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Total absolute curvature estimation

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Abstract

Total (absolute) curvature is defined for any curve in a metric space. Its properties, finiteness, local boundedness, Lipschitzness, depending whether there are satisfied or not, permit a classification of curves alternative to the classical regularity classes. In this paper, we are mainly interested in the total curvature estimation. Under the sole assumption of curve simpleness, we prove the convergence, as $\epsilon \rightarrow \mathbf{0}$, of the *naive turn estimators* which are families of polygonal lines whose vertices are at distance at most ϵ from the curve and whose edges are in $\Omega(\epsilon^\alpha) \cap O(\epsilon^\beta)$ with $\mathbf{0} < \beta \leq \alpha < \frac{1}{2}$. Besides, we give lower bounds of the speed of convergence under an additional assumption that can be summarized as being “convex-or-Lipschitz”.

Keywords: total curvature, geometric feature estimate, digital geometry

1 Introduction

Total curvature is the arithmetic sum of direction changes along a curve. As such, it is an indicator of the complexity of the curve [1, 2]. For this reason, it is used as a regularization term in segmentation processes [3–6] or as a characterization of digital straightness [7]. Moreover, due to Fenchel’s Theorem which states that any closed curve has a total curvature greater than 2π if not convex and equal to 2π if convex ([8] for the regular case and [9] for the general case), it is also a characterization of convexity [7]. Besides, locally limiting total curvature makes it possible to isotopically link the curve and its digitization [10]. Let us add that, contrary to the differential curvature, total curvature can be defined for any curve, that is without smoothness assumption. As reported in [11], this advantage was already underlined by J. Steiner at the end of the nineteenth century about the total mean curvature of surfaces. Another interesting

property when estimating the total curvature from a digital image is its scale invariance (it is dimensionless). Thus, one does not need the image resolution to get a total curvature estimate.

The definition of total curvature used in our paper is that suggested by R. H. Fox to Milnor [12] and extensively developed by Alexandrov and Reshetnyak [9] (see also the shorter paper of [13]). Let E^n be an n -dimensional Euclidean space. The total curvature $\kappa(P)$ of a polygonal line $P \subset E^n$ is the sum of its exterior angles (see Figure 1) and the total curvature of a curve $\mathcal{C} \subset E^n$ is the supremum of the total curvatures of its inscribed polygonal lines:

$$\kappa(\mathcal{C}) = \sup\{\kappa(P) \mid P \text{ inscribed in } \mathcal{C}\}. \quad (1)$$

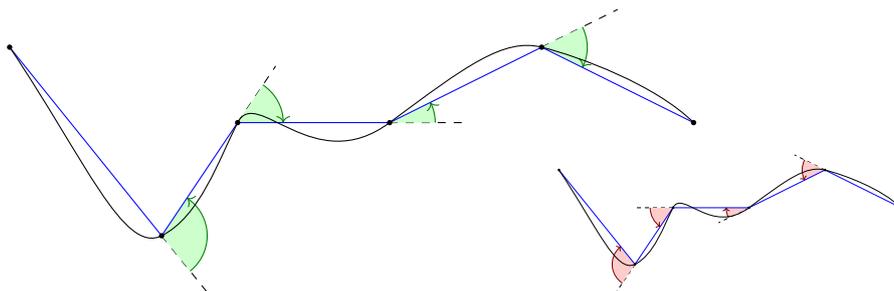


Fig. 1: Left: the curve and its inscribed polygonal line are traversed from left to right. Right: the curve and the polygonal line are traversed from right to left. The total curvature of the polygonal line is the sum of its absolute “turns” at each of its interior vertices. These turns are marked in green on the left and in red on the right. Their values are independent of the curve orientation. The total curvature of the curve is the supremum of the total curvatures of its inscribed polygonal lines.

Following Alexandrov, we call *turn* the total curvature when referring to Formula (1). The turn generalizes the total curvature of differential geometry: for curves of class C^2 , both definitions coincide. The total curvature may also be defined by means of circumscribing polygons as done in [14] for finite unions of convex arcs¹. Nevertheless, the definition is less general than that of Milnor.

In this paper, we aim to investigate whether the definition of the turn given by Formula (1) yields an easy but convergent way of estimating the total curvature, especially in the case of imprecise data like digitized sample data. Specifically, do we have the assurance that sequences of polygonal lines with increasingly shorter edges and vertices closer and closer to the curve would yield a convergent sequence of estimates? In the sequel, we will refer to these estimates by means of polygonal line total curvatures as total curvature *naive estimators*. To our knowledge, though there an important literature on estimating the differential curvature (see for instance

¹More precisely, the authors define the total curvature of convex curves then derive the total signed curvature of curves that are finite unions of convex arcs

the introductory sections in [15–17]) there is no reference on curve total curvature estimation, a fortiori from imprecise data (see the introductory section in [11] for surface total mean curvature estimation). Of course, total curvature may be computed from differential curvature estimates. But, doing so implies to jump from a C^0 to a C^2 assumption and to add a summation on the curvature estimation error. In view of the simplicity of the total curvature definition compared to the curvature one, directly estimating total curvature seems preferable. On the contrary, a total curvature estimate could serve as a basis for the differential curvature estimate. Indeed, in the same calculation loop one can compute the turn and the length of a polygonal line interpolating a small piece of curve. The quotient of these two values provides an estimate of the average differential curvature on the curve arc which could be used as an estimate of the differential curvature at some point of the arc. This is not so far from what is done by the integral based curvature estimator [18] which assume a constant curvature inside a ball of given radius. As for how to deal with data imprecision, error bounds on differential curvature estimates have been obtained directly by fitting digital circular arcs with quantified data [19–21]. Another way to obtain a bound on the error of the estimate is to decompose it into two terms, the first one due to discretization and the second one due to quantization (or data imprecision). This is the strategy followed, for instance, in [17, 18] for differential curvature estimation based on integral invariants. With regard to the estimation of the total curvature, without any assumptions on the shape of the curve, we need to proceed by decomposition.

Given exact sample points, thanks to a lemma from [9], we prove in Section 2.2 the convergence of the naive estimators for simple curves. But this result is purely qualitative and says nothing about the convergence speed. In the general case, the estimation of the total curvature can be tricky because sample points may be aligned in the worst case. This is also the case with the differential curvature and lead generally to exclude the possibility of such alignments. For instance, the integral invariant based curvature estimator [17, 22, 23] relies on a Taylor expansion shown in [24] where the alignment configuration is excluded by the first Lemma of the article. In the case of total curvature, excluding alignments, that is considering convex shapes, greatly simplifies the estimation. Indeed, it is easily derived from Fenchel’s theorem that the total curvature of a convex curve only depends on the angle between the tangent directions at its ends (Lemma 2). This property was also noticed by Latecki and Rosenfeld in [14] for their definition of total curvature using circumscribing polygons. Observe that the existence of tangents is ensured by convexity. Nevertheless, bounding the tangent estimation error requires extra regularity. Since we are interested in measuring the total curvature and we mainly control the Euclidean distance between the sample points, the regularity property used here is the turn Lipschitz continuity with respect to the Euclidean distance. More specifically, given a curve \mathcal{C} in E^n , we say that its turn is *k-strongly Lipschitz* if

$$\forall x, y \in \mathcal{C}, \kappa(\mathcal{C}_x^y) \leq k\|x - y\|, \quad (2)$$

where $\kappa(\mathcal{C}_x^y)$ denotes the curve arc between the points $x \in \mathcal{C}$ and $y \in \mathcal{C}$. Actually, the class of curves with strongly Lipschitz turn is well-known by its other characteristic properties, see Table 1 and Figure 2. In particular, in Appendix A and Appendix B, we give the proof that a curve has a strongly Lipschitz turn if and only if its turn is both

locally bounded and Lipschitz with respect to the geodesic distance and if and only if the curve is of class $C^{1,1}$. In Section 4, the strong Lipschitz continuity assumption is invoked to obtain speeds of convergence of the naive turn estimators.

	Property	Description	Equivalence
(A)	Strongly Lipschitz turn	A curve has a <i>strongly Lipschitz turn</i> if its turn is Lipschitz with respect to the Euclidean distance.	
(B)	$C^{1,1}$	A curve $t \mapsto \gamma(t)$ is $C^{1,1}$ if γ is C^1 and its derivative γ' is Lipschitz.	(B) \Leftrightarrow (A) Appendix B
(C)	Positive reach [25]	The <i>medial axis</i> MA of \mathcal{C} is the sets of space points that have at least two nearest points on \mathcal{C} . The <i>reach</i> is the infimum distance between the points of \mathcal{C} and MA.	(B) \Leftrightarrow “ γ , as set, has a positive reach” [25]
(D)	Par(r)-regularity [26–29]	Basically, a plane curve is <i>par(r)-regular</i> if a ball with radius r can roll smoothly on both sides of the curve	(D) \Leftrightarrow (C) [30]
(E)	Locally bounded turn [31]	A curve has a δ - <i>locally bounded turn</i> if its turn between any two points at Euclidean distance less than δ is at most $\pi/2$.	(E) \Leftrightarrow (D) [32]
	Lipschitz turn [31]	A curve has a <i>Lipschitz turn</i> if its turn is Lipschitz with respect to the geodesic distance.	(E) \Leftrightarrow (A) Appendix A

Table 1: Characteristic properties of curves with strongly Lipschitz turn.

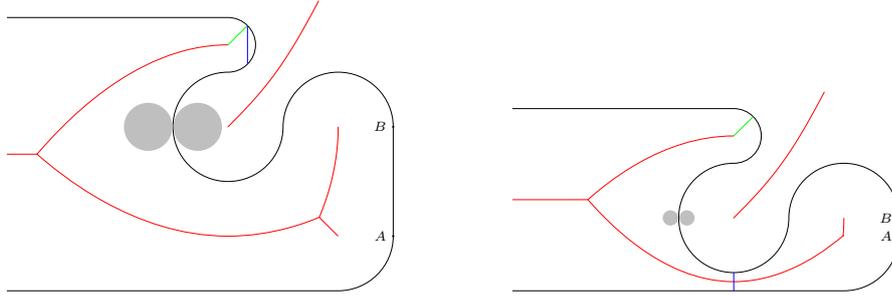


Fig. 2: Representation of the properties described in Table 1. Apart from the length of the straight segment AB , the left and right black curves are identical. Their medial axes are depicted in red. On each picture, the gray balls have radii strictly smaller than the reach of the curve. Then, they can roll along the curve. The blue segments indicate the maximum values of δ , the turn local boundedness parameter and the green segments indicate the smallest radii of curvature of the curves, which are related to the Lipschitz parameters.

In the general case, that is with non convex curves, the alignment configuration produces maximum estimation errors. Indeed, by definition of the turn, the error

caused by the naive estimator is at most equal to the curve total curvature. Equality happens when the turn of the polygonal interpolation is null. For instance, a sine curve may be interpolated by a straight segment. Nevertheless, when studying the convergence of a curvature estimator as curve sampling densifies, resampling with more and more points should reduce the proportion of the alignment configuration occurrences and tend to make it vanishing. This is true when the curve is composed of a finite number of pair of convex arcs with opposite convexities, that is when there is finitely many inflection points. It can be extended to curves with finitely many limit points of inflection points (see Section 4.2.4) but is not true in the general case (see Figure 6 which describes a nowhere convex curve with finite turn). Rather than “*ruling out pathological arcs which have infinitely many inflections or which turn infinitely often*” [14], here we just exclude curves that are difficult to travel, that is non simple curves (see Section 2.2). We prove the correctness of the turn limit for simple curve polygonal interpolations when edge lengths vanish. In the case of simple curves with finite turns, we propose to measure the non convex parts relative to a resolution and derive a bound on the turn estimation error. In Section 4.2, we deal with finite and infinite concatenations of convex curves.

The turn estimation error due to data imprecision is more easy to handle than the one due to the discretization. In this paper (Section 3), we bound the former geometrically. Our proof is not optimal but very simple. We decompose the error due to data imprecision in a sum of elementary errors caused by each triplet of consecutive sample points.

Taking into account both errors, the one due to the discretization of the curve and the one due to the uncertainty on the data, the global error depends on the part of the total curvature outside convex interpolated subarcs. For curves with finitely many inflection points, the best asymptotic error bound—equal to $O(\epsilon^{1/3})$ —is obtained when the edge lengths of the ϵ -approximation polygonal line belong to $O(\epsilon^{1/3})$ (Proposition 6). Shortest edges increase the error due to the imprecision of the data whereas longest edges increase the error due to the discretization. This error is comparable to differential curvature estimation error found in [17, 21]. For curves with an inflection limit point as $x^n \sin(\pi/x)$, we find a best error bound in $O(\epsilon^{1/5})$ taking edges in $O(\epsilon^{2/5})$ (Section 4.2.4). In Section 5, in order to give an insight on the constants hidden by the Landau notations, we present the results of a small set of tests using naive turn estimators on three plane curves. Section 6 concludes the paper and presents some further researches and developments in link with the article.

2 Discretization error

In order to estimate the turn of a shape boundary arc, we use a polygonal line build from the shape digitization. The difference between the true turn of the arc and the polygonal line turn is split into two terms, the discretization error and the quantization error. The discretization error is the difference between the turn of the curve and the turn of a polygonal line built from a sequence of sample points. These points lie *on* the curve unlike those coming from the digitization process which lie *close to* the curve.

Let us therefore describe these polygonal lines in our setting which is that of the book “General Theory of Irregular Curves” [9].

2.1 Vocabulary and previous results

2.1.1 Linear interpolations

Most of the vocabulary presented here is borrowed from [9]. Given a curve \mathcal{C} in E^n , a *chain* ξ is an ordered sequence of points on that curve. A polygonal line $P = [x_i]_{i=0}^m$ is *inscribed into the curve* \mathcal{C} if on the latter there can be found a chain $\xi = \langle y_i \rangle_{i=0}^m$ such that for any i , y_i coincides with x_i as a space point. Finally, an inscribed polygonal line P is a *linear interpolation* of \mathcal{C} if its endpoints are the endpoints of \mathcal{C} . If the curve \mathcal{C} is not simple, it may happen that several chains spatially coincide with a linear interpolation. For instance, consider a straight segment $[a, b]$ which is traveled three times, from a to b , then from b to a and again from a to b . Divide this segment by means of m equally spaced inner points. We get a simple linear interpolation P_m of the curve. The reader can check that there exist $2m + 3$ chains of that curve that share with P_m the same sequence of ordered space points. If, instead of traveling three times the segment $[a, b]$, we go from point a to point b following the Peano curve [33], thus filling a square of which $[a, b]$ is a diagonal, the polygonal line P_m is no more inscribed in the curve but for a few small values of m . Well, P_2 is inscribed in this Peano curve with 3 corresponding chains whereas P_4 is not inscribed there (due to the space order in which the Peano curve fills the square).

We will now focus on sequences of linear interpolations with vanishing edge lengths.

2.1.2 Fréchet convergence and turn limit

Before examining the question of the convergence of linear interpolations sequences towards the interpolated curve, let us give some vocabulary, again from [9]. Given a curve \mathcal{C} in E^n and a chain ξ , the biggest of the diameters of the arcs into which \mathcal{C} is divided by the vertices of ξ is called the *module* of the chain and is denoted through $\lambda(\xi)$. Let P be a linear interpolation of \mathcal{C} . The supremum of the modules of the chains that spatially coincide with P is called the *module of P relative \mathcal{C}* and is designated through $\lambda_{\mathcal{C}}(P)$. Notice that, contrary to [9], we will omit the index \mathcal{C} in the following if and only if $P = \mathcal{C}$. Hence, $\lambda(P)$ denotes the length of the largest edge of P . Eventually, the Fréchet distance between two curves \mathcal{C} and \mathcal{C}' is written $\rho(\mathcal{C}, \mathcal{C}')$.

Let us now recall three results of Alexandrov and Reshetnyak on the Fréchet convergence and the turn limit. The first one gives a sufficient condition for a sequence of curves to converge towards a given curve.

Lemma 1 ([9], page 23). *Let \mathcal{C} be a curve in the metric space (M, χ) and (\mathcal{C}_m) , $m = 1, 2, \dots$ be a sequence of curves in M . Let us assume that for any $\epsilon > 0$ there is a positive integer m_ϵ , such that at $m \geq m_\epsilon$ one can construct on the curves \mathcal{C} and \mathcal{C}_m the sequences of points x_1, x_2, \dots, x_k on the curve \mathcal{C} and x'_1, x'_2, \dots, x'_k on the curve \mathcal{C}_m , such that $\chi(x_i, x'_i) < \epsilon$ for all i and the diameters of the arcs into which the curves \mathcal{C} and \mathcal{C}_m are divided by the points are less than ϵ . In this case, the curves \mathcal{C}_m converge to the curve \mathcal{C} as $m \rightarrow \infty$ [for the Fréchet distance].*

In [9], the following corollary is immediately deduced from Lemma 1

Corollary 1 ([9], page 23). *Let \mathcal{C} be a curve in an n -dimensional Euclidean space, and let P_m , $m = 1, 2, \dots$, be an arbitrary sequence of polygonal lines inscribed in it. In this case, if at $m \rightarrow \infty$ $\lambda_{\mathcal{C}}(P_m) \rightarrow 0$, then the polygonal lines P_m converge to the curve \mathcal{C} .*

The third result make the link with the turn.

Theorem 1 ([9], page 121). *Let \mathcal{C} be a curve in an n -dimensional Euclidean space, and let P_m , $m = 1, 2, \dots$, be an arbitrary sequence of polygonal lines inscribed into it converging to the curve \mathcal{C} at $m \rightarrow \infty$. Then $\kappa(\mathcal{C}) = \lim_{m \rightarrow \infty} \kappa(P_m)$.*

In Section 2.2, we will replace Corollary 1 by a new result using the modules $\lambda(P_m)$ instead of the modules $\lambda_{\mathcal{C}}(P_m)$, which are unknown, at the cost of a curve simpleness assumption.

2.2 Simple curves

When estimating the turn of a curve \mathcal{C} in E^n by mean of a polygonal line P , we have control over the module $\lambda_P(P)$, that is over the edge lengths of P , not over $\lambda_{\mathcal{C}}(P)$. Hence, Corollary 1 is not useful to find converging sequences of polygonal lines. Now, the question is: can we replace the module $\lambda_{\mathcal{C}}(P)$ by the module $\lambda_P(P)$ in the corollary to Lemma 1? Unfortunately, the answer is negative in the general case. Indeed, consider the segment $[a, b]$, which is traveled three times, and the sequence of linear uniform interpolations $(P_m)_{m=0}^{\infty}$ described in Section 2.1.1. Then, the edge length of P_m , which is $\|b - a\|/(m + 1)$, tends towards 0 as $m \rightarrow \infty$ but the Fréchet distance between the curve and P_m does not vanish, staying equal to $\|b - a\|/2$. In the case of the Peano curve, also described in Section 2.1.1, we saw that $(P_m)_{m=0}^{\infty}$ is not a valid sequence of linear interpolations. But, it can be seen that a polygonal line P'_m from a to b that follow the edges of the square filled by the Peano curve with, for instance, $2m + 1$ equally spaced inner vertices is actually a linear interpolation of the curve. Nevertheless, it is plain that the sequence $(P'_m)_{m=0}^{\infty}$ does not converge to the Peano curve. In both previous examples, the reason convergence does not occur is that the considered linear interpolations may avoid significant parts of the curve. This is due to the fact that the curve is not simple. And indeed, as stated by the following proposition, it is enough to add the simpleness hypothesis to obtain the convergence of a sequence of linear interpolations whose edge lengths tends to 0.

Proposition 1. *Let \mathcal{C} be a simple curve in an n -dimensional Euclidean space, and let P_m , $m = 1, 2, \dots$, be a sequence of linear interpolations of \mathcal{C} . In this case, if at $m \rightarrow \infty$ $\lambda(P_m) \rightarrow 0$, then the polygonal lines P_m converge to the curve \mathcal{C} .*

Proof. Let \mathcal{C} be a curve in an n -dimensional Euclidean space and let P_m , $m = 1, 2, \dots$, be a sequence of polygonal lines inscribed in \mathcal{C} with $\lambda(P_m) \rightarrow 0$ as $m \rightarrow \infty$. We assume that (P_m) does not converge to the curve \mathcal{C} . We shall prove that \mathcal{C} is not simple. Since the sequence (P_m) does not converge, according to Lemma 1, there exists a subsequence (P'_m) and a positive real ϵ such that, for any m , $\lambda_{\mathcal{C}}(P'_m) \geq \epsilon$. On each polygonal line P'_m , let us choose a vertex X_i^m such that the diameter of the arc of \mathcal{C} between X_i^m and X_{i+1}^m is greater than or equal to ϵ . Observe that such vertices of P'_m exist because we assume that \mathcal{C} and P'_m share their endpoints. Otherwise, X_i^m or X_{i+1}^m may be an endpoint of \mathcal{C} without being a vertex of P'_m . By compactness of \mathcal{C} ,

there exists a pair of converging subsequences $X_{i_k}^k, X_{i_k+1}^k$. Let $X = \lim_{k \rightarrow \infty} X_{i_k}^k \in \mathcal{C}$ and $Y = \lim_{k \rightarrow \infty} X_{i_k+1}^k \in \mathcal{C}$. Since we assume $\lambda(P_m) \rightarrow 0$ as $m \rightarrow \infty$, we derive that $\|X_{i_k}^k - X_{i_k+1}^k\| \rightarrow 0$ as $k \rightarrow \infty$. Thus, X and Y share the same spatial support. Nevertheless, X and Y need not be equal as curve points. Actually, since the diameters of the arcs $[X_{i_k}^k, X_{i_k+1}^k]$ of \mathcal{C} are not less than ϵ , by continuity, the diameter of the arc $[X, Y]$ of \mathcal{C} is greater than, or equal to, ϵ . We derive that $X \neq Y$: the curve \mathcal{C} is not simple. \square

From the previous proposition and Theorem 1, we immediately derive the following result.

Corollary 2. *Let \mathcal{C} be a simple curve in an n -dimensional Euclidean space and let $(P_m)_{m=0}^\infty$ be a sequence of linear interpolations of \mathcal{C} . Then, $\lim_{m \rightarrow \infty} \kappa(P_m) = \kappa(\mathcal{C})$.*

Now that we have proved the convergence of the turns of linear interpolations with vanishing edges, let us look at the discretization error. In the case where the curve has infinite turn, the error is infinite. Indeed, the turn of any polygonal line with finitely many vertices is finite. It is upper bounded by $(m-2)\pi$ where m is the number of vertices. We will from now focus on the case of curves with finite turns.

2.3 Simple curves with finite turn

Given a curve \mathcal{C} in E^n and a linear interpolation P , the definition of the turn implies that $0 \leq \kappa(P) \leq \kappa(\mathcal{C})$. Therefore,

$$0 \leq \kappa(\mathcal{C}) - \kappa(P) \leq \kappa(\mathcal{C}). \quad (3)$$

The lower bound 0 of the discretization error is reached for instance when \mathcal{C} is a polygonal line and $P = \mathcal{C}$. The upper bound, $\kappa(\mathcal{C})$, is reached for instance if P is just a straight segment joining the ends of \mathcal{C} but also if \mathcal{C} is the sine curve $y = \sin(x)$ where $x \in [0, k\pi]$ (k being any positive integer) and the vertices of P are the points of the x axis with abscissa $i\pi$ for $0 \leq i \leq k$. On a 2D convex curve, the upper bound in (3) can be improved. This is the purpose of the following section.

2.3.1 Discretization error on a 2D convex curve

By convex curve, we refer to a connected subset of the boundary of a plane convex set. Observe that an arc of spiral may have everywhere a positive (or negative) differential curvature without being a convex curve in the above sense.

First, we note that, due to Fenchel's Theorem, a convex curve has a finite turn, equal to at most 2π . Our first statement regarding convex curves, showcased by Figure 3, expresses that the turn of a plane convex arc only depends on the tangent lines at its ends (recall that a convex curve has left and right tangent everywhere [34]).

Lemma 2. *Let \mathcal{C} be a non degenerated curve from the point a to the point b in E^n . Let \mathbf{t}_a and \mathbf{t}_b be respectively the right unit tangent vector at a and the left unit tangent vector at b . Then,*

$$\kappa(\mathcal{C}) \geq \angle(\mathbf{t}_a, \vec{ab}) + \angle(\mathbf{t}_b, \vec{ab}).$$

Moreover, the equality occurs if and only if \mathcal{C} is a plane convex curve.

Proof. The lemma is a direct consequence of two theorems proved in [9]:

(a) (Th. 5.1.5) The turn of a closed curve is at least 2π . The equality is valid only for convex curves.

Thus, with the hypotheses of the lemma

$$\kappa(\mathcal{C} \cup [a, b]) \geq 2\pi,$$

and the equality occurs if and only if \mathcal{C} is convex.

(b) (Th. 5.1.3) Let X be a curve with finite turn and x a point on X which is not an extremity of X . Let \mathbf{t}_l and \mathbf{t}_r be the left and right unit tangent vectors to X at x . Then,

- if X is a closed curve, $\kappa(X) = \kappa(X \setminus \{x\}) + \angle(\mathbf{t}_l, \mathbf{t}_r)$;
- otherwise, $\kappa(X) = \kappa(X_l) + \kappa(X_r) + \angle(\mathbf{t}_l, \mathbf{t}_r)$ where X_l and X_r are the left and right part of $X \setminus \{x\}$.

Thus, $\kappa(\mathcal{C} \cup [a, b]) = \kappa(\mathcal{C}) + \angle(\mathbf{t}_b, \vec{ba}) + \angle(\vec{ba}, \mathbf{t}_a)$. That is,

$$\kappa(\mathcal{C} \cup [a, b]) = \kappa(\mathcal{C}) + 2\pi - \angle(\mathbf{t}_b, \vec{ab}) - \angle(\vec{ab}, \mathbf{t}_a).$$

Putting together Items (a) and (b) straightforwardly ends the proof. \square

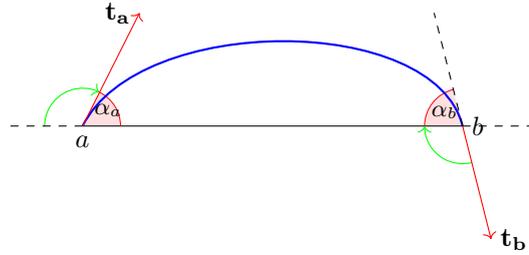


Fig. 3: Lemma 2 states that the turn of the blue convex curve \mathcal{C} is the sum of the red angles α_a and α_b (the red vectors are tangent to \mathcal{C} at a and b).

From the previous lemma, we derive an error bound for any linear interpolation.

Corollary 3. Let \mathcal{C} be a plane convex curve. Let $P = [a_i]_{i=0}^m$ ($m \geq 1$) be a linear interpolation of \mathcal{C} . Then,

$$0 \leq \kappa(\mathcal{C}) - \kappa(P) \leq \kappa(\mathcal{C}_{a_0}^{a_1}) + \kappa(\mathcal{C}_{a_{m-1}}^{a_m}).$$

Proof. We set $a = a_0$, $b = a_m$. Let us write α_a and α_b , resp. β_a and β_b , for the angles between the straight segment $[a, b]$ and the curve \mathcal{C} , resp. the polygonal line P , at the points a and b (Figure 4 shows the configuration and the notations near the point a). The result comes from Lemma 2 applied to the four curves \mathcal{C} , P , $\mathcal{C}_a^{a_1}$ and $\mathcal{C}_{a_{m-1}}^b$:

$$\kappa(\mathcal{C}) = \alpha_a + \alpha_b, \tag{4}$$

$$\kappa(P) = \beta_a + \beta_b, \quad (5)$$

$$\kappa(\mathcal{C}_a^{a_1}) \geq \alpha_a - \beta_a \quad (\text{see the figure}), \quad (6)$$

$$\kappa(\mathcal{C}_{a_{m-1}}^b) \geq \alpha_b - \beta_b. \quad (7)$$

Subtracting Equation (5) to Equation (4) and using Inequalities (6) and (7) gives the stated result. \square

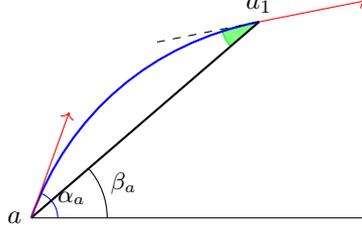


Fig. 4: Blue, thick: a convex curve \mathcal{C} with endpoint a and b (the point b is outside the figure). Black, thick: the first edge of a polygonal line $P = [a_i]_{i=0}^n$ that interpolates the curve \mathcal{C} . Black, thin: the end of the straight segment $[a, b]$. Red: the unit vector right-tangent to \mathcal{C} at a and the unit vector left-tangent to \mathcal{C} at a_1 . The turn of the arc $\mathcal{C}_a^{a_1}$ is the sum of the angle $\alpha_a - \beta_a$ and the green angle at a_1 . Then, $\kappa(\mathcal{C}_a^{a_1}) \geq \alpha_a - \beta_a$.

Provided the convex arc \mathcal{C}_a^b has a strongly Lipschitz turn, at least at its ends, Corollary 3 makes it possible to obtain an estimation of the turn of \mathcal{C}_a^b with a given precision by using a linear interpolation with sufficiently small edges at its ends (see Section 4.2.2).

2.3.2 General case

In the general case, we cut the curve in maximal convex parts and sewing parts. Let us define both parts. Let \mathcal{C} be a curve in E^n and $P = [a_i]_{i=0}^m$ be a linear interpolation of \mathcal{C} . An open convex arc $\hat{\mathcal{C}}_{a_i}^{a_j}$, $0 \leq i \leq j-2 \leq m-2$ is *maximal convex* if $j = m$ or $j < m$ and $\hat{\mathcal{C}}_{a_i}^{a_{j+1}}$ is not convex. By traversing the curve from a_0 to a_m , we iteratively select the non overlapping maximal convex arcs encountered so as to obtain a disjoint union of maximal convex arcs that we call \mathcal{C}_{\sim}^P . The sewing part, \mathcal{C}_{\sim}^P includes $\mathcal{C} \setminus \mathcal{C}_{\sim}^P$ and all the first and last subarcs $\mathcal{C}_{a_k}^{a_{k+1}}$ of the maximal convex arcs that compose \mathcal{C}_{\sim}^P . It can also be defined as follows:

$$\mathcal{C}_{\sim}^P = \bigcup \{ \mathcal{C}_{a_{\max(i-1,0)}^{\min(i+1,m)}} \mid a_i \notin \mathcal{C}_{\sim}^P \}. \quad (8)$$

Figure 5 illustrates this definition. It is plain that the convex arcs included in \mathcal{C}_{\sim}^P may be different depending on the travel direction. Also, notice that it may happen that \mathcal{C}_{\sim}^P is itself an union of convex arcs. This is in particular the case when \mathcal{C} is a spiral.

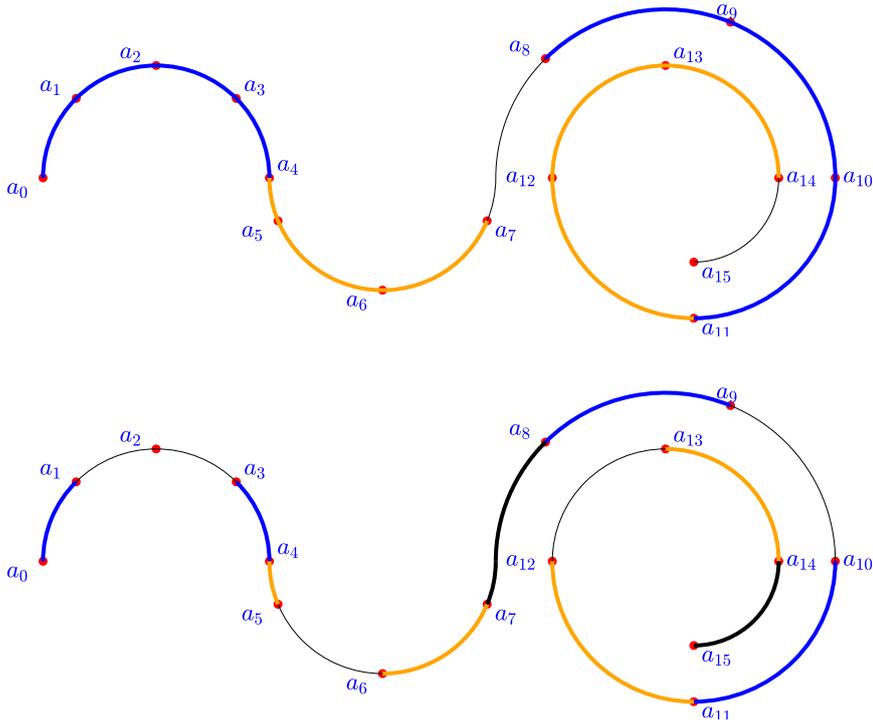


Fig. 5: Maximal convex arcs and sewing arcs relative to a linear interpolation. Top: denoting by \mathcal{C} and P the curve and its linear interpolation, the “convex part” of \mathcal{C} , \mathcal{C}_{\sim}^P is the union of the four maximal convex open arcs depicted in blue and orange. Bottom: the sewing part, \mathcal{C}_{\sim}^P , depicted by thick lines, includes the first and last arcs, $\mathcal{C}_{a_0}^{a_1}$ and $\mathcal{C}_{a_{m-1}}^{a_m}$, and the arcs $\mathcal{C}_{a_i}^{a_{i+1}}$ that are not included in \mathcal{C}_{\sim}^P . Numbering the sample points in the opposite direction would have given another decomposition.

Generally none of \mathcal{C}_{\sim}^P and \mathcal{C}_{\sim}^P is connected. By $\kappa(\mathcal{C}_{\sim}^P)$ and $\kappa(\mathcal{C}_{\sim}^P)$ we will denote the sum of the turns of their connected parts. Besides, an arc $\mathcal{C}_{a_i}^{a_{i+1}}$ is included in $\mathcal{C}_{\sim}^P \cap \mathcal{C}_{\sim}^P$ if and only if it is an end of a maximal convex arc of \mathcal{C}_{\sim}^P .

Since we cut the curve in overlapping parts, we will use the following equality.

Lemma 3. *Let $[a, b, c, d]$ be an ordered sequence of points on a curve \mathcal{C} of E^n whose turn is finite. Then,*

$$\kappa(\mathcal{C}_a^d) = \kappa(\mathcal{C}_a^c) + \kappa(\mathcal{C}_b^d) - \kappa(\mathcal{C}_b^c).$$

Proof. According to [9, Theorem 5.1.3 p. 122]

$$\kappa(\mathcal{C}_a^d) = \kappa(\mathcal{C}_a^c) + \kappa_c + \kappa(\mathcal{C}_c^d), \quad (9)$$

where κ_c denotes the turn at the point c , that is the angle between the left and right unit tangent vectors. Since we also have $\kappa(\mathcal{C}_b^d) = \kappa(\mathcal{C}_b^c) + \kappa_c + \kappa(\mathcal{C}_c^d)$, Equation 9 can

be rewritten as

$$\kappa(\mathcal{C}_a^d) = \kappa(\mathcal{C}_a^e) + \kappa(\mathcal{C}_b^d) - \kappa(\mathcal{C}_b^e). \quad \square$$

We conclude this section on the discretization error by showing that $\kappa(\mathcal{C}_\sim^P)$ is an upper bound of this error.

Proposition 2. *Let \mathcal{C} be a curve in E^n with finite turn and $P = [a_i]_{i=0}^m$ be a linear interpolation of \mathcal{C} . Then,*

$$0 \leq \kappa(\mathcal{C}) - \kappa(P) \leq \kappa(\mathcal{C}_\sim^P). \quad (10)$$

Proof. Thanks to Lemma 3, the turn of \mathcal{C} can be split as follows:

$$\kappa(\mathcal{C}) = \kappa(\mathcal{C}_\frown^P) + \kappa(\mathcal{C}_\smile^P) - \kappa(\mathcal{C}_\frown^P \cap \mathcal{C}_\smile^P). \quad (11)$$

Moreover, denoting by P_\frown and P_\smile the parts of P that interpolate respectively \mathcal{C}_\frown^P and \mathcal{C}_\smile^P , we have

$$\kappa(P) = \kappa(P_\frown) + \kappa(P_\smile). \quad (12)$$

Besides, from the turn definition,

$$0 \leq \kappa(\mathcal{C}_\smile^P) - \kappa(P_\smile) \leq \kappa(\mathcal{C}_\smile^P), \quad (13)$$

and, thanks to Corollary 3,

$$0 \leq \kappa(\mathcal{C}_\frown^P) - \kappa(P_\frown) \leq \kappa(\mathcal{C}_\frown^P \cap \mathcal{C}_\smile^P). \quad (14)$$

Eventually, subtracting Equation 12 to Equation 11 and using the bounds provided by Equations 13 and 14, we get

$$0 \leq \kappa(\mathcal{C}) - \kappa(P) \leq \kappa(\mathcal{C}_\sim^P). \quad \square$$

Proposition 2 together with a strongly Lipschitz assumption helps to upper bound the discretization error when estimating the turn with a polygonal line provided the curve is an union of, possibly infinitely many, convex arcs (see Section 4.2). Nevertheless, when the curve has no convex subarcs, Proposition 2 comes down to the obvious upper bound $\kappa(\mathcal{C}) - \kappa(P) \leq \kappa(\mathcal{C})$. Figure 6 showcases such a curve for which Proposition 2 is useless.

Finally, let us note that all the results obtained in Section 2.3 are easily interpreted using the *tangent indicatrix* also known as *tantrix*. Curves of finite turn are rectifiable and have everywhere left and right tangents [9]. Then, the tantrix is the curve described by the unit tangent vectors of such a curve on the unit sphere S^{n-1} [36, Chapter 5]. When the left and right tangent vectors differ, they are connected on the tantrix by means of a great circle arc [9, Chapters 3 and 5]. Due to the Mean Value Theorem, the tantrix of a linear interpolation of a curve is a spherical polygonal line inscribed into the tantrix of that curve. If finite, the turn of a curve is equal to the length of its tantrix. Thereby, the discretization error is the difference between the lengths of the two tantrices, the tantrix of the curve and that of the linear interpolation. Convex plane curves have monotonic tantrices on the unit circle. Clearly, the lengths of these

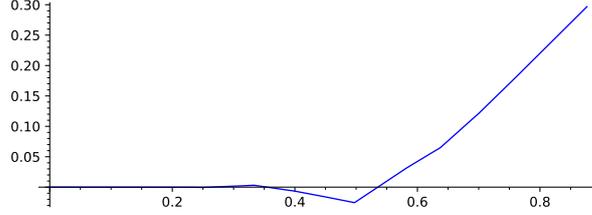


Fig. 6: A curve \mathcal{C} in \mathbb{R}^2 that has no convex subarc. At each rational arc length —the total length of the curve is 1, the curve turns to the left or to the right so that the turn of \mathcal{C} is $\pi/2$ and no arc of \mathcal{C} is convex. This curve is the limit of a sequence of polygonal lines whose vertex geodesic abscissa correspond to a line of the Stern-Brocot tree [35] and the turn at each rational geodesic abscissa r is $\pi/4^h$, h being the minimal height of r in the Stern-Brocot tree. The turn at r is oriented in the opposite direction to that of the turn of the vertex ahead.

tantrices only depend on their ends. Figure 7-left illustrates this situation and makes the link with Corollary 3. Figure 7-right showcases the case of a non-convex plane curve and interprets Proposition 2 thanks to the tantrices.

3 Quantization error

Given a curve \mathcal{C} in E^n and a positive real ϵ , we say that a polygonal line $Q = [b_i]_{i=0}^m$ is a *linear ϵ -approximation* of \mathcal{C} if there exists a linear interpolation of \mathcal{C} , $P = [a_i]_{i=0}^m$ such that $\|a_i - b_i\| \leq \epsilon$ for any $i \in [0, m]$. We say that a polygonal line Q is a *linear approximation* of \mathcal{C} if Q is a linear ϵ -approximation of \mathcal{C} for some $\epsilon > 0$. When estimating the turn of \mathcal{C} , the *quantization error* is the difference between the turn of a linear approximation of \mathcal{C} and the turn of the related linear interpolation.

Firstly, we compare the turn of two elementary polygonal lines, only having three vertices, and then we extend the result to any pair of polygonal lines.

3.1 Elementary error

Let us introduce new notations. Given a positive real r and points x, y, z in E^n , we denote by $B_r(x)$ the closed ball of center x and radius r and we denote by $\angle_x(y, z)$ the angle at x in the triangle xyz , that is the angle between the vectors \vec{xy} and \vec{xz} .

Let $\epsilon > 0$. We consider three balls in E^n with radius ϵ and centers a, b, c and three points a', b', c' belonging respectively to these three balls: $a' \in B_\epsilon(a)$, $b' \in B_\epsilon(b)$, $c' \in B_\epsilon(c)$. In order to upper bound the difference between the turn of the polygonal line $[a, b, c]$ and the turn of the polygonal line $[a', b', c']$, that is the difference $|\angle_b(a, c) - \angle_{b'}(a', c')|$, we consider two pairs of tangents shared by the circles $\partial B_\epsilon(a)$ and $\partial B_\epsilon(b)$ on the one hand and the circles $\partial B_\epsilon(c)$ and $\partial B_\epsilon(b)$ on the other hand. The slopes of these tangents bound the slopes of both sides of the angle $\angle_{b'}(a', c')$ (see Figure 8). This gives an easy, but not optimal, upper-bound. Indeed, in the general case, the line segments $[a', b']$ and $[b', c']$ cannot be both tangent to the circle $\partial B_\epsilon(b)$. In other words, the point z in Figure 8 is outside the circle $\partial B_\epsilon(b)$. Nevertheless, as the

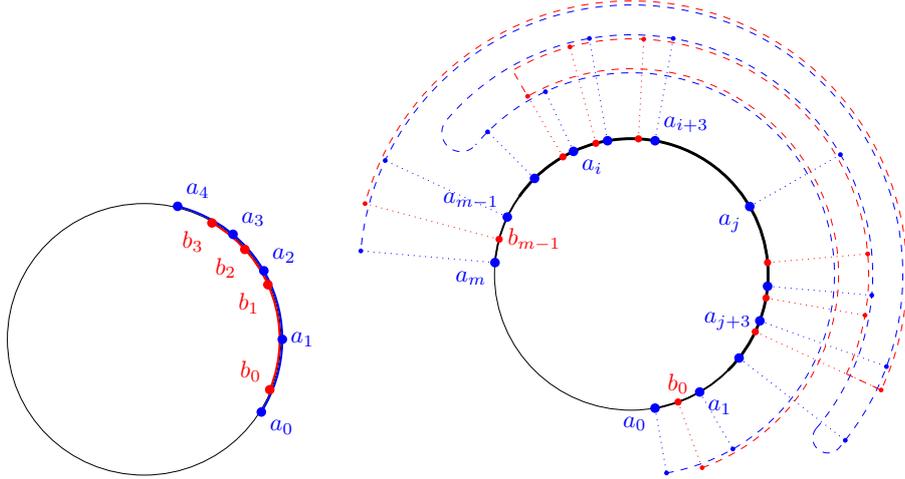


Fig. 7: Tantrices. Left: The circle arc from a_0 to a_4 is the tantrix of a plane convex curve \mathcal{C} . The circle arc from b_0 to b_3 is the tantrix of a linear interpolation P of \mathcal{C} . The points a_i correspond to the unit tangent vectors of \mathcal{C} at the vertices of P whereas the points b_j correspond to the edge directions of P . Corollary 3 asserts that the difference between the lengths of the two circular arcs is upper bounded by the sum of the lengths of the arcs from a_0 to a_1 and from a_3 to a_4 . Right: The circle arc from a_0 to a_m is the tantrix T of a plane non-convex curve \mathcal{C} . Since \mathcal{C} is non-convex, T is non-simple. We use an artificial dashed blue line to show how T is traveled. The circle arc from b_0 to b_{m-1} is the tantrix of a linear interpolation of \mathcal{C} . The dashed red line shows how this latter tantrix is traveled. The blue points a_i and the red points b_j are defined in the same way as in the figure on the left. Proposition 2 asserts the difference between the two circular arcs are upper bounded by the sum of the lengths of, on the one hand, the ends of T , $T_{a_0}^{a_1}$ and $T_{a_{m-1}}^{a_m}$ and, on the other hand, the U-turns of T , $T_{a_i}^{a_{i+3}}$ and $T_{a_j}^{a_{j+3}}$.

edge lengths of the linear interpolations tends towards zero, and provided the curve is sufficiently regular, that is its turn is strongly Lipschitz, the turns at the interpolation vertices vanish. Furthermore, optimal choices for the ratio linear interpolation edge length to ϵ make the ratio tends towards infinity as $\epsilon \rightarrow 0$ (see Section 4). In this case, the point z of Figure 8 tends towards the circle $\partial B_\epsilon(b)$ making the proposed upper bound more and more tight.

Looking at Figure 8, upper bounding the difference between the turns of the polygonal lines $[a, b, c]$ and $[a', b', c']$ should be easily geometrically handled. But, there is many other geometric contingencies and it is not easy to be sure not to forget a specific case. That is why our proof moves away from intuition to use a more formal way.

First, let us state two properties of right circular solid vector cones. We denote by V^n the vector space associated to E^n .

Lemma 4. *Let \mathbf{u} be a unit vector of V^n and $\theta \in [0, \pi/2]$. Let \mathbf{a} and \mathbf{b} be two vectors belonging respectively to the right circular solid vector cones with aperture 2θ and axes*

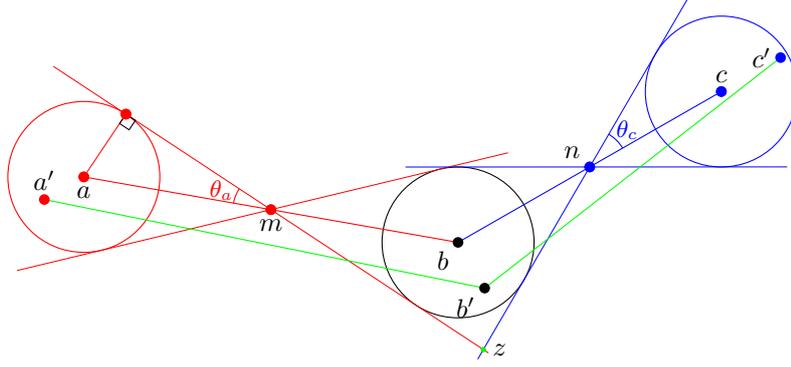


Fig. 8: The three circles have the same radius. Lemma 6 states that the difference between the curvature of the polygonal line $[a', b', c']$ (in green) and the curvature of the polygonal line $[a, b, c]$ (in red and blue) is at most the sum of the angles θ_a and θ_c .

\mathbf{u} and $-\mathbf{u}$. Then, the vector $\mathbf{a} - \mathbf{b}$ belongs to the solid vector cone with axis \mathbf{u} and aperture 2θ .

Proof. If $\mathbf{a} = \mathbf{b}$, the result is obvious. From now on, we assume $\mathbf{a} \neq \mathbf{b}$. In the limit case $\theta = \pi/2$, denoting by p the linear projection from V^n to the line $\mathbb{R}u$, the hypotheses are $\langle p(\mathbf{a}) | \mathbf{u} \rangle \geq 0$ and $\langle p(\mathbf{b}) | \mathbf{u} \rangle \leq 0$. Then, $\langle p(\mathbf{a} - \mathbf{b}) | \mathbf{u} \rangle \geq 0$ which is the desired result. Now, assuming $\theta < \pi/2$, we derive from the main hypotheses that there exist four real numbers $\lambda_a, \lambda_b, \mu_a, \mu_b$ and two unit vectors \mathbf{v}, \mathbf{w} orthogonal to the vector \mathbf{u} such that

$$\mathbf{a} = \lambda_a \mathbf{u} + \mu_a \tan(\theta) \mathbf{v} \quad \text{and} \quad \mathbf{b} = \lambda_b \mathbf{u} + \mu_b \tan(\theta) \mathbf{w},$$

where $|\mu_a| \leq |\lambda_a|$, $|\mu_b| \leq |\lambda_b|$, $\lambda_a \geq 0$ and $\lambda_b \leq 0$.

Let $\theta_{a,b}$ be the (unoriented) angle between the vector \mathbf{u} and the vector $\mathbf{a} - \mathbf{b}$. We have,

$$\cos(\theta_{a,b}) = \frac{\langle \mathbf{a} - \mathbf{b} | \mathbf{u} \rangle}{\|\mathbf{a} - \mathbf{b}\| \|\mathbf{u}\|} = \frac{\lambda_a - \lambda_b}{\sqrt{(\lambda_a - \lambda_b)^2 + \tan^2(\theta) N^2}},$$

where $N = \|\mu_a \mathbf{v} - \mu_b \mathbf{w}\| \leq |\mu_a| + |\mu_b| \leq |\lambda_a| + |\lambda_b| = \lambda_a - \lambda_b$. Then,

$$\cos(\theta_{a,b}) \geq \frac{1}{\sqrt{1 + \tan^2(\theta)}} = \cos(\theta).$$

We derive that $0 \leq \theta_{a,b} \leq \theta$, that is, $\mathbf{a} - \mathbf{b}$ belongs to the solid cone with aperture 2θ and axis \mathbf{u} . \square

Lemma 5. Let C_1 , resp. C_2 , be a right circular solid vector cone of V^n whose axis is directed by the unit vector \mathbf{u}_1 , resp. \mathbf{u}_2 , and whose aperture is $2\theta_1$, resp. $2\theta_2$, with θ_1 , resp. θ_2 , $\in [0, \pi/2]$. Then, for any $\mathbf{v}_1 \in C_1$, $\mathbf{v}_2 \in C_2$,

$$|\angle(\mathbf{v}_1, \mathbf{v}_2) - \angle(\mathbf{u}_1, \mathbf{u}_2)| \leq \theta_1 + \theta_2.$$

Proof. Thanks to the triangle inequality for angles of vectors [37], one has

$$\angle(\mathbf{u}_1, \mathbf{u}_2) \leq \angle(\mathbf{u}_1, \mathbf{v}_1) + \angle(\mathbf{v}_1, \mathbf{v}_2) + \angle(\mathbf{v}_2, \mathbf{u}_2),$$

that is,

$$\angle(\mathbf{u}_1, \mathbf{u}_2) - \angle(\mathbf{v}_1, \mathbf{v}_2) \leq \angle(\mathbf{u}_1, \mathbf{v}_1) + \angle(\mathbf{v}_2, \mathbf{u}_2),$$

and

$$\angle(\mathbf{v}_1, \mathbf{v}_2) \leq \angle(\mathbf{v}_1, \mathbf{u}_1) + \angle(\mathbf{u}_1, \mathbf{u}_2) + \angle(\mathbf{u}_2, \mathbf{v}_2),$$

that is

$$\angle(\mathbf{v}_1, \mathbf{v}_2) - \angle(\mathbf{u}_1, \mathbf{u}_2) \leq \angle(\mathbf{v}_1, \mathbf{u}_1) + \angle(\mathbf{u}_2, \mathbf{v}_2).$$

Since $\angle(\mathbf{u}_1, \mathbf{v}_1)$ and $\angle(\mathbf{v}_2, \mathbf{u}_2)$ are respectively upper bounded by θ_1 and θ_2 , we get

$$|\angle(\mathbf{v}_1, \mathbf{v}_2) - \angle(\mathbf{u}_1, \mathbf{u}_2)| \leq \theta_1 + \theta_2. \quad \square$$

Thanks to the preceding lemmas, we can now bound the quantization error in the basic case of a linear ϵ -approximation having only three vertices.

Lemma 6. *Let $\epsilon > 0$ and a, b, c in E^n such that $\|b - a\|$ and $\|c - b\|$ are both greater than or equal to 2ϵ . Then, for any $a' \in B_\epsilon(a)$, $b' \in B_\epsilon(b)$ and $c' \in B_\epsilon(c)$,*

$$|\kappa([a', b', c']) - \kappa([a, b, c])| \leq \arcsin \frac{2\epsilon}{\|b - a\|} + \arcsin \frac{2\epsilon}{\|b - c\|}.$$

Proof. Let m and n be the midpoints of the line segments $[ba]$ and $[bc]$. From Lemma 4, the vectors $a' - b'$ and $c' - b'$ respectively belong to the solid vector cones C_a and C_c , with axis $\mathbf{u}_a = a - m$ and aperture $\theta_a = \arcsin \frac{2\epsilon}{\|b - a\|}$ for the former and with axis $\mathbf{u}_b = c - n$ and aperture $\theta_c = \arcsin \frac{2\epsilon}{\|b - c\|}$ for the latter (see Figure 8). Thanks to Lemma 5, we derive that

$$|\angle_{b'}(a', c') - \angle(\mathbf{u}_a, \mathbf{u}_b)| \leq \theta_a + \theta_c.$$

Since $\angle_{b'}(a', c') = \pi - \kappa([a', b', c'])$ and $\angle(\mathbf{u}_a, \mathbf{u}_b) = \angle_b(a, c) = \pi - \kappa([a, b, c])$, we obtain

$$|\kappa([a', b', c']) - \kappa([a, b, c])| \leq \theta_a + \theta_c. \quad \square$$

3.2 Global quantization error

Applying Lemma 6 at each of the non terminal vertices of an ϵ -approximation yields a quantization error upper bound.

Proposition 3. *Let ϵ be a positive real and $P = [p_i]_{i=0}^m$, $Q = [q_i]_{i=0}^m$ be two polygonal lines in E^n such that $d(p_i, q_i) \leq \epsilon$ for any $i \in [0, m]$ ($m > 0$). Let ℓ be the minimum length of the edges of Q . If $\ell \geq 2\epsilon$, then*

$$|\kappa(P) - \kappa(Q)| \leq (2m - 2) \arcsin \frac{2\epsilon}{\ell}.$$

Proof. We apply Lemma 6 at each vertex q_i , $1 \leq i \leq m - 1$. □

4 Convergence of naive turn estimators

In this section, we prove the convergence of the total curvature naive estimators in the case of simple curves and we compute the convergence speeds under various assumptions.

4.1 Simple curves

Given a polygonal line P , we denote by $\mu(P)$ the length of its smaller side (recall that $\lambda(P)$ denotes the length of its longest side).

To ensure that both the discretization error and the quantization error vanish as $\epsilon \rightarrow 0$, the edge lengths of the naive turn estimators need to obey laws that may seem antagonistic. Reducing the discretization error implies decreasing these lengths whereas reducing the quantization error implies increasing —with respect to the ϵ parameter— the same lengths. Let us first look at the quantization error. The following lemma provides a simple ratio that dominates as $\epsilon \rightarrow 0$ the difference between the turn of an ϵ -approximation and the turn of an associated linear interpolation.

Lemma 7. *Let \mathcal{C} be a simple curve in E^n . Let $(Q_\epsilon)_{\epsilon>0}$ be a family of linear ϵ -approximations of \mathcal{C} . If the minimum edge length $\mu(Q_\epsilon)$ of Q_ϵ is such that $\mu(Q_\epsilon) \in \omega(\epsilon)$, then, for any family $(P_\epsilon)_{\epsilon>0}$ of linear interpolations of \mathcal{C} whose vertices are at distance at most ϵ from the vertices of Q_ϵ ,*

$$|\kappa(P_\epsilon) - \kappa(Q_\epsilon)| = O\left(\frac{\epsilon}{\mu(Q_\epsilon)^2}\right).$$

Proof. Let $(P_\epsilon)_{\epsilon>0}$ be a family of linear interpolations whose vertices are at distance at most ϵ from the vertices of Q_ϵ . From the hypotheses, we have

$$\frac{\epsilon}{\mu(Q_\epsilon)} = o(1) \quad \text{as } \epsilon \rightarrow 0.$$

Then, for sufficiently small ϵ , the hypothesis $\mu(Q_\epsilon) \geq 2\epsilon$ of Proposition 3 is satisfied and,

$$|\kappa(P_\epsilon) - \kappa(Q_\epsilon)| \leq (2m - 2) \arcsin \frac{2\epsilon}{\mu(Q_\epsilon)},$$

where m is the edge number of Q_ϵ . Since $\arcsin(x) \sim x$ as $x \rightarrow 0$, we derive that

$$|\kappa(P_\epsilon) - \kappa(Q_\epsilon)| = O\left(\frac{m\epsilon}{\mu(Q_\epsilon)}\right). \quad (15)$$

Besides, denoting by $\mathcal{L}(\cdot)$ the length, we have

$$m \times \mu(P_\epsilon) \leq \mathcal{L}(P) \leq \mathcal{L}(C).$$

Hence,

$$m \in O\left(\frac{1}{\mu(P_\epsilon)}\right) \subset O\left(\frac{1}{\mu(Q_\epsilon)}\right).$$

By substituting $O(\frac{1}{\mu(Q_\epsilon)})$ to n in (15), we get the result:

$$|\kappa(P_\epsilon) - \kappa(Q_\epsilon)| = O\left(\frac{\epsilon}{\mu(Q_\epsilon)^2}\right). \quad \square$$

We can now obtain sufficient conditions so that naive turn estimates tend towards the right value as $\epsilon \rightarrow 0$.

Proposition 4. *Let \mathcal{C} be a simple curve in E^n . Let $(Q_\epsilon)_{\epsilon>0}$ be a family of linear ϵ -approximations of \mathcal{C} . If the maximum edge length $\lambda(Q_\epsilon)$ of Q_ϵ tends to 0 as $\epsilon \rightarrow 0$ and the minimum edge length $\mu(Q_\epsilon)$ of Q_ϵ is such that $\mu(Q_\epsilon) \in \omega(\epsilon^{1/2})$, then*

$$\lim_{\epsilon \rightarrow 0} \kappa(Q_\epsilon) = \kappa(\mathcal{C}).$$

Proof. For any $\epsilon > 0$, there exists a linear interpolation, which we denote by P_ϵ , whose vertices are at distance at most ϵ from the vertices of Q_ϵ . Then,

$$|\lambda(P_\epsilon) - \lambda(Q_\epsilon)| \leq 2\epsilon.$$

Since, by hypothesis, $\lambda(Q_\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$, we derive

$$\lim_{\epsilon \rightarrow 0} \lambda(P_\epsilon) = 0,$$

and, according to Corollary 2,

$$\lim_{\epsilon \rightarrow 0} \kappa(P_\epsilon) = \kappa(\mathcal{C}). \quad (16)$$

Furthermore, according to Lemma 7,

$$|\kappa(P_\epsilon) - \kappa(Q_\epsilon)| = O\left(\frac{\epsilon}{\mu(Q_\epsilon)^2}\right).$$

Taking into account the hypothesis $\mu(Q_\epsilon) \in \omega(\epsilon^{1/2})$, we get

$$|\kappa(P_\epsilon) - \kappa(Q_\epsilon)| = o(1), \quad (17)$$

Eventually, from Equations (16) and (17), we conclude straightforwardly:

$$\lim_{\epsilon \rightarrow 0} \kappa(Q_\epsilon) = \kappa(\mathcal{C}). \quad \square$$

Examining the proof of Proposition 4, we can see that it is sufficient to assume $\mu(Q_\epsilon) \in \Omega(\epsilon^2)$ to decide whether $\kappa(\mathcal{C}) = \infty$ or $\kappa(\mathcal{C}) < \infty$ (for $\infty + O(1) = \infty$). Nevertheless, we need the assumption $\mu(Q_\epsilon) \in \omega(\epsilon^2)$ to obtain the correct limit when $\kappa(\mathcal{C}) < \infty$.

The simpleness assumption being too weak to allow convergence speed estimation, we examine in the remainder of this section some examples of concatenations of convex plane arcs whose turns are strongly Lipschitz, at least near the end of the arcs.

4.2 Simple curves with finite and Lipschitz turn

In order to upper bound the difference between a naive turn estimation and the true turn value, we need to add a regularity assumption to the simpleness hypothesis on curves. In Section 1, we introduced the notion of *strongly Lipschitz* turn and justified its use in this work. Recall that a curve has a strongly Lipschitz turn if its turn is Lipschitz with respect to the Euclidean distance, that is, if there exists a real $k > 0$ such that, for any points $a, b \in C$, $\kappa(C_a^b) \leq k\|a - b\|$. Since the turn of a convex arc only depends on the direction of the curve at its ends (Lemma 2), in the best case (finite concatenations of convex arcs) we only need the strongly Lipschitz property near the ends of the convex subarcs of the curve. Hence, given a positive real k , we say that a curve \mathcal{C} has a k -strongly Lipschitz turn at $a \in \mathcal{C}$ if \mathcal{C} is smooth at a and there exists a positive real r such that $\kappa(C_a^b) \leq k\|a - b\|$ for any $b \in \mathcal{C} \cap B_r(a)$. We start by a general result about the order of convergence of the discretization error and then we will deduce lower bounds for the naive estimator convergence speed when applied to curves that are composed of convex arcs.

4.2.1 Discretization error convergence order

Lemma 8. *Let \mathcal{C} in E^n be a simple curve of finite turn and $(P_\epsilon)_{\epsilon > 0}$ be a family of linear interpolations of \mathcal{C} . If, for some positive k and for sufficiently small ϵ , the turn of \mathcal{C} is k -strongly Lipschitz on each connected component of $\mathcal{C}_{\sim}^{P_\epsilon}$, then*

$$\kappa(\mathcal{C}) - \kappa(P_\epsilon) \leq k n_\epsilon \lambda(P_\epsilon),$$

where n_ϵ denotes the number of edges of P_ϵ that interpolate an arc of $\mathcal{C}_{\sim}^{P_\epsilon}$. In particular, if $\lambda(P_\epsilon) \in O(\epsilon^\beta)$ where β is a positive real, then

$$\kappa(\mathcal{C}) - \kappa(P_\epsilon) \in n_\epsilon O(\epsilon^\beta).$$

Proof. Let $k > 0$ and $\epsilon_0 > 0$ such that the turn of \mathcal{C} is k -strongly Lipschitz on each connected component of $\mathcal{C}_{\sim}^{P_\epsilon}$ where $0 < \epsilon < \epsilon_0$. Thanks to Proposition 2,

$$\kappa(\mathcal{C}) - \kappa(P_\epsilon) \leq \kappa(\mathcal{C}_{\sim}^{P_\epsilon}).$$

For any $\epsilon \in (0, \epsilon_0)$, the components of $\mathcal{C}_{\sim}^{P_\epsilon}$ are smooth for the turn \mathcal{C} is k -Lipschitz on $\mathcal{C}_{\sim}^{P_\epsilon}$ (Proposition 8). Thereby,

$$\kappa(\mathcal{C}_{\sim}^{P_\epsilon}) = \sum \left\{ \kappa\left(C_{a_{i-1}}^{a_i}\right) \mid C_{a_{i-1}}^{a_i} \subset \mathcal{C}_{\sim}^{P_\epsilon} \right\}.$$

where a_0, \dots, a_m are the vertices of P_ϵ . Since κ is k -strongly Lipschitz on $\mathcal{C}_{\sim}^{P_\epsilon}$, we derive

$$\kappa(\mathcal{C}_{\sim}^{P_\epsilon}) \leq k \sum \left\{ \|a_{i-1} - a_i\| \mid C_{a_{i-1}}^{a_i} \subset \mathcal{C}_{\sim}^{P_\epsilon} \right\} \leq k n_\epsilon \lambda(P_\epsilon).$$

If, furthermore, $\lambda(P_\epsilon) \in O(\epsilon^\beta)$, we get

$$\kappa(\mathcal{C}_{\sim}^{P_\epsilon}) \in n_\epsilon O(\epsilon^\beta). \quad \square$$

4.2.2 Convex arcs

The case of convex curves is very particular. Indeed, we saw in Section 2.3.1 that, for estimating the turn, only the ends of the curve matter. Then, the number of edges of the polygonal approximations need not increase when $\epsilon \rightarrow 0$. On the contrary, the smaller the number of edges, the smaller the quantization error, especially since a polygonal approximation of a convex curve need not be convex.

Proposition 5. *Let \mathcal{C} be a plane convex curve whose turn is strongly Lipschitz at its ends. Let $(Q_\epsilon)_{\epsilon>0} = ([a_0^\epsilon, a_1^\epsilon, a_2^\epsilon, a_3^\epsilon])_{\epsilon>0}$ be a family of linear ϵ -approximations of \mathcal{C} . If $\|a_0^\epsilon - a_1^\epsilon\|$ and $\|a_2^\epsilon - a_3^\epsilon\|$ are both in $\Omega(\epsilon^\alpha) \cap O(\epsilon^\beta)$ with $0 < \beta \leq \alpha < 1$, then*

$$\kappa(\mathcal{C}) - \kappa(Q_\epsilon) = O(\epsilon^\beta) + O(\epsilon^{1-\alpha}).$$

In particular, the best convergence speed is obtain taking $\alpha = \beta = \frac{1}{2}$.

Proof. Let $(P_\epsilon)_{\epsilon>0}$ be a family of linear interpolations of \mathcal{C} associated to the ϵ -approximations Q_ϵ . On the one hand, thanks to Proposition 3 and the hypotheses, we have

$$|\kappa(P_\epsilon) - \kappa(Q_\epsilon)| = O(\epsilon^{1-\alpha}). \quad (18)$$

On the other hand, thanks to Lemma 8, we know that

$$|\kappa(\mathcal{C}) - \kappa(P_\epsilon)| = O(\epsilon^\beta). \quad (19)$$

By adding both previous equations, we get the result. \square

4.2.3 Finite concatenations of convex arcs

When the curve is supposed to be a finite concatenations of convex arcs, but the locations and perhaps even the number of convex subarcs are unknown, we cannot use Proposition 6 to upper bound the turn estimation error. Another upper bound should be use instead. Indeed, Proposition 6 assumes an adequate linear approximation which cannot be found in the general case of a convex arc finite concatenation.

Proposition 6. *Let \mathcal{C} be a curve in E^n which is a finite concatenation of convex plane arcs and whose turn is strongly Lipschitz at the ends of \mathcal{C} and at each gluing point. Let $(Q_\epsilon)_{\epsilon>0}$ be a family of linear ϵ -approximations of \mathcal{C} . If the minimum and maximum edge lengths $\mu(Q_\epsilon)$ and $\lambda(Q_\epsilon)$ are such that $\mu(Q_\epsilon) \in \Omega(\epsilon^\alpha)$ and $\lambda(Q_\epsilon) \in O(\epsilon^\beta)$ with $0 < \beta \leq \alpha < 1/2$, then*

$$\kappa(\mathcal{C}) - \kappa(Q_\epsilon) = O(\epsilon^\beta) + O(\epsilon^{1-2\alpha}).$$

In particular, the best convergence speed is obtain taking $\alpha = \beta = \frac{1}{3}$.

Proof. Let $(P_\epsilon)_{\epsilon>0}$ be a family of linear interpolations of \mathcal{C} associated to $(Q_\epsilon)_{\epsilon>0}$. Then, $\lambda(P_\epsilon) \in O(\epsilon^\beta)$. Since \mathcal{C} is a finite concatenation of convex arcs —say m convex arcs ($m \geq 2$), its turn is finite. Indeed, the turn of each convex arc is at most 2π [9, Theo. 5.1.5, p. 125] and a convex arc having right and left tangent everywhere, the

turn at each point of concatenation is well-defined and at most 2π . Thereby, the turn of C is at most $(4m - 2)\pi$. The turn of C being finite, we derive from Lemma 8 that

$$\kappa(C) - \kappa(P_\epsilon) = n_\epsilon O(\epsilon^\beta),$$

where n_ϵ is the number of edges of P_ϵ that interpolate an arc of $\mathcal{C}_{\sim}^{P_\epsilon}$. By definition of $\mathcal{C}_{\sim}^{P_\epsilon}$ (Formula (8)), the value of n_ϵ is at most three times the value of m (see Figure 5). Since m is a constant, we derive an upper bound of the discretization error convergence order:

$$\kappa(C) - \kappa(P_\epsilon) = O(\epsilon^\beta). \quad (20)$$

We bound the quantization error thanks to Lemma 7 and the hypothesis $\mu(Q_\epsilon) \in O(\epsilon^\alpha)$:

$$\kappa(P_\epsilon) - \kappa(Q_\epsilon) = O(\epsilon^{1-2\alpha}). \quad (21)$$

Eventually, adding Equations (20) and (21), we get the result:

$$\kappa(C) - \kappa(Q_\epsilon) = O(\epsilon^\beta) + O(\epsilon^{1-2\alpha}). \quad \square$$

4.2.4 Example of infinite concatenation of convex arcs

In this section, we do not give a general result. We just give an example with a curve that contain an accumulation point for the changes of convexity.

We consider the real-valued function f defined on $[0, 1]$ by $f(0) = 0$ and, for any $x \neq 0$, by $f(x) = x^\eta \sin(\pi/x)$ where $\eta \geq 4$ so that f is of class $C^{1,1}$. We denote by \mathcal{C} , the graph of f . Since f is of class $C^{1,1}$, the turn of \mathcal{C} is strongly Lipschitz (Proposition 8) and finite [9, Theorem 5.4.3].

Let $\epsilon > 0$ and P_ϵ be a linear interpolation of \mathcal{C} whose maximum edge length, $\lambda(P_\epsilon)$, is in $O(\epsilon^\beta)$ where $\beta \in (0, 1)$. In order to upper bound $\kappa(\mathcal{C}_{\sim}^{P_\epsilon})$, we do not use directly Lemma 8. Instead, we firstly split the curve \mathcal{C} into two parts, $\mathcal{C}_o^{c_\epsilon}$ and $\mathcal{C}_{c_\epsilon}^e$ where o and e are the left and right ends of \mathcal{C} and c_ϵ is an inner point of the curve chosen among the vertices of P_ϵ (see Figure 9). We denote by x_ϵ the abscissa of the point c_ϵ and by P_ϵ^l and P_ϵ^r the parts of P_ϵ that interpolate respectively $\mathcal{C}_o^{c_\epsilon}$ and $\mathcal{C}_{c_\epsilon}^e$.

On the left arc $\mathcal{C}_o^{c_\epsilon}$, there are so much inflection points that, likely, the subarcs of \mathcal{C} interpolated by the edges of P_ϵ contain at least one inflection point. Hence, we just use the Lipschitz hypothesis:

$$\kappa(\mathcal{C}_o^{c_\epsilon}) - \kappa(P_\epsilon^l) \leq \kappa(\mathcal{C}_o^{c_\epsilon}) \leq k\sqrt{2}x_\epsilon,$$

where k is the Lipschitz parameter. The factor $\sqrt{2}$ stems from the inequality $\|c_\epsilon - o\| \leq \sqrt{2}x_\epsilon$. Thereby, we have:

$$\kappa(\mathcal{C}_o^{c_\epsilon}) - \kappa(P_\epsilon^l) = O(x_\epsilon). \quad (22)$$

In the right part, there is a finite number of convexity changes and for sufficiently small ϵ there are only a few subarcs of \mathcal{C} interpolated by the edges of P_ϵ that contain an inflection point. Then, we apply Lemma 8:

$$\kappa(\mathcal{C}_{c_\epsilon}^e) - \kappa(P_\epsilon^r) \leq k n_\epsilon \lambda(P_\epsilon),$$

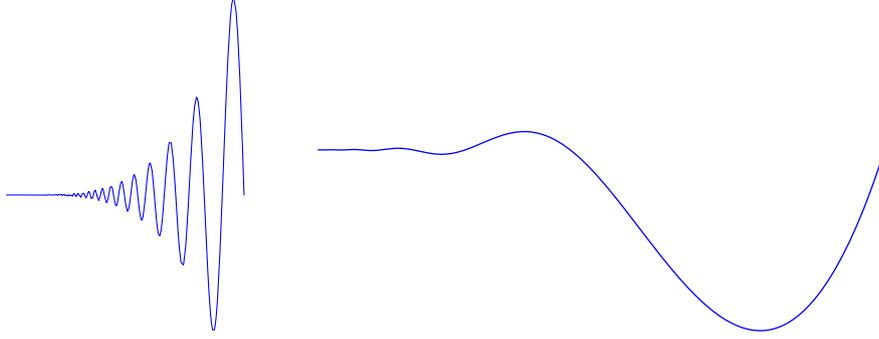


Fig. 9: The graph \mathcal{C} of $f(x) = x^4 \sin(\pi/x)$, $x \in [0, 1]$. Left: the left part of \mathcal{C} , $\mathcal{C}_o^{c_\epsilon}$ where the abscissa of c_ϵ is 0.1 and the scale of the y -axis is 1000 times that of the x -axis. Right: the right part of \mathcal{C} , $\mathcal{C}_{c_\epsilon}^e$. The x -axis and the y -axis have the same scale.

where n_ϵ denotes the number of edges of P_ϵ that interpolate an arc of $\mathcal{C}_{\tilde{c}_\epsilon}^{P_\epsilon} \cap \mathcal{C}_{c_\epsilon}^e$. By the definition of $\mathcal{C}_{\tilde{c}_\epsilon}^{P_\epsilon}$, the value of n_ϵ is at most 3 times the number of inflection points on $\mathcal{C}_{c_\epsilon}^e$. Moreover, to within one unit, the number of inflection points in the arc $\mathcal{C}_{c_\epsilon}^e$ is the number of intersections between the arc and the abscissa axis, that is about $1/x_\epsilon$. Then, using the hypothesis $\lambda(P_\epsilon) \in O(\epsilon^\beta)$,

$$\kappa(\mathcal{C}_{c_\epsilon}^e) - \kappa(P_\epsilon^r) \in O(x_\epsilon^{-1} \epsilon^\beta). \quad (23)$$

Since \mathcal{C} is smooth, we derive from Equations 22 and (23) that

$$\kappa(\mathcal{C}) - \kappa(P_\epsilon) = O(x_\epsilon) + O(x_\epsilon^{-1} \epsilon^\beta). \quad (24)$$

Equation (24) is valid for any $x_\epsilon \in (0, 1)$. The best choice is to take $x_\epsilon \approx x_\epsilon^{-1} \epsilon^\beta$ which gives $x_\epsilon \in \Theta(\epsilon^{\beta/2})$ —which is compatible with the fact that c_ϵ is chosen among the vertices of P_ϵ — Then,

$$\kappa(\mathcal{C}) - \kappa(P_\epsilon) = O(\epsilon^{\beta/2}). \quad (25)$$

Eventually, adding the quantification error, we conclude that, given a family $(Q_\epsilon)_{\epsilon>0}$ of linear ϵ -approximations of \mathcal{C} whose minimum and maximum edge lengths $\mu(Q_\epsilon)$ and $\lambda(Q_\epsilon)$ are such that $\mu(Q_\epsilon) \in \Omega(\epsilon^\alpha)$ and $\lambda(Q_\epsilon) \in O(\epsilon^\beta)$ with $0 < \beta \leq \alpha < 1/2$, we have

$$\kappa(\mathcal{C}) - \kappa(Q_\epsilon) = O(\epsilon^{\beta/2}) + O(\epsilon^{1-2\alpha}).$$

In particular, the best convergence speed is $O(\epsilon^{1/5})$ obtained taking $\alpha = \beta = \frac{2}{5}$.

5 Tests

Let us present a set of turn estimations for three plane curves. The first curve is an arc of focal circular cubic whose Cartesian equation is $(x-2)(x^2+y^2)+4y=0$ with $x \leq 2, y \leq 2+\sqrt{3}$. The curve has two inflection points. The second curve is an arc of spiral whose polar equation is $\rho = \theta/10$, where $\theta \in [0, 4\pi]$. It is not convex though it

has no inflection point. The last curve belongs to the family studied in Section 4.2.4. For the experiment, we took the curve $x \mapsto x^4 \sin(\pi/x)$ with $x \in [0, 1]$. It is of class $C^{1,1}$, thus it is of finite turn, and has infinitely many inflection points. The three curves are depicted in Figure 10.

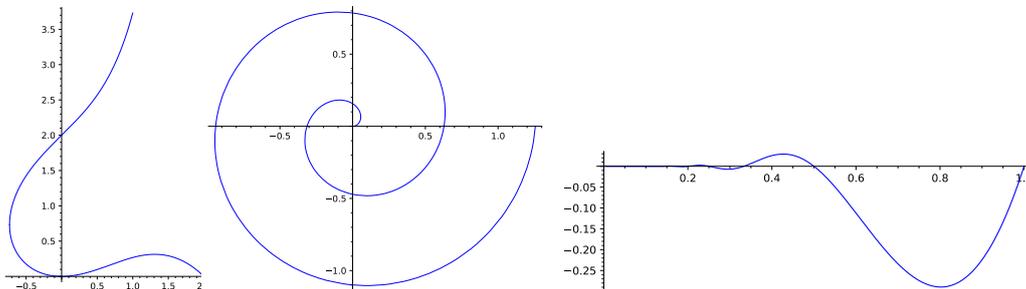


Fig. 10: The three curves on which the naive turn estimators are tested. Their turns are numerically calculated by integrating their differential curvatures. Then, these turns are estimated by means of ϵ -linear approximations with ϵ decreasing from 0.05 to about $5e^{-9}$ and with edge sizes belonging to $\Theta(\epsilon^\alpha)$ where $\alpha \in \{\frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}\}$.

For each curve, we compute 4 sequences of ϵ -approximations. In a given sequence, the edge lengths of the polygonal approximations belong to $\Theta(\epsilon^\alpha)$ where α is one of $\{\frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}\}$. The values of ϵ are $2^{-k}/10$, $1 \leq k \leq 24$. Eventually, for each ϵ -approximation, we calculate

1. the discretization error, that is the difference between the turn of the curve and the turn of a curve interpolation associated to the given linear approximation;
2. the absolute quantization error, that is the absolute difference between the turn of the ϵ -approximation and the turn of its associated curve interpolation;
3. the global error, that is the difference between the turn of the curve and the turn of the ϵ -approximation.

The results are gathered in Table 2.

Let us now examine the test results against the error upper bounds found in Section 4.2. The observed convergence orders are given in Table 3. In our test, we use ϵ -linear approximations with edge lengths in $\Theta(\epsilon^\alpha)$. Hence, we have to look at formulas of Sections 4.2.3 and 4.2.4 taking $\beta = \alpha$.

- Regarding the discretization error, the calculated convergence order lower bound is α for the cubic curve and the spiral (Formula 20) and $\alpha/2$ for the sinusoidal curve (Formula 25). The observed results fit these lower bounds. Note that the scatter plots for the third curve are slightly concave indicating that the coefficient in front of $\epsilon^{\alpha/2}$ increases (towards some finite limit) as ϵ goes to zero. We could not explain this behavior.
- Regarding the quantization error, data points are more scattered around the linear model. Furthermore, the convergence speed is generally much better than our lower bound, which is $1 - 2\alpha$, except for $\alpha = \frac{1}{2}$. Actually, when we evaluated the upper

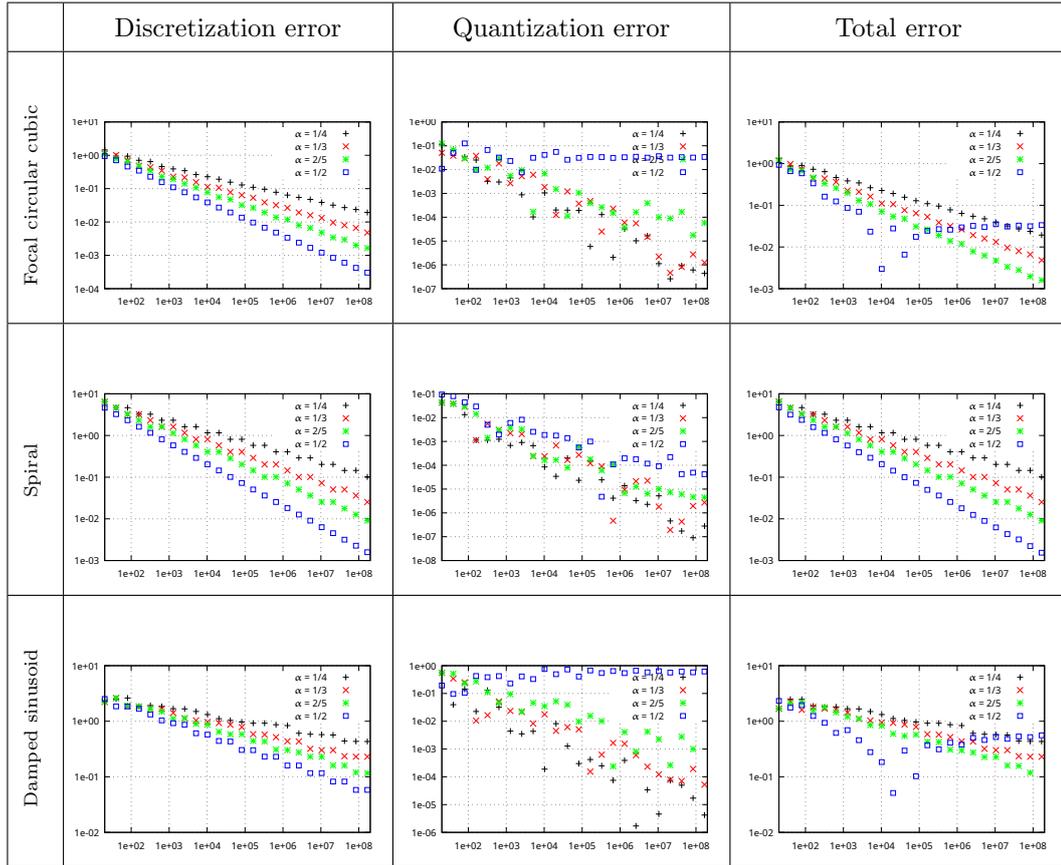


Table 2: Errors in naive turn estimations in function of the inverse of the precision parameter ϵ . The parameter α controls the edge sizes of the polygonal lines used by the estimator: the minimum and maximum edge lengths belong to $\Theta(\epsilon^\alpha)$.

bound in Section 3, we placed ourselves in the worst case scenario in which each turn at a linear approximation vertex increases the difference between the curve turn and the interpolation turn. It is likely that it is rarely the case (though it can occur, for instance if a straight segment is approximated by a staircase function).

- Finally, at the tested values of the parameter ϵ , the quantization error is several order of magnitude smaller than the discretization error. This results in a total error which is generally better with smaller linear approximation edge lengths that is, larger values of α . Nevertheless, we can also observe dramatic changes of behavior of the yellow plots in the third column of Table 2. Indeed, for the first and third curves the quantization error is almost constant when $\alpha = \frac{1}{2}$. Therefore, at some point, the quantization error overrides the discretization error and the estimator fails to converge. Observe that in these special cases, both errors have opposite signs when their magnitudes coincide, thus temporarily collapsing the total error.

	Discretization error				Quantization error			
	$\alpha = \frac{1}{4}$	$\alpha = \frac{1}{3}$	$\alpha = \frac{2}{5}$	$\alpha = \frac{1}{2}$	$\alpha = \frac{1}{4}$	$\alpha = \frac{1}{3}$	$\alpha = \frac{2}{5}$	$\alpha = \frac{1}{2}$
Cubic	0.26	0.34	0.41	0.51	0.79	0.73	0.48	0.01
Spiral	0.25	0.34	0.40	0.50	0.75	0.71	0.61	0.48
Simusoid	0.12	0.16	0.20	0.24	0.67	0.54	0.44	0.08

Table 3: Absolute slopes of the linear models in the log-log plots.

6 Conclusion

In this paper, we propose a convergent turn estimator for simple curves. This so called *naive estimator* (because of its simplicity) rely on polygonal lines ϵ -closed to linear interpolations of the curve. The convergence is ensured provided the edge lengths of the ϵ -approximations tend sufficiently slowly toward 0 as $\epsilon \rightarrow 0$. More specifically the edge lengths have to be in $\Omega(\epsilon^{1/2})$. Such ϵ -approximations may be obtained from the digitization process, in particular through Gauss digitizations on cubic grids. In the latter case, we would say that the naive turn estimator is “multigrid convergent”. Nevertheless, it should be paid attention to the fact that we do not address the problem of finding ϵ -approximations from digitizations. This issue is not easy and often overlooked. Indeed the digitization process yields an unordered finite set of digital points. Building a correct digital chain from this set, that is a digital polygonal line whose vertices are closed to a chain of the digitized shape boundary may prove impossible without assumptions on the digitized shape (see Figure 11). This was a motivation for the definitions of par-regularity and turn local boundedness (see Table 1 for definitions and references). The curve simpleness assumption of our result is therefore somewhat misleading in the sense that it does not take into account the hypotheses necessary to obtain those ϵ -approximations.

With supplementary assumptions, our paper gives lower bounds on the convergence speeds of the naive estimators. Regarding convex plane curves, assuming their ends are strongly Lipschitz, that is of class $C^{1,1}$, ϵ -approximations with edges in $\Theta(\epsilon^{1/2})$ ensure a convergence speed of order 1/2. With curves that are assumed to be finite concatenations of convex arcs, the edges of the polygonal lines have to be longer, in $\Theta(\epsilon^{1/3})$, to get the best speed of convergence which is $\epsilon^{1/3}$. With sinusoidal curves $x \mapsto x^n \sin(\pi/x)$ having infinitely many inflection points, the best convergence speed goes down to $O(\epsilon^{1/5})$ using edges in $\Theta(\epsilon^{2/5})$.

As said in the article introduction, in the future we would like to test the differential curvature estimation by mean of total curvature and length estimations. It is remarkable that the results presented in [17] for differential curvature estimation of C^3 boundaries having positive reach coincide with our own results about total curvature estimation on finite concatenation of convex arcs. Indeed, in both cases, the better convergence speed lower bound is $O(\epsilon^{1/3})$, obtained taking an estimator parameter itself in $O(\epsilon^{1/3})$ (the parameter is the radius of the estimation window for the differential curvature and the edge length for the total curvature). We will also have to extend the curve decomposition in plane convex arcs (here written $\mathcal{C}_{\sim}^{P_\epsilon}$) to nD curve

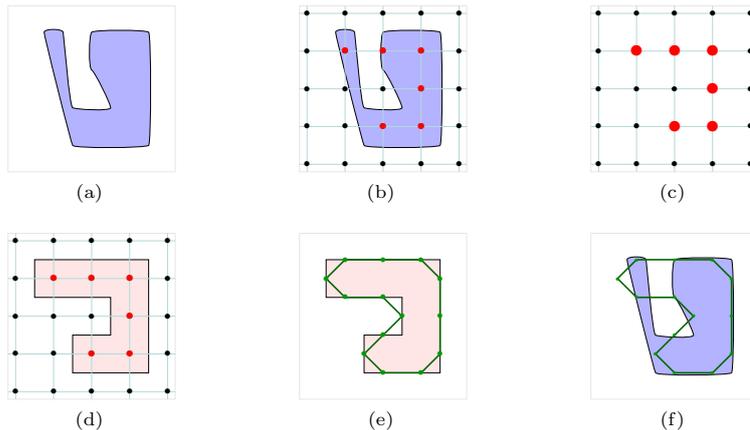


Fig. 11: A wrong polygonal approximation from a Gauss digitization. (a) Original shape. (b) Gauss digitization process on a (2ϵ) wide grid. (c) Digitized shape (in red). (d) Cubic reconstruction. (e-f) Wrong ϵ -linear approximation of the shape boundary: there is no linear interpolation ϵ -closed to the green polygon.

chunks on which the alignment configuration cannot occur. Besides, some more theoretical questions about the turn Lipschitz continuity are worth to be explored. It is easy to build a spiral curve of Lipschitz (with respect to the geodesic distance) and infinite turn which is either unbounded or non closed. But, can we find a compact curve of Lipschitz and infinite turn? If not, is the Lipschitz continuity equivalent to the strongly Lipschitz continuity in the case of compact curves? Equivalently, does the Lipschitz continuity implies the local boundedness of the turn (though neither the parameter of the former property nor the parameter of the latter one can be inferred from the other as shown by Figure A1)? The link with the reach in nD , $n \geq 3$, as also to be further investigated.

Acknowledgments. We would like to warmly thank É. Le Quentrec and É. Baudrier for their pertinent remarks on an early version of this paper.

Appendix A Curves with strongly Lipschitz turns

The notion of strong Lipschitz turn proposed in this article—that is, Lipschitz continuity of the turn with respect to the Euclidean distance—originates from E. Le Quentrec’s thesis. Two properties of the turn were introduced into his article [31] in order to then handle length estimation [10], *local boundedness* and *Lipschitz continuity* (with respect to the arc length).

- A curve \mathcal{C} in E^n has a locally bounded turn if there exists a positive real δ such that for any pair of points $a, b \in \mathcal{C}$ at Euclidean distance from each other less than δ , the turn of the arc of \mathcal{C} between a and b is at most $\pi/2$.

- A curve \mathcal{C} has a Lipschitz turn if there exists a positive real k such that, for any arc of \mathcal{C} , the ratio between the turn and the length of the arc is at most k . When the curve is the boundary of a compact domain of E^2 , the local boundedness can be seen as a control over the thickness of this domain whereas the Lipschitz continuity controls the local curvature of the boundary (see Figure A1).

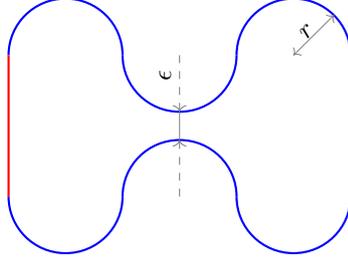


Fig. A1: [32, Figure 4] This simple closed curve $F_{r,\epsilon}$ is composed of six half-circles with radius r (in blue) and two line segments with length $2r + \epsilon$ (in red) where r and ϵ are any positive reals. The turn of the curve $F_{r,\epsilon}$ is $(1/r)$ -Lipschitz and δ -locally bounded where $\delta = \min(\epsilon, \sqrt{2}r)$.

In this appendix, we show that the strong Lipschitz property is equivalent to the set of two turn properties introduced by Le Quentrec *et al.* .

Proposition 7. *A curve in E^n has a strongly Lipschitz turn if and only if this curve has a Lipschitz and locally bounded turn.*

Proof. In this proof, we denote the length of a curve X by $\mathcal{L}(X)$.

Let \mathcal{C} be a curve in E^n having a k -strongly Lipschitz turn: $\kappa(C_a^b) \leq k\|a - b\|$ for any $a, b \in C$. On the one hand, since $\|a - b\| \leq \mathcal{L}(C_a^b)$ for any points $a, b \in C$, we directly have that the turn of C is Lipschitz. On the other hand, for any points $a, b \in C$ such that $\|a - b\| < \pi/(2k)$, we have $\kappa(C_a^b) \leq \pi/2$. Thus, the turn of C is locally bounded.

Conversely, assume that the turn of C is k -Lipschitz and δ -locally bounded. Since the turn of C is locally bounded, the length of C is finite [31]. Then, given two points $a, b \in C$ such that $\|a - b\| \geq \delta$, one has

$$\kappa(C_a^b) \leq k\mathcal{L}(C_a^b) \leq k\frac{\mathcal{L}(C)}{\delta}\|a - b\|. \quad (\text{A1})$$

If $\|a - b\| < \delta$, by the locally boundedness assumption, the turn of C_a^b is at most $\pi/2$. Then according to [9, Theo. 5.8.1, p. 151], $\mathcal{L}(C_a^b) \cos(\pi/4) \leq \|a - b\|$. Hence, because the turn of C is k -Lipschitz,

$$\kappa(C_a^b) \leq k\mathcal{L}(C_a^b) \leq k\sqrt{2}\|a - b\|. \quad (\text{A2})$$

Eventually, putting $k' = k \max(\mathcal{L}(\mathcal{C})/\delta, \sqrt{2})$, we derive from Equations (A1) and (A2) that

$$\kappa(\mathcal{C}_a^b) \leq k' \|a - b\|.$$

We have proved that the turn of \mathcal{C} is k' -strongly Lipschitz. \square

Appendix B The class of curves of strongly Lipschitz turn is the regularity class $\mathbf{C}^{1,1}$

Recall that a function $f: \mathbb{R} \rightarrow E^n$ belongs to the class $\mathbf{C}^{1,1}$ if f is \mathbf{C}^1 and its derivative is Lipschitz continuous on any compact interval. We show that the images of these functions are the curves whose turn is Lipschitz with respect to the Euclidean distance.

Proposition 8. *A curve in E^n has a strongly Lipschitz turn if and only if it admits a $\mathbf{C}^{1,1}$ parametrization.*

Proof. Let \mathcal{C} be a curve in E^n whose turn is strongly Lipschitz. From Proposition 7, we derive that \mathcal{C} has a locally bounded turn. Then, according to [31, Corollary 4], the curve \mathcal{C} has a finite turn and is rectifiable. Thanks to [9, Theorems 2.1.4, 3.3.3, 5.1.2], we get that \mathcal{C} has an arc length parametrisation, $\gamma: [0, L] \rightarrow E^n$ (where L is the length of \mathcal{C}) which has everywhere left and right derivative. Furthermore, by [9, Theorems 5.1.3], given a chain (x, y, z) of \mathcal{C} ,

$$\kappa(\mathcal{C}_x^z) = \kappa(\mathcal{C}_x^y) + \kappa(\mathcal{C}_y^z) + \angle(\mathbf{t}_l, \mathbf{t}_r)$$

where \mathbf{t}_l and \mathbf{t}_r are the left and right unit tangent vectors to \mathcal{C} at y . Then, the strongly Lipschitz assumption implies that $\angle(\mathbf{t}_l, \mathbf{t}_r) = 0$. In other words, γ is differentiable. Let $s_1, s_2 \in [0, L]$ and $\mathbf{t}_1 = \gamma'(s_1)$, $\mathbf{t}_2 = \gamma'(s_2)$. Putting $a_1 = \gamma(s_1)$ and $a_2 = \gamma(s_2)$, we have,

$$\|\mathbf{t}_1 - \mathbf{t}_2\| \leq \angle(\mathbf{t}_1, \mathbf{t}_2) \leq \angle(\mathbf{t}_1, a_1 \vec{a}_2) + \angle(a_1 \vec{a}_2, \mathbf{t}_2).$$

Thanks to Lemma 2, we derive that

$$\|\mathbf{t}_1 - \mathbf{t}_2\| \leq \angle(\mathbf{t}_1, \mathbf{t}_2) \leq \kappa(\mathcal{C}_{a_1}^{a_2}),$$

and, since κ is Lipschitz with respect to the Euclidean distance —thus a fortiori with respect to the geodesic distance,

$$\|\mathbf{t}_1 - \mathbf{t}_2\| \leq k |s_1 - s_2|.$$

Therefore, γ is $\mathbf{C}^{1,1}$.

Conversely, we now assume that the curve \mathcal{C} is of class $\mathbf{C}^{1,1}$. Then, \mathcal{C} is rectifiable. Let $\gamma: [0, L] \rightarrow \mathcal{C}$ be a parametrisation by arc length of \mathcal{C} . Let s_1, s_2 be two reals such that $0 \leq s_1 < s_2 \leq L$ and set $a_1 = \gamma(s_1)$ and $a_2 = \gamma(s_2)$. By [9, Theorem 5.2.2], the turn of $\mathcal{C}_{a_1}^{a_2}$ is equal to the length of the tangent indicatrix of $\mathcal{C}_{s_1}^{s_2}$, that is, to the total variation of the derivative γ' between s_1 and s_2 . Let k be the Lipschitz parameter of γ' and γ_0 be the parametrisation by arc length of a circle with radius $r = 1/k$. Since

γ' is k -Lipschitz, the total variation of γ' between s_1 and s_2 is upper bounded by $|s_1 - s_2|/r$, which is the turn of the arc $\gamma_0([s_1, s_2])$. Hence,

$$\kappa(\mathcal{C}_{a_1}^{a_2}) \leq \kappa(\gamma_0([s_1, s_2])). \quad (\text{B3})$$

This last result being valid for any interval $[s_1, s_2] \subset [0, L]$, the hypotheses of Schur's Comparison Theorem [38] are fulfilled. Thus,

$$\|\gamma_0(s_1) - \gamma_0(s_2)\| \leq \|\gamma(s_1) - \gamma(s_2)\|. \quad (\text{B4})$$

Let us now assume that the turn of $\gamma_0([s_1, s_2])$ is at most π , that is $|s_1 - s_2| \leq r\pi$. Then, this turn is linked to the chord of $\gamma_0([s_1, s_2])$ by

$$\|\gamma_0(s_1) - \gamma_0(s_2)\| = 2r \sin\left(\frac{\kappa(\gamma_0([s_1, s_2]))}{2}\right).$$

Hence, for the sine function is convave on $[0, \frac{\pi}{2}]$,

$$\|\gamma_0(s_1) - \gamma_0(s_2)\| \geq \frac{2r}{\pi} \kappa(\gamma_0([s_1, s_2])). \quad (\text{B5})$$

Putting Equations (B3), (B4), (B5) together, we obtain

$$\forall s_1, s_2 \in [0, L], |s_1 - s_2| \leq r\pi \implies \kappa\left(\mathcal{C}_{\gamma(s_1)}^{\gamma(s_2)}\right) \leq \frac{k\pi}{2} \|\gamma(s_1) - \gamma(s_2)\|. \quad (\text{B6})$$

We now examine the case $|s_1 - s_2| > r\pi$. For any $s \in [0, L]$, we set $I_s = [0, L] \setminus (s - r\pi, s + r\pi)$, $\mathcal{C}_s = \gamma(I_s)$ and $\delta(s) = d(\gamma(s), \mathcal{C}_s)$ where $d(\cdot, \cdot)$ denotes the Euclidean distance. Since \mathcal{C}_s is a compact set and \mathcal{C} is simple, $\delta(s)$ is a positive real for any s . Let us show that δ has a positive infimum on $[0, L]$. Since the curve \mathcal{C} need not be close, the map $\delta: [0, L] \rightarrow [0, +\infty)$ may have two points of discontinuity. Then, we have to split the proof into three parts.

Firstly, let us prove that the map δ is continuous on $[r\pi, L - r\pi]$. For any $s \in [r\pi, L - r\pi]$, I_s is the union of two non empty intervals, $[0, s - r\pi]$ and $[s + r\pi, L]$. We have, for any s, s_0 in $[r\pi, L - r\pi]$,

$$d(\gamma(s), \mathcal{C}_s) \leq d(\gamma(s), \gamma(s_0)) + d(\gamma(s_0), \mathcal{C}_{s_0}) + d_H(\mathcal{C}_{s_0}, \mathcal{C}_s),$$

where d_H denotes the Hausdorff distance. Hence,

$$|\delta(s) - \delta(s_0)| \leq d(\gamma(s), \gamma(s_0)) + d_H(\mathcal{C}_{s_0}, \mathcal{C}_s).$$

Since \mathcal{C}_{s_0} and \mathcal{C}_s are the images by γ of the intervals $[s_0 - r\pi, s_0 + r\pi]$ and $[s - r\pi, s + r\pi]$ and γ is uniformly continuous on $[0, L]$, sufficiently small differences $s - s_0$ yield $d(\gamma(s), \gamma(s_0)) < \epsilon$ and $d_H(\mathcal{C}_{s_0}, \mathcal{C}_s) < \epsilon$ for any given ϵ . Thereby, we have proved that δ is continuous on $[r\pi, L - r\pi]$. Let $s_0 \in [r\pi, L - r\pi]$ such that δ_0 be the minimum

value of δ on the compact set $[r\pi, L - r\pi]$. Then, $\delta_0 = d(\gamma(s_0), \gamma(t_0))$ for some t_0 in \mathcal{C}_{s_0} . Because \mathcal{C} is simple, we have $\delta_0 > 0$.

To obtain a similar result on the left part $[0, r\pi]$ of the interval $[0, L]$, we need to slightly modify the definition of I_s . Now, we put $I_s = [0, L] \setminus [(s - r\pi, s + r\pi)$. Hence, $I_s = [s + r\pi, L]$ for any $s \in [0, r\pi]$. This definition coincide with the former one except when $s = r\pi$ where we would have had $I_{r\pi} = \{0\} \cup [2r\pi, L]$. Notably, this implies that both meanings of δ coincide on $[0, r\pi]$. The rest of the reasoning is identical as before: with this modification, δ_s is continuous on $[0, r\pi]$ and we find a positive real δ_1 such that $\delta \geq \delta_1$ on $[0, r\pi]$. In particular, $\delta \geq \delta_1$ on $[0, r\pi)$ with the first acceptance of δ . Alike, we find positive real δ_2 such that $\delta \geq \delta_2$ on $(L - r\pi, L]$.

Eventually, we have found a positive real $\delta_3 = \min(\delta_0, \delta_1, \delta_2)$ such that $d(\gamma(s), \gamma([0, L] \setminus (s - r\pi, s + r\pi))) \geq \delta_3$ for any $s \in [0, L]$. We end the proof by putting $k_2 = \kappa(\mathcal{C})/\delta_3$ which is finite since \mathcal{C} is $C^{1,1}$ ([9, Theorem 5.4.3]). Let s_1, s_2 in $[0, L]$ such that $|s_1 - s_2| > r\pi$. We have

$$\kappa \left(\mathcal{C}_{\gamma(s_1)}^{\gamma(s_2)} \right) \leq \kappa(\mathcal{C}) \leq \frac{\kappa(\mathcal{C})}{\delta_3} \delta(s_1) \leq \frac{\kappa(\mathcal{C})}{\delta_3} \|\gamma(s_1) - \gamma(s_2)\|. \quad (\text{B7})$$

In view of Formulas (B6) and (B7), we conclude that the turn of \mathcal{C} is strongly Lipschitz with parameter $\min(k\pi/2, \kappa(\mathcal{C})/\delta_3)$. \square

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