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► **To cite this version:**

Yanni Pei, Jiang Zeng. Counting derangements with signed right-to-left minima and excedances. *Advances in Applied Mathematics*, 2024, 152, pp.102599. 10.1016/j.aam.2023.102599 . hal-04442944

HAL Id: hal-04442944

<https://hal.science/hal-04442944>

Submitted on 15 Mar 2024

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COUNTING DERANGEMENTS WITH SIGNED RIGHT-TO-LEFT MINIMA AND EXCEDANCES

YANNI PEI AND JIANG ZENG

ABSTRACT. Recently Alexandersson and Getachew proved some multivariate generalizations of a formula for enumerating signed excedances in derangements. In this paper we first relate their work to a recent continued fraction for permutations and confirm some of their observations. Our second main result is two refinements of their multivariate identities, which clearly explain the meaning of each term in their main formulas. We also explore some similar formulas for permutations of type B.

1. INTRODUCTION

Over the last thirty years much work has been done on enumeration over the symmetric group (and more generally Coxeter or Weyl groups), taking into account also the sign (or one dimensional character) of each permutation, see [DF92, Wa92, AGR05, Zh13, ELL21, KZ03, Mo15, Si16, AG21] and the references therein.

In this paper we mainly study signed statistics over derangements. Let σ be a permutation of the set $[n] := \{1, \dots, n\}$, written as a word $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$. Then σ is called *derangement* if $\sigma(i) \neq i$ for $i \in [n]$. The index i (resp. $\sigma(i)$) is an *excedance index* (resp. value) if $\sigma(i) > i$. Similarly, we say that i (resp. $\sigma(i)$) is a *right-to-left minimum index* (resp. value) if $\sigma(i) < \sigma(j)$ for all $j > i$. Let \mathfrak{S}_n (resp. \mathcal{D}_n) be the set of permutations (resp. derangements) of $[n]$. For $\sigma \in \mathfrak{S}_n$, the sets of indices and values of excedances and left-to-right minima are denoted, respectively, by

$$\text{EXCi}(\sigma) = \{i \in [n] \mid \text{and } \sigma(i) > i\};$$

$$\text{EXCv}(\sigma) = \{\sigma(i) \mid i \in \text{EXCi}(\sigma)\};$$

$$\text{RLMi}(\sigma) = \{i \in [n] \mid \sigma(i) < \sigma(j) \text{ for } j \in \{i+1, \dots, n\}\};$$

$$\text{RLMv}(\sigma) = \{\sigma(i) \mid i \in \text{RLMi}(\sigma)\}.$$

Note that $\text{EXCi}(\sigma) \cap \text{RLMi}(\sigma) = \emptyset$ and $\text{EXCv}(\sigma) \cap \text{RLMv}(\sigma) = \emptyset$. Let $\text{exc}(\sigma)$, $\text{rlm}(\sigma)$ and $\text{cyc}(\sigma)$ be respectively the number of excedances, right-to-left minima and cycles of the

Date: October 10, 2023.

Key words and phrases. Derangements, Right-to-left minima, Excedances, Cycles, Laguerre polynomials.

permutation σ . It is well-known, see [Ze93, Br00, KZ03], that

$$\sum_{\sigma \in \mathcal{D}_n} (-1)^{\text{cyc}(\sigma)} x^{\text{exc}(\sigma)} = -(x + x^2 + \cdots + x^{n-1}). \quad (1.1a)$$

As $\text{inv}(\sigma) \equiv n - \text{cyc}(\sigma) \pmod{2}$, formula (1.1a) can be rephrased as

$$\sum_{\sigma \in \mathcal{D}_n} (-1)^{\text{inv}(\sigma)} x^{\text{exc}(\sigma)} = (-1)^{n-1} (x + x^2 + \cdots + x^{n-1}). \quad (1.1b)$$

which is in [MR03, Proposition 4.3]. In this paper we shall freely switch from the inv-version to cyc-version of an identity and vice versa. In 2003 Ksavrelof and the second author [KZ03] proved the following refinement of (1.1):

$$\sum_{\sigma \in \mathcal{D}_n, \sigma(n)=j} (-1)^{\text{cyc}(\sigma)} x^{\text{exc}(\sigma)} = -x^{n-j} \quad (1 \leq j \leq n-1). \quad (1.2)$$

Similar results about excedances for Coxeter groups and colored permutation groups have been achieved in [BG05, Si11, Zh13, Si16].

In 2021, Alexandersson and Getachew [AG21, AG21a] came up with two multivariate refinements of (1.1):

$$\sum_{\sigma \in \mathcal{D}_n} (-1)^{\text{inv}(\sigma)} \left(\prod_{j \in \text{RLMv}(\sigma)} x_j \right) \left(\prod_{j \in \text{EXCv}(\sigma)} y_j \right) = (-1)^{n-1} \sum_{j=1}^{n-1} x_1 \cdots x_j y_{j+1} \cdots y_n, \quad (1.3a)$$

and

$$\sum_{\sigma \in \mathcal{D}_n} (-1)^{\text{inv}(\sigma)} \left(\prod_{j \in \text{RLMi}(\sigma)} x_j \right) \left(\prod_{j \in \text{EXCi}(\sigma)} y_j \right) = (-1)^{n-1} \sum_{j=1}^{n-1} y_1 \cdots y_j x_{j+1} \cdots x_n. \quad (1.3b)$$

These identities prompted us to seek for possible multivariate analogues of (1.2), which would refine (1.3) consequently. It is somehow surprising that our findings do fulfill this requirement. More precisely, for $j \in [n-1]$ our two multivariate analogues of (1.2) read as follows:

$$\sum_{\sigma \in \mathcal{D}_n, \sigma(n)=j} (-1)^{\text{cyc}(\sigma)} \left(\prod_{i \in \text{RLMv}(\sigma)} x_i \right) \left(\prod_{i \in \text{EXCv}(\sigma)} y_i \right) = -x_1 \cdots x_j y_{j+1} \cdots y_n, \quad (1.4a)$$

and

$$\sum_{\sigma \in \mathcal{D}_n, \sigma(n+1-j)=1} (-1)^{\text{cyc}(\sigma)} \left(\prod_{i \in \text{RLMi}(\sigma)} x_i \right) \left(\prod_{i \in \text{EXCi}(\sigma)} y_i \right) = -y_1 \cdots y_{n-j} x_{n+1-j} \cdots x_n \quad (1.4b)$$

from which eqs. (1.3) follow by summing j on $[n-1]$.

On the other hand, Sokal and the second author [SZ22] studied some multivariate Eulerian polynomials whose ordinary generating functions have nice J-fractions formulas. We note that the specialization of the polynomial \widehat{Q}_n in [SZ22, (2.29)] with $x_1 = xy$, $x_2 = x$,

$u_2 = v_1 = y_1 = 1$, $u_1 = v_2 = y_2 = y$, $w_\ell = 0$ turns out to be the enumerative polynomial of derangements

$$D_n(x, y, \lambda) := \sum_{\sigma \in \mathcal{D}_n} \lambda^{\text{cyc}(\sigma)} x^{\text{rlm}(\sigma)} y^{\text{exc}(\sigma)}, \quad (1.5)$$

and Theorem 2.4 in [SZ22] reduces to the following result

$$\sum_{n=0}^{\infty} D_n(x, y, \lambda) t^n = \frac{1}{1 - \frac{\lambda xy t^2}{1 - (x+y)t - \frac{(\lambda+1)(x+1)yt^2}{1 - (x+2y+1)t - \frac{(\lambda+2)(x+2)yt^2}{\dots}}}}, \quad (1.6a)$$

where the coefficients under the $(n+1)$ -th row are $\gamma_0 = 0$ and for $n \geq 1$,

$$\gamma_n = x + ny + n - 1, \quad (1.6b)$$

$$\beta_n = (\lambda + n - 1)(x + n - 1)y. \quad (1.6c)$$

It is interesting to note that a special case of (1.3a) does follow from (1.6a). Indeed, setting $\lambda = -1$ in (1.6a) yields

$$\sum_{n=0}^{\infty} D_n(x, y, -1) t^n = \frac{1 - (x+y)t}{(1-xt)(1-yt)} = \frac{\alpha}{1-xt} + \frac{\beta}{1-yt} \quad (1.7)$$

with $\alpha = -y/(x-y)$ and $\beta = x/(x-y)$, equivalently,

$$D_n(x, y, -1) = \frac{xy}{x-y} (y^{n-1} - x^{n-1}) = - \sum_{j=1}^{n-1} x^j y^{n-j}. \quad (1.8)$$

which is (1.3a) with $x_i = x$ and $y_i = y$ for $i \in [n]$.

This paper is organized as follows. In Section 2, we shall exploit the J-fraction (1.6a) to confirm several observations in [AG21] on $\mu_n(x) := D_n(x, 1, 1)$ and prove that the polynomials $\mu_n(x)$ are moments of *co-recursive Laguerre polynomials*. We then prove (1.4) (see Theorem 1) by constructing *weight-preserving and sign-reversing* (WPSR) involutions on derangements in Section 3. Finally, we establish two similar multivariate identities for permutations of type B, which unify some results in [AG21, Zh13] for permutations of both type A and type B in Section 4.

2. DERANGEMENTS WITH RIGHT-TO-LEFT MINIMA AND LAGUERRE POLYNOMIALS

Let $d_{n,k}$ be the number of derangements in \mathcal{D}_n with k right-to-left minima, namely

$$\mu_n(x) := \sum_{\sigma \in \mathcal{D}_n} x^{\text{rlm}(\sigma)} = \sum_{k=1}^n d_{n,k} x^k. \quad (2.1)$$

At the end of [AG21] Alexandersson and Getachew speculate some open problems about the coefficients $d_{n,k}$. We note that a more general multivariate polynomial of $\mu_n(x)$ had been studied in a recent paper by Sokal and the second author [SZ22]. More precisely, the polynomial \widehat{Q}_n in [SZ22, (2.29)] with $x_1 = x_2 = x$, $u_2 = v_1 = y_1 = u_1 = v_2 = y_2 = \lambda = 1$ and $w_\ell = 0$ becomes $\mu_n(x)$, therefore we derive from [SZ22, Theorem 2.4] that

$$\sum_{n=0}^{\infty} \mu_n(x) t^n = \frac{1}{1 - \frac{x t^2}{1 - (x+1)t - \frac{2(x+1)t^2}{1 - (x+3)t - \frac{3(x+2)t^2}{\dots}}}}, \quad (2.2a)$$

where the coefficients under the $(n+1)$ -th row are $\gamma_0 = 0$ and for $n \geq 1$,

$$\gamma_n = x + 2n - 1, \quad (2.2b)$$

$$\beta_n = n(x + n - 1). \quad (2.2c)$$

The above fraction generates immediately the first few values of $\mu_n(x)$ below (see [AG21, Table 2]).

$$\mu_2 = x$$

$$\mu_3 = x^2 + x$$

$$\mu_4 = x^3 + 5x^2 + 3x$$

$$\mu_5 = x^4 + 11x^3 + 21x^2 + 11x$$

$$\mu_6 = x^5 + 19x^4 + 79x^3 + 113x^2 + 53x$$

$$\mu_7 = x^6 + 29x^5 + 211x^4 + 589x^3 + 715x^2 + 309x$$

$$\mu_8 = x^7 + 41x^6 + 461x^5 + 2141x^4 + 4835x^3 + 5235x^2 + 2119x$$

$$\mu_9 = x^8 + 55x^7 + 883x^6 + 6175x^5 + 22357x^4 + 43831x^3 + 43507x^2 + 16687x$$

Clearly, for $n \geq 2$ we have $d_{n,n-1} = 1$ as $\sigma = n12 \cdots (n-1)$ is the only derangement $\sigma \in \mathcal{D}_n$ with $\text{rlm}(\sigma) = n-1$. The following recurrence for $d_n := |\mathcal{D}_n|$ is well-known,

$$d_n = (n-1)(d_{n-1} + d_{n-2}) \quad \text{with} \quad d_0 = 1, \quad d_1 = 0. \quad (2.3)$$

This can be verified as in the following. For $\sigma \in \mathcal{D}_n$ let $C = (n, \sigma(n), \sigma^2(n), \dots, \sigma^k(n))$ be the cycle containing n with $k \geq 1$. If $k = 1$, deleting the 2-cycle C in σ yields a derangement σ' of $[n-1] \setminus \{\sigma(n)\}$ and there are $(n-1)d_{n-2}$ such derangements; if $k \geq 2$, then replacing the cycle C by $C^* = (\sigma(n), \sigma^2(n), \dots, \sigma^k(n))$ in σ yields a derangement σ^* of $[n-1]$ and there are $(n-1)d_{n-1}$ such derangements σ .

Proposition 1. *For $n \geq 2$, let $\bar{d}_n = d_{n,1}$. Then*

$$\bar{d}_n = d_{n-1} + d_{n-2}, \quad (2.4)$$

$$d_n = (n-1)\bar{d}_n, \quad (2.5)$$

$$\bar{d}_n = (n-2)\bar{d}_{n-1} + (n-3)\bar{d}_{n-2} \quad (n \geq 3). \quad (2.6)$$

Proof. Consider a $\sigma \in \mathcal{D}_n$ with $\sigma(n) = 1$. Then either $\sigma(1) = n$ or $\sigma(1) \neq n$. In the first case the restriction of σ on $\{2, \dots, n-1\}$ is a derangement, in the second case define the permutation σ' of $\{2, \dots, n\}$ by $\sigma'(i) = \sigma(i)$ if $i \neq n$ and $\sigma'(n) = \sigma(1)$. Then σ' is a derangement of $\{2, \dots, n\}$. It follows that $\bar{d}_n = d_{n-1} + d_{n-2}$. Next we obtain (2.5) from (2.4) and (2.3). Finally we derive (2.6) from (2.4) and (2.5). \square

Remark 1. *Eq. (2.6) was observed in [AG21, Section 5.3], see A000255 [SI19]. As $\sum_{n \geq 0} d_n \frac{x^n}{n!} = \frac{e^{-x}}{1-x}$, we derive from (2.4) the exponential generating function*

$$\sum_{n \geq 0} d_{n+2,1} \frac{x^n}{n!} = \frac{e^{-x}}{(1-x)^2}. \quad (2.7)$$

By applying the Stieltjes-Rogers addition formula to (2.7) one can derive the following J-fraction, which also follows from (2.2a),

$$\sum_{n \geq 0} d_{n+2,1} t^n = \frac{1}{1 - t - \frac{2t^2}{1 - 3t - \frac{6t^2}{\dots}}}, \quad (2.8)$$

with $\gamma_0 = 1$, $\gamma_n = 2n + 1$ and $\beta_n = n(n+1)$ for $n \geq 1$.

It was observed in [AG21] that the sequence $d_{n,n-2}$ seems to be A028387 in [SI19]. We provide a proof of this observation by manipulating the continued fraction (2.2a), though there must be a simpler proof.

Proposition 2. *We have $d_{3,1} = 1$ and for $n \geq 3$,*

$$d_{n,n-2} = (n-3) + (n-2)^2. \quad (2.9)$$

Proof. In what follows we use $[x^k t^n] \sum_{n \geq 0} a_n(x) t^n$ to denote the coefficient of $x^k t^n$ in the formal series $\sum_{n \geq 0} a_n(x) t^n$ with $a_n(x) \in \mathbb{R}[x]$. Hence $d_{n,n-2} = [x^{n-2} t^n] \sum_{n=0}^{\infty} \mu_n(x) t^n$ for $n \geq 3$. By (2.2a) we have

$$\begin{aligned} d_{n,n-2} &= [x^{n-2} t^n] \frac{1}{1 - \frac{x t^2}{1 - (x+1)t - \frac{2(x+1)t^2}{1 - (x+3)t}}} \\ &= [x^{n-2} t^n] \frac{x t^2}{1 - (x+1)t} + [x^{n-2} t^n] \frac{x^2 t^4}{(1 - (x+1)t)^2} \\ &\quad + [x^{n-3} t^{n-2}] \sum_{j \geq 1} \left((x+1)t + \frac{2(x+1)t^2}{1 - (x+3)t} \right)^j \\ &= (n-2) + (n-3) + 2 \sum_{j=1}^{n-3} j \end{aligned}$$

which is equal to $(n-2)^2 + n - 3$. □

Remark 2. *The generating function is*

$$\sum_{n \geq 3} d_{n,n-2} x^{n-3} = \frac{1 + 2x - x^2}{(1-x)^3} = 1 + 5x + 11x^2 + 19x^3 + 29x^4 + \dots \quad (2.10)$$

By (2.9) we derive $d_{n+2,n} = d_{n+1,n-1} + 2n$ ($n \geq 2$) It should be possible to verify this recurrence directly.

2.1. The polynomials $\mu_n(a)$'s as moments. The Laguerre polynomials $L_n^\alpha(x)$ of degree n is

$$L_n^\alpha(x) = \sum_{i=0}^n (-1)^i \binom{n+\alpha}{n-i} \frac{x^i}{i!}.$$

The monic Laguerre polynomials $P_n^\alpha(x) = (-1)^n n! L_n^\alpha(x)$, whose weight function is

$$x^\alpha e^{-x}, \quad x \geq 0,$$

may be defined by the recurrence relation

$$P_{n+1}^\alpha(x) = (x - (\alpha + 2n + 1))P_n^\alpha(x) - n(n + \alpha)P_{n-1}^\alpha(x), \quad P_0^\alpha(x) = 1, \quad P_{-1}^\alpha(x) = 0. \quad (2.11)$$

The associated Laguerre polynomials $P_n^\alpha(x; c)$ are defined by the recurrence

$$P_{n+1}^\alpha(x; c) = (x - (\alpha + 2n + 2c + 1))P_n^\alpha(x; c) - (n+c)(n+c+\alpha)P_{n-1}^\alpha(x; c) \quad (2.12)$$

with $P_{-1}^\alpha(x; c) = 0$ and $P_0^\alpha(x; c) = 1$. It is well known that the moments of Laguerre polynomials are the enumerative polynomials of permutations by cycles or left-to-right

minima. Kim and Stanton [KS15] described the moments of $P_n^\alpha(x; c)$ using certain statistics on permutations and permutation tableaux. In the following we determine the orthogonal polynomials whose moments are the enumerative polynomials of derangements by right-to-left minima.

Proposition 3. *The polynomials $p_n(x)$ orthogonal with respect to the moment sequence $\{\mu_n(a)\}$ are given by*

$$p_n(x) = (-1)^n n! L_n^{a-1}(x+1) + (-1)^{n-1} (n-1)! (a-1) L_{n-1}^{a-1}(x+1; 1), \quad (2.13)$$

with $p_{-1}(x) = 0$ and $p_0(x) = 1$.

Proof. By (2.2) the orthogonal polynomials $P_n^*(x)$ with respect to the moments $\{\mu_n(a)\}$ satisfy the recurrence, for $n \geq 1$,

$$P_{n+1}^*(x) = (x - (a + 2n - 1))P_n^*(x) - n(n + a - 1)P_{n-1}^*(x) \quad (2.14)$$

with $P_0^*(x) = 1$, $P_1^*(x) = x$. By (2.11) and (2.12) we verify straightforwardly that the polynomials

$$p_n(x) := P_n^{a-1}(x+1) + (a-1)P_{n-1}^{a-1}(x+1; 1) \quad (2.15)$$

satisfy (2.14) with

$$p_1(x) = (x + 1 - a) + (a - 1) = x.$$

This establishes (2.13). \square

Remark 3. *One can also derive the above result from Chihara's theory about the co-recursive orthogonal polynomials [Chi57], see also [SS96]. The associated Laguerre polynomials have an explicit double sum formula, see [KS15].*

3. RIGHT-TO-LEFT MINIMA AND EXCEDANCES IN DERANGEMENTS

We partition \mathcal{D}_n ($n \geq 2$) according to the value of n and the position of 1 in derangements, respectively,

$$\mathcal{D}_{n,j} = \{\sigma \in \mathcal{D}_n : \sigma(n) = j\}, \quad (3.1a)$$

$$\tilde{\mathcal{D}}_{n,j} = \{\sigma \in \mathcal{D}_n : \sigma(n+1-j) = 1\} \quad (j \in [n-1]). \quad (3.1b)$$

The following is the main theorem of this section.

Theorem 1. *For $n \geq 2$ and $j \in [n-1]$, we have*

$$\sum_{\sigma \in \mathcal{D}_{n,j}} (-1)^{\text{cyc}(\sigma)} \left(\prod_{i \in \text{RLMv}(\sigma)} x_i \right) \left(\prod_{i \in \text{EXCv}(\sigma)} y_i \right) = -x_1 \cdots x_j y_{j+1} \cdots y_n, \quad (3.2a)$$

and

$$\sum_{\sigma \in \tilde{\mathcal{D}}_{n,j}} (-1)^{\text{cyc}(\sigma)} \left(\prod_{i \in \text{RLMi}(\sigma)} x_i \right) \left(\prod_{i \in \text{EXCi}(\sigma)} y_i \right) = -y_1 \cdots y_{n-j} x_{n+1-j} \cdots x_n. \quad (3.2b)$$

We first give a lemma which reduces the proof of the above two identities (3.2a) and (3.2b) by half. Note that the properties about FIX, EXC and RLM in the lemma appeared already in [AG21, Lemma 21].

Lemma 1. *The mapping $\zeta : \sigma \mapsto \rho \circ \sigma^{-1} \circ \rho$ on \mathfrak{S}_n with $\rho(i) = n + 1 - i$ for $i \in [n]$ is an involution on \mathfrak{S}_n such that for $j \in [n]$,*

$$\begin{aligned} \text{cyc}(\sigma) &= \text{cyc}(\zeta(\sigma)), \\ \sigma(n) = j &\iff \zeta(\sigma)(n + 1 - j) = 1, \\ j \in \text{FIX}(\sigma) &\iff \rho(j) \in \text{FIX}(\zeta(\sigma)), \\ j \in \text{EXCi}(\sigma) &\iff \rho(j) \in \text{EXCv}(\zeta(\sigma)), \\ j \in \text{RLMi}(\sigma) &\iff \rho(j) \in \text{RLMv}(\zeta(\sigma)). \end{aligned}$$

Moreover, the restriction $\zeta : \mathcal{D}_{n,j} \rightarrow \tilde{\mathcal{D}}_{n,j}$ (resp. $\zeta : \tilde{\mathcal{D}}_{n,j} \rightarrow \mathcal{D}_{n,j}$) is a bijection for $j \in [n - 1]$.

Proof. For $\sigma \in \mathfrak{S}_n$, the two permutations σ and $\zeta(\sigma)$ are conjugate in \mathfrak{S}_n , so they have the same cycle type and a fortiori $\text{cyc}(\sigma) = \text{cyc}(\zeta(\sigma))$. Next,

$$\zeta(\sigma)(n + 1 - j) = 1 \iff \sigma^{-1}(j) = n \iff \sigma(n) = j;$$

$$\rho(j) \in \text{FIX}(\zeta(\sigma)) \iff \zeta(\sigma)(\rho(j)) = \rho(j) \iff \rho \circ \sigma^{-1}(j) = \rho(j) \iff \sigma(j) = j.$$

For the fourth and fifth properties, as $(\zeta(\sigma))^{-1}(\rho(j)) = \rho \circ \sigma(j)$, we have

$$\rho(j) \in \text{EXCv}(\zeta(\sigma)) \iff \rho(j) > \rho(\sigma(j)) \iff j < \sigma(j) \iff j \in \text{EXCi}(\sigma);$$

$$\begin{aligned} j \in \text{RLMi}(\sigma) &\iff \sigma(j) < \sigma(k) \quad \text{for } k > j \\ &\iff (\zeta(\sigma))^{-1}(\rho(j)) > (\zeta(\sigma))^{-1}(\rho(k)) \quad \text{for } \rho(k) < \rho(j) \\ &\iff \rho(j) \in \text{RLMv}(\zeta(\sigma)). \end{aligned}$$

The last property follows from the second and third properties. \square

For convenience, we illustrate the properties of mapping ζ by an example.

Example 1. If $\sigma = (1, 3, 6, 8, 4)(5, 9)(2)(7)$, then $\sigma^{-1} = (1, 4, 8, 6, 3)(5, 9)(2)(7)$ and $\zeta(\sigma) = (9, 6, 2, 4, 7)(5, 1)(8)(3)$. Hence

$$\sigma = 326198745 \in \mathcal{D}_{9,5}, \quad \zeta(\sigma) = 543712986 \in \widetilde{\mathcal{D}}_{9,5}.$$

and

$$\begin{aligned} \text{RLMi}(\sigma) &= \{4, 8, 9\}, & \text{RLMv}(\zeta(\sigma)) &= \{1, 2, 6\}, \\ \text{EXCi}(\sigma) &= \{1, 3, 5, 6\}, & \text{EXCv}(\zeta(\sigma)) &= \{4, 5, 7, 9\}, \\ \text{FIX}(\sigma) &= \{2, 7\}, & \text{FIX}(\zeta(\sigma)) &= \{3, 8\}. \end{aligned}$$

Lemma 2. The two identities (3.2a) and (3.2b) are equivalent.

Proof. For $i \in [n]$, the mapping $\rho : i \mapsto n + 1 - i$ is a permutation of $[n]$. Now, making substitution $x_i \mapsto x_{\rho(i)}$ and $y_i \mapsto y_{\rho(i)}$ for $i \in [n]$ in (3.2b), we obtain

$$\sum_{\sigma \in \mathcal{D}_{n,j}} (-1)^{\text{cyc}(\sigma)} \left(\prod_{i \in \text{RLMi}(\sigma)} x_{\rho(i)} \right) \left(\prod_{i \in \text{EXCi}(\sigma)} y_{\rho(i)} \right) = -x_1 \cdots x_j y_{j+1} \cdots y_n. \quad (3.3)$$

By Lemma 1, the restriction of ζ on $\widetilde{\mathcal{D}}_{n,j}$ sets up a bijection from $\widetilde{\mathcal{D}}_{n,j}$ to $\mathcal{D}_{n,j}$ such that

$$\begin{aligned} j \in \text{EXCi}(\sigma) &\iff \rho(j) \in \text{EXCv}(\zeta(\sigma)), \\ j \in \text{RLMi}(\sigma) &\iff \rho(j) \in \text{RLMv}(\zeta(\sigma)). \end{aligned}$$

Therefore, letting $\zeta(\sigma) = \tau$ for $\sigma \in \widetilde{\mathcal{D}}_{n,j}$, eq. (3.3) is equivalent to

$$\sum_{\tau \in \mathcal{D}_{n,j}} (-1)^{\text{cyc}(\tau)} \left(\prod_{i \in \text{RLMv}(\tau)} x_i \right) \left(\prod_{i \in \text{EXCv}(\tau)} y_i \right) = -x_1 \cdots x_j y_{j+1} \cdots y_n,$$

which is exactly (3.2a). \square

For any derangement $\sigma \in \mathcal{D}_n$, as $\text{EXCi}(\sigma) = [n] \setminus \text{EXCv}(\sigma^{-1})$, we derive from (3.2a) the following result.

Corollary 1. Let $\widehat{\mathcal{D}}_{n,j} = \{\sigma \in \mathcal{D}_n : \sigma(j) = n\}$ for $j \in [n-1]$. Then

$$\sum_{\sigma \in \widehat{\mathcal{D}}_{n,j}} (-1)^{\text{cyc}(\sigma)} \prod_{i \in \text{EXCi}(\sigma)} x_i = -x_1 \cdots x_j. \quad (3.4)$$

Note that multiplying (3.4) by t^j and summing over $j \in [n-1]$ yields [Si16, Theorem 7].

3.1. Proof of Theorem 1. Thanks to Lemma 2 we only need to prove (3.2a). For $j \in \{1, 2, \dots, n-1\}$ with $n \geq 3$, we divide $\mathcal{D}_{n,j}$ into two parts:

$$U_{n,j} = \begin{cases} \{(1, 2, \dots, n-1, n)\} & \text{if } j = 1, \\ \{\sigma \in \mathcal{D}_{n,j} \mid \sigma(1) \neq 2, \sigma(2) = 1\} & \text{if } j \geq 2, \end{cases} \quad (3.5a)$$

and

$$\bar{U}_{n,j} = \mathcal{D}_{n,j} \setminus U_{n,j} \quad (1 \leq j \leq n-1). \quad (3.5b)$$

Let $\mathbf{x} = (x_k)_{k \geq 1}$ and $\mathbf{y} = (y_k)_{k \geq 1}$ be two sequences of variables. For $\sigma \in \mathcal{D}_n$ define the weight

$$w(\sigma; \mathbf{x}, \mathbf{y}) = (-1)^{\text{cyc}(\sigma)} \prod_{i \in \text{RLMv}(\sigma)} x_i \prod_{i \in \text{EXCv}(\sigma)} y_i. \quad (3.6)$$

Lemma 3. For $\sigma \in \bar{U}_{n,j}$ with $j \in [n-1]$, let $\varphi(\sigma) = (\sigma(i_\sigma), \sigma(i_\sigma + 1)) \circ \sigma$, where i_σ is the smallest integer i such that $\sigma(i) \neq i+1$. Then φ is a weight-preserving and sign-reversing (WPSR) involution on $\bar{U}_{n,j}$, i.e., $w(\sigma; \mathbf{x}, \mathbf{y}) = -w(\varphi(\sigma); \mathbf{x}, \mathbf{y})$.

Proof. Let $\sigma \in \bar{U}_{n,j}$. Obviously permutations $\varphi(\sigma)$ and σ have opposite signs. To show that φ is an involution on $\bar{U}_{n,j}$, we verify the following three points.

- $\varphi(\sigma) \in \mathcal{D}_{n,j}$. If $j = 1$, as $\sigma \neq (1, 2, \dots, n-1, n)$, there is an integer $i \in [n-2]$ such that $\sigma(i) \neq i+1$, so $1 \leq i_\sigma \leq n-2$; if $j > 1$, as $\sigma(n) = j$, we have $\sigma(j-1) \neq j$, so $1 \leq i_\sigma \leq j-1$. In any case, we have $n > i_\sigma + 1$, so $\varphi(\sigma)(n) = \sigma(n) = j$. Besides $\{\sigma(i_\sigma), \sigma(i_\sigma + 1)\} \cap \{i_\sigma, i_\sigma + 1\} = \emptyset$. Indeed, as σ is a derangement, we have $\sigma(i) \neq i$ for any $i \in [n]$; if $i_\sigma = 1$, then $\sigma(1) \neq 2$ and $\sigma(2) \neq 1$ because $\sigma \in \bar{U}_{n,j}$; if $i_\sigma > 1$, then $\sigma(i_\sigma) \neq i_\sigma + 1$ and $\sigma(i_\sigma + 1) \neq i_\sigma$ because $\sigma(i_\sigma - 1) = i_\sigma$. Hence, by (1) we have $\varphi(\sigma) \in \mathcal{D}_{n,j}$.
- $\varphi(\sigma) \notin U_{n,j}$. If $j = 1$, then $i_\sigma < n-1$ and $\varphi(\sigma)(i_\sigma) = \sigma(i_\sigma + 1) \neq i_\sigma + 1$. Hence $\varphi(\sigma) \notin U_{n,1}$; If $j > 1$ and $i_\sigma = 1$, then $\varphi(\sigma)(2) = \sigma(1) \neq 1$, if $j > 1$ and $i_\sigma > 1$, then $\sigma(1) \notin \{\sigma(i_\sigma), \sigma(i_\sigma + 1)\}$, so $\varphi(\sigma)(1) = \sigma(1) = 2$ because $1 < i_\sigma$. Hence $\varphi(\sigma) \notin U_{n,j}$ for $j > 1$.
- $\varphi^2(\sigma) = \sigma$. If $i < i_\sigma$, then $\varphi(\sigma)(i) = \sigma(i) = i+1$ and $\varphi(\sigma)(i_\sigma) = \sigma(i_\sigma + 1) \neq i_\sigma + 1$. Thus $i_\sigma = i_{\varphi(\sigma)}$ and $(\varphi(\sigma)(i_\sigma), \varphi(\sigma)(i_\sigma + 1)) = (\sigma(i_\sigma + 1), \sigma(i_\sigma))$. So $\varphi^2(\sigma) = \sigma$.

Next, we show that φ is weight-preserving by verifying the following two points.

- $\text{RLMv}(\sigma) = \text{RLMv}(\varphi(\sigma))$: If $j = 1$, then $\text{RLMv}(\sigma) = \text{RLMv}(\varphi(\sigma)) = \{1\}$. So we consider the case $j > 1$ in the following.
 - (1) If $\sigma(i_\sigma) = 1$, then the first i_σ letters of σ are $2, 3, \dots, i_\sigma, 1$, so $\sigma(i_\sigma + 1) > i_\sigma + 1$ and there must exist an integer k satisfied $k > i_\sigma + 1$ and $\sigma(k) = i_\sigma + 1$,

thus $\sigma(i_\sigma + 1) \notin \text{RLMv}(\sigma)$. We have $\varphi(\sigma)(i_\sigma + 1) = 1 \in \text{RLMv}(\varphi(\sigma))$ and $\varphi(\sigma)(i_\sigma) = \sigma(i_\sigma + 1) \notin \text{RLMv}(\varphi(\sigma))$.

(2) If $\sigma(i_\sigma) > 1$ and $\sigma(i_\sigma + 1) = 1 \in \text{RLMv}(\sigma)$, there is an integer $k > i_\sigma + 1$ such that $\sigma(k) = i_\sigma + 1$, so $\sigma(i_\sigma) \notin \text{RLMv}(\sigma)$, $\varphi(\sigma)(i_\sigma + 1) = \sigma(i_\sigma) \notin \text{RLMv}(\varphi(\sigma))$ and $\varphi(\sigma)(i_\sigma) = 1 \in \text{RLMv}(\varphi(\sigma))$.

(3) If $\sigma(i_\sigma) > 1$ and $\sigma(i_\sigma + 1) > 1$, then 1 appears at the right of the letter $\sigma(i_\sigma + 1)$ in σ , hence $\varphi(\sigma)(i_\sigma + 1) \notin \text{RLMv}(\varphi(\sigma))$.

- $\text{EXCv}(\sigma) = \text{EXCv}(\varphi(\sigma))$: As $\sigma(i_\sigma) \neq i_\sigma, i_\sigma + 1$, it is clear that $\sigma(i_\sigma) > i_\sigma$ if and only if $\varphi(\sigma)(i_\sigma + 1) = \sigma(i_\sigma) > i_\sigma + 1$. Similarly, we see that $\sigma(i_\sigma + 1) > i_\sigma + 1$ if and only if $\varphi(\sigma)(i_\sigma) = \sigma(i_\sigma + 1) > i_\sigma$. Hence $\text{EXCv}(\sigma) = \text{EXCv}(\varphi(\sigma))$.

Summarizing we conclude that φ is the mapping we wished. \square

Now, we are ready to complete the proof of Theorem 1.

Proof of Theorem 1. By Lemma 3, for $j \in [n - 1]$ we have

$$P_{n,j}(\mathbf{x}, \mathbf{y}) := \sum_{\sigma \in \mathcal{D}_{n,j}} w(\sigma; \mathbf{x}, \mathbf{y}) = \sum_{\sigma \in U_{n,j}} w(\sigma; \mathbf{x}, \mathbf{y}).$$

By (3.5a), the only element in $U_{n,1}$ is $\sigma = 23 \dots n1$, therefore

$$P_{n,1}(\mathbf{x}, \mathbf{y}) = -x_1 y_2 \dots y_n. \quad (3.7)$$

For $\sigma \in \mathcal{D}_n$ and $m \in \mathbb{N}$ define the shifted version of (3.6) by

$$w(\sigma; \delta^m(\mathbf{x}), \delta^m(\mathbf{y})) := (-1)^{\text{cyc}(\sigma)} \prod_{i \in \text{RLMv}(\sigma)} x_{i+m} \prod_{i \in \text{EXCv}(\sigma)} y_{i+m}. \quad (3.8)$$

If $\sigma \in U_{n,j}$ with $j > 1$, then $\sigma(2) = 1$, let $\psi(\sigma) = \sigma'(1)\sigma'(3) \dots \sigma'(n)$ with $\sigma'(i) = \sigma(i) - 1$ for $i \in \{1, 3, \dots, n\}$. It is easy to see that the mapping $\psi : U_{n,j} \rightarrow \mathcal{D}_{n-1,j-1}$ is a bijection such that $w(\sigma; \mathbf{x}, \mathbf{y}) = x_1 w(\sigma'; \delta(\mathbf{x}), \delta(\mathbf{y}))$, thus

$$P_{n,j}(\mathbf{x}, \mathbf{y}) = x_1 P_{n-1,j-1}(\delta(\mathbf{x}), \delta(\mathbf{y}))$$

and by iterating, we obtain the equation

$$P_{n,j}(\mathbf{x}, \mathbf{y}) = x_1 \dots x_{j-1} P_{n-j+1,1}(\delta^{j-1}(\mathbf{x}), \delta^{j-1}(\mathbf{y}))$$

from which follows (1.4a) by invoking (3.7). \square

4. IDENTITIES FOR PERMUTATIONS OF TYPE B

Let B_n be the set of permutations σ of $[n] \cup \{-n, \dots, -1\}$ such that $\sigma(-i) = -\sigma(i)$ for each $i \in [n]$. As usual, we denote the negative element $-i$ by \bar{i} , identify σ with the word $\sigma = \sigma(1) \cdots \sigma(n)$ and write $|\sigma| := |\sigma(1)| \dots |\sigma(n)|$ the associated permutation of $[n]$. We follow Brenti [Br94] for definition of permutation statistics of type B. The *cycle decomposition* of $\sigma \in B_n$ is accomplished by first writing $|\sigma|$ as the disjoint union of cycles, then turning each element $|\sigma(i)|$ to $\sigma(i)$. Let $\text{cyc}(\sigma)$ be the number of cycles of $|\sigma|$. For example, if $\sigma = \bar{6}42\bar{3}158\bar{7}$, then $|\sigma| = (1, 6, 5)(3, 2, 4)(7, 8)$ and $\sigma = (1, \bar{6}, 5)(\bar{3}, 2, 4)(\bar{7}, 8)$, hence $\text{cyc}(\sigma) = 3$.

4.1. Excedance of type B. If $\sigma \in B_n$, the index $|\sigma(i)|$ with $i \in [n]$ is a type B *excedance* of σ if $\sigma(|\sigma(i)|) = -|\sigma(i)|$ or $\sigma(|\sigma(i)|) > \sigma(i)$, and the index $|\sigma(i)|$ is a type B *anti-excedance* of σ if $\sigma(|\sigma(i)|) = |\sigma(i)|$ or $\sigma(|\sigma(i)|) < \sigma(i)$. Let

$$\text{EXC}_B(\sigma) = \{|\sigma(i)| : \sigma(|\sigma(i)|) = -|\sigma(i)| \text{ or } \sigma(|\sigma(i)|) > \sigma(i), i \in [n]\},$$

$$\text{AnEXC}_B(\sigma) = \{|\sigma(i)| : \sigma(|\sigma(i)|) = |\sigma(i)| \text{ or } \sigma(|\sigma(i)|) < \sigma(i), i \in [n]\}.$$

For example, if $\sigma = \bar{6}42\bar{3}158\bar{7}\bar{9} = (1, \bar{6}, 5)(\bar{3}, 2, 4)(\bar{7}, 8)(\bar{9})$, then

$$\text{EXC}_B(\sigma) = \{1, 2, 7, 9\},$$

$$\text{AnEXC}_B(\sigma) = \{3, 4, 5, 6, 8\}.$$

Remark 4. The above definition of excedance is given by Brenti [Br94], see another notion of excedance of type B in [St94].

If $\sigma \in B_n$ let $\text{Neg}(\sigma) = \{i \in [n] : \sigma(i) < 0\}$, and define the statistics

$$\text{neg}(\sigma) = \#\text{Neg}(\sigma), \quad \text{nsum}(\sigma) = \sum_{i \in \text{Neg}(\sigma)} |\sigma(i)|.$$

We define the enumerative polynomial of a subset $T \subseteq B_n$ by

$$P_T(\mathbf{x}, s, t) = \sum_{\sigma \in T} w_\sigma(\mathbf{x}, s, t)$$

with

$$w_\sigma(\mathbf{x}, s, t) = (-1)^{\text{cyc}(\sigma)} s^{\text{neg}(\sigma)} t^{\text{nsum}(\sigma)} \prod_{i \in \text{EXC}_B(\sigma)} x_i.$$

Alexandersson and Getachew [AG21, Proposition 29] proved the identity:

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\text{cyc}(\sigma)} \prod_{i \in \text{EXC}_i(\sigma)} x_i = - \prod_{j=1}^{n-1} (x_j - 1). \quad (4.1)$$

The special $x_1 = \cdots = x_n = x$ case of this identity is well known, see [Br00, KZ03] and a type B analogue of the latter case was given by Zhao [Zh13, Theorem 3] as follows:

$$P_{B_n}(\mathbf{x}, 1, 1) = \begin{cases} -(x+1)(x-1)^{n-1} & \text{if } n \text{ is odd;} \\ (x-1)^n & \text{if } n \text{ is even.} \end{cases} \quad (4.2)$$

We can unify (4.1) and (4.2) by the following theorem.

Theorem 2. *Let $n \geq 1$, then we have*

$$P_{B_n}(\mathbf{x}, s, t) = - \left(1 + (-1)^{n-1} x_n s^n t^{\frac{n(n+1)}{2}} \right) \prod_{j=1}^{n-1} (x_j - 1). \quad (4.3)$$

In order to prove this identity we divide the set B_n into three subsets:

$$\begin{aligned} B_n^+ &= \{\sigma \in B_n : \sigma(i) > 0 \text{ for all } i \in [n]\}, \\ B_n^- &= \{\sigma \in B_n : \sigma(i) < 0 \text{ for all } i \in [n]\}, \\ B_n^\pm &= \{\sigma \in B_n : \sigma(i)\sigma(j) < 0 \text{ for some } i, j \in [n]\} \end{aligned}$$

and compute their enumerative polynomials respectively.

Clearly the subset B_n^+ is \mathfrak{S}_n , so the corresponding identity is (4.1). We give an alternative proof of (4.1) by adapting the proof in [KZ03] for a special case.

Lemma 4 ([AG21]). *Let $n \geq 1$, then*

$$P_{B_n^+}(\mathbf{x}, s, t) = - \prod_{j=1}^{n-1} (x_j - 1). \quad (4.4)$$

Proof. Obviously the identity is true for $n = 1, 2$. Assume $n \geq 3$ and divide B_n^+ into three subsets:

$$\begin{aligned} B'_n &= \{\sigma \in B_n^+ : \sigma(n-1) \neq n \text{ and } \sigma(n) \neq n\}, \\ B''_n &= \{\sigma \in B_n^+ : \sigma(n) = n\}, \\ B'''_n &= \{\sigma \in B_n^+ : \sigma(n-1) = n\}. \end{aligned}$$

It is easy to verify that the mapping $\varphi_1 : \sigma \mapsto \sigma' = (\sigma(n-1), \sigma(n)) \circ \sigma$ is a WPSR involution on B'_n , i.e., $w_\sigma(\mathbf{x}, s, t) = -w_{\sigma'}(\mathbf{x}, s, t)$; the mapping $\varphi_2 : \sigma \mapsto \sigma' = \sigma(1) \cdots \sigma(n-1)$ is a bijection from B''_n to B_{n-1}^+ such that $w_\sigma(\mathbf{x}, s, t) = -w_{\sigma'}(\mathbf{x}, s, t)$, and the mapping $\varphi_2 \circ \varphi_1 : \sigma \mapsto \sigma'$ is a bijection from B'''_n to B_{n-1}^+ such that $w_\sigma(\mathbf{x}, s, t) = x_{n-1} w_{\sigma'}(\mathbf{x}, s, t)$. Combining the three cases, we have

$$P_{B_n^+}(\mathbf{x}, s, t) = (x_{n-1} - 1) P_{B_{n-1}^+}(\mathbf{x}, s, t).$$

As $P_{B_2^+}(\mathbf{x}, s, t) = 1 - x_1$, eq. (4.4) follows by iterating the above recurrence. \square

Lemma 5. *Let $n \geq 1$, then*

$$P_{B_n^-}(\mathbf{x}, s, t) = (-1)^n s^n t^{n(n+1)/2} x_n \prod_{j=1}^{n-1} (x_j - 1). \quad (4.5)$$

Proof. As $P_{B_n^-}(\mathbf{x}, s, t) = s^n t^{n(n+1)/2} P_{B_n^-}(\mathbf{x}, 1, 1)$, we assume that $s = t = 1$ in the following. For $\sigma \in B_n^+$, the mapping $\varphi : \sigma \rightarrow \bar{\sigma}$, where $\bar{\sigma}$ is the permutation defined by $\bar{\sigma}(i) := -\sigma(i)$ for $i \in [n]$, is a bijection from B_n^+ to B_n^- . It is clear that $i \in \text{AnEXC}_B(\sigma)$ if and only if $i \in \text{EXC}_B(\bar{\sigma})$. For $n \geq 1$, we have

$$\begin{aligned} P_{B_n^-}(\mathbf{x}, 1, 1) &= \sum_{\sigma \in B_n^-} (-1)^{\text{cyc}(\sigma)} \prod_{i \in \text{EXC}_B(\sigma)} x_i \\ &= \sum_{\sigma \in B_n^+} (-1)^{\text{cyc}(\sigma)} \prod_{i \in \text{AnEXC}_B(\sigma)} x_i \\ &= x_1 \cdots x_n \cdot \sum_{\sigma \in B_n^+} (-1)^{\text{cyc}(\sigma)} \prod_{i \in \text{EXC}_B(\sigma)} x_i^{-1} \end{aligned}$$

from which eq. (4.5) follows from (4.4). \square

For $a = n$ or \bar{n} , $k \in [n]$ and $\sigma \in B_{n-1}$ we define the permutation $\sigma' = \Psi(a, k, \sigma)$ in B_n by $\sigma'(k) = a$, $\sigma'(n) = \sigma(k)$, $\sigma'(i) = \sigma(i)$ for $i \in [n-1] \setminus \{k\}$. Note that if $k = |a| = n$ the operation amounts to add the cycle (a) to σ . The mapping Ψ will be called the *insertion operation* on B_{n-1} in what follows. For a permutation $\sigma \in B_n$, let $\text{Im}(\sigma) = \{\sigma(i) : i \in [n]\}$.

Lemma 6. *Let $n \geq 1$, then*

$$P_{B_n^\pm}(\mathbf{x}, s, t) = 0. \quad (4.6)$$

Proof. By induction on $n \geq 1$, we construct a WPSR involution without fixed point on B_n^\pm . For convenience, we shall describe such an involution by matching all the elements of B_n^\pm two by two $\{\sigma, \sigma'\}$ such that $w_\sigma(\mathbf{x}, s, t) = -w_{\sigma'}(\mathbf{x}, s, t)$.

Obviously $B_1^\pm = \emptyset$ and $\{\{\bar{1}2, 2\bar{1}\}, \{1\bar{2}, \bar{2}1\}\}$ is a WPSR perfect matching of B_2^\pm , i.e.,

$$\begin{aligned} w_{\bar{1}2}(\mathbf{x}, s, t) + w_{2\bar{1}}(\mathbf{x}, s, t) &= x_1 s t - x_1 s t = 0, \\ w_{1\bar{2}}(\mathbf{x}, s, t) + w_{\bar{2}1}(\mathbf{x}, s, t) &= x_2 s t^2 - x_2 s t^2 = 0. \end{aligned}$$

Hence $P_{B_2^\pm}(\mathbf{x}, s, t) = 0$. Note that $\text{Im}(\sigma) = \text{Im}(\sigma')$.

Let $n \geq 3$. It is easy to see that any $\sigma \in B_n^\pm$ can be obtained by applying the insertion operation $\Psi(a, k, \tau)$ with $a = n$ or \bar{n} , $k \in [n]$ and $\tau \in B_{n-1}$. More precisely, there are four cases:

- (1) $a = \bar{n}$ and $\tau \in B_{n-1}^+$, (1') $a = n$ and $\tau \in B_{n-1}^-$;
- (2) $a = \bar{n}$ and $\tau \in B_{n-1}^\pm$, (2') $a = n$ and $\tau \in B_{n-1}^\pm$.

It suffices to show that the sum of weights of each of the four cases is zero. We just consider the cases (1) and (2) since the other two cases are similar.

- (1) Let C_n be the set of permutations obtained by procedure (1). We show that there is such a WPSR perfect matching on C_n . For any $\sigma \in C_n$ let $\sigma' = (\sigma(n-1), \sigma(n)) \circ \sigma$. It is easy to check that $\{\{\sigma, \sigma'\} \mid \sigma \in C_n\}$ is a WPSR perfect matching of C_n .
- (2) Let C'_n be the sets of permutations obtained by procedure (2). We show that there are such WPSR perfect matching on C_n and C'_n . Assume that there exists a WPSR perfect matching M_{n-1} of B_{n-1}^\pm and $(\tau, \tau') \in M_{n-1}$ is a doubleton such that $\text{Im}(\tau) = \text{Im}(\tau')$. We claim that applying the insertion operation Φ on C'_{n-1} yields a perfect matching of C'_n , where $k \in [n]$.

- If $k = n$, the operation amounts to add the cycle (\bar{n}) to τ and τ' . So $\text{cyc}(\sigma) = \text{cyc}(\tau) + 1$, $\text{cyc}(\sigma') = \text{cyc}(\tau') + 1$, $\text{EXC}_B(\sigma) = \text{EXC}_B(\tau) \cup \{n\}$ and $\text{EXC}_B(\sigma') = \text{EXC}_B(\tau') \cup \{n\}$.
- If $k \neq n$, it is equivalent to insert \bar{n} in cycles $c = (\dots \epsilon, \tau(k) \dots)$ and $c' = (\dots \epsilon, \tau'(k) \dots)$ of τ and τ' as image of $\epsilon = k$ or \bar{k} , then we obtain the corresponding cycles $(\dots \epsilon, \bar{n}, \tau(k) \dots)$ and $(\dots \epsilon, \bar{n}, \tau'(k) \dots)$ in σ and σ' . Thus $\text{cyc}(\sigma) = \text{cyc}(\tau)$ and $\text{cyc}(\sigma') = \text{cyc}(\tau')$. If $k \in \text{EXC}_B(\tau)$, then $\text{EXC}_B(\sigma) = \text{EXC}_B(\sigma') = \text{EXC}_B(\tau) \cup \{n\} \setminus \{k\}$. If $k \notin \text{EXC}_B(\tau)$, then $\text{EXC}_B(\sigma) = \text{EXC}_B(\sigma') = \text{EXC}_B(\tau) \cup \{n\}$.

As $\text{Im}(\tau) = \text{Im}(\tau')$ we have a WPSR perfect matching M_n on C'_n .

Combining the above two cases and the initial conditions, we derive (4.6). \square

Example 2. Inserting $\bar{3}$ into matchings $\{\bar{1}2, 2\bar{1}\}$ and $\{1\bar{2}, \bar{2}1\}$ results

$$\begin{aligned} w_{1\bar{2}\bar{3}}(\mathbf{x}, s, t) + w_{2\bar{1}\bar{3}}(\mathbf{x}, s, t) &= -x_1x_3s^2t^4 + x_1x_3s^2t^4 = 0, \\ w_{\bar{1}32}(\mathbf{x}, s, t) + w_{2\bar{3}1}(\mathbf{x}, s, t) &= x_1x_3s^2t^4 - x_1x_3s^2t^4 = 0, \\ w_{\bar{3}12}(\mathbf{x}, s, t) + w_{\bar{3}21}(\mathbf{x}, s, t) &= -x_3s^2t^4 + x_3s^2t^4 = 0. \end{aligned}$$

Proof of Theorem 2. Clearly, Theorem 2 follows from Lemmas 4 through 6. \square

4.2. Right-to-left minima of type B. Petersen [Pe11] defined the set of *right-to-left-minimum values* of $\sigma \in B_n$ by

$$\text{RLM}_B(\sigma) = \{\sigma(i) : 0 < \sigma(i) < |\sigma(j)| \text{ for all } j > i\}.$$

Consider the enumerative polynomial

$$Q_{B_n}(\mathbf{y}, s, t) := \sum_{\sigma \in B_n} w_\sigma(\mathbf{y}, s, t)$$

where

$$w_\sigma(\mathbf{y}, s, t) = (-1)^{\text{cyc}(\sigma)} s^{\text{neg}(\sigma)} t^{\text{nsum}(\sigma)} \prod_{j \in \text{RLM}_B(\sigma)} y_j.$$

The following is a right-to-left minima analogue of Theorem 2.

Theorem 3. *For $n \geq 1$, we have*

$$Q_{B_n}(\mathbf{y}, s, t) = (-1)^n \prod_{i \in [n]; i \text{ even}} (y_i - 1) \prod_{i \in [n]; i \text{ odd}} (y_i + st^i). \quad (4.7)$$

Proof. The formula is obvious for $n = 1$, i.e., $Q_{B_1}(\mathbf{y}, s, t) = -(y_1 + st)$. For $n \geq 2$, partition B_n in two subsets,

$$\begin{aligned} B'_n &= \{\sigma \in B_n : |\sigma(n)| = n\}, \\ B''_n &= \{\sigma \in B_n : |\sigma(n)| \neq n\}. \end{aligned}$$

For $\sigma \in B'_n$, let σ' be the restriction of σ on $[n-1]$. Then $\phi_1 : \sigma \mapsto \sigma'$ is a bijection from B'_n to $\{n, \bar{n}\} \times B_{n-1}$ satisfying

$$w_\sigma(\mathbf{y}, s, t) = \begin{cases} -y_n w_{\sigma'}(\mathbf{y}, s, t) & \text{if } \sigma(n) = n, \\ -st^n w_{\sigma'}(\mathbf{y}, s, t) & \text{if } \sigma(n) = \bar{n}. \end{cases}$$

It follows that

$$\sum_{\sigma \in B'_n} w_\sigma(\mathbf{y}, s, t) = -(y_n + st^n) Q_{B_{n-1}}(\mathbf{y}, s, t). \quad (4.8)$$

Let

$$C_n^k = \{\sigma \in B''_n : |\sigma(2k-1)| = n\},$$

where $k \in \{1, \dots, (n-1)/2\}$ if n is odd and $k \in \{1, \dots, (n-2)/2\}$ if n is even. Then the mapping $\sigma \mapsto (\sigma(2k-1), \sigma(2k)) \circ \sigma$ is a WPSR involution on C_n^k . Therefore

$$\sum_{\sigma \in B''_n} w_\sigma(\mathbf{y}, s, t) = \sum_{\sigma \in E_n} w_\sigma(\mathbf{y}, s, t) \quad (4.9)$$

with $E_n = \{\sigma \in B''_n : |\sigma(n-1)| = n\}$ if n is even and $E_n = \emptyset$ if n is odd.

If n is even, for $\sigma \in E_n$ let σ' be the restriction of σ on $[n] \setminus \{n-1\}$. Then $\phi_2 : \sigma \mapsto \sigma'$ is a bijection from E_n to $\{n, \bar{n}\} \times B_{n-1}$ satisfying

$$w_\sigma(\mathbf{y}, s, t) = \begin{cases} w_{\sigma'}(\mathbf{y}, s, t) & \text{if } \sigma(n-1) = n, \\ st^n w_{\sigma'}(\mathbf{y}, s, t) & \text{if } \sigma(n-1) = \bar{n}. \end{cases}$$

Hence, if n is even, then

$$\sum_{\sigma \in E_n} w_\sigma(\mathbf{y}, s, t) = (1 + st^n) \cdot Q_{B_{n-1}}(\mathbf{y}, s, t). \quad (4.10)$$

It follows from (4.8), (4.9) and (4.10) that

$$Q_{B_n}(\mathbf{y}, s, t) = \begin{cases} -(y_n + st^n) \cdot Q_{B_{n-1}}(\mathbf{y}, s, t) & \text{if } n \text{ is odd;} \\ -(y_n - 1) \cdot Q_{B_{n-1}}(\mathbf{y}, s, t) & \text{if } n \text{ is even.} \end{cases}$$

Invoking the initial value of $Q_{B_1}(\mathbf{y}, s, t)$, this recurrence implies (4.7). \square

Remark 5. When $s = 0$ eq. (4.7) reduces to [AG21a, Corollary 22], see also [AG21, Corollary 36]:

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\text{cyc}(\sigma)} \prod_{i \in \text{RLMv}(\sigma)} y_i = (-1)^n \prod_{\substack{j \in [n] \\ j \text{ odd}}} y_j \cdot \prod_{\substack{j \in [n] \\ j \text{ even}}} (y_j - 1) \quad (4.11)$$

which is also the $q = -1$ case of a formula due to Björner and Wachs [BW91, (5.3)],

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} \prod_{j \in \text{RLMv}(\sigma)} y_j = \prod_{i=1}^n (y_i + q + \cdots + q^{i-1}). \quad (4.12)$$

Note that Poznanović [Po14] and Eu et al. [ELW15] have given far-reaching generalizations of (4.12) to other types of permutations and colored permutations.

5. ACKNOWLEDGEMENTS

The first author was supported by the *China Scholarship Council*. This work was done during her visit at Université Claude Bernard Lyon 1 in 2021-2022.

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(Yanni Pei) SCHOOL OF MATHEMATICAL SCIENCES, DALIAN UNIVERSITY OF TECHNOLOGY, DALIAN 116024, P. R. CHINA

Email address: peiyanni@hotmail.com

(Jiang Zeng) UNIV LYON, UNIVERSITÉ CLAUDE BERNARD LYON 1, CNRS UMR 5208, INSTITUT CAMILLE JORDAN, 43 BLVD. DU 11 NOVEMBRE 1918, F-69622 VILLEURBANNE CEDEX, FRANCE

Email address: zeng@math.univ-lyon1.fr