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Florence Fauquant-Millet

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# SYMMETRIC SEMI-INVARIANTS FOR SOME INÖNÜ-WIGNER CONTRACTIONS.

FLORENCE FAUQUANT-MILLET

ABSTRACT. Let  $\mathfrak{p}$  be a proper parabolic subalgebra of a simple Lie algebra  $\mathfrak{g}$ . Writing  $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{m}$  with  $\mathfrak{r}$  being the Levi factor of  $\mathfrak{p}$  and  $\mathfrak{m}$  the nilpotent radical of  $\mathfrak{p}$ , the semi-direct product  $\tilde{\mathfrak{p}} = \mathfrak{r} \ltimes (\mathfrak{m})^a$ , where  $(\mathfrak{m})^a$  is an abelian ideal of  $\tilde{\mathfrak{p}}$ , isomorphic to  $\mathfrak{m}$  as an  $\mathfrak{r}$ -module, is a Lie algebra. This is a special case of Inönü-Wigner contraction and may be considered as a degeneration of  $\mathfrak{p}$ . For any Lie algebra  $\mathfrak{a}$ , denote by  $Sy(\mathfrak{a})$  the algebra of symmetric semi-invariants in the symmetric algebra  $S(\mathfrak{a})$  of  $\mathfrak{a}$  under the adjoint action of  $\mathfrak{a}$ . In this paper we are interested in the polynomiality of the algebra  $Sy(\tilde{\mathfrak{p}})$ . Inspired by our method in [Fauquant-Millet F., Joseph A.: Semi-centre de l'algèbre enveloppante d'une sous-algèbre parabolique d'une algèbre de Lie semi-simple, Ann. Sci. Éc. Norm. Sup. 38 (2005) 155–191] where we studied the polynomiality of  $Sy(\mathfrak{p})$  (the nondegenerate case), we obtain in this paper a lower bound for the formal character of the algebra  $Sy(\tilde{\mathfrak{p}})$ , when the latter is well defined. The method in the nondegenerate case does not apply directly in the degenerate case : in the present paper we define a so-called generalized PBW filtration on a highest weight irreducible representation of  $\mathfrak{g}$  to provide the lower bound. Combined with an upper bound we will construct in the near future for particular contractions  $\tilde{\mathfrak{p}}$ , our goal is to show that the algebra  $Sy(\tilde{\mathfrak{p}})$  is a polynomial algebra, by showing that both bounds coincide.

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*Key words* : Inönü-Wigner contraction, parabolic subalgebra, symmetric invariants, semi-invariants.

## 1. INTRODUCTION.

The base field  $\mathbb{k}$  is algebraically closed of characteristic zero.

**1.1. The aim of the paper.** Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{k}$  and fix a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . Then choose a set  $\pi$  of simple roots for  $(\mathfrak{g}, \mathfrak{h})$  and denote by  $\mathfrak{b}$  the Borel subalgebra of  $\mathfrak{g}$  associated with it. Let  $\mathfrak{p} \supset \mathfrak{b}$  be a proper parabolic subalgebra of  $\mathfrak{g}$ . Denote by  $\mathfrak{n}$ , resp.  $\mathfrak{n}^-$ , the maximal nilpotent subalgebra of  $\mathfrak{g}$  generated by all positive, resp. negative, root vectors, so that  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$  and  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ . Let  $\mathfrak{r}$  denote the Levi factor of  $\mathfrak{p}$  (so that  $\mathfrak{r}$  is a reductive Lie algebra) and  $\mathfrak{m}$  the nilpotent radical of  $\mathfrak{p}$ . Then one has that  $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{m}$ . Now consider the semi-direct product  $\tilde{\mathfrak{p}} = \mathfrak{r} \ltimes (\mathfrak{m})^a$  where  $(\mathfrak{m})^a$  is isomorphic to  $\mathfrak{m}$  as an  $\mathfrak{r}$ -module, the superscript  $a$  meaning that  $(\mathfrak{m})^a$  is an abelian ideal of  $\tilde{\mathfrak{p}}$ . The semi-direct product  $\tilde{\mathfrak{p}}$  is

still a Lie algebra which may be viewed as a *degeneration* of the parabolic subalgebra  $\mathfrak{p}$  (of course when  $\mathfrak{m}$  is already abelian, we have that  $\tilde{\mathfrak{p}} = \mathfrak{p}$  as a Lie algebra). The Lie algebra  $\tilde{\mathfrak{p}}$  is called an Inönü-Wigner contraction, or a one-parameter contraction of  $\mathfrak{p}$  (see [31, Sect. 4]). Denoting by  $\mathfrak{a}'$  the derived subalgebra of any Lie algebra  $\mathfrak{a}$ , one has that  $\tilde{\mathfrak{p}}' = \mathfrak{r}' \ltimes (\mathfrak{m})^{\mathfrak{a}}$ .

In this paper we are interested in the algebra  $Sy(\tilde{\mathfrak{p}})$  of symmetric semi-invariants in the symmetric algebra  $S(\tilde{\mathfrak{p}})$  of  $\tilde{\mathfrak{p}}$  under the adjoint action of  $\tilde{\mathfrak{p}}$ , which is also equal to the algebra  $S(\tilde{\mathfrak{p}})^{\tilde{\mathfrak{p}}'}$  of symmetric invariants under the adjoint action of  $\tilde{\mathfrak{p}}'$ . In some cases (especially when  $\mathfrak{p}$  is a maximal parabolic subalgebra), we have that  $Sy(\tilde{\mathfrak{p}}) = S(\tilde{\mathfrak{p}}')^{\tilde{\mathfrak{p}}'}$ . For the natural Poisson structure on  $S(\tilde{\mathfrak{p}})$ , the algebra  $Sy(\tilde{\mathfrak{p}})$  is also equal to the Poisson semicentre of  $S(\tilde{\mathfrak{p}})$ .

The study of the algebra  $Sy(\tilde{\mathfrak{p}})$  will be called the *degenerate case* while the study of the algebra  $Sy(\mathfrak{p}) = S(\mathfrak{p})^{\mathfrak{p}'}$  of symmetric semi-invariants in the symmetric algebra  $S(\mathfrak{p})$  of  $\mathfrak{p}$  under the adjoint action of  $\mathfrak{p}$  will be called the *nondegenerate case*. The aim of the present paper is to construct a lower bound for the formal character of  $Sy(\tilde{\mathfrak{p}})$  (when the latter is well defined), by adapting methods in the nondegenerate case [9, Prop. 3.1] or [10, Sect. 6].

We then hope to get an upper bound for the formal character of  $Sy(\tilde{\mathfrak{p}})$  (at least for the contraction of a parabolic subalgebra  $\mathfrak{p}$  such that  $\dim \mathfrak{m}$  is not too big with respect to  $\dim \mathfrak{r}'$ ) which would be equal to this lower bound. This would imply that the algebra  $Sy(\tilde{\mathfrak{p}})$  is a polynomial algebra, for which we could give the number of algebraically independent generators, their weight and degree.

Motivated by computation for several examples of contractions  $\tilde{\mathfrak{p}}$  of maximal parabolic subalgebras in a simple Lie algebra  $\mathfrak{g}$ , where we get an upper bound for the formal character of  $Sy(\tilde{\mathfrak{p}})$  which coincides with the lower bound in the present paper, we hope to extend in the near future the construction for the contraction of every maximal parabolic subalgebra, at least in classical cases.

Results about one-parameter contractions, and especially about their ring of invariants (or semi-invariants) are reviewed below. Feigin in [13] introduced the contraction  $\tilde{\mathfrak{g}}$  of a semi-simple Lie algebra  $\mathfrak{g}$  given by the decomposition  $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$ , motivated by some problem in representation theory (see [15]). He then studied degenerations of flag varieties related to this contraction in [14]. For such a contraction  $\tilde{\mathfrak{g}}$ , Panyushev and Yakimova studied in [20] the algebra of symmetric invariants  $S(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$  and showed that it is always a polynomial algebra in  $\text{rank}(\mathfrak{g})$  generators. In [30] Yakimova showed that the algebra of symmetric semi-invariants  $Sy(\tilde{\mathfrak{g}})$  is also always polynomial.

Extending Feigin's construction, Panyushev and Yakimova studied the contraction  $\tilde{\mathfrak{g}}$  of a semi-simple Lie algebra  $\mathfrak{g}$  given by the decomposition  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m}^-$  (where  $\mathfrak{m}^-$  is the nilpotent radical of the opposite parabolic subalgebra of  $\mathfrak{p}$ ). They showed in [21] that the algebra of symmetric invariants  $S(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$  is always polynomial when  $\mathfrak{g}$  is simple of type A or C and for particular parabolic subalgebras in type B, using results on centralisers in [19].

Based on results of Panyushev and Yakimova, Phommady [25] studied the polynomiality of the algebra of symmetric semi-invariants  $Sy(\mathfrak{g})$ : he proved that it is always a polynomial algebra in type A and gave a counter-example in type C.

All the previous contractions are semi-direct products of the form  $\mathfrak{q} = \mathfrak{s} \ltimes V$  with  $\mathfrak{s}$  being a Lie algebra (not necessarily reductive) and  $V$  being a finite-dimensional representation of  $\mathfrak{s}$ . It was observed ([31, Sect. 3]) that, if  $S(\mathfrak{q})^{\mathfrak{q}}$  is polynomial, then it is the same for  $S(V)^{\mathfrak{s}}$ . When  $\mathfrak{q}$  is the contraction of a  $\mathbb{Z}_2$ -grading of a reductive Lie algebra, the polynomiality of the ring of invariants  $S(\mathfrak{q})^{\mathfrak{q}}$  was studied in [18]. On the other hand when  $\mathfrak{s}$  is a semisimple Lie algebra, Panyushev and Yakimova used the classification in [5], [6], [27], [29] to give a classification (see [22], [23], [24]) of all finite-dimensional representations of simple Lie groups (that is, for  $\mathfrak{s}$  being simple) and also of all irreducible representations of arbitrary semisimple groups with toral generic stabilisers, for which  $S(\mathfrak{q})^{\mathfrak{q}}$  is a polynomial algebra. However when  $\mathfrak{s}$  is simple of type A, their classification was partial (see [32]).

Observe that in our paper we deal with a semi-direct product  $\mathfrak{q} = \mathfrak{s} \ltimes V$  with  $\mathfrak{s} = \mathfrak{r}'$  being semisimple (and not necessarily simple in general) and  $V = \mathfrak{m}$ . However even when  $\mathfrak{r}'$  is simple, the study of Panyushev and Yakimova does not seem to always give the answer whether  $Sy(\tilde{\mathfrak{p}})$  is or not polynomial. For example if we consider the contraction  $\tilde{\mathfrak{p}} = \mathfrak{r} \ltimes (\mathfrak{m})^a$  of the maximal parabolic subalgebra  $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{m}$  of  $\mathfrak{g}$  simple of type  $B_4$ , whose Levi subalgebra  $\mathfrak{r}'$  is simple of type  $A_3$ , then  $\mathfrak{m}$  does not occur among the representations of  $\mathfrak{sl}_4$  treated by Panyushev and Yakimova in [22] or in [32]. In this case, we have constructed an adapted pair for  $\tilde{\mathfrak{p}}'$  which provides an upper bound for the formal character of  $Sy(\tilde{\mathfrak{p}}) = S(\tilde{\mathfrak{p}}')^{\tilde{\mathfrak{p}}'}$  and this upper bound coincides with the lower bound we have built in this paper. Thus we obtain in this case that the algebra of symmetric semi-invariants  $Sy(\tilde{\mathfrak{p}})$  is a polynomial algebra in three generators, for which we can give their weight and degree.

Note also that it is shown in [20, Th. 1.1] that the bi-homogeneous components of highest degree relative to  $\mathfrak{m}$  of homogeneous elements in  $S(\mathfrak{p}')^{\mathfrak{p}'}$  lie in  $S(\tilde{\mathfrak{p}}')^{\tilde{\mathfrak{p}}'} = S(\mathfrak{p}')^{\tilde{\mathfrak{p}}'}$ . Moreover [30, Th. 3.8] may give a criterion to know whether  $S(\mathfrak{p}')^{\tilde{\mathfrak{p}}'}$  is polynomial, based on the degrees relative to  $\mathfrak{m}$  of generators of  $S(\mathfrak{p}')^{\mathfrak{p}'}$ , if the latter is finitely generated. Unfortunately, even when  $S(\mathfrak{p}')^{\mathfrak{p}'}$  is known to be polynomial and when the degree is known for each homogeneous generator of  $S(\mathfrak{p}')^{\mathfrak{p}'}$ , it does not seem to be easy to compute its degree relative to  $\mathfrak{m}$ .

**1.2. The method.** The method we use in this paper is completely different from this of Panyushev and Yakimova. Our method is partly inspired by this used in [9], [10] to study the polynomiality of the algebra of symmetric semi-invariants  $Sy(\mathfrak{p})$ . The study of the latter algebra will be called *the nondegenerate case*, while we will call the study of  $Sy(\tilde{\mathfrak{p}})$  *the degenerate case*. In the degenerate case, the method to build a lower bound is more complicated than in the nondegenerate case, and then we cannot just rewrite

the proof used in the nondegenerate case. In particular we will construct some degeneration of an irreducible representation of the simple Lie algebra  $\mathfrak{g}$  and we will consider matrix coefficients associated with it. We also have to define a suitable action of some associative algebra  $A$ , which provides the adjoint action of  $\tilde{\mathfrak{p}}$  on  $S(\tilde{\mathfrak{p}})$  after some identifications we will explain below.

Denote by  $P^+(\pi)$  the set of dominant integral weights of  $(\mathfrak{g}, \mathfrak{h})$  and let  $V(\lambda)$  be the irreducible highest weight  $\mathfrak{g}$ -module of highest weight  $\lambda$ , for  $\lambda \in P^+(\pi)$  (so that  $V(\lambda)$  is finite-dimensional).

Let us describe our method and our main result more precisely.

- In subsections 3.1 and 3.2 we fix  $\lambda \in P^+(\pi)$  and denote by  $\mathfrak{p}^- = \mathfrak{r} \oplus \mathfrak{m}^- \supset \mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$  the opposite parabolic subalgebra of  $\mathfrak{p}$ , where  $\mathfrak{m}^-$  is the nilpotent radical of  $\mathfrak{p}^-$  and by  $\tilde{\mathfrak{p}}^- = \mathfrak{r} \ltimes (\mathfrak{m}^-)^a$  the one-parameter contraction of  $\mathfrak{p}^-$ . Then, inspired by the construction in [15], we define what we call a *generalized PBW filtration*  $(\mathcal{F}_k(V(\lambda)))_{k \in \mathbb{N}}$  on  $V(\lambda)$ , which is an increasing and exhaustive filtration on  $V(\lambda)$ , induced by the canonical (or standard degree) filtration  $(U_k(\mathfrak{m}^-))_{k \in \mathbb{N}}$  on the enveloping algebra  $U(\mathfrak{m}^-)$  of  $\mathfrak{m}^-$ .

The associated graded space, that we call the *degenerate highest weight module associated with  $\lambda$* , is denoted by

$$\tilde{V}(\lambda) := gr_{\mathcal{F}}(V(\lambda)) = \bigoplus_{k \in \mathbb{N}} gr_k(V(\lambda))$$

where  $gr_k(V(\lambda)) = \frac{\mathcal{F}_k(V(\lambda))}{\mathcal{F}_{k-1}(V(\lambda))}$  for all  $k \in \mathbb{N}$  with  $\mathcal{F}_{-1}(V(\lambda)) := \{0\}$ .

If  $v_\lambda$  is a nonzero vector of highest weight  $\lambda$  in  $V(\lambda)$ , we denote by  $V'(\lambda)$  the irreducible  $U(\mathfrak{r})$ -submodule of  $V(\lambda)$  generated by  $v_\lambda$  and by  $\tilde{V}'(\lambda) \subset \tilde{V}(\lambda)$  the canonical image of  $V'(\lambda)$  in  $\tilde{V}(\lambda)$ . We will observe that, as  $U(\mathfrak{r})$ -modules, we have  $\tilde{V}(\lambda) \simeq V(\lambda)$ . Set  $\tilde{v}_\lambda$  the canonical image of  $v_\lambda$  in  $\tilde{V}(\lambda)$ . We define a left  $U(\tilde{\mathfrak{p}}^-)$ -module structure on  $\tilde{V}(\lambda)$ , for which we have that  $\tilde{V}'(\lambda) = U(\mathfrak{r}).\tilde{v}_\lambda$  and that

$$\tilde{V}(\lambda) = U(\tilde{\mathfrak{p}}^-).\tilde{v}_\lambda = S(\mathfrak{m}^-).\tilde{V}'(\lambda) = U(\tilde{\mathfrak{p}}^-).\tilde{V}'(\lambda).$$

- In subsections 4.1, 4.2, 4.4, denoting by  $T(\mathfrak{m})$  the tensor algebra of  $\mathfrak{m}$ , we consider the associative algebra  $A = T(\mathfrak{m}) \# U(\mathfrak{r})$ , which is the Hopf smash product of the left  $U(\mathfrak{r})$ -algebra  $T(\mathfrak{m})$  by the Hopf algebra  $U(\mathfrak{r})$ , as defined for example in [17, 1.1.8]. As  $T(\mathfrak{m})$  is also equipped with a coproduct, we obtain that this smash product  $A$  also inherits a structure of a bialgebra and indeed of a Hopf algebra.

We then consider the coadjoint action, which we denote by  $ad^*$ , of  $U(\tilde{\mathfrak{p}})$  on  $\mathfrak{p}^- \simeq \tilde{\mathfrak{p}}^*$  (as vector spaces). Then  $ad^*$  extends uniquely by derivation to an action of  $U(\tilde{\mathfrak{p}})$  on  $S(\mathfrak{p}^-)$ . From this action  $ad^*$ , we define what we call a *generalized adjoint action  $ad^{**}$*  of  $A$  on  $U(\tilde{\mathfrak{p}}^-)$ , which coincides with the adjoint action on  $U(\tilde{\mathfrak{p}}^-)$ , when restricted to  $U(\mathfrak{r})$ .

- In subsections 5.1 and 5.3, we consider spaces of matrix coefficients.

For  $\lambda \in P^+(\pi)$ , we set  $\tilde{v}_{w_0\lambda}$  the canonical image in  $\tilde{V}(\lambda)$  of a chosen nonzero lowest weight vector in  $V(\lambda)$  and by  $\tilde{V}''(\lambda)$  the  $U(\mathfrak{r})$ -submodule of  $\tilde{V}(\lambda)$  generated by  $\tilde{v}_{w_0\lambda}$ . We denote by  $\tilde{V}(\lambda)^*$  the dual space of  $\tilde{V}(\lambda)$ . For all  $\xi \in \tilde{V}(\lambda)^*$  and  $v \in \tilde{V}'(\lambda)$ , the matrix coefficient  $c_{\xi,v} \in U(\tilde{\mathfrak{p}}^-)^*$  is defined by :

$$c_{\xi,v}(u) = \xi(u.v) \text{ for all } u \in U(\tilde{\mathfrak{p}}^-).$$

Then we define  $\tilde{C}_{\mathfrak{p}}(\lambda)$  to be the subspace of  $U(\tilde{\mathfrak{p}}^-)^*$  generated by

$$\{c_{\xi,v} \mid \xi \in \tilde{V}(\lambda)^*, v \in \tilde{V}'(\lambda)\}$$

and  $\tilde{C}_{\mathfrak{r}}(\lambda)$  to be the subspace of  $\tilde{C}_{\mathfrak{p}}(\lambda)$  generated by

$$\{c_{\xi,v} \mid \xi \in \tilde{V}''(\lambda)^*, v \in \tilde{V}'(\lambda)\}.$$

We set  $\tilde{C}_{\mathfrak{p}} = \sum_{\lambda \in P^+(\pi)} \tilde{C}_{\mathfrak{p}}(\lambda)$  and  $\tilde{C}_{\mathfrak{r}} = \sum_{\lambda \in P^+(\pi)} \tilde{C}_{\mathfrak{r}}(\lambda)$ . We show that these are direct sums and that  $\tilde{C}_{\mathfrak{r}}$  is a subalgebra of  $U(\tilde{\mathfrak{p}}^-)^*$ .

- In subsection 5.5, we consider the dual representation of  $ad^{**}$ , which defines a left  $A$ -module structure on  $U(\tilde{\mathfrak{p}}^-)^*$ . When restricted to  $U(\mathfrak{r})$ , the dual representation of  $ad^{**}$  defines a left  $U(\mathfrak{r})$ -module structure on every  $\tilde{C}_{\mathfrak{r}}(\lambda)$ ,  $\lambda \in P^+(\pi)$ , and then on  $\tilde{C}_{\mathfrak{r}}$ , which coincides with the coadjoint representation.
- In subsections 6.1 and 6.2, for all  $\lambda \in P^+(\pi)$ , we denote by  $\tilde{C}_{\mathfrak{r}}(\lambda)^{U(\mathfrak{r}'')}$ , resp.  $\tilde{C}_{\mathfrak{r}}^{U(\mathfrak{r}'')}$  the vector space, resp. the algebra, of invariants in  $\tilde{C}_{\mathfrak{r}}(\lambda)$ , resp. in  $\tilde{C}_{\mathfrak{r}}$ , under the coadjoint representation of  $U(\mathfrak{r}'')$ . We have that

$$\tilde{C}_{\mathfrak{r}}^{U(\mathfrak{r}'')} = \bigoplus_{\lambda \in P^+(\pi)} \tilde{C}_{\mathfrak{r}}(\lambda)^{U(\mathfrak{r}'')}.$$

Denote by  $\pi' \subset \pi$  the subset of simple roots of  $(\mathfrak{g}, \mathfrak{h})$  associated with the parabolic subalgebra  $\mathfrak{p}$ , set  $\mathfrak{h}_{\pi'} = \mathfrak{h} \cap \mathfrak{p}'$ , and denote by  $(\cdot, \cdot)$  the nondegenerate symmetric bilinear form on  $\mathfrak{h}^* \times \mathfrak{h}^*$  induced by the Killing form on  $\mathfrak{g}$ . Since, for all  $\lambda \in P^+(\pi)$ ,  $\tilde{V}'(\lambda)$  is an irreducible  $U(\mathfrak{r})$ -module, the Jacobson density theorem implies that the  $U(\mathfrak{r})$ -module  $\tilde{C}_{\mathfrak{r}}(\lambda)$  is isomorphic to the  $U(\mathfrak{r})$ -module  $\tilde{V}''(\lambda)^* \otimes \tilde{V}'(\lambda)$  where the latter is endowed with the diagonal action of  $U(\mathfrak{r})$ . It follows that, for all  $\lambda \in P^+(\pi)$ ,  $\tilde{C}_{\mathfrak{r}}(\lambda)^{U(\mathfrak{r}'')}$  is of dimension less or equal to one, and equal to one if and only if

$$(w'_0\lambda - w_0\lambda, \pi') = 0$$

where  $w'_0$ , resp.  $w_0$ , is the longest element in the Weyl group of  $(\mathfrak{r}', \mathfrak{h}_{\pi'})$ , resp. of  $(\mathfrak{g}, \mathfrak{h})$ . As a consequence we show (as in [9, prop. 3.1]) that  $\tilde{C}_{\mathfrak{r}}^{U(\mathfrak{r}'')}$  is a polynomial algebra, for which we can compute the number of algebraically independent generators and their weight.

- In subsections 7.1, 7.3, 7.4 and 7.5, inspired by [10, 6.1], one defines on the algebra  $U(\tilde{\mathfrak{p}}^-)^*$  what we call *the generalized Kostant filtration*  $(\mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*))_{k \in \mathbb{N}}$ , which is a decreasing, exhaustive and separated ring filtration. This filtration is invariant under the action of  $A$  given by the dual representation of  $ad^{**}$ .

One denotes by  $gr_K(U(\tilde{\mathfrak{p}}^-)^*) = \bigoplus_{k \in \mathbb{N}} gr_K^k(U(\tilde{\mathfrak{p}}^-)^*)$  the graded algebra associated with this filtration where, for all  $k \in \mathbb{N}$ ,

$$gr_K^k(U(\tilde{\mathfrak{p}}^-)^*) = \frac{\mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*)}{\mathcal{F}_K^{k+1}(U(\tilde{\mathfrak{p}}^-)^*)}.$$

The dual representation of  $ad^{**}$  induces a left action of  $A$  on this graded algebra and one checks that, for all  $x \in \mathfrak{m}$ , for all  $f \in \tilde{C}_\tau \cap \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*)$ , one has for this action

$$x.f \in \mathcal{F}_K^{k+1}(U(\tilde{\mathfrak{p}}^-)^*)$$

that is, that

$$x.gr_K^k(f) = 0$$

where  $gr_K^k(f)$  denotes the canonical image of  $f$  in  $gr_K(U(\tilde{\mathfrak{p}}^-)^*)$ .

Then for all  $k \in \mathbb{N}$  and all vector space  $V$ , denoting by  $S_k(V)$  the vector subspace of the symmetric algebra  $S(V)$  of  $V$  formed by all homogeneous polynomials of degree  $k$ , one defines a morphism  $\psi_k : gr_K^k(U(\tilde{\mathfrak{p}}^-)^*) \rightarrow S_k(\mathfrak{p}^-)^*$ . It is easily checked that actually  $\psi_k$  is an isomorphism of left  $U(\tilde{\mathfrak{p}})$ -modules, where the left structure on  $gr_K^k(U(\tilde{\mathfrak{p}}^-)^*)$  is induced by the dual representation of  $ad^{**}$  and where the left structure on  $S_k(\mathfrak{p}^-)^*$  is given by the dual representation of  $ad^*$ . With this structure, it is easily checked that  $S_k(\mathfrak{p}^-)^*$  is isomorphic to the  $U(\tilde{\mathfrak{p}})$ -module  $S_k(\tilde{\mathfrak{p}}) = S_k(\mathfrak{p})$  where the action of  $\tilde{\mathfrak{p}}$  is the adjoint action which extends by derivation the Lie bracket in  $\tilde{\mathfrak{p}}$ . Thus we obtain an isomorphism of  $U(\tilde{\mathfrak{p}})$ -modules and of algebras from  $gr_K(U(\tilde{\mathfrak{p}}^-)^*)$  to  $S(\tilde{\mathfrak{p}})$ .

- Denote by  $gr_K(\tilde{C}_\tau^{U(\mathfrak{p}')}})$  the graded algebra associated with the induced generalized Kostant filtration on  $\tilde{C}_\tau^{U(\mathfrak{p}')}$ . The former may be viewed as a subalgebra of  $gr_K(U(\tilde{\mathfrak{p}}^-)^*)$ , which by equation in the above alinea is invariant under the action of  $U(\tilde{\mathfrak{p}}')$  induced by the action of  $A$  on  $U(\tilde{\mathfrak{p}}^-)^*$  given by the dual representation of  $ad^{**}$ . Finally one can establish the main result of our paper (see subsection 7.6).

**Theorem.** *There is an injection of algebras and of  $U(\mathfrak{h})$ -modules from  $gr_K(\tilde{C}_\tau^{U(\mathfrak{p}')}})$  into the Poisson semicentre  $Sy(\tilde{\mathfrak{p}}) = S(\tilde{\mathfrak{p}})^{\mathfrak{p}'}$ . This implies a lower bound for the formal character of  $Sy(\tilde{\mathfrak{p}})$ , when the latter is well defined.*

## 2. NOTATION.

**2.1. General notation.** Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{k}$ ,  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$  and choose a set  $\pi$  of simple roots for  $(\mathfrak{g}, \mathfrak{h})$ . Denote by  $\Delta^\pm$  the set of positive, resp. negative, roots of  $(\mathfrak{g}, \mathfrak{h})$  with respect to  $\pi$  and  $\Delta = \Delta^+ \sqcup \Delta^-$  the set of roots of  $(\mathfrak{g}, \mathfrak{h})$ . Denote by  $[\cdot, \cdot]$  the Lie bracket in  $\mathfrak{g}$  and by  $\langle \cdot, \cdot \rangle$  the natural duality between  $\mathfrak{h}$  and  $\mathfrak{h}^*$ . Then for all root  $\alpha \in \Delta$ , set  $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \forall h \in \mathfrak{h}, [h, x] = \langle h, \alpha \rangle x\}$  and fix a nonzero root vector  $x_\alpha$  in  $\mathfrak{g}_\alpha$ .

Denote by  $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ , resp.  $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$ , the maximal nilpotent subalgebra of  $\mathfrak{g}$  generated by positive, resp. negative, root vectors, so that  $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ . Let  $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$  be the Borel subalgebra of  $\mathfrak{g}$ .

For each subset  $\pi'$  of  $\pi$ , we denote by  $\Delta_{\pi'}^\pm$  the subset of  $\Delta^\pm$  generated by  $\pi'$  that is,  $\Delta_{\pi'}^\pm = (\pm \mathbb{N}\pi') \cap \Delta^\pm$ . Set  $\mathfrak{n}_{\pi'}^+ = \bigoplus_{\alpha \in \Delta_{\pi'}^+} \mathfrak{g}_\alpha$ , resp.  $\mathfrak{n}_{\pi'}^- = \bigoplus_{\alpha \in \Delta_{\pi'}^-} \mathfrak{g}_\alpha$ . Then the (standard) parabolic subalgebra  $\mathfrak{p} \supset \mathfrak{b}$  of  $\mathfrak{g}$  associated with  $\pi'$  is

$$\mathfrak{p} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_{\pi'}^-.$$

The Levi factor  $\mathfrak{r}$  of  $\mathfrak{p}$  is

$$\mathfrak{r} = \mathfrak{n}_{\pi'}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_{\pi'}^+$$

and its derived subalgebra (which is semisimple) is  $\mathfrak{r}' = \mathfrak{n}_{\pi'}^- \oplus \mathfrak{h}_{\pi'} \oplus \mathfrak{n}_{\pi'}^+$ , where  $\mathfrak{h}_{\pi'} = \mathfrak{h} \cap \mathfrak{p}'$ , with  $\mathfrak{p}' = [\mathfrak{p}, \mathfrak{p}]$  being the derived subalgebra of  $\mathfrak{p}$ . If for all  $\alpha \in \pi$ ,  $\check{\alpha}$  denotes the coroot associated with  $\alpha$ , we have that  $\mathfrak{h}_{\pi'}$  is the  $\mathbb{k}$ -vector space generated by the coroots  $\check{\alpha}$ , with  $\alpha \in \pi'$ .

The longest element of the Weyl group  $W$ , resp.  $W'$ , of  $(\mathfrak{g}, \mathfrak{h})$ , resp. of  $(\mathfrak{r}', \mathfrak{h}_{\pi'})$ , is denoted by  $w_0$ , resp.  $w'_0$ .

Set  $\mathfrak{h}^{\pi \setminus \pi'} = \{h \in \mathfrak{h} \mid \langle h, \pi' \rangle = 0\}$  so that  $\mathfrak{h} = \mathfrak{h}_{\pi'} \oplus \mathfrak{h}^{\pi \setminus \pi'}$ . Denote by  $\mathfrak{m}$  the nilpotent radical of  $\mathfrak{p}$ , so that  $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{m}$ . We have that  $\mathfrak{n} = \mathfrak{n}_{\pi'}^- \oplus \mathfrak{m}$  and that  $\mathfrak{m} = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_{\pi'}^+} \mathfrak{g}_\alpha$ . The opposite subalgebra  $\mathfrak{p}^-$  of  $\mathfrak{p}$  is the parabolic subalgebra of  $\mathfrak{g}$  defined by

$$\mathfrak{p}^- = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_{\pi'}^+.$$

We denote by  $\mathfrak{m}^-$  the nilpotent radical of  $\mathfrak{p}^-$  (so that  $\mathfrak{p}^- = \mathfrak{r} \oplus \mathfrak{m}^-$ ). The Killing form  $K$  on  $\mathfrak{g} \times \mathfrak{g}$  induces an isomorphism between the dual space  $\mathfrak{p}^*$  of  $\mathfrak{p}$  and the vector space  $\mathfrak{p}^-$ , since  $K$  is non degenerate on  $\mathfrak{p} \times \mathfrak{p}^-$ . Moreover since  $K$  is also non degenerate on  $\mathfrak{h} \times \mathfrak{h}$ , it induces a non degenerate symmetric bilinear form  $(\cdot, \cdot)$  on  $\mathfrak{h}^* \times \mathfrak{h}^*$  which is invariant under the action of  $W$  (see for instance [11, 5.2.2]).

For all  $\alpha \in \pi$ , resp.  $\alpha \in \pi'$ , let  $\varpi_\alpha$ , resp.  $\varpi'_\alpha$ , be the fundamental weight associated with  $\alpha$  with respect to  $(\mathfrak{g}, \mathfrak{h})$ , resp. with respect to  $(\mathfrak{r}', \mathfrak{h}_{\pi'})$ . Then  $P(\pi) = \sum_{\alpha \in \pi} \mathbb{Z}\varpi_\alpha$ , resp.  $P(\pi') = \sum_{\alpha \in \pi'} \mathbb{Z}\varpi'_\alpha$ , is the weight lattice of  $(\mathfrak{g}, \mathfrak{h})$ , resp.  $(\mathfrak{r}', \mathfrak{h}_{\pi'})$ . Moreover  $P^+(\pi) = \sum_{\alpha \in \pi} \mathbb{N}\varpi_\alpha$ , resp.  $P^+(\pi') = \sum_{\alpha \in \pi'} \mathbb{N}\varpi'_\alpha$ , is the set of dominant integral weights of  $(\mathfrak{g}, \mathfrak{h})$ , resp.  $(\mathfrak{r}', \mathfrak{h}_{\pi'})$ . By [10, 2.5], there exists some positive integer  $r$  such that



$$(1) \quad P(\pi) \subset P(\pi') \oplus \frac{1}{r} \sum_{\alpha \in \pi \setminus \pi'} \mathbb{Z} \varpi_\alpha$$

and for all  $\alpha \in \pi'$ , the projection of  $\varpi_\alpha$  in  $P(\pi')$  with respect to this decomposition (1) is  $\varpi'_\alpha$ . For  $\lambda = \sum_{\alpha \in \pi} m_\alpha \varpi_\alpha \in P(\pi)$  ( $m_\alpha \in \mathbb{Z}$  for each  $\alpha \in \pi$ ), we denote by  $\lambda' = \sum_{\alpha \in \pi'} m_\alpha \varpi'_\alpha$  its projection in  $P(\pi')$  with respect to the decomposition (1).

For any finite-dimensional Lie algebra  $\mathfrak{a}$ , we denote by  $U(\mathfrak{a})$  its universal enveloping algebra and by  $S(\mathfrak{a})$  its symmetric algebra, which may be viewed as the (commutative) graded algebra associated with the canonical filtration  $(U_k(\mathfrak{a}))_{k \in \mathbb{N}}$  on  $U(\mathfrak{a})$  (see [4, 2.3]). We may also identify  $S(\mathfrak{a})$  with the algebra  $\mathbb{k}[\mathfrak{a}^*]$  of polynomial functions on the dual space  $\mathfrak{a}^*$  of  $\mathfrak{a}$ . For all  $k \in \mathbb{N}$ , we denote by  $S_k(\mathfrak{a})$  the vector subspace of  $S(\mathfrak{a})$  formed by all homogeneous polynomials of degree  $k$ .

For all  $\lambda \in P^+(\pi)$ , the irreducible highest weight  $\mathfrak{g}$ -module of highest weight  $\lambda$  (which is obtained by quotienting the corresponding Verma module by its largest proper sub- $\mathfrak{g}$ -module, as defined for example in [4, 7.1.11]) is denoted by  $V(\lambda)$ : recall ([4, 7.2.6]) that this is a finite-dimensional  $U(\mathfrak{g})$ -module. We may pay attention that (unlike the notation in [4, 7.1.4, 7.1.12]) the highest weight of  $V(\lambda)$  in our paper is  $\lambda$  and not  $\lambda - \rho$ , where  $\rho$  is the sum of all fundamental weights of  $(\mathfrak{g}, \mathfrak{h})$ .

**2.2. Semi-direct product.** Recall the parabolic subalgebra  $\mathfrak{p} = \mathfrak{t} \oplus \mathfrak{m}$  and its opposite parabolic subalgebra  $\mathfrak{p}^- = \mathfrak{t} \oplus \mathfrak{m}^-$ , with  $\mathfrak{m}$ , resp.  $\mathfrak{m}^-$ , the nilpotent radical of  $\mathfrak{p}$ , resp.  $\mathfrak{p}^-$ .

We now consider the semi-direct product  $\tilde{\mathfrak{p}} = \mathfrak{t} \ltimes (\mathfrak{m})^a$ , resp.  $\tilde{\mathfrak{p}}^- = \mathfrak{t} \ltimes (\mathfrak{m}^-)^a$ , where  $(\mathfrak{m})^a$ , resp.  $(\mathfrak{m}^-)^a$ , is isomorphic to  $\mathfrak{m}$ , resp.  $\mathfrak{m}^-$ , as a  $\mathfrak{t}$ -module, but where the superscript  $a$  means that  $(\mathfrak{m})^a$ , resp.  $(\mathfrak{m}^-)^a$ , is an abelian ideal of  $\tilde{\mathfrak{p}}$ , resp. of  $\tilde{\mathfrak{p}}^-$ . Such a semi-direct product is still a Lie algebra by [31, Sect. 4] for example, called an Inönü-Wigner contraction, or a one-parameter contraction of  $\mathfrak{p}$ , resp. of  $\mathfrak{p}^-$ . The  $\mathbb{k}$ -vector space  $\tilde{\mathfrak{p}}$ , resp.  $\tilde{\mathfrak{p}}^-$ , is equal to  $\mathfrak{p}$ , resp.  $\mathfrak{p}^-$ , as a vector space and if we denote by  $[\cdot, \cdot]_{\tilde{\mathfrak{p}}}$ , resp.  $[\cdot, \cdot]_{\tilde{\mathfrak{p}}^-}$  the Lie bracket in  $\tilde{\mathfrak{p}}$ , resp.  $\tilde{\mathfrak{p}}^-$ , and by  $[\cdot, \cdot]$  the Lie bracket in  $\mathfrak{g}$ , then one has that

$$(2) \quad \forall z, z' \in \mathfrak{t}, \forall x, x' \in \mathfrak{m}, \quad [z, x]_{\tilde{\mathfrak{p}}} = [z, x], \quad [z, z']_{\tilde{\mathfrak{p}}} = [z, z'], \quad [x, x']_{\tilde{\mathfrak{p}}} = 0$$

$$(3) \quad \forall z, z' \in \mathfrak{t}, \forall y, y' \in \mathfrak{m}^-, \quad [z, y]_{\tilde{\mathfrak{p}}^-} = [z, y], \quad [z, z']_{\tilde{\mathfrak{p}}^-} = [z, z'], \quad [y, y']_{\tilde{\mathfrak{p}}^-} = 0.$$

### 3. THE DEGENERATE HIGHEST WEIGHT MODULE.

In this section, we fix  $\lambda \in P^+(\pi)$  and we will define, from the irreducible highest weight module  $V(\lambda)$  of highest weight  $\lambda$ , some vector space denoted by  $\tilde{V}(\lambda)$  which can be endowed with a left  $U(\tilde{\mathfrak{p}}^-)$ -module structure, so that it is isomorphic to  $V(\lambda)$  as a left  $U(\mathfrak{t})$ -module.

**3.1. The generalized PBW filtration.** Consider  $V(\lambda)$  the irreducible highest weight  $\mathfrak{g}$ -module of highest weight  $\lambda$  as defined in subsection 2.1.

Generalizing the PBW filtration on a highest weight irreducible  $\mathfrak{g}$ -module introduced in [15], when  $\mathfrak{p} = \mathfrak{b}$  is a Borel subalgebra of  $\mathfrak{g}$  (that is, when  $\pi' = \emptyset$ ), we define what we call *the generalized Poincaré-Birkhoff-Witt filtration* on  $V(\lambda)$  as follows.

Choose  $v_\lambda$  a nonzero weight vector in  $V(\lambda)$  of highest weight  $\lambda$  and  $v_{w_0\lambda}$  a nonzero weight vector in  $V(\lambda)$  of lowest weight  $w_0\lambda$ . Since  $\mathfrak{n}^- = \mathfrak{n}_{\pi'}^- \oplus \mathfrak{m}^-$ , the multiplication in the enveloping algebra gives, by the Poincaré-Birkhoff-Witt theorem [4, 2.1.11], an isomorphism of vector spaces  $U(\mathfrak{n}_{\pi'}^-) \otimes U(\mathfrak{m}^-) \simeq U(\mathfrak{n}^-)$ . Then we have that

$$V(\lambda) = U(\mathfrak{n}_{\pi'}^-).(U(\mathfrak{m}^-).v_\lambda) = U(\mathfrak{r}).(U(\mathfrak{m}^-).v_\lambda) = U(\mathfrak{m}^-).(U(\mathfrak{n}_{\pi'}^-).v_\lambda)$$

since  $\mathfrak{m}^-$  is an ideal of  $\mathfrak{p}^-$ . In other words, every element in  $V(\lambda)$  is a finite sum of vectors of the form  $u'u.v_\lambda$  or of the form  $uu'.v_\lambda$  where  $u' \in U(\mathfrak{r})$  and  $u \in U(\mathfrak{m}^-)$ . Set  $V'(\lambda) = U(\mathfrak{n}_{\pi'}^-).v_\lambda$ . The latter is an irreducible  $U(\mathfrak{r})$ -module.

Recall  $(U_k(\mathfrak{m}^-))_{k \in \mathbb{N}}$  the canonical filtration (also called standard degree filtration in [15]) on the enveloping algebra  $U(\mathfrak{m}^-)$  of  $\mathfrak{m}^-$ . More precisely  $U_k(\mathfrak{m}^-)$  is the vector subspace of  $U(\mathfrak{m}^-)$  generated by the products  $y_1 \cdots y_p$  where  $y_i \in \mathfrak{m}^-$  for all  $i$ ,  $1 \leq i \leq p$ , and  $p \leq k$ .

For all  $k \in \mathbb{N}$ , let  $\mathcal{F}_k(V(\lambda))$  be the vector subspace of  $V(\lambda)$  generated by

$$\{v \in V(\lambda) \mid \exists p \in \mathbb{N}, p \leq k, \exists y_1, \dots, y_p \in \mathfrak{m}^-, \exists u' \in U(\mathfrak{r}); \\ v = u' y_1 \cdots y_p.v_\lambda\}.$$

where  $u' y_1 \cdots y_p$  denotes an element in  $U(\mathfrak{p}^-)$ . Observe that we also have that  $\mathcal{F}_k(V(\lambda))$  is the vector subspace of  $V(\lambda)$  generated by

$$\{v \in V(\lambda) \mid \exists p \in \mathbb{N}, p \leq k, \exists y_1, \dots, y_p \in \mathfrak{m}^-, \exists u' \in U(\mathfrak{r}); \\ v = y_1 \cdots y_p u'.v_\lambda\}$$

since  $[\mathfrak{r}, \mathfrak{m}^-] \subset \mathfrak{m}^-$ .

In other words, one has that  $\mathcal{F}_0(V(\lambda)) = U(\mathfrak{r}).v_\lambda = U(\mathfrak{n}_{\pi'}^-).v_\lambda = V'(\lambda)$  and for all  $k \in \mathbb{N}$ ,  $\mathcal{F}_k(V(\lambda)) = U_k(\mathfrak{m}^-).V'(\lambda)$  is a left  $U(\mathfrak{r})$ -module. We have the lemma.

**Lemma.** *Set  $\mathcal{F} := (\mathcal{F}_k(V(\lambda)))_{k \in \mathbb{N}}$ . Then  $\mathcal{F}$  is an increasing and exhaustive filtration on  $V(\lambda)$ .*

*Proof.*  $\mathcal{F}$  is increasing since the canonical filtration  $(U_k(\mathfrak{m}^-))_{k \in \mathbb{N}}$  on  $U(\mathfrak{m}^-)$  is increasing. Moreover  $\bigcup_{k \in \mathbb{N}} \mathcal{F}_k(V(\lambda)) = V(\lambda)$  since  $V(\lambda) = U(\mathfrak{m}^-).V'(\lambda)$  and since the canonical filtration on  $U(\mathfrak{m}^-)$  is exhaustive.  $\square$

We call  $\mathcal{F}$  the generalized Poincaré-Birkhoff-Witt filtration on  $V(\lambda)$  since when  $\pi' = \emptyset$ , it coincides with the PBW filtration on  $V(\lambda)$  introduced in [15]. The associated graded space is denoted by

$$\tilde{V}(\lambda) := gr_{\mathcal{F}}(V(\lambda)) = \bigoplus_{k \in \mathbb{N}} \frac{\mathcal{F}_k(V(\lambda))}{\mathcal{F}_{k-1}(V(\lambda))}$$

where  $\mathcal{F}_{-1}(V(\lambda)) := \{0\}$  and we call  $\tilde{V}(\lambda)$  the *degenerate highest weight module associated with  $\lambda$* . For all  $v \in \mathcal{F}_k(V(\lambda))$ , we denote by  $gr_k(v)$  its canonical image in  $gr_k(V(\lambda)) := \frac{\mathcal{F}_k(V(\lambda))}{\mathcal{F}_{k-1}(V(\lambda))}$ . Denote by  $\tilde{V}'(\lambda)$  the canonical image of  $V'(\lambda)$  in  $\tilde{V}(\lambda)$  that is,  $\tilde{V}'(\lambda) = gr_0(V'(\lambda)) = gr_0(V(\lambda)) \subset \tilde{V}(\lambda)$ .

**3.2. Structure of module on the degenerate highest weight module.** Recall that, for all  $k \in \mathbb{N}$ ,  $\mathcal{F}_k(V(\lambda))$  is a finite-dimensional left  $U(\mathfrak{r})$ -module and that the Lie algebra  $\mathfrak{r}$  is reductive and the elements of its centre act reductively in  $\mathcal{F}_k(V(\lambda))$ . Then by [4, 1.6.4] one has that  $\mathcal{F}_k(V(\lambda))$  is a semisimple  $U(\mathfrak{r})$ -module. Moreover  $\mathcal{F}_{k-1}(V(\lambda))$  is a submodule of  $\mathcal{F}_k(V(\lambda))$ . Then there exists a left  $U(\mathfrak{r})$ -submodule  $\mathcal{F}^k(V(\lambda))$  of  $\mathcal{F}_k(V(\lambda))$  such that  $\mathcal{F}_k(V(\lambda)) = \mathcal{F}^k(V(\lambda)) \oplus \mathcal{F}_{k-1}(V(\lambda))$  and we have that

$$\mathcal{F}_k(V(\lambda)) = \bigoplus_{i=0}^k \mathcal{F}^i(V(\lambda))$$

where  $\mathcal{F}^0(V(\lambda)) = \mathcal{F}_0(V(\lambda))$ . One deduces that

$$V(\lambda) = \bigoplus_{k \in \mathbb{N}} \mathcal{F}^k(V(\lambda)).$$

It allows us to define, for all  $k \in \mathbb{N}$ , an isomorphism of vector spaces

$$\beta_\lambda^k : gr_k(V(\lambda)) \longrightarrow \mathcal{F}^k(V(\lambda))$$

such that, for all  $v \in \mathcal{F}_k(V(\lambda))$ ,  $v = \sum_{i=0}^k v_i$  with  $v_i \in \mathcal{F}^i(V(\lambda))$ , for all  $0 \leq i \leq k$ ,

$$\beta_\lambda^k(gr_k(v)) = v_k.$$

Then the direct sum  $\beta_\lambda = \bigoplus_{k \in \mathbb{N}} \beta_\lambda^k$  is an isomorphism between the vector spaces  $\tilde{V}(\lambda)$  and  $V(\lambda)$ .

Set, for all  $y \in \mathfrak{m}^-$ ,  $z \in \mathfrak{r}$  and  $v \in \mathcal{F}_k(V(\lambda))$ ,

$$(4) \quad y.gr_k(v) = gr_{k+1}(y.v)$$

and

$$(5) \quad z.gr_k(v) = gr_k(z.v).$$

We will see below that equations (4) and (5) extend to a left  $U(\tilde{\mathfrak{p}}^-)$ -module structure on  $\tilde{V}(\lambda)$  and that  $\beta_\lambda$  is an isomorphism of  $U(\mathfrak{r})$ -modules.

Set  $\tilde{\mathfrak{n}}^- = \mathfrak{n}_{\pi'}^- \ltimes (\mathfrak{m}^-)^a$  : it is a Lie subalgebra of  $\tilde{\mathfrak{p}}^-$ . Set also  $\tilde{v}_\lambda = gr_0(v_\lambda)$ .

Denote by  $\theta : S(\mathfrak{p}^-) \longrightarrow U(\mathfrak{p}^-)$  the symmetrisation, as defined in [4, 2.4.6]. More precisely for  $k \in \mathbb{N}^*$ , and for all  $y_1, \dots, y_k \in \mathfrak{p}^-$ ,

$$\theta(y_1 \cdots y_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} y_{\sigma(1)} \cdots y_{\sigma(k)}$$

where  $\mathfrak{S}_k$  is the set of permutations of  $k$  elements, the product in the left hand side lying in  $S_k(\mathfrak{p}^-)$  and the product in the right hand side lying in

$U_k(\mathfrak{p}^-)$ . Endow the symmetric algebra  $S(\mathfrak{p}^-)$ , resp. the enveloping algebra  $U(\mathfrak{p}^-)$ , with the adjoint action of  $U(\mathfrak{r})$ , denoted by  $ad$ , which extends uniquely by derivation the adjoint action of  $\mathfrak{r}$  on  $\mathfrak{p}^-$  given by Lie bracket. By [4, 2.4.10] the map  $\theta$  is an isomorphism of  $ad U(\mathfrak{r})$ -modules. For all  $k \in \mathbb{N}$ , set  $U^k(\mathfrak{m}^-) = \theta(S_k(\mathfrak{m}^-))$ . Then  $U^k(\mathfrak{m}^-)$  is a left  $ad U(\mathfrak{r})$ -submodule of  $U_k(\mathfrak{m}^-)$  and actually one has that  $U_k(\mathfrak{m}^-) = U^k(\mathfrak{m}^-) \oplus U_{k-1}(\mathfrak{m}^-)$  by [4, 2.4.4, 2.4.5]. Denote by  $pr_{U^k(\mathfrak{m}^-)}$  the projection onto  $U^k(\mathfrak{m}^-)$  with respect to the above decomposition. We have the following.

**Lemma.** *Let  $\lambda \in P^+(\pi)$  and  $k \in \mathbb{N}$ .*

(i) *Equations (4) and (5) extend to a left  $U(\tilde{\mathfrak{p}}^-)$ -action on the vector space  $\tilde{V}(\lambda)$  and for this structure we have the following equalities :*

$$(6) \quad \tilde{V}'(\lambda) = U(\mathfrak{r}).\tilde{v}_\lambda$$

$$(7) \quad \tilde{V}(\lambda) = U(\tilde{\mathfrak{p}}^-).\tilde{v}_\lambda = U(\tilde{\mathfrak{n}}^-).\tilde{v}_\lambda = S(\mathfrak{m}^-).\tilde{V}'(\lambda) = U(\tilde{\mathfrak{p}}^-).\tilde{V}'(\lambda).$$

(ii) *For all  $s \in S_k(\mathfrak{m}^-)$ ,  $u' \in U(\mathfrak{r})$  and  $u \in U_k(\mathfrak{m}^-)$  one has :*

$$(8) \quad su' . \tilde{v}_\lambda = gr_k(\theta(s)u' . v_\lambda)$$

$$(9) \quad gr_k(uu' . v_\lambda) = gr_k(pr_{U^k(\mathfrak{m}^-)}(u)u' . v_\lambda)$$

$$(10) \quad gr_k(V(\lambda)) = S_k(\mathfrak{m}^-).\tilde{V}'(\lambda).$$

(iii) *The map  $\beta_\lambda$  is an isomorphism of  $U(\mathfrak{r})$ -modules between  $\tilde{V}(\lambda)$  and  $V(\lambda)$ . Then  $\tilde{V}'(\lambda)$  is a left irreducible  $U(\mathfrak{r})$ -module and  $\tilde{V}(\lambda)$  has the same set of weights as  $V(\lambda)$ , especially  $\lambda$  is the highest weight of  $\tilde{V}(\lambda)$  and  $w_0\lambda$  is its lowest weight.*

(iv) *One may choose  $\mathcal{F}^k(V(\lambda))$  to be included in  $U^k(\mathfrak{m}^-).V'(\lambda)$ .*

*Proof.* By [4, 2.1.1] and (3) of subsection 2.2, one may observe that the algebra  $U(\tilde{\mathfrak{p}}^-)$  is the quotient of the tensor algebra  $T(\tilde{\mathfrak{p}}^-) = T(\mathfrak{p}^-)$  of the vector space  $\tilde{\mathfrak{p}}^- = \mathfrak{p}^-$  by the two-sided ideal generated by the set

$$\{z \otimes z' - z' \otimes z - [z, z'], z \otimes y - y \otimes z - [z, y], y \otimes y' - y' \otimes y; z, z' \in \mathfrak{r}, y, y' \in \mathfrak{m}^-\}$$

and that, by the Poincaré-Birkhoff-Witt theorem [4, 2.1.11], the multiplication is an isomorphism between the  $\mathbb{k}$ -vector spaces  $U(\mathfrak{r}) \otimes S(\mathfrak{m}^-)$  and  $U(\tilde{\mathfrak{p}}^-)$ .

Fix  $k \in \mathbb{N}$ . For all  $x \in \mathfrak{m}^- \oplus \mathfrak{r} = \mathfrak{p}^-$ , denote by  $x.\mathcal{F}_k(V(\lambda))$  the vector subspace of  $V(\lambda)$  formed by all the vectors  $x.v$ , with  $v \in \mathcal{F}_k(V(\lambda))$  (where  $x.v$  denotes the action of  $x$  on  $v$  by the left  $U(\mathfrak{g})$ -module structure on  $V(\lambda)$ ).

Then for all  $y \in \mathfrak{m}^-$ , one has that  $y.\mathcal{F}_k(V(\lambda)) \subset \mathcal{F}_{k+1}(V(\lambda))$ , and for all  $z \in \mathfrak{r}$ , one has that  $z.\mathcal{F}_k(V(\lambda)) \subset \mathcal{F}_k(V(\lambda))$ . It follows that equation (4) extends to a left action of  $S(\mathfrak{m}^-)$  on  $\tilde{V}(\lambda)$  since moreover, for  $y, y' \in \mathfrak{m}^-$  and  $v \in \mathcal{F}_k(V(\lambda))$ , we have:

$$y.(y' . (gr_k(v))) - y' . (y . gr_k(v)) = gr_{k+2}((yy' - y'y) . v) = gr_{k+2}([y, y'] . v) = 0.$$

Similarly equation (5) extends to a left action of  $U(\mathfrak{r})$  on  $\widetilde{V}(\lambda)$  induced by the left action of  $U(\mathfrak{r})$  on  $V(\lambda)$ . Finally both equations (4) and (5) extend to a left action of  $U(\widetilde{\mathfrak{p}}^-)$  on  $\widetilde{V}(\lambda)$  (by say, [4, 2.2.1, 2.2.2]). Equation (6) follows since  $\widetilde{V}'(\lambda) = gr_0(V'(\lambda)) = gr_0(U(\mathfrak{r}).v_\lambda)$ .

Let  $\tilde{v} \in \widetilde{V}(\lambda)$ . There exists  $k \in \mathbb{N}$  and  $v_i \in \mathcal{F}_i(V(\lambda))$ , for  $0 \leq i \leq k$ , such that  $\tilde{v} = \sum_{i=0}^k gr_i(v_i)$  with, for all  $i$ ,  $v_i = \sum_{j=1}^{n_i} u'_{ij} u_{ij}.v_\lambda$  where  $u'_{ij} \in U(\mathfrak{r})$  and  $u_{ij} \in U_i(\mathfrak{m}^-)$ . Then by equation (5), one has

$$gr_i(v_i) = \sum_{j=1}^{n_i} u'_{ij}.gr_i(u_{ij}.v_\lambda)$$

and by equation (4),

$$gr_i(u_{ij}.v_\lambda) \in S_i(\mathfrak{m}^-).gr_0(v_\lambda).$$

Actually we may take the  $u'_{ij}$  in  $U(\mathfrak{n}_{\pi'}^-)$ , since

$$V(\lambda) = U(\mathfrak{n}^-).v_\lambda = U(\mathfrak{n}_{\pi'}^-).(U(\mathfrak{m}^-).v_\lambda).$$

We then have  $\widetilde{V}(\lambda) = U(\widetilde{\mathfrak{p}}^-).\tilde{v}_\lambda = U(\widetilde{\mathfrak{n}}^-).\tilde{v}_\lambda$ . Since  $\widetilde{V}'(\lambda) = U(\mathfrak{r}).\tilde{v}_\lambda$  and since the multiplication gives the isomorphism of vector spaces  $U(\widetilde{\mathfrak{p}}^-) \simeq S(\mathfrak{m}^-) \otimes U(\mathfrak{r})$ , we also have that  $\widetilde{V}'(\lambda) = S(\mathfrak{m}^-).\widetilde{V}'(\lambda) = U(\widetilde{\mathfrak{p}}^-).\widetilde{V}'(\lambda)$ . Hence equation (7).

Let  $k \in \mathbb{N}^*$  and set  $s = y_1 \cdots y_k \in S_k(\mathfrak{m}^-)$  with  $y_i \in \mathfrak{m}^-$  for all  $1 \leq i \leq k$ . Then  $\theta(s) = y_1 \cdots y_k + u \in U^k(\mathfrak{m}^-)$  with  $u \in U_{k-1}(\mathfrak{m}^-)$  and  $y_1 \cdots y_k \in U_k(\mathfrak{m}^-)$ . Then equations (4) and (5) and the fact that  $U_{k-1}(\mathfrak{m}^-)U(\mathfrak{r}).v_\lambda = \mathcal{F}_{k-1}(V(\lambda))$  give equation (8). Equation (9) is obvious by the decomposition  $U_k(\mathfrak{m}^-) = U^k(\mathfrak{m}^-) \oplus U_{k-1}(\mathfrak{m}^-)$ . Both equations imply equation (10).

By equation (5), we have that  $gr_k(V(\lambda))$  is an  $U(\mathfrak{r})$ -module. Moreover if  $\tilde{v} \in gr_k(V(\lambda))$  is such that  $\tilde{v} = gr_k(v_k)$  with  $v_k \in \mathcal{F}^k(V(\lambda))$ , we have that  $\beta_\lambda^k(\tilde{v}) = v_k$ . Let  $z \in \mathfrak{r}$ . Then by equation (5), one has that  $z.\tilde{v} = gr_k(z.v_k)$  which implies that

$$\beta_\lambda^k(z.\tilde{v}) = z.v_k = z.\beta_\lambda^k(\tilde{v}),$$

since  $z.v_k \in \mathcal{F}^k(V(\lambda))$  because  $\mathcal{F}^k(V(\lambda))$  is an  $U(\mathfrak{r})$ -module. This shows (iii). Finally to prove (iv) it suffices to observe that  $U^k(\mathfrak{m}^-).V'(\lambda)$  is a finite dimensional  $U(\mathfrak{r})$ -module. Set  $W_k = U^k(\mathfrak{m}^-).V'(\lambda) \cap \mathcal{F}_{k-1}(V(\lambda))$ . Then  $W_k$  is a left  $U(\mathfrak{r})$ -submodule of  $U^k(\mathfrak{m}^-).V'(\lambda)$  and then there exists a left  $U(\mathfrak{r})$ -submodule  $W'_k$  such that  $U^k(\mathfrak{m}^-).V'(\lambda) = W_k \oplus W'_k$ . Now  $W'_k \cap \mathcal{F}_{k-1}(V(\lambda)) = \{0\}$  and then one may choose the  $U(\mathfrak{r})$ -module  $\mathcal{F}^k(V(\lambda))$  to contain  $W'_k$ . But  $\mathcal{F}_k(V(\lambda)) = \mathcal{F}_{k-1}(V(\lambda)) \oplus \mathcal{F}^k(V(\lambda)) = U_k(\mathfrak{m}^-).V'(\lambda) \subset U^k(\mathfrak{m}^-).V'(\lambda) + U_{k-1}(\mathfrak{m}^-).V'(\lambda)$  since  $U_k(\mathfrak{m}^-) = U^k(\mathfrak{m}^-) \oplus U_{k-1}(\mathfrak{m}^-)$ . It follows that  $W'_k = \mathcal{F}^k(V(\lambda))$ , which completes the proof.  $\square$

**3.3. Another structure of module on the degenerate highest weight module.** Recall the isomorphism  $\beta_\lambda$  of  $U(\mathfrak{r})$ -modules from  $\widetilde{V}(\lambda)$  into  $V(\lambda)$

(lemma 3.2 (iii)). For all  $\tilde{v} \in \tilde{V}(\lambda)$  and all  $x \in \mathfrak{p}$ , one sets

$$(11) \quad \rho_\lambda(x)(\tilde{v}) = \beta_\lambda^{-1}(x.\beta_\lambda(\tilde{v}))$$

where  $x.\beta_\lambda(\tilde{v})$  stands for the left action of  $x \in \mathfrak{g}$  on  $\beta_\lambda(\tilde{v}) \in V(\lambda)$ .

It is easily checked that  $\rho_\lambda$  is a morphism of Lie algebras from  $\mathfrak{p}$  to  $\mathfrak{gl}(\tilde{V}(\lambda))$ , hence that it extends to a left action of  $U(\mathfrak{p})$  on  $\tilde{V}(\lambda)$  (again by [4, 2.2.1, 2.2.2]). Moreover for all  $x \in \mathfrak{r}$  and  $\tilde{v} \in \tilde{V}(\lambda)$ , since  $\beta_\lambda$  is a morphism of  $U(\mathfrak{r})$ -modules, one has that  $\rho_\lambda(x)(\tilde{v}) = x.\tilde{v}$  where the right hand side denotes the left action of  $\mathfrak{r}$  on  $\tilde{V}(\lambda)$  defined in subsection 3.2.

**Remark.** By [10, 2.7], one has that  $V'(\lambda) = \{v \in V(\lambda) \mid \mathfrak{m}.v = 0\}$ . Hence

$$(12) \quad \tilde{V}'(\lambda) = \{\tilde{v} \in \tilde{V}(\lambda) \mid \rho_\lambda(\mathfrak{m})(\tilde{v}) = 0\}$$

since  $\tilde{V}'(\lambda) = \beta_\lambda^{-1}(V'(\lambda))$ .

#### 4. ACTION OF A SMASH PRODUCT.

In this section, we will define a smash product  $A = T(\mathfrak{m})\#U(\mathfrak{r})$ , containing the enveloping algebra  $U(\mathfrak{r})$  and the tensor algebra  $T(\mathfrak{m})$ , where the action of  $U(\mathfrak{r})$  on  $T(\mathfrak{m})$  derives from the adjoint action of  $\mathfrak{r}$  in  $T(\mathfrak{m})$  which extends uniquely by derivation the adjoint action given by Lie bracket. This algebra  $A$  is an associative algebra, which is actually a Hopf algebra. We will define what we call a generalized adjoint action (denoted by  $ad^{**}$ ) of the algebra  $A$  on the enveloping algebra  $U(\tilde{\mathfrak{p}}^-)$  and another left action of  $A$  on  $U(\tilde{\mathfrak{p}}^-)$ , where the latter is simply left multiplication when restricted to  $U(\mathfrak{r})$ . The action  $ad^{**}$  derives from the coadjoint action, denoted by  $ad^*$ , of  $\tilde{\mathfrak{p}}$  on  $\mathfrak{p}^-$  (note that, as vector spaces, one has  $\tilde{\mathfrak{p}}^* \simeq \mathfrak{p}^-$ ). We will see in subsection 7.3 why we need to take this coadjoint action  $ad^*$ .

**4.1. A smash product.** Recall that  $\mathfrak{m}$  denotes the nilpotent radical of  $\mathfrak{p}$  and that  $T(\mathfrak{m})$  denotes the tensor algebra of  $\mathfrak{m}$ . Since  $[\mathfrak{r}, \mathfrak{m}] \subset \mathfrak{m}$ , the algebra  $T(\mathfrak{m})$  is an  $U(\mathfrak{r})$ -algebra (in the sense of [17, 1.1.6]) with the adjoint action of  $\mathfrak{r}$  on  $T(\mathfrak{m})$  (denoted by  $ad$ ) extending by derivation the adjoint action of  $\mathfrak{r}$  on  $\mathfrak{m}$  given by the Lie bracket in  $\mathfrak{g}$ . Then we may consider the Hopf smash product  $A = T(\mathfrak{m})\#U(\mathfrak{r})$  in the sense of [17, 1.1.8]. More precisely  $A$  is equal as a vector space to the tensor product  $T(\mathfrak{m}) \otimes U(\mathfrak{r})$ , with multiplication given by  $(s \otimes u)(s' \otimes u') = s ad u_1(s') \otimes u_2 u'$  where  $\Delta(u) = u_1 \otimes u_2$  (Sweedler notation),  $\Delta$  being the coproduct in  $U(\mathfrak{r})$ ,  $s, s' \in T(\mathfrak{m})$  and  $u, u' \in U(\mathfrak{r})$ .

For example for all  $z \in \mathfrak{r}$ ,  $s, s' \in T(\mathfrak{m})$  and  $u \in U(\mathfrak{r})$ , one has that  $(s' \otimes z)(s \otimes u) = s' ad z(s) \otimes u + s' s \otimes zu$ . By setting  $s \otimes 1 = s$  and  $1 \otimes u = u$  we may view  $T(\mathfrak{m})$  and  $U(\mathfrak{r})$  as subalgebras of  $A$ . Then one has in  $A$  that  $s \otimes u = (s \otimes 1)(1 \otimes u) = su$  and that

$$(13) \quad \forall z \in \mathfrak{r}, \forall s \in T(\mathfrak{m}), ad z(s) = zs - sz$$

and in particular

$$(14) \quad \forall z \in \mathfrak{r}, \forall x \in \mathfrak{m}, [z, x] = zx - xz.$$

Observe that  $A$  is an associative unitary algebra (see [17, 1.1.8]) which is also a bialgebra thanks to the coproducts in  $T(\mathfrak{m})$  and in  $U(\mathfrak{r})$ . More precisely denoting also by  $\Delta$  the coproduct in  $T(\mathfrak{m})$ , and by  $\Delta_A$  the coproduct in  $A$ , we set for  $s \in T(\mathfrak{m})$  and  $u \in U(\mathfrak{r})$ ,  $\Delta_A(s \otimes u) = (s_1 \otimes u_1) \otimes (s_2 \otimes u_2)$  if  $\Delta(s) = s_1 \otimes s_2$  and  $\Delta(u) = u_1 \otimes u_2$  with Sweedler notation. We then have that  $\Delta_A((s \otimes 1)(1 \otimes u)) = \Delta_A(s \otimes 1)\Delta_A(1 \otimes u)$  and more generally for  $s, s' \in T(\mathfrak{m})$  and  $u, u' \in U(\mathfrak{r})$ ,  $\Delta_A((s \otimes u)(s' \otimes u')) = \Delta_A(s \otimes u)\Delta_A(s' \otimes u')$  by the cocommutativity of  $\Delta$ . Note that the coproduct  $\Delta_A$  extends the coproduct  $\Delta$  in  $T(\mathfrak{m})$  and in  $U(\mathfrak{r})$ . Actually the bialgebra  $A$  is a Hopf algebra with the coidentity  $\varepsilon$  given by  $\varepsilon(x) = 0$  for all  $x \in \mathfrak{p}$  and the antipode given by  $a \in A \mapsto a^\top \in A$ , where

$$(15) \quad a^\top = (-1)^r x_r \cdots x_1 \in A$$

if  $a = x_1 \cdots x_r \in A$  (product in  $A$ ) with  $x_1, \dots, x_r \in \mathfrak{p}$  extended by linearity to every element in  $A$ . One checks easily that the coidentity and the antipode (which coincide respectively with the coidentity and the antipode on  $T(\mathfrak{m})$  and on  $U(\mathfrak{r})$ , see for instance [17, 1.2.5]) are compatible with equation (14) which defines the smash product  $A$ .

Roughly speaking, the Hopf algebra  $A$  coincides with the enveloping algebra  $U(\mathfrak{p})$  or even  $U(\tilde{\mathfrak{p}})$ , except that no relations are required for the associative product of elements in  $\mathfrak{m}$ .

**4.2. Coadjoint action.** Recall the opposite parabolic subalgebra  $\mathfrak{p}^-$  of  $\mathfrak{p}$ . Thanks to the Killing form on  $\mathfrak{g}$ , we have the isomorphism of vector spaces  $\tilde{\mathfrak{p}}^* \simeq \mathfrak{p}^-$ . As it was already mentioned in [18, 2],  $\mathfrak{p}^-$  is a  $\tilde{\mathfrak{p}}$ -module, by the so-called coadjoint representation (denoted by  $ad^*$ ) of  $\tilde{\mathfrak{p}} = \mathfrak{r} \ltimes (\mathfrak{m})^a$  in  $\mathfrak{p}^-$  defined as follows.

$$(16) \quad \forall x \in \mathfrak{r}, \forall y \in \mathfrak{p}^-, ad^*x(y) = [x, y].$$

$$(17) \quad \forall x \in \mathfrak{m}, \forall y \in \mathfrak{p}^-, ad^*x(y) = pr_{\mathfrak{r}}([x, y])$$

where  $pr_{\mathfrak{r}}$  is the projection of  $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{m} \oplus \mathfrak{m}^-$  onto  $\mathfrak{r}$ . In particular

$$(18) \quad \forall x \in \mathfrak{m}, \forall y \in \mathfrak{r}, ad^*x(y) = 0.$$

**Lemma.** *The map  $ad^* : \tilde{\mathfrak{p}} \rightarrow \mathfrak{gl}(\mathfrak{p}^-)$  is a morphism between the Lie algebras  $\tilde{\mathfrak{p}}$  and  $\mathfrak{gl}(\mathfrak{p}^-)$ . In other words it gives a representation of  $\tilde{\mathfrak{p}}$  in  $\mathfrak{p}^-$ , which extends uniquely to a representation of  $U(\tilde{\mathfrak{p}})$  in  $\mathfrak{p}^-$ . This representation also extends uniquely by derivation to a representation of  $U(\tilde{\mathfrak{p}})$  in the symmetric algebra  $S(\mathfrak{p}^-)$ , which we still denote by  $ad^*$ .*

*Proof.* We give a proof of the lemma for the reader's convenience. It suffices to prove that, for all  $x, x' \in \tilde{\mathfrak{p}}$ , and for all  $y \in \mathfrak{p}^-$ , we have

$$(19) \quad (ad^*x \circ ad^*x')(y) - (ad^*x' \circ ad^*x)(y) - ad^*[x, x']_{\tilde{\mathfrak{p}}}(y) = 0.$$

Assume that  $x, x' \in \mathfrak{m}$ . Then  $[x, x']_{\tilde{\mathfrak{p}}} = 0$  by equation (2) in subsection 2.2. Moreover for all  $y \in \mathfrak{p}^-$ ,

$$(ad^*x \circ ad^*x')(y) = ad^*x(pr_{\mathfrak{r}}([x', y])) = 0$$

by (17) and (18). Then equality (19) follows in this case.

Assume that  $x, x' \in \mathfrak{r}$ . Then equality (19) follows from equations (2) and (16).

It remains to prove equality (19) for  $x \in \mathfrak{m}$  and  $x' \in \mathfrak{r}$ . By equations (2), (16) and (17) one has that, for all  $y \in \mathfrak{p}^-$ ,

$$\begin{aligned} & (ad^*x \circ ad^*x')(y) - (ad^*x' \circ ad^*x)(y) - ad^*[x, x']_{\tilde{\mathfrak{p}}}(y) \\ &= ad^*x([x', y]) - ad^*x'(pr_{\mathfrak{r}}([x, y])) - pr_{\mathfrak{r}}([[x, x'], y]) \\ &= pr_{\mathfrak{r}}([x, [x', y]]) - [x', pr_{\mathfrak{r}}([x, y])] - pr_{\mathfrak{r}}([[x, x'], y]). \end{aligned}$$

Denote by  $pr_{\mathfrak{m}}$ , resp.  $pr_{\mathfrak{m}^-}$ , the projection of  $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{m} \oplus \mathfrak{m}^-$  onto  $\mathfrak{m}$ , resp. onto  $\mathfrak{m}^-$ .

Then

$$(20) \quad [x', pr_{\mathfrak{r}}([x, y])] = [x', [x, y] - pr_{\mathfrak{m}^-}([x, y]) - pr_{\mathfrak{m}}([x, y])]$$

and we have

$$(21) \quad [x', pr_{\mathfrak{m}^-}([x, y])] \in \mathfrak{m}^-, [x', pr_{\mathfrak{m}}([x, y])] \in \mathfrak{m}, [x', pr_{\mathfrak{r}}([x, y])] \in \mathfrak{r}.$$

Then by (20) and (21) we have that

$$(22) \quad [x', pr_{\mathfrak{r}}([x, y])] = pr_{\mathfrak{r}}([x', [x, y]]).$$

It follows by (22) that

$$\begin{aligned} & (ad^*x \circ ad^*x')(y) - (ad^*x' \circ ad^*x)(y) - ad^*[x, x']_{\tilde{\mathfrak{p}}}(y) \\ &= pr_{\mathfrak{r}}([x, [x', y]] - [x', [x, y]] - [[x, x'], y]) = 0 \end{aligned}$$

by Jacobi identity in  $\mathfrak{g}$ .

Applying [4, 2.2.1] and [4, 1.2.14] completes the proof of the lemma.  $\square$

**4.3. Partial symmetrisation.** Recall the symmetrisation  $\theta : S(\mathfrak{p}^-) \rightarrow U(\mathfrak{p}^-)$  which is an isomorphism of  $adU(\mathfrak{r})$ -modules, when  $S(\mathfrak{p}^-)$  and  $U(\mathfrak{p}^-)$  are endowed with the adjoint action  $ad$  (see 3.2). Denote also by  $ad$  the adjoint action of  $U(\mathfrak{r})$  on  $U(\tilde{\mathfrak{p}}^-)$ , extending by derivation the Lie bracket of  $\mathfrak{r}$  on  $\tilde{\mathfrak{p}}^-$  (see equation (3)).

Set  $\tilde{\theta} : S(\mathfrak{p}^-) \simeq S(\mathfrak{m}^-) \otimes S(\mathfrak{r}) \rightarrow U(\tilde{\mathfrak{p}}^-) \simeq S(\mathfrak{m}^-) \otimes U(\mathfrak{r})$

defined by the following.

$$(23) \quad \forall s \in S(\mathfrak{m}^-), \forall s' \in S(\mathfrak{r}), \tilde{\theta}(ss') = s\theta(s').$$

We call the map  $\tilde{\theta}$  a *partial symmetrisation*. Observe that  $\tilde{\theta}$  does not coincide with the symmetrisation  $\tilde{\tilde{\theta}}$  of  $S(\tilde{\mathfrak{p}}^-) = S(\mathfrak{p}^-)$  in  $U(\tilde{\mathfrak{p}}^-)$ . For instance, for  $y \in \mathfrak{m}^-$  and  $z \in \mathfrak{r}$ , one has that  $\tilde{\theta}(yz) = yz$ , while  $\tilde{\tilde{\theta}}(yz) = \frac{1}{2}(yz + zy) = yz + \frac{1}{2}[z, y]$ .



**Lemma.** *The map  $\tilde{\theta}$  is an isomorphism of  $adU(\mathfrak{r})$ -modules, when  $S(\mathfrak{p}^-)$  and  $U(\tilde{\mathfrak{p}}^-)$  are endowed with the adjoint action.*

*Proof.* Since  $Id_{S(\mathfrak{m}^-)}$  and  $\theta|_{S(\mathfrak{r})} : S(\mathfrak{r}) \rightarrow U(\mathfrak{r})$  are isomorphisms, it follows that  $\tilde{\theta} = Id_{S(\mathfrak{m}^-)} \otimes \theta|_{S(\mathfrak{r})}$  is an isomorphism too.

Let  $z \in \mathfrak{r}$ ,  $s \in S(\mathfrak{m}^-)$  and  $s' \in S(\mathfrak{r})$ . Observe that one has that  $adz(s) \in S(\mathfrak{m}^-)$ , and this element may be viewed equally as an element in  $S(\mathfrak{p}^-)$  or in  $U(\tilde{\mathfrak{p}}^-)$ . Moreover  $adz(s') \in S(\mathfrak{r})$  and  $adz(\theta(s')) \in U(\mathfrak{r})$ . Since  $adz$  is a derivation and since  $\theta$  is a morphism of  $adU(\mathfrak{r})$ -modules we have, by equation (23):

$$\begin{aligned} \tilde{\theta}(adz(ss')) &= \tilde{\theta}(adz(s)s' + sadz(s')) \\ &= adz(s)\theta(s') + s\theta(adz(s')) \\ &= adz(s)\theta(s') + sadz(\theta(s')) \\ &= adz(s\theta(s')) \\ &= adz(\tilde{\theta}(s's')) \end{aligned}$$

This proves the lemma.  $\square$

**4.4. Generalized adjoint action.** Recall the isomorphism of vector spaces  $U(\tilde{\mathfrak{p}}^-) \simeq S(\mathfrak{m}^-) \otimes U(\mathfrak{r})$ . Then one has that

$$(24) \quad U(\tilde{\mathfrak{p}}^-) \simeq \bigoplus_{k \in \mathbb{N}} S_k(\mathfrak{m}^-) \otimes U(\mathfrak{r})$$

and for all  $k \in \mathbb{N}$ ,

$$(25) \quad U_k(\tilde{\mathfrak{p}}^-) \simeq \bigoplus_{0 \leq j \leq k} S_j(\mathfrak{m}^-) \otimes U_{k-j}(\mathfrak{r})$$

as vector spaces. Recall also the coadjoint representation of  $\tilde{\mathfrak{p}}$  in the symmetric algebra  $S(\mathfrak{p}^-)$ , which we have denoted by  $ad^*$  (subsection 4.2). Fix  $k$  and  $j$  in  $\mathbb{N}$  and set  $S_{-1}(\mathfrak{m}^-) = \{0\}$ . One has by equation (17) that

$$(26) \quad \forall x \in \mathfrak{m}, \forall s \in S_k(\mathfrak{m}^-), ad^*x(s) \in S_{k-1}(\mathfrak{m}^-)\mathfrak{r} \subset S_k(\mathfrak{p}^-).$$

Then one has that

$$(27) \quad \forall x \in \mathfrak{m}, \forall s \in S_k(\mathfrak{m}^-), \forall u' \in U_j(\mathfrak{r}), \\ \tilde{\theta}(ad^*x(s))u' \in S_{k-1}(\mathfrak{m}^-)U_{j+1}(\mathfrak{r}) \subset U_{k+j}(\tilde{\mathfrak{p}}^-)$$

and that

$$(28) \quad \forall z \in \mathfrak{r}, \forall s \in S_k(\mathfrak{m}^-), \forall u' \in U(\mathfrak{r}), adz(su') \in S_k(\mathfrak{m}^-)U(\mathfrak{r}) \subset U(\tilde{\mathfrak{p}}^-).$$

We set

$$(29) \quad \forall x \in \mathfrak{m}, \forall s \in S_k(\mathfrak{m}^-), \forall u' \in U(\mathfrak{r}), ad^{**}x(su') = \tilde{\theta}(ad^*x(s))u' \in U(\tilde{\mathfrak{p}}^-)$$

and

$$(30) \quad \forall z \in \mathfrak{r}, \forall s \in S_k(\mathfrak{m}^-), \forall u' \in U(\mathfrak{r}), \quad ad^{**}z(su') = adz(su') \in U(\tilde{\mathfrak{p}}^-).$$

Observe that

$$(31) \quad \forall x \in \mathfrak{m}, \forall u' \in U(\mathfrak{r}), \quad ad^{**}x(u') = 0$$

and

$$(32) \quad \forall x \in \mathfrak{m}, \forall s \in S_k(\mathfrak{m}^-), \forall u' \in U(\mathfrak{r}), \quad ad^{**}x(su') = ad^{**}x(s)u'.$$

**Lemma.** *Equations (29) and (30) extend to a left action of  $A$  on the enveloping algebra  $U(\tilde{\mathfrak{p}}^-)$ . We call this action the generalized adjoint action of  $A$  on  $U(\tilde{\mathfrak{p}}^-)$ .*

*Proof.* Since equation (30) is just the adjoint action, it extends to a left action of  $U(\mathfrak{r})$  on  $U(\tilde{\mathfrak{p}}^-)$  by [4, 2.2.1, 2.4.9].

Now consider  $x \in \mathfrak{m}$ . One can extend equation (29) by linearity so that  $ad^{**}x \in \text{End}(U(\tilde{\mathfrak{p}}^-))$ . Let us explain why this is well defined.

Note first that, for  $y \in \mathfrak{m}^-$  and  $z \in \mathfrak{r}$ , one sets

$$ad^{**}x(zy) = ad^{**}x(yz) + ad^{**}x([z, y]).$$

More generally by equation (24) every element in  $U(\tilde{\mathfrak{p}}^-)$  may be written in the form  $\sum_{i \in I} s_i u'_i$  with  $I$  a finite set and for all  $i \in I$ ,  $s_i \in S(\mathfrak{m}^-)$  and  $u'_i \in U(\mathfrak{r})$ , with the  $s_i$ ,  $i \in I$ , linearly independent. Then if such an element is zero, we have that  $u'_i = 0$  for all  $i \in I$ , and then  $ad^{**}x(\sum_{i \in I} s_i u'_i) = \sum_{i \in I} \tilde{\theta}(ad^*x(s_i))u'_i = 0$ .

Moreover for  $s, s' \in S(\mathfrak{m}^-)$ , one has

$$ad^{**}x(ss') = \tilde{\theta}(ad^*x(ss')) = \tilde{\theta}(ad^*x(s's)) = ad^{**}x(s's)$$

since  $ad^*x$  is an endomorphism of  $S(\mathfrak{p}^-)$ . Then  $ad^{**}x$  is well defined on  $U(\tilde{\mathfrak{p}}^-)$ .

Finally  $ad^{**}$  is a linear map from  $\mathfrak{m}$  to  $\text{End}(U(\tilde{\mathfrak{p}}^-))$ , which extends naturally to a  $\mathbb{k}$ -algebras morphism from  $T(\mathfrak{m})$  to  $\text{End}(U(\tilde{\mathfrak{p}}^-))$ . We then obtain left  $U(\mathfrak{r})$  and  $T(\mathfrak{m})$ -module structures on  $U(\tilde{\mathfrak{p}}^-)$ .

It remains to check that both structures imply a left  $A$ -module structure on  $U(\tilde{\mathfrak{p}}^-)$ , that is, are compatible with equation (13), which defines the smash product  $A$ . For this it suffices to prove that

$$(33) \quad \forall x \in \mathfrak{m}, \forall z \in \mathfrak{r}, \quad ad^{**}z \circ ad^{**}x - ad^{**}x \circ ad^{**}z = ad^{**}[z, x].$$

Let  $x \in \mathfrak{m}$ ,  $z \in \mathfrak{r}$ ,  $s \in S(\mathfrak{m}^-)$  and  $u' \in U(\mathfrak{r})$ . Since  $adz$  is a derivation in  $U(\tilde{\mathfrak{p}}^-)$ , one has that

$$\begin{aligned}
ad^{**}(zx - xz)(su') &= (ad^{**}z \circ ad^{**}x - ad^{**}x \circ ad^{**}z)(su') \\
&= adz(\tilde{\theta}(ad^*x(s))u') \\
&\quad - ad^{**}x(adz(s)u' + sadz(u')) \\
&= adz(\tilde{\theta}(ad^*x(s))u') + \tilde{\theta}(ad^*x(s))adz(u') \\
&\quad - \tilde{\theta}(ad^*x(adz(s))u') - \tilde{\theta}(ad^*x(s))adz(u')
\end{aligned}$$

since moreover  $adz(s) \in S(\mathfrak{m}^-)$  and  $adz(u') \in U(\mathfrak{r})$ . Then

$$ad^{**}(zx - xz)(su') = \tilde{\theta}((adz \circ ad^*x)(s))u' - \tilde{\theta}(ad^*x(adz(s)))u'$$

by lemma 4.3 and then

$$ad^{**}(zx - xz)(su') = \tilde{\theta}((ad^*z \circ ad^*x)(s))u' - \tilde{\theta}((ad^*x \circ ad^*z)(s))u'$$

since, for all  $t \in S(\mathfrak{p}^-)$ , one has that  $adz(t) = ad^*z(t)$  by equation (16).

Since

$$ad^*z \circ ad^*x - ad^*x \circ ad^*z = ad^*[z, x]$$

in  $End(S(\mathfrak{p}^-))$  by the proof of Lemma 4.2, the required equation (33) follows.  $\square$

**Remark.** We will see in subsection 4.5 why we call this left action  $ad^{**}$  of  $A$  on  $U(\tilde{\mathfrak{p}}^-)$  a generalized adjoint action.

**4.5. Left and right actions.** Here we will define a right, resp. a left action, of  $A$  on  $U(\tilde{\mathfrak{p}}^-)$  as follows.

$$(34) \quad \forall u' \in U(\mathfrak{r}), \forall u \in U(\tilde{\mathfrak{p}}^-), R(u')(u) = uu'$$

(product in  $U(\tilde{\mathfrak{p}}^-)$ ). Then  $R|_{U(\mathfrak{r})}$  is a right action of  $U(\mathfrak{r})$  on  $U(\tilde{\mathfrak{p}}^-)$  called the regular right action (see [4, 2.2.21]). We extend this right action by setting

$$(35) \quad \forall x \in \mathfrak{m}, R(x) = 0.$$

One checks immediately that the map  $R$  induces a right action of  $A$  on  $U(\tilde{\mathfrak{p}}^-)$  (still denoted by  $R$ ). It follows that the map  $a \in A \mapsto R(a^\top)$  is a left action of  $A$  on  $U(\tilde{\mathfrak{p}}^-)$ .

One also sets:

$$(36) \quad \forall u' \in U(\mathfrak{r}), \forall u \in U(\tilde{\mathfrak{p}}^-), L(u')(u) = u'u$$

(product in  $U(\tilde{\mathfrak{p}}^-)$ ).

Then  $L|_{U(\mathfrak{r})}$  is a left action of  $U(\mathfrak{r})$  on  $U(\tilde{\mathfrak{p}}^-)$  called the regular left action (see [4, 2.2.21]). We extend this left action by setting

$$(37) \quad \forall x \in \mathfrak{m}, L(x) = ad^{**}x$$

(see equation (29)), which extends by the proof of lemma 4.4 to a left action of  $T(\mathfrak{m})$  on  $U(\tilde{\mathfrak{p}}^-)$ .

**Lemma.** *The map  $L$  extends to a left action of  $A$  on  $U(\tilde{\mathfrak{p}}^-)$  (still denoted by  $L$ ). Note that this is not in general a left action of  $U(\mathfrak{p})$  nor of  $U(\tilde{\mathfrak{p}})$  on  $U(\tilde{\mathfrak{p}}^-)$ .*

*Proof.* We have to check that the map  $L$  preserves equation (13) which defines the smash product  $A$  and for this it suffices to check that  $L$  preserves equation (14). In other words we have to check that

$$(38) \quad \forall z \in \mathfrak{r}, \forall x \in \mathfrak{m}, L([z, x]) = L(z) \circ L(x) - L(x) \circ L(z).$$

Let  $x \in \mathfrak{m}$ ,  $z \in \mathfrak{r}$ ,  $s \in S_k(\mathfrak{m}^-)$  and  $u' \in U(\mathfrak{r})$ . One has that

$$\begin{aligned} & (L(z) \circ L(x) - L(x) \circ L(z) - L([z, x]))(su') \\ &= z \operatorname{ad}^{**} x(su') - \operatorname{ad}^{**} x(zsu') - \operatorname{ad}^{**}[z, x](su') \\ &= z \operatorname{ad}^{**} x(su') - (\operatorname{ad}^{**} x \circ \operatorname{ad}^{**} z)(su') - \operatorname{ad}^{**} x(su'z) \\ & \quad - \operatorname{ad}^{**}[z, x](su') \end{aligned}$$

since  $\operatorname{ad}^{**} z(su') = \operatorname{ad} z(su') = zsu' - su'z$  in  $U(\tilde{\mathfrak{p}}^-)$ .

Recall that  $\operatorname{ad}^{**} x(su'z) = \theta(\operatorname{ad}^* x(s))u'z = \operatorname{ad}^{**} x(su')z$ . Then

$$\begin{aligned} & (L(z) \circ L(x) - L(x) \circ L(z) - L([z, x]))(su') \\ &= z \operatorname{ad}^{**} x(su') - (\operatorname{ad}^{**} x \circ \operatorname{ad}^{**} z)(su') - \operatorname{ad}^{**} x(su')z \\ & \quad - \operatorname{ad}^{**}[z, x](su') \\ &= (\operatorname{ad}^{**} z \circ \operatorname{ad}^{**} x)(su') - (\operatorname{ad}^{**} x \circ \operatorname{ad}^{**} z)(su') - \operatorname{ad}^{**}[z, x](su') \end{aligned}$$

since in  $U(\tilde{\mathfrak{p}}^-)$  we have  $(\operatorname{ad}^{**} z \circ \operatorname{ad}^{**} x)(su') = z \operatorname{ad}^{**} x(su') - \operatorname{ad}^{**} x(su')z$ . Now equation (33) gives equation (38).  $\square$

Recall (see [17, 1.3.1] for instance) that adjoint action in a Hopf algebra  $A$  may be expanded by using the right action  $R$  and the left action  $L$  of  $A$  as in the following proposition. Hence we may view  $\operatorname{ad}^{**}$  as a generalized adjoint action (here the Hopf algebra  $A$  does not act on itself but on  $U(\tilde{\mathfrak{p}}^-)$ ).

**Proposition.** *One has the following.*

$$(39) \quad \forall a \in A, \operatorname{ad}^{**} a = L(a_1) \circ R(a_2^\top)$$

where  $\Delta_A(a) = a_1 \otimes a_2$  (with Sweedler notation).

*Proof.* Observe that, as vector spaces, one has

$$A \simeq U(\mathfrak{r}) \oplus \mathfrak{m} \otimes T(\mathfrak{m}) \otimes U(\mathfrak{r}).$$

Let  $a \in U(\mathfrak{r})$ . Then in this case,  $\operatorname{ad}^{**} a = \operatorname{ad} a$  and for all  $u \in U(\tilde{\mathfrak{p}}^-)$  one has  $(L(a_1) \circ R(a_2^\top))(u) = a_1 u a_2^\top$  (product in  $U(\tilde{\mathfrak{p}}^-)$ ). Hence the required equation (39) in this case, by [17, 1.3.1].

Assume now that  $a = uu'$  with  $u \in \mathfrak{m} \otimes T(\mathfrak{m})$  and  $u' \in U(\mathfrak{r})$ . Set  $\Delta_A(u) = u_1 \otimes u_2$  and  $\Delta_A(u') = u'_1 \otimes u'_2$ . We have that  $u'_1, u'_2 \in U(\mathfrak{r})$  and  $u_1, u_2 \in T(\mathfrak{m})$  and  $\Delta_A(a) = u_1 u'_1 \otimes u_2 u'_2$ . Moreover one has that  $\Delta_A(u) = u \otimes 1 + \sum_{i \in I} u_{1i} \otimes$

$u_{2i}$  with for all  $i \in I$ ,  $u_{2i} \in \mathfrak{m} \otimes T(\mathfrak{m})$ . But if  $u_{2i} \in \mathfrak{m} \otimes T(\mathfrak{m})$  then  $R(u_{2i}^\top) = 0$ . It follows that, for all  $v \in U(\tilde{\mathfrak{p}}^-)$ , one has

$$\begin{aligned} (L(a_1) \circ R(a_2^\top))(v) &= (L(uu'_1) \circ R(u'_2{}^\top))(v) \\ &= (L(u) \circ L(u'_1) \circ R(u'_2{}^\top))(v) \\ &= (L(u) \circ \text{ad}^{**}u')(v) \\ &= (\text{ad}^{**}u \circ \text{ad}^{**}u')(v) \\ &= \text{ad}^{**}a(v) \end{aligned}$$

which completes the proof.  $\square$

**Remark.** *Actually one also has that*

$$(40) \quad \forall a \in A, \forall b \in A, R(b) \circ L(a) = L(a) \circ R(b).$$

Indeed if  $a, b \in U(\mathfrak{r})$  or if  $b \in \mathfrak{m} \otimes T(\mathfrak{m}) \otimes U(\mathfrak{r})$  then equation (40) is immediate by the associativity of the product in  $U(\tilde{\mathfrak{p}}^-)$  or because, in the second case, that  $R(b) = 0$ . Finally when  $b \in U(\mathfrak{r})$  and  $a \in \mathfrak{m} \otimes T(\mathfrak{m}) \otimes U(\mathfrak{r})$ , equation (40) follows from equation (32).

## 5. MATRIX COEFFICIENTS.

**5.1. Definitions and further notation.** Let  $\lambda \in P^+(\pi)$ . Here we use the notation and results of subsection 3.2. By Lemma 3.2 the degenerate highest weight module  $\tilde{V}(\lambda)$  is endowed with a left  $U(\tilde{\mathfrak{p}}^-)$ -module structure. Denote by  $\tilde{V}(\lambda)^*$  its dual vector space. Let  $\tilde{v} \in \tilde{V}(\lambda)$ ,  $\tilde{\xi} \in \tilde{V}(\lambda)^*$ ,  $u \in U(\tilde{\mathfrak{p}}^-)$ . Denoting by  $u.\tilde{v}$  the action of  $u$  on  $\tilde{v}$  for this left  $U(\tilde{\mathfrak{p}}^-)$ -module structure on  $\tilde{V}(\lambda)$ , we denote by  $\tilde{\xi}.u$  the right action it implies on  $\tilde{V}(\lambda)^*$ , namely  $(\tilde{\xi}.u)(\tilde{v}) = \tilde{\xi}(u.\tilde{v})$ . Recall  $\tilde{v}_\lambda = gr_0(v_\lambda)$  and the isomorphism  $\beta_\lambda : \tilde{V}(\lambda) \rightarrow V(\lambda)$  which is an isomorphism of left  $U(\mathfrak{r})$ -modules. In particular this isomorphism preserves the weights. Its dual map  ${}^t\beta_\lambda : V(\lambda)^* \rightarrow \tilde{V}(\lambda)^*$  is also an isomorphism of right  $U(\mathfrak{r})$ -modules. By definition of  $\beta_\lambda$ , one has that  $\beta_\lambda(\tilde{v}_\lambda) = v_\lambda$  and then  $\beta_\lambda(\tilde{V}'(\lambda)) = \beta(U(\mathfrak{r}).\tilde{v}_\lambda) = U(\mathfrak{r}).v_\lambda = V'(\lambda)$ .

Recall  $\tilde{\mathfrak{n}}^- = \mathfrak{n}_{\tilde{\pi}}^- \ltimes (\mathfrak{m}^-)^a \subset \tilde{\mathfrak{p}}^-$ . Set  $\tilde{\xi}_\lambda = {}^t\beta_\lambda(\xi_\lambda)$  where  $\xi_\lambda$  is the unique vector in  $V(\lambda)^*$  of (right) weight  $\lambda$  such that  $\xi_\lambda(v_\lambda) = 1$ . Then  $\tilde{\xi}_\lambda$  is the unique vector in  $\tilde{V}(\lambda)^*$  of weight  $\lambda$  such that  $\tilde{\xi}_\lambda(\tilde{v}_\lambda) = 1$  and by weight considerations one has that, for all  $y \in \tilde{\mathfrak{n}}^-$ ,  $\tilde{\xi}_\lambda.y = 0$  since  $\xi_\lambda$  is a vector of lowest weight in  $V(\lambda)^*$ . Recall that  $v_{w_0\lambda}$  is a chosen nonzero vector in  $V(\lambda)$  of weight  $w_0\lambda$  and that it is a lowest weight vector in  $V(\lambda)$ . Then set  $\xi_{w_0\lambda} \in V(\lambda)^*$  the vector of (right) weight  $w_0\lambda$  such that  $\xi_{w_0\lambda}(v_{w_0\lambda}) = 1$  :  $\xi_{w_0\lambda}$  is a highest weight vector in  $V(\lambda)^*$ . Set also  $\tilde{v}_{w_0\lambda} = \beta_\lambda^{-1}(v_{w_0\lambda}) \in \tilde{V}(\lambda)$ . By weight considerations one has that  $y.\tilde{v}_{w_0\lambda} = 0$  for all  $y \in \tilde{\mathfrak{n}}^-$ . Set also  $\tilde{\xi}_{w_0\lambda} = {}^t\beta_\lambda(\xi_{w_0\lambda}) \in \tilde{V}(\lambda)^*$ . Then  $\tilde{\xi}_{w_0\lambda}(\tilde{v}_{w_0\lambda}) = 1$ . Since all vectors in  $V(\lambda)$  of weight  $w_0\lambda$  are proportional, one may observe that there exists  $k_0 \in \mathbb{N}$

such that

$$(41) \quad v_{w_0\lambda} \in \mathcal{F}^{k_0}(V(\lambda)).$$

Then one has that

$$(42) \quad \tilde{v}_{w_0\lambda} = gr_{k_0}(v_{w_0\lambda}).$$

Set  $V''(\lambda) = U(\mathfrak{r}).v_{w_0\lambda} = U(\mathfrak{n}_{\pi'}).v_{w_0\lambda}$  : it is an irreducible  $U(\mathfrak{r})$ -module of lowest weight  $w_0\lambda$ . Setting  $\tilde{V}''(\lambda) = \beta_\lambda^{-1}(V''(\lambda)) \subset \tilde{V}(\lambda)$ , we have that  $\tilde{V}''(\lambda) = U(\mathfrak{r}).\tilde{v}_{w_0\lambda}$ . Then its dual space  $\tilde{V}''(\lambda)^*$  is such that  $\tilde{V}''(\lambda)^* = {}^t\beta_\lambda(V''(\lambda)^*) = \tilde{\xi}_{w_0\lambda}.U(\mathfrak{n}_{\pi'}^-) = \tilde{\xi}_{w_0\lambda}.U(\mathfrak{r}) \subset \tilde{V}(\lambda)^*$ .

Since  $U(\tilde{\mathfrak{p}}^-)$  is a representation in  $\tilde{V}(\lambda)$  by (i) of Lemma 3.2, we may consider by [4, 2.7.8] the space  $C(\tilde{V}(\lambda))$  of matrix coefficients of  $\tilde{V}(\lambda)$  which is the  $\mathbb{k}$ -vector subspace of  $U(\tilde{\mathfrak{p}}^-)^*$  generated by the linear forms  $c_{\xi, v}^\lambda$  or simply  $c_{\xi, v}$  defined by

$$c_{\xi, v} : u \in U(\tilde{\mathfrak{p}}^-) \mapsto \xi(u.v) \in \mathbb{k}$$

for all  $\xi \in \tilde{V}(\lambda)^*$  and  $v \in \tilde{V}(\lambda)$ . By equation (7) of Lemma 3.2, we may also define the  $\mathbb{k}$ -vector subspace of  $C(\tilde{V}(\lambda))$  generated by the matrix coefficients  $c_{\xi, v'}$  with  $\xi \in \tilde{V}(\lambda)^*$  and  $v' \in \tilde{V}'(\lambda) \subset \tilde{V}(\lambda)$ , which we will denote by  $\tilde{C}_{\mathfrak{p}}(\lambda)$ .

Finally denote by  $\tilde{C}_{\mathfrak{r}}(\lambda)$  the subspace of  $\tilde{C}_{\mathfrak{p}}(\lambda)$  generated by the matrix coefficients  $c_{\xi, v}$  where  $\xi \in \tilde{V}''(\lambda)^*$  and  $v \in \tilde{V}'(\lambda)$ .

## 5.2. Tensor decomposition.

**Lemma.** *Let  $\lambda, \mu \in P^+(\pi)$ . Then  $\tilde{V}'(\lambda) \otimes \tilde{V}'(\mu)$ , resp.  $\tilde{V}''(\lambda)^* \otimes \tilde{V}''(\mu)^*$ , is a direct sum of some copies of  $\tilde{V}'(\nu)$ , resp.  $\tilde{V}''(\nu)^*$ , for  $\nu \in P^+(\pi)$ . Each of them contains the unique copy of  $\tilde{V}'(\lambda + \mu)$ , resp. of  $\tilde{V}''(\lambda + \mu)^*$ .*

*Proof.* The proof is similar as the proof of [9, lemma 2.2]. We give it for the reader's convenience. Let  $\nu$  be an  $\mathfrak{h}$ -weight of  $\tilde{V}'(\lambda)$ . Then  $\nu \in \lambda - \mathbb{N}\pi'$ . Since  $\lambda \in P^+(\pi)$  we have that, for all  $\alpha \in \pi \setminus \pi'$ ,  $\langle \alpha, \nu \rangle \in \mathbb{N}$ . Moreover every vector in  $\tilde{V}'(\lambda)$  is annihilated by  $\rho_\lambda(\mathfrak{m})$  by remark 12. Since  $\mathfrak{r}'$  is a semisimple Lie algebra, the finite dimensional  $U(\mathfrak{r}')$ -module (for diagonal action)  $\tilde{V}'(\lambda) \otimes \tilde{V}'(\mu)$  decomposes into a direct sum of irreducible  $U(\mathfrak{r}')$ -modules, each of them being generated by a highest weight nonzero vector whose  $\mathfrak{h}$ -weight actually belongs to  $P^+(\pi)$  : this highest weight nonzero vector  $\tilde{v}$  is indeed such that  $(\rho_\lambda \otimes \rho_\mu)(x)(\tilde{v}) = 0$  for all  $x \in \mathfrak{n}$ , where  $\rho_\lambda \otimes \rho_\mu$  is the tensor product of the representations  $\rho_\lambda$  and  $\rho_\mu$  as defined for instance in [4, 1.2.14] and its  $\mathfrak{h}$ -weight belongs to  $\lambda + \mu - \mathbb{N}\pi'$ . Thus the tensor product  $\tilde{V}'(\lambda) \otimes \tilde{V}'(\mu)$  is a direct sum of some copies of  $\tilde{V}'(\nu)$ , for  $\nu \in P^+(\pi)$  such that  $\nu \in \lambda + \mu - \mathbb{N}\pi'$ . Moreover  $U(\mathfrak{r}).(\tilde{v}_\lambda \otimes \tilde{v}_\mu)$  is the unique copy of  $\tilde{V}'(\lambda + \mu)$  which occurs in this tensor product.

Observe that  $\tilde{V}(\lambda)^*$  may be viewed as a left  $U(\mathfrak{r})$ -module, by setting for all  $u \in U(\mathfrak{r})$ , for all  $\xi \in \tilde{V}(\lambda)^*$ ,  $u.\xi = \xi.u^\top$ . Then  $\tilde{V}''(\lambda)^* \simeq \tilde{V}'(-w_0\lambda)$  as

left  $U(\mathfrak{r})$ -modules. Then one obtains similarly the second part of the lemma,  $\widetilde{V}''(\lambda)^* \otimes \widetilde{V}''(\mu)^*$  being a direct sum of some copies of  $\widetilde{V}''(\nu)^*$ , with  $\nu \in P^+(\pi)$  such that  $w_0\nu \in w_0\lambda + w_0\mu + \mathbb{N}\pi'$ . Finally  $(\xi_{w_0\lambda} \otimes \xi_{w_0\mu}) \cdot U(\mathfrak{r})$  is the unique copy of  $\widetilde{V}''(\lambda + \mu)^*$  which occurs in the tensor product  $\widetilde{V}''(\lambda)^* \otimes \widetilde{V}''(\mu)^*$ .  $\square$

**5.3. Direct sums.** Recall that the dual vector space  $U(\tilde{\mathfrak{p}}^-)^*$  of  $U(\tilde{\mathfrak{p}}^-)$  is an associative algebra with product given by the dual map of the coproduct in  $U(\tilde{\mathfrak{p}}^-)$  (see for instance [4, 2.7.4]).

**Lemma.** *The sum  $\widetilde{C}_{\mathfrak{p}} = \sum_{\lambda \in P^+(\pi)} \widetilde{C}_{\mathfrak{p}}(\lambda)$  is a direct sum. The same holds for  $\widetilde{C}_{\mathfrak{r}} = \sum_{\lambda \in P^+(\pi)} \widetilde{C}_{\mathfrak{r}}(\lambda)$ . Moreover the latter is a subalgebra of  $U(\tilde{\mathfrak{p}}^-)^*$ .*

*Proof.* Let  $\lambda \in P^+(\pi)$  and for all  $\xi \in \widetilde{V}(\lambda)^*$ , denote by  $h_\xi : \widetilde{V}'(\lambda) \rightarrow U(\tilde{\mathfrak{p}}^-)^*$  the map such that  $h_\xi(v) = c_{\xi, v}$  for all  $v \in \widetilde{V}'(\lambda)$ . For all  $u' \in U(\mathfrak{r})$ , recall  $R(u')$  the (regular) right action of  $u'$  on  $U(\tilde{\mathfrak{p}}^-)$  defined by  $R(u')(u) = uu'$  for all  $u \in U(\tilde{\mathfrak{p}}^-)$ , where  $uu'$  is the product in  $U(\tilde{\mathfrak{p}}^-)$  (see subsection 4.5). Then its dual map  ${}^tR(u') : U(\tilde{\mathfrak{p}}^-)^* \rightarrow U(\tilde{\mathfrak{p}}^-)^*$  defines a left action on  $U(\tilde{\mathfrak{p}}^-)^*$ , called the coregular right representation of  $U(\mathfrak{r})$  on  $U(\tilde{\mathfrak{p}}^-)^*$  (see [4, 2.7.7]). One sees easily that  $h_\xi$  is a morphism of  $U(\mathfrak{r})$ -modules, when  $U(\tilde{\mathfrak{p}}^-)^*$  is endowed with the coregular right representation of  $U(\mathfrak{r})$  (see also [4, 2.7.11]). When  $\xi \neq 0$ , one checks that  $h_\xi(\widetilde{V}'(\lambda)) \neq \{0\}$  and then  $h_\xi(\widetilde{V}'(\lambda))$  is an irreducible  $U(\mathfrak{r})$ -module for the coregular right representation. Then  $\widetilde{C}_{\mathfrak{p}}(\lambda) = \sum_{\xi \in \widetilde{V}(\lambda)^* \setminus \{0\}} h_\xi(\widetilde{V}'(\lambda))$  is a sum of irreducible  $U(\mathfrak{r})$ -modules all isomorphic to  $\widetilde{V}'(\lambda)$ . Since, for  $\lambda \neq \mu \in P^+(\pi)$ ,  $\widetilde{V}'(\lambda)$  and  $\widetilde{V}'(\mu)$  are not isomorphic as  $U(\mathfrak{r})$ -modules, it follows that  $\sum_{\lambda \in P^+(\pi)} \widetilde{C}_{\mathfrak{p}}(\lambda)$  is a direct sum. Obviously  $\widetilde{C}_{\mathfrak{r}} = \sum_{\lambda \in P^+(\pi)} \widetilde{C}_{\mathfrak{r}}(\lambda)$  is also a direct sum, since  $\widetilde{C}_{\mathfrak{r}}(\lambda) \subset \widetilde{C}_{\mathfrak{p}}(\lambda)$  for all  $\lambda \in P^+(\pi)$ . Finally  $\widetilde{C}_{\mathfrak{r}}$  is an algebra by [4, 2.7.10], as a consequence of lemma 5.2.  $\square$

**5.4. Isomorphisms.** Let  $\lambda \in P^+(\pi)$  and set  $\Phi_{\mathfrak{p}}^\lambda : \widetilde{V}(\lambda)^* \otimes \widetilde{V}'(\lambda) \rightarrow \widetilde{C}_{\mathfrak{p}}(\lambda)$  defined by  $\xi \otimes v' \mapsto c_{\xi, v'}$  and extended by linearity. Similarly one sets  $\Phi_{\mathfrak{r}}^\lambda : \widetilde{V}''(\lambda)^* \otimes \widetilde{V}'(\lambda) \rightarrow \widetilde{C}_{\mathfrak{r}}(\lambda)$  defined by  $\xi \otimes v' \mapsto c_{\xi, v'}$  extended by linearity.

**Lemma.** *The maps  $\Phi_{\mathfrak{p}}^\lambda$  and  $\Phi_{\mathfrak{r}}^\lambda$  are isomorphisms of vector spaces.*

*Proof.* Firstly these maps are obviously well defined and onto. It remains to verify the injectivity. Assume that there exists  $I$  a finite set, and for all  $i \in I$ ,  $\xi_i \in \widetilde{V}(\lambda)^*$  and  $v'_i \in \widetilde{V}'(\lambda) = U(\mathfrak{r}) \cdot \tilde{v}_\lambda$  such that  $\sum_{i \in I} c_{\xi_i, v'_i} = 0$ . We can also assume that the  $v'_i$ ,  $i \in I$ , are linearly independent. We want to show that for all  $i \in I$ ,  $\xi_i = 0$ . Assume that there exists  $i_0 \in I$  such that  $\xi_{i_0} \neq 0$  and complete  $\xi_{i_0}$  in a basis of  $\widetilde{V}(\lambda)^*$ . By taking the dual basis, there exists  $v_{i_0} \in \widetilde{V}(\lambda)$  such that  $\xi_{i_0}(v_{i_0}) = 1$ . By (i) of lemma 3.2 there exists  $u_0 \in U(\tilde{\mathfrak{p}}^-)$  such that  $v_{i_0} = u_0 \cdot \tilde{v}_\lambda$ . Now recall that  $\widetilde{V}'(\lambda)$  is a left irreducible

$U(\mathfrak{r})$ -module. Then by Jacobson density theorem (see [28, Chap. 3, § 3, 2]), there exists  $a \in U(\mathfrak{r})$  such that for all  $i \in I \setminus \{i_0\}$ ,  $a.v'_i = 0$  and  $a.v'_{i_0} = \tilde{v}_\lambda$ . Since  $u_0 a \in U(\tilde{\mathfrak{p}}^-)$  we obtain that  $\sum_{i \in I} c_{\xi_i, v'_i}(u_0 a) = 0 = \xi_{i_0}(u_0.(a.v'_{i_0})) = \xi_{i_0}(u_0.\tilde{v}_\lambda) = \xi_{i_0}(v_{i_0}) = 1$  which is a contradiction. Hence the lemma for the map  $\Phi_{\mathfrak{p}}^\lambda$  and of course also for  $\Phi_{\mathfrak{r}}^\lambda$ .  $\square$

**5.5. The dual representation of the generalized adjoint action.** Recall the left representation of  $A$  in  $U(\tilde{\mathfrak{p}}^-)$  defined in subsection 4.4 we have denoted by  $ad^{**}$ . Then the dual representation of  $A$  in  $U(\tilde{\mathfrak{p}}^-)^*$  is defined as follows.

$$(43) \quad \forall a \in A, \forall f \in U(\tilde{\mathfrak{p}}^-)^*, \quad a.f = f \circ ad^{**} a^\top$$

(where recall  $a^\top$  was defined in equation (15)).

This defines a left action of  $A$  on  $U(\tilde{\mathfrak{p}}^-)^*$  (see for instance [4, 2.2.19]) and by proposition 4.5, we deduce that one has

$$(44) \quad \forall a \in A, \forall f \in U(\tilde{\mathfrak{p}}^-)^*, \quad a.f = f \circ L(a_2^\top) \circ R(a_1) = ({}^t R(a_1) \circ {}^t L(a_2^\top))(f)$$

where  $\Delta_A(a) = a_1 \otimes a_2$ .

In particular one has that

$$(45) \quad \forall x \in \mathfrak{p}, \forall f \in U(\tilde{\mathfrak{p}}^-)^*, \quad x.f = {}^t R(x)(f) - {}^t L(x)(f).$$

Then one deduces the following lemma.

**Lemma.** *Let  $\lambda \in P^+(\pi)$ . One has that*

$$(46) \quad \forall x \in \mathfrak{r}, \forall \xi \in \tilde{V}(\lambda)^*, \forall v \in \tilde{V}'(\lambda), \quad x.c_{\xi, v} = c_{\xi, x.v} - c_{\xi, x}.v.$$

*Proof.* Let  $x \in \mathfrak{r}$ ,  $\xi \in \tilde{V}(\lambda)^*$  and  $v \in \tilde{V}'(\lambda)$ . One checks easily that  ${}^t R(x)(c_{\xi, v}) = c_{\xi, x.v}$ , resp. that  ${}^t L(x)(c_{\xi, v}) = c_{\xi, x}.v$ , by definition of  $R$ , resp. of  $L$ , given in equation (34), resp. in equation (36). Then the lemma follows from equation (45).  $\square$

Let  $\lambda \in P^+(\pi)$ . Endow  $U(\tilde{\mathfrak{p}}^-)^*$  with the dual representation of  $A$  given by equation (43) and in particular with the dual representation of  $U(\mathfrak{r}) \subset A$ , which coincides in the latter case with the coadjoint representation of  $U(\mathfrak{r})$ . By equation (46) every  $\tilde{C}_{\mathfrak{p}}(\lambda)$ , resp.  $\tilde{C}_{\mathfrak{r}}(\lambda)$ , is a left  $U(\mathfrak{r})$ -module for the coadjoint representation.

On the other hand, endow  $\tilde{V}(\lambda)$  with the left action of  $U(\mathfrak{r})$  described in subsection 3.2 and  $\tilde{V}(\lambda)^*$  with the left action of  $U(\mathfrak{r})$  corresponding with its dual representation, namely for all  $\xi \in \tilde{V}(\lambda)^*$ , for all  $u \in U(\mathfrak{r})$ ,  $u.\xi = \xi.u^\top$ . Then endow the tensor product  $\tilde{V}(\lambda)^* \otimes \tilde{V}'(\lambda)$  with the diagonal action of  $U(\mathfrak{r})$ , namely for  $u \in U(\mathfrak{r})$  such that  $\Delta(u) = u_1 \otimes u_2$ , for all  $\xi \in \tilde{V}(\lambda)^*$ , for all  $v' \in \tilde{V}'(\lambda)$ ,

$$(47) \quad u.(\xi \otimes v') = u_1.\xi \otimes u_2.v' = \xi.u_1^\top \otimes u_2.v'.$$

In particular, one has that

$$(48) \quad \forall x \in \mathfrak{r}, \forall \xi \in \tilde{V}(\lambda)^*, \forall v \in \tilde{V}'(\lambda), \quad x.(\xi \otimes v) = -\xi.x \otimes v + \xi \otimes x.v.$$



Recall the isomorphisms of vector spaces  $\Phi_{\mathfrak{p}}^\lambda$  and  $\Phi_{\mathfrak{r}}^\lambda$  defined in subsection 5.4.

**Proposition.** *Let  $\lambda \in P^+(\pi)$ . With the left actions of  $U(\mathfrak{r})$  given by equation (47) and equation (43) respectively, the isomorphisms of vector spaces  $\Phi_{\mathfrak{p}}^\lambda$  and  $\Phi_{\mathfrak{r}}^\lambda$  are isomorphisms of left  $U(\mathfrak{r})$ -modules.*

*Proof.* It is immediate by equations (46) and (48).  $\square$

## 6. A POLYNOMIAL ALGEBRA.

**6.1. A semigroup.** Recall  $\mathfrak{r}'$  the derived subalgebra of  $\mathfrak{r}$  : the former is a semi-simple Lie algebra. Denote by  $(U(\tilde{\mathfrak{p}}^-)^*)^{U(\mathfrak{r}')}$  the set of elements in  $U(\tilde{\mathfrak{p}}^-)^*$  which are invariant under the coadjoint action of  $U(\mathfrak{r}')$ . Since for all  $z \in \mathfrak{r}'$ , for all  $u \in U(\tilde{\mathfrak{p}}^-)$  such that  $\Delta(u) = u_1 \otimes u_2$ , we have that  $\Delta(ad z(u)) = ad z(u_1) \otimes u_2 + u_1 \otimes ad z(u_2)$ , the set  $(U(\tilde{\mathfrak{p}}^-)^*)^{U(\mathfrak{r}')}$  is an algebra.

For all  $\lambda \in P^+(\pi)$ , recall that  $\tilde{C}_{\mathfrak{r}}(\lambda)$  is a left  $U(\mathfrak{r}')$ -module (for the coadjoint representation of  $U(\mathfrak{r}')$ ) by equation (46). Then define  $\tilde{C}_{\mathfrak{r}}(\lambda)^{U(\mathfrak{r}')}$  as the set of elements in  $\tilde{C}_{\mathfrak{r}}(\lambda)$  which are invariant under the coadjoint action of  $U(\mathfrak{r}')$  : this is of course a vector space.

Denote by  $\tilde{C}_{\mathfrak{r}}^{U(\mathfrak{r}')} \subset (U(\tilde{\mathfrak{p}}^-)^*)^{U(\mathfrak{r}')}$  the set of elements in  $\tilde{C}_{\mathfrak{r}}$  which are invariant under the coadjoint action of  $U(\mathfrak{r}')$ . Since  $\tilde{C}_{\mathfrak{r}}$  is an algebra by lemma 5.3 and by what we said above,  $\tilde{C}_{\mathfrak{r}}^{U(\mathfrak{r}'})$  is an algebra.

Since moreover the sum of the  $\tilde{C}_{\mathfrak{r}}(\lambda)$ 's is a direct sum by lemma 5.3, we have that

$$(49) \quad \tilde{C}_{\mathfrak{r}}^{U(\mathfrak{r}')} = \bigoplus_{\lambda \in P^+(\pi)} \tilde{C}_{\mathfrak{r}}(\lambda)^{U(\mathfrak{r}')}.$$

Let  $\mathcal{D}$  be the set of all  $\lambda \in P^+(\pi)$  such that

$$(50) \quad (w'_0 \lambda - w_0 \lambda, \pi') = 0.$$

**Proposition.** *One has that, for all  $\lambda \in P^+(\pi)$ ,  $\dim \tilde{C}_{\mathfrak{r}}(\lambda)^{U(\mathfrak{r}')} \leq 1$  with equality if and only if  $\lambda \in \mathcal{D}$  and then*

$$(51) \quad \tilde{C}_{\mathfrak{r}}^{U(\mathfrak{r}')} = \bigoplus_{\lambda \in \mathcal{D}} \tilde{C}_{\mathfrak{r}}(\lambda)^{U(\mathfrak{r}')}.$$

*Proof.* The proof is quite similar as the proof in [8, Thm. §3]. We give it for the reader's convenience. Fix  $\lambda \in P^+(\pi)$ . Denote by  $\text{Hom}(\tilde{V}'(\lambda)^*, \tilde{V}''(\lambda)^*)$  the set of all morphisms between the vector spaces  $\tilde{V}'(\lambda)^*$  and  $\tilde{V}''(\lambda)^*$ , endowed with the  $U(\mathfrak{r}')$ -module structure given by

$$(52) \quad \forall u \in U(\mathfrak{r}'), \forall \varphi \in \text{Hom}(\tilde{V}'(\lambda)^*, \tilde{V}''(\lambda)^*), \forall \xi \in \tilde{V}'(\lambda)^*, \\ (u \cdot \varphi)(\xi) = u_2 \cdot \varphi(u_1^\top \cdot \xi)$$

where  $\Delta(u) = u_1 \otimes u_2$ . Denote by  $\Phi$  the following morphism between  $\tilde{V}''(\lambda)^* \otimes \tilde{V}'(\lambda)$ , endowed with the diagonal action of  $U(\mathfrak{r}')$  given by equation

(47) and  $\text{Hom}(\widetilde{V}'(\lambda)^*, \widetilde{V}''(\lambda)^*)$ , endowed with the action given by equation (52).

$$(53) \quad \Phi : \xi \otimes v' \in \widetilde{V}''(\lambda)^* \otimes \widetilde{V}'(\lambda) \mapsto (\xi' \in \widetilde{V}'(\lambda)^* \mapsto \xi'(v')\xi)$$

Then by [17, A.2.16] for instance,  $\Phi$  is an isomorphism of  $U(\mathfrak{r}')$ -modules.

Denote by  $\text{Hom}_{U(\mathfrak{r}')}(\widetilde{V}'(\lambda)^*, \widetilde{V}''(\lambda)^*)$  the set of all  $U(\mathfrak{r}')$ -morphisms between  $\widetilde{V}'(\lambda)^*$  and  $\widetilde{V}''(\lambda)^*$  and by  $(\widetilde{V}''(\lambda)^* \otimes \widetilde{V}'(\lambda))^{U(\mathfrak{r}')}$  the set of elements in the tensor product  $\widetilde{V}''(\lambda)^* \otimes \widetilde{V}'(\lambda)$  which are invariant under the diagonal action of  $U(\mathfrak{r}')$  given by equation (47). Then we have

$$(54) \quad \Phi((\widetilde{V}''(\lambda)^* \otimes \widetilde{V}'(\lambda))^{U(\mathfrak{r}')} ) = \text{Hom}_{U(\mathfrak{r}')}(\widetilde{V}'(\lambda)^*, \widetilde{V}''(\lambda)^*).$$

Moreover the  $U(\mathfrak{r}')$ -modules  $\widetilde{V}''(\lambda)^*$  and  $\widetilde{V}'(\lambda)$  (and also  $\widetilde{V}'(\lambda)^*$ ) are irreducible. Then by Schur lemma (see for instance [28, Chap. 3, § 3, 1]),

$$(55) \quad \dim \text{Hom}_{U(\mathfrak{r}')}(\widetilde{V}'(\lambda)^*, \widetilde{V}''(\lambda)^*) \leq 1$$

with equality if and only if the irreducible  $U(\mathfrak{r}')$ -modules  $\widetilde{V}''(\lambda)^*$  and  $\widetilde{V}'(\lambda)^*$  are isomorphic that is, if and only if

$$(56) \quad w'_0\lambda - w_0\lambda = \sum_{\alpha \in \pi \setminus \pi'} m_\alpha \varpi_\alpha \text{ with } m_\alpha \in \mathbb{N}, \forall \alpha \in \pi \setminus \pi'$$

or equivalently if and only if  $\lambda$  verifies equation (50) that is, if and only if  $\lambda \in \mathcal{D}$ .

Indeed we have that  $\widetilde{V}''(\lambda)^* \simeq \widetilde{V}'(-w_0\lambda)$  as left  $U(\mathfrak{r})$ -modules by what we already said in the proof of Lemma 5.2. Similarly since  $\widetilde{V}'(\lambda) = U(\mathfrak{n}_{\pi'}^-) \cdot \tilde{v}_\lambda = U(\mathfrak{n}_{\pi'}^-) \cdot \tilde{v}_{w'_0\lambda}$  where  $\tilde{v}_{w'_0\lambda}$  is a chosen nonzero weight vector in  $\widetilde{V}'(\lambda)$  of weight  $w'_0\lambda$ , we have that  $\widetilde{V}'(\lambda)^* \simeq \widetilde{V}'(-w'_0\lambda)$  as left  $U(\mathfrak{r})$ -modules. Then the irreducible  $U(\mathfrak{r}')$ -modules  $\widetilde{V}''(\lambda)^*$  and  $\widetilde{V}'(\lambda)^*$  are isomorphic if and only if  $(-w_0\lambda)' = (-w'_0\lambda)'$  where recall that the superscript “prime” denotes the projection in  $P(\pi')$  of an element in  $P(\pi)$  with respect to the decomposition (1).

By proposition 5.5 one has that

$$(57) \quad \widetilde{C}_{\mathfrak{r}}(\lambda)^{U(\mathfrak{r}')} = \Phi_{\mathfrak{r}}^\lambda \left( (\widetilde{V}''(\lambda)^* \otimes \widetilde{V}'(\lambda))^{U(\mathfrak{r}')} \right).$$

Then by equations (54) and (55) we have that  $\dim \widetilde{C}_{\mathfrak{r}}(\lambda)^{U(\mathfrak{r}')} \leq 1$  with equality if and only if  $\lambda \in \mathcal{D}$ . For all  $\lambda \in \mathcal{D}$  set  $c_\lambda \in \widetilde{C}_{\mathfrak{r}}(\lambda)^{U(\mathfrak{r}')} \setminus \{0\}$  so that  $\widetilde{C}_{\mathfrak{r}}(\lambda)^{U(\mathfrak{r}')} = \mathbb{k}c_\lambda$ . By equation (49) we also have

$$(58) \quad \widetilde{C}_{\mathfrak{r}}^{U(\mathfrak{r}')} = \bigoplus_{\lambda \in \mathcal{D}} \mathbb{k}c_\lambda.$$

This completes the proof.  $\square$

Let  $\lambda \in \mathcal{D}$ . Choose  $\varphi_\lambda \in \text{Hom}_{U(\mathfrak{r})}(\widetilde{V}'(\lambda)^*, \widetilde{V}''(\lambda)^*) \setminus \{0\}$  and denote by  $U(\mathfrak{r})_+$  the kernel of the coidentity in the enveloping algebra  $U(\mathfrak{r})$ . By [17, 7.1.16] we have that

$$(59) \quad \Phi^{-1} : \text{Hom}(\widetilde{V}'(\lambda)^*, \widetilde{V}''(\lambda)^*) \xrightarrow{\sim} U(\mathfrak{r})_+ \cdot (\tilde{\xi}_{w_0\lambda} \otimes \tilde{v}_{w'_0\lambda}) \oplus \mathbb{k}\Phi^{-1}(\varphi_\lambda).$$

It follows that we have, up to a nonzero scalar

$$(60) \quad (\Phi_\mathfrak{r}^\lambda)^{-1}(c_\lambda) = \tilde{\xi}_{w_0\lambda} \otimes \tilde{v}_{w'_0\lambda} + \sum_{i \in I} u_i^- \cdot \tilde{\xi}_{w_0\lambda} \otimes u_i^+ \cdot \tilde{v}_{w'_0\lambda}$$

where  $u_i^\pm \in \mathfrak{n}_{\pi'}^\pm U(\mathfrak{n}_{\pi'}^\pm)$  for all  $i \in I$ ,  $I$  a finite set, since moreover  $\tilde{\xi}_{w_0\lambda} \otimes \tilde{v}_{w'_0\lambda}$  is a cyclic vector for the  $U(\mathfrak{r})$ -module  $\widetilde{V}''(\lambda)^* \otimes \widetilde{V}'(\lambda)$  endowed with the diagonal action. Hence the  $\mathfrak{h}$ -weight of  $c_\lambda$  is equal to

$$(61) \quad w'_0\lambda - w_0\lambda.$$

By equation (56) this weight belongs to  $P^+(\pi)$  and by equation (50) it annihilates on  $\pi'$ .

Let  $i$  and  $j$  denote the permutations in  $\pi$  defined below.

$$(62) \quad \forall \alpha \in \pi, j(\alpha) = -w_0(\alpha)$$

$$(63) \quad \forall \alpha \in \pi', i(\alpha) = -w'_0(\alpha)$$

$$(64) \quad \forall \alpha \in \pi \setminus \pi', \begin{cases} i(\alpha) = j(\alpha) & \text{if } j(\alpha) \notin \pi' \\ i(\alpha) = j(ij)^{r_\alpha}(\alpha) & \text{otherwise} \end{cases}$$

where  $r_\alpha$  is the smallest integer such that  $j(ij)^{r_\alpha}(\alpha) \notin \pi'$ . Let  $E(\pi')$  be the set of  $\langle ij \rangle$ -orbits in  $\pi$ , where  $\langle ij \rangle$  denotes the subgroup generated by the composition map  $ij$ .

For instance, if  $\mathfrak{p}$  is a maximal parabolic subalgebra of  $\mathfrak{g}$  that is, if  $\pi \setminus \pi' = \{\alpha\}$ , then  $i(\alpha) = \alpha$  and the  $\langle ij \rangle$ -orbit of  $\alpha$  is  $\Gamma_\alpha = \{(ji)^s j(\alpha), 0 \leq s \leq r_\alpha\}$ .

Recall [9, Thm. 1] (see also [7, 4.1]):

**Theorem.** *The set  $\mathcal{D}$  is a free additive semigroup generated by the  $\mathbb{Z}$ -linearly independent elements  $d_\Gamma = \sum_{\gamma \in \Gamma} \varpi_\gamma$ ,  $\Gamma \in E(\pi')$ .*

**6.2. A filtration.** Assume that  $\pi = \{\alpha_1, \dots, \alpha_n\}$ . Then for all  $\lambda \in P^+(\pi)$ , there exist  $k_i \in \mathbb{Q}_+$  for all  $i$ ,  $1 \leq i \leq n$ , such that  $\lambda = \sum_{i=1}^n k_i \alpha_i$ . Set  $\text{deg}(\lambda) = 2 \sum_{i=1}^n k_i$ . By [17, 7.1.25],  $\text{deg}(\lambda) \in \mathbb{N}$ . For all  $m \in \mathbb{N}$ , we set  $\mathcal{F}'_m(\tilde{C}_\mathfrak{r}) = \bigoplus_{\lambda \in P^+(\pi) | \text{deg}(\lambda) \leq m} \tilde{C}_\mathfrak{r}(\lambda)$ , which is a left  $U(\mathfrak{r})$ -submodule of  $\tilde{C}_\mathfrak{r}$  for coadjoint action. Then  $(\mathcal{F}'_m(\tilde{C}_\mathfrak{r}))_{m \in \mathbb{N}}$  is an increasing filtration  $\mathcal{F}'$  of the algebra  $\tilde{C}_\mathfrak{r}$  since for all  $\lambda, \mu \in P^+(\pi)$ ,

$$(65) \quad \tilde{C}_\mathfrak{r}(\lambda)\tilde{C}_\mathfrak{r}(\mu) \subset \bigoplus_{\nu \in \mathbb{N}\pi' | \lambda + \mu - \nu \in P^+(\pi)} \tilde{C}_\mathfrak{r}(\lambda + \mu - \nu)$$

by the proof of lemma 5.2. Then denote by  $gr_{\mathcal{F}'}(\tilde{C}_\tau)$  the associated graded algebra and for all  $c \in \mathcal{F}'_m(\tilde{C}_\tau)$ , denote by  $gr_{m, \mathcal{F}'}(c)$  its canonical image in  $gr_{\mathcal{F}'}(\tilde{C}_\tau)$ . Recall the notation in subsection 6.1.

**Lemma.** *Let  $\lambda, \mu \in \mathcal{D}$ . Set  $m = \deg(\lambda)$  and  $m' = \deg(\mu)$ .*

*Then  $gr_{m, \mathcal{F}'}(c_\lambda)gr_{m', \mathcal{F}'}(c_\mu)$  is a nonzero multiple of  $gr_{m+m', \mathcal{F}'}(c_{\lambda+\mu})$ .*

*Proof.* By definition of the multiplication in the graded algebra  $gr_{\mathcal{F}'}(\tilde{C}_\tau)$ , one has that

$$gr_{m, \mathcal{F}'}(c_\lambda)gr_{m', \mathcal{F}'}(c_\mu) = gr_{m+m', \mathcal{F}'}(c_\lambda c_\mu).$$

Now by equation (60) in the product  $c_\lambda c_\mu$  appears, up to a nonzero scalar, the term

$$c_{\tilde{\xi}_{w_0\lambda} \otimes \tilde{\xi}_{w_0\mu}, \tilde{v}_{w'_0\lambda} \otimes \tilde{v}_{w'_0\mu}} = c_{\tilde{\xi}_{w_0(\lambda+\mu)}, \tilde{v}_{w'_0(\lambda+\mu)}} \in \tilde{C}_\tau(\lambda + \mu).$$

Indeed this term cannot be annihilated by the other terms, by lemmas 5.3 and 5.4.

Since moreover  $c_\lambda c_\mu \in \tilde{C}_\tau^{U(\tau')}$ , equations (58), (60) and (65) imply that

$$\begin{aligned} gr_{m+m', \mathcal{F}'}(c_\lambda c_\mu) &= \sum_{\nu \in \mathcal{D}, \deg(\nu)=m+m', \nu \in \lambda+\mu - \mathbb{N}\pi'} gr_{m+m', \mathcal{F}'}(c_\nu) \\ &= gr_{m+m', \mathcal{F}'}(c_{\lambda+\mu}) \end{aligned}$$

up to multiplication by a nonzero scalar.  $\square$

Now we can conclude the following.

**Proposition.** *The algebra of invariants  $\tilde{C}_\tau^{U(\tau')}$  is a polynomial algebra over  $\mathbb{k}$ , whose number of algebraically independent generators is equal to the cardinality of the set  $E(\pi')$ .*

*Proof.* It follows as in the proof of [9, Thm. 1] (see also [9, Prop. 3.1]). Let  $\lambda_i, i \in I$ , be a set of  $\mathbb{Z}$ -linearly independent generators of  $\mathcal{D}$  and set  $m_i = \deg(\lambda_i)$  for all  $i \in I$  (one has that  $|I| = |E(\pi')|$  by Thm 6.1). Denote by  $gr_{\mathcal{F}'}(\tilde{C}_\tau^{U(\tau')})$  the graded algebra of the algebra  $\tilde{C}_\tau^{U(\tau')}$  associated with the induced filtration. Note that the above lemma also holds in this graded algebra. Then equation (58) and the above lemma imply that  $gr_{m_i, \mathcal{F}'}(c_{\lambda_i}), i \in I$ , are  $\mathbb{k}$ -algebraically independent and generate  $gr_{\mathcal{F}'}(\tilde{C}_\tau^{U(\tau')})$ . Hence  $gr_{\mathcal{F}'}(\tilde{C}_\tau^{U(\tau')})$  is a polynomial algebra over  $\mathbb{k}$  in  $|E(\pi')|$  generators and it follows (see [3, Chap. III, § 2, n° 9, Prop. 10]) that the algebra  $\tilde{C}_\tau^{U(\tau')}$  is also a polynomial algebra over  $\mathbb{k}$  in  $|E(\pi')|$  generators  $c_{\lambda_i}, i \in I$ , whose  $\mathfrak{h}$ -weight is equal to  $\delta_i = w'_0\lambda_i - w_0\lambda_i$  by equation (61).  $\square$

## 7. GENERALIZED KOSTANT FILTRATION AND MORPHISM.

**7.1. Generalized Kostant filtration.** In [10, 6.1] we have defined what we called the Kostant filtration (denoted by  $\mathcal{F}_K$ ) on the Hopf dual of the enveloping algebra of the simple Lie algebra  $\mathfrak{g}$ . Here we will consider what we

call the *generalized Kostant filtration* on the dual algebra  $U(\tilde{\mathfrak{p}}^-)^*$  of  $U(\tilde{\mathfrak{p}}^-)$ . More precisely, we set

$$(66) \quad \forall k \in \mathbb{N}, \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*) = \{f \in U(\tilde{\mathfrak{p}}^-)^* \mid f(U_{k-1}(\tilde{\mathfrak{p}}^-)) = 0\}$$

where  $(U_k(\tilde{\mathfrak{p}}^-))_{k \in \mathbb{N} \cup \{-1\}}$  is the canonical filtration on  $U(\tilde{\mathfrak{p}}^-)$ , with  $U_{-1}(\tilde{\mathfrak{p}}^-) = \{0\}$ .

**Lemma.** *The generalized Kostant filtration  $(\mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*))_{k \in \mathbb{N}}$  is a decreasing, exhaustive and separated filtration on the algebra  $U(\tilde{\mathfrak{p}}^-)^*$ . Moreover this filtration is invariant by the left action of  $A$  defined by equation (43).*

*Proof.* The first assertions are obvious.

Let us show the invariance by the left action of  $A$ . Let  $a \in A$ ,  $k \in \mathbb{N}$  and  $f \in \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*)$ .

If  $a = z \in \mathfrak{t}$ , then  $ad^{**}z = adz$  by equation (30) and for all  $u \in U_{k-1}(\tilde{\mathfrak{p}}^-)$ ,  $adz(u) \in U_{k-1}(\tilde{\mathfrak{p}}^-)$ . Then  $z.f \in \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*)$ .

Now assume that  $a = x \in \mathfrak{m}$ , and let  $u \in U_{k-1}(\tilde{\mathfrak{p}}^-)$ . Recall equation (25). Then  $u = \sum_{j=0}^{k-1} s_j u'_j$  with  $s_j \in S_j(\mathfrak{m}^-)$  and  $u'_j \in U_{k-1-j}(\mathfrak{t})$ , for all  $0 \leq j \leq k-1$ .

Then by equation (29) one has :

$$ad^{**}x(u) = \sum_{j=0}^{k-1} \tilde{\theta}(ad^*x(s_j))u'_j \in \sum_{j=0}^{k-1} S_{j-1}(\mathfrak{m}^-)U_{k-j}(\mathfrak{t}) \subset U_{k-1}(\tilde{\mathfrak{p}}^-)$$

by equation (27).

It follows that  $x.f(u) = 0$  and the lemma.  $\square$

**7.2. The graded algebra associated with the generalized Kostant filtration.** Set

$$(67) \quad gr_K(U(\tilde{\mathfrak{p}}^-)^*) = \bigoplus_{k \in \mathbb{N}} gr_K^k(U(\tilde{\mathfrak{p}}^-)^*)$$

where, for all  $k \in \mathbb{N}$ ,

$$(68) \quad gr_K^k(U(\tilde{\mathfrak{p}}^-)^*) = \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*) / \mathcal{F}_K^{k+1}(U(\tilde{\mathfrak{p}}^-)^*).$$

Then  $gr_K(U(\tilde{\mathfrak{p}}^-)^*)$  is the graded algebra associated with the generalized Kostant filtration  $\mathcal{F}_K$  on the algebra  $U(\tilde{\mathfrak{p}}^-)^*$ . For all  $f \in \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*)$ , one denotes by  $gr_K^k(f)$  its canonical image in  $gr_K^k(U(\tilde{\mathfrak{p}}^-)^*)$ .

By lemma 7.1 the dual representation of  $A$  on  $U(\tilde{\mathfrak{p}}^-)^*$  (given by equation (43)) induces a left action on  $gr_K(U(\tilde{\mathfrak{p}}^-)^*)$  defined by

$$(69) \quad \forall a \in A, \forall f \in \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*), a.gr_K^k(f) = gr_K^k(a.f).$$

**Proposition.** *Let  $x \in \mathfrak{m}$  and  $f \in \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*) \cap \tilde{C}_\mathfrak{t}$ ,  $k \in \mathbb{N}$ . Then  $x.f \in \mathcal{F}_K^{k+1}(U(\tilde{\mathfrak{p}}^-)^*)$  and therefore*

$$(70) \quad x.gr_K^k(f) = 0$$

where recall  $gr_K^k(g) = g + \mathcal{F}_K^{k+1}(U(\tilde{\mathfrak{p}}^-)^*)$ , for all  $g \in \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*)$ .

*Proof.* Take

$$f = \sum_{\lambda \in \Lambda} \sum_{i \in I_\lambda} c_{\xi_{w_0\lambda}, a_i, b_i, \tilde{v}_\lambda}^\lambda \in \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*) \cap \tilde{C}_\tau$$

with  $\Lambda \subset P^+(\pi)$  a finite set and for all  $\lambda \in \Lambda$ ,  $I_\lambda$  a finite set,  $a_i, b_i \in U(\mathfrak{n}_{\pi'}^-)$ , for all  $i \in I_\lambda$ . Moreover one may assume, if  $f \neq 0$ , that  $a_i, b_i$ , for all  $i \in I_\lambda$ , are nonzero weight vectors. We need the lemma below.

**Lemma.** *Let  $k \in \mathbb{N}$  and  $0 \leq j \leq k$ . With the above hypotheses, we have that*

$$(71) \quad \forall u \in U_{j-1}(\mathfrak{g}), \forall u' \in U_{k-j}(\mathfrak{r}), \sum_{\lambda \in \Lambda} \sum_{i \in I_\lambda} \xi_{w_0\lambda}(a_i u u' b_i \cdot v_\lambda) = 0.$$

*Proof.* The lemma is obvious for  $j = 0$  since  $U_{-1}(\mathfrak{g}) = \{0\}$ . Assume that  $j \in \mathbb{N}^*$ . Take  $u \in U_{j-1}(\mathfrak{g})$  and  $u' \in U_{k-j}(\mathfrak{r})$ . Since  $\mathfrak{m} \cdot V'(\lambda) = \{0\}$  and since  $U_{j-1}(\mathfrak{g}) = U_{j-1}(\mathfrak{p}^-) \oplus U_{j-2}(\mathfrak{g})\mathfrak{m}$ , one may assume that  $u \in U_{j-1}(\mathfrak{p}^-)$ . One also may assume that  $u$  and  $u'$  are nonzero weight vectors.

Since  $\tilde{\xi}_{w_0\lambda}$  vanishes on the weight vectors  $\beta_\lambda^{-1}(a_i u u' b_i \cdot v_\lambda)$  which are not of weight  $w_0\lambda$  and by equation (42), one has that

$$\begin{aligned} \sum_{\lambda \in \Lambda} \sum_{i \in I_\lambda} \xi_{w_0\lambda}(a_i u u' b_i \cdot v_\lambda) &= \sum_{\lambda \in \Lambda} \sum_{i \in I_\lambda} \tilde{\xi}_{w_0\lambda}(\beta_\lambda^{-1}(a_i u u' b_i \cdot v_\lambda)) \\ &= \sum_{\lambda \in \Lambda} \sum_{i \in I'_\lambda} \tilde{\xi}_{w_0\lambda}(gr_{k_0}(a_i u u' b_i \cdot v_\lambda)) \end{aligned}$$

where for all  $\lambda \in \Lambda$ ,  $I'_\lambda = \{i \in I_\lambda \mid \beta_\lambda^{-1}(a_i u u' b_i \cdot v_\lambda) \text{ is of weight } w_0\lambda\}$ .

Now write  $u = \sum_{t=0}^{j-1} u_t v_t$  with  $u_t = \theta(s_t) \in U^t(\mathfrak{m}^-) = \theta(S_t(\mathfrak{m}^-))$ ,  $s_t \in S_t(\mathfrak{m}^-)$  and  $v_t \in U_{j-1-t}(\mathfrak{r})$  for all  $0 \leq t \leq j-1$ .

Then

$$gr_{k_0}(a_i u u' b_i \cdot v_\lambda) = a_i \left( \sum_{t=0}^{j-1} s_t v_t \right) u' b_i \cdot \tilde{v}_\lambda$$

by equation (8).

But  $\sum_{t=0}^{j-1} s_t v_t \in U_{j-1}(\tilde{\mathfrak{p}}^-)$  and recall that  $u' \in U_{k-j}(\mathfrak{r})$ . Then

$$\left( \sum_{t=0}^{j-1} s_t v_t \right) u' \in U_{k-1}(\tilde{\mathfrak{p}}^-)$$

and we obtain the required equation (71) since  $f(U_{k-1}(\tilde{\mathfrak{p}}^-)) = 0$ .  $\square$

Fix  $x \in \mathfrak{m}$  being a nonzero weight vector and for all  $0 \leq j \leq k$ , take  $s \in S_j(\mathfrak{m}^-)$  and  $u' \in U_{k-j}(\mathfrak{r})$  being weight vectors. If  $j \geq 1$ , one may assume that  $s = y_1 \cdots y_j \in S_j(\mathfrak{m}^-)$  with  $y_t \in \mathfrak{m}^-$  being a weight vector for all  $1 \leq t \leq j$ .

Recall that

$$(72) \quad ad^*x(s) = \sum_{1 \leq t \leq j \mid [x, y_t] \in \mathfrak{r}} y_1 \cdots y_{t-1} [x, y_t] y_{t+1} \cdots y_j \in S_j(\mathfrak{p}^-).$$

Set

$$(73) \quad ad_{\mathfrak{m}^-}x(s) = \sum_{1 \leq t \leq j | [x, y_t] \in \mathfrak{m}^-} y_1 \cdots y_{t-1} [x, y_t] y_{t+1} \cdots y_j \in S_j(\mathfrak{m}^-)$$

and

$$(74) \quad ad_{\mathfrak{m}}x(s) = \sum_{1 \leq t \leq j | [x, y_t] \in \mathfrak{m}} y_1 \cdots y_{t-1} [x, y_t] y_{t+1} \cdots y_j \in S_j(\mathfrak{g}).$$

By equations (43) and (29), one has that

$$-x.f(su') = f(ad^{**}x(su')) = f(\tilde{\theta}(ad^*x(s))u').$$

Then if  $j = 0$  one has obviously that  $x.f(su') = 0$ , by equation (31).

From now on, assume that  $j \geq 1$ . By the above and by what we said in the proof of the previous lemma we have that

$$(75) \quad -x.f(su') = \sum_{\lambda \in \Lambda} \sum_{i \in I'_\lambda} \tilde{\xi}_{w_0\lambda} \left( a_i \tilde{\theta}(ad^*x(s))u'b_i.\tilde{v}_\lambda \right)$$

where for all  $\lambda \in \Lambda$ ,  $I'_\lambda = \{i \in I_\lambda \mid \exists c_i \in \mathbb{k}^*; a_i \tilde{\theta}(ad^*x(s))u'b_i.\tilde{v}_\lambda = c_i \tilde{v}_{w_0\lambda}\}$ .

Fix  $\lambda \in \Lambda$  and  $i \in I'_\lambda$ . One has that  $a_i \tilde{\theta}(ad^*x(s))u'b_i.\tilde{v}_\lambda \in gr_{j-1}(V(\lambda))$  by equations (27) and (10), and by equation (42) we have that  $j-1 = k_0$ .

Consider  $\theta : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  the symmetrization (which is an isomorphism of  $adU(\mathfrak{g})$ -modules) and the adjoint action (denoted by  $ad$ ) of  $\mathfrak{m}$  on  $S(\mathfrak{g})$  which extends uniquely by derivation the adjoint action of  $\mathfrak{m}$  on  $\mathfrak{g}$  given by Lie bracket. Observe that

$$(76) \quad adx(s) = ad^*x(s) + ad_{\mathfrak{m}^-}x(s) + ad_{\mathfrak{m}}x(s).$$

Moreover for all  $1 \leq t \leq j$ , one has that

$$(77) \quad \begin{aligned} & \theta(y_1 \cdots y_{t-1} [x, y_t] y_{t+1} \cdots y_j) = \\ & \theta(y_1 \cdots y_{t-1} y_{t+1} \cdots y_j) [x, y_t] \pmod{U_{j-1}(\mathfrak{g})}. \end{aligned}$$

By equations (75), (77) and the previous lemma, it follows that

$$(78) \quad -x.f(su') = \sum_{\lambda \in \Lambda} \sum_{i \in I'_\lambda} \xi_{w_0\lambda} \left( a_i \theta(ad^*x(s))u'b_i.v_\lambda \right)$$

By equation (77) and the previous lemma, and since  $\mathfrak{m}.V'(\lambda) = \{0\}$  one also has that

$$(79) \quad \sum_{\lambda \in \Lambda} \sum_{i \in I'_\lambda} \xi_{w_0\lambda} (a_i \theta(ad_{\mathfrak{m}}x(s))u'b_i.v_\lambda) = 0.$$

We claim that

$$(80) \quad -x.f(su') = \sum_{\lambda \in \Lambda} \sum_{i \in I'_\lambda} \xi_{w_0\lambda} \left( a_i \theta(adx(s))u'b_i.v_\lambda \right).$$

By equation (76) it remains to show that

$$(81) \quad \sum_{\lambda \in \Lambda} \sum_{i \in I'_\lambda} \xi_{w_0\lambda}(a_i \theta(ad_{\mathfrak{m}^-} x(s)) u' b_i \cdot v_\lambda) = 0.$$

Fix an index  $t$ ,  $1 \leq t \leq j$ , with  $[x, y_t] \in \mathfrak{m}^-$  and set

$$u_t = \theta(y_1 \cdots y_{t-1} [x, y_t] y_{t+1} \cdots y_j).$$

Then  $u_t \in U^j(\mathfrak{m}^-)$ . Take  $\lambda \in \Lambda$  and  $i \in I'_\lambda$  (if there exist) such that

$$\xi_{w_0\lambda}(a_i u_t u' b_i \cdot v_\lambda) \neq 0.$$

Since all weight vectors non vanishing on  $\xi_{w_0\lambda}$  are proportional to  $v_{w_0\lambda}$ , one has that  $a_i u_t u' b_i \cdot v_\lambda$  is proportional to  $v_{w_0\lambda}$ .

On the other hand, one knows that  $v_{w_0\lambda} \in \mathcal{F}^{k_0}(V(\lambda)) \subset U^{k_0}(\mathfrak{m}^-) \cdot V'(\lambda)$  by (iv) of Lemma 3.2 and that  $k_0 = j - 1$  (otherwise  $I'_\lambda = \emptyset$ ). Then by the irreducibility of the  $U(\mathfrak{r})$ -module  $V'(\lambda)$ , there exists a nonzero weight vector  $u_{i,t} \in U^{j-1}(\mathfrak{m}^-) U(\mathfrak{n}_{\pi'}^-) U(\mathfrak{n}_{\pi'})$  such that

$$(82) \quad (a_i u_t - u_{i,t}) \cdot (u' b_i \cdot v_\lambda) = 0.$$

In other words, we have that

$$(83) \quad a_i u_t - u_{i,t} \in \text{Ann}_{U(\mathfrak{m}^-) U(\mathfrak{n}_{\pi'}^-) U(\mathfrak{n}_{\pi'})}(u' b_i \cdot v_\lambda)$$

For all  $\gamma \in \Delta^+ \setminus \Delta_{\pi'}^+$ , denote by  $r_{\gamma,i}$  the smallest positive integer such that

$$(84) \quad x_{-\gamma}^{r_{\gamma,i}} \cdot (u' b_i \cdot v_\lambda) = 0.$$

If we denote by  $\mu_i$  the weight of the vector  $u' b_i \cdot v_\lambda$ , we have that  $r_{\gamma,i} = \langle \gamma, \mu_i \rangle + 1$ , since  $x_\gamma \cdot (u' b_i \cdot v_\lambda) = 0$ .

Similarly for all  $\beta \in \Delta_{\pi'}^+$ , denote by  $r_{\beta,i}^\pm$  the smallest positive integer such that

$$(85) \quad x_{\pm\beta}^{r_{\beta,i}^\pm} \cdot (u' b_i \cdot v_\lambda) = 0.$$

See [16, 21] for more details.

Then one has that

$$(86) \quad \begin{aligned} a_i u_t - u_{i,t} \in & \sum_{\gamma \in \Delta^+ \setminus \Delta_{\pi'}^+} U(\mathfrak{m}^-) U(\mathfrak{n}_{\pi'}^-) U(\mathfrak{n}_{\pi'}) x_{-\gamma}^{r_{\gamma,i}} \\ & + \sum_{\beta \in \Delta_{\pi'}^+} U(\mathfrak{m}^-) U(\mathfrak{n}_{\pi'}^-) U(\mathfrak{n}_{\pi'}) x_{\beta}^{r_{\beta,i}^+} \\ & + \sum_{\beta \in \Delta_{\pi'}^+} U(\mathfrak{m}^-) U(\mathfrak{n}_{\pi'}^-) U(\mathfrak{n}_{\pi'}) x_{-\beta}^{r_{\beta,i}^-}. \end{aligned}$$

By the Poincaré-Birkhoff-Witt theorem ([4, 2.1.11]) setting  $\Delta^+ \setminus \Delta_{\pi'}^+ = \{\gamma_1, \dots, \gamma_r\}$ , we have that

$$(87) \quad u_t = \sum_{\vec{\nu}} c_{\vec{\nu}} x_{-\gamma_1}^{\nu_1} \cdots x_{-\gamma_r}^{\nu_r}$$



where the sum over the  $\vec{\nu} = (\nu_1, \dots, \nu_r) \in \mathbb{N}^r$  is finite,  $r \leq j$ , and with, for all  $\vec{\nu}$ ,  $c_{\vec{\nu}} \in \mathbb{k}$ . Moreover there exists only one  $\vec{\nu} \in \mathbb{N}^j$  with  $c_{\vec{\nu}} \neq 0$ .

Observe that

$$U(\mathfrak{r})U^j(\mathfrak{m}^-) = U^j(\mathfrak{m}^-)U(\mathfrak{r}) \text{ and } U^j(\mathfrak{m}^-)U(\mathfrak{r}) \cap U^{j-1}(\mathfrak{m}^-)U(\mathfrak{r}) = \{0\}.$$

Comparing equations (86) and (87) and using the above observation, one deduces that, for all  $\gamma \in \Delta^+ \setminus \Delta_{\pi'}^+$ , there exists  $w_{\gamma, i} \in U(\mathfrak{m}^-)$  and that, for all  $\beta \in \Delta_{\pi'}^+$ , there exists  $w_{\pm\beta, i} \in U^j(\mathfrak{m}^-)U(\mathfrak{n}_{\pi'}^-)U(\mathfrak{n}_{\pi'})$  such that

$$(88) \quad a_i u_t = \sum_{\gamma \in \Delta^+ \setminus \Delta_{\pi'}^+} a_i w_{\gamma, i} x_{-\gamma}^{r_{\gamma, i}} + \sum_{\beta \in \Delta_{\pi'}^+} w_{\beta, i} x_{\beta}^{r_{\beta, i}^+} + \sum_{\beta \in \Delta_{\pi'}^+} w_{-\beta, i} x_{-\beta}^{r_{-\beta, i}^-}.$$

Then by equations (84) and (85) one has that

$$(89) \quad \sum_{\lambda \in \Lambda} \sum_{i \in I'_\lambda} \xi_{w_0 \lambda} (a_i u_t u' b_i \cdot v_\lambda) = 0.$$

Hence

$$\begin{aligned} & \sum_{\lambda \in \Lambda} \sum_{i \in I'_\lambda} \xi_{w_0 \lambda} (a_i \theta(ad_{\mathfrak{m}^-} x(s)) u' b_i \cdot v_\lambda) = \\ & \sum_{1 \leq t \leq j} \sum_{[x, y_t] \in \mathfrak{m}^-} \sum_{\lambda \in \Lambda} \sum_{i \in I'_\lambda} \xi_{w_0 \lambda} (a_i u_t u' b_i \cdot v_\lambda) = 0 \end{aligned}$$

which is the required equation (81). We then obtain equation (80) and therefore, since  $\theta$  is a morphism of  $adU(\mathfrak{g})$ -modules,

$$\begin{aligned} -x \cdot f(su') &= \sum_{\lambda \in \Lambda} \sum_{i \in I'_\lambda} \xi_{w_0 \lambda} (a_i ad_x(\theta(s)) u' b_i \cdot v_\lambda) \\ &= \sum_{\lambda \in \Lambda} \sum_{i \in I'_\lambda} \xi_{w_0 \lambda} (a_i (x\theta(s) - \theta(s)x) u' b_i \cdot v_\lambda) \\ &= 0 \end{aligned}$$

since moreover  $\mathfrak{m} \cdot V'(\lambda) = \{0\}$  and  $V''(\lambda)^* \cdot \mathfrak{m} = \{0\}$ .  $\square$

**7.3. An important isomorphism.** Recall subsection 4.2 and in particular the representation, denoted by  $ad^*$ , of  $U(\tilde{\mathfrak{p}})$  in  $S(\mathfrak{p}^-)$  (see lemma 4.2) and also in every  $S_k(\mathfrak{p}^-)$  ( $k \in \mathbb{N}$ ). We then can endow  $S_k(\mathfrak{p}^-)^*$  with the dual representation of  $U(\tilde{\mathfrak{p}})$  given by

$$(90) \quad \forall u \in U(\tilde{\mathfrak{p}}), \forall f \in S_k(\mathfrak{p}^-)^*, u \cdot f = f \circ ad^* u^\top,$$

where  $u^\top$  denotes the image of  $u$  by the antipode defined similarly as in equation (15).

We have the following.

**Lemma.** *Let  $k \in \mathbb{N}$ . Then the  $U(\tilde{\mathfrak{p}})$ -module  $S_k(\mathfrak{p}^-)^*$  is isomorphic to the  $U(\tilde{\mathfrak{p}})$ -module  $S_k(\tilde{\mathfrak{p}}) = S_k(\mathfrak{p})$  when the latter is endowed with the adjoint action of  $\tilde{\mathfrak{p}}$  which extends by derivation the Lie bracket in  $\tilde{\mathfrak{p}}$ .*

*Proof.* For  $k = 0$ , the assertion is obvious. Recall (subsection 2.1) that we denote by  $K$  the Killing form on  $\mathfrak{g} \times \mathfrak{g}$ . Then the vector space  $\tilde{\mathfrak{p}} \simeq \mathfrak{p}$  is isomorphic to the dual space  $(\mathfrak{p}^-)^*$  through the map

$$f : x \in \tilde{\mathfrak{p}} \mapsto K(x, -)|_{\mathfrak{p}^-}.$$

When  $(\mathfrak{p}^-)^*$  is endowed with the action of  $U(\tilde{\mathfrak{p}})$  given by equation (44) and  $\tilde{\mathfrak{p}}$  with the adjoint action of  $\tilde{\mathfrak{p}}$ , the map  $f$  is an isomorphism of  $U(\tilde{\mathfrak{p}})$ -modules.

Indeed assume firstly that  $x' \in \mathfrak{m}$ . Then for all  $x \in \tilde{\mathfrak{p}}$  and for all  $y \in \mathfrak{p}^-$ ,  $x'.f(x)(y) = -K(x, pr_{\mathfrak{r}}([x', y]))$  by equations (17) and (44). If moreover  $x \in \mathfrak{m}$ , then

$$K(x, pr_{\mathfrak{r}}([x', y])) = 0.$$

But one also has  $[x', x]_{\tilde{\mathfrak{p}}} = 0$  by equation (2). Then

$$x'.f(x) = f([x', x]_{\tilde{\mathfrak{p}}})$$

in this case. If  $x \in \mathfrak{r}$ , then

$$x'.f(x)(y) = -K(x, pr_{\mathfrak{r}}([x', y])) = -K(x, [x', y])$$

since  $[x', y] = pr_{\mathfrak{r}}([x', y]) + pr_{\mathfrak{m}}([x', y]) + pr_{\mathfrak{m}^-}([x', y])$ . Then by the invariance of the Killing form, we obtain that

$$x'.f(x)(y) = K([x', x], y) = f([x', x]_{\tilde{\mathfrak{p}}})(y)$$

by equation (2). Now if  $x' \in \mathfrak{r}$ , then the assertion follows immediatly from the invariance of the Killing form and equation (16). This proves the lemma for  $k = 1$ .

Let now  $k \in \mathbb{N}^*$ . Consider the map  $f_k : S_k(\tilde{\mathfrak{p}}) \longrightarrow S_k(\mathfrak{p}^-)^*$  defined by

$$f_k(x_1 \cdots x_k) = K_k(x_1 \cdots x_k, -)|_{S_k(\mathfrak{p}^-)}$$

for all  $x_1, \dots, x_k \in \tilde{\mathfrak{p}}$  where  $K_k$  is defined as in [11, 2.7], namely : for  $y_1, \dots, y_k \in \mathfrak{p}^-$ ,

$$K_k(x_1 \cdots x_k, y_1 \cdots y_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \prod_{i=1}^k K(x_i, y_{\sigma(i)})$$

where we denote by  $\mathfrak{S}_k$  the group of permutations of  $k$  elements.

By [11, 2.7] we have that  $f_k$  is an isomorphism of vector spaces. It remains to show that  $f_k$  is an isomorphism of  $U(\tilde{\mathfrak{p}})$ -modules.

Let  $x_1, \dots, x_k \in \tilde{\mathfrak{p}}, y_1, \dots, y_k \in \mathfrak{p}^-$  and  $x \in \tilde{\mathfrak{p}}$ . Then one has

$$\begin{aligned} & x \cdot f_k(x_1 \cdots x_k)(y_1 \cdots y_k) \\ &= -K_k(x_1 \cdots x_k, ad^*x(y_1 \cdots y_k)) \\ &= -\sum_{i=1}^k K_k(x_1 \cdots x_k, y_1 \cdots y_{i-1} ad^*x(y_i) y_{i+1} \cdots y_k) \\ &= -\frac{1}{k!} \sum_{i=1}^k \sum_{\sigma \in \mathfrak{S}_k} \prod_{t \neq \sigma^{-1}(i)} K(x_t, y_{\sigma(t)}) K(x_{\sigma^{-1}(i)}, ad^*x(y_i)) \\ &= -\frac{1}{k!} \sum_{j=1}^k \sum_{\sigma \in \mathfrak{S}_k} \prod_{t \neq j} K(x_t, y_{\sigma(t)}) K(x_j, ad^*x(y_{\sigma(j)})) \end{aligned}$$

by exchanging both sums and setting  $j = \sigma^{-1}(i)$ .

On the other hand, one has

$$\begin{aligned} f_k(ad_{\tilde{\mathfrak{p}}}x(x_1 \cdots x_k))(y_1 \cdots y_k) &= \sum_{i=1}^k f_k(x_1 \cdots x_{i-1} [x, x_i]_{\tilde{\mathfrak{p}}} x_{i+1} \cdots x_k)(y_1 \cdots y_k) \\ &= \sum_{i=1}^k \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \prod_{t \neq i} K(x_t, y_{\sigma(t)}) K([x, x_i]_{\tilde{\mathfrak{p}}}, y_{\sigma(i)}) \end{aligned}$$

By the case  $k = 1$  one obtains that, for all  $1 \leq i \leq k$ , and all  $\sigma \in \mathfrak{S}_k$ ,

$$K([x, x_i]_{\tilde{\mathfrak{p}}}, y_{\sigma(i)}) = -K(x_i, ad^*x(y_{\sigma(i)})).$$

This completes the lemma by the above.  $\square$

**7.4. Kostant morphism.** Recall subsection 4.3 and let  $k \in \mathbb{N}$ .

We define

$$\psi_k : \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*) \longrightarrow S_k(\mathfrak{p}^-)^*$$

by the following. For all  $f \in \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*)$ , we set :

(91)

$$\forall j \in \mathbb{N}, 0 \leq j \leq k, \forall s \in S_j(\mathfrak{m}^-), \forall s' \in S_{k-j}(\mathfrak{r}), \psi_k(f)(ss') = f(s\theta(s'))$$

that we extend by linearity, so that  $\psi_k$  is a linear map. As in [10, 6.2] we call  $\psi_k$  the Kostant map.

**Proposition.** *Let  $k \in \mathbb{N}$ . The kernel of the linear map  $\psi_k$  is equal to  $\mathcal{F}_K^{k+1}(U(\tilde{\mathfrak{p}}^-)^*)$ . Moreover  $\psi_k$  is onto.*

*Proof.* It follows from the fact that  $\bigoplus_{j=0}^k S_j(\mathfrak{m}^-) \otimes \theta(S_{k-j}(\mathfrak{r}))$  is a complement of  $U_{k-1}(\tilde{\mathfrak{p}}^-)$  in  $U_k(\tilde{\mathfrak{p}}^-)$ .  $\square$

Endow  $U(\tilde{\mathfrak{p}}^-)^*$  with the dual representation of  $A$  given by equation (43). Let  $k \in \mathbb{N}$ . Then  $\mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*)$  is a left  $A$ -module by lemma 7.1 and  $S_k(\mathfrak{p}^-)^*$  is a left  $U(\tilde{\mathfrak{p}}^-)$ -module (see subsection 7.3).

**Lemma.** *Let  $k \in \mathbb{N}$ . Then the Kostant map  $\psi_k$  is a morphism from the left  $A$ -module  $\mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*)$  to the left  $U(\tilde{\mathfrak{p}})$ -module  $S_k(\mathfrak{p}^-)^*$ .*

*Proof.* Let  $f \in \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*)$  and  $0 \leq j \leq k$ ,  $j \in \mathbb{N}$ ,  $s \in S_j(\mathfrak{m}^-)$  and  $s' \in S_{k-j}(\mathfrak{t})$ .

Assume firstly that  $x \in \mathfrak{m}$ .

Then by equation (91)

$$\begin{aligned} \psi_k(x.f)(ss') &= (x.f)(s\theta(s')) \\ &= -f(ad^{**}x(s\theta(s'))) \\ &= -f(\tilde{\theta}(ad^*x(s))\theta(s')) \end{aligned}$$

by equations (43) and (29).

Write  $ad^*x(s) = \sum_{i \in I} s_i z_i$  with  $s_i \in S_{j-1}(\mathfrak{m}^-)$  and  $z_i \in \mathfrak{t}$  for all  $i \in I$ , by equation (26).

Then

$$(92) \quad \psi_k(x.f)(ss') = - \sum_{i \in I} f(s_i \theta(z_i) \theta(s'))$$

by definition of  $\tilde{\theta}$  (see subsection 4.3).

On the other hand, one has by equation (44) :

$$\begin{aligned} (x.\psi_k(f))(ss') &= -\psi_k(f)(ad^*x(ss')) \\ &= -\psi_k(f)(ad^*x(s)s') \end{aligned}$$

since  $ad^*x(s') = 0$  by equation (18).

Then by equation (91)

$$(93) \quad (x.\psi_k(f))(ss') = - \sum_{i \in I} f(s_i \theta(z_i s'))$$

But, for all  $i \in I$ ,  $s_i \theta(z_i s') = s_i \theta(z_i) \theta(s') \pmod{U_{k-1}(\tilde{\mathfrak{p}}^-)}$ . Equations (92) and (93) imply that

$$(\psi_k(x.f) - (x.\psi_k(f)))(ss') = 0$$

since  $f(U_{k-1}(\tilde{\mathfrak{p}}^-)) = 0$ .

Now assume that  $x \in \mathfrak{t}$ .

$$\begin{aligned} \psi_k(x.f)(ss') &= (x.f)(s\theta(s')) \\ &= -f(ad x(s\theta(s'))) \\ &= -f(ad x(s)\theta(s') + s ad x(\theta(s'))) \end{aligned}$$

by equation (16).

Then

$$\psi_k(x.f)(ss') = -f(ad x(s)\theta(s') + s\theta(ad x(s')))$$

since  $\theta$  is a morphism of  $U(\mathfrak{r})$ -modules for the adjoint action.

On the other hand

$$\begin{aligned} x.\psi_k(f)(ss') &= -\psi_k(f)(ad x(ss')) \\ &= -\psi_k(f)(ad x(s)s' + s ad x(s')) \\ &= -f(ad x(s)\theta(s') + s\theta(ad x(s'))) \end{aligned}$$

This completes the lemma.  $\square$

7.5. Recall subsections 7.1 and 7.4. By Proposition and Lemma 7.4 we obtain the following lemma.

**Lemma.** *For all  $k \in \mathbb{N}$ , the induced morphism (still denoted by  $\psi_k$ ) is an isomorphism of left  $U(\tilde{\mathfrak{p}})$ -modules from  $gr_K^k(U(\tilde{\mathfrak{p}}^-)^*)$  to  $S_k(\mathfrak{p}^-)^*$ .*

*Proof.* We already know that the left  $A$ -module structure on  $U(\tilde{\mathfrak{p}}^-)^*$  given by equation (43) induces a left  $A$ -module structure on  $gr_K^k(U(\tilde{\mathfrak{p}}^-)^*)$  by the invariance of the Kostant filtration under the left action of  $A$  (see equation (69)). Then the induced morphism  $\psi_k$  is an isomorphism from the left  $A$ -module  $gr_K^k(U(\tilde{\mathfrak{p}}^-)^*)$  to the left  $U(\tilde{\mathfrak{p}})$ -module  $S_k(\mathfrak{p}^-)^*$ . Moreover since  $S_k(\mathfrak{p}^-)^*$  is a left  $U(\tilde{\mathfrak{p}})$ -module, it follows that it is the same for  $gr_K^k(U(\tilde{\mathfrak{p}}^-)^*)$ . Let us verify directly that  $gr_K^k(U(\tilde{\mathfrak{p}}^-)^*)$  is indeed a left  $U(\tilde{\mathfrak{p}})$ -module. Let  $x, x' \in \mathfrak{m}$  and  $u \in U_k(\tilde{\mathfrak{p}}^-)$ . One checks that

$$(94) \quad (ad^{**}x \circ ad^{**}x' - ad^{**}x' \circ ad^{**}x)(u) \in U_{k-1}(\tilde{\mathfrak{p}}^-).$$

Indeed write  $u = su'$  with  $s \in S_j(\mathfrak{m}^-)$  and  $u' \in U_{k-j}(\mathfrak{r})$  for  $0 \leq j \leq k$ . If  $j = 0$  then  $ad^{**}x \circ ad^{**}x'(su') = 0 = ad^{**}x' \circ ad^{**}x(su')$  by equation (31). Now assume that  $1 \leq j \leq k$  and take  $s = y_1 \cdots y_j \in S_j(\mathfrak{m}^-)$ , with  $y_i \in \mathfrak{m}^-$  for all  $1 \leq i \leq j$ . By definition of  $ad^{**}$  (see equation (29)), we obtain that

$$\begin{aligned} & ad^{**}x \circ ad^{**}x'(su') \\ &= \sum_{1 \leq i \neq k \leq j} \prod_{t \notin \{i, k\}} y_t \theta(ad^*x(y_k)) \theta(ad^*x'(y_i)) u' \\ &= \sum_{1 \leq i \neq k \leq j} \prod_{t \notin \{i, k\}} y_t \theta(ad^*x(y_k) ad^*x'(y_i)) u' \quad \text{mod } U_{k-1}(\tilde{\mathfrak{p}}^-) \\ &= \sum_{1 \leq i \neq k \leq j} \prod_{t \notin \{i, k\}} y_t \theta(ad^*x(y_i) ad^*x'(y_k)) u' \quad \text{mod } U_{k-1}(\tilde{\mathfrak{p}}^-) \\ &= ad^{**}x' \circ ad^{**}x(su') \quad \text{mod } U_{k-1}(\tilde{\mathfrak{p}}^-) \end{aligned}$$

since for all  $a, b \in \mathfrak{r}$ , one has  $\theta(a)\theta(b) = \theta(ab) \quad \text{mod } U_1(\mathfrak{r})$ .

One deduces that, for all  $f \in \mathcal{F}_K^k(U(\tilde{\mathfrak{p}}^-)^*)$ , for all  $x, x' \in \mathfrak{m}$ ,

$$x.(x'.f) - x'.(x.f) \in \mathcal{F}_K^{k+1}(U(\tilde{\mathfrak{p}}^-)^*)$$

and then

$$x.(x'.gr_K^k(f)) = x'.(x.gr_K^k(f)).$$

□

Recall the notation in the proof of Lemma 7.3. Let  $k \in \mathbb{N}$  and set  $j_k = f_k^{-1}$ , which is an isomorphism of  $U(\tilde{\mathfrak{p}})$ -modules from  $S_k(\mathfrak{p}^-)^*$  to  $S_k(\tilde{\mathfrak{p}})$ . Moreover set  $\psi_k^0 = j_k \circ \psi_k$  and  $\psi^0 = \bigoplus_{k \in \mathbb{N}} \psi_k^0$ : this is by the above an isomorphism of  $U(\tilde{\mathfrak{p}})$ -modules from  $gr_K(U(\tilde{\mathfrak{p}}^-)^*)$  to  $S(\tilde{\mathfrak{p}})$ . Now as in [10, 6.6] set  $\tilde{\psi}_k = \frac{1}{k!} \psi_k^0$  and  $\tilde{\psi} = \bigoplus_{k \in \mathbb{N}} \tilde{\psi}_k$ . One deduces the following, as in [10, 6.6].

**Proposition.**  $\tilde{\psi}$  is an isomorphism of  $U(\tilde{\mathfrak{p}})$ -modules and of algebras from  $gr_K(U(\tilde{\mathfrak{p}}^-)^*)$  to  $S(\tilde{\mathfrak{p}})$ .

7.6. Denote by  $gr_K(\tilde{C}_\tau)$  the graded algebra associated to the induced generalized Kostant filtration on  $\tilde{C}_\tau$ , and by  $gr_K(\tilde{C}_\tau^{U(\tau')})$  the graded algebra associated to the induced generalized Kostant filtration on  $\tilde{C}_\tau^{U(\tau')}$ . Denote by  $(gr_K(U(\tilde{\mathfrak{p}}^-)^*))^{U(\tilde{\mathfrak{p}}')}$  the algebra of invariants in  $gr_K(U(\tilde{\mathfrak{p}}^-)^*)$  by the action of  $U(\tilde{\mathfrak{p}}')$  given by equation (69). We have that

$$gr_K(\tilde{C}_\tau) \subset gr_K(U(\tilde{\mathfrak{p}}^-)^*)$$

and by Proposition 7.2 that

$$gr_K(\tilde{C}_\tau^{U(\tau')}) \subset (gr_K(U(\tilde{\mathfrak{p}}^-)^*))^{U(\tilde{\mathfrak{p}}')}.$$

Denote also by  $S(\tilde{\mathfrak{p}})^{U(\tilde{\mathfrak{p}}')}$  the algebra of invariants in  $S(\tilde{\mathfrak{p}})$  by the adjoint action of  $U(\tilde{\mathfrak{p}}')$ : this is also the algebra of semi-invariants in  $S(\tilde{\mathfrak{p}})$ , which we denote by  $Sy(\tilde{\mathfrak{p}})$ . From Proposition 7.5, one deduces the following.

**Theorem.** One has that

$$\tilde{\psi}(gr_K(\tilde{C}_\tau^{U(\tau')})) \subset Sy(\tilde{\mathfrak{p}}).$$

7.7. Let  $M$  denote a left  $\mathfrak{h}$ -module such that each of its weight spaces  $M_\nu = \{m \in M \mid \forall h \in \mathfrak{h}, h.m = \nu(h)m\}$ , for all  $\nu \in \mathfrak{h}^*$ , is finite dimensional. Then one may define ([17, 3.4.7]) the formal character  $\text{ch } M$  of  $M$  as follows.

$$\text{ch } M = \sum_{\nu \in \mathfrak{h}^*} \dim M_\nu e^\nu.$$

Recall the set  $E(\pi')$  in subsection 6.1, and for all  $\Gamma \in E(\pi')$ , set  $\delta_\Gamma = w'_0 d_\Gamma - w_0 d_\Gamma$ , with  $d_\Gamma = \sum_{\gamma \in \Gamma} \varpi_\gamma$ . By Prop. 6.2 and Theorem 6.1, the algebra of invariants  $\tilde{C}_\tau^{U(\tau')}$  is a polynomial algebra in  $|E(\pi')|$  variables, each of them having  $\delta_\Gamma$ ,  $\Gamma \in E(\pi')$ , as an  $\mathfrak{h}$ -weight. Moreover for  $\mathfrak{g}$  simple and  $\pi' \subsetneq \pi$  that is, for  $\mathfrak{p} \subsetneq \mathfrak{g}$ , one has that  $\delta_\Gamma \in P^+(\pi) \setminus \{0\}$  for all  $\Gamma \in E(\pi')$  by [10, 5.4.3]. Then in this case  $\text{ch } \tilde{C}_\tau^{U(\tau')}$  is well defined.

**Proposition.** *Assume, for all  $\nu \in \mathfrak{h}^*$ , that the weight vectorspace  $Sy(\tilde{\mathfrak{p}})_\nu$  is finite-dimensional, so that the formal character  $\text{ch } Sy(\tilde{\mathfrak{p}})$  is well defined. Then*

$$\text{ch } \tilde{C}_\tau^{U(\tau')} \leq \text{ch } Sy(\tilde{\mathfrak{p}})$$

namely

$$\prod_{\Gamma \in E(\pi')} (1 - e^{\delta_\Gamma})^{-1} \leq \text{ch } Sy(\tilde{\mathfrak{p}}).$$

*Proof.* Recall that the generalized Kostant filtration  $\mathcal{F}_K$  is decreasing, separated and that  $\mathcal{F}_K^0(U(\tilde{\mathfrak{p}}^-)^*) = U(\tilde{\mathfrak{p}}^-)^*$ . Then, for every finite dimensional subspace  $V$  of  $U(\tilde{\mathfrak{p}}^-)^*$ , there exists  $N \in \mathbb{N}$  such that  $V \cap \mathcal{F}_K^N(U(\tilde{\mathfrak{p}}^-)^*) = \{0\}$ . One deduces easily that the graded vector space  $gr_K(V)$  associated to the induced generalized Kostant filtration on  $V$  is isomorphic to  $V$ . Using Theorem 7.6 and a same argument as in [10, 7.1] completes the proof.  $\square$

## REFERENCES

- [1] Bourbaki N.: *Eléments de mathématique, Groupes et Algèbres de Lie, Chapitres 4, 5 et 6*, Masson, Paris et al. (1981).
- [2] Bourbaki N.: *Eléments de mathématique, Groupes et algèbres de Lie, Chapitres 7 et 8*, Springer-Verlag, Berlin Heidelberg, 2006.
- [3] Bourbaki N.: *Eléments de mathématique, Algèbre commutative*, Hermann, Paris 1968.
- [4] Dixmier J.: *Algèbres Enveloppantes*, Editions Jacques Gabay, les Grands Classiques Gauthier-Villars, Paris (1974).
- [5] Elashvili A.G.: *Canonical form and stationary subalgebras of points of general position for simple linear Lie groups*, *Funct. Anal. Appl.* **6** (1972) 44–53.
- [6] Elashvili A.G.: *Stationary subalgebras of points of general position for irreducible linear Lie groups*, *Funct. Anal. Appl.* **6** (1972) 139–148.
- [7] Fauquant-Millet F.: *Weierstrass sections for some truncated parabolic subalgebras*, *Journal of Algebra* **580** (2021) 299–365.
- [8] Fauquant-Millet F.: *Sur une algèbre parabolique  $\mathbf{P}$  de  $\check{U}_q(\mathfrak{sl}_{n+1})$  et ses semi-invariants par l'action adjointe de  $\mathbf{P}$* , *Bull. Sci. math.* **122** (1998) 495–519.
- [9] Fauquant-Millet F., Joseph A.: *Sur les semi-invariants d'une sous-algèbre parabolique d'une algèbre enveloppante quantifiée*, *Transform. Groups* **6** (2001) 125–142.
- [10] Fauquant-Millet F., Joseph A.: *Semi-centre de l'algèbre enveloppante d'une sous-algèbre parabolique d'une algèbre de Lie semi-simple*, *Ann. Sci. Éc. Norm. Sup.* **38** (2005) 155–191.
- [11] Fauquant-Millet F., Joseph A.: *La somme des faux degrés - un mystère en théorie des invariants*, *Adv. Math.* **217** (2008) 1476–1520.
- [12] Fauquant-Millet F., Joseph A.: *Adapted pairs and Weierstrass sections*, <https://arxiv.org/abs/1503.02523>.
- [13] Feigin E.:  $\mathbb{G}_A^M$ -*degeneration of flag varieties*, *Selecta Math. (N.S.)* **18**, 513–537 (2012).
- [14] Feigin E.: *Degenerate flag varieties and the median Genocchi numbers*, *Math. Res. Lett.* **18**, 1163–1178 (2011).
- [15] Feigin E., Fourier G., Littelmann P.: *PBW filtration and bases for irreducible modules in type  $A_n$* , *Transformation Groups* **16**, 71–89 (2011). <https://doi.org/10.1007/s00031-010-9115-4>

- [16] Humphreys J.E. : *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics, Springer-Verlag New-York, 1972.
- [17] Joseph A.: *Quantum Groups and Their Primitive Ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer Verlag, Berlin-Heidelberg-New-York, 1995.
- [18] Panyushev D.: *On the coadjoint representation of  $\mathbb{Z}_2$ -contractions of reductive Lie algebras*, Adv. Math. **213** (2007), no.1, 343–404.
- [19] Panyushev D., Premet A., Yakimova O.: *On symmetric invariants of centralisers in reductive Lie algebras*, J. Algebra **313** (2007) 343–391.
- [20] Panyushev D., Yakimova O.: *A remarkable contraction of semisimple Lie algebras*, Ann. Inst. Fourier (Grenoble) **62** (2012) 2053–2068.
- [21] Panyushev D., Yakimova O.: *Parabolic contractions of semisimple Lie algebras and their invariants*, Sel. Math. New Ser. **19** (2013) 699–717.
- [22] Panyushev D., Yakimova O.: *Semi-direct products of Lie algebras and covariants*, J. Algebra, **490**, 2017, 283–315.
- [23] Panyushev D., Yakimova O.: *Semi-direct products involving  $Sp_{2n}$  or  $Spin_n$  with free algebras of symmetric invariants*, in Representations and Nilpotent Orbits of Lie Algebraic Systems, 441-470, part of the Progress in Mathematics book series (PM, volume 330), 2019, Birkhäuser, Springer.
- [24] Panyushev D., Yakimova O.: *Symmetric invariants related to representations of exceptional simple groups*, Trans. Moscow Math. Soc. **78** (2017) 161–170.
- [25] Phommady K., *Semi-invariants symétriques de contractions paraboliques*, <https://arxiv.org/abs/2007.14185>.
- [26] Popov V. L. *Sections in invariant theory*. The Sophus Lie Memorial Conference (Oslo 1992), Scand. Univ. Press, Oslo (1994) 315-361.
- [27] Popov V. L., Vinberg E.B. *Invariant Theory*, in Encyclopaedia of Mathematical Sciences, Vol. 55, Algebraic Geometry IV, Springer-Verlag Berlin, Heidelberg 1994.
- [28] Renault G.: *Algèbre non commutative*, Collection “Varia Mathematica”, Gauthier-Villars, 1975.
- [29] Schwarz G.: *Representations of simple Lie groups with regular rings of invariants*, Invent. Math. 49 (1978) 167–191.
- [30] Yakimova O.: *One-parameter contractions of Lie-Poisson brackets*, J. Eur. Math. Soc. **16** (2014), 387–407.
- [31] Yakimova O.: *Symmetric invariants of  $\mathbb{Z}_2$ -contractions and other semi-direct products*, Int. Math. Res. Notices **2017** (2017)1674–1716.
- [32] Yakimova O.: *Some semi-direct products with free algebras of symmetric invariants*, In: Perspectives in Lie Theory, F. Callegaro, G. Carnovale, F. Caselli, C. De Concini, A. De Sole (eds.), Springer, Cham **19** (2017) 267–279.

UNIVERSITÉ JEAN MONNET, SAINT-ETIENNE, INSTITUT CAMILLE JORDAN, CNRS  
UMR 5208, F- 42023 SAINT-ETIENNE, FRANCE

*E-mail address:* [florence.millet@univ-st-etienne.fr](mailto:florence.millet@univ-st-etienne.fr)