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► **To cite this version:**

Lalit Vaishya. Identities among some combinatorial objects involving special values of multiple zeta functions. Hardy-Ramanujan Journal, 2024, 46, pp.42 – 50. hal-04443007v1

**HAL Id: hal-04443007**

**<https://hal.science/hal-04443007v1>**

Submitted on 7 Feb 2024 (v1), last revised 11 Feb 2024 (v2)

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# Identities among some combinatorial objects involving special values of multiple zeta functions

Lalit Vaishya

**Abstract.** In the article, we establish some identities involving special values of multiple zeta functions among the counting functions of number of representations of an integer by a linear combination of figurate numbers such as triangular numbers, square numbers, pentagonal numbers, etc. More precisely, we provide our result for  $\delta_k(n)$ ,  $r_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ), the number of representations of  $n$  as a sum of  $k$ -triangular numbers, as a sum of  $k$ -square numbers and as a sum of  $k$ -higher figurate numbers (for a fixed  $a \geq 3$ ), respectively. Moreover, these identities also occur when one of  $\delta_k(n)$ ,  $r_k(n)$  and  $\mathcal{N}_k^a(n)$  is replaced by the  $k$ -colored partition functions.

**Keywords.**  $q$ -series,  $k$ -coloured partition function, Trigonometric functions, Integer composition.

**2010 Mathematics Subject Classification.** Primary 33B10, 40B05, 40C15, Secondary 17A42

## 1. Preliminaries

We define the  $q$ -Pochhammer symbol, also known as the  $q$ -shifted factorial, given as the product of the form

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k) = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}), \quad (1.1)$$

with  $(a; q)_0 = 1$ . It is a  $q$ -analogue of the Pochhammer symbol  $(x)_n = x(x + 1) \cdots (x + n - 1)$  for  $x \in \mathbb{R}^+$  and  $n \in \mathbb{N}$ , in the sense that

$$\lim_{q \rightarrow 1} \frac{(q^x; q)_n}{(1 - q)^n} = (x)_n. \quad (1.2)$$

The  $q$ -Pochhammer symbol is a major building block in the theory of hypergeometric series and the theory of integer partitions. The  $q$ -Pochhammer symbol can be extended to an infinite product given by

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k). \quad (1.3)$$

Moreover, for  $a = q$ ,  $(a; q)_\infty$  is the Dedekind eta function  $\eta(\tau)$  upto a multiple of  $q$ -power, where  $q = e^{2\pi i \tau}$  and  $\tau \in \mathbb{H}$  (complex upper half-plane), i.e.,  $\eta(\tau) = q^{1/24} (q; q)_\infty$ . A product of the form  $\prod_{d|N} \eta^{r_d}(\tau)$ ; with  $r_d \in \mathbb{Z}$ , is commonly known as a eta-quotient of level  $N$  and weight  $\frac{1}{2} \sum_{d|N} r_d$ , in the theory of modular forms. These are the building blocks for finding the formulas for  $r_k(n)$ ,  $\delta_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ) (See [RV23]).

For a fixed integer  $a \geq 1$ , we define the  $n^{\text{th}}$ -figurate number by  $f_a(n) := \frac{an^2 + (a-2)n}{2}$ , following Ono-Robins-Wahl [ORW95]. Note that the function  $f_a(n)$  denotes the  $n^{\text{th}}$ -triangular number (resp. square number and pentagonal number) when  $a = 1$  (resp. 2 and 3). In the literature, the  $n^{\text{th}}$ -triangular number is denoted by  $T_n$ , and for a fixed  $a \geq 3$ , the integer  $f_a(n)$  is known as the  $n^{\text{th}}$ -higher figurate number. These functions are associated with counting the number of vertices of some

geometric objects. We denote the generating functions for these figurate numbers as  $\Psi(\tau)$  (for triangular numbers),  $\theta(\tau)$  (for square numbers) and  $\Phi_a(\tau)$  (for  $a \geq 3$ ), and they are given respectively by

$$\theta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2}, \quad \Psi(\tau) := \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}, \quad \text{and} \quad \text{for } a \geq 3, \quad \Phi_a(\tau) := \sum_{n \in \mathbb{Z}} q^{f_a(n)}. \quad (1.4)$$

where  $q \in \mathbb{H}$ . It is known from [ORW95, Proposition 1, Theorem 10] that the generating function for the figurate numbers defined in (1.4) are explicitly obtained from Jacobi triple product identity given by

$$\theta(\tau) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^5}{(1 - q^n)^2(1 - q^{4n})^2} = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2(q^4; q^4)_{\infty}^2}, \quad \Psi(\tau) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^n)} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}} \quad (1.5)$$

and for  $a \geq 3$ ,

$$\Phi_a(\tau) = \prod_{n=1}^{\infty} (1 - q^{an})(1 - q^{an-1})(1 - q^{an-(a-1)}) = (q^a; q^a)_{\infty}(q^{-1}; q^a)_{\infty}(q^{-(a-1)}; q^a)_{\infty}. \quad (1.6)$$

Let  $\delta_k(n)$ ,  $r_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ) denote the number of representations of  $n$  as a sum of  $k$ -triangular numbers, as a sum of  $k$ -square numbers and as a sum of  $k$ -higher figurate numbers (for a fixed  $a \geq 3$ ) respectively, i.e.,

$$r_k(n) = \# \left\{ (x_1, x_2, \dots, x_k) \in \mathbb{Z}^k \mid n = \sum_{i=1}^k x_i^2 \right\},$$

$$\delta_k(n) = \# \left\{ (x_1, x_2, \dots, x_k) \in \mathbb{Z}^k \mid n = \sum_{i=1}^k \frac{x_i(x_i + 1)}{2} \right\}$$

and for a fixed integer  $a \geq 3$ ,  $\mathcal{N}_k^a(n) = \# \left\{ (x_1, x_2, \dots, x_k) \in \mathbb{Z}^k \mid n = \sum_{i=1}^k f_a(x_i) \right\}.$

It is easy to see that

$$\theta^k(\tau) = 1 + \sum_{n=1}^{\infty} r_k(n) q^n, \quad \Psi^k(\tau) = 1 + \sum_{n=1}^{\infty} \delta_k(n) q^n \quad \text{and} \quad \Phi_a^k(\tau) = 1 + \sum_{n=1}^{\infty} \mathcal{N}_k^a(n) q^n$$

for  $a \geq 3$ . For more details on these arithmetical functions, we refer to [ORW95] and [RV23].

For positive integers  $k_1, k_2, \dots, k_w \geq 2$ , we define the multiple zeta function as follows:

$$\zeta(k_1, k_2, \dots, k_w) = \sum_{m_1 < m_2 < \dots < m_w} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_w^{k_w}}$$

and  $\zeta^{odd}(k_1, k_2, \dots, k_w) = \sum_{\substack{m_1 < m_2 < \dots < m_w \\ m_i - \text{odd}}} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_w^{k_w}}.$  (1.7)

When  $k_i = k$  for all  $i$ , we simply denote it as  $\zeta_w(k)$  and  $\zeta_w^{odd}(k)$ , respectively. By convention, we write  $\zeta(k) = \zeta_1(k)$ ,  $\zeta_0(k) = 1$ ,  $\zeta^{odd}(k) = \zeta_1^{odd}(k)$  and  $\zeta_0^{odd}(k) = 1$ .

## Trigonometric identities

For each  $z \in \mathbb{C}$ , we have the following trigonometric function with their product expansion and Taylor series expansion given as follows

$$\begin{aligned} \sin z &= z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \\ \cos z &= \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2\pi^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \end{aligned} \quad (1.8)$$

$$\begin{aligned} \sin^2 z &= z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)^2 = \sum_{n=1}^{\infty} \left(\sum_{r=0}^{n-1} \binom{2n}{2r+1}\right) \frac{(-1)^{n-1}}{(2n)!} z^{2n} \\ \text{and } \cos^2 z &= \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2\pi^2}\right)^2 = \sum_{n=0}^{\infty} \left(\sum_{r=0}^n \binom{2n}{2r}\right) \frac{(-1)^n}{(2n)!} z^{2n}. \end{aligned} \quad (1.9)$$

Here,  $\binom{n}{r}$  denotes the number of ways to choose  $r$  objects from a set of  $n$  objects.

## 2. Identities for $\delta_k(n)$ , $r_k(n)$ and $\mathcal{N}_k^a(n)$

In this section, we establish certain identities for  $r_k(n)$ ,  $\delta_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ) which involves the special values of multiple zeta function. We make use of trigonometric functions to prove our results. Before stating our results, we define the set  $S(n)$  as a collection of all possible compositions of non-zero positive integers  $\{a_1, a_2, \dots, a_r\}$  such that  $a_1 + a_2 + \dots + a_r = n$ , i.e.,

$$S(n) = \{\{a_1, a_2, \dots, a_r\} \mid a_i > 0 \text{ for all } i, \text{ and } a_1 + a_2 + \dots + a_r = n\}.$$

For example, given  $n = 6$ , the elements  $\{3, 2, 1\}$ ,  $\{2, 3, 1\}$  and  $\{1, 2, 3\}$  are different integer compositions of  $S(n)$ .

Now, we present our results.

**Theorem 2.1.** *For a given positive integer  $n$ , we have the following identities for  $r_k(n)$ ,  $\delta_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ), respectively involving the special values of multiple zeta function  $\zeta_k(2)$  given by*

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} r_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} r_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-distinct}}} \frac{(-1)^\ell r_2(s_1) r_2(s_2) \cdots r_2(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{m \neq m_1, m_2, \dots, m_\ell} \left(1 - \frac{1}{m^2 \pi^2}\right), \end{aligned} \quad (2.10)$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \delta_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} \delta_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-distinct}}} \frac{(-1)^\ell \delta_2(s_1) \delta_2(s_2) \cdots \delta_2(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{m \neq m_1, m_2, \dots, m_\ell} \left(1 - \frac{1}{m^2 \pi^2}\right) \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \mathcal{N}_{2k}^a(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} \mathcal{N}_{2k}^a(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-distinct}}} \frac{(-1)^\ell \mathcal{N}_2^a(s_1) \mathcal{N}_2^a(s_2) \cdots \mathcal{N}_2^a(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{m \neq m_1, m_2, \dots, m_\ell} \left(1 - \frac{1}{m^2 \pi^2}\right). \end{aligned} \quad (2.12)$$

*Proof.* We prove the above result for  $r_k(\cdot)$  and other identities can be obtained following similar arguments. Let us consider the product formula of  $\sin z$  given by  $\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$ . To get the required result, we consider the product  $\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$  and expand it in two different ways by substituting the  $q$ -product of generating function for square numbers, i.e.,  $z = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^5}{(1 - q^n)^2(1 - q^{4n})^2}$  (given in (1.5)) and observe that  $z^k = \sum_{n=0}^{\infty} r_k(n)q^n$ . More precisely, the expansion of  $\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$  gives that

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{\pi^{2k}} \left( \sum_{\substack{n_1, n_2, \dots, n_k \\ n_i \text{-distinct}}} \frac{1}{n_1^2 n_2^2 \cdots n_k^2} \right) z^{2k} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} z^{2k}.$$

Now, we substitute  $z^{2k} = \sum_{n=0}^{\infty} r_{2k}(n)q^n$  and see that the constant term ( $q^0$ -th coefficient) and the coefficient of  $q^n$  ( $n > 0$ ) is given by;

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} r_{2k}(n), \quad (2.13)$$

respectively. On the other hand, we first substitute  $z^2 = \sum_{n=0}^{\infty} r_2(n)q^n$  in  $\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$  to get

$$\prod_{n=1}^{\infty} \left( \left(1 - \frac{1}{n^2\pi^2}\right) q^0 - \sum_{k=1}^{\infty} \frac{r_2(k)}{n^2\pi^2} q^k \right). \quad (2.14)$$

We expand the product in (2.14) using the Cauchy product of infinite series to obtain the coefficients of  $q^n$ . The coefficients of  $q^n$  in this process is given by

$$\sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-distinct}}} \frac{(-1)^\ell r_2(s_1) r_2(s_2) \cdots r_2(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{m \neq m_1, m_2, \dots, m_\ell} \left(1 - \frac{1}{m^2\pi^2}\right). \quad (2.15)$$

Since the  $q^n$ -coefficients given in (2.13) and (2.15) are the  $q^n$ -coefficients of the expansions of the product  $\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$  by putting the value of  $z = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^5}{(1 - q^n)^2(1 - q^{4n})^2}$ . Hence, we have the required identity. The first equality is obvious by the Taylor series expansion of  $\sin z$ . Thus, we have the required result.

**Theorem 2.2.** For a given positive integer  $n$ , we have the following identities for  $r_k(n)$ ,  $\delta_k(n)$  and  $N_k^a(n)$  (for a fixed  $a \geq 3$ ), respectively involving the special values of multiple zeta function  $\zeta_k^{\text{odd}}(2)$  given by

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} r_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k 4^k \zeta_k^{\text{odd}}(2)}{\pi^{2k}} r_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-odd \& distinct}}} \frac{(-4)^\ell r_2(s_1) r_2(s_2) \cdots r_2(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left(1 - \frac{1}{m^2\pi^2}\right), \end{aligned} \quad (2.16)$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \delta_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k 4^k \zeta_k^{odd}(2)}{\pi^{2k}} \delta_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-odd} \ \& \ \text{distinct}}} \frac{(-4)^\ell \delta_2(s_1) \delta_2(s_2) \cdots \delta_2(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left(1 - \frac{1}{m^2 \pi^2}\right) \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \mathcal{N}_{2k}^a(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k 4^k \zeta_k^{odd}(2)}{\pi^{2k}} \mathcal{N}_{2k}^a(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-odd} \ \& \ \text{distinct}}} \frac{(-4)^\ell \mathcal{N}_2^a(s_1) \mathcal{N}_2^a(s_2) \cdots \mathcal{N}_2^a(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left(1 - \frac{1}{m^2 \pi^2}\right). \end{aligned} \quad (2.18)$$

*Proof.* We use the same arguments to prove the above result for  $\delta_k(\cdot)$  and the other identities. Let us consider the product formula of  $\cos z$  given by  $\cos z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2 \pi^2}\right)$ . We expand the product  $\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2 \pi^2}\right)$  in two different ways by substituting the  $q$ -product of generating function for triangular numbers, i.e.,  $z = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)}$  and observe that  $z^k = \sum_{n=0}^{\infty} \delta_k(n) q^n$  to get the required result. More precisely, a simple expansion of  $\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2 \pi^2}\right)$  gives that

$$\begin{aligned} \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2 \pi^2}\right) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 4^k}{\pi^{2k}} \left( \sum_{\substack{n_1, n_2, \dots, n_k \\ n_i \text{-odd} \ \& \ \text{distinct}}} \frac{1}{n_1^2 n_2^2 \cdots n_k^2} \right) z^{2k} \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 4^k \zeta_k^{odd}(2)}{\pi^{2k}} z^{2k}. \end{aligned}$$

Now, we substitute  $z^{2k} = \sum_{n=0}^{\infty} \delta_{2k}(n) q^n$  and see that the coefficient of  $q^n$  is

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k 4^k \zeta_k^{odd}(2)}{\pi^{2k}} \delta_{2k}(n). \quad (2.19)$$

On the other hand, first we substitute  $z^2 = \sum_{n=0}^{\infty} \delta_2(n) q^n$  in  $\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2 \pi^2}\right)$  and then we expand the product to obtain the coefficients of  $q^n$ . The coefficients of  $q^n$  in this process is given by

$$\sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-odd} \ \& \ \text{distinct}}} \frac{(-4)^\ell \delta_2(s_1) \delta_2(s_2) \cdots \delta_2(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left(1 - \frac{1}{m^2 \pi^2}\right). \quad (2.20)$$

Hence, comparing the  $q^n$ -coefficients of the expansions of the product  $\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2 \pi^2}\right)$  (in two ways) by substituting  $z = \frac{(q^2; q^2)_\infty}{(q; q)_\infty}$ , we obtain the required result. The first equality is obvious from the Taylor series expansion of  $\cos z$ .

Following the same arguments and using the product and sum expression for  $\sin^2 z$  and  $\cos^2 z$  respectively (given in (1.9)), we have the following results.

**Theorem 2.3.** For a given positive integer  $n$ , we have the following identities for  $r_k(n)$ ,  $\delta_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ) respectively.

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \sum_{r=0}^{k-1} \binom{2k}{2r+1} \right) \frac{(-1)^{k-1}}{(2k)!} r_{2k}(n) \\ &= \sum_{f=1}^n r_2(f) \sum^* \left( \prod_{1 \leq e \leq \ell} \left( -\frac{2}{\pi^2 m_e^2} r_2(s_e) + \frac{1}{\pi^4 m_e^4} r_2(s_e) \right) \prod_{m \neq m_1, m_2, \dots, m_\ell} \left( 1 - \frac{2}{m^2 \pi^2} + \frac{1}{m^4 \pi^4} \right) \right), \end{aligned} \quad (2.21)$$

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \sum_{r=0}^{k-1} \binom{2k}{2r+1} \right) \frac{(-1)^{k-1}}{(2k)!} \delta_{2k}(n) \\ &= \sum_{f=1}^n \delta_2(f) \sum^* \left( \prod_{1 \leq e \leq \ell} \left( -\frac{2}{\pi^2 m_e^2} \delta_2(s_e) + \frac{1}{\pi^4 m_e^4} \delta_2(s_e) \right) \prod_{m \neq m_1, m_2, \dots, m_\ell} \left( 1 - \frac{2}{m^2 \pi^2} + \frac{1}{m^4 \pi^4} \right) \right), \end{aligned} \quad (2.22)$$

and

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \sum_{r=0}^{k-1} \binom{2k}{2r+1} \right) \frac{(-1)^{k-1}}{(2k)!} \mathcal{N}_{2k}^a(n) \\ &= \sum_{f=1}^n \mathcal{N}_2^a(f) \sum^* \left( \prod_{1 \leq e \leq \ell} \left( -\frac{2}{\pi^2 m_e^2} \mathcal{N}_2^a(s_e) + \frac{1}{\pi^4 m_e^4} \mathcal{N}_2^a(s_e) \right) \prod_{m \neq m_1, m_2, \dots, m_\ell} \left( 1 - \frac{2}{m^2 \pi^2} + \frac{1}{m^4 \pi^4} \right) \right). \end{aligned} \quad (2.23)$$

where  $\sum^*$  denotes the double summation  $\sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n-f)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-distinct}}}$ .

**Theorem 2.4.** For a given positive integer  $n$ , we have the following identities for  $r_k(n)$ ,  $\delta_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ) respectively.

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \sum_{r=0}^k \binom{2k}{2r} \right) \frac{(-1)^k}{(2k)!} r_{2k}(n) \\ &= \sum^\# \prod_{1 \leq e \leq \ell} \left( -\frac{2}{\pi^2 m_e^2} r_2(s_e) + \frac{1}{\pi^4 m_e^4} r_2(s_e) \right) \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left( 1 - \frac{2}{m^2 \pi^2} + \frac{1}{m^4 \pi^4} \right), \end{aligned} \quad (2.24)$$

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \sum_{r=0}^k \binom{2k}{2r} \right) \frac{(-1)^k}{(2k)!} \delta_{2k}(n) \\ &= \sum^\# \prod_{1 \leq e \leq \ell} \left( -\frac{2}{\pi^2 m_e^2} \delta_2(s_e) + \frac{1}{\pi^4 m_e^4} \delta_2(s_e) \right) \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left( 1 - \frac{2}{m^2 \pi^2} + \frac{1}{m^4 \pi^4} \right). \end{aligned} \quad (2.25)$$

and

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \sum_{r=0}^k \binom{2k}{2r} \right) \frac{(-1)^k}{(2k)!} \mathcal{N}_{2k}^a(n) \\ &= \sum^\# \left( \prod_{1 \leq e \leq \ell} \left( -\frac{2}{\pi^2 m_e^2} \mathcal{N}_2^a(s_e) + \frac{1}{\pi^4 m_e^4} \mathcal{N}_2^a(s_e) \right) \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left( 1 - \frac{2}{m^2 \pi^2} + \frac{1}{m^4 \pi^4} \right) \right). \end{aligned} \quad (2.26)$$

where  $\sum^\#$  denotes the summation  $\sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-odd \& distinct}}}$ .

### 3. Identities for $k$ -colored partition functions:

Let  $n$  be a positive integer. For each  $k \geq 1$ , let  $p_k(n)$  denote the number of  $k$ -colored partition of  $n$ . For  $k = 1$ , it is the usual partition function  $p(n)$ , first considered by Ramanujan. We define  $\overline{p}_k(n)$ , as the counting function for the number of  $k$ -colored over-partition of a positive integer  $n$ , and  $pd_k(n)$  denotes the number of  $k$ -colored partition of  $n$  into distinct parts, and  $po_k(n)$  denotes the number of  $k$ -colored partition of  $n$  into odd parts. The generating function for these  $k$ -colored partition functions is given in terms of  $q$ -product as follows.

$$\begin{aligned} \sum_{n=0}^{\infty} p_k(n)q^n &= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k} = (q; q)_{\infty}^{-k}, \\ \sum_{n=0}^{\infty} \overline{p}_k(n)q^n &= \prod_{n=1}^{\infty} \frac{(1+q^n)^k}{(1-q^n)^k} = \frac{(-q; q)_{\infty}^k}{(q; q)_{\infty}^k}, \\ \sum_{n=0}^{\infty} pd_k(n)q^n &= \prod_{n=1}^{\infty} (1+q^n)^k = (-q; q)_{\infty}^k, \\ \text{and } \sum_{n=0}^{\infty} po_k(n)q^n &= \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})^k} = (q^{-1}; q^2)_{\infty}^k. \end{aligned} \tag{3.27}$$

For details on the partition functions and their combinatorial interpretations, we refer to [And98]. Similar identities obtained in Theorem 2.1 and Theorem 2.2, can be given for these  $k$ -colored partition functions. The proof of these identities take place by substituting  $z = (-q; q)_{\infty}$  ( for  $pd_k(n)$ ),  $z = (q^{-1}; q^2)_{\infty}$  (for  $po_k(n)$ ),  $z = \frac{1}{(q; q)_{\infty}}$  ( for  $p_k(n)$ ) and  $z = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}}$  (for  $\overline{p}_k(n)$ ) in the expression of  $\sin z$  and  $\cos z$  and following the same arguments as in the proofs of Theorem 2.1 and Theorem 2.2. Below, we state these identities only for  $pd_k(n)$ , and identities for others can be stated analogously.

**Theorem 3.1.** *For a given positive integer  $n$ , we have the following identities for  $pd_k(n)$  involving the special values of multiple zeta function  $\zeta_k(2)$  given by*

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} pd_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} pd_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_{\ell}\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_{\ell} \\ m_i \text{-distinct}}} \frac{(-1)^{\ell} pd_2(s_1) pd_2(s_2) \cdots pd_2(s_{\ell})}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_{\ell}^2} \prod_{m \neq m_1, m_2, \dots, m_{\ell}} \left(1 - \frac{1}{m^2 \pi^2}\right). \end{aligned} \tag{3.28}$$

*These identities also valid if  $pd_k(n)$  is replaced by one of these  $k$ -colored partition functions among  $p_k(n)$ ,  $\overline{p}_k(n)$  and  $po_k(n)$ .*

**Theorem 3.2.** *For a given positive integer  $n$ , we have the following identities for  $pd_k(n)$  involving the special values of multiple zeta function  $\zeta_k^{\text{odd}}(2)$  given by*

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} pd_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k 4^k \zeta_k^{\text{odd}}(2)}{\pi^{2k}} pd_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_{\ell}\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_{\ell} \\ m_i \text{-odd \& distinct}}} \frac{(-4)^{\ell} pd_2(s_1) pd_2(s_2) \cdots pd_2(s_{\ell})}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_{\ell}^2} \prod_{\substack{m \neq m_1, m_2, \dots, m_{\ell} \\ m \text{-odd}}} \left(1 - \frac{1}{m^2 \pi^2}\right). \end{aligned} \tag{3.29}$$

*These identities also valid if  $pd_k(n)$  is replaced by one of these  $k$ -colored partition functions among  $p_k(n)$ ,  $\overline{p}_k(n)$  and  $po_k(n)$ .*

**Remark 3.1.** *The identities appearing in the Theorem 2.3 and Theorem 2.4 can be given for each of these  $k$ -colored partition functions  $p_k(n)$ ,  $\overline{p}_k(n)$ ,  $pd_k(n)$  and  $po_k(n)$  by replacing the arithmetical function  $r_k(n)$  by one of the above  $k$ -colored partition functions.*



### 3.A. $k$ -colored identities associated to Rogers-Ramanujan partition functions:

For a positive integer  $n$ , Let  $[\lambda_1, \lambda_2, \dots, \lambda_r]$  such that  $n = \lambda_1 + \lambda_2 + \dots + \lambda_r$ ,  $r \in \mathbb{N}$  and each  $\lambda_i \geq \lambda_{i+1} > 0$ , denote a partition of  $n$ . We define the following partition functions given by

$$\begin{aligned} RR(n) &:= \#\{\text{partiton } [\lambda_1, \lambda_2, \dots, \lambda_r] \text{ of } n \mid \lambda_i - \lambda_{i+1} \geq 2 \text{ for each } i\} \quad \text{and} \\ RS(n) &:= \#\{\text{partiton } [\lambda_1, \lambda_2, \dots, \lambda_r] \text{ of } n \mid \lambda_i - \lambda_{i+1} \geq 2, \text{ and } \lambda_i \geq 2 \text{ for each } i\}. \end{aligned} \quad (3.30)$$

The generating function for these is given by

$$\sum_{n=0}^{\infty} RR(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q, q)_n} \quad \text{and} \quad \sum_{n=0}^{\infty} RS(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q, q)_n}. \quad (3.31)$$

Let  $R1(n)$  denote the numbers of partition  $[\lambda_1, \lambda_2, \dots, \lambda_r]$  of  $n$  such that each  $\lambda_i \equiv \pm 1 \pmod{5}$ , and  $R2(n)$  denote the numbers of partition  $[\lambda_1, \lambda_2, \dots, \lambda_r]$  of  $n$  such that each  $\lambda_i \equiv \pm 2 \pmod{5}$ , respectively. The generating function of these is given by

$$\begin{aligned} \sum_{n=0}^{\infty} R1(n)q^n &= \prod_{m=1}^{\infty} \frac{1}{(1 - q^{5m-4})(1 - q^{5m-1})} = \frac{1}{(q; q^5)_{\infty}(q^4; q^5)_{\infty}} \\ \text{and} \quad \sum_{n=0}^{\infty} R2(n)q^n &= \prod_{m=1}^{\infty} \frac{1}{(1 - q^{5m-2})(1 - q^{5m-3})} = \frac{1}{(q^2; q^5)_{\infty}(q^3; q^5)_{\infty}}, \end{aligned} \quad (3.32)$$

respectively. Then, Rogers-Ramanujan identities say that

$$RR(n) = R1(n) \quad \text{and} \quad RS(n) = R2(n). \quad (3.33)$$

To prove the Rogers-Ramanujan identities, it is equivalent to show the following

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q, q)_n} = \frac{1}{(q; q^5)_{\infty}(q^4; q^5)_{\infty}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q, q)_n} = \frac{1}{(q^2; q^5)_{\infty}(q^3; q^5)_{\infty}}, \quad (3.34)$$

respectively. For details, we refer to [And98]. Now, we define the following  $k$ -colored Rogers-Ramanujan type partitioned functions as follows:  $R1_k(n)$  denote the numbers of partition  $[\lambda_1, \lambda_2, \dots, \lambda_r]$  of  $n$  such that each  $\lambda_i \equiv \pm 1 \pmod{5}$  and its part is colored by at most  $k$ -colours, and  $R2_k(n)$  denote the numbers of partition  $[\lambda_1, \lambda_2, \dots, \lambda_r]$  of  $n$  such that each  $\lambda_i \equiv \pm 2 \pmod{5}$  and its part is colored by at most  $k$ -colours, respectively. The generating functions are given by

$$\sum_{n=0}^{\infty} R1_k(n)q^n = \frac{1}{(q; q^5)_{\infty}^k (q^4; q^5)_{\infty}^k} \quad \text{and} \quad \sum_{n=0}^{\infty} R2_k(n)q^n = \frac{1}{(q^2; q^5)_{\infty}^k (q^3; q^5)_{\infty}^k}, \quad (3.35)$$

respectively. Here, we state our result without proof as it follows exactly the same arguments as in the proofs of Theorem 2.1 and Theorem 2.2, respectively.

**Theorem 3.3.** *For a given positive integer  $n$ , we have the following identities for  $R1_k(n)$  involving the special values of multiple zeta function  $\zeta_k(2)$  given by*

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} R1_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} R1_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_{\ell}\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_{\ell} \\ m_i \text{ - distinct}}} \frac{(-1)^{\ell} R1_2(s_1) R1_2(s_2) \cdots R1_2(s_{\ell})}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_{\ell}^2} \prod_{m \neq m_1, m_2, \dots, m_{\ell}} \left(1 - \frac{1}{m^2 \pi^2}\right). \end{aligned} \quad (3.36)$$

These identities also valid if  $R1_k(n)$  is replaced by  $R2_k(n)$ .

**Theorem 3.4.** For a given positive integer  $n$ , we have the following identities for  $R1_k(n)$  involving the special values of multiple zeta function  $\zeta_k^{odd}(2)$  given by

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} R1_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k 4^k \zeta_k^{odd}(2)}{\pi^{2k}} R1_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-odd \& distinct}}} \frac{(-4)^\ell}{\pi^{2\ell}} \frac{R1_2(s_1) R1_2(s_2) \cdots R1_2(s_\ell)}{m_1^2 m_2^2 \cdots m_\ell^2} \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left(1 - \frac{1}{m^2 \pi^2}\right). \end{aligned} \quad (3.37)$$

These identities also valid if  $R1_k(n)$  is replaced by  $R2_k(n)$ .

**Remark 3.2.** The identities appearing in the Theorem 2.3 and Theorem 2.4 can be given for  $R1_k(n)$  and  $R2_k(n)$ .

**Acknowledgement.** The author would like to thank IMSc, Chennai for providing an excellent academic atmosphere and financial support through an institute fellowship. Finally, the author thanks the referee for meticulously going through the manuscript and making numerous suggestions which improved the texture of the article.

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