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# On certain partial sums involving squares of Hecke eigenvalues

Winfried Kohnen and Florian Luca

**Abstract.** Let  $a(n)$  be the  $n$ th Fourier coefficient of a cuspidal Hecke eigenform of even integral weight  $k \geq 2$  and trivial character that is a normalized new form for some level  $N$ . We show that the partial sums

$$H_n = \sum_{m=1}^n a(m)^2/m^k$$

are not integral for  $n \geq n_0$ .

**Keywords.** Fourier coefficients of automorphic forms.

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## 1. Introduction and Results

Let

$$h_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad (n \in \mathbf{N})$$

be the  $n$ -th harmonic number. Harmonic numbers have been studied since antiquity and are important in various branches of number theory. They also have interesting arithmetic properties. For example, a result of Theisinger [The1915] asserts that  $h_n$  for  $n \geq 2$  never is an integer. Except for the use of “Bertrand’s postulate” (cf. below) the proof is completely elementary.

If one seeks for a generalization in the context of modular forms one naturally is led to the partial sums formally attached to the Rankin-Selberg  $L$ -series at  $s = k$  where  $k$  is the weight of the form in question. (For properties of the Rankin-Selberg zeta function we refer, e.g. to [Bu98, sect. 1.6].) Indeed, more precisely let

$$f(z) = \sum_{n \geq 1} a(n)e^{2\pi inz}$$

( $z \in \mathcal{H} =$  complex upper half-plane) be a cuspidal Hecke eigenform of even integral weight  $k \geq 2$  and trivial character that is a normalized new form of some level  $N$ . Recall that  $a(1) = 1$  and  $a(n)$  is the eigenvalue of the  $n$ -th Hecke operator  $T(n)$ . It is well-known that the  $a(n)$  are algebraic integers and generate a totally real number field  $K$ . Let

$$D_f(s) := \sum_{n \geq 1} a(n)^2 n^{-s} \quad (\Re(s) \gg 1)$$

be the Rankin-Selberg  $L$ -series attached to  $f$ . Recall that  $\sigma_0 = k$  is the abscissa of convergence of  $D_f(s)$ . We define

$$H_n = H_n(f) := \sum_{m=1}^n a(m)^2 m^{-k} \quad (n \in \mathbf{N}). \quad (1.1)$$

Then  $H_n \in K$  and the  $H_n$  could be viewed as (at least) formal generalizations of the numbers  $h_n$ .

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We prove:

**Theorem 1.** *Assume that  $n \geq 8$  is an integer such that  $a(p) \neq 0$  for some prime  $p \in (n/2, n]$ . Then  $H_n$  is not an algebraic integer.*

The proof of Theorem 1 works in a similar way as the proof in [The1915] with an additional use of Deligne's deep result

$$|a(p)| \leq 2p^{\frac{k-1}{2}}. \quad (1.2)$$

The assumption  $n \geq 8$  (which implies that  $p \geq 5$ ) comes from the fact that seemingly we do not have good information about the exact  $p$ -power dividing  $a(p)$  for  $p = 2, 3$ , as will be clear from the proof.

## 2. Proof of Theorem 1

Assume that  $n \geq 8$ . Let  $p$  be any prime number in the interval  $(n/2, n]$  which is guaranteed to exist by the hypothesis of Theorem 1. Further,  $p \geq 5$ . We denote by  $\mathcal{O} \subset K$  the ring of algebraic integers of  $K$ . Suppose that  $H_n \in \mathcal{O}$ .

Let  $C(n, p)$  be the  $n$ -th factorial, with the factor  $p$  omitted; i.e.:

$$C(n, p) := (p-1)! \cdot \prod_{\nu=1}^{n-p} (p+\nu).$$

We multiply both sides of (1.1) with  $C(n, p)^k$ . Our assumption implies that  $C(n, p)^k H_n \in \mathcal{O}$ . Hence, we infer that

$$C(n, p)^k \cdot \frac{a(p)^2}{p^k} \in \mathcal{O}$$

since the denominators of all other terms of the resulting sum on the right-hand side cancel. Therefore also

$$C(n, p)^{k/2} \cdot \frac{a(p)}{p^{k/2}} \in \mathcal{O}. \quad (2.3)$$

Now note that  $p$  does not divide any of the numbers  $p+1, p+2, \dots, n$ , because the next multiple of  $p$ , namely  $2p$  is larger than  $n$ .

By the unique factorization of integral ideals into products of prime ideals in the Dedekind domain  $\mathcal{O}$ , applied to the principal ideals in question and using that  $p$  does not divide  $C(n, p)$ , we deduce from (2.3) that  $p^{k/2} | a(p)$ ; i.e.,  $a(p) = p^{k/2} \alpha_p$  with  $\alpha_p \in \mathcal{O}$ .

From (1.2) we therefore find that

$$|\alpha_p| \leq \frac{2}{\sqrt{p}} < 1 \quad (2.4)$$

where for the last inequality in (2.4) we have used that  $p \geq 5$ .

Let  $\sigma$  be one of the finitely many embeddings of  $K$  over  $\mathbf{Q}$  into an algebraic closure of  $\mathbf{Q}$ . Then as is well-known

$$f^\sigma(z) := \sum_{n \geq 1} a(n)^\sigma e^{2\pi i n z} \quad (z \in \mathcal{H})$$

is a normalized cuspidal Hecke eigenform of weight  $k$ , trivial character and level  $N$ . We have  $a(p)^\sigma = p^{k/2} \alpha_p^\sigma$  and

$$|\alpha_p^\sigma| \leq \frac{2}{\sqrt{p}} < 1.$$

From the above we therefore find that the norm of  $\alpha_p$  satisfies

$$|N(\alpha_p)| = \left| \prod_{\sigma} \alpha_p^{\sigma} \right| < 1.$$

However,  $N(\alpha_p)$  is a rational integer. Hence, we should have that  $a(p) = 0$ , which contradicts our assumption. This proves Theorem 1.

*Remarks.* i) For the Ramanujan  $\tau$ -function, which are the Fourier coefficients of the discriminant function  $\Delta$  of weight  $k = 12$  (see [Leh47]), Lehmer conjectured that  $a(n) \neq 0$  for all  $n$ . If this were true, then Theorem 1 together with a well-known theorem of Tchebyshev (also known under the name of Bertrand's postulate, see [HaWr59, p. 373, notes on chap. XXII]) would apply immediately to show that  $H_n$  is not algebraic integer for all  $n \geq 8$ .

ii) Theorems about the vanishing of  $a(p)$  for primes  $p$  were proved by Serre in his seminal paper [Ser81]. Theorem 15 on Page 174 and Proposition 18 on page 180 there, give the following:

**Serre's Theorem** *Assume that  $k \geq 2$  and  $x \geq 2$ .*

*i) In case  $f$  is non-CM (there is no complex quadratic field  $\mathbf{L}$  such that  $a(p) = 0$  for all primes  $p$  which are inert in  $\mathbf{L}$ ), we have that*

$$\#\{p \leq x : a(p) = 0\} = O\left(\frac{x}{(\log x)^{4/3}}\right).$$

*ii) In case  $f$  is CM, we have that*

$$\#\{p \leq x : a(p) = 0\} = \frac{x}{2 \log x} + O\left(\frac{x}{(\log x)^2}\right).$$

In either case, with  $n$  large and  $x = n$ , we conclude that the interval  $(x/2, x]$  contains

$$\begin{aligned} \pi(x) - \pi(x/2) - \#\{p \in (x/2, x] : a(p) = 0\} &\geq \frac{x}{\log x} - \frac{x/2}{\log(x/2)} - \left(\frac{x/2}{\log(x/2)} - \frac{x/4}{\log(x/4)}\right) \\ &+ O\left(\frac{x}{(\log x)^{4/3}}\right) \geq \frac{x}{4 \log x} + O\left(\frac{x}{(\log x)^{4/3}}\right) \end{aligned} \quad (2.5)$$

primes  $p$  such that  $a(p) \neq 0$ . In the above, we used the Prime Number Theorem with the error term

$$\pi(x) = \frac{x}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right) \text{ as } x \rightarrow \infty. \quad (2.6)$$

Since the last quantity indicated in the right-hand side of (2.5) is positive for large  $x$ , we conclude that for large  $n$ , the interval  $(n/2, n]$  contains a prime  $p$  with  $a(p) \neq 0$  and thanks to Theorem 1, we have that  $H_n$  is not an algebraic integer. Thus, we may write our conclusion as:

**Theorem 2.** *There exists  $n_0 = n_0(f)$  such that  $H_n$  is not an algebraic integer for  $n \geq n_0$ .*

### 3. Comments and a challenge problem

We would like to suggest the following challenge question. Let  $\ell, j$  be integers and put

$$H_{n,\ell,j} = \sum_{m=1}^n a(m)^\ell m^j.$$

Our paper discussed the case  $\ell = 2$  and  $j = -k$ . We ask under what conditions on  $\ell, j$  one may guarantee that  $H_{n,\ell,j}$  is an algebraic integer for only finitely many  $n$ . When  $j$  is negative and large in absolute value (say  $-j > \ell(k-1)/2$ ) the above argument applies. Indeed, the argument used in Theorem 1 creates an algebraic integer all whose conjugates have absolute values at most  $2/p^{-j-\ell(p-1)/2}$  and since the exponent in the denominator is positive, this bounds  $p$ ; hence,  $n$ . If  $\ell \geq 1$  and  $j \geq 0$ , then  $H_{n,\ell,j}$  is an algebraic integer for all  $n$ , and the case when  $\ell < 0$  is difficult since  $a(m)$  might be zero (although we may decide to sum only over the values of  $a(m)$  which are nonzero). The case  $\ell \geq 1$  and  $j \in [-\ell(k-1)/2, -1]$  is different since the argument of Theorem 1 creates an algebraic integer with large conjugates so it does not lead to a contradiction. As a toy example, we took  $a(m) = \tau(m)$ , the Ramanujan function of  $m$  and studied the borderline case  $\ell = 2, j = -1$ . Let  $n$  be large and assume  $s \geq 1$  is such that  $13^s$  is the largest power of 13 smaller than or equal to  $n$ . Then the only numbers  $m \leq n$  divisible by  $13^s$  are of the form  $\ell \times 13^s$  where  $\ell \in \{1, 2, \dots, K\}$  and  $K \leq 12$  is some integer. Separating those ones out and multiplying everything by  $A = 13^{s-1} \cdot \text{lcm}[p^t \leq n, p \neq 13]$ , and using the multiplicativity of  $\tau(m)^2$ , we get that

$$AH(n, 2, -1) = \frac{(A/13^{s-1})\tau(13^s)^2}{13} \sum_{\ell=1}^K \frac{\tau(\ell)^2}{\ell} \pmod{\mathbf{Z}}.$$

But  $13 \nmid \tau(13^s)$ ,  $13 \nmid A/13^{s-1}$  and a small calculation with Mathematica reveals that

$$\sum_{\ell=1}^K \frac{\tau(\ell)^2}{\ell}$$

is not a rational number whose numerator is divisible by 13 for any  $K \in \{1, \dots, 12\}$ . Thus,  $H(n, 2, -1) \notin \mathbf{Z}$  for any  $n \geq 13$ . Presumably there is an integer  $n_0$  such that  $H_{n,\ell,j}$  is not an integer for any  $n \geq n_0, \ell \geq 1$  and  $j < 0$ . Note that since  $p \mid \tau(p)$  for  $p \in \{2, 3, 5, 7\}$ , we have that  $H_{n,\ell,j}$  is an integer for all  $n \leq 10, \ell \geq 1$  and  $j$  negative and small in absolute value. Knowledge of non-ordinary primes  $p$ , namely primes  $p$  such that  $p \mid \tau(p)$  would be useful. It has been checked that in addition to the above four, 2411, 7758337633 are the only ones below  $10^{10}$ . See [MoSm13] for some results on this problem.

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