 + a_3^2) = c_1 x^3 (\log x)^2 + c_2 x^3 \log x + c_3 x^3 + O\left(x^{\frac{11}{4} + \epsilon}\right),$$

where c_1, c_2, c_3 are some constants and ϵ is any positive number. Finally, Blomer [Blo18] proved an asymptotic formula for $D_k(f(x_1, x_2, \dots, x_n), x)$ for any $k \geq 2$ where $f(x_1, x_2, \dots, x_n)$ is a form of degree k in $n = k - 1$ variables, coming from incomplete norm form. Very recently, Lapkova and Zhou [LaZh21] investigated the average sum of the k^{th} divisor function over values of quadratic polynomials f , not necessarily homogenous, in $n \geq 3$ variables for any $k \geq 2$.

Let \mathbb{K}_3 be a non-normal cubic extension of a rational field \mathbb{Q} given by an irreducible polynomial $f(x) = x^3 + ax^2 + bx + c$ of discriminant $D(< 0)$. It is natural to study the k^{th} integral power of Dedekind zeta function, i.e.,

$$(\zeta_{\mathbb{K}_3}(s))^k = \sum_{n=1}^{\infty} \frac{a_{k, \mathbb{K}_3}(n)}{n^s}, \quad (1.1)$$

for $\Re(s) > 1$, where $a_{k,\mathbb{K}_3}(n) = \sum_{n=n_1 n_2 \dots n_k} a_{\mathbb{K}_3}(n_1) a_{\mathbb{K}_3}(n_2) \dots a_{\mathbb{K}_3}(n_k)$.

In 2012, Lü [Lü13] was able to refine the previously known results (by Fomenko [Fom08]) of mean square and third power of $a_{\mathbb{K}_3}(n)$ to

$$\sum_{n \leq x} a_{\mathbb{K}_3}(n)^2 = a_1 x \log x + a_2 x + O\left(x^{\frac{23}{31} + \epsilon}\right)$$

where a_1 and a_2 are constants and

$$\sum_{n \leq x} a_{\mathbb{K}_3}(n)^3 = x P_3(\log x) + O\left(x^{\frac{235}{259} + \epsilon}\right),$$

where $P_3(t)$ is a suitable polynomial in t of degree 4.

In this paper, we will consider the average of the Dirichlet coefficients $a_{k,\mathbb{K}_3}(n)$ of the k^{th} power $(\zeta_{\mathbb{K}_3}(s))^k$ of the Dedekind zeta-function of a non-normal cubic extension \mathbb{K}_3 of \mathbb{Q} over the sequence of values of a binary quadratic form $F(x_1, x_2, \dots, x_6) = \sum_{k=1}^6 x_k^2$. More precisely, we are interested in the asymptotic formula for the sum

$$\sum_{\substack{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 \leq x \\ (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{Z}^6}} a_{k,\mathbb{K}_3}(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2),$$

for any integer $k \geq 1$, where

$$a_{k,\mathbb{K}_3}(n) = \sum_{n=n_1 n_2 \dots n_k} a_{\mathbb{K}_3}(n_1) a_{\mathbb{K}_3}(n_2) \dots a_{\mathbb{K}_3}(n_k).$$

Note that, $a_{1,\mathbb{K}_3}(n) = a_{\mathbb{K}_3}(n)$.

First, we make the following remark.

Remark 1. Let $|t| \geq 1$ and $\epsilon > 0$ be any small constant. Then we have

$$\zeta\left(\frac{1}{2} + it\right) \ll (|t| + 1)^{\mu + \epsilon},$$

where $\mu = \mu(\frac{1}{2})$. Moreover, Phragmén Lindelöf principle leads to

$$\zeta(\sigma + it) \ll (|t| + 1)^{2\mu(1-\sigma) + \epsilon}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $|t| \geq 1$.

For any integer $k \geq 1$, writing,

$$\begin{aligned} \sum_{\substack{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 \leq x \\ (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{Z}^6}} a_{k,\mathbb{K}_3}(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2) &= \sum_{n \leq x} a_{k,\mathbb{K}_3}(n) \sum_{\substack{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 \leq x \\ (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{Z}^6}} 1 \\ &= \sum_{n \leq x} a_{k,\mathbb{K}_3}(n) r_6(n) \\ &= M_{k,\mathbb{K}_3}(x) + \tilde{E}_{k,\mathbb{K}_3}(x), \end{aligned} \tag{1.2}$$

where $M_{k,\mathbb{K}_3}(x)$ is the main term which is of the form $x^3 P_{k-1}(\log x)$, where $P_{k-1}(t)$ is a polynomial in t of degree $k - 1$. We prove the following theorem.

Theorem 1. *Let $\epsilon > 0$ (be any small constant) and define (for $k \geq 1$) $\tilde{\lambda}_k = \max(\lambda_k, \lambda'_k)$, where λ_k, λ'_k are defined in Theorems 2, 3 resp. Then we have for any integer $k \geq 1$,*

$$\tilde{E}_{k, \mathbb{K}_3}(x) \ll x^{3 - \frac{1}{2(1+\tilde{\lambda}_k)} + 3k\epsilon}.$$

To prove Theorem 1, first, we demonstrate the following theorems:

Theorem 2. *Let $\epsilon > 0$ (be any small constant) and define $\lambda_1 = 3\epsilon$, $\lambda_2 = \min(2\mu, \frac{1}{4})$, $\lambda_3 = \min(\mu + \frac{1}{2}, \frac{5}{8})$, $\lambda_4 = \min(2\mu + \frac{3}{4}, 1)$, $\lambda_5 = \min(3\mu + 1, \frac{3}{2})$ and $\lambda_k = \mu(k - 6) + \frac{k}{3}$ for $k \geq 6$. Then we have for any integer $k \geq 1$,*

$$E_{k, \mathbb{K}_3}(x) \ll x^{3 - \frac{1}{2(1+\lambda_k)} + 3k\epsilon},$$

where

$$\sum_{n \leq x} a_{k, \mathbb{K}_3}(n) l_1(n) = M_{k, \mathbb{K}_3}(x) + E_{k, \mathbb{K}_3}(x),$$

and $l_1(n)$ is defined in Section 2.

Remark 2. From [Bou17] of Bourgain, we can very well take $\mu = \frac{13}{84}$. Thus, the theorem is unconditional with $\mu = \frac{13}{84}$.

Theorem 3. *Let $\epsilon > 0$ (be any small constant) and define $\lambda'_1 = 3\epsilon$, $\lambda'_2 = \frac{1}{4}$, and $\lambda'_k = \frac{3k-5}{6}$ for $k \geq 3$. Then we have for any integer $k \geq 1$,*

$$E'_{k, \mathbb{K}_3}(x) \ll x^{3 - \frac{1}{2(1+\lambda'_k)} + 3k\epsilon},$$

where

$$\sum_{n \leq x} a_{k, \mathbb{K}_3}(n) v_1(n) = E'_{k, \mathbb{K}_3}(x),$$

and $v_1(n)$ is defined in Section 2.

From (1.2), one can easily see that the proof of Theorem 1 follows from the proof of Theorems 2 and 3.

2. Preliminaries and some important lemmas

Let $r_k(n) := \#\{(n_1, n_2, \dots, n_k) \in \mathbb{Z}^k : n_1^2 + n_2^2 + \dots + n_k^2 = n\}$ allowing zeros, distinguishing signs, and order. We will be concerned with the function $r_6(n)$.

Lemma 1. *For any positive integer n , we have*

$$r_6(n) = 16 \sum_{d|n} \chi(d') d^2 - 4 \sum_{d|n} \chi(d) d^2, \quad (2.3)$$

where $dd' = n$, and χ is the non-principal Dirichlet character modulo 4, i.e.,

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{4} \\ -1 & \text{if } n \equiv -1 \pmod{4} \\ 0 & \text{if } n \equiv 0 \pmod{2} \end{cases}.$$

Proof. See, for instance, Lemma 1 of [ShSa22c].

We can reframe the equation (2.3) as

$$\begin{aligned} r_6(n) &= 16 \sum_{d|n} \chi(d) \frac{n^2}{d^2} - 4 \sum_{d|n} \chi(d) d^2 \\ &=: 16l(n) - 4v(n). \end{aligned}$$

We write $l_1(n) = 16l(n)$, and $v_1(n) = 4v(n)$.

The functions $\chi(d)$ and $\frac{n^2}{d^2}$ are completely multiplicative functions. This implies that $\chi(d) \frac{n^2}{d^2}$ is multiplicative. If $g(d)$ is any multiplicative function, then $\sum_{d|n} g(d)$ is also multiplicative. Therefore, $l(n)$ is a multiplicative function. Similarly, $v(n)$ is also multiplicative.

Note that

$$\begin{aligned} l(p) &= p^2 + \chi(p), \\ l(p^2) &= p^4 + p^2\chi(p) + \chi(p^2), \end{aligned}$$

and

$$\begin{aligned} v(p) &= 1 + p^2\chi(p), \\ v(p^2) &= 1 + p^2\chi(p) + p^4\chi(p^2). \end{aligned}$$

Lemma 2. ([Lü13]) For $\Re(s) > 1$, we have

$$\zeta_{\mathbb{K}_3}(s) = \zeta(s)L(s, f),$$

where f is a holomorphic cusp form of weight 1 with respect to the congruence group $\Gamma_0(|D|)$ and $D(< 0)$ be the discriminant of $f(x) = x^3 + ax^2 + bx + c$.

From Lemma 2, we can write

$$a_{\mathbb{K}_3}(n) = \sum_{d|n} \lambda_f(d).$$

Also, note that

$$a_{\mathbb{K}_3}(p) = 1 + \lambda_f(p).$$

Lemma 3. For any $\epsilon > 0$, we have

$$\int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \sim \frac{T(\log T)^4}{2\pi} \quad (2.4)$$

and

$$\int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{12} dt \ll T^{2+\epsilon} \quad (2.5)$$

uniformly for $T \geq 1$.

Proof. For the proof of (2.4) see (Theorem 5.1 of [Ivi12]), and (2.5) result is due to Heath-Brown [Hea78].

Lemma 4. ([Go82]) For any $\epsilon > 0$, we have

$$\int_1^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 dt \ll T \log T,$$

uniformly for $T \geq 1$ and

$$L(\sigma + it, \chi) \ll_{\epsilon} (1 + |t|)^{\frac{1}{3}(1-\sigma)+\epsilon},$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$, and $|t| \geq t_0$ (where t_0 is sufficiently large).

Lemma 5. ([Rama74]) For any $\epsilon > 0$, we have

$$\int_1^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \ll T^{1+\epsilon},$$

uniformly for $T \geq 1$.

Lemma 6. For any $\epsilon > 0$ and for any $T \geq 1$ uniformly, we have

$$\int_1^T \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \sim cT \log T \quad (2.6)$$

and

$$\int_1^T \left| L\left(\frac{1}{2} + it, f\right) \right|^6 dt \ll T^{2+\epsilon}. \quad (2.7)$$

Proof. Proofs of (2.6) and (2.7) follow by A. Good [Go82] and Jutila [Jut87], respectively.

Lemma 7. For any $\epsilon > 0$, we have

$$L(\sigma + it, f) \ll (1 + |t|)^{\max(\frac{2(1-\sigma)}{3}, 0) + \epsilon}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$, $|t| \geq 1$.

Proof. Proof follows from a result of A. Good [Go82] on using maximum-modulus principle to a suitable function.

Lemma 8. Let f be defined as in Lemma 2 and $a_{k, \mathbb{K}_3}(n)$ be defined as in equation (1.1). If

$$F_k(s) = \sum_{n=1}^{\infty} \frac{a_{k, \mathbb{K}_3}(n) l(n)}{n^s},$$

for $\Re(s) > 3$, then

$$F_k(s) = G_k(s) H_k(s),$$

where

$$G_k(s) = \zeta(s-2)^k L(s, \chi)^k L(s-2, f)^k L(s, f \otimes \chi)^k,$$

and χ is the non-principal character modulo 4. Here, $H_k(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $H_k(s) \neq 0$ on $\Re(s) = 3$.

Proof. We observe that $a_{k, \mathbb{K}_3}(n)l(n)$ is multiplicative, and hence

$$F_k(s) = \prod_p \left(1 + \frac{a_{k, \mathbb{K}_3}(p)l(p)}{p^s} + \dots + \frac{a_{k, \mathbb{K}_3}(p^m)l(p^m)}{p^{ms}} + \dots \right).$$

Note that

$$\begin{aligned} a_{k, \mathbb{K}_3}(n)l(p) &= ka_{\mathbb{K}_3}(p)l(p) \\ &= k(1 + \lambda_f(p))(p^2 + \chi(p)) \\ &= kp^2 + k\chi(p) + kp^2\lambda_f(p) + k\lambda_f(p)\chi(p) \\ &=: b(p). \end{aligned}$$

From the structure of $b(p)$, we define the coefficients $b(n)$ as

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \zeta(s-2)^k L(s, \chi)^k L(s-2, f)^k L(s, f \otimes \chi)^k,$$

which is absolutely convergent in $\Re(s) > 3$. We also note that

$$\begin{aligned} \prod_p \left(1 + \frac{b(p)}{p^s} + \dots + \frac{b(p^m)}{p^{ms}} + \dots \right) \\ = \zeta(s-2)^k L(s, \chi)^k L(s-2, f)^k L(s, f \otimes \chi)^k \\ =: G_k(s), \end{aligned}$$

for $\Re(s) > 3$. Observe that $b(n) \ll_{\epsilon} n^{2+\epsilon}$ for any small positive constant ϵ .

Now, we note that in the half plane $\Re(s) \geq 3 + 2\epsilon$, we have

$$\begin{aligned} \left| \frac{b(p)}{p^s} + \frac{b(p^2)}{p^{2s}} + \dots + \frac{b(p^m)}{p^{ms}} + \dots \right| &\ll \sum_{m=1}^{\infty} \frac{p^{(2+\epsilon)m}}{p^{ms}} \\ &\leq \sum_{m=1}^{\infty} \frac{p^{(2+\epsilon)m}}{p^{(3+2\epsilon)m}} \\ &= \sum_{m=1}^{\infty} \frac{1}{p^{(1+\epsilon)m}} \\ &= \frac{1}{p^{1+\epsilon}} \\ &= \frac{1}{1 - \frac{1}{p^{1+\epsilon}}} \\ &= \frac{1}{p^{1+\epsilon} - 1} \\ &< 1. \end{aligned}$$

Let us write

$$A = \frac{a_{k, \mathbb{K}_3}(p)l(p)}{p^s} + \dots + \frac{a_{k, \mathbb{K}_3}(p^m)l(p^m)}{p^{ms}} + \dots, \quad \text{and} \quad B = \frac{b(p)}{p^s} + \dots + \frac{b(p^m)}{p^{ms}} + \dots.$$

From the above calculations, we observe that $|B| < 1$ in $\Re(s) \geq 3 + 2\epsilon$.

We note that in the half plane $\Re(s) \geq 3 + 2\epsilon$, we have

$$\begin{aligned} \frac{1+A}{1+B} &= (1+A)(1-B+B^2-B^3+\dots) \\ &= 1+A-B-AB+\text{higher terms} \\ &= 1 + \frac{a_{k,\mathbb{K}_3}(p^2)l(p^2) - b(p^2)}{p^{2s}} + \dots + \frac{c_m(p^m)}{p^{ms}} + \dots, \end{aligned}$$

with $c_m(n) \ll_\epsilon n^{2+\epsilon}$. So, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\prod_p \left(\frac{1+A}{1+B} \right) = \prod_p \left(1 + \frac{a_{k,\mathbb{K}_3}(p^2)l(p^2) - b(p^2)}{p^{2s}} + \dots + \frac{c_m(p^m)}{p^{ms}} + \dots \right) \ll_\epsilon 1.$$

Thus, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned} H_k(s) &:= \frac{F_k(s)}{G_k(s)} \\ &= \prod_p \left(\frac{1+A}{1+B} \right) \\ &\ll_\epsilon 1, \end{aligned}$$

and also $H_k(s) \neq 0$ on $\Re(s) = 3$.

Lemma 9. *Let f be defined as in Lemma 2 and $a_{k,\mathbb{K}_3}(n)$ be defined as in equation (1.1). If*

$$\tilde{F}_k(s) = \sum_{n=1}^{\infty} \frac{a_{k,\mathbb{K}_3}(n)v(n)}{n^s},$$

for $\Re(s) > 3$, then

$$\tilde{F}_k(s) = \tilde{G}_k(s)\tilde{H}_k(s),$$

where

$$\tilde{G}_k(s) = \zeta(s)^k L(s-2, \chi)^k L(s, f)^k L(s-2, f \otimes \chi)^k,$$

and χ is the non-principal character modulo 4. Here, $\tilde{H}_k(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $\tilde{H}_k(s) \neq 0$ on $\Re(s) = 3$.

Proof. The proof of Lemma 9 follows along similar lines as the proof of Lemma 8.

Lemma 10. ([JiLü14]) *Let χ be a primitive character modulo q and $\mathfrak{L}_{m,n}^d(s, \chi)$ be a general L -function of degree $2A$. For any $\epsilon > 0$, we have*

$$\int_T^{2T} \left| \mathfrak{L}_{m,n}^d(\sigma + it, \chi) \right|^2 dt \ll (qT)^{2A(1-\sigma)+\epsilon},$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$, and $T \geq 1$. Also,

$$\mathfrak{L}_{m,n}^d(\sigma + it, \chi) \ll (q(1+|t|))^{\max\{A(1-\sigma), 0\}+\epsilon},$$

uniformly for $-\epsilon \leq \sigma \leq 1 + \epsilon$.

3. Proof of Theorem 2

Let $k \geq 1$ be an integer. Firstly, we consider the sum $\sum_{n \leq x} a_{k, \mathbb{K}_3}(n) l_1(n)$. We begin by applying Perron's formula (see [GrSo14, Chapter 2.4]) to $F_k(s)$ with $\eta = 3 + \epsilon$ and $10 \leq T \leq x$. Thus, we have

$$\begin{aligned} \sum_{n \leq x} a_{k, \mathbb{K}_3}(n) l_1(n) &= 16 \sum_{n \leq x} a_{k, \mathbb{K}_3}(n) l(n) \\ &= \frac{16}{2\pi i} \int_{\eta-iT}^{\eta+iT} F_k(s) \frac{x^s}{s} ds + O\left(\frac{x^{3+3\epsilon}}{T}\right). \end{aligned}$$

We move the line of integration to $\Re(s) = \frac{5}{2} + \epsilon$. By Cauchy's residue theorem there is only one pole at $s = 3$ of order k , coming from the factor $\zeta(s-2)^k$.

So, we obtain

$$\begin{aligned} \sum_{n \leq x} a_{k, \mathbb{K}_3}(n) l_1(n) &= \operatorname{Res}_{s=3} \left\{ F_k(s) \frac{x^s}{s} \right\} + \frac{16}{2\pi i} \left\{ \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} + \int_{3+\epsilon-iT}^{\frac{5}{2}+\epsilon-iT} + \int_{\frac{5}{2}+\epsilon+iT}^{3+\epsilon+iT} \right\} F_k(s) \frac{x^s}{s} ds \\ &\quad + O\left(\frac{x^{3+3\epsilon}}{T}\right) \\ &=: x^3 P_{k-1}(\log x) + \frac{16}{2\pi i} (J_1(k) + J_2(k) + J_3(k)) + O\left(\frac{x^{3+3\epsilon}}{T}\right), \end{aligned}$$

where $P_{k-1}(t)$ is a polynomial in t of degree $k-1$.

Note that the horizontal lines ($J_2(k)$ and $J_3(k)$) contribute (for any fixed integer $k \geq 1$), using Lemma 6, Lemma 7 and Remark 1

$$\begin{aligned} J_2(k) + J_3(k) &\ll (x^2)^{\max_{\frac{1}{2}+\epsilon \leq \sigma \leq 1+\epsilon}} x^\sigma T^{(2k\mu + \frac{2k}{3})(1-\sigma) + \epsilon} T^{-1} \\ &\ll (x^{2+\epsilon})^{\max_{\frac{1}{2}+\epsilon \leq \sigma \leq 1+\epsilon}} \left(\frac{x}{T^{2k\mu + \frac{2k}{3}}} \right)^\sigma T^{2k\mu + \frac{2k}{3} - 1 + \epsilon}. \end{aligned}$$

For any fixed $k, \mu (> 0)$, $\left(\frac{x}{T^{2k\mu + \frac{2k}{3}}} \right)^\sigma$ is monotonic as a function of σ for $\frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon$ and hence the maximum is attained at the extremities of the interval $[\frac{1}{2} + \epsilon, 1 + \epsilon]$. Thus,

$$J_2(k) + J_3(k) \ll \frac{x^{3+3\epsilon}}{T} + x^{\frac{5}{2}+3\epsilon} T^{\frac{1}{2}(2k\mu + \frac{2k}{3}) - 1}.$$

Vertical line contributions:

1. **For $k=1$:**

$$J_1(1) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_1(s) \frac{x^s}{s} ds.$$

Using Lemma 5, Lemma 6, Lemma 7 and Cauchy-Schwarz inequality,

$$\begin{aligned}
J_1(1) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta\left(\frac{1}{2} + it\right) \right| \left| L\left(\frac{1}{2} + it, f\right) \right| dt \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left(\int_{\frac{U}{2}}^U \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \right)^{\frac{1}{2}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\frac{1}{2}+\epsilon} U^{\frac{1}{2}+\epsilon} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} T^{3\epsilon},
\end{aligned}$$

which dominates over $J_2(1) + J_3(1)$.

2. For $k=2$:

$$J_1(2) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_2(s) \frac{x^s}{s} ds.$$

Using Lemma 7, Remark 1 and Lemma 5,

$$\begin{aligned}
J_1(2) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{2\mu+2\epsilon} U \log U \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} T^{2\mu+4\epsilon}.
\end{aligned}$$

Note that, by Lemma 5

$$\begin{aligned}
\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f\right) \right|^4 dt &\ll \left(\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f\right) \right|^6 dt \right)^{\frac{1}{2}} \\
&\ll U^{\frac{3}{2}+\epsilon}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
J_1(2) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left(\int_{\frac{U}{2}}^U \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \right)^{\frac{1}{2}} \left(\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f\right) \right|^4 dt \right)^{\frac{1}{2}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\frac{1}{2}+\epsilon} U^{\frac{1}{2}(\frac{3}{2}+\epsilon)} \right\} \text{ (using Lemma 2 and above observation)} \\
&\ll x^{\frac{5}{2}+\epsilon} T^{\frac{1}{4}+2\epsilon}.
\end{aligned}$$

Thus, we have

$$J_1(2) \ll x^{\frac{5}{2}+4\epsilon} T^{\min(2\mu, \frac{1}{4})},$$

which dominates over $J_2(2) + J_3(2)$.

3. For $k=3$:

$$J_1(3) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_3(s) \frac{x^s}{s} ds.$$

Using Lemma 2, Lemma 5, Cauchy-Schwarz Inequality and Remark 1,

$$\begin{aligned}
J_1(3) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^3 \left| L \left(\frac{1}{2} + it, f \right) \right|^3 dt \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{\mu+\epsilon} \left(\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^4 dt \right)^{\frac{1}{2}} \right. \\
&\quad \left. \times \left(\int_{\frac{U}{2}}^U \left| L \left(\frac{1}{2} + it, f \right) \right|^6 dt \right)^{\frac{1}{2}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\mu+\epsilon} U^{\frac{1}{2}(1+\epsilon)} U^{\frac{1}{2}(2+\epsilon)} \right\} \\
&\ll x^{\frac{5}{2}+4\epsilon} T^{\mu+\frac{1}{2}}.
\end{aligned}$$

Also, we have (using Lemma 2, Lemma 5 and above observation)

$$\begin{aligned}
J_1(3) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left(\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^{12} dt \right)^{\frac{1}{4}} \left(\int_{\frac{U}{2}}^U \left| L \left(\frac{1}{2} + it, f \right) \right|^4 dt \right)^{\frac{3}{4}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\frac{1}{2}+\epsilon} U^{\frac{3}{4}(\frac{3}{2}+\epsilon)} \right\} \\
&\ll x^{\frac{5}{2}+5\epsilon} T^{\frac{5}{8}}.
\end{aligned}$$

Thus, we have

$$J_1(3) \ll x^{\frac{5}{2}+5\epsilon} T^{\min(\mu+\frac{1}{2}, \frac{5}{8})},$$

which dominates over $J_2(3) + J_3(3)$.

4. **For $k=4$:** First we observe, (using Lemma 2 and Cauchy-Schwarz inequality)

$$\begin{aligned}
\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^8 dt &\ll \left(\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^4 dt \right)^{\frac{1}{2}} \left(\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^{12} dt \right)^{\frac{1}{2}} \\
&\ll U^{\frac{1}{2}(1+\epsilon)} U^{\frac{1}{2}(2+\epsilon)} \\
&\ll U^{\frac{3}{2}+\epsilon}.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^6 dt &\ll \left(\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^4 dt \right)^{\frac{1}{2}} \left(\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^8 dt \right)^{\frac{1}{2}} \\
&\ll U^{\frac{1}{2}(1+\epsilon)} T^{\frac{1}{2}(\frac{3}{2}+\epsilon)} \\
&\ll U^{\frac{5}{4}+\epsilon}.
\end{aligned}$$

Now,

$$J_1(4) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_4(s) \frac{x^s}{s} ds.$$

Using Lemma 2, Lemma 5, Hölder's inequality and Remark 1,

$$\begin{aligned}
J_1(4) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^4 \left| L \left(\frac{1}{2} + it, f \right) \right|^4 dt \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{2\mu+2\epsilon} \left(\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^6 dt \right)^{\frac{1}{3}} \right. \\
&\quad \left. \times \left(\int_{\frac{U}{2}}^U \left| L \left(\frac{1}{2} + it, f \right) \right|^6 dt \right)^{\frac{2}{3}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{2\mu+2\epsilon} U^{\frac{1}{3}(\frac{5}{4}+\epsilon)} U^{\frac{2}{3}(2+\epsilon)} \right\} \text{ (using above observation)} \\
&\ll x^{\frac{5}{2}+5\epsilon} T^{2\mu+\frac{3}{4}}.
\end{aligned}$$

Also, we have (using Lemma 7, Lemma 2, Lemma 5, and Hölder's inequality)

$$\begin{aligned}
J_1(4) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left(\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^{12} dt \right)^{\frac{1}{3}} \left(\int_{\frac{U}{2}}^U \left| L \left(\frac{1}{2} + it, f \right) \right|^6 dt \right)^{\frac{2}{3}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\frac{1}{3}(2+\epsilon)} U^{\frac{2}{3}(2+\epsilon)} \right\} \\
&\ll x^{\frac{5}{2}+5\epsilon} T.
\end{aligned}$$

Thus, we have

$$J_1(4) \ll x^{\frac{5}{2}+5\epsilon} T^{\min(2\mu+\frac{3}{4}, 1)},$$

which dominates over $J_2(4) + J_3(4)$.

5. For $k=5$:

$$J_1(5) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_5(s) \frac{x^s}{s} ds.$$

Using Lemma 2, Lemma 5, Hölder's inequality and Remark 1,

$$\begin{aligned}
J_1(5) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^5 \left| L \left(\frac{1}{2} + it, f \right) \right|^5 dt \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{3\mu+3\epsilon} \left(\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^{12} dt \right)^{\frac{1}{6}} \right. \\
&\quad \left. \times \left(\int_{\frac{U}{2}}^U \left| L \left(\frac{1}{2} + it, f \right) \right|^6 dt \right)^{\frac{5}{6}} \right\} \\
&\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{3\mu+3\epsilon} U^{\frac{1}{6}(2+\epsilon)} U^{\frac{5}{6}(2+\epsilon)} \right\} \\
&\ll x^{\frac{5}{2}+6\epsilon} T^{3\mu+1}.
\end{aligned}$$

Also, we have

$$\begin{aligned} J_1(5) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left(\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^{12} dt \right)^{\frac{5}{12}} \left(\int_{\frac{U}{2}}^U \left| L \left(\frac{1}{2} + it, f \right) \right|^{5 \cdot \frac{12}{7}} dt \right)^{\frac{7}{12}} \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{\frac{5}{12}(2+\epsilon)} U^{\frac{7}{12}(2+\epsilon)} U^{\frac{18}{7} \cdot \frac{1}{3} \cdot \frac{7}{12} + 2\epsilon} \right\} \\ &\ll x^{\frac{5}{2}+5\epsilon} T^{\frac{3}{2}}. \end{aligned}$$

Thus, we have

$$J_1(5) \ll x^{\frac{5}{2}+5\epsilon} T^{\min(3\mu+1, \frac{3}{2})},$$

which dominates over $J_2(5) + J_3(5)$.

6. For $k \geq 6$:

$$J_1(k) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} F_k(s) \frac{x^s}{s} ds.$$

Using Lemma 2, Cauchy-Schwarz Inequality, Lemma 5, Lemma 6, and Remark 1 we get

$$\begin{aligned} J_1(k) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^k \left| L \left(\frac{1}{2} + it, f \right) \right|^k dt \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{(k-6)(\mu+\epsilon)} U^{(k-3)(\frac{1}{3}+\epsilon)} \int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^6 \left| L \left(\frac{1}{2} + it, f \right) \right|^3 dt \right\} \\ &\ll x^{\frac{5}{2}+2k\epsilon} \left\{ \max_{1 \leq U \leq T} U^{\mu(k-6)+\frac{1}{3}(k-3)-1} \left(\int_{\frac{U}{2}}^U \left| \zeta \left(\frac{1}{2} + it \right) \right|^{12} dt \right)^{\frac{1}{2}} \right. \\ &\quad \left. \times \left(\int_{\frac{U}{2}}^U \left| L \left(\frac{1}{2} + it, f \right) \right|^6 dt \right)^{\frac{1}{2}} \right\} \\ &\ll x^{\frac{5}{2}+2k\epsilon} \left\{ \max_{1 \leq U \leq T} U^{\mu(k-6)+\frac{1}{3}(k-3)-1} U^{\frac{1}{2}(2+\epsilon)} U^{\frac{1}{2}(2+\epsilon)} \right\} \\ &\ll x^{\frac{5}{2}+3k\epsilon} T^{\mu(k-6)+\frac{k}{3}}. \end{aligned}$$

Define

$$\lambda_k := \mu(k-6) + \frac{k}{3},$$

for $k \geq 6$, then

$$J_1(k) \ll x^{\frac{5}{2}+3k\epsilon} T^{\lambda_k}.$$

Therefore, we have (for $k \geq 1$)

$$\sum_{n \leq x} a_{k, \mathbb{K}_3}(n) l_1(n) = x^3 P_{k-1}(\log x) + E_{k, \mathbb{K}_3}(x),$$

where $E_{k, \mathbb{K}_3}(x) \ll \frac{x^{3+3\epsilon}}{T} + x^{\frac{5}{2}+3k\epsilon} T^{\lambda_k}$. We choose T such that $\frac{x^3}{T} \sim x^{\frac{5}{2}} T^{\lambda_k}$, i.e., $T^{1+\lambda_k} \sim x^{\frac{1}{2}}$, i.e., $T \sim x^{\frac{1}{2(1+\lambda_k)}}$.

So finally, we have

$$E_{k, \mathbb{K}_3}(x) \ll x^{3-\frac{1}{2(1+\lambda_k)}+3k\epsilon}.$$

This proves the theorem.

4. Proof of Theorem 3

Let $k \geq 1$ be an integer. Now, we consider the sum $\sum_{n \leq x} a_{k, \mathbb{K}_3}(n) v_1(n)$. We begin by applying Perron's formula (see [GrSo14, Chapter 2.4]) to $\tilde{F}_k(s)$ with $\eta = 3 + \epsilon$ and $10 \leq T \leq x$. Thus, we have

$$\begin{aligned} \sum_{n \leq x} a_{k, \mathbb{K}_3}(n) v_1(n) &= 4 \sum_{n \leq x} a_{k, \mathbb{K}_3}(n) v(n) \\ &= \frac{4}{2\pi i} \int_{\eta-iT}^{\eta+iT} \tilde{F}_k(s) \frac{x^s}{s} ds + O\left(\frac{x^{3+3\epsilon}}{T}\right). \end{aligned}$$

We move the line of integration to $\Re(s) = \frac{5}{2} + \epsilon$. There is no singularity in the rectangle obtained and the function $\tilde{F}_k(s) \frac{x^s}{s}$ is analytic in this region. Thus, using Cauchy's theorem for rectangles pertaining to analytic functions, we get

$$\begin{aligned} \sum_{n \leq x} a_{k, \mathbb{K}_3}(n) v_1(n) &= \operatorname{Res}_{s=3} \left\{ \tilde{F}_k(s) \frac{x^s}{s} \right\} + \frac{4}{2\pi i} \left\{ \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} + \int_{3+\epsilon-iT}^{\frac{5}{2}+\epsilon-iT} + \int_{\frac{5}{2}+\epsilon+iT}^{3+\epsilon+iT} \right\} \tilde{F}_k(s) \frac{x^s}{s} ds \\ &\quad + O\left(\frac{x^{3+3\epsilon}}{T}\right) \\ &=: \frac{4}{2\pi i} (J'_1(k) + J'_2(k) + J'_3(k)) + O\left(\frac{x^{3+3\epsilon}}{T}\right). \end{aligned}$$

Note that the horizontal lines ($J'_2(k)$ and $J'_3(k)$) contribute (for any fixed integer $k \geq 1$), using Lemma 9, Lemma 10 and Lemma 6

$$\begin{aligned} J'_2(k) + J'_3(k) &\ll (x^2) \max_{\frac{1}{2}+\epsilon \leq \sigma \leq 1+\epsilon} x^\sigma T^{(\frac{k}{3} + \frac{2k}{3})(1-\sigma) + \epsilon} T^{-1} \\ &\ll (x^{2+\epsilon}) \max_{\frac{1}{2}+\epsilon \leq \sigma \leq 1+\epsilon} \left(\frac{x}{T^k}\right)^\sigma T^{k-1+\epsilon}. \end{aligned}$$

For any fixed k , $\mu(> 0)$, $\left(\frac{x}{T^k}\right)^\sigma$ is monotonic as a function of σ for $\frac{1}{2} + \epsilon \leq \sigma \leq 1 + \epsilon$ and hence the maximum is attained at the extremities of the interval $[\frac{1}{2} + \epsilon, 1 + \epsilon]$. Thus,

$$J'_2(k) + J'_3(k) \ll \frac{x^{3+3\epsilon}}{T} + x^{\frac{5}{2}+3\epsilon} T^{\frac{k}{2}-1}.$$

Vertical line contributions:

1. For $k=1$:

$$J'_1(1) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} \tilde{F}_1(s) \frac{x^s}{s} ds.$$

Using Lemma 9, Lemma 3, Lemma 10 and Cauchy-Schwarz inequality,

$$\begin{aligned} J'_1(1) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right| \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right| dt \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left(\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^2 dt \right)^{\frac{1}{2}} \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\frac{1}{2}+\epsilon} U^{\frac{1}{2}+\epsilon} \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} T^{3\epsilon}, \end{aligned}$$

which dominates over $J'_2(1) + J'_3(1)$.

2. For $k=2$

$$J'_1(2) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} \tilde{F}_2(s) \frac{x^s}{s} ds.$$

Note that, by Lemma 4

$$\begin{aligned} \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^4 dt &\ll \left(\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^6 dt \right)^{\frac{1}{2}} \\ &\ll U^{\frac{3}{2}+\epsilon}. \end{aligned}$$

Using Lemma 4 and Lemma 10,

$$\begin{aligned} J'_1(2) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^2 dt \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \left(\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \right)^{\frac{1}{2}} \left(\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^4 dt \right)^{\frac{1}{2}} \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \log T \max_{1 \leq U \leq T} \left\{ \frac{1}{U} U^{\frac{1}{2}+\epsilon} U^{\frac{1}{2}(\frac{3}{2}+\epsilon)} \right\} \text{ (using above observation)} \\ &\ll x^{\frac{5}{2}+\epsilon} T^{\frac{1}{4}+2\epsilon}, \end{aligned}$$

which dominates over $J'_2(2) + J'_3(2)$.

3. For $k \geq 3$:

$$J'_1(k) := \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} \tilde{F}_k(s) \frac{x^s}{s} ds.$$

Using Lemma 8, Cauchy-Schwarz Inequality, Lemma 4, Lemma 5, and Lemma 6 we get

$$\begin{aligned} J'_1(k) &\ll x^{\frac{5}{2}+\epsilon} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right|^k \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^k dt \right\} \\ &\ll x^{\frac{5}{2}+\epsilon} \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{(k-2)(1-\sigma)} U^{\frac{2}{3}(k-3)(1-\sigma)} \int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^3 dt \right\} \\ &\ll x^{\frac{5}{2}+2k\epsilon} \left\{ \max_{1 \leq U \leq T} U^{\frac{k-2}{6} + \frac{k-3}{3} - 1} \left(\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \right)^{\frac{1}{2}} \right. \\ &\quad \left. \times \left(\int_{\frac{U}{2}}^U \left| L\left(\frac{1}{2} + it, f \otimes \chi\right) \right|^6 dt \right)^{\frac{1}{2}} \right\} \\ &\ll x^{\frac{5}{2}+2k\epsilon} \left\{ \max_{1 \leq U \leq T} U^{\frac{k-2}{6} + \frac{k-3}{3} - 1} U^{\frac{1}{2}(1+\epsilon)} U^{\frac{1}{2}(2+\epsilon)} \right\} \\ &\ll x^{\frac{5}{2}+3k\epsilon} T^{\frac{k}{2} - \frac{5}{6}}, \end{aligned}$$

which dominates over $J'_2(k) + J'_3(k)$.

Therefore, we have (for $k \geq 1$)

$$\sum_{n \leq x} a_{k, \mathbb{K}_3}(n) v_1(n) = E'_{k, \mathbb{K}_3}(x),$$

where $E'_{k, \mathbb{K}_3}(x) \ll \frac{x^{3+3\epsilon}}{T} + x^{\frac{5}{2}+3k\epsilon} T^{\lambda'_k}$. We choose T such that $\frac{x^3}{T} \sim x^{\frac{5}{2}} T^{\lambda'_k}$, i.e., $T^{1+\lambda'_k} \sim x^{\frac{1}{2}}$, i.e., $T \sim x^{\frac{1}{2(1+\lambda'_k)}}$.

So finally, we have

$$E'_{k, \mathbb{K}_3}(x) \ll x^{3 - \frac{1}{2(1+\lambda'_k)} + 3k\epsilon}.$$

This proves the theorem.

Acknowledgements. The first author, Anubhav Sharma, wishes to express his gratitude to IIT Delhi IRD project MI02455 for its financial support. The authors are grateful to the anonymous referee for carefully reading the manuscript and fruitful comments.

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