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CYCLES OF EVEN-ODD DROP PERMUTATIONS AND CONTINUED FRACTIONS OF GENOCCHI NUMBERS

QIONGQIONG PAN AND JIANG ZENG

ABSTRACT. Recently, Lazar and Wachs (arXiv:1910.07651) showed that the (median) Genocchi numbers play a fundamental role in the study of the homogenized Linial arrangement and obtained two new permutation models (called D-permutations and E-permutations) for (median) Genocchi numbers. They further conjecture that the distributions of cycle numbers over the two models are equal. In a follow-up, Eu et al. (arXiv:2103.09130) further proved the gamma-positivity of the descent polynomials of even-odd descent permutations, which are in bijection with E-permutations by Foata's fundamental transformation. This paper merges the above two papers by considering a general moment sequence which encompasses the number of cycles and number of drops of E-permutations. Using the combinatorial theory of continued fraction, the moment connection enables us to confirm Lazar-Wachs' conjecture and obtain a natural (p, q) -analogue of Eu et al's descent polynomials. Furthermore, we show that the γ -coefficients of our (p, q) -analogue of descent polynomials have the same factorization flavor as the γ -coefficients of Brändén's (p, q) -Eulerian polynomials.

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1. INTRODUCTION

The Genocchi numbers G_{2n} and median Genocchi numbers H_{2n+1} are well studied, and have seen recent attention and new combinatorial interpretations, see [2, 3, 8, 11, 13, 21,

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[24, 29] and <https://oeis.org/A005439>. These two allied sequences of numbers can be easily defined by the *Seidel triangle* [27] through a boustrophedon algorithm as follows:

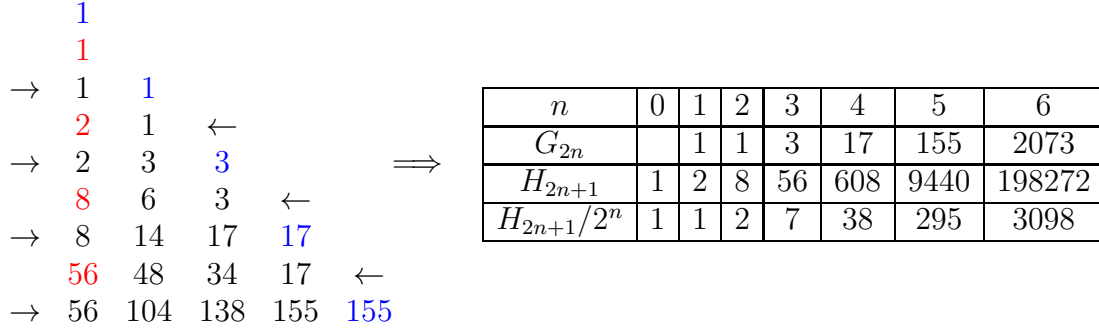


FIGURE 1. The first values of the Genocchi numbers G_{2n} , median Genocchi numbers H_{2n+1} and normalized median Genocchi numbers $H_{2n+1}/2^n$ are tabulated in the right table.

In 2019 Hetyei [21] introduced a hyperplane arrangement (called the homogenized Linial arrangement) and showed that its number of regions is a median Genocchi number. Lazar and Wachs [23] refined Hetyei's result by obtaining a combinatorial interpretation of the Möbius function of this lattice in terms of variants of the Dumont permutations.

A permutation $\sigma \in \mathfrak{S}_{2n}$ is a *D-permutation* (resp. *E-permutation*) if $i \leq \sigma(i)$ whenever i is odd and $i \geq \sigma(i)$ whenever i is even (resp. if $i > \sigma(i)$ implies i is even and $\sigma(i)$ is odd). An E-permutation is also called *even-odd drop permutation*. Introduce the set notations

$$\begin{aligned} \mathfrak{D}_{2n} &= \{\text{D-permutations on } [2n]\}; & \mathfrak{DC}_{2n} &= \{\text{D-cycles on } [2n]\}; \\ \mathfrak{E}_{2n} &= \{\text{E-permutations on } [2n]\}; & \mathfrak{EC}_{2n} &= \{\text{E-cycles on } [2n]\}. \end{aligned}$$

For example,

$$\begin{aligned} \mathfrak{D}_4 = \{ & (1)(2)(3)(4), (1, 2)(3)(4), (1, 4)(2)(3), (3, 4)(1)(2), \\ & (1, 2)(3, 4), (1, 3, 4)(2), (1, 4, 2)(3), (1, 3, 4, 2)\}; \end{aligned}$$

and

$$\begin{aligned} \mathfrak{E}_4 = \{ & (1)(2)(3)(4), (1, 2)(3)(4), (1, 4)(2)(3), (3, 4)(1)(2), \\ & (1, 2)(3, 4), (1, 3, 4)(2), (1, 2, 4)(3), (1, 2, 3, 4)\}. \end{aligned}$$

Here the permutations are written in cycle notations.

Theorem 1 (Lazar and Wachs [23]). *For integer $n \geq 1$ we have*

$$\begin{aligned} H_{2n+1} &= |\mathfrak{D}_{2n}| = |\mathfrak{E}_{2n}|, \\ G_{2n} &= |\mathfrak{DC}_{2n}|. \end{aligned}$$

Lazar and Wachs [23, Conjecture 6.5] also make the following conjecture.

Conjecture 2 (Lazar and Wachs). *The number of D -permutations on $[2n]$ with k cycles equals the number of E -permutations on $[2n]$ with k cycles for all k . Consequently*

$$G_{2n} = |\mathcal{EC}_{2n}|.$$

In this paper we will take a different approach to enumerate the cycles and drops of the E -permutations and prove Lazar-Wachs' conjecture. A *descent* of a permutation σ on a finite set of positive integers is a pair (σ_i, σ_{i+1}) for which $\sigma_i > \sigma_{i+1}$. We call σ_i (resp. σ_{i+1}) the descent top (resp. descent bottom). A descent pair (σ_i, σ_{i+1}) is called *even-odd* if σ_i is even and σ_{i+1} is odd. The number of descents of σ is denoted by $\text{des } \sigma$. Moreover, the value σ_i is said to be

- a *left-to-right maxima*, if $\sigma_j < \sigma_i$ for all $j < i$;
- a *right-to-left minimum*, if $\sigma_i < \sigma_j$ for all $j > i$.

Let $\text{lma } \sigma$ and $\text{rmi } \sigma$ be the numbers of left-to-right maxima and right-to-left minima. We further denote by $\text{lema } \sigma$ and $\text{loma } \sigma$ the numbers of *even* and *odd* left-to-right maxima. Similarly we define $\text{remi } \sigma$ and $\text{romi } \sigma$ to be the numbers of *even* and *odd* right-to-left minima.

Let \mathcal{X}_{2n} be the set of permutations in \mathfrak{S}_{2n} that contain only even-odd descents. A permutation in \mathcal{X}_{2n} is called an *even-odd descent permutation*. For $n = 1, 2$, we have $\mathcal{X}_2 = \{12, 21\}$ and

$$\mathcal{X}_4 = \{1234, 2134, 1243, 3412, 4123, 2341, 2413, 2143\}.$$

Note that one version of Foata's first fundamental transformation φ takes drops to descents and maxima of cycles to left-to-right maxima, i.e.,

$$(\text{drop}, \text{cyc})\sigma = (\text{des}, \text{lma})\varphi(\sigma) \quad \text{for } \sigma \in \mathfrak{S}_n. \quad (1.1)$$

Here $\text{drop } \sigma$ and $\text{cyc } \sigma$ denote the numbers of drops and cycles of $\sigma \in \mathfrak{S}_n$, respectively. Hence, by restriction φ sets up a bijection from \mathcal{E}_{2n} to \mathcal{X}_{2n} , and the cardinality of \mathcal{X}_{2n} equals the median Genocchi number H_{2n+1} .

An even-odd descent permutation $\sigma \in \mathcal{X}_{2n}$ is called \mathcal{E} -permutation if for any integer $j \in \{1, \dots, n\}$ the two entries $2j - 1$ and $2j$ are not simultaneously a descent bottom and a descent top of σ .¹ Let $\overline{\mathcal{X}}_{2n}$ be the set of \mathcal{E} -permutations in \mathcal{X}_{2n} and $\overline{\mathcal{X}}_{2n,k}$ be the subset consisting of permutations with k descents. The first few \mathcal{E} -permutations are

$$\begin{aligned} \overline{\mathcal{X}}_{2,0} &= \{12\}, & \overline{\mathcal{X}}_{4,0} &= \{1234\}, & \overline{\mathcal{X}}_{4,1} &= \{3412, 4123, 2341, 2413\} \\ \overline{\mathcal{X}}_{6,0} &= \{123456\}, & \overline{\mathcal{X}}_{6,1} &= \{124563, 124635, 125634, 126345, 234156, 241356, 412356, \\ & & & & & 341256, 261345, 361245, 461235, 561234, 236145, 246135, 256134, 346125, 356124, \\ & & & & & 456123, 234615, 235614, 245613, 345612, 234561, 612345\}. \end{aligned}$$

Note that the even-odd descent permutation $\sigma = 24163785$ is not an \mathcal{E} -permutation.

¹It is easy to see that our definition of \mathcal{E} -permutations is equivalent to the *primary even-odd descent permutations* in Eu et al. [15].

Eu-Fu-Lai-Lo [15, Theorem 1.2] recently studied the descent polynomials of even-odd descent permutations, and proved, among other things, the following result.

Theorem 3 (Eu et al. [15]). *We have*

$$X_n(t) := \sum_{\sigma \in \mathcal{X}_{2n}} t^{\text{des } \sigma} = \sum_{k=0}^{\lfloor n/2 \rfloor} |\bar{\mathcal{X}}_{2n,k}| t^k (1+t)^{n-2k}. \quad (1.2)$$

An expansion of type (1.2) is called the γ -expansion of the polynomial $X_n(t)$ in the literature [1, 7]. Indeed, a polynomial with real coefficients $h(t) = \sum_{i=0}^n h_i t^i$ is said to be *palindromic* if $h_i = h_{n-i}$ for $0 \leq i \leq n/2$. It is known that any palindromic polynomial can be written uniquely in the form $\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_i t^i (1+t)^{n-2i}$. The coefficients γ_i are called the γ -coefficients of $h(t)$. If the γ -coefficients γ_i are all nonnegative then we say that $h(t)$ is γ -positive. Note that the γ -positivity of $h(t)$ implies the symmetry and unimodal property of the coefficients (h_0, \dots, h_n) . A prototype of γ -positive polynomials is the Eulerian polynomials $A_n(t)$, which are the descent polynomials of permutations in \mathfrak{S}_n . The following expansion of Eulerian polynomials is known [17, 25]:

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des } \sigma} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} 2^k d_{n,k} t^k (1+t)^{n-1-2k}, \quad (1.3)$$

where $d_{n,k}$ is the number of André permutations in \mathfrak{S}_n with k descents.

This paper stems from the observation that the generating function of the descent polynomials $X_n(t)$ has a similar continued fraction expansion as the Eulerian polynomials and the cardinality $|\bar{\mathcal{X}}_{2n,k}|$ in (1.2) is divisible by 4^k , consequently (1.2) would imply a refinement of the normalized median Genocchi numbers

$$H_{2n+1}/2^n = \sum_{k=0}^{\lfloor n/2 \rfloor} |\bar{\mathcal{X}}_{2n,k}|/4^k. \quad (1.4)$$

In 2008 Brändén [6] studied a (p, q) -analogue of Eulerian polynomials and proved that his generalized Eulerian polynomials have a γ -positive expansion by modifying Foata and Strehl's *valley-hopping action*. At the end of his paper Brändén [6] speculated that the corresponding γ -coefficients have a factor $(p+q)^k$, which is a (p, q) -analogue of 2^k . Shin and Zeng [28] confirmed the conjectured divisibility of γ -coefficients by using Flajolet-Viennot's combinatorial theory of orthogonal polynomials [16, 30] and proved that the corresponding quotient $d_{n,k}(p, q)$ is a polynomial in p and q with nonnegative integer coefficients. Finally, motivated by an open problem of Han [19] about q -analogue of Euler numbers, the two authors [25] came up with a combinatorial interpretation for $d_{n,k}(p, q)$ by refining the André permutation interpretation for $d_{n,k}$.

In this paper we shall consider a natural (p, q) -analogue of the descent polynomials $X_n(p, q, t)$ and prove that the γ -coefficient $\gamma_{n,k}(p, q)$ of $X_n(p, q, t)$ has a factor $(p+q)^{2k}$, and provide a combinatorial interpretation for $\gamma_{n,k}(p, q)/(p+q)^{2k}$, see Theorem 13. It turns out that our approach can catch more permutation statistics over even-odd descent

permutations. For $j \in [n]$, the doubleton $\{2j - 1, 2j\}$ is called a *domino* of $\sigma \in \mathcal{X}_{2n}$ if $2j$ and $2j - 1$ are descent top and descent bottom of σ , respectively. Denote by $\text{dom}(\sigma)$ the number of dominos of σ . For $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathfrak{S}_n$, the statistic (31-2) σ is the number of pairs (i, j) such that $2 \leq i < j \leq n$ and $\sigma_{i-1} > \sigma_j > \sigma_i$. The pair (σ_{i-1}, σ_i) is called a *left-embracing* of σ_j . Similarly, the statistic (2-31) σ is the number of pairs (i, j) such that $1 \leq i < j \leq n - 1$ and $\sigma_j > \sigma_i > \sigma_{j+1}$. The pair (σ_j, σ_{j+1}) is called a *right-embracing* of σ_i .

We refine the median Genocchi numbers by the octuple-variable polynomials

$$\begin{aligned} X_n &:= X_n(a, \bar{a}, b, \bar{b}, p, q, y, t) \\ &= \sum_{\sigma \in \mathcal{X}_{2n}} a^{\text{lema } \sigma} \bar{a}^{\text{loma } \sigma} b^{\text{romi } \sigma} \bar{b}^{\text{remi } \sigma} p^{(2-31) \sigma} q^{(31-2) \sigma} y^{\text{dom } \sigma} t^{\text{des } \sigma}, \end{aligned} \quad (1.5)$$

which reduces to $X_n(t)$ when $a = \bar{a} = b = \bar{b} = y = 1$. For example,

$$\begin{aligned} X_1 &= ab(\bar{a}\bar{b} + ty), \\ X_2 &= a^2\bar{a}^2b^2\bar{b}^2 + 2a^2\bar{a}b^2\bar{b}ty + a\bar{a}bb\bar{b}pqt + ab^2\bar{b}q^2t + a^2\bar{a}bp^2t + a^2b^2pqt + a^2b^2t^2y^2. \end{aligned}$$

We will prove an explicit J-fraction expansion (Theorem 4) for the ordinary generating function of X_n , and prove (p, q) -analogue of Theorem 3 and Lazar-Wachs' conjecture as applications.

The Genocchi numbers G_{2n} can also be defined by the exponential generating function $\sum_{n=1}^{\infty} G_{2n} \frac{x^{2n}}{(2n)!} = x \tan \frac{x}{2}$, while the median Genocchi numbers do not seem to have a reasonable exponential generating function. Our method relies on the S-fraction expansions of the ordinary generating functions of Genocchi and median Genocchi numbers (see [14, 29]):

$$\sum_{n=0}^{\infty} G_{2n+2} x^n = \frac{1}{1 - \frac{1^2 \cdot x}{1 - \frac{1 \cdot 2 \cdot x}{1 - \frac{2^2 \cdot x}{1 - \frac{2 \cdot 3 \cdot x}{1 - \dots}}}}} = 1 + x + 3x^2 + 17x^3 + \dots, \quad (1.6a)$$

$$1 + \sum_{n=0}^{\infty} H_{2n+1} x^{n+1} = \frac{1}{1 - \frac{1^2 \cdot x}{1 - \frac{1^2 \cdot x}{1 - \frac{2^2 \cdot x}{1 - \frac{2^2 \cdot x}{1 - \dots}}}}} = 1 + x + 2x^2 + 8x^3 \dots. \quad (1.6b)$$

By the general theory of orthogonal polynomials the above continued fraction expansions mean that the Genocchi and median Genocchi numbers are moments of orthogonal polynomials. More precisely, as shown in [18], they are indeed moments of special or shifted continuous dual Hahn polynomials, see also [5,9,10,18] for recent papers on combinatorial aspects related to the moments of Askey-Wilson polynomials.

For reader's convenience, we recall two standard *contraction formulae* transforming an S-fraction to J-fractions, see [14].

Lemma 1 (Contraction formula). *For any sequence $\{\alpha_n\}$ of elements in an arbitrary ring containing \mathbb{Z} , the following holds*

$$\frac{1}{1 - \frac{\alpha_1 x}{1 - \frac{\alpha_2 x}{1 - \frac{\alpha_3 x}{\ddots}}}} = \frac{1}{1 - b_0 x - \frac{\lambda_1 x^2}{1 - b_1 x - \frac{\lambda_2 x^2}{1 - b_2 x - \frac{\lambda_3 x^2}{\ddots}}}} \quad (1.7)$$

if $b_0 = \alpha_1$, $b_n = \alpha_{2n} + \alpha_{2n+1}$, $\lambda_n = \alpha_{2n-1}\alpha_{2n}$ for $n \geq 1$; and

$$\frac{1}{1 - \frac{\alpha_1 x}{1 - \frac{\alpha_2 x}{1 - \frac{\alpha_3 x}{\ddots}}}} = 1 + \frac{\alpha_1 x}{1 - b_0 x - \frac{\lambda_1 x^2}{1 - b_1 x - \frac{\lambda_2 x^2}{1 - b_2 x - \frac{\lambda_3 x^2}{\ddots}}}}. \quad (1.8)$$

if $b_{n-1} = \alpha_{2n-1} + \alpha_{2n}$, $\lambda_n = \alpha_{2n}\alpha_{2n+1}$ for $n \geq 1$.

From (1.6b) we derive the J-fraction

$$\sum_{n=0}^{\infty} H_{2n+1} x^n = \frac{1}{1 - 2 \cdot 1^2 x - \frac{1^2 \cdot 2^2 x^2}{1 - 2 \cdot 2^2 x - \frac{2^2 \cdot 3^2 x^2}{1 - 2 \cdot 3^2 x - \frac{3^2 \cdot 4^2 x^2}{\ddots}}}}. \quad (1.9)$$

Let $h_n = H_{2n+1}/2^n$ ($n \in \mathbb{N}$) be the normalized median Genocchi numbers. By replacing x by $x/2$, we obtain

$$\sum_{n=0}^{\infty} h_n x^n = \frac{1}{1 - 1^2 x - \frac{1^2 \cdot x^2}{1 - 2^2 x - \frac{\binom{3}{2}^2 \cdot x^2}{1 - 3^2 x - \frac{\binom{4}{2}^2 \cdot x^2}{\ddots}}}}. \quad (1.10)$$

Dumont [11] initiated the combinatorial characterization of Genocchi numbers. A permutation $\sigma \in \mathfrak{S}_{2n}$ is a *Dumont permutation* if $\sigma(2i-1) \geq 2i-1$ and $\sigma(2i) < 2i$ for $i \in [n]$. A permutation $\sigma \in \mathfrak{S}_{2n}$ is a *Dumont derangement* if $\sigma(2i-1) > 2i-1$ and $\sigma(2i) < 2i$ for $i \in [n]$. Dumont [11] and Dumont-Randrianarivony [13] proved respectively that the Dumont permutations in \mathfrak{S}_{2n} is the Genocchi number G_{2n} and the Dumont derangements in \mathfrak{S}_{2n} is the median Genocchi number H_{2n+1} . Kitaev and Remmel [22] conjectured and Burstein, Josuat-Vergès, and Stromquist [8] proved that the set of permutations in \mathfrak{S}_{2n} with only even-even descents has cardinality equal to G_{2n} . Clearly the set of even-even descent permutations in \mathfrak{S}_{2n} is in bijection with the set of odd-odd descent permutations in \mathfrak{S}_{2n-1} .

Let \mathcal{Y}_{2n+1} be the set of permutations in \mathfrak{S}_{2n+1} that contain only odd-odd descents, notice that $|\mathcal{Y}_{2n+1}| = G_{2n+2}$. Let \mathcal{Y}_{2n+1}^* be the subset of \mathcal{Y}_{2n+1} consisting of the permutations ending with an odd element. The first three sets are $\mathcal{Y}_1^* = \{1\}$, $\mathcal{Y}_3^* = \{123, 231\}$ and

$$\mathcal{Y}_5^* = \{12345, 12453, 23451, 24531, 24513, 23145, 31245, 45123\}.$$

Eu et al. [15] proved that the cardinality of \mathcal{Y}_{2n+1}^* is equal to the median Genocchi number H_{2n+1} . They also considered a q -analogue of the descent polynomials of \mathcal{Y}_{2n+1}^* and proved a similar γ -expansion for the q -descent polynomials. We show Section 7 that it is possible to refine their results and prove similar results as for the descent polynomials of even-odd descent permutations.

2. MAIN RESULTS

For any integer $n \geq 1$ we define the generalized (p, q) -analogue of n by

$$[x, n, y]_{p,q} = xp^{n-1} + \sum_{i=1}^{n-2} p^{n-1-i} q^i + yq^{n-1}, \quad (2.1a)$$

$$[x, n]_{p,q} = xp^{n-1} + \sum_{i=1}^{n-1} p^{n-1-i} q^i, \quad (2.1b)$$

with $[x, 1, y]_{p,q} = xy$ and $[x, 1]_{p,q} = x$. In particular,

$$[n]_{p,q} = [1, n]_{p,q} = \frac{p^n - q^n}{p - q},$$

and the (p, q) -analogue of binomial coefficient $\binom{n}{k}$ is defined by

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q} \cdots [n-k+1]_{p,q}}{[1]_{p,q} \cdots [k]_{p,q}} \quad (1 \leq k \leq n).$$

Theorem 4. *We have*

$$1 + \sum_{n=1}^{\infty} X_n(a, \bar{a}, b, \bar{b}, p, q, y, t)x^n = \frac{1}{1 - ab(\bar{a}\bar{b} + ty)x - \frac{abt(ap + \bar{b}q)(\bar{a}p + bq)x^2}{\dots}} \quad (2.2)$$

with coefficients for $n \geq 1$

$$\begin{cases} b_{n-1} &= [\bar{a}, n, b]_{p,q}[a, n, \bar{b}]_{p,q} + ty[a, n]_{p,q}[b, n]_{q,p}, \\ \lambda_n &= t[a, n]_{p,q}[a, n+1, \bar{b}]_{p,q}[b, n]_{q,p}[\bar{a}, n+1, b]_{p,q}. \end{cases}$$

We first record some special cases of Theorem 4. When $b = \bar{b} = 1$ we have the more compact formula.

Corollary 5.

$$1 + \sum_{n=1}^{\infty} X_n(a, \bar{a}, 1, 1, p, q, y, t)x^n = \frac{1}{1 - (\bar{a} + ty)ax - \frac{at(\bar{a}p + q)(ap + q)x^2}{1 - ([\bar{a}, 2]_{p,q} + ty[2]_{p,q})[a, 2]_{p,q}x - \frac{t[a, 2]_{p,q}[a, 3]_{p,q}[2]_{p,q}[\bar{a}, 3]_{p,q}x^2}{\dots}}} \quad (2.3)$$

with coefficients for $n \geq 1$

$$\begin{cases} b_{n-1} &= ([\bar{a}, n]_{p,q} + ty[n]_{p,q})[a, n]_{p,q}, \\ \lambda_n &= t[a, n]_{p,q}[a, n+1]_{p,q}[n]_{p,q}[\bar{a}, n+1]_{p,q}. \end{cases}$$

Let $y = 1$ in Theorem 4, by the contraction formula we can rewrite the above J-fraction as an S-fraction.

Corollary 6. *We have the S-fraction expansion*

$$1 + a\bar{a} \sum_{n=0}^{\infty} X_n(a, \bar{a}, 1, 1, p, q, 1, t)x^{n+1} = \frac{1}{1 - \frac{a\bar{a} \cdot x}{1 - \frac{at \cdot x}{1 - \frac{[\bar{a}, 2]_{p,q}[a, 2]_{p,q} \cdot x}{1 - \frac{[a, 2]_{p,q}[2]_{p,q} t \cdot x}{\dots}}}}} \quad (2.4)$$

with $X_0 = 1$ and coefficients

$$\begin{cases} \alpha_{2n-1} &= [\bar{a}, n]_{p,q}[a, n]_{p,q}, & n \geq 1, \\ \alpha_{2n} &= [a, n]_{p,q}[n]_{p,q}t. \end{cases}$$

By Foata's fundamental transformation (see (1.1)) we have

$$P_n(t, z) := \sum_{\sigma \in \mathcal{X}_{2n}} z^{\text{des } \sigma} z^{\text{lma } \sigma} = \sum_{\sigma \in \mathcal{E}_{2n}} z^{\text{drop } \sigma} z^{\text{cyc } \sigma}.$$

Thus, letting $a = \bar{a} = z$, $p = q = y = 1$ in Corollary 5 we derive the continued fraction for the ordinary generating function of $P_n(t, z)$.

Corollary 7.

$$\begin{aligned} & 1 + \sum_{n=1}^{\infty} P_n(t, z) x^n \\ &= \frac{1}{1 - z(z+t)x - \frac{z(z+1)^2 t x^2}{\dots}} \\ & \quad \frac{\dots}{1 - (z+n)(z+n+(n+1)t)x - \frac{(n+1)(z+n)(z+n+1)^2 t x^2}{\dots}} \end{aligned} \quad (2.5)$$

Taking $y = 0$ in Theorem 4 we obtain the corresponding formula for \mathcal{E} -permutations.

Corollary 8. *We have*

$$\begin{aligned} & 1 + \sum_{n=1}^{\infty} \sum_{\sigma \in \bar{\mathcal{X}}_{2n}} a^{\text{lema } \sigma} \bar{a}^{\text{loma } \sigma} p^{(2-31)\sigma} q^{(31-2)\sigma} t^{\text{des } \sigma} x^n = \\ & \frac{1}{1 - a\bar{a}x - \frac{at[2]_{p,q}[a, 2]_{p,q} x^2}{1 - (ap+q)(\bar{a}p+q)x - \frac{t[2]_{p,q}[a, 2]_{p,q}[3]_{p,q}[\bar{a}, 3]_{p,q} x^2}{\dots}}} \end{aligned} \quad (2.6)$$

with coefficients for $n \geq 1$

$$\begin{cases} b_{n-1} &= [a, n]_{p,q} [\bar{a}, n]_{p,q}, \\ \lambda_n &= t[a, n]_{p,q} [a, n+1]_{p,q} [n]_{p,q} [\bar{a}, n+1]_{p,q}. \end{cases}$$

We shall prove Theorem 4 in Section 3 by constructing a bijection from the even-odd descent permutations to lattice path diagrams. Combining Theorem 4, the surjective pistol theory developed by Dumont-Randrianarivony and Lazar-Wachs and Randrianarivony-Zeng's continued fraction expansion for the ordinary generating function of generalized Dumont-Foata polynomials we prove the conjecture of Lazar-Wachs [23, Conjecture 6.4] in section 4.

Theorem 9. *Conjecture 2 is true, i.e., for all $n \geq 1$,*

$$\sum_{\sigma \in \mathcal{E}_{2n}} z^{\text{cyc } \sigma} = \sum_{\sigma \in \mathcal{D}_{2n}} z^{\text{cyc}(\sigma)}. \quad (2.7)$$

Consequently $|\mathcal{EC}_{2n}| = |\mathcal{DC}_{2n}| = G_{2n+2}$.

Our third result is a quasi-gamma decomposition of $X_n(a, 1, b, 1, p, q, y, t)$, which provides a neat (p, q) -analogue of the γ -formula (1.2) by setting $a = b = y = 1$.

Theorem 10. *We have*

$$1 + \sum_{n=1}^{\infty} X_n(a, 1, b, 1, p, q, y, t)x^n = \frac{1}{1 - ab(\bar{a}\bar{b} + ty)x - \frac{abt(\bar{a}p + \bar{q})(\bar{a}p + bq)x^2}{\dots}} \quad (2.8)$$

with coefficients for $n \geq 1$

$$\begin{cases} b_{n-1} &= (1 + ty)[a, n]_{p,q}[b, n]_{q,p}, \\ \lambda_n &= t[a, n]_{p,q}[a, n+1]_{p,q}[b, n]_{q,p}[b, n+1]_{q,p}. \end{cases}$$

Moreover the following γ -formula holds

$$X_n(a, 1, b, 1, p, q, y, t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k}(a, b, p, q) t^k (1 + yt)^{n-2k}, \quad (2.9a)$$

with coefficients

$$\gamma_{n,k}(a, b, p, q) = \sum_{\sigma \in \mathcal{X}_{2n,k}} a^{\text{lema } \sigma} b^{\text{romi } \sigma} p^{(2-31) \sigma} q^{(31-2) \sigma}. \quad (2.9b)$$

Clearly Theorem 4 reduces to (2.8) by setting $\bar{a} = \bar{b} = 1$. By comparing Corollary 8 with formula (2.9b) we derive (2.9a). We shall give another proof of (2.9a) using *Inter-hopping* action on \mathcal{X}_{2n} in Section 5.

A *weak signature* of order k in $[2n]$ is a subset $S \subseteq [2n]$ consisting of k odd and k even numbers such that for each $i \in [2n]$, the number of odd numbers is greater than or equal to the number of even numbers in $S \cap \{1, \dots, i\}$. Note that our weak signature is different with the *signature* in [8, 15]. Denote by S_o (resp. S_e) the set of odd (resp. even) elements of S . Let $S_o(i)$ and $S_e(i)$ be the numbers of odd and even elements in S less than i . The associated signature function s has domain $[2n]$ and is defined by

$$s(x) = S_o(x) - S_e(x) + (1 \text{ if } x \in S_o \text{ or } x \notin S). \quad (2.10)$$

We use $l_\sigma(i)$ (resp. $r_\sigma(\sigma_i)$) to denote the number of left-embracings (resp. right-embracings) of σ_i in σ , thus

$$(31-2) \sigma = \sum_{i=1}^n l_\sigma(\sigma_i); \quad (2.11a)$$

$$(2-31) \sigma = \sum_{i=1}^n r_\sigma(\sigma_i). \quad (2.11b)$$

Lemma 2. *For $\sigma \in \mathcal{X}_{2n}$, if S be the set of descent tops and bottoms of σ , then S is a weak signature of $[2n]$ and the associated signature function s is given by*

$$s(i) = l_\sigma(i) + r_\sigma(i) + 1 \quad \forall i \in [2n]. \quad (2.12)$$

Proof. Let $\sigma \in \mathcal{X}_{2n}$. For any $i \in [2n]$, there is an injection $\phi : S_e \cap [i] \rightarrow S_o \cap [i]$ which maps each decent top $t \in S_e \cap [i]$ to a descent bottom $b \in S_o \cap [i]$ such that (t, b) forms a descent pair of σ . Hence $S_o(i) \geq S_e(i)$ and S is a weak signature of $[2n]$.

Next we compute the embracing numbers $l_\sigma(i) + r_\sigma(i)$ of $i \in [2n]$.

- (1) If $i \in S$ is odd or $i \notin S$, the number of even-odd descents (t, b) with $t < i$ is equal to $S_e(i)$, so the embracing number of i is $S_o(i) - S_e(i)$.
- (2) If $i \in S$ is even, the number of even-odd descents (t, b) with $t \leq i$ is equal to $S_e(i) + 1$, so i has $S_o(i) - S_e(i) - 1$ embraces.

Combining the two cases and comparing with (2.10) we obtain (2.12). \square

In what follows, for any $\sigma \in \mathcal{X}_{2n}$ we use S_σ and s_σ to denote the weak signature of σ and the associated signature function, respectively.

Definition 11. *An \mathcal{E} -permutation $\sigma \in \overline{\mathcal{X}}_{2n}$ is normalized if it satisfies the following conditions: for $j \in [n]$,*

- (a) *if $(2j - 1, s(2j - 1)) \in S_\sigma \times (2\mathbb{N} + 1)$, then $l_\sigma(2j)$ is even;*
- (b) *if $(2j - 1, s(2j - 1)) \in S_\sigma \times 2\mathbb{N}$, then $l_\sigma(2j - 1)$ is even;*
- (c) *if $(2j, s(2j)) \in S_\sigma \times (2\mathbb{N} + 1)$, then $r_\sigma(2j - 1)$ is even;*
- (d) *if $(2j, s(2j)) \in S_\sigma \times 2\mathbb{N}$, then $r_\sigma(2j)$ is even.*

Let $\widehat{\mathcal{X}}_{2n}$ be the set of normalized \mathcal{E} -permutations in $\overline{\mathcal{X}}_{2n}$ and let

$$\widehat{\mathcal{X}}_{2n,k} = \{\sigma \in \widehat{\mathcal{X}}_{2n} : \text{des } \sigma = k\}.$$

Example 1. *The following are the first few normalized \mathcal{E} -permutations:*

$$\widehat{\mathcal{X}}_2 = \{12\}$$

$$\widehat{\mathcal{X}}_4 = \{1234, 2413\}$$

$$\widehat{\mathcal{X}}_6 = \{123456, 124635, 241356, 234615, 261345, 236145, 246135\}.$$

Proposition 12. *For $\sigma \in \widehat{\mathcal{X}}_{2n,k}$, we have $(31-2)\sigma \geq k$ and $(2-31)\sigma \geq k$.*

Proof. As $\sigma \in \widehat{\mathcal{X}}_{2n,k}$, according to the definition of normalized \mathcal{E} -permutations,

- (a) if $2i - 1 \in S_\sigma$ and $s_\sigma(2i - 1)$ is odd, then $s_\sigma(2i) = s_\sigma(2i - 1) + 1$, so by Lemma 2, we have $r_\sigma(2i)$ is odd.
- (b) if $2i - 1 \in S_\sigma$ and $s_\sigma(2i - 1)$ is even, then $l_\sigma(2i - 1)$ is even, by Lemma 2, $r_\sigma(2i - 1)$ is odd.
- (c) if $2i \in S_\sigma$ and $s_\sigma(2i)$ is odd then $r_\sigma(2i - 1)$ is even, and $s_\sigma(2i - 1) = s_\sigma(2i) + 1$ is even, by Lemma 2, $l_\sigma(2i - 1)$ is odd.
- (d) if $2i \in S_\sigma$ and $s_\sigma(2i)$ is even, then $\gamma_\sigma(2i)$ is even, also by Lemma 2, we have $l_\sigma(2i)$ is odd.

So, for a descent bottom $2i - 1$ of σ , either $2i - 1$ or $2i$ contributes at least one (2-31) pattern. And for a descent top $2j$ of σ , either $2j$ or $2j - 1$ contributes at least one (31-2) pattern. Therefore, $(31-2)\sigma \geq k$ and $(2-31)\sigma \geq k$. \square

Theorem 13. *We have the (p, q) -analogue of (1.2)*

$$X_n(p, q, t) := \sum_{\sigma \in \mathcal{X}_{2n}} p^{(2-31)\sigma} q^{(31-2)\sigma} t^{\text{des } \sigma} = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k}(p, q) t^k (1+t)^{n-2k}, \quad (2.13a)$$

with coefficients

$$\gamma_{n,k}(p, q) = \sum_{\sigma \in \widehat{\mathcal{X}}_{2n,k}} p^{(2-31)\sigma} q^{(31-2)\sigma}. \quad (2.13b)$$

Moreover, the γ -coefficients $\gamma_{n,k}(p, q)$ have the factorization

$$\gamma_{n,k}(p, q) = (p+q)^{2k} \sum_{\sigma \in \widehat{\mathcal{X}}_{2n,k}} q^{(31-2)\sigma-k} p^{(2-31)\sigma-k}. \quad (2.13c)$$

Clearly Theorem 10 implies (2.13a). We shall prove Theorem 13 (2.13c) in Section 6. From Theorem 13 and Corollary 8 we derive the following J-fraction.

Corollary 14. *We have*

$$\begin{aligned} & \sum_{n \geq 0} \sum_{\sigma \in \widehat{\mathcal{X}}_{2n}} p^{(2-31)\sigma} q^{(31-2)\sigma} t^{\text{des } \sigma} x^n \\ &= \frac{1}{1 - [1]_{p,q}^2 x - \frac{(2)_{p,q}^2 t \cdot x^2}{1 - [2]_{p,q}^2 x - \frac{(3)_{p,q}^2 t \cdot x^2}{1 - [3]_{p,q}^2 x - \frac{(4)_{p,q}^2 t \cdot x^2}{\dots}}}}}. \end{aligned} \quad (2.14)$$

Note that (2.14) reduces to (1.10) when $p = q = t = 1$. Thus $|\widehat{\mathcal{X}}_{2n}| = h_n$ for $n \geq 1$.

3. EVEN-ODD PERMUTATIONS AND LABELLED MOTZKIN PATHS

A *Motzkin path* of length n is a sequence of points $\omega := (\omega_0, \dots, \omega_n)$ in the integer plan $\mathbb{Z} \times \mathbb{Z}$ such that

- $\omega_0 = (0, 0)$ and $\omega_n = (n, 0)$,
- $\omega_i - \omega_{i-1} \in \{(1, 0), (1, 1), (1, -1)\}$,
- $\omega_i := (x_i, y_i) \in \mathbb{N} \times \mathbb{N}$ for $i = 0, \dots, n$.

In other words, a Motzkin path of length $n \geq 0$ is a lattice path in the right quadrant $\mathbb{N} \times \mathbb{N}$ starting at $(0, 0)$ and ending at $(n, 0)$, each step $s_i = \omega_i - \omega_{i-1}$ is a rise $\mathbf{U} = (1, 1)$, fall $\mathbf{D} = (1, -1)$ or level $\mathbf{L} = (1, 0)$. A 2-Motzkin path is a Motzkin path with two types of level-steps \mathbf{L}_1 and \mathbf{L}_2 . Let \mathcal{MP}_n be the set of 2-Motzkin paths of length n . Clearly we can identify 2-Motzkin paths of length n with words w on $\{\mathbf{U}, \mathbf{L}_1, \mathbf{L}_2, \mathbf{D}\}$ of length n such that all prefixes of w contain no more \mathbf{D} 's than \mathbf{U} 's and the number of \mathbf{D} 's equals the number of \mathbf{U} 's. The *height* of ω_i is the ordinate of ω_i and denoted by $h(\omega_i)$, which is also called the height of the step s_i .

Let $\mathbf{a} = (a_i)_{i \geq 0}$, $\mathbf{b} = (b_i)_{i \geq 0}$, $\mathbf{b}' = (b'_i)_{i \geq 0}$ and $\mathbf{c} = (c_i)_{i \geq 1}$ be four sequences of indeterminates; we will work in the ring $\mathbb{Z}[[\mathbf{a}, \mathbf{b}, \mathbf{b}', \mathbf{c}]]$. To each Motzkin path ω we assign a weight $W(\omega)$ that is the product of the weights for the individual steps, where a up-step (resp. down-step) at height i gets weight a_i (resp. λ_i), and a level-step of type 1 (resp. 2) at height i gets weight b_i (resp. b'_i). The following result of Flajolet [16] is folklore.

Lemma 3 (Flajolet). *We have*

$$\sum_{n=0}^{\infty} \left(\sum_{\omega \in \mathcal{MP}_n} W(\omega) \right) x^n = \frac{1}{1 - (b_0 + b'_0)x - \frac{a_0 c_1 x^2}{1 - (b_1 + b'_1)x - \frac{a_1 c_2 x^2}{1 - (b_2 + b'_2)x - \dots}}}$$

Definition 15. *A path diagram of length n is a triplet $(\omega, (\xi, \xi'))$, where ω is a Motzkin path of length n , $\xi = (\xi_1, \dots, \xi_n)$ and $\xi' = (\xi'_1, \dots, \xi'_n)$ are two integer sequences satisfying the following conditions:*

- *If the k -th step of ω is a rise with height $h \geq 0$, then*

$$(\xi_i, \xi'_i) \in \{0, \dots, h\} \times \{0, \dots, h+1\}. \quad (3.1a)$$

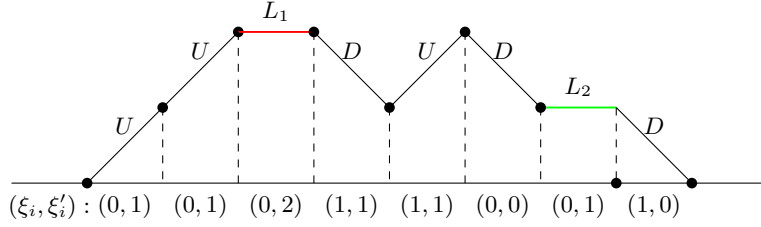
- *If the k -th step of ω is a fall with height $h > 0$, then*

$$(\xi_i, \xi'_i) \in \{0, \dots, h\} \times \{0, \dots, h-1\}. \quad (3.1b)$$

- *If the k -th step of ω is a level of type 1 or 2 with height $h \geq 0$, then*

$$(\xi_i, \xi'_i) \in \{0, \dots, h\} \times \{0, \dots, h\}. \quad (3.1c)$$

We denote by \mathcal{PD}_n the set of path diagrams of length n .

FIGURE 2. Illustration of $\Phi : \sigma \mapsto (\omega, (\xi, \xi'))$.

For $\sigma \in \mathcal{X}_{2n}$, we construct the path diagram $\Phi(\sigma) = (\omega, (\xi, \xi'))$ in \mathcal{PD}_n as in the following. Let $\overline{S}_\sigma = [2n] \setminus S_\sigma$. For $j \in \{1, \dots, n\}$ we define the j -th step $s_j = \omega_j - \omega_{j-1}$ of the path $\omega = (\omega_0, \omega_1, \dots, \omega_n)$ by

$$s_j = \begin{cases} U & \text{if } (2j-1, 2j) \in S_\sigma \times \overline{S}_\sigma \\ D & \text{if } (2j-1, 2j) \in \overline{S}_\sigma \times S_\sigma \\ L_1 & \text{if } (2j-1, 2j) \in \overline{S}_\sigma \times \overline{S}_\sigma \\ L_2 & \text{if } (2j-1, 2j) \in S_\sigma \times S_\sigma \end{cases} \quad (3.2a)$$

and the bi-sequence (ξ, ξ') by

$$(\xi_j, \xi'_j) = (r_\sigma(2j-1), r_\sigma(2j)). \quad (3.2b)$$

The *restriction* of σ on $[2j]$ is the word π_j obtained from σ by replacing each subword of σ consisting of consecutive letters greater than $2j$ by a *slot* \sqcup . It is easier to describe the above construction by looking at the successive restrictions of σ on $\{1, \dots, 2j\}$ for $j \in [n]$. For example, if $\sigma = 2 \ 6 \ 8 \cdot 1 \ 4 \ 7 \ 14 \cdot 9 \ 10 \ 12 \cdot 3 \ 5 \ 11 \ 15 \ 16 \cdot 13 \in \mathcal{X}_{16}$, then $S = \{1, 3, 9, 13\} \cup \{8, 12, 14, 16\}$. Hence $\pi_0 = \sqcup$, the successive restrictions of σ read as follows:

i	π_j	$(2j-1, 2j)$	(ξ_j, ξ'_j)	$(s_\sigma(2j-1), s_\sigma(2j))$
1	2 \sqcup 1 \sqcup	$(1, 2) \in S_\sigma \times \overline{S}_\sigma$	$(0, 1)$	$(1, 2)$
2	2 \sqcup 1 4 \sqcup 3 \sqcup	$(3, 4) \in S_\sigma \times \overline{S}_\sigma$	$(0, 1)$	$(2, 3)$
3	2 6 \sqcup 1 4 \sqcup 3 5 \sqcup	$(5, 6) \in \overline{S}_\sigma \times \overline{S}_\sigma$	$(0, 2)$	$(3, 3)$
4	2 6 8 1 4 7 \sqcup 3 5 \sqcup	$(7, 8) \in \overline{S}_\sigma \times S_\sigma$	$(1, 1)$	$(3, 2)$
5	2 6 8 1 4 7 \sqcup 9 10 \sqcup 3 5 \sqcup	$(9, 10) \in S_\sigma \times \overline{S}_\sigma$	$(1, 1)$	$(2, 3)$
6	2 6 8 1 4 7 \sqcup 9 10 12 3 5 11 \sqcup	$(11, 12) \in \overline{S}_\sigma \times S_\sigma$	$(0, 0)$	$(3, 2)$
7	2 6 8 1 4 7 14 9 10 12 3 5 11 \sqcup 13 \sqcup	$(13, 14) \in S_\sigma \times S_\sigma$	$(0, 1)$	$(2, 2)$
8	2 6 8 1 4 7 14 9 10 12 3 5 11 15 16 13 \sqcup	$(15, 16) \in \overline{S}_\sigma \times S_\sigma$	$(1, 0)$	$(2, 1)$

FIGURE 3. The restrictions π_j of $\sigma \in \mathcal{X}_{16}$ for $j \in [8]$.

Hence the corresponding step sequence is $(U, U, L_1, D, U, D, L_2, D)$ with height sequence $(0, 1, 2, 2, 1, 2, 1, 1)$. The corresponding path diagram $(\omega, (\xi, \xi'))$ is illustrated in Figure 2.

Lemma 4. *The mapping $\Phi : \sigma \mapsto (\omega, (\xi, \xi'))$ is a bijection from \mathcal{X}_{2n} to \mathcal{PD}_n such that*

$$\text{dom } \sigma = |L_2|_\omega, \quad (3.3a)$$

$$\text{des } \sigma = |U|_\omega + |L_2|_\omega, \quad (3.3b)$$

$$(2-31) \sigma = \sum_{i=1}^n (\xi_i + \xi'_i), \quad (3.3c)$$

$$(31-2) \sigma = |U|_\omega + |D|_\omega + \sum_{i=1}^n (h(\omega_i) - \xi_i + h(\omega_i) - \xi'_i), \quad (3.3d)$$

$$(h(\omega_{j-1}), h(\omega_j)) = (s_\sigma(2j-1) - 1, s_\sigma(2j) - 1) \quad j \in [n], \quad (3.3e)$$

where $|A|_\omega$ is the number of steps of type A in the path ω .

Proof. First of all, Lemma 2 ensures that $(\omega, (\xi, \xi'))$ is a path diagram in \mathcal{PD}_n . The trivial verification of (3.3a)-(3.3e) is omitted. Note that

$$0 \leq \xi_j \leq s_\sigma(2j-1) - 1, \quad 0 \leq \xi'_j \leq s_\sigma(2j) - 1$$

To show that Φ is a bijection, we construct the inverse mapping Φ^{-1} . Starting from a path diagram $(\omega, (\xi, \xi')) \in \mathcal{PD}_n$, by ω and (3.2a), we determine the weak signature S_σ . For $j \in [n]$ we construct the permutation σ step by step by inserting $2j-1$ and $2j$ in the word π_{j-1} with $\pi_0 = \sqcup$ as follows:

- (1) if the j -th step is *rise*, we replace the $\xi_j + 1$ -th slot from right to left in π_{j-1} by $\sqcup(2j-1)\sqcup$ and then replace the $\xi'_j + 1$ -th slot by $2j\sqcup$;
- (2) if the j -th step is *fall*, we replace the $\xi_j + 1$ -slot in π_{j-1} by $(2j-1)\sqcup$ and then the $\xi'_j + 2$ -th slot in by $(2j)$;
- (3) if the j -th step is *level of type 1*, we replace the $\xi_j + 1$ -th slot in π_{j-1} by $(2j-1)\sqcup$ and then replace the $\xi'_j + 1$ -slot by $2j\sqcup$;
- (4) if the j -th step is *level of type 2*, we replace the $\xi_j + 1$ -th slot in π_{j-1} by $\sqcup(2j-1)\sqcup$ and then replace the $\xi'_j + 2$ -th slot by $2j$.

The permutation σ is obtained by deleting the final slot \sqcup in π_n . □

Lemma 5. For $\sigma \in \mathcal{X}_{2n}$ let $\Phi(\sigma) = (\omega, (\xi, \xi'))$ be the path diagram in \mathcal{PD}_n . Then

$$\text{lema } \sigma = \sum_{i=1}^n \chi(\xi'_i = h(w_i)), \quad (3.4a)$$

$$\text{loma } \sigma = \sum_{i=1}^n \chi(\xi_i = h(w_i)) \chi(s_i \in \{D, L_1\}), \quad (3.4b)$$

$$\text{remi } \sigma = \sum_{i=1}^n \chi(\xi_i = 0) \chi(s_i \in \{U, L_1\}), \quad (3.4c)$$

$$\text{romi } \sigma = \sum_{i=1}^n \chi(\xi_i = 0) \chi(s_i \in \{U, D, L_1\}), \quad (3.4d)$$

Proof. The trivial verification is omitted. \square

Proof of Theorem 4. For $\sigma \in \mathcal{X}_{2n}$, if $\Phi(\sigma) = (\omega, (\xi, \xi')) \in \mathcal{PD}_n$, we derive from the above lemma

$$a^{\text{lema } \sigma} \bar{a}^{\text{loma } \sigma} b^{\text{romi } \sigma} \bar{b}^{\text{remi } \sigma} p^{(2-31)\sigma} q^{(31-2)\sigma} y^{\text{dom } \sigma} t^{\text{des } \sigma} = \prod_{i=1}^n w(s_i, (\xi_i, \xi'_i)) \quad (3.5)$$

with h being the height of the step s_i ,

$$w(s_i, (\xi_i, \xi'_i)) = \begin{cases} a^{\chi(\xi'_i=h+1)} b^{\chi(\xi_i=0)} p^{\xi_i} q^{h-\xi_i} \cdot p^{\xi'_i} q^{h+1-\xi'_i} \cdot t & \text{if } s_i = U; \\ a^{\chi(\xi'_i=h)} \bar{a}^{\chi(\xi_i=h)} b^{\chi(\xi_i=0)} p^{\xi_i} q^{h-\xi_i} \cdot p^{\xi'_i} q^{h-1-\xi'_i} & \text{if } s_i = D; \\ a^{\chi(\xi'_i=h)} \bar{a}^{\chi(\xi_i=h)} b^{\chi(\xi'_i=0)} p^{\xi_i} q^{h-\xi_i} \cdot p^{\xi'_i} q^{h-\xi'_i} & \text{if } s_i = L_1; \\ a^{\chi(\xi'_i=h)} p^{\xi_i} q^{h-\xi_i} \cdot p^{\xi'_i} q^{h-\xi'_i} ty & \text{if } s_i = L_2. \end{cases} \quad (3.6)$$

Therefore, the corresponding polynomial and weights become

$$\begin{aligned} X_n(a, \bar{a}, b, \bar{b}, p, q, y, t) &= \sum_{\sigma \in \mathcal{X}_{2n}} a^{\text{lema } \sigma} \bar{a}^{\text{loma } \sigma} b^{\text{romi } \sigma} \bar{b}^{\text{remi } \sigma} p^{(2-31)\sigma} q^{(31-2)\sigma} y^{\text{dom } \sigma} t^{\text{des } \sigma} \\ &= \sum_{\omega \in \mathcal{MP}_n} \sum_{(\xi_i, \xi'_i)} w(s_i, (\xi_i, \xi'_i)) \end{aligned}$$

with

$$\sum_{(\xi_i, \xi'_i)} w(s_i, (\xi_i, \xi'_i)) = \begin{cases} [b, h+1]_{q,p} \cdot [a, h+2, \bar{b}]_{p,q} \cdot t & \text{if } s_i = U; \\ [\bar{a}, h+2, b]_{p,q} \cdot [a, h+1]_{p,q} & \text{if } s_i = D; \\ [\bar{a}, h+1, b]_{p,q} \cdot [a, h+1, \bar{b}]_{p,q} & \text{if } s_i = L_1; \\ [a, h+1]_{p,q} \cdot [b, h+1]_{q,p} ty & \text{if } s_i = L_2. \end{cases} \quad (3.7)$$

with $[x, 1, y]_{p,q} = xy$ and

$$[x, n, y]_{p,q} = xp^{n-1} + \sum_{i=1}^{n-2} p^{n-1-i} q^i + yq^{n-1}.$$

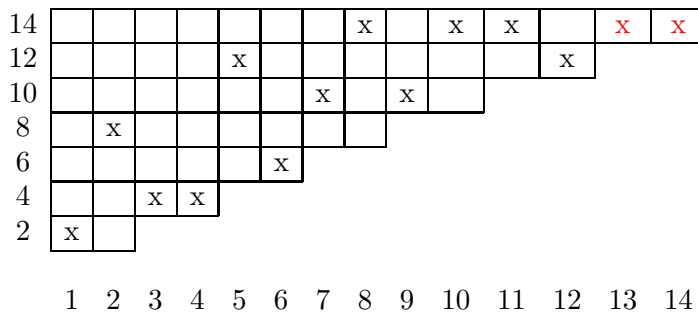


FIGURE 4. $f = 2\ 8\ 4\ 4\ 12\ 6\ 10\ 14\ 10\ 14\ 14\ 12\ 14\ 14 \in \mathcal{P}_{14}$

By Lemma 3 we derive the corresponding continued fraction for the generating function of $X_n(a, \bar{a}, b, \bar{b}, p, q, y, t)$. \square

4. CONTINUED FRACTIONS FOR CYCLES IN D-PERMUTATIONS AND E-PERMUTATIONS

A *surjective pistol* on $[2n]$ is the graph of a surjective mapping $f : [2n] \rightarrow [2n]$ such that

$$f(i) \geq i \quad \text{for } i \in [2n].$$

Let \mathcal{P}_{2n} be the set of surjective pistols on $[2n]$. A surjective pistol is depicted in Figure 4. A point $(k, f(k))$ of a pistol $f \in \mathcal{P}_{2n}$ is

- *maximum* if $f(k) = 2n$ and $k \leq 2n - 2$,
- *fixed* if $f(k) = k$ and $k < 2n$;
- *surfixed* if $f(k) = k + 1 < 2n$;
- *doubled* if $\exists j \neq k$ such that $f(j) = f(k)$.

A maximum $(k, f(k))$ is even (resp. odd) if k is even (resp. odd). The number of even (resp. odd) maxima of f is denoted by $\text{me}(f)$ (resp. $\text{mo}(f)$). The number of doubled (resp. isolated) fixed points is denoted by $\text{fd}(f)$ (resp. $\text{fi}(f)$). The number of doubled (resp. isolated) surfixed points is denoted by $\text{sd}(f)$ (resp. $\text{si}(f)$). The six statistics of the pistol f in Figure 4 are:

$$\text{mo}(f) = 1, \text{me}(f) = 2, \text{fd}(f) = 2, \text{fi}(f) = 1, \text{sd}(f) = 2, \text{si}(f) = 1.$$

Let

$$\Gamma_n(\alpha, \beta, \gamma, \bar{\alpha}, \bar{\gamma}, \bar{z}) = \sum_{f \in \mathcal{P}_{2n}} \alpha^{\text{mo}(f)} \beta^{\text{fd}(f)} \gamma^{\text{si}(f)} \bar{\alpha}^{\text{me}(f)} \bar{\beta}^{\text{fi}(f)} \bar{\gamma}^{\text{sd}(f)}.$$

Dumont [12] conjectured, and Randrianarivony [26] and Zeng [31] proved the following result.

Lemma 6 (Randrianarivony-Zeng). *We have*

$$\sum_{n \geq 0} \Gamma_{n+1}(\alpha, \bar{\alpha})x^n = \frac{1}{1 - (\alpha\bar{\beta} + \beta\bar{\gamma} + \gamma\bar{\alpha})x - \frac{(\bar{\alpha} + \beta)(\bar{\beta} + z)(\bar{\gamma} + \alpha)x^2}{\dots}} \quad (4.1)$$

where the coefficients under the $(n+1)$ -th row of fraction is

$$1 - [(\alpha + n)(\bar{\beta} + n) + (\beta + n)(\bar{\gamma} + n) + (\gamma + n)(\bar{\alpha} + n) - n(n + 1)]x - \frac{(n + 1)(\bar{\alpha} + \beta + n)(\bar{\beta} + \gamma + n)(\bar{\gamma} + \alpha + n)x^2}{\dots} \quad (4.2)$$

Lazar-Wachs [23, Lemma 5.2] proved the following result.

Lemma 7 (Lazar-Wachs). *There is a bijection*

$$\phi : \mathfrak{D}_{2n} \mapsto \{f \in \mathcal{P}_{2n+2} : f \text{ has no even maximal}\}$$

such that for all $\sigma \in \mathfrak{D}_{2n}$ and $j \in [2n]$, the following properties hold:

- (1) j is an even cycle maximum of σ iff it is a fixed point of $\phi(\sigma)$,
- (2) j is an even fixed point of σ iff it is an isolated fixed point of $\phi(\sigma)$,
- (3) j is an odd fixed point of σ iff it is an odd maximum of $\phi(\sigma)$.

Proof of Theorem 9. By Lemma 7 we obtain immediately

$$\sum_{\sigma \in \mathfrak{D}_{2n}} x_0^{\text{fix}_e(\sigma)} x_1^{\text{fix}_o(\sigma)} z^{\text{cyc}(\sigma)} = \Gamma_{n+1}(x_1 z, z, 1, 0, x_0 z, 1) \quad \text{for } n \geq 1,$$

where $\text{fix}_e(\sigma)$ (resp. $\text{fix}_o(\sigma)$) is the number of even (resp. odd) fixed points of σ .

It follows from Lemma 6 that

$$1 + \sum_{n \geq 1} \sum_{\sigma \in \mathfrak{D}_{2n}} x_0^{\text{fix}_e(\sigma)} x_1^{\text{fix}_o(\sigma)} z^{\text{cyc}(\sigma)} x^n = \frac{1}{1 - (x_0 x_1 z^2 + z)x - \frac{z(x_0 z + 1)(x_1 z + 1)x^2}{\dots}} \quad (4.3a)$$

where the coefficients under the $(n+1)$ -th row of fraction is

$$1 - [(x_1 z + n)(x_0 z + n) + (z + n)(1 + n)]x - \frac{(n + 1)(z + n)(x_0 z + 1 + n)(x_1 z + 1 + n)x^2}{\dots} \quad (4.3b)$$

Comparing Corollary 7 with $t = 1$ and (4.3a) with $x_0 = x_1 = 1$ we see that $\sum_{\sigma \in \mathcal{E}_{2n}} z^{\text{cyc}(\sigma)} = \sum_{\sigma \in \mathfrak{D}_{2n}} z^{\text{cyc}(\sigma)}$. \square

Note that setting $x_0 = x_1 = 0$ in (4.3a) yields the S-fraction

$$1 + \sum_{n \geq 1} \sum_{\sigma \in \mathfrak{D}_{2n}^*} z^{\text{cyc}(\sigma)} t^n = \frac{1}{1 - \frac{1 \cdot z t}{1^2 t}} \quad (4.4)$$

$$1 - \frac{2 \cdot (z+1) t}{1 - \frac{2^2 t}{\dots}}$$

where \mathfrak{D}_{2n}^* is the set of derangements in \mathfrak{D}_{2n} .

5. GAMMA-DECOMPOSITION AND GROUP ACTIONS

We give another proof of (2.9a) using *Inter-hopping* action on \mathcal{X}_{2n} , see [8, 15]. For $\sigma \in \mathcal{X}_{2n}$ and $j \in [n]$, a doubleton $\{2j-1, 2j\}$ is called *free* (in σ) if both $2j-1$ and $2j$ are in S_σ , or neither $2j-1$ nor $2j$ is in S_σ , namely $(2j-1, 2j) \in (S_\sigma \times S_\sigma) \cup (\overline{S}_\sigma \times \overline{S}_\sigma)$.

Let $w = w_1 \cdots w_{2n} \in \mathcal{X}_{2n}$ with a free pair $\{w_i, w_j\} = \{2r-1, 2r\}$ for some $i < j$. We define the action ϕ_r on w such that $|\text{des } w - \text{des } \phi_r(w)| = 1$.

Inter-hopping action ϕ_r

(A1) $(2j-1, 2j) \in \overline{S}_\sigma \times \overline{S}_\sigma$. If the elements $2r-1$ and $2r$ are adjacent in w then $\phi_r(w)$ is obtained by switching $2r-1$ and $2r$. Otherwise, we factorize w as

$$w = \cdots \beta_0 w_i \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_d \beta_d w_j \alpha_{d+1} \cdots,$$

where α_j (resp. β_j) is a maximal sequence of consecutive entries greater than $2r$ (resp. less than $2r-1$). Note that neither α_j nor β_j is empty for $1 \leq j \leq d$, but β_0 and α_{d+1} are possibly empty.

– If $w_i = 2r$ and $w_j = 2r-1$, then

$$\phi_r(w) := \cdots \beta_0 \alpha_1 (2r-1) \alpha_2 \beta_1 \alpha_3 \beta_2 \cdots \alpha_d \beta_{d-1} (2r) \beta_d \alpha_{d+1} \cdots.$$

– If $w_i = 2r-1$ and $w_j = 2r$, then

$$\phi_r(w) := \cdots \beta_0 (2r) \beta_1 \alpha_1 \beta_2 \alpha_2 \cdots \beta_d \alpha_d (2r-1) \alpha_{d+1} \cdots.$$

(A2) $(2j-1, 2j) \in S_\sigma \times S_\sigma$. Then $\phi_r(w)$ is obtained by reversing the process of (A1).

For $\sigma \in \mathcal{X}_{2n}$, let $\text{Orb}(\sigma) = \{g(\sigma) : g \in \mathbb{Z}_2^n\}$ be the orbit of σ under the Inter-hopping action ϕ . Let $\tilde{\sigma}$ be the unique \mathcal{E} -permutation in $\text{Orb}(\sigma)$. We have the following lemma.

Lemma 8. *For any $\tilde{\sigma} \in \overline{\mathcal{X}}_{2n}$, we have*

$$\sum_{\sigma \in \text{Orb}(\tilde{\sigma})} p^{(31-2)\sigma} q^{(2-31)\sigma} t^{\text{des } \sigma} y^{\text{dom}(\sigma)} = p^{(31-2)\tilde{\sigma}} q^{(2-31)\tilde{\sigma}} t^{\text{des } \tilde{\sigma}} (1+yt)^{n-2\text{des } \tilde{\sigma}}. \quad (5.1)$$

Proof. For $w \in \mathcal{X}_{2n}$ with a free pair $\{2r-1, 2r\}$, if $2r-1$ and $2r$ are adjacent in w , then obviously,

$$(2-31)\phi_r(w) = (2-31)w, \quad (31-2)\phi_r(w) = (31-2)w.$$

Otherwise, we factorize w as

$$w = \cdots \beta_0 w_i \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_d \beta_d w_j \alpha_{d+1} \cdots .$$

We assume that $w_i = 2r$ and $w_j = 2r - 1$ with $i < j$ (the $i > j$ case is similarly). Then

$$\phi_r(w) := \cdots \beta_0 \alpha_1 (2r - 1) \alpha_2 \beta_1 \alpha_3 \beta_2 \cdots \alpha_d \beta_{d-1} (2r) \beta_d \alpha_{d+1} \cdots .$$

We see that the Inter-hopping action ϕ_r does not change the relative orders of α_i and β_j respectively, so

- (i) if there is a descent pair in α_i (β_j , respectively), then this descent pair can only form (2-31) or (31-2) patterns with a letter in some α_k (β_l , respectively);
- (ii) in w , for $a \in \alpha_i$, a forms a (2-31) pattern with some descent pair say $(\text{last}(\alpha_j), \text{first}(\beta_j))$, if and only if in $\phi_r(w)$, a forms a (2-31) pattern either with descent

$$(\text{last}(\alpha_j), \text{first}(\beta_{j-1})) \quad \text{or} \quad (\text{last}(\alpha_j), \text{first}(\beta_j)).$$

For $b \in \beta_i$, b formed a (2-31) pattern with some descent pairs $(\text{last}(\alpha_j), \text{first}(\beta_j))$ if and only if b forms a (2-31) pattern either with descent pair $(\text{last}(\alpha_{j-1}), \text{first}(\beta_j))$ or with descent pair $(\text{last}(\alpha_j), \text{first}(\beta_j))$;

- (iii) the number of (2-31) patterns formed by $2r - 1$ and $2r$ with some descent pairs in w is equal to the number of (2-31) patterns formed by $2r - 1$ and $2r$ with some descent pairs in $\phi_r(w)$.

Combining (i), (ii) and (iii), we obtain $(2\text{-}31)w = (2\text{-}31)\phi_r(w)$. Similarly, we have $(31\text{-}2)w = (31\text{-}2)\phi_r(w)$. Setting each free pair of w not in S_w yields the unique \mathcal{E} -permutation, say $\tilde{\sigma}$, in $\text{Orb}(\sigma)$. Moreover, if $(2j, 2j - 2k - 1)$ is a descent pair, then the doubletons $\{2j, 2j - 1\}$ and $\{2j - 2k, 2j - 2k - 1\}$ are not free. So, the number of free doubletons in $\tilde{\sigma}$ is $n - 2 \text{des}(\tilde{\sigma})$, which implies the identity (5.1). \square

For convenience, we call an element *changed its type*, if it changed from an even left-to-right maxima (resp. odd right-to-left minima) to a non even left-to-right maxima (resp. a non odd right-to-left minima), or from a non even left-to-right maxima to an even left-to-right maxima (resp. an odd right-to-left minima).

Involution θ_r

For $\sigma \in \mathcal{X}_{2n}$, and $\{2r - 1, 2r\}$ is free with $(2r - 1, 2r) \in \overline{S}_\sigma \times \overline{S}_\sigma$. Then $\theta_r(\sigma) = \tau$, where $\tau \in \mathcal{X}_{2n}$ is obtained by exchanging $2r - 1$ and $2r$ in σ .

Lemma 9. For $w \in \mathcal{X}_{2n}$, if $\{2r - 1, 2r\}$ is free with $(2r - 1, 2r) \in \overline{S}_\sigma \times \overline{S}_\sigma$, then

$$\text{lema } \theta_r(w) = \text{lema } \phi_r(w) \tag{5.2}$$

$$\text{romi } \theta_r(w) = \text{romi } \phi_r(w), \tag{5.3}$$

and

$$\text{lema}(\phi_r \circ \theta_r)(w) = \text{lema } w \tag{5.4}$$

$$\text{romi}(\phi_r \circ \theta_r)(w) = \text{romi } w. \tag{5.5}$$

If $\{2r - 1, 2r\}$ is a free pair with $(2r - 1, 2r) \in S_\sigma \times S_\sigma$, then

$$\text{lema}(\theta_r \circ \phi_r)(w) = \text{lema } w \quad (5.6)$$

$$\text{romi}(\theta_r \circ \phi_r)(w) = \text{romi } w. \quad (5.7)$$

Proof. We take the factorization of w as in (A1), i.e.,

$$w = \cdots \beta_0 w_i \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_d \beta_d w_j \alpha_{d+1} \cdots, \quad (5.8)$$

here we assume $w_i = 2r - 1$, $w_j = 2r$. Then,

$$\phi_r(w) = \cdots \beta_0(2r) \beta_1 \alpha_1 \beta_2 \alpha_2 \cdots \beta_d \alpha_d (2r - 1) \alpha_{d+1} \cdots \quad (5.9)$$

$$\theta_r(w) = \cdots \beta_0(2r) \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_d \beta_d (2r - 1) \alpha_{d+1} \cdots. \quad (5.10)$$

Since all the letters in β_i are smaller than $2r - 1$ and all the letters in α_j are larger than $2r$. So, the even left-to-right maxima element, if exists, it must be in one of α_i or the element before β_0 , or it is $2r$. And the odd right-to-left minima element, if exists, it must be in one of β_j or the element after α_{d+1} or it is $2r - 1$. Compare (5.9) with (5.8) and (5.10) with (5.8), we can see that under the actions θ_r and ϕ_r , if some elements changed their type, they must in $\{2r - 1, 2r\}$. And compare (5.9) and (5.10), we have (5.2) and (5.3). By the same reason, we can derive (5.4), (5.5), (5.6) and (5.7). \square

Note that for $\sigma \in \mathcal{X}_{2n}$, neither θ_r and ϕ_r changes the number of (31-2) or (2-31) patterns when them action on σ . Then combine Lemma 8 and Lemma 9, we can finally derive Theorem 10.

6. FACTORIZATION OF γ -COEFFICIENTS AND GROUP ACTIONS

We prove Theorem 13. Clearly it suffices to prove

$$\sum_{\sigma \in \overline{\mathcal{X}}_{2n,k}} p^{(2-31)\sigma} q^{(31-2)\sigma} = (p + q)^{2k} \sum_{\sigma \in \widehat{\mathcal{X}}_{2n,k}} q^{(31-2)\sigma-k} p^{(2-31)\sigma-k} \quad (6.1)$$

with $\overline{\mathcal{X}}_{2n,k} = \{\sigma \in \overline{\mathcal{X}}_{2n} : \text{des } \sigma = k\}$ and $\widehat{\mathcal{X}}_{2n,k} = \{\sigma \in \widehat{\mathcal{X}}_{2n} : \text{des } \sigma = k\}$. We define an action φ_x on $\overline{\mathcal{X}}_{2n}$ as the following.

For $\sigma \in \overline{\mathcal{X}}_{2n}$, and $x \in S_\sigma$, we define $\varphi_x(\sigma)$ as follows:

- (A) If $x = 2i - 1$ and $s_\sigma(2i - 1)$ is odd, we factorize $\sigma = \tau'_3 \tau'_2 \tau'_1(2i) \tau_1 \tau_2 \tau_3$, where τ_1 and τ'_2 is the maximal sequence of consecutive entries greater than $2i$, τ_2 and τ'_1 is the maximal sequence of consecutive entries smaller than $2i$. We define

$$\varphi_x(\sigma) = \begin{cases} \tau'_3 \tau'_2 \tau'_1 \tau_1 \tau_2(2i) \tau_3 & \text{if } l_\sigma(2i) \text{ is even;} \\ \tau'_3(2i) \tau'_2 \tau'_1 \tau_1 \tau_2 \tau_3 & \text{if } l_\sigma(2i) \text{ is odd.} \end{cases}$$

- (B) If $x = 2j$, and $s_\sigma(2j)$ is odd, we factorize $\sigma = \tau'_3 \tau'_2 \tau'_1(2j - 1) \tau_1 \tau_2 \tau_3$, where τ_1 and τ'_2 is the maximal sequence of consecutive entries greater than $2j - 1$, τ_2 and τ'_1 is

the maximal sequence of consecutive entries smaller than $2j - 1$. Define

$$\varphi_x(\sigma) = \begin{cases} \tau'_3(2j-1)\tau'_2\tau'_1\tau_1\tau_2\tau_3 & \text{if } r_\sigma(2j-1) \text{ is even;} \\ \tau'_3\tau'_2\tau'_1\tau_1\tau_2(2j-1)\tau_3 & \text{if } r_\sigma(2j-1) \text{ is odd.} \end{cases}$$

- (C) If $x = 2i - 1$ and $s_\sigma(2i - 1)$ is even,
 (i) if $l_\sigma(2i - 1) = 2a$ for some $a \in \mathbb{N}$, then

$$\varphi_x(\sigma) = w \in \overline{\mathcal{X}}_{2n},$$

where w is constructed as follows.

Let w_0 be the subsequence of σ consisting of those elements which are smaller than $2i - 1$ in S_σ . And w_1 (σ_1 , respectively) is the subsequence of w (σ , respectively) which consists of all the elements in S_σ . We obtain w_{01} by inserting $2i - 1$ in w_0 such that the number of elements at the left side of $2i - 1$ in w_{01} minus two times of the number of even-odd descents at the left side of $2i - 1$ in w_{01} is $2a + 1$. Then we insert the rest elements of S_σ in w_{01} one by one in increasing order to obtain w_1 , such that the order of the rest elements in S_σ appear in w_1 is the same as in σ_1 . (The same order means when restrict w_1 and σ_1 on the elements $\{1, 2, \dots, k\}$ for $|S_\sigma| \geq k \geq 2i$ then the elements at left side of k minus two times of the number of even-odd descents are the same in the restrictions of w_1 and σ_1) Last, we insert the elements of $[2n] \setminus S_\sigma$ one by one in increasing order in w_1 to obtain w such that $(l_w(i), r_w(i)) = (l_\sigma(i), r_\sigma(i))$.

- (ii) if $l_\sigma(2i - 1) = 2a + 1$ for some $a \in \mathbb{N}$, then

$$\varphi_x(\sigma) = w \in \overline{\mathcal{X}}_{2n},$$

where w is constructed by following almost the process in (i), except when obtain w_{01} by inserting $2i - 1$ in w_0 such that the number of the elements at the left side of $2i - 1$ in w_{01} minus two times the number of even-odd descents at the left side of $2i - 1$ in w_{01} is $2a$.

- (D) If $x = 2j$ and $s_\sigma(2j)$ is even,
 (i) if $r_\sigma(2j) = 2a$ for some $a \in \mathbb{N}$, then

$$\varphi_x(\sigma) = w \in \overline{\mathcal{X}}_{2n},$$

Let w_0 be the subsequence of σ which consists of the elements in S_σ and smaller than $2j$. And w_1 (σ_1 , respectively) be the subsequence of w (σ , respectively) which consists of all the elements in S_σ . We obtain w_{01} by inserting $2j$ in w_0 such that the number of elements at the right side of $2j$ in w_{01} minus two times of the number of even-odd descents at the right side of $2j$ in w_{01} is $2a + 1$. Then we insert the rest elements of S_σ in w_{01} one by one in increasing order to obtain w_1 , such that the order of the rest elements in S_σ appear in w_1 is the same as in σ_1 . Last, we insert the elements of $[2n] \setminus S_\sigma$

one by one in increasing order in w_1 to obtain w such that $(l_w(i), r_w(i)) = (l_\sigma(i), r_\sigma(i))$.

(ii) if $r_\sigma(2j) = 2a + 1$ for some $a \in \mathbb{N}$, then

$$\varphi_x(\sigma) = w \in \overline{\mathcal{X}}_{2n},$$

where w is constructed by following almost the process in (i), except when obtain w_{01} by inserting $2j$ in w_0 such that the number of the elements at the right side of $2j$ in w_{01} minus two times of the number of even-odd descents at the right side of $2j$ in w_{01} is $2a$.

Example 2. For $\sigma = 56347812910 \in \mathcal{X}_{10}$, the set of descent tops and descent bottoms is $S_\sigma = \{1, 3, 6, 8\}$. So we have, $\varphi_1(\sigma) = 25634781910$, $\varphi_3(\sigma) = 56124783910$, $\varphi_6(\sigma) = 57834612910$, $\varphi_8(\sigma) = 56348127910$. We illustrate the φ act on 1 and 8 in Figure 5.

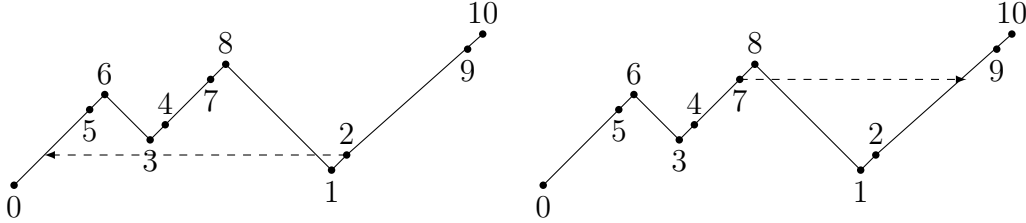


FIGURE 5. $\varphi_1(\sigma) = 25634781910$ and $\varphi_8(\sigma) = 56348127910$

We extend φ_x to all $x \in [2n]$ by $\varphi_x(\sigma) = \sigma$ for $x \notin S_\sigma$. Clearly the action φ_x is an involution on $\overline{\mathcal{X}}_{2n}$, besides two actions φ_x and φ_y commute for all $x, y \in [2n]$. Hence for any subset $X \subset [2n]$ we may define the action φ_X on $\sigma \in \overline{\mathcal{X}}_{2n}$ by

$$\varphi_X(\sigma) = \prod_{x \in X} \varphi_x(\sigma). \tag{6.2}$$

Lemma 10. Let $\sigma \in \overline{\mathcal{X}}_{2n}$ with weak signature S_σ . For any $x \in S_\sigma$, let

$$\Delta_{\varphi_x}(\sigma) = ((2-31)\sigma', (31-2)\sigma') - ((2-31)\sigma, (31-2)\sigma)$$

with $\sigma' := \varphi_x(\sigma)$, then $S_\sigma = S_{\sigma'}$ and

(a) If x and $s_\sigma(x)$ are odd, then

$$\Delta_{\varphi_x}(\sigma) = \begin{cases} (-1, 1) & \text{if } l_\sigma(x+1) \text{ is even} \\ (1, -1) & \text{if } l_\sigma(x+1) \text{ is odd;} \end{cases}$$

(b) If x is even but $s_\sigma(x)$ is odd, then

$$\Delta_{\varphi_x}(\sigma) = \begin{cases} (1, -1) & \text{if } l_\sigma(x-1) \text{ is even} \\ (-1, 1) & \text{if } l_\sigma(x-1) \text{ is odd;} \end{cases}$$

(c) If x is odd but $s_\sigma(x)$ is even, then

$$\Delta_{\varphi_x}(\sigma) = \begin{cases} (-1, 1) & \text{if } l_\sigma(x) \text{ is even} \\ (1, -1) & \text{if } l_\sigma(x) \text{ is odd;} \end{cases}$$

(d) If x and $s_\sigma(x)$ are even, then

$$\Delta_{\varphi_x}(\sigma) = \begin{cases} (1, -1) & \text{if } r_\sigma(x) \text{ is even} \\ (-1, 1) & \text{if } r_\sigma(x) \text{ is odd.} \end{cases}$$

Proof. For case (a), as the action φ_x moves $2i$ to the slot between two adjacent descent pairs, so the number of (2-31) patterns reduces by 1 and the number of (31-2) patterns increases by 1 when $2i$ moves to the slot between a descent pair at right. And the number of (2-31) patterns increase by 1 and the number of (31-2) patterns decreases by 1 when $2i$ moves to the slot between a descent pair at left. Case (b) is similar to case (a).

For case (c), it corresponds to (C) of the action φ_x . In this case, it is easy to see that

- if $l_\sigma(x)$ is even, the construction of $\sigma' = \varphi_x(\sigma)$ decreases the right embracing number of x by 1 and increases the left embracing number of x by 1;
- if $l_\sigma(x)$ is odd, it increases the right embracing number of x by 1 and decreases the left embracing number of x by 1.

Case (d) is similar to case (c). □

Lemma 11. For each $\hat{\sigma} \in \widehat{\mathcal{X}}_{2n,k}$ let $\text{Orb}(\hat{\sigma})$ be the orbit of $\hat{\sigma}$ under φ_S . Then

$$\sum_{\sigma \in \text{Orb}(\hat{\sigma})} p^{(2-31)\sigma} q^{(31-2)\sigma} = (p+q)^{2k} \cdot q^{(31-2)\hat{\sigma}-k} p^{(2-31)\hat{\sigma}-k}. \quad (6.3)$$

Proof. We rewrite (6.3) as

$$\sum_{\sigma \in \text{Orb}(\hat{\sigma})} p^{(2-31)\sigma} q^{(31-2)\sigma} = (1+p^{-1}q)^k \cdot (1+pq^{-1})^k \cdot q^{(31-2)\hat{\sigma}} p^{(2-31)\hat{\sigma}}. \quad (6.4)$$

The four cases (a)-(3.4a), (b)-(3.4c), (c)-(3.4e) and (d)-(3.4g) correspond to the four cases of normalized \mathcal{E} -permutations in Definition 11. If $\hat{\sigma} \in \widehat{\mathcal{X}}_{2n}$, by Lemma 10, for any $X \subset S_{\hat{\sigma}}$, applying action φ_X on a descent top (resp. bottom) of $\hat{\sigma}$ decreases the number of (31-2) (resp. (2-31)) patterns by 1 and increases the number of (2-31) (resp. (31-2)) patterns by 1, namely, it modifies the weight of $\hat{\sigma}$ by the factor pq^{-1} (resp. qp^{-1}). Since $\hat{\sigma}$ has exactly k descent tops (resp. bottoms), we obtain (6.4). □

Example 3. If $\sigma = 124635 \in \widehat{\mathcal{X}}_{6,1}$ with $S = \{3, 6\}$, then $\varphi_3(\sigma) = 126345$, $\varphi_6(\sigma) = 124563$ and $\varphi_S(\sigma) = 125634$. Thus

$$\text{Orb}(\sigma) = \{124635, 126345, 124563, 125634\}.$$

By Lemma 11, summing over all the orbits yields (6.1). □

7. OTHER INTERPRETATIONS OF GENOCCHI NUMBERS

For $\sigma = \sigma_1\sigma_2\dots\sigma_n \in \mathfrak{S}_n$, the statistics (2-13) σ is the number of pairs (i, j) such that $1 \leq i < j \leq n-1$ and $\sigma_{j+1} > \sigma_i > \sigma_j$.²

Definition 16. For $\sigma = \sigma_1\dots\sigma_n \in \mathfrak{S}_n$ with $\sigma_0 = \sigma_{n+1} = 0$, the value σ_i is

- a peak if $\sigma_{i-1} < \sigma_i$ and $\sigma_i > \sigma_{i+1}$;
- a valley if $\sigma_{i-1} > \sigma_i$ and $\sigma_i < \sigma_{i+1}$;
- a double ascent if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$;
- a double descent if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

Let $\text{dd}\sigma$ be the number of double descents in σ . Define the enumerative polynomials

$$Y_n(a, p, q, y, t) = \sum_{\sigma \in \mathcal{Y}_{2n+1}^*} a^{\text{lema}\sigma} p^{(2-13)\sigma} q^{(31-2)\sigma} y^{\text{dd}\sigma} t^{\text{des}\sigma} \quad (7.1)$$

Theorem 17. For $n \geq 1$ we have

$$Y_n(a, p, q, y, t) = X_n(a, 1, 1, 1, p, q, y, t). \quad (7.2)$$

Proof. It suffices to show that the two polynomials have the same ordinary generating functions. In view of Corollary 5 we need only to prove that

$$\sum_{n=0}^{\infty} Y_n(a, p, q, y, t)x^n = \frac{1}{1 - (1 + yt)[a, 1]_{p,q}x - \frac{t[a, 1]_{p,q}[a, 2]_{p,q}[2]_{p,q} \cdot x^2}{\dots}} \quad (7.3)$$

with coefficients

$$\begin{cases} b_{n-1} &= (1 + ty)[a, n]_{p,q}[n]_{p,q} \quad \text{for } n \geq 1, \\ \lambda_n &= t[a, n]_{p,q}[a, n+1]_{p,q}[n]_{p,q}[n+1]_{p,q}. \end{cases}$$

As the proof is similar to that of Theorem 4, we just indicate the mapping $\psi : \mathcal{Y}_{2n+1}^* \rightarrow \mathcal{PD}_n$ with the associated weight and omit the details.

Let $\sigma \in \mathcal{Y}_{2n+1}^*$ and $\psi(\sigma) = (w, (\xi, \xi'))$. For $j \in [n]$ we define the steps $s_j = w_j - w_{j-1}$ of the path $w = (w_0, w_1, \dots, w_n)$ as follows:

$$s_j = \begin{cases} U & \text{if } 2j-1 \text{ is a valley of } \sigma; \\ D & \text{if } 2j-1 \text{ is a peak of } \sigma; \\ L_1 & \text{if } 2j-1 \text{ is a double ascent of } \sigma; \\ L_2 & \text{if } 2j-1 \text{ is a double descent of } \sigma, \end{cases} \quad (7.4)$$

²The (2-13) statistic is called *artificial statistic* and denoted by “art” in [15].

and let $(\xi_j, \xi'_j) = (l_\sigma(2j-1), l_\sigma(2j))$. Thus, the weight of the path diagram $(w, (\xi, \xi')) = \prod_{i=1}^n w(w_i, (\xi_i, \xi'_i))$ is defined by

$$w(w_i, (\xi_i, \xi'_i)) = \begin{cases} a^{\chi(\xi'_i=0)} q^{\xi_i} p^{h-\xi_i} \cdot q^{\xi'_i} p^{h+1-\xi'_i} \cdot t & \text{if } \omega_i = U; \\ a^{\chi(\xi'_i=0)} q^{\xi_i} p^{h+1-\xi_i} \cdot q^{\xi'_i} p^{h-\xi'_i} & \text{if } \omega_i = D; \\ a^{\chi(\xi'_i=0)} q^{\xi_i} p^{h-\xi_i} \cdot q^{\xi'_i} p^{h-\xi'_i} & \text{if } \omega_i = L_1; \\ a^{\chi(\xi'_i=0)} q^{\xi_i} p^{h-\xi_i} \cdot q^{\xi'_i} p^{h-\xi'_i} \cdot yt & \text{if } \omega_i = L_2. \end{cases} \quad (7.5)$$

The J-fraction (7.3) follows then from Lemma 3. \square

A permutation $\sigma \in \mathcal{Y}_{2n+1}^*$ is called a \mathcal{F} -permutation if σ contains no double descent, and the last entry of σ is a peak, i.e., $\sigma_{2n} < \sigma_{2n+1}$. Denote $\overline{\mathcal{Y}}_{2n+1}^*$ as the set of \mathcal{F} -permutations with length \mathfrak{S}_{2n+1} , and $\overline{\mathcal{Y}}_{2n+1,k}^*$ the subset of permutations in $\overline{\mathcal{Y}}_{2n+1}^*$ with k descents.

Theorem 18. *We have*

$$Y_n(a, p, q, y, t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k}(a, p, q) t^k (1 + yt)^{n-2k}, \quad (7.6)$$

with coefficients

$$\gamma_{n,k}(a, p, q) = \sum_{\sigma \in \overline{\mathcal{Y}}_{2n+1,k}^*} a^{\text{lema}\sigma} p^{(2-13)\sigma} q^{(31-2)\sigma}.$$

Proof. Note that $Y_n(a, p, q, 0, t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k}(a, p, q) t^k$ and

$$\sum_{n=0}^{\infty} Y_n(a, p, q, 0, t) x^n = \frac{1}{1 - [a, 1]_{p,q} x - \frac{t[a, 1]_{p,q}[a, 2]_{p,q}[2]_{p,q} \cdot x^2}{\dots}} \quad (7.7)$$

with coefficients

$$\begin{cases} b_{n-1} &= [a, n]_{p,q} [n]_{p,q} \quad \text{for } n \geq 1, \\ \lambda_n &= t[a, n]_{p,q} [a, n+1]_{p,q} [n]_{p,q} [n+1]_{p,q}. \end{cases}$$

We derive (7.6) by comparing (7.3) and (7.7). \square

Remark 1. *We can also prove the above theorem by applying Brändén's modified Foata-Strehl action (a.k.a. MFS-action) on \mathcal{Y}_{2n+1}^* [6]. More precisely, we move every double descent to a double ascent position. It is known that that MFS-action does not change the number of (2-13) patterns and the number of (31-2) patterns. It is also clear that MFS-action does not change the even left-to-right maxima.*

For $\sigma \in \overline{\mathcal{Y}}_{2n+1}^*$ and $i \in [n]$, the doubleton $\{2i-1, 2i\}$ is called a *VOP pair* of σ if $2i-1$ is either a valley or peak of σ . As for \mathcal{E} -permutations, we associate a sequence $(a(1), a(2), \dots, a(2n))$ with σ by

$$a(j) = f(j) - g(j) + 1 \quad \text{for } j \in [2n].$$

where $f(i)$ (resp. $g(i)$) is the number of valleys (resp. peaks) less than i in σ .

Fact . If $\{2i - 1, 2i\}$ is a VOP pair of $\sigma \in \overline{\mathcal{Y}}_{2n+1}^*$, then there is one and only one even element in $\{a(2i - 1), a(2i)\}$.

For convenience, we define the embracing number of a VOP pair $x = \{2i - 1, 2i\}$ of σ by

$$l_\sigma(x) = \begin{cases} (2-31)(2i - 1) & \text{if } a(2i - 1) \text{ is even;} \\ (31-2)(2i) & \text{if } a(2i - 1) \text{ is odd.} \end{cases}$$

Definition 19. A \mathcal{F} -permutation $\sigma \in \overline{\mathcal{Y}}_{2n+1}^*$ is called *normalized*, if for any VOP pair $x = \{2i - 1, 2i\}$ of σ , the embracing number $l_\sigma(x)$ is even. Let $\widehat{\mathcal{Y}}_{2n+1}^*$ be the set of normalized \mathcal{F} -permutations in $\overline{\mathcal{Y}}_{2n+1}^*$ and $\widehat{\mathcal{Y}}_{2n+1,k}^*$ the subset of normalized \mathcal{F} -permutations in $\widehat{\mathcal{Y}}_{2n+1}^*$ with k descents by.

Action $\bar{\varphi}_x$ on $\overline{\mathcal{Y}}_{2n+1}^*$. For a VOP pair $x = \{2i - 1, 2i\}$ of $\sigma \in \overline{\mathcal{Y}}_{2n+1,k}^*$. We define $\bar{\varphi}_x$ as follows.

- (1) If $a(2i)$ is even then we factorize $\sigma = \tau_3' \tau_2' \tau_1'(2i) \tau_1 \tau_2 \tau_3$, where τ_1 and τ_2' is the maximal sequence of consecutive entries greater than $2i$, τ_2 and τ_1' is the maximal sequence of consecutive entries smaller than $2i$. We define

$$\bar{\varphi}_x(\sigma) = \begin{cases} \tau_3' \tau_2' \tau_1' \tau_1 \tau_2(2i) \tau_3 & \text{if } l_\sigma(x) \text{ is even;} \\ \tau_3'(2i) \tau_2' \tau_1' \tau_1 \tau_2 \tau_3 & \text{if } l_\sigma(x) \text{ is odd.} \end{cases}$$

- (2) If $a(2i - 1)$ is even.
 (a) If $l_\sigma(2i - 1) = 2b$ for some $b \in \mathbb{N}$, then

$$\bar{\varphi}_{2i-1}(\sigma) = w \in \overline{\varphi}_{2n+1,k}^*,$$

where w is constructed as follows.

If $2i - 1$ is a valley (resp. peak) of σ , then insert $2i - 1$ in w_0 such that at the left side of $2i - 1$ in w_{01} , the number of valleys minus the numbers of peaks equals to $l_{2i-1}(\sigma) + 1$ (here, valleys and peaks are those element smaller than $2i - 1$ in σ). Then we insert one by one the rest peaks and valleys are greater than $2i - 1$ in σ into w_{01} in increasing order to obtain w_1 , such that those peaks and valleys say $2j - 1$ s are still peaks and valleys in w_1 and for every $l_\sigma(2j - 1) = l_{w_1}(2j - 1)$. Last we insert the rest element of σ in w_1 one by one in increasing order such that $l_w(i) = l_\sigma(i)$ for all $[2n + 1] \setminus \{2i - 1\}$.

- (b) If $l_\sigma(2i - 1) = 2b + 1$ for some $b \in \mathbb{N}$, then

$$\bar{\varphi}_{2i-1}(\sigma) = w \in \overline{\mathcal{Y}}_{2n+1}^*,$$

where w is constructed by following almost the process in (a), except when obtain w_{01} by inserting $2i - 1$ in w_0 such that at the left side of $2i - 1$ in w_{01} , the number of valleys minus the number of peaks equals to $l_{2i-1}(\sigma) - 1$.

Clearly the action $\bar{\varphi}_x$ is an involution on $\bar{\mathcal{Y}}_{2n+1}^*$, besides two actions $\bar{\varphi}_x$ and $\bar{\varphi}_y$ commute for all x, y are VOP pairs. Hence for any subset $X \subset H$ where H is the set of VOP pairs of σ , we may define the action φ_X on $\sigma \in \bar{\mathcal{X}}_{2n}$ by

$$\bar{\varphi}_X(\sigma) = \prod_{x \in X} \bar{\varphi}_x(\sigma). \quad (7.8)$$

Following the similar proof of Lemma 10 and Lemma 11, we also have the following two Lemmas for \mathcal{F} -permutations.

Lemma 12. *Let $\sigma \in \bar{\mathcal{Y}}_{2n+1}^*$, for any x a VOP pair of σ , the permutation $\bar{\varphi}_x(\sigma)$ has the same set of valleys and peaks as σ . Let*

$$\Delta_{\bar{\varphi}_x}(\sigma) = ((2-31) \bar{\varphi}_x(\sigma), (31-2) \bar{\varphi}_x(\sigma)) - ((2-31) \sigma, (31-2) \sigma).$$

Then we have,

$$\Delta_{\bar{\varphi}_x}(\sigma) = \begin{cases} (-1, 1) & \text{if } l_\sigma(x) \text{ is even} \\ (1, -1) & \text{if } l_\sigma(x) \text{ is odd} \end{cases}$$

Lemma 13. *For each $\hat{\sigma} \in \widehat{\mathcal{Y}}_{2n+1}^*$, let $\text{Orb}(\hat{\sigma})$ be the orbit of $\hat{\sigma}$ under $\bar{\varphi}_H$. Then*

$$\sum_{\sigma \in \text{Orb}(\hat{\sigma})} p^{(2-13)\sigma} q^{(31-2)\sigma} = (p+q)^{2k} p^{(2-13)\sigma-2k} q^{(31-2)\sigma}.$$

Combine Lemma 12 and Lemma 13, we obtain the following Theorem.

Theorem 20. *We have*

$$\gamma_{n,k}(1, p, q) = (p+q)^{2k} \sum_{\sigma \in \widehat{\mathcal{Y}}_{2n+1,k}^*} p^{(2-13)\sigma-2k} q^{(31-2)\sigma}.$$

Let

$$\bar{Y}_n(a, p, q, t) = \sum_{\sigma \in \bar{\mathcal{Y}}_{2n+1}^*} a^{\text{lema}\sigma} p^{(2-13)\sigma} q^{(31-2)\sigma} t^{\text{des}\sigma} \quad (7.9a)$$

$$\widehat{Y}_n(p, q, t) = \sum_{\sigma \in \widehat{\mathcal{Y}}_{2n+1}^*} p^{(2-13)\sigma} q^{(31-2)\sigma} t^{\text{des}\sigma}. \quad (7.9b)$$

From Theorems 17, 18 and 20 we derive immediately the following identities.

Corollary 21. *We have*

$$\bar{Y}_n(a, p, q, t) = \bar{X}_n(a, 1, p, q, t), \quad (7.9c)$$

$$\widehat{Y}_n(p, q, t) = \widehat{X}_n(1, 1, p, q, t). \quad (7.9d)$$

Remark 2. *The normalized \mathcal{E} -permutations and normalized \mathcal{F} -permutations are two new models for the normalized median Genocchi numbers. It would be interesting to establish connections of these models with the well-known Dellac configurations for normalized Genocchi numbers in [4, 20].*

REFERENCES

- [1] Christos A. Athanasiadis. Gamma-positivity in combinatorics and geometry. *Sém. Lothar. Combin.*, 77:Art. B77i, 64, [2016-2018]. [4](#)
- [2] Daniel Barsky and Dominique Dumont. Congruences pour les nombres de Genocchi de 2e espèce. In *Study Group on Ultrametric Analysis. 7th–8th years: 1979–1981 (Paris, 1979/1981) (French)*, pages Exp. No. 34, 13. Secrétariat Math., Paris, 1981. [1](#)
- [3] Beáta Bényi and Gábor V. Nagy. Lonesum and Γ -free 0-1 fillings of Ferrers shapes. *European J. Combin.*, 89:103180, 15, 2020. [1](#)
- [4] Ange Bigeni. Combinatorial study of Dellac configurations and q -extended normalized median Genocchi numbers. *Electron. J. Combin.*, 21(2):Paper 2.32, 27, 2014. [28](#)
- [5] Natasha Blitvić and Einar Steingrímsson. Permutations, moments, measures. *Transactions of the American Mathematical Society*, 2021. [6](#)
- [6] Petter Brändén. Actions on permutations and unimodality of descent polynomials. *European J. Combin.*, 29(2):514–531, 2008. [4](#), [26](#)
- [7] Petter Brändén. Unimodality, log-concavity, real-rootedness and beyond. In *Handbook of enumerative combinatorics*, Discrete Math. Appl. (Boca Raton), pages 437–483. CRC Press, Boca Raton, FL, 2015. [4](#)
- [8] Alexander Burstein, Matthieu Josuat-Vergès, and Walter Stromquist. New Dumont permutations. *Pure Math. Appl. (P.U.M.A.)*, 21(2):177–206, 2010. [1](#), [7](#), [10](#), [19](#)
- [9] Sylvie Corteel, Matthieu Josuat-Vergès, and Lauren K. Williams. The matrix ansatz, orthogonal polynomials, and permutations. *Adv. in Appl. Math.*, 46(1-4):209–225, 2011. [6](#)
- [10] Sylvie Corteel, Jang Soo Kim, and Dennis Stanton. Moments of orthogonal polynomials and combinatorics. In *Recent trends in combinatorics*, volume 159 of *IMA Vol. Math. Appl.*, pages 545–578. Springer, [Cham], 2016. [6](#)
- [11] Dominique Dumont. Interprétations combinatoires des nombres de Genocchi. *Duke Math. J.*, 41:305–318, 1974. [1](#), [7](#)
- [12] Dominique Dumont. Conjectures sur des symétries ternaires liées aux nombres de Genocchi. volume 139, pages 469–472. 1995. Formal power series and algebraic combinatorics (Montreal, PQ, 1992). [17](#)
- [13] Dominique Dumont and Arthur Randrianarivony. Dérangements et nombres de Genocchi. *Discrete Math.*, 132(1-3):37–49, 1994. [1](#), [7](#)
- [14] Dominique Dumont and Jiang Zeng. Further results on the Euler and Genocchi numbers. *Aequationes Math.*, 47(1):31–42, 1994. [5](#), [6](#)
- [15] Sen-Peng Eu, Tung-Shan Fu, Hsin-Hao Lai, and Yuan-Hsun Lo. Gamma-positivity for a refinement of median genocchi numbers. *arXiv preprint arXiv:2103.09130*, 2021. [3](#), [4](#), [7](#), [10](#), [19](#), [25](#)
- [16] P. Flajolet. Combinatorial aspects of continued fractions. *Discrete Math.*, 32(2):125–161, 1980. [4](#), [13](#)
- [17] Dominique Foata and Marcel-P. Schützenberger. *Théorie géométrique des polynômes eulériens*. Lecture Notes in Mathematics, Vol. 138. Springer-Verlag, Berlin-New York, 1970. [4](#)
- [18] Ira M Gessel and Jiang Zeng. Moments of orthogonal polynomials and exponential generating functions. *arXiv preprint arXiv:2107.00255*, 2021. [6](#)
- [19] Guo-Niu Han. Hankel continued fractions and Hankel determinants of the Euler numbers. *Trans. Amer. Math. Soc.*, 373(6):4255–4283, 2020. [4](#)
- [20] Guo-Niu Han and Jiang Zeng. On a q -sequence that generalizes the median Genocchi numbers. *Ann. Sci. Math. Québec*, 23(1):63–72, 1999. [28](#)
- [21] Gábor Hetyei. Alternation acyclic tournaments. *European J. Combin.*, 81:1–21, 2019. [1](#), [2](#)
- [22] Sergey Kitaev and Jeffrey Remmel. Classifying descents according to parity. *Ann. Comb.*, 11(2):173–193, 2007. [7](#)

- [23] Alexander Lazar and Michelle L Wachs. The homogenized linial arrangement and genocchi numbers. *arXiv preprint arXiv:1910.07651*, 2019. [2](#), [3](#), [9](#), [18](#)
- [24] Alexander Lazar and Michelle L. Wachs. On the homogenized Linial arrangement: intersection lattice and Genocchi numbers. *Sém. Lothar. Combin.*, 82B:Art. 93, 12, 2020. [1](#)
- [25] Qiong Qiong Pan and Jiang Zeng. Brändén’s (p, q) -Eulerian polynomials, André permutations and continued fractions. *J. Combin. Theory Ser. A*, 181:105445, 2021. [4](#)
- [26] Arthur Randrianarivony. Polynômes de Dumont-Foata généralisés. *Sém. Lothar. Combin.*, 32:Art. B32d, approx. 12, 1994. [17](#)
- [27] Ludwig Seidel. Über eine einfache entstehungsweise der bernoulli’schen zahlen u. einiger verwandten reihen. *Sitzungsberichte der mathematisch-physikalischen Classe der königlich bayerischen Akademie der Wissenschaften zu München*, 7:157–187, 1877. [2](#)
- [28] Heesung Shin and Jiang Zeng. The symmetric and unimodal expansion of Eulerian polynomials via continued fractions. *European J. Combin.*, 33(2):111–127, 2012. [4](#)
- [29] Gérard Viennot. Interprétations combinatoires des nombres d’Euler et de Genocchi. In *Seminar on Number Theory, 1981/1982*, pages Exp. No. 11, 94. Univ. Bordeaux I, Talence, 1982. [1](#), [5](#)
- [30] Gérard Viennot. Une théorie combinatoire des polynômes orthogonaux généraux, univ. quebec, montreal. *Que.(unpublished notes)*, 1984. [4](#)
- [31] Jiang Zeng. Sur quelques propriétés de symétrie des nombres de Genocchi. In *Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993)*, volume 153, pages 319–333, 1996. [17](#)

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