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# Identities among some combinatorial objects involving special values of multiple zeta functions

Lalit Vaishya

**Abstract.** In the article, we establish some identities involving special values of multiple zeta functions among the counting functions of number of representations of an integer by a linear combination of figurate numbers such as triangular numbers, square numbers, pentagonal numbers, etc. More precisely, we provide our result for  $\delta_k(n)$ ,  $r_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ), the number of representations of  $n$  as a sum of  $k$ -triangular numbers, as a sum of  $k$ -square numbers and as a sum of  $k$ -higher figurate numbers (for a fixed  $a \geq 3$ ), respectively. Moreover, these identities also occur when one of  $\delta_k(n)$ ,  $r_k(n)$  and  $\mathcal{N}_k^a(n)$  is replaced by the  $k$ -colored partition functions.

**Keywords.**  $q$ -series,  $k$ -coloured partition function, Trigonometric functions, Integer composition.

**2010 Mathematics Subject Classification.** Primary 33B10, 40B05, 40C15, Secondary 17A42

## 1. Preliminaries

We define the  $q$ -Pochhammer symbol, also known as the  $q$ -shifted factorial, given as the product of the form

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k) = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}), \quad (1.1)$$

with  $(a; q)_0 = 1$ . It is a  $q$ -analogue of the Pochhammer symbol  $(x)_n = x(x + 1) \cdots (x + n - 1)$  for  $x \in \mathbb{R}^+$  and  $n \in \mathbb{N}$ , in the sense that

$$\lim_{q \rightarrow 1} \frac{(q^x; q)_n}{(1 - q)^n} = (x)_n. \quad (1.2)$$

The  $q$ -Pochhammer symbol is a major building block in the theory of hypergeometric series and the theory of integer partitions. The  $q$ -Pochhammer symbol can be extended to an infinite product given by

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k). \quad (1.3)$$

Moreover, for  $a = q$ ,  $(a; q)_\infty$  is the Dedekind eta function  $\eta(\tau)$  upto a multiple of  $q$ -power, where  $q = e^{2\pi i\tau}$  and  $\tau \in \mathbb{H}$  (complex upper half-plane), i.e.,  $\eta(\tau) = q^{1/24}(q; q)_\infty$ . A product of the form  $\prod_{d|N} \eta^{r_d}(d\tau)$ ; with  $r_d \in \mathbb{Z}$ , is commonly known as a eta-quotient of level  $N$  and weight  $\frac{1}{2} \sum_{d|N} r_d$ , in the theory of modular forms. These are the building blocks for finding the formulas for  $r_k(n)$ ,  $\delta_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ) (See [RV23]).

For a fixed integer  $a \geq 1$ , we define the  $n^{\text{th}}$ -figurate number by  $f_a(n) := \frac{an^2 + (a-2)n}{2}$ , following Ono-Robins-Wahl [ORW95]. Note that the function  $f_a(n)$  denotes the  $n^{\text{th}}$ -triangular number (resp. square number and pentagonal number) when  $a = 1$  (resp. 2 and 3). In the literature, the  $n^{\text{th}}$ -triangular number is denoted by  $T_n$ , and for a fixed  $a \geq 3$ , the integer  $f_a(n)$  is known as the  $n^{\text{th}}$ -higher figurate number. These functions are associated with counting the number of vertices of some

geometric objects. We denote the generating functions for these figurate numbers as  $\Psi(\tau)$  (for triangular numbers),  $\theta(\tau)$  (for square numbers) and  $\Phi_a(\tau)$  (for  $a \geq 3$ ), and they are given respectively by

$$\theta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2}, \quad \Psi(\tau) := \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}, \quad \text{and} \quad \text{for } a \geq 3, \quad \Phi_a(\tau) := \sum_{n \in \mathbb{Z}} q^{f_a(n)}. \quad (1.4)$$

where  $q \in \mathbb{H}$ . It is known from [ORW95, Proposition 1, Theorem 10] that the generating function for the figurate numbers defined in (1.4) are explicitly obtained from Jacobi triple product identity given by

$$\theta(\tau) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^5}{(1 - q^n)^2(1 - q^{4n})^2} = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2(q^4; q^4)_{\infty}^2}, \quad \Psi(\tau) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^n)} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}} \quad (1.5)$$

and for  $a \geq 3$ ,

$$\Phi_a(\tau) = \prod_{n=1}^{\infty} (1 - q^{an})(1 - q^{an-1})(1 - q^{an-(a-1)}) = (q^a; q^a)_{\infty}(q^{-1}; q^a)_{\infty}(q^{-(a-1)}; q^a)_{\infty}. \quad (1.6)$$

Let  $\delta_k(n)$ ,  $r_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ) denote the number of representations of  $n$  as a sum of  $k$ -triangular numbers, as a sum of  $k$ -square numbers and as a sum of  $k$ -higher figurate numbers (for a fixed  $a \geq 3$ ) respectively, i.e.,

$$r_k(n) = \# \left\{ (x_1, x_2, \dots, x_k) \in \mathbb{Z}^k \mid n = \sum_{i=1}^k x_i^2 \right\},$$

$$\delta_k(n) = \# \left\{ (x_1, x_2, \dots, x_k) \in \mathbb{Z}^k \mid n = \sum_{i=1}^k \frac{x_i(x_i + 1)}{2} \right\}$$

and for a fixed integer  $a \geq 3$ ,  $\mathcal{N}_k^a(n) = \# \left\{ (x_1, x_2, \dots, x_k) \in \mathbb{Z}^k \mid n = \sum_{i=1}^k f_a(x_i) \right\}$ .

It is easy to see that

$$\theta^k(\tau) = 1 + \sum_{n=1}^{\infty} r_k(n) q^n, \quad \Psi^k(\tau) = 1 + \sum_{n=1}^{\infty} \delta_k(n) q^n \quad \text{and} \quad \Phi_a^k(\tau) = 1 + \sum_{n=1}^{\infty} \mathcal{N}_k^a(n) q^n$$

for  $a \geq 3$ . For more details on these arithmetical functions, we refer to [ORW95] and [RV23].

For positive integers  $k_1, k_2, \dots, k_w \geq 2$ , we define the multiple zeta function as follows:

$$\zeta(k_1, k_2, \dots, k_w) = \sum_{m_1 < m_2 < \dots < m_w} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_w^{k_w}}$$

and  $\zeta^{odd}(k_1, k_2, \dots, k_w) = \sum_{\substack{m_1 < m_2 < \dots < m_w \\ m_i - \text{odd}}} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_w^{k_w}}.$  (1.7)

When  $k_i = k$  for all  $i$ , we simply denote it as  $\zeta_w(k)$  and  $\zeta_w^{odd}(k)$ , respectively. By convention, we write  $\zeta(k) = \zeta_1(k)$ ,  $\zeta_0(k) = 1$ ,  $\zeta^{odd}(k) = \zeta_1^{odd}(k)$  and  $\zeta_0^{odd}(k) = 1$ .

## Trigonometric identities

For each  $z \in \mathbb{C}$ , we have the following trigonometric function with their product expansion and Taylor series expansion given as follows

$$\begin{aligned} \sin z &= z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \\ \cos z &= \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2\pi^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \end{aligned} \quad (1.8)$$

$$\begin{aligned} \sin^2 z &= z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)^2 = \sum_{n=1}^{\infty} \left(\sum_{r=0}^{n-1} \binom{2n}{2r+1}\right) \frac{(-1)^{n-1}}{(2n)!} z^{2n} \\ \text{and } \cos^2 z &= \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2\pi^2}\right)^2 = \sum_{n=0}^{\infty} \left(\sum_{r=0}^n \binom{2n}{2r}\right) \frac{(-1)^n}{(2n)!} z^{2n}. \end{aligned} \quad (1.9)$$

Here,  $\binom{n}{r}$  denotes the number of ways to choose  $r$  objects from a set of  $n$  objects.

## 2. Identities for $\delta_k(n)$ , $r_k(n)$ and $\mathcal{N}_k^a(n)$

In this section, we establish certain identities for  $r_k(n)$ ,  $\delta_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ) which involves the special values of multiple zeta function. We make use of trigonometric functions to prove our results. Before stating our results, we define the set  $S(n)$  as a collection of all possible compositions of non-zero positive integers  $\{a_1, a_2, \dots, a_r\}$  such that  $a_1 + a_2 + \dots + a_r = n$ , i.e.,

$$S(n) = \{\{a_1, a_2, \dots, a_r\} \mid a_i > 0 \text{ for all } i, \text{ and } a_1 + a_2 + \dots + a_r = n\}.$$

For example, given  $n = 6$ , the elements  $\{3, 2, 1\}$ ,  $\{2, 3, 1\}$  and  $\{1, 2, 3\}$  are different integer compositions of  $S(n)$ .

Now, we present our results.

**Theorem 2.1.** *For a given positive integer  $n$ , we have the following identities for  $r_k(n)$ ,  $\delta_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ), respectively involving the special values of multiple zeta function  $\zeta_k(2)$  given by*

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} r_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} r_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-distinct}}} \frac{(-1)^\ell r_2(s_1) r_2(s_2) \cdots r_2(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{m \neq m_1, m_2, \dots, m_\ell} \left(1 - \frac{1}{m^2 \pi^2}\right), \end{aligned} \quad (2.10)$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \delta_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} \delta_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-distinct}}} \frac{(-1)^\ell \delta_2(s_1) \delta_2(s_2) \cdots \delta_2(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{m \neq m_1, m_2, \dots, m_\ell} \left(1 - \frac{1}{m^2 \pi^2}\right) \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \mathcal{N}_{2k}^a(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} \mathcal{N}_{2k}^a(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-distinct}}} \frac{(-1)^\ell \mathcal{N}_2^a(s_1) \mathcal{N}_2^a(s_2) \cdots \mathcal{N}_2^a(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{m \neq m_1, m_2, \dots, m_\ell} \left(1 - \frac{1}{m^2 \pi^2}\right). \end{aligned} \quad (2.12)$$

*Proof.* We prove the above result for  $r_k(\cdot)$  and other identities can be obtained following similar arguments. Let us consider the product formula of  $\sin z$  given by  $\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$ . To get the required result, we consider the product  $\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$  and expand it in two different ways by substituting the  $q$ -product of generating function for square numbers, i.e.,  $z = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^5}{(1 - q^n)^2(1 - q^{4n})^2}$  (given in (1.5)) and observe that  $z^k = \sum_{n=0}^{\infty} r_k(n)q^n$ . More precisely, the expansion of  $\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$  gives that

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{\pi^{2k}} \left( \sum_{\substack{n_1, n_2, \dots, n_k \\ n_i \text{-distinct}}} \frac{1}{n_1^2 n_2^2 \cdots n_k^2} \right) z^{2k} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} z^{2k}.$$

Now, we substitute  $z^{2k} = \sum_{n=0}^{\infty} r_{2k}(n)q^n$  and see that the constant term ( $q^0$ -th coefficient) and the coefficient of  $q^n$  ( $n > 0$ ) is given by;

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} r_{2k}(n), \quad (2.13)$$

respectively. On the other hand, we first substitute  $z^2 = \sum_{n=0}^{\infty} r_2(n)q^n$  in  $\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$  to get

$$\prod_{n=1}^{\infty} \left( \left(1 - \frac{1}{n^2\pi^2}\right) q^0 - \sum_{k=1}^{\infty} \frac{r_2(k)}{n^2\pi^2} q^k \right). \quad (2.14)$$

We expand the product in (2.14) using the Cauchy product of infinite series to obtain the coefficients of  $q^n$ . The coefficients of  $q^n$  in this process is given by

$$\sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-distinct}}} \frac{(-1)^\ell r_2(s_1) r_2(s_2) \cdots r_2(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{m \neq m_1, m_2, \dots, m_\ell} \left(1 - \frac{1}{m^2\pi^2}\right). \quad (2.15)$$

Since the  $q^n$ -coefficients given in (2.13) and (2.15) are the  $q^n$ -coefficients of the expansions of the product  $\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$  by putting the value of  $z = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^5}{(1 - q^n)^2(1 - q^{4n})^2}$ . Hence, we have the required identity. The first equality is obvious by the Taylor series expansion of  $\sin z$ . Thus, we have the required result.

**Theorem 2.2.** For a given positive integer  $n$ , we have the following identities for  $r_k(n)$ ,  $\delta_k(n)$  and  $N_k^a(n)$  (for a fixed  $a \geq 3$ ), respectively involving the special values of multiple zeta function  $\zeta_k^{\text{odd}}(2)$  given by

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} r_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k 4^k \zeta_k^{\text{odd}}(2)}{\pi^{2k}} r_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-odd \& distinct}}} \frac{(-4)^\ell r_2(s_1) r_2(s_2) \cdots r_2(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left(1 - \frac{1}{m^2\pi^2}\right), \end{aligned} \quad (2.16)$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \delta_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k 4^k \zeta_k^{odd}(2)}{\pi^{2k}} \delta_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-odd} \ \& \ \text{distinct}}} \frac{(-4)^\ell \delta_2(s_1) \delta_2(s_2) \cdots \delta_2(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left(1 - \frac{1}{m^2 \pi^2}\right) \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \mathcal{N}_{2k}^a(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k 4^k \zeta_k^{odd}(2)}{\pi^{2k}} \mathcal{N}_{2k}^a(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-odd} \ \& \ \text{distinct}}} \frac{(-4)^\ell \mathcal{N}_2^a(s_1) \mathcal{N}_2^a(s_2) \cdots \mathcal{N}_2^a(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left(1 - \frac{1}{m^2 \pi^2}\right). \end{aligned} \quad (2.18)$$

*Proof.* We use the same arguments to prove the above result for  $\delta_k(\cdot)$  and the other identities. Let us consider the product formula of  $\cos z$  given by  $\cos z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2 \pi^2}\right)$ . We expand the product  $\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2 \pi^2}\right)$  in two different ways by substituting the  $q$ -product of generating function for triangular numbers, i.e.,  $z = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)}$  and observe that  $z^k = \sum_{n=0}^{\infty} \delta_k(n) q^n$  to get the required result. More precisely, a simple expansion of  $\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2 \pi^2}\right)$  gives that

$$\begin{aligned} \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2 \pi^2}\right) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 4^k}{\pi^{2k}} \left( \sum_{\substack{n_1, n_2, \dots, n_k \\ n_i \text{-odd} \ \& \ \text{distinct}}} \frac{1}{n_1^2 n_2^2 \cdots n_k^2} \right) z^{2k} \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 4^k \zeta_k^{odd}(2)}{\pi^{2k}} z^{2k}. \end{aligned}$$

Now, we substitute  $z^{2k} = \sum_{n=0}^{\infty} \delta_{2k}(n) q^n$  and see that the coefficient of  $q^n$  is

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k 4^k \zeta_k^{odd}(2)}{\pi^{2k}} \delta_{2k}(n). \quad (2.19)$$

On the other hand, first we substitute  $z^2 = \sum_{n=0}^{\infty} \delta_2(n) q^n$  in  $\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2 \pi^2}\right)$  and then we expand the product to obtain the coefficients of  $q^n$ . The coefficients of  $q^n$  in this process is given by

$$\sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-odd} \ \& \ \text{distinct}}} \frac{(-4)^\ell \delta_2(s_1) \delta_2(s_2) \cdots \delta_2(s_\ell)}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_\ell^2} \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left(1 - \frac{1}{m^2 \pi^2}\right). \quad (2.20)$$

Hence, comparing the  $q^n$ -coefficients of the expansions of the product  $\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2 \pi^2}\right)$  (in two ways) by substituting  $z = \frac{(q^2; q^2)_\infty}{(q; q)_\infty}$ , we obtain the required result. The first equality is obvious from the Taylor series expansion of  $\cos z$ .

Following the same arguments and using the product and sum expression for  $\sin^2 z$  and  $\cos^2 z$  respectively (given in (1.9)), we have the following results.

**Theorem 2.3.** For a given positive integer  $n$ , we have the following identities for  $r_k(n)$ ,  $\delta_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ) respectively.

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \sum_{r=0}^{k-1} \binom{2k}{2r+1} \right) \frac{(-1)^{k-1}}{(2k)!} r_{2k}(n) \\ &= \sum_{f=1}^n r_2(f) \sum^* \left( \prod_{1 \leq e \leq \ell} \left( -\frac{2}{\pi^2 m_e^2} r_2(s_e) + \frac{1}{\pi^4 m_e^4} r_2(s_e) \right) \prod_{m \neq m_1, m_2, \dots, m_\ell} \left( 1 - \frac{2}{m^2 \pi^2} + \frac{1}{m^4 \pi^4} \right) \right), \end{aligned} \quad (2.21)$$

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \sum_{r=0}^{k-1} \binom{2k}{2r+1} \right) \frac{(-1)^{k-1}}{(2k)!} \delta_{2k}(n) \\ &= \sum_{f=1}^n \delta_2(f) \sum^* \left( \prod_{1 \leq e \leq \ell} \left( -\frac{2}{\pi^2 m_e^2} \delta_2(s_e) + \frac{1}{\pi^4 m_e^4} \delta_2(s_e) \right) \prod_{m \neq m_1, m_2, \dots, m_\ell} \left( 1 - \frac{2}{m^2 \pi^2} + \frac{1}{m^4 \pi^4} \right) \right), \end{aligned} \quad (2.22)$$

and

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \sum_{r=0}^{k-1} \binom{2k}{2r+1} \right) \frac{(-1)^{k-1}}{(2k)!} \mathcal{N}_{2k}^a(n) \\ &= \sum_{f=1}^n \mathcal{N}_2^a(f) \sum^* \left( \prod_{1 \leq e \leq \ell} \left( -\frac{2}{\pi^2 m_e^2} \mathcal{N}_2^a(s_e) + \frac{1}{\pi^4 m_e^4} \mathcal{N}_2^a(s_e) \right) \prod_{m \neq m_1, m_2, \dots, m_\ell} \left( 1 - \frac{2}{m^2 \pi^2} + \frac{1}{m^4 \pi^4} \right) \right). \end{aligned} \quad (2.23)$$

where  $\sum^*$  denotes the double summation  $\sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n-f)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-distinct}}}$ .

**Theorem 2.4.** For a given positive integer  $n$ , we have the following identities for  $r_k(n)$ ,  $\delta_k(n)$  and  $\mathcal{N}_k^a(n)$  (for a fixed  $a \geq 3$ ) respectively.

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \sum_{r=0}^k \binom{2k}{2r} \right) \frac{(-1)^k}{(2k)!} r_{2k}(n) \\ &= \sum^\# \prod_{1 \leq e \leq \ell} \left( -\frac{2}{\pi^2 m_e^2} r_2(s_e) + \frac{1}{\pi^4 m_e^4} r_2(s_e) \right) \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left( 1 - \frac{2}{m^2 \pi^2} + \frac{1}{m^4 \pi^4} \right), \end{aligned} \quad (2.24)$$

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \sum_{r=0}^k \binom{2k}{2r} \right) \frac{(-1)^k}{(2k)!} \delta_{2k}(n) \\ &= \sum^\# \prod_{1 \leq e \leq \ell} \left( -\frac{2}{\pi^2 m_e^2} \delta_2(s_e) + \frac{1}{\pi^4 m_e^4} \delta_2(s_e) \right) \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left( 1 - \frac{2}{m^2 \pi^2} + \frac{1}{m^4 \pi^4} \right). \end{aligned} \quad (2.25)$$

and

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \sum_{r=0}^k \binom{2k}{2r} \right) \frac{(-1)^k}{(2k)!} \mathcal{N}_{2k}^a(n) \\ &= \sum^\# \left( \prod_{1 \leq e \leq \ell} \left( -\frac{2}{\pi^2 m_e^2} \mathcal{N}_2^a(s_e) + \frac{1}{\pi^4 m_e^4} \mathcal{N}_2^a(s_e) \right) \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left( 1 - \frac{2}{m^2 \pi^2} + \frac{1}{m^4 \pi^4} \right) \right). \end{aligned} \quad (2.26)$$

where  $\sum^\#$  denotes the summation  $\sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-odd \& distinct}}}$ .

### 3. Identities for $k$ -colored partition functions:

Let  $n$  be a positive integer. For each  $k \geq 1$ , let  $p_k(n)$  denote the number of  $k$ -colored partition of  $n$ . For  $k = 1$ , it is the usual partition function  $p(n)$ , first considered by Ramanujan. We define  $\overline{p}_k(n)$ , as the counting function for the number of  $k$ -colored over-partition of a positive integer  $n$ , and  $pd_k(n)$  denotes the number of  $k$ -colored partition of  $n$  into distinct parts, and  $po_k(n)$  denotes the number of  $k$ -colored partition of  $n$  into odd parts. The generating function for these  $k$ -colored partition functions is given in terms of  $q$ -product as follows.

$$\begin{aligned} \sum_{n=0}^{\infty} p_k(n)q^n &= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k} = (q; q)_{\infty}^{-k}, \\ \sum_{n=0}^{\infty} \overline{p}_k(n)q^n &= \prod_{n=1}^{\infty} \frac{(1+q^n)^k}{(1-q^n)^k} = \frac{(-q; q)_{\infty}^k}{(q; q)_{\infty}^k}, \\ \sum_{n=0}^{\infty} pd_k(n)q^n &= \prod_{n=1}^{\infty} (1+q^n)^k = (-q; q)_{\infty}^k, \\ \text{and } \sum_{n=0}^{\infty} po_k(n)q^n &= \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})^k} = (q^{-1}; q^2)_{\infty}^k. \end{aligned} \tag{3.27}$$

For details on the partition functions and their combinatorial interpretations, we refer to [And98]. Similar identities obtained in Theorem 2.1 and Theorem 2.2, can be given for these  $k$ -colored partition functions. The proof of these identities take place by substituting  $z = (-q; q)_{\infty}$  ( for  $pd_k(n)$ ),  $z = (q^{-1}; q^2)_{\infty}$  (for  $po_k(n)$ ),  $z = \frac{1}{(q; q)_{\infty}}$  ( for  $p_k(n)$ ) and  $z = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}}$  (for  $\overline{p}_k(n)$ ) in the expression of  $\sin z$  and  $\cos z$  and following the same arguments as in the proofs of Theorem 2.1 and Theorem 2.2. Below, we state these identities only for  $pd_k(n)$ , and identities for others can be stated analogously.

**Theorem 3.1.** *For a given positive integer  $n$ , we have the following identities for  $pd_k(n)$  involving the special values of multiple zeta function  $\zeta_k(2)$  given by*

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} pd_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} pd_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_{\ell}\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_{\ell} \\ m_i \text{-distinct}}} \frac{(-1)^{\ell} pd_2(s_1) pd_2(s_2) \cdots pd_2(s_{\ell})}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_{\ell}^2} \prod_{m \neq m_1, m_2, \dots, m_{\ell}} \left(1 - \frac{1}{m^2 \pi^2}\right). \end{aligned} \tag{3.28}$$

*These identities also valid if  $pd_k(n)$  is replaced by one of these  $k$ -colored partition functions among  $p_k(n)$ ,  $\overline{p}_k(n)$  and  $po_k(n)$ .*

**Theorem 3.2.** *For a given positive integer  $n$ , we have the following identities for  $pd_k(n)$  involving the special values of multiple zeta function  $\zeta_k^{\text{odd}}(2)$  given by*

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} pd_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k 4^k \zeta_k^{\text{odd}}(2)}{\pi^{2k}} pd_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_{\ell}\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_{\ell} \\ m_i \text{-odd \& distinct}}} \frac{(-4)^{\ell} pd_2(s_1) pd_2(s_2) \cdots pd_2(s_{\ell})}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_{\ell}^2} \prod_{\substack{m \neq m_1, m_2, \dots, m_{\ell} \\ m \text{-odd}}} \left(1 - \frac{1}{m^2 \pi^2}\right). \end{aligned} \tag{3.29}$$

*These identities also valid if  $pd_k(n)$  is replaced by one of these  $k$ -colored partition functions among  $p_k(n)$ ,  $\overline{p}_k(n)$  and  $po_k(n)$ .*

**Remark 3.1.** *The identities appearing in the Theorem 2.3 and Theorem 2.4 can be given for each of these  $k$ -colored partition functions  $p_k(n)$ ,  $\overline{p}_k(n)$ ,  $pd_k(n)$  and  $po_k(n)$  by replacing the arithmetical function  $r_k(n)$  by one of the above  $k$ -colored partition functions.*



### 3.A. $k$ -colored identities associated to Rogers-Ramanujan partition functions:

For a positive integer  $n$ , Let  $[\lambda_1, \lambda_2, \dots, \lambda_r]$  such that  $n = \lambda_1 + \lambda_2 + \dots + \lambda_r$ ,  $r \in \mathbb{N}$  and each  $\lambda_i \geq \lambda_{i+1} > 0$ , denote a partition of  $n$ . We define the following partition functions given by

$$\begin{aligned} RR(n) &:= \#\{\text{partiton } [\lambda_1, \lambda_2, \dots, \lambda_r] \text{ of } n \mid \lambda_i - \lambda_{i+1} \geq 2 \text{ for each } i\} \quad \text{and} \\ RS(n) &:= \#\{\text{partiton } [\lambda_1, \lambda_2, \dots, \lambda_r] \text{ of } n \mid \lambda_i - \lambda_{i+1} \geq 2, \text{ and } \lambda_i \geq 2 \text{ for each } i\}. \end{aligned} \quad (3.30)$$

The generating function for these is given by

$$\sum_{n=0}^{\infty} RR(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q, q)_n} \quad \text{and} \quad \sum_{n=0}^{\infty} RS(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q, q)_n}. \quad (3.31)$$

Let  $R1(n)$  denote the numbers of partition  $[\lambda_1, \lambda_2, \dots, \lambda_r]$  of  $n$  such that each  $\lambda_i \equiv \pm 1 \pmod{5}$ , and  $R2(n)$  denote the numbers of partition  $[\lambda_1, \lambda_2, \dots, \lambda_r]$  of  $n$  such that each  $\lambda_i \equiv \pm 2 \pmod{5}$ , respectively. The generating function of these is given by

$$\begin{aligned} \sum_{n=0}^{\infty} R1(n)q^n &= \prod_{m=1}^{\infty} \frac{1}{(1 - q^{5m-4})(1 - q^{5m-1})} = \frac{1}{(q; q^5)_{\infty}(q^4; q^5)_{\infty}} \\ \text{and} \quad \sum_{n=0}^{\infty} R2(n)q^n &= \prod_{m=1}^{\infty} \frac{1}{(1 - q^{5m-2})(1 - q^{5m-3})} = \frac{1}{(q^2; q^5)_{\infty}(q^3; q^5)_{\infty}}, \end{aligned} \quad (3.32)$$

respectively. Then, Rogers-Ramanujan identities say that

$$RR(n) = R1(n) \quad \text{and} \quad RS(n) = R2(n). \quad (3.33)$$

To prove the Rogers-Ramanujan identities, it is equivalent to show the following

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q, q)_n} = \frac{1}{(q; q^5)_{\infty}(q^4; q^5)_{\infty}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q, q)_n} = \frac{1}{(q^2; q^5)_{\infty}(q^3; q^5)_{\infty}}, \quad (3.34)$$

respectively. For details, we refer to [And98]. Now, we define the following  $k$ -colored Rogers-Ramanujan type partitioned functions as follows:  $R1_k(n)$  denote the numbers of partition  $[\lambda_1, \lambda_2, \dots, \lambda_r]$  of  $n$  such that each  $\lambda_i \equiv \pm 1 \pmod{5}$  and its part is colored by at most  $k$ -colours, and  $R2_k(n)$  denote the numbers of partition  $[\lambda_1, \lambda_2, \dots, \lambda_r]$  of  $n$  such that each  $\lambda_i \equiv \pm 2 \pmod{5}$  and its part is colored by at most  $k$ -colours, respectively. The generating functions are given by

$$\sum_{n=0}^{\infty} R1_k(n)q^n = \frac{1}{(q; q^5)_{\infty}^k (q^4; q^5)_{\infty}^k} \quad \text{and} \quad \sum_{n=0}^{\infty} R2_k(n)q^n = \frac{1}{(q^2; q^5)_{\infty}^k (q^3; q^5)_{\infty}^k}, \quad (3.35)$$

respectively. Here, we state our result without proof as it follows exactly the same arguments as in the proofs of Theorem 2.1 and Theorem 2.2, respectively.

**Theorem 3.3.** *For a given positive integer  $n$ , we have the following identities for  $R1_k(n)$  involving the special values of multiple zeta function  $\zeta_k(2)$  given by*

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} R1_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k \zeta_k(2)}{\pi^{2k}} R1_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_{\ell}\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_{\ell} \\ m_i \text{ - distinct}}} \frac{(-1)^{\ell} R1_2(s_1) R1_2(s_2) \cdots R1_2(s_{\ell})}{\pi^{2\ell} m_1^2 m_2^2 \cdots m_{\ell}^2} \prod_{m \neq m_1, m_2, \dots, m_{\ell}} \left(1 - \frac{1}{m^2 \pi^2}\right). \end{aligned} \quad (3.36)$$

These identities also valid if  $R1_k(n)$  is replaced by  $R2_k(n)$ .

**Theorem 3.4.** For a given positive integer  $n$ , we have the following identities for  $R1_k(n)$  involving the special values of multiple zeta function  $\zeta_k^{odd}(2)$  given by

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} R1_{2k}(n) &= \sum_{k=0}^{\infty} \frac{(-1)^k 4^k \zeta_k^{odd}(2)}{\pi^{2k}} R1_{2k}(n) \\ &= \sum_{\{s_1, s_2, \dots, s_\ell\} \in S(n)} \sum_{\substack{m_1, m_2, \dots, m_\ell \\ m_i \text{-odd \& distinct}}} \frac{(-4)^\ell}{\pi^{2\ell}} \frac{R1_2(s_1) R1_2(s_2) \cdots R1_2(s_\ell)}{m_1^2 m_2^2 \cdots m_\ell^2} \prod_{\substack{m \neq m_1, m_2, \dots, m_\ell \\ m \text{-odd}}} \left(1 - \frac{1}{m^2 \pi^2}\right). \end{aligned} \quad (3.37)$$

These identities also valid if  $R1_k(n)$  is replaced by  $R2_k(n)$ .

**Remark 3.2.** The identities appearing in the Theorem 2.3 and Theorem 2.4 can be given for  $R1_k(n)$  and  $R2_k(n)$ .

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