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## Notes on Milnor-Witt K-theory

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# NOTES ON MILNOR-WITT K-THEORY

F. DÉGLISE

ABSTRACT. These notes are devoted to the foundations of Milnor-Witt K-theory of fields of arbitrary characteristic and without any perfectness assumptions. Extending the fundamental work of Morel, we establish all its functorial properties as stated in Feld’s theory of Milnor-Witt modules, with a special attention about twists. The main new result is a computation of transfers in the general (in particular inseparable) case in terms of Grothendieck (differential) trace maps. These notes are used as the foundation for an expository work on Chow-Witt groups. They are built upon a series of talks given at the Spring School “Invariants in Algebraic Geometry”, organized by Daniele Faenzi, Adrien Dubouloz and Ronan Terpereau.

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## 1. INTRODUCTION

**1.1. Milnor's K-theory and conjectures.** One is retrospectively amazed by John Milnor's cornerstone work [Mil70] on K-theory and quadratic forms. At first sight, Milnor failed to define higher K-groups of a field  $K$  as one now knows that the K-theory he defines only agrees with higher K-theory up to degree 2, by Matsumoto's theorem.<sup>1</sup> On the other hand, the graded ring defined by Milnor was to be recognized more than 20 years later<sup>2</sup> as an invariant as fundamental as algebraic K-theory: it is the  $(n, n)$ -part of the motivic cohomology ring of  $k$ . So that Milnor's definition was truly the first appearance of motivic cohomology, in its symbolic guise.

Moreover, Milnor had the brilliant insight of relating his new K-theory ring to two apparently unconnected classical invariants: Galois cohomology of  $K$  with coefficients in the 2-torsion ring  $\mathbb{Z}/2$ , and the graded algebra associated with the fundamental ideal of the Witt ring of  $K$ . One motivation came from the theory of characteristic classes [MS74]<sup>3</sup>, and in particular from the so-called Stiefel-Whitney classes  $w_n(\xi) \in H^n(X, \mathbb{Z}/2)$  of a real vector bundle  $\xi$  over a smooth manifold  $X$ . Among the inspirations of Milnor was a very short note [Del62] of Delzant defining an algebraic analogue of these classes, for a field of characteristic not 2:

$$w_n^D : \mathrm{GW}(K) \rightarrow H^n(G_K, \mathbb{Z}/2)$$

where  $\mathrm{GW}(K)$  is the Grothendieck group of quadratic  $K$ -vector spaces<sup>4</sup> and the right hand-side is the  $\mathbb{Z}/2$ -cohomology of the absolute Galois group of  $K$ , or equivalently the étale  $\mathbb{Z}/2$ -cohomology of  $\mathrm{Spec}(K)$ .

<sup>1</sup>Among his inspiration, Milnor cites Moore and Matsumoto's works. See 2.2.1 and Remark 2.2.2 for recall.

<sup>2</sup>By a famous theorem of Totaro [Tot92].

<sup>3</sup>Recall that this book is based on Milnor's 1957 lectures!

<sup>4</sup>Though Delzant was obviously inspired by the Witt ring introduced in [Wit37], this is the first occurrence of  $\mathrm{GW}(K)$  in the literature. See 2.1.2 more generally.

Milnor remarks that Delzant's Stiefel-Whitney classes factorize through his K-groups modulo 2, thus obtaining a factorization of  $w_n^D$  as:

$$\mathrm{GW}(K) \xrightarrow{w_n} \mathrm{K}_n^M(K)/2 \xrightarrow{h_K} H^n(G_K, \mathbb{Z}/2)$$

where  $h_K$  is sometimes called the *norm residue homomorphism* or the *Galois symbol*. The first of the Milnor conjectures states that  $h_K$  is an isomorphism for all  $n > 0$ .<sup>5</sup>

Even more remarkably, Milnor recognizes a way to go backward the map  $w_n$ , and formulates another conjecture<sup>6</sup> relating the Witt ring with his K-theory ring. This question was solved more than 30 years later by Orlov, Vishik, Voevodsky in [OVV07], after Voevodsky's proof of the first Milnor conjecture and his revolutionary idea of introducing *motivic homotopy theory*.

**1.2. Barge and Morel obstruction theory.** Extending Milnor's ideas on characteristic classes, and building on the ideas of motivic homotopy, Jean Barge and Fabien Morel introduced in [BM00] an algebraic-geometrical analogue of the Euler class of real oriented vector bundles (see [MS74, §9, Def. p. 98]). The technical innovation of their definition is the construction of an appropriate algebraic analogue of the integral singular cohomology of a real manifold, a cohomology ring that they call the *Chow groups of oriented cycles*, which one now simply calls the *Chow-Witt groups*.

While Milnor K-theory modulo 2 is a suitable receptacle for Stiefel-Whitney classes (over fields), as seen above, Barge and Morel had the idea of gluing above this 2-torsion ring the integral information coming from the fundamental ideal  $I(K)$  and the Milnor K-ring  $\mathrm{K}_*^M(K)$ : see [BM00, section 1] or Corollary 2.3.7 here. The resulting graded ring, denoted by  $J^*(E)$  in *loc. cit.* is now called the *Milnor-Witt K-theory* of  $K$  and denoted by  $\mathrm{K}_*^{MW}(K)$ . The study of this functor on fields is the main subject of the present expository notes.

Before diving into the motivation for this work, let us come back briefly to the work of Barge and Morel: using works of Rost [Ros96] and Schmid [Sch97], they define the Chow-Witt groups of a smooth  $k$ -scheme as the cohomology of a Gersten-like complex with coefficients in  $\mathrm{K}_*^{MW}$ , which is now called the *Rost-Schmid complex*. Then they define the Euler class of an *oriented* algebraic vector bundle and make various conjectures about it (Conjecture p. 289, Rem. 2.4 in *loc. cit.*).<sup>7</sup>

<sup>5</sup>This was later generalized by Bloch and Kato by replacing 2 with an arbitrary prime. A complete proof of the Bloch-Kato conjecture is available in [HW19], and a detailed account of the history of the Milnor conjecture can be found in Section 1.7 of *loc. cit.*

<sup>6</sup>see question 4.3 of [Mil70] or Theorem 2.2.3 of the present paper

<sup>7</sup>The conjecture was partially solved by Morel in [Mor12, Th. 3.12] and was pushed much further by works of Asok and Fasel; see [AF21] for a survey, and in particular §4.2 for a review on (Barge-Morel) Euler classes.

**1.3. Basic aim and scope of these notes.** The aim of these notes is to lay the ground for the theory of quadratic cycles and Chow-Witt groups, based on Feld's axiomatic approach [Fel20a]. We will therefore present the theory of the Milnor-Witt ring of fields, with an emphasis on its functoriality. As a motivation, the hasty reader can take a look at Section 5, and in particular the first paragraph, for the functoriality properties we are aiming. We improve the theory known so far by removing any assumption on the fields considered.<sup>8</sup>

Our presentation of Milnor-Witt K-theory mixes two historical approaches. The first one, due to Barge and Morel as already mentioned above, gives the construction as a gluing of Witt's theory of quadratic forms<sup>9</sup> and Milnor's K-ring. See Corollary 2.3.7 for the statement. The second one, due to Hopkins and Morel, is much closer to Milnor's viewpoint, and gives a beautiful presentation of the Milnor-Witt K-theory ring in terms of explicit generators and relations (Definition 2.2.7). In fact, the richness of the theory comes from the comparison of these two approaches. This is based on Milnor's conjecture: here we refer to [GSZ16] in characteristic not 2, and to [Car] in characteristic 2. Except for references to these two papers, and to basics on Witt rings ([MH73]) and Milnor K-theory ([BT73], [Kat82]), these notes are self-contained.

As explained in the previous section, the justification for defining Milnor-Witt K-theory in addition to Milnor K-theory is to be able to develop an algebraic orientation theory. The concrete trace of this orientation theory is the existence of twists on the former K-theory which we consider as part of the structure almost from the start. The first phenomenon that demonstrates the need to consider twists is the construction of residues associated to a discretely valued field  $(K, v)$  with residue field  $\kappa_v$ :

$$\partial_v : K_*^{MW}(K) \rightarrow K_*^{MW}(\kappa_v, \omega_v).$$

Here the twist is given by the *normal space*  $\omega_v$  associated with the valuation  $v$ ; see Definition 2.5.3 and therein for more. In our opinion, the consideration of twists shades lights also to Witt's theory as explained in Example 2.5.7.

**1.4. Transfers.** In fact, most of the paper is devoted to the study of transfer maps on Milnor-Witt K-theory. This is no wonder as it was a famous problem left open by Bass and Tate for Milnor K-theory ([BT73, I.§5]), only resolved by Kato in [Kat80, §1.7, Prop. 5]. Given a finite field extension  $E/k$ , or  $\varphi : k \rightarrow E$ , the transfer map has the form (see Definition 4.4.6):

$$(1.4.0.a) \quad \mathrm{Tr}_{E/k}^{MW} = \varphi^* : K_n^{MW}(E, \omega_{E/k}) \rightarrow K_n^{MW}(k),$$

---

<sup>8</sup>Usually, one considers fields of characteristic not 2, and sometimes one assumes they are finitely generated over some perfect base field.

<sup>9</sup>Or rather Grothendieck-Witt theory of inner product spaces in order to allow fields of characteristic two;

where  $\omega_{E/k}$  is the determinant of the cotangent complex of  $E/k$  (we call this the canonical module see Definition 4.1.5). Here again, the twist is essential, though it is trivial for (and only for) separable extensions.

For finitely generated field extension over a perfect base field of characteristic not 2, these transfer maps were introduced by Morel in [Mor12, Chap. 4], in the more general context of strongly  $\mathbb{A}^1$ -invariant sheaves but mostly avoiding twists. The theory was recast for Milnor-Witt K-theory, still with the same restriction on fields, by Feld in [Fel20b].

There are two methods to define transfers on functors defined on fields. The first one is to follow the approach of Bass and Tate via residue maps and what is called after Rost the *Weil reciprocity formula* (see Remark 4.2.4(2) for the case of the projective line). The second one is by gluing known transfer maps. This is closer to the approach of Fasel for defining pushforward on Chow-Witt groups (see [Fas08, Cor. 10.4.5]).

*Bass-Tate method.* In these notes, we develop the two approaches. For the Bass-Tate method, we have chosen to introduce Chow-Witt groups of Dedekind schemes. This serves both as an illustration of the theory and as convenient language to express the Weil reciprocity formula: this amounts to a computation of Chow-Witt groups of quadratic 0-cycles of the projective line. We state it here for the convenience of the reader:

**Theorem.** (see Theorem 3.4.4) *Let  $k$  be a field, and  $\mathcal{L}$  be an invertible sheaf on the projective line  $\mathbb{P}_k^1$ . We let  $\mathcal{L}_\infty$  be the restriction of  $\mathcal{L}$  over the point at  $\infty$  and  $\omega_\infty$  be the conormal sheaf of the immersion  $i_\infty : \{\infty\} \rightarrow \mathbb{P}_k^1$ .*

*Then the Chow-Witt group (in  $\mathbb{G}_m$ -degree 0) of quadratic divisors of  $\mathbb{P}_k^1$  with coefficients in  $\mathcal{L}$  are given (induced by pushforward along  $i_\infty$ ) by:*

$$\widetilde{\text{CH}}^1(\mathbb{P}_k^1, \mathcal{L}) \simeq \begin{cases} \text{GW}(k, \omega_\infty \otimes \mathcal{L}_\infty) & \deg(\mathcal{L}) \text{ even,} \\ \mathbb{Z} & \deg(\mathcal{L}) \text{ odd.} \end{cases}$$

This theorem perfectly illustrates the role of twists in Milnor-Witt K-theory and Chow-Witt groups, which is the major difference with Milnor K-theory and usual Chow groups. It was first proved by Fasel in [Fas13], though we lift any restriction on the base field. Applying the above result when  $\mathcal{L}$  is the canonical sheaf  $\omega = \mathcal{O}(-2)$  on  $\mathbb{P}_k^1$  gives the degree map on Chow-Witt groups:

$$\text{deg} : \widetilde{\text{CH}}^1(\mathbb{P}_k^1, \omega) \rightarrow \text{GW}(k).$$

This degree map actually encompasses all the transfer maps for Milnor-Witt K-theory for any monogenic extension field of  $k$ .<sup>10</sup> This is the geometric interpretation

<sup>10</sup>We have formulated here the results for  $\mathbb{G}_m$ -degree 1 for the sake of clarity. It is important to note that to get transfer maps in other degrees, one needs to consider the whole gradation on the Rost-Schmid complex, corresponding to the fact that Chow-Witt groups are the 0-th  $\mathbb{G}_m$ -graded part of a bigraded group (see Definition 3.1.5 for more details).

of the method of Bass and Tate.<sup>11</sup> The problem with this approach is to show that these transfers are independent of the chosen generators, and that one can extend the definition to any finite field extension.

*Glueing and differential traces.* Instead of proving this directly, as is done by Morel and Feld, we use the second mentioned approach. On the one hand, one has well-defined transfer maps on Milnor K-theory after Kato. For the Grothendieck-Witt part (based on inner product forms to deal with characteristic 2), we show that one can define twisted transfer maps by directly using the trace morphism that follow from Grothendieck duality formalism, and which we call the differential trace map (see Definition 6.2.4):

$$\mathrm{Tr}_{E/k}^{\omega} : \omega_{E/k} \rightarrow k.$$

Using this map and a classical method of Scharlau, one deduces transfers for twisted Grothendieck-Witt groups. The advantage of these transfers is that they do not depend on any choice, and yet can be compared precisely to Scharlau's ones (see Remark 4.3.5).<sup>12</sup> Moreover, they can be glued appropriately to Kato's transfers on Milnor K-theory and induces the desired transfers (1.4.0.a) on twisted Milnor-Witt K-theory.

The important result is to compare these glued transfers with the one obtained by the Bass-Tate method. According to the uniqueness property of the latter, this involves to check a *twisted form of the quadratic reciprocity law* for the transfers based on the differential trace maps. This is an enhancement of the classical result of Scharlau, now valid in arbitrary characteristics (see Theorem 4.3.7 for the exact statement). As a consequence, our definition agrees with that of Morel, and we also reprove the independence theorem of Morel and Feld (see Proposition 4.4.13).

The nice feature of the second way of defining transfers is that it is well-suited for computing trace maps. In particular, the differential trace map can be computed explicitly using a method of Sheja and Storch. This links the trace maps of Milnor-Witt K-theory with the method of Kass and Wickelgren for computing (local)  $\mathbb{A}^1$ -brouwer degree ([KW19]). One can express these computations in term of *Bezoutian* (see 6.3.4 and 6.3.8) and this is deeply linked with methods developed in [BMP21].<sup>13</sup> As an example let us state the following formula which complete the étale case which was already known (from [Hoy14, Lemma 5.8]):

**Theorem** (see Example 4.4.9(3)). *Let  $k$  be a field of positive characteristic  $p > 0$ , and  $a \in k^{\times}$  be an element which is not a power of  $p$ . Consider the inseparable extension  $E = k[\sqrt[q]{a}]$  of  $k$ ,  $q = p^n$ .*

<sup>11</sup>see [BT73, (5.4)], §(5.4), and especially diagram (3)

<sup>12</sup>Note that this kind of construction of twisted transfers for Witt groups (of  $\mathbb{Z}[\frac{1}{2}]$ -schemes) has been previously considered by several authors including [Gil02, Nen07, CH11]. Our treatment is simple and direct, and is well-suited for explicit computations, as explained below.

<sup>13</sup>This will be made more precise in the second part of these notes [Dég23].

Let  $\tau_{E/k}^\alpha$  be Tate trace map associated with  $E/k$  and the choice of  $\alpha = \sqrt[\alpha]{a}$  (see [Tat52, §1, (2)] and Remark 6.3.10). Let  $w = dt \otimes (\overline{t^q - a})^*$  be the non zero element of the canonical module  $\omega_{E/k}$ . Then for any unit  $u \in E^\times$ , the following formula holds in  $\text{GW}(k)$ :

$$\text{Tr}_{E/k}^{MW}(\langle u \rangle \otimes w) = [\tau_{E/k}^\alpha(u \cdot -)]$$

where the right hand-side denotes the class of the inner product:  $(x, y) \mapsto \tau_{E/k}^\alpha(uxy)$ .

This formula was actually our main motivation for writing these notes. It illustrates again the importance of the twists in computation with Milnor-Witt K-theory. In particular, in the above formula, changing  $w$  usually completely modify the result of the computation. We give further examples of this phenomenon in Example 4.4.9, as well as an analogue of the degree formula for Milnor K-theory: Corollary 4.4.10.

**1.5. Reading guide.** For readers who like axiomatic presentations, it is a good idea to start with the list of structural maps of Milnor-Witt K-theory (Section 5.1) and the basic rules they satisfy (Section 5.2).

This work is roughly divided into three parts, with some preliminaries.

The preliminaries consist first of recall on the theory of quadratic forms over fields, but more precisely on inner products spaces to deal with arbitrary characteristics. The main results and computations concerning the Grothendieck-Witt and Witt rings are recalled in Section 2.1.

Secondly, after a quick recollection on cotangent complexes and canonical sheaves (or modules) in Section 4.1, we have given in an appendix (Section 6) some reminders on Grothendieck coherent duality theory, and explain the link with the work of Sheja and Storch which allows explicit computations. In particular, we define explicitly what we call the *differential trace map*, and give various interesting properties and formulas: the expression of Grothendieck residue symbols (Definition 6.3.2), the computations in term of explicit presentations and Bezoutians: see Proposition 6.3.12 and (6.3.12.a).

The first part concerns the definition and basic properties of the Milnor-Witt K-ring of a field<sup>14</sup>: this is essentially Section 2. As mentioned, we start with the presentation in term of generators and relation (Definition 2.2.7), which is the more elementary and then relate it to the presentation in terms of Milnor K-theory and the fundamental ideal (Corollary 2.3.7). Recall this relation is a (non-trivial) consequence of the second Milnor conjecture (stated in Theorem 2.2.3).

Explicit computations are given in 2.3.1, 2.3.8, 2.3.9, 2.3.10 and 2.3.11. The main specificity of Milnor-Witt K-theory are twists. We introduce them in a second step, and use them to define residue maps in the end of Section 2.

<sup>14</sup>See Remark 2.2.8 for the case of rings.



The remaining of the paper is mainly devoted to transfers. However, we start by introducing in Section 3 the Chow-Witt groups of arbitrary Dedekind schemes, both as an illustration (for example of the use of twists) and as an essential tool for the Bass-Tate method. Recall that compare to classical intersection theory, Chow-Witt groups are twisted by a line bundle, and comes with a bigraduation: the first one is codimension<sup>15</sup> and the second one is a  $\mathbb{G}_m$ -grading, which can be explained by the existence of Tate twists for motives.<sup>16</sup> The main result of the section is the computation of the twisted Chow-Witt groups of the projective line (partially stated above): Theorem 3.4.4. The two essential tools used here are  $\mathbb{A}^1$ -invariance of Chow-Witt groups and the localization long exact sequence. The link with usual Chow groups is explained in 3.1.13 and 3.1.14.

The core study of transfers is done in Section 4. As a quick glance at the table of contents shows, it follows the plan explained above, based on the own hand on the quadratic degree map of Definition 4.2.2, and on the other hand on the differential trace map, used for defining transfers of Grothendieck-Witt rings in Definition 4.3.2.

The fourth part (Section 5) is a synthetic account of the previous parts, making use of Feld’s axiomatic of Milnor-Witt modules. The only formula left open is rule R3b whose proof is more sound using the full axiomatic of Rost-Schmid complexes, and therefore will be treated in [Dég23] (but see Remark 5.2.2 for references to the literature). Note finally that we formulate and prove stronger rules in Section 5.3, following and improving formulas due to Feld, which are very relevant for quadratic intersection theory, and in particular for the notion of *quadratic multiplicities*.

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## 2. MILNOR-WITT K-THEORY AND GROTHENDIECK-WITT GROUPS

### 2.1. Grothendieck-Witt groups and symmetric bilinear forms.

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<sup>15</sup>only 0 and 1 for Dedekind schemes!

<sup>16</sup>Beware that  $\mathbb{G}_m$ -twists refers in practice to twists by  $\mathbb{Z}(1)[1]$  in motivic notation; moreover, the bigraded Chow-Witt groups do not correspond to the bigraded Milnor-Witt motivic cohomology of [BCD<sup>+</sup>22]: it is only related to these groups, relation that one can see by looking at the  $E_2$ -page of the coniveau filtration.

**2.1.1.** (cf. [MH73]) Let  $K$  be a field. An inner product space or simply *inner space*  $(E, \phi)$  over  $K$  is a finite  $K$ -vector space  $E$  with a bilinear form

$$\phi : E \otimes_K E \rightarrow K$$

which is symmetric and non degenerate:  $E \rightarrow E^\vee, x \mapsto \phi(x, -)$  is an isomorphism. The dimension of  $E/K$  is called the *rank* of the inner space  $(E, \phi)$ . A morphism  $(E, \phi) \rightarrow (F, \psi)$  of inner spaces is a  $K$ -linear morphism  $f : E \rightarrow F$  such that  $\phi(f(u), f(v)) = \psi(u, v)$ .

The category of inner spaces admits direct sums and tensor products:

$$\begin{aligned} (E, \phi) \perp (F, \psi) &\rightarrow (E \oplus F, \phi + \psi) \\ (E, \phi) \otimes (F, \psi) &\rightarrow (E \otimes_K F, \phi.\psi). \end{aligned}$$

Therefore the set  $I_K$  of isomorphism classes<sup>17</sup> of inner spaces over  $K$  is commutative monoid for  $\oplus$ , and a commutative semi-ring for  $\oplus, \otimes$ . The following definition comes from Milnor and Husemoller [MH73]. It is a variant of the fundamental definition of Witt [Wit37]<sup>18</sup>, using the Grothendieck construction, that apparently first appeared in the short work of Delzant [Del62].

**Definition 2.1.2.** The *Grothendieck-Witt ring*  $\text{GW}(K)$  of  $K$  is the group completion of the monoid  $(I_K, \oplus)$ , with products induced by the tensor product  $\otimes$ .

The rank of inner spaces induces a ring map:

$$(2.1.2.a) \quad \text{GW}(K) \xrightarrow{\text{rk}} \mathbb{Z}.$$

*Remark 2.1.3.* If the characteristic of  $K$  is different from 2, for any  $K$ -vector space  $V$ , there is a one-to-one correspondence between the symmetric bilinear form  $\phi$  on  $V$  and the quadratic forms  $q$ .<sup>19</sup> Then the Grothendieck-Witt ring can be defined in terms of isomorphism classes of quadratic forms.

This is no longer true in characteristic 2, but the definition based on inner spaces is the correct one for  $\mathbb{A}^1$ -homotopy. Nevertheless, one abusively use terms such as *quadratic* intersection theory, in any characteristic.

**Example 2.1.4.** (1) Let  $u$  be a unit in  $K$ . Then  $K \otimes K \rightarrow K, (x, y) \mapsto u.xy$  is an inner space of rank 1. Its class in the Grothendieck-Witt ring is denoted by  $\langle u \rangle$ . Obviously,  $\langle uv^2 \rangle = \langle u \rangle$ . Therefore, one has a canonical map:

$$Q(K) := K^\times / (K^\times)^2 \rightarrow \text{GW}(K).$$

The group  $Q(K)$  will be called the group of *quadratic classes* of  $K$ .

<sup>17</sup>This is indeed a set, in bijection with

$$\sqcup_{n \geq 0} \text{Sym}_n(K) / \sim$$

where  $\text{Sym}_n(K)$  is the set of invertible symmetric  $(n \times n)$ -matrices with coefficients in  $K$ , and  $\sim$  is the congruence relation on such matrices:  $M \sim N$  if  $M = PNP^t$ ;

<sup>18</sup>now called the Witt group, see below

<sup>19</sup> $q(x) = \phi(x, x), \phi(x, y) = \frac{1}{2}(q(x+y) - q(x) - q(y)) !$

- (2) Given units  $u_i \in K^\times$ , we put  $\langle u_1, \dots, u_n \rangle = \langle u_1 \rangle + \dots + \langle u_n \rangle$ .

A bilinear form on a framed  $K$ -vector space is defined by a symmetric invertible matrix. The above element of  $\text{GW}(K)$  is represented by the  $K$ -vector space  $K^n$  and the diagonal matrix with coefficients  $u_i$ .

**Example 2.1.5.** (1) If  $K$  is an algebraically closed field the rank map  $\text{rk} : \text{GW}(K) \rightarrow \mathbb{Z}$  is an isomorphism.<sup>20</sup> More generally,  $\text{rk}$  is an isomorphism whenever  $(-1)$  is a square in  $K$  (see Example 2.1.13).

- (2) It is well-known that a quadratic form over a real vector space is determined by its signature. In other words, any  $\sigma \in \text{GW}(\mathbb{R})$  can be uniquely written as  $\sigma = p \cdot \langle 1 \rangle + q \cdot \langle -1 \rangle$ ,  $\text{rk}(\sigma) = p + q$  and the signature of  $\sigma$  is defined as the pair  $(p, q)$ . The map  $\text{GW}(\mathbb{R}) \rightarrow \mathbb{Z} \oplus \mathbb{Z}, \sigma \mapsto (p, q)$  is an isomorphism.
- (3) Let  $K = \mathbb{F}_q$  be a finite field,  $q = p^n$ . Then the following sequence of abelian groups is exact:

$$0 \rightarrow Q(\mathbb{F}_q) \rightarrow \text{GW}(\mathbb{F}_q) \xrightarrow{\text{rk}} \mathbb{Z} \rightarrow 0$$

$$\bar{u} \mapsto 1 - \langle u \rangle$$

where  $Q(\mathbb{F}_q)$  is the group quadratic classes of  $\mathbb{F}_q$  (Example 2.1.4(1)). Note that this fits with item (1) above !

The preceding sequence is obviously split. Moreover, the abelian group  $\mathbb{F}_q^\times$  is cyclic of order  $(q - 1)$ . In particular, Lagrange's theorem implies that  $Q(\mathbb{F}_q)$  is zero if  $q$  is even, and  $\mathbb{Z}/2$  if  $q$  is odd. Consequently:

$$\text{GW}(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & q \text{ even,} \\ \mathbb{Z}/2 \oplus \mathbb{Z} & q \text{ odd.} \end{cases}$$

Consider the notations of Example 2.1.4. The element  $h = \langle 1, -1 \rangle$  is called the (class of the) *hyperbolic form*. One can recover the following famous definition (and extension in arbitrary characteristic) of Witt.

**Definition 2.1.6.** One defines the Witt ring of a field  $K$  as the quotient ring:

$$W(K) = \text{GW}(K)/(h).$$

The hyperbolic form being of rank 2, the map (2.1.2.a) induces a morphism of rings:

$$W(K) \rightarrow \mathbb{Z}/2$$

which is again called the *rank map*.

*Remark 2.1.7.* Definition 2.1.2, as well as the previous one, can be extended to an arbitrary (commutative) ring  $A$  instead of a field  $K$  (see [Kne77, I.§4, Prop. 1] for the Grothendieck-Witt ring, and [MH73, I. 7.1] for the Witt ring): instead of finite

<sup>20</sup>This is obvious in characteristic not 2, as any inner space admits an orthogonal base, and every element in  $K$  is a square.

$K$ -vector spaces, one considers finitely generated projective  $R$ -modules  $M$  equipped with a non-degenerate symmetric bilinear form

$$\phi : M \otimes_R M \rightarrow R \mid \phi' : M \xrightarrow{\sim} \text{Hom}_R(M, R) = M^\vee$$

and considers the Grothendieck group  $\text{GW}(R)$  associated with the monoid of isomorphism classes of  $(M, \phi)$ .

It follows from Minkowski's convex body theorem (see [MH73, 4.4]) that one can define an isomorphism of rings, called the *signature*,

$$\sigma : \text{W}(\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}.$$

The map  $\sigma$  associates to the class of  $(M, \phi)$  the signature of  $(M \otimes_{\mathbb{Z}} \mathbb{R}, \phi \otimes_{\mathbb{Z}} \mathbb{R})$ .

As  $h$  is non- $\mathbb{Z}$ -torsion in  $\text{GW}(\mathbb{Z})$  (because it is not so in  $\text{GW}(\mathbb{R})$ ), one obtains that  $\text{GW}(\mathbb{Z})$  is a free rank 2 abelian group, and:

$$\text{GW}(\mathbb{Z}) = \mathbb{Z} \cdot \langle 1 \rangle \oplus \mathbb{Z} \cdot h.$$

To get a presentation as a ring, we consider the element:  $\epsilon = -\langle -1 \rangle$ . Then one deduces from the above isomorphism an isomorphism of rings:

$$\text{GW}(\mathbb{Z}) = \mathbb{Z}[\epsilon]/(\epsilon^2 - 1).$$

We will retain that this ring always acts (by functoriality) on rings of the form  $\text{GW}(K)$ , and more generally on the invariants of  $\mathbb{A}^1$ -homotopy theory such as the Milnor-Witt K-theory.

**Definition 2.1.8.** We define the *fundamental ideal* of  $\text{W}(K)$  as:

$$I(K) := \text{Ker}(\text{rk} : \text{GW}(K) \rightarrow \mathbb{Z}) \simeq \text{Ker}(\text{rk} : \text{W}(K) \rightarrow \mathbb{Z}/2).$$

Typical elements of  $I(K)$  are given by the following *Pfister forms* associated with  $u \in K^\times$ :

$$\langle\langle u \rangle\rangle := 1 - \langle u \rangle.$$

*Remark 2.1.9.* (1) According to [MH73, 3.3],  $I(F)$  is the only (prime) ideal of  $\text{W}(F)$  with residue field  $\mathbb{F}_2$ .

(2) This ideal is of fundamental (historical) importance as it is central to the Milnor conjecture on quadratic forms: see Theorem 2.2.3.

**Example 2.1.10.** Consider the case of a finite field  $K = \mathbb{F}_q$ ,  $q = p^n$ . According to Example 2.1.5, one gets that

$$I(\mathbb{F}_q) = Q(\mathbb{F}_q) = \begin{cases} 0 & q \text{ even,} \\ \mathbb{Z}/2 & q \text{ odd.} \end{cases}$$

So if  $q$  is even,  $\text{W}(\mathbb{F}_q) = \mathbb{Z}/2$ , via the rank morphism. If  $q$  is odd, one has (applying again the preceding example) a short exact sequence:

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \text{W}(\mathbb{F}_q) \xrightarrow{\text{rk}} \mathbb{Z}/2 \rightarrow 0$$

which is split if  $q \equiv 1 \pmod{4}$ , and non split if  $q \equiv 3 \pmod{4}$ . In fact, as a ring, one gets more precisely:

$$\mathrm{GW}(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/2 & q \text{ even,} \\ \mathbb{Z}/2[t]/(t-1)^2 & q \equiv 1 \pmod{4}, \\ \mathbb{Z}/4 & q \equiv 3 \pmod{4}. \end{cases}$$

In any case, one deduces that  $I^n(\mathbb{F}_q) = 0$  if  $n > 1$ .

The following result is an elaboration of Witt's theorems on quadratic forms ([Wit37]).

**Theorem 2.1.11.** *The abelian group  $\mathrm{GW}(K)$  admits a presentation whose generators are given by symbols  $\langle u \rangle$  for  $u \in K^\times$  (mapping to the elements of Example 2.1.4) with relations:*

$$(GW1) \quad \langle uv^2 \rangle = \langle u \rangle.$$

$$(GW2) \quad \langle u, v \rangle = \langle u + v, (u + v)uv \rangle, \quad u + v \neq 0$$

where we have used the notation  $\langle u, v \rangle := \langle u \rangle + \langle v \rangle$ .

Moreover, the relation (GW2) implies the following one:

$$(GW3) \quad \langle u, -u \rangle = \langle 1, -1 \rangle.$$

The analogous presentation for the Witt group is well-known (see [MH73, Lem. 1.1]). We refer the reader to [Car, Th. 1.6] for a direct proof.<sup>21</sup>

*Remark 2.1.12.* (1) If one wants a presentation of  $\mathrm{GW}(K)$  as a ring, one needs only adding the relation  $\langle uv \rangle = \langle u \rangle \langle v \rangle$ .

(2) One can take as generators of the abelian group  $\mathrm{GW}(K)$  the symbols  $\langle \bar{u} \rangle$  where  $\bar{u} \in Q(K)$  is a quadratic class of a unit  $u \in K^\times$ . Then one has only a single relation, given by (GW2) (with  $u, v$  replaced by  $\bar{u}, \bar{v}$ ).

(3) Modulo the hyperbolic plane, one recovers the classical presentation of the Witt group  $W(K)$ : it is generated by symbols  $\langle u \rangle$  for a unit  $u \in K^\times$  subject to the relations:

$$(W0) \quad \langle 1, -1 \rangle = 0$$

$$(W1) \quad \langle uv^2 \rangle = \langle u \rangle$$

$$(W2) \quad \langle u, v \rangle = \langle u + v, (u + v)uv \rangle, \quad u + v \neq 0.$$

Again, one can start with symbols  $\langle \bar{u} \rangle$  of a quadratic class  $\bar{u} \in Q(K)$ , in which case relation (W1) is unnecessary.

**Example 2.1.13.** One deduces from the above presentation the following remarkable property of the Witt group of a field  $K$ . The following conditions are equivalent:

<sup>21</sup>Here is the trick to get relation (GW3) from (GW2). One can assume  $u \neq -1$ , and one writes using (GW2):

$$\langle -u, u + 1 \rangle = \langle 1, -u(u + 1) \rangle, \langle -1, 1 + u \rangle = \langle u, -u(1 + u) \rangle.$$

Taking the difference of these two equalities gives (GW3).

- (1) the rank map  $\text{rk} : W(K) \rightarrow \mathbb{Z}/2$  is an isomorphism;
- (2)  $Q(K) = 1$ , *i.e.* every unit in  $K$  is a square.

**2.1.14. Twists.** In what follows, it will be crucial to consider  $\mathcal{L}$ -valued inner space for an arbitrary invertible  $K$ -vector space  $\mathcal{L}$ . These are  $K$  vector spaces  $V$  with a symmetric bilinear forms  $V \otimes_K V \rightarrow \mathcal{L}$  such that the adjoint map  $V \rightarrow \text{Hom}_K(V, \mathcal{L})$  is an isomorphism.

Then one can define as above, using again the orthogonal sum, a  $\text{GW}(K)$ -module  $\text{GW}(K, \mathcal{L})$ , called the  $\mathcal{L}$ -twisted Grothendieck-Witt group of  $K$ . After modding out by  $h$ , one obtains a  $W(K)$ -module  $W(K, \mathcal{L})$ , the  $\mathcal{L}$ -twisted Witt group of  $K$ .

The tensor product of  $K$ -vector spaces induces an action of the ring  $\text{GW}(K)$  on  $\text{GW}(K, \mathcal{L})$ , and more generally an exterior product:

$$\text{GW}(K, \mathcal{L}) \otimes \text{GW}(K, \mathcal{L}') \rightarrow \text{GW}(K, \mathcal{L} \otimes \mathcal{L}'),$$

and similarly for the twisted Witt ring.

*Remark 2.1.15.* Both abelian groups  $\text{GW}(K, \mathcal{L})$  and  $W(K, \mathcal{L})$  are *non canonically* isomorphic to their untwisted counter-parts. However, these twists can be interpreted as local orientations in the theory of Chow-Witt groups.

**2.2. Definition by generators and relations.**

**2.2.1. Milnor K-theory.** Let us first recall that the Milnor K-theory  $K_*^M(K)$  of a field  $K$  is defined as the  $\mathbb{Z}$ -graded algebra generated by symbols  $\{a\}$  in degree  $+1$  for  $a \in K^\times$  modulo the relations:

- (M1)  $\{a, 1 - a\} = 0$
- (M2)  $\{ab\} = \{a\} + \{b\}$

where we have put  $\{a_1, \dots, a_n\} = \{a_1\} \dots \{a_n\}$ .

Note in particular that:  $K_0^M(K) = \mathbb{Z}$ ,  $K_1^M(K) = K^\times$ .

*Remark 2.2.2.* In general, there is a canonical *symbol map* with values in (Quillen) algebraic K-theory:

$$K_n^M(K) \rightarrow K_n(K)$$

which is an isomorphism if  $n \leq 2$ . The case  $n \leq 1$  is easy, but the case  $n = 2$  is a difficult theorem due to Matsumoto (see [Mat69]). The cokernel of the symbol map is called the *indecomposable part* of algebraic K-theory.

We now have all the tools to formulate the Milnor conjecture, now a theorem due to Orlov, Vishik and Voevodsky:

**Theorem 2.2.3** (Kato, Orlov-Vishik-Voevodsky). *Let  $K$  be an arbitrary field and consider the notation of Definition 2.1.8.*

*Then for any  $n \geq 0$ , the map  $K^\times \rightarrow I(K), u \mapsto \langle\langle u \rangle\rangle$  induces a ring morphism:*

$$\mu : K_*^M(K)/2K_*^M(K) \rightarrow \bigoplus_{n \geq 0} I^n(K)/I^{n+1}(K)$$

*which is an isomorphism.*

See [Mil70, Question 4.3] for the statement of the conjecture, [Kat82] for the proof when  $K$  is of characteristic 2 and [OVV07] (or [Mor05]) for the proof in the other cases.

**2.2.4. Notation.**— It is customary to denote by  $I^*(K)$  the  $\mathbb{Z}$ -graded  $W(K)$ -algebra where we conventionally put  $I^n(K) = W(K)$  for  $n \leq 0$ , and the product is induced by that of  $W(K)$ .

Then  $I(K)$  induces an ideal in  $I^*(K)$  and we denote by  $\bar{I}^*(K)$  the quotient  $\mathbb{Z}$ -graded  $W(K)$ -algebra,<sup>22</sup> so that  $\bar{I}^n(K) = I^n(K)/I^{n+1}(K)$  if  $n \geq 0$ , and 0 otherwise. Note that it is clear that the action of  $W(K)$  factors through the rank map so that  $\bar{I}^*(K)$  is actually a  $\mathbb{Z}/2$ -algebra.

With this notation and the previous theorem, the morphism  $\mu$  defined by Milnor takes the form of an isomorphism of  $\mathbb{Z}$ -graded algebras over  $\mathbb{Z}/2$ :

$$\mu : K_*^M(K)/2 \rightarrow \bar{I}^*(K).$$

**Example 2.2.5.** The case  $n = 0$  is trivial. In the case  $n = 1$ , the map takes the form  $\mu_1 : Q(K) \rightarrow \bar{I}^1(K) = I(K)/I^2(K)$ , where  $Q(K)$  is the group of quadratic classes. Then an explicit inverse is given by the discriminant map

$$d : \bar{I}^1(K) \rightarrow Q(K), [(E, \phi)] \mapsto (-1)^{r(r-1)/2} \cdot \det(M_\phi)$$

where  $(E, \phi)$  is an inner space of even rank  $r$ , and  $M_\phi$  is any matrix that represents it. See [Mil70, Th. 4.1] and [MH73, def. p.77]. For the case  $n = 2$ , and the interpretation of  $\bar{I}^2(K)$  in terms of Clifford invariants of quadratic forms, we refer the reader to [MH73], Theorem III.5.8 and its proof.

**2.2.6.** The following definition, due to Hopkins and Morel (see [Mor04, Section 5]), gives an extension of Milnor's theory which mixes generators and relations of Milnor K-theory and of the Grothendieck-Witt ring:

**Definition 2.2.7.** Let  $K$  be any field. We define the *Milnor-Witt ring*, or *Milnor-Witt K-theory*,  $K_*^{MW}(K)$  of  $K$  as the  $\mathbb{Z}$ -graded ring with the following presentations:

Generators are given by symbols  $[a]$  of degree +1 for  $a \in K^\times$ , and a symbol  $\eta$  of degree  $-1$  called the *Hopf element*. Let us introduce the following notations to formulate the relations:

$$\begin{aligned} [a_1, \dots, a_n] &= [a_1] \dots [a_n] \\ h &= 2 + \eta[-1] \end{aligned}$$

Relations are given as follows, whenever they make sense:

- (MW1)  $[a, 1 - a] = 0$
- (MW2)  $[ab] = [a] + [b] + \eta \cdot [a, b]$
- (MW3)  $\eta[a] = [a]\eta$

---

<sup>22</sup>One also finds the notation  $i^n(K)$  for  $\bar{I}^n(K)$ ;

(MW4)  $\eta h = 0$

Obviously, Milnor-Witt K-theory is a covariant functor with respect to inclusion morphisms of fields. Given such a map  $\varphi : K \rightarrow L$ , the natural morphism of  $\mathbb{Z}$ -graded ring (homogeneous of degree 0):

$$\varphi_* : K_*^{MW}(K) \rightarrow K_*^{MW}(L).$$

This map is sometimes called the *corestriction*.

*Remark 2.2.8.* Given any ring  $A$ , the preceding definition makes sense so that we can define the ring  $K_*^{MW}(A)$ .<sup>23</sup> The resulting  $\mathbb{Z}$ -graded ring is covariantly functorial in the ring  $A$ . This extended definition is useful for example when  $A$  is a local ring as we will see in Theorem 2.5.9.

Note that one can directly compute this ring when  $A = \mathbb{Z}$ :

$$K_*^{MW}(\mathbb{Z}) = \mathbb{Z}[\epsilon, \eta, [-1]] / (\epsilon^2 - 1, \eta[-1] - \epsilon + 1)$$

where  $\epsilon, \eta, [-1]$  are respectively in degree 0,  $-1$  and 1. In particular,

$$K_0^{MW}(\mathbb{Z}) = \mathbb{Z}[\epsilon] / (\epsilon^2 - 1) = \text{GW}(\mathbb{Z})$$

according to Remark 2.1.7. The ring  $K_*^{MW}(\mathbb{Z})$  always acts on rings  $K_*^{MW}(K)$  (and more generally on invariants of  $\mathbb{A}^1$ -homotopy theory).

**2.2.9. Relation with Milnor K-theory.** One immediately remark that if one adds  $\eta = 0$  to the above relations (MW\*), one recovers the relations (M\*) of Milnor K-theory. In other words, sending the generators  $\{a\}$  to the class of  $[a]$  in  $K_*^{MW}(K)/(\eta)$  induces an isomorphism of  $\mathbb{Z}$ -graded algebras:

$$F : K_*^M(K) \xrightarrow{\sim} K_*^{MW}(K)/(\eta).$$

In particular, for any integer  $q \in \mathbb{Z}$ , one deduces an exact sequence of abelian groups:

$$(2.2.9.a) \quad K_{q+1}^{MW}(K) \xrightarrow{\gamma_\eta} K_q^{MW}(K) \xrightarrow{F} K_q^M(K) \rightarrow 0.$$

In the other direction, one can look at the morphism of  $\mathbb{N}$ -graded algebras

$$(K^\times)^{\otimes, *} \mapsto K_*^{MW}(K), u_1 \otimes \dots \otimes u_q \mapsto h.[u_1, \dots, u_q] (q \geq 0).$$

Because of relation (MW4), this maps factors through relation (M1) and (M2), and therefore induces a well-defined morphism of  $\mathbb{Z}$ -graded algebras:

$$H : K_*^M(K) \rightarrow K_*^{MW}(K).$$

Following [BCD<sup>+</sup>22, Chap. 2, §1]), we introduce the following terminology for these two maps:

<sup>23</sup>See [Sch17, Def. 4.10] for more developments.



**Definition 2.2.10.** The morphisms of  $\mathbb{Z}$ -graded algebras<sup>24</sup>  $F : K_*^{MW}(K) \rightarrow K_*^M(K)$  and  $H : K_*^M(K) \rightarrow K_*^{MW}(K)$  are respectively called the *forgetful* and *hyperbolic maps*.

**2.2.11.** By definition, each of the above maps are uniquely characterized by the properties:

$$F(\eta) = 0, \quad F([a]) = \{a\}.$$

$$H(\{a\}) = h.[a].$$

Moreover, one deduces the following relations<sup>25</sup>:

$$F \circ H = 2 \cdot \text{Id}$$

$$H \circ F = \gamma_h.$$

*Remark 2.2.12.* . In particular, one can remark that the forgetful map induces a split epimorphism:

$$K_*^{MW}(K)[1/2] \rightarrow K_*^M(K)[1/2].$$

This fact will be made more precise in Example 2.3.9.

Let us come back to the study of the general groups  $K_*^{MW}(K)$ . One obtains the following presentation of each individual graded parts, as abelian groups:

**Proposition 2.2.13.** *Consider an arbitrary field  $K$  and an integer  $n \in \mathbb{Z}$ . Then the abelian group  $K_n^{MW}(K)$  is generated by symbols of the form:*

$$[\eta^r, a_1, \dots, a_{n+r}], r \geq 0, a_i \in K^\times$$

*modulo the following three relations:*

- (MW1ab)  $[\eta^r, a_1, \dots, a_{n+r}] = 0$  if  $a_i + a_{i+1} = 1$  for some  $i$
- (MW2ab)  $[\eta^r, a_1, \dots, a_i b_i, \dots, a_{n+r}] = [\eta^r, a_1, \dots, a_i, \dots, a_{n+r}] + [\eta^r, a_1, \dots, b_i, \dots, a_{n+r}] + [\eta^{r+1}, a_1, \dots, a_i, b_i, \dots, a_{n+r}]$
- (MW4ab)  $[\eta^r, a_1, \dots, -1, \dots, a_{n+r-1}] = -2[\eta^r, a_1, \dots, \cancel{1}, \dots, a_{n+r-1}]$  for  $r \geq 2$

See [Car] for the proof.

**Corollary 2.2.14.** *Assume that  $n \geq 1$ , then the abelian group  $K_n^{MW}(K)$  is generated by the elements  $[u_1, \dots, u_n]$  for an  $n$ -uplet of units  $u_i \in K^\times$ .*

This simply follows from the previous proposition by using relation (MW2ab).

*Remark 2.2.15.* In particular, the abelian group  $K_1^{MW}(K)$  is generated by symbols  $[u]$  for  $u \in K^\times$ . However, beware that the map  $\iota : K^\times \rightarrow K_1^{MW}(K), u \mapsto [u]$  is not a morphism of group, except when  $K = \mathbb{F}_2$ . Indeed, one can express the addition law in  $K_1^{MW}(K)$  by the formula:

$$[u] + [v] = [uv] - \eta.[u, v],$$

<sup>24</sup>both homogeneous of degree 0

<sup>25</sup>Use that  $h$  modulo  $\eta$  is equal to 2 in  $K_*^{MW}(K)$  for the first one

and  $\eta.[u, v]$  is not zero in general. Note also that the forgetful map

$$K_1^{MW}(K) \xrightarrow{F} K_1^M(K) = K^\times$$

is a surjective morphism of abelian group, but  $\iota$  is a splitting of  $F$  only after forgetting the group structure. In fact, we will give an explicit description of this group in Example 2.3.8.

**2.2.16.** Following Morel, one considers the following important element in Milnor-Witt K-theory:

$$\epsilon = -\langle -1 \rangle \in K_0^{MW}(K).$$

Then relation (MW4) can be rewritten as  $\epsilon.\eta = \eta$ . Moreover, the defect of commutativity of the multiplicative structure of Milnor-Witt K-theory can be precisely expressed in terms of  $\epsilon$  as follows.

**Proposition 2.2.17.** *For any field  $K$ , one has the relation:*

$$\forall(\alpha, \beta) \in K_n^{MW}(K) \times K_m^{MW}(K), \alpha\beta = \epsilon^{nm}.\beta\alpha.$$

One says that the  $\mathbb{Z}$ -graded algebra  $K_*^{MW}(K)$  is  $\epsilon$ -commutative. To prove this formula, one reduces to the case  $\alpha = [a]$ ,  $\beta = [b]$  for units  $a, b$  (see [Car, Cor. 1.5]).

**2.2.18. Quadratic multiplicities.** One associates to any integer  $n \in \mathbb{Z}$  the following element of  $K_0^{MW}(K)$ :

$$n_\epsilon = \begin{cases} \sum_{i=0}^{n-1} (-\epsilon)^i & n \geq 0 \\ \epsilon.(-n)_\epsilon & n < 0. \end{cases}$$

An equivalent computation:

$$n_\epsilon = \begin{cases} m.h & n = 2m \\ m.h + 1 & n = 2m + 1 \end{cases}$$

Beware that the induced arrow  $\mathbb{Z} \rightarrow K_0^{MW}(K), n \mapsto n_\epsilon$  is a monoid morphism for multiplication

$$(nm)_\epsilon = n_\epsilon m_\epsilon$$

but not for the addition (compute  $3_\epsilon$  and  $4_\epsilon$ ).

*Remark 2.2.19.* (1) A principle of quadratic enumerative geometry (see [Lev20]) is that, under a careful choice of orientations, degrees of classical enumerative geometry should be replaced by  $\epsilon$ -degrees as defined above.

(2) With the previous notation, relation (MW4) translates to:

$$2_\epsilon.\eta = 0$$

This should remind the reader of the fact the classical/topological Hopf map  $\eta : S^3 \rightarrow S^2$  induces a 2-torsion element in the stable homotopy groups of spheres, which account for the isomorphism:

$$\pi_3^{st}(S^2) = \mathbb{Z}/2.\eta$$

where the left hand-side group is the third stable homotopy group of  $S^2$ .

(3) In negative degree, the quadratic multiplicities  $n_\epsilon$  became drastically simpler! Indeed, modulo  $h$  or equivalently in  $W(K)$ ,

$$n_\epsilon = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

**2.3. Relations with quadratic forms.** Using the presentation obtained in the lemma just above, together with the presentation of Grothendieck-Witt groups Theorem 2.1.11, Morel deduces the following computation (for full details, see [Car, Prop. 1.9, Lem. 1.3]):

**Proposition 2.3.1.** *The following map is well-defined*

$$\text{GW}(K) \rightarrow K_0^{MW}(K), \langle a \rangle \mapsto 1 + \eta \cdot [a]$$

and induces an isomorphism of rings.

For any  $n > 0$ , the multiplication map:  $K_0^{MW}(K) \xrightarrow{\eta^n} K_{-n}^{MW}(K)$  induces an isomorphism:

$$W(K) = \text{GW}(K)/(h) \rightarrow K_{-n}^{MW}(K).$$

Finally, for any  $n \geq 0$ , the abelian group  $K_n^{MW}(K)$  is generated by symbols of the form  $[a_1, \dots, a_n]$  for units  $a_i \in K^\times$ .

As a consequence, we will view the elements of  $\text{GW}(K)$  as elements in degree 0 of Milnor-Witt K-theory. Note moreover, that  $\text{GW}(K)$  lands in the center of the ring  $K_*^{MW}(K)$ .

**Example 2.3.2.** The notation  $h \in K_0^{MW}(K)$  in relation (MW4) was therefore justified, as it corresponds to the hyperbolic form in  $\text{GW}(K)$ . Note that relation (GW3) in Theorem 2.1.11 can be written as:

$$(2.3.2.a) \quad \forall u \in K^\times, \langle u \rangle \cdot h = h.$$

Remark also that  $h^2 = 2 \cdot h$  (direct computation).

Recall from 2.2.3 that given a unit  $u \in K^\times$ , one defines the *Pfister form* associated with  $u$  as the element  $\langle\langle u \rangle\rangle = 1 - \langle u \rangle$  of  $W(K)$ .

**Corollary 2.3.3.** *Let  $W(K)[t, t^{-1}]$  be the  $t$ -periodic  $\mathbb{Z}$ -graded algebra with  $t$  a formal variable in degree 1.*

*Then there exists a unique morphism of  $\mathbb{Z}$ -graded algebra*

$$\phi : K_*^{MW}(K)[\eta^{-1}] \rightarrow W(K)[t, t^{-1}], [u] \mapsto -\langle\langle u \rangle\rangle \cdot t, \eta \mapsto t^{-1}$$

and it is an isomorphism.

*Proof.* The uniqueness of  $\phi$  is obvious. We need to show that it is well-defined. First note that relation (MW4) implies that  $h = 0$  in  $K_*^{MW}(K)[\eta^{-1}]$ . Thus, it suffices to show that the elements  $-\langle\langle u \rangle\rangle \cdot t$  and  $t^{-1}$  of  $W(K)[t, t^{-1}]$  satisfy the

relation (MW1), (MW2) and (MW3). Relation (MW1) follows from relation (W2) in the Witt ring (see Remark 2.1.12(2)). Relation (MW2) follows from the rule  $\langle u \rangle \langle v \rangle = \langle uv \rangle$  in the Witt ring, while relation (MW3) is obvious.

Finally, the preceding proposition shows that multiplication by  $\eta$  induces an isomorphism on the negative part of the  $\mathbb{Z}$ -graded algebra  $K_*^{MW}(K)$ . In particular, the canonical map  $K_*^{MW}(K) \rightarrow K_*^{MW}(K)[\eta^{-1}]$  is an isomorphism in negative degree. On the other hand,  $\phi(1 + \eta[u]) = 1 - t^{-1} \langle \langle u \rangle \rangle . t = 1 - (1 - \langle u \rangle) = \langle u \rangle$ . Therefore, applying again the preceding proposition, one deduces that  $\phi$  is an isomorphism in negative degree. As both the source and target of  $\phi$  are  $\mathbb{Z}$ -periodic, one deduces that  $\phi$  is an isomorphism in all degrees.  $\square$

**2.3.4.** As in [Mor04], one can define the Witt K-theory of  $K$  as the quotient  $\mathbb{Z}$ -graded algebra:

$$K_*^W(K) = K_*^{MW}(K)/(h).$$

Indeed, the relations (MW\*) correspond to the relations of *loc. cit.*, Definition 3.1.

On the other hand, one can consider the sub-algebra  $I^*(K)$  of  $W(K)[t, t^{-1}]$  generated by  $I(K).t$  (recall Definition 2.1.8 and the notation of 2.2.4). The main result of *loc. cit.* is that Theorem 2.2.3 implies the following finer comparison result.

**Theorem 2.3.5.** *Consider the preceding notation. Then there exists a unique morphism  $\psi$  of  $\mathbb{Z}$ -graded algebras that fits into the commutative diagram*

$$\begin{array}{ccc} K_*^W(K) & \xrightarrow{\psi} & I^*(K) \\ \nu \downarrow & & \downarrow \\ K_*^{MW}(K)[\eta^{-1}] & \xrightarrow{\phi} & W(K)[t, t^{-1}] \end{array}$$

where  $\nu$  is the canonical map (use relation (MW4)), and the right-hand vertical one is the obvious inclusion.

Moreover,  $\psi$  is an isomorphism.

This theorem was first proved in [Mor04] when the characteristic of  $K$  is different from 2. We refer the reader to [GSZ16, Th. 3.8] for a proof in the latter case, and to [Car] for the proof in the characteristic 2 case.

**2.3.6.** As an application of the previous theorem, one deduces a canonical map:

$$\mu'_n : K_n^{MW}(K) \longrightarrow K_n^W(K) \xrightarrow{(-1)^n \cdot \psi_n} I^n(K)$$

which can be uniquely characterized, as a morphism  $\mu' : K_*^{MW}(K) \rightarrow I^*(K)$  of  $\mathbb{Z}$ -graded algebras, as the map which sends  $[u]$  to the Pfister form  $\langle \langle u \rangle \rangle \in I^1(K)$  and the element  $\eta$  to the class  $\langle 1 \rangle$  in  $I^{-1}(K) = W(K)$ .

**Corollary 2.3.7.** *The following commutative square of  $\mathbb{Z}$ -graded algebras is cartesian:*

$$\begin{array}{ccc} K_*^{MW}(K) & \xrightarrow{F} & K_*^M(K) \\ \mu' \downarrow & & \downarrow \mu \\ I^*(K) & \xrightarrow{\pi} & \bar{I}^*(K) \end{array}$$

Here  $F$  is the forgetful map (Definition 2.2.10) and  $\mu$  is the map defined by Milnor (Theorem 2.2.3 and 2.2.4).<sup>26</sup>

*Proof.* Indeed, the above square in degree  $n$  fits into the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_{n+1}^W(K) & \xrightarrow{\tilde{\gamma}_{n,n}} & K_n^{MW}(K) & \xrightarrow{F_n} & K_n^M(K) & \longrightarrow & 0 \\ & & (-1)^n \cdot \psi_n \downarrow & & \mu'_n \downarrow & & \downarrow \mu_n & & \\ 0 & \longrightarrow & I^{n+1}(K) & \longrightarrow & I^n(K) & \longrightarrow & \bar{I}^n(K) & \longrightarrow & 0 \end{array}$$

and the result follows as  $\mu'$  and  $F$  are surjective and  $\psi$  is an isomorphism.  $\square$

**Example 2.3.8.** Looking at degree 1, we deduce the following explicit description of  $K_1^{MW}(K)$ , for any field  $K$ . The group  $K_1^{MW}(K)$  is made of pairs  $([\varphi], u)$  where  $\varphi$  is the Witt-class of an inner space  $\phi : V \otimes_L V \rightarrow K$  of even rank,  $u \in K^\times$  is a unit such that  $d(\varphi) = \bar{u} \in Q(K)$  where  $d$  is the discriminant of  $\varphi$  (see Example 2.2.5). In other words, an element of  $K_1^{MW}(K)$  is given by the Witt class of an inner space over  $K$  of even rank and a lift of its discriminant in  $K^\times$ .

In this description, for any unit  $u \in K^\times$ , the symbol  $[u] \in K_1^{MW}(K)$  is sent to the pair  $(\langle\langle u \rangle\rangle, \{u\})$ .

**Example 2.3.9.** . As  $\bar{I}^*(K)$  is 2-torsion, one deducing from the previous corollary the following interesting fact which extends Remark 2.2.12. After inverting 2, the canonical maps  $F$  and  $\mu'$  of the previous corollary induces an isomorphism of  $\mathbb{Z}$ -graded rings:

$$K_*^{MW}(K)[1/2] \xrightarrow{F \times \mu'} K_*^M(K)[1/2] \times I^*(K)[1/2].$$

**Corollary 2.3.10.** (1) *One has an equality of ideals of  $K_*^{MW}(K)$ :*

$$\text{Ker}(\gamma_\eta) = (h) = \text{Im}(H).$$

*In particular the sequence (2.2.9.a) can be extended into a long exact sequence:*

$$(2.3.10.a) \quad K_*^M(K) \xrightarrow{H} K_*^{MW}(K) \xrightarrow{\gamma_\eta} K_*^{MW}(K) \xrightarrow{F} K_*^M(K) \rightarrow 0$$

*which can be truncated and gives the short exact sequence:*

$$0 \rightarrow I^*(K) \xrightarrow{\tilde{\gamma}_\eta} K_*^{MW}(K) \xrightarrow{F} K_*^M(K) \rightarrow 0$$

*such that  $\tilde{\gamma}_\eta$  is homogeneous of degree  $-1$ .*

<sup>26</sup>recall it sends a generator  $\{u\}$  to the class of the Pfister form  $\langle\langle u \rangle\rangle \in I^1(K)$  modulo  $I^2(K)$

(2) Moreover, the forgetful map  $F : K_*^{MW}(K) \rightarrow K_*^M(K)$  identifies the principal ideal  $(h)$  with the principal ideal  $2 \cdot K_*^M(K)$  generated by 2 in the Milnor ring of  $K$ . One deduces a short exact sequence:

$$0 \rightarrow 2 \cdot K_*^M(K) \xrightarrow{\tilde{H}} K_*^{MW}(K) \xrightarrow{\mu'} I^*(K) \rightarrow 0$$

where  $\tilde{H}$  sends  $2 \in K_0^M(K)$  to  $h$ , and for  $n > 0$ , sends a 2-divisible symbol  $\{a_1, \dots, a_n\} \in K_n^M(K)$  to the element  $[u_1, \dots, u_n] \in K_n^{MW}(K)$ .

Finally,  $\mu' \circ \bar{\gamma}_\eta$  is equal in degree  $n$  to  $(-1)^n \cdot i_n$  where  $i_n : I^{n+1}(K) \rightarrow I^n(K)$  is the canonical inclusion.

*Proof.* Indeed, the preceding theorem implies that  $\nu$  is injective, which implies that  $\text{Ker}(\gamma_\eta) = (h)$  as ideals of  $K_*^{MW}(K)$ . This concludes the first assertion as, by construction, the image of  $H$  is the ideal  $(h)$ . The first two exact sequences follows directly, taking into account the isomorphism  $\psi : K_*^W(K) \rightarrow I^*(K)$ . The last exact sequence follows from the preceding corollary.  $\square$

**Example 2.3.11.** We finish this subsection with a computation that easily follows from Corollary 2.3.7 and Example 2.1.13. If every unit in  $K$  admits a square root, one has:

$$K_n^{MW}(K) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2 & n < 0 \\ K_n^M(K) & n > 0. \end{cases}$$

Recall also that if  $K$  is algebraically closed, for all  $n > 1$ ,  $K_n^{MW}(K) = K_n^M(K)$  is divisible. These are therefore huge groups !

**2.4. Twists.** We now introduce twists on Milnor-Witt K-theory, along the lines of 2.1.14. As already mentioned (see Section 1.2), they will account for the local orientations that appears on quadratic cycles (see Section 3.1 and in particular 3.1.8). Moreover, they are necessary to obtain canonical residue maps (see Definition 2.5.3).

**Definition 2.4.1.** Let  $K$  be a field, and  $\mathcal{L}$  be an invertible (*i.e.* of dimension one over  $K$ )  $K$ -vector space. Consider the set:  $\mathcal{L}^\times := \mathcal{L} - \{0\}$ . The actions of  $K^\times$  on  $K_*^{MW}(K)$ , via the map  $K^\times \rightarrow K_0^{MW}(K), a \mapsto \langle a \rangle$ , and on  $\mathcal{L}^\times$  by scalar multiplication, give two morphisms of rings:

$$\begin{aligned} \mathbb{Z}[K^\times] &\rightarrow K_*^{MW}(K), \\ \mathbb{Z}[K^\times] &\rightarrow \mathbb{Z}[\mathcal{L}^\times]. \end{aligned}$$

We define the  $\mathcal{L}$ -twisted Milnor-Witt K-theory of  $K$  (or simply the Milnor-Witt K-theory of the pair  $(K, \mathcal{L})$ ) in degree  $n \in \mathbb{Z}$  as the following abelian group:

$$K_n^{MW}(K, \mathcal{L}) : K_n^{MW}(K) \otimes_{\mathbb{Z}[K^\times]} \mathbb{Z}[\mathcal{L}^\times].$$

Elements of  $K_n^{MW}(K, \mathcal{L})$  are therefore formal sums of elements of the form  $\sigma \otimes l$  where  $\sigma \in K_n^{MW}(K)$  and  $l \in \mathcal{L}^\times$ .

**2.4.2.** We will identify the untwisted group  $K_*^{MW}(K)$  with  $K_*^{MW}(K, K)$  via the obvious isomorphism:

$$K_*^{MW}(K) \rightarrow K_*^{MW}(K, K), \sigma \mapsto \sigma \otimes 1.$$

Further, given any choice of  $l \in \mathcal{L}^\times$ , we get an isomorphism of invertible  $K$ -vector spaces  $\Theta_l : K \rightarrow \mathcal{L}, \lambda \mapsto \lambda.l$  and therefore an isomorphism:

$$\text{ev}_l = (\Theta_l^{-1})_* : K_*^{MW}(K, \mathcal{L}) \rightarrow K_*^{MW}(K, K) = K_*^{MW}(K).$$

According to definition, for any  $u \in K^*$ , one has:

$$\text{ev}_{ul} = \langle u \rangle . \text{ev}_l.$$

Given an element  $\alpha \in K_*^{MW}(K, \mathcal{L})$ , one obtains a function:

$$\underline{\alpha} : \mathcal{L}^\times \rightarrow K_*^{MW}(K), l \mapsto \text{ev}_l(\alpha)$$

which is  $K^*$ -equivariant:  $\underline{\alpha}(ul) = \langle u \rangle . \underline{\alpha}(l)$ . In other words, one further deduces the following isomorphism<sup>27</sup> of  $\mathbb{Z}$ -graded rings:

$$\begin{aligned} K_*^{MW}(K, \mathcal{L}) &\rightarrow \text{Hom}_{K^\times}(\mathcal{L}^\times, K_*^{MW}(K)) \\ \alpha &\mapsto \underline{\alpha}. \end{aligned}$$

*Remark 2.4.3.* In particular, the twisted groups  $K_*^{MW}(K, \mathcal{L})$  are all abstractly isomorphic, but via a *non-canonical* isomorphism.

In the theory of quadratic cycles, the invertible vector space  $\mathcal{L}$  will be the space of local parameters (see eg. 3.1.8). Then one has two interpretations of the elements of the twisted groups, in view of the preceding isomorphism:

- in the form  $\alpha = \sigma \otimes l$ ,  $\sigma$  is some coefficient, and  $l$  is a choice of a local parameter;
- in the form  $\underline{\alpha} : \mathcal{L}^\times \rightarrow K_*^{MW}(K)$ , we have a functional coefficient which to any choice of a local parametrization associates some symbol in a  $K^\times$ -equivariant way.

Both point of views are useful.

**Example 2.4.4.** Let  $(K, \mathcal{L})$  be as above. Then for any  $n \geq 0$ , the isomorphism of Proposition 2.3.1 induces a *canonical* isomorphism:

$$\begin{aligned} \text{GW}(K, \mathcal{L}) &\xrightarrow{\cong} K_0^{MW}(K, \mathcal{L}) \\ \text{W}(K, \mathcal{L}) &\xrightarrow{\cong} K_n^{MW}(K, \mathcal{L}), n < 0. \end{aligned}$$

where the left hand-side was defined in 2.1.14. Indeed, it suffices to use the isomorphism:

$$\text{GW}(K) \otimes_{\mathbb{Z}[K^\times]} \mathbb{Z}[\mathcal{L}^\times] \rightarrow \text{GW}(K, \mathcal{L}), [\phi] \otimes l \mapsto [\phi.l].$$

<sup>27</sup>To get the reciproc, choose an arbitrary  $l \in \mathcal{L}^*$ , and consider  $f \mapsto f(l) \otimes l$ ;

*Remark 2.4.5.* We consider again the situation of Remark 2.2.8, and assume that  $A$  is regular and semi-local (thus noetherian). Let  $\mathcal{L}$  be an invertible<sup>28</sup>  $A$ -module. As  $A$  is regular semi-local,  $\mathcal{L}$  is trivializable: in fact, for any  $l \in \mathcal{L}^\times := \mathcal{L} - \{0\}$ , the map  $\Theta_l : A \rightarrow \mathcal{L}, \lambda \mapsto \lambda.l$  is an isomorphism.<sup>29</sup>

Moreover, the definition of  $\langle a \rangle = 1 + \eta.[a]$  makes sense for any unit  $a \in A^\times$ . Thus we can define:

$$K_n^{MW}(A, \mathcal{L}) = K_*^{MW}(A) \otimes_{\mathbb{Z}[A^\times]} \mathbb{Z}[\mathcal{L}^\times]$$

**2.4.6. Basic operations on twisted Milnor-Witt K-theory.** We have the following structure on twisted Milnor-Witt K-theory:

(1) Products:

$$K_n^{MW}(K, \mathcal{L}) \otimes K_m^{MW}(K, \mathcal{L}') \rightarrow K_{n+m}^{MW}(K, \mathcal{L} \otimes \mathcal{L}'), (\sigma \otimes l, \tau \otimes l') \mapsto (\sigma.\tau, l \otimes l').$$

(2) First functoriality: given a morphism of field  $\varphi : K \rightarrow L$ , one gets:

$$\varphi_* : K_n^{MW}(K, \mathcal{L}) \rightarrow K_n^{MW}(L, \mathcal{L} \otimes_K L), (\sigma, l) \mapsto (\varphi_*(\sigma), l \otimes_K 1_L).$$

(3) Second functoriality: given an isomorphism of  $K$ -vector spaces  $\theta : \mathcal{L} \rightarrow \mathcal{L}'$  one gets:

$$\theta_* : K_n^{MW}(K, \mathcal{L}) \rightarrow K_n^{MW}(L, \mathcal{L}'), (\sigma, l) \mapsto (\sigma, \Theta(l)).$$

which is an isomorphism of abelian groups.

*Remark 2.4.7.* It is possible to unite the first and second functorialities. One considers the category of *twisted fields*  $\mathcal{T}\mathcal{F}$  whose objects are pairs  $(K, \mathcal{L})$  where  $K$  is a field and  $\mathcal{L}$  of an invertible  $K$ -vector space. Morphisms are given by

$$(\varphi, \Theta) : (K, \mathcal{L}) \rightarrow (L, \mathcal{L}')$$

where  $\varphi : K \rightarrow L$  is a morphism of fields, and  $\Theta : \mathcal{L} \otimes_K L \rightarrow \mathcal{L}'$  is an isomorphism. Composition is defined in the obvious way. Then  $K_*^{MW}$  becomes a covariant functor from the category of twisted fields to the category of graded abelian groups.

The category of twisted fields is *cofibred* over the category of fields  $\mathcal{F}$ . To interpret correctly the tensor product, via a *symmetric* monoidal structure, one has to consider the graded category of twisted fields. This is obtained via the Grothendieck construction applied to the graded Picard category over fields (see [Del87] for this category and [Fas20] for the monoidal structure).

**Example 2.4.8.** Consider  $(K, \mathcal{L})$  as above. Remark that the action of  $K^\times$  on  $K_*^{MW}(K)/\eta$  via the map  $u \mapsto \langle u \rangle$  is trivial: indeed,  $\langle u \rangle = 1 \pmod{\eta}$ . This implies that  $K_*^{MW}(K, \mathcal{L})/\eta$  is canonically isomorphic to  $K_*^{MW}(K)/\eta = K_*^M(K)$ , which I recall is just the Milnor K-theory of  $K$ .

We further extends Definition 2.2.10 as follows.

<sup>28</sup>*i.e.* locally free of rank 1

<sup>29</sup>This comes from the isomorphism  $\text{Pic}(A) = \mathbb{Z}$ .



**Definition 2.4.9.** Let  $(K, \mathcal{L})$  be a twisted field. Then one defines the twisted forgetful (resp. hyperbolic) maps:

$$\begin{aligned} F : K_*^{MW}(K, \mathcal{L}) &\rightarrow K_*^M(K), (\sigma \otimes l) \mapsto F(\sigma) \\ H : K_*^M(K) &\rightarrow K_*^{MW}(K, \mathcal{L}), \sigma \mapsto (h\sigma) \otimes l \end{aligned}$$

where the last formula does not depend on the choice of  $l \in \mathcal{L}^\times$ , given Equation (2.3.2.a).

Obviously, the two relations of 2.2.11 still holds with twists.

**2.4.10.** Consider a twisted field  $(K, \mathcal{L})$ . There exists an action of  $K^\times$  on the graded algebra  $I^*(K)$  associated with the fundamental ideal  $I(K) \subset W(K)$  (see 2.2.4), via its  $W(K)$ -module structure. This allows us to define

$$I^*(K, \mathcal{L}) := I^*(K) \otimes_{\mathbb{Z}[K^\times]} \mathbb{Z}[\mathcal{L}^\times]$$

as in Definition 2.4.1. In fact, one also has  $I^*(K, \mathcal{L}) \subset W(K, \mathcal{L})[t, t^{-1}]$  where  $t$  is a formal variable as in Corollary 2.3.3. The isomorphism of Theorem 2.3.5 induces an isomorphism (of  $\mathbb{Z}$ -graded  $W(K)$ -algebras):

$$K_*^W(K, \mathcal{L}) \xrightarrow{\psi} I^*(K, \mathcal{L}).$$

As remarked in 2.2.4, the action of  $W(K)$  on the quotient algebra  $\bar{I}^*(K, \mathcal{L})$  is trivial. Therefore, one deduces as in Example 2.4.8 a canonical identification:  $\bar{I}^*(K, \mathcal{L}) = \bar{I}^*(K)$ .

These considerations allow to extend Corollary 2.3.7 as follows:

**Proposition 2.4.11.** *The following commutative square of  $\mathbb{Z}$ -graded algebras is cartesian:*

$$\begin{array}{ccc} K_*^{MW}(K, \mathcal{L}) & \xrightarrow{F} & K_*^M(K) \\ \mu' \downarrow & & \downarrow \mu \\ I^*(K, \mathcal{L}) & \xrightarrow{\pi} & \bar{I}^*(K). \end{array}$$

Here  $F$  is the twisted forgetful map (Definition 2.2.10),  $\mu'$  is the  $\mathcal{L}$ -twisted version of the map defined in 2.3.6, and  $\mu$  is the map defined by Milnor (Theorem 2.2.3).

## 2.5. Residues.

**2.5.1.** Residues are a famous part of the functoriality of Milnor K-theory (see [BT73, §4]). A *discretely valued field* will be a pair  $(K, v)$  of a field  $K$  with a discrete valuation  $v$ . We let  $\mathcal{O}_v$  be its ring of integers,  $\mathcal{M}_v$  the maximal ideal of  $\mathcal{O}_v$  and  $\kappa_v = \mathcal{O}_v/\mathcal{M}_v$  its residue field.

Given a valuation  $v : K^\times \rightarrow \mathbb{Z}$ , with residue field  $\kappa_v$ , one deduces for any  $n > 0$  a canonical morphism:

$$\partial_v : K_n^M(K) \rightarrow K_{n-1}^M(\kappa_v)$$

uniquely characterized by the property:

$$\partial_v(\{u_1, \dots, u_n\}) = m \cdot \{\bar{u}_2, \dots, \bar{u}_n\}$$

for units  $u_i \in K^\times$  such that  $v(u_1) = m$  and for  $i > 1$ ,  $v(u_i) = 0$ ,  $\bar{u}_i$  being the residue class of  $u_i$ .

The analogous construction exists on Milnor-Witt K-theory, but the twists are now necessary.

**Theorem 2.5.2.** *Consider as above a discretely valued field  $(K, v)$ . The  $\kappa_v$ -space  $\mathcal{C}_v := \mathcal{M}_v/\mathcal{M}_v^2$  is the conormal cone associated with  $(K, v)$ . It is an invertible  $\kappa_v$ -space (i.e. of dimension 1) and we let  $\omega_v := (\mathcal{M}_v/\mathcal{M}_v^2)^\vee$  be its  $\kappa_v$ -dual – in other words, the normal cone associated with  $(K, v)$ .*

*Then for any integer  $n \in \mathbb{Z}$ , there exists a unique morphism of abelian groups:*

$$\partial_v : K_n^{MW}(K) \rightarrow K_{n-1}^{MW}(\kappa_v, \omega_v)$$

*satisfying the two following properties:*

*(Res1)  $\partial_v(\eta.\sigma) = \eta.\partial_v(\sigma)$ , for all  $\sigma \in K_{n+1}^{MW}(K)$ .*

*(Res2) For any uniformizer  $\pi \in K$  and any units  $u_1, \dots, u_n \in K^\times$  such that  $u_1 = v_1\pi^m$ , and  $v(u_i) = 0$  for  $i > 1$ , one has:*

$$\partial_v([u_1, u_2, \dots, u_n]) = m_\epsilon \langle \bar{v}_1 \rangle [\bar{u}_2, \dots, \bar{u}_n] \otimes \bar{\pi}^*.$$

*Proof.* (See [Mor12, 3.15, 3.21]) We first choose some uniformizing parameter  $\pi \in \mathcal{M}_v$  of  $v$ . Then we introduce the following quotient ring:

$$A_* = K_*^{MW}(\kappa_v)[\xi]/(\xi - [-1].\xi)$$

which we view as a graded ring by putting  $\xi$  in degree 1. Then the proof reduces to show that the canonical map:

$$K^\times \rightarrow A_*, (u = a\pi^m) \mapsto [\bar{a}] + m_\epsilon \langle \bar{a} \rangle . \xi$$

extends uniquely to a morphism of  $\mathbb{Z}$ -graded ring

$$\Theta_\pi : K_*^{MW}(K) \rightarrow A_*$$

such that  $\Theta_\pi(\eta) = \eta$ .

Then given  $\sigma \in K_n^{MW}(K)$ , one can write uniquely:

$$\Theta_\pi(\sigma) = s_v^\pi(\sigma) + \partial_v^\pi(\sigma).\xi$$

so that we get two maps

$$s_v^\pi : K_*^{MW}(K) \rightarrow K_*^{MW}(\kappa_v)$$

$$\partial_v^\pi : K_*^{MW}(K) \rightarrow K_{*-1}^{MW}(\kappa_v)$$

such that  $s_v^\pi$  is a (homogeneous) morphism of  $\mathbb{Z}$ -graded ring (obvious).

Both maps depend on the choice of  $\pi$  in general. We then get the desired canonical map by the formula:

$$(2.5.2.a) \quad \partial_v(\sigma) = \partial_v^\pi(\sigma) \otimes \bar{\pi}^*.$$

□

**Definition 2.5.3.** Consider the notation of the above theorem. The homogeneous morphism of  $\mathbb{Z}$ -graded abelian groups

$$\partial_v : K_*^{MW}(K) \rightarrow K_*^{MW}(\kappa_v, \omega_v)$$

of degree  $-1$  is called the residue map associated with the valued field  $(K, v)$ .

Given a prime  $\pi$  of  $(K, v)$ , one also defines the residue map specialized at  $\pi$  as the map

$$\partial_v^\pi = \text{ev}_{\bar{\pi}^*} \circ \partial_v : K_*^{MW}(K) \rightarrow K_*^{MW}(\kappa_v)$$

with the notation of 2.4.2. Equivalently, this is the unique homogeneous morphism of  $\mathbb{Z}$ -graded abelian groups of degree  $-1$  such that relation (2.5.2.a) holds.

Finally, one defines the specialization map associated with of  $(K, v, \pi)$  as the morphism of  $\mathbb{Z}$ -graded rings

$$s_v^\pi : K_*^{MW}(K) \rightarrow K_*^{MW}(\kappa_v)$$

defined in the above proof.

**2.5.4.** Let  $(K, v)$  be a valued field,  $\pi$  a prime of  $v$  and  $u \in \mathcal{O}_v$  a unit. One can derive from the previous formula the following rule to compute residues for any symbol  $\sigma \in K_*^{MW}(K)$ :

$$\begin{aligned} \partial_v^{u\pi}(\sigma) &= \langle u \rangle \partial_v^\pi(\sigma) \\ \partial_v(\langle u \rangle \sigma) &= \langle \bar{u} \rangle \partial_v(\sigma) \\ \partial_v([u]\sigma) &= \epsilon[\bar{u}]\partial_v(\sigma) \end{aligned}$$

The first statement follows from 2.4.2, and the other ones follows by using the formula of the previous theorem and Proposition 2.2.13.

The specialization map can be computed by the following formulas (similar proof):

$$s_v^\pi(\sigma) = \partial_v^\pi([\pi] \cdot \sigma) - [-1]\partial_v^\pi(\sigma) = -\epsilon \partial_v^\pi([- \pi] \cdot \sigma).$$

Consider finally a morphism of rings  $R \rightarrow \mathcal{O}_v$ , and let  $\varphi : R \rightarrow K$ ,  $\bar{\varphi} : R \rightarrow \kappa_v$  be the induced morphisms. Then, for any symbol  $\alpha \in K_*^{MW}(R)$  (notation of Remark 2.2.8), one deduces the relation (use the same argument as for the previous relations):

$$\partial_v(\varphi_*(\alpha)\sigma) = \bar{\varphi}_*(\alpha)\partial_v(\sigma).$$

This implies that  $\partial_v$  is  $K_*^{MW}(\mathbb{Z})$ -linear. In particular, it commutes with multiplication not only by  $\eta$  but also by  $\epsilon$  and  $h$ .

**2.5.5.** Consider the assumptions of the previous theorem. One can further define, for any invertible  $\mathcal{O}_v$ -module  $\mathcal{L}$ , a twisted version:

$$\partial_v^\mathcal{L} : K_n^{MW}(K, \mathcal{L}_K) \rightarrow K_{n-1}^{MW}(K, \omega_v \otimes_{\kappa_v} \mathcal{L}_{\kappa_v})$$

where  $\mathcal{L}_E = \mathcal{L} \otimes_{\mathcal{O}_v} E$  for  $E = K, \kappa_v$ . The procedure is a bit intricate: take an element  $\sigma \otimes l$  on the left hand-side:  $\sigma \in K_n^{MW}(K)$  and  $l \in (\mathcal{L}_K)^\times$ . By definition,

there exists elements  $l_0 \in (\mathcal{L} - \{0\})$  (recall  $\mathcal{L}$  is an  $\mathcal{O}_v$ -module) and  $a \in K^\times$  such that  $l = l_0 \otimes_K a$ . Then one deduces by definition:

$$(2.5.5.a) \quad \sigma \otimes l = (\langle a \rangle \sigma) \otimes (l_0 \otimes_K 1_K).$$

One puts:

$$\partial_v^{\mathcal{L}}(\sigma \otimes l) = \partial_v(\langle a \rangle \sigma) \otimes (l_0 \otimes_K 1_{\kappa_v})$$

or simply  $\partial_v$  when  $\mathcal{L}$  is clear from the context.

*Remark 2.5.6.* The necessity to “renormalize” the parameter, as in (2.5.5.a), when considering residues makes the computation in quadratic intersection theory sometime quite cumbersome! Intuitively, we will be following a given orientation from open subschemes to the complementary (reduced) closed subscheme.

**Example 2.5.7.** We can specialize the definition of the above residue map to negative degree. Then according to Proposition 2.3.1, we get a canonical residue map:

$$\partial_v : W(K) \rightarrow W(\kappa_v, \omega_v)$$

such that

$$\partial_v(\langle u \rangle) = \begin{cases} 0 & v(u) \text{ even,} \\ \langle u\pi^{-v(u)} \rangle \otimes \bar{\pi}^* & v(u) \text{ odd, } \pi \text{ any prime.} \end{cases}$$

(Use Remark 2.2.19, point (3)!) Despite untwisted, this residue map is well-known in Witt theory: after the choice of a prime  $\pi$ , one has  $\partial_v^\pi = \psi^1$  in the notation of [MH73, IV, §1], and it is called the *second residue class morphism*.<sup>30</sup>

Note also that in degree 0, we get a more regular formula:

$$\partial_v : GW(K) \rightarrow W(\kappa_v, \omega_v), \partial_v(\langle u \rangle) = m_\epsilon \langle \bar{a} \rangle \otimes \bar{\pi}^*,$$

for  $u = a\pi^m$ ,  $v(a) = 0$ ,  $v(\pi) = 1$ .

*Remark 2.5.8.* Comparing the formulas in 2.5.1 and Theorem 2.5.2, it is clear that the residue in Milnor-Witt K-theory “modulo  $\eta$ ” coincides with the residue map in Milnor K-theory. One can be more precise using the maps of Definition 2.4.9. Given a discretely valued field  $(K, v)$ , and an invertible  $\mathcal{O}_v$ -module  $\mathcal{L}$ , one gets a commutative diagram:

$$\begin{array}{ccccc} K_*^M(K) & \xrightarrow{H} & K_*^{MW}(K, \mathcal{L}_K) & \xrightarrow{F} & K_*^M(K) \\ \partial_v^M \downarrow & & \downarrow \partial_v & & \downarrow \partial_v^M \\ K_*^M(\kappa_v) & \xrightarrow{H} & K_*^{MW}(\kappa_v, \omega_v \otimes \mathcal{L}_v) & \xrightarrow{F} & K_*^M(\kappa_v) \end{array}$$

where, for clarity,  $\partial_v^M$  is the residue on Milnor K-theory. The commutativity of the right-hand square was just explained, while the second one follows from the formula  $\partial_v(h.\sigma) = h.\partial_v^M(\sigma)$  (indeed  $h$  is unramified with respect to  $v$ ).

<sup>30</sup>The first residue class morphism is defined by the formula  $\psi^0 = \psi^1 \circ \gamma_{\langle \pi \rangle}$ .

Similarly, the second residue morphism on (twisted) Witt  $K$ -theory of the previous example obviously induces a canonical residue map:

$$\partial_v^I : \Gamma^n(K, \mathcal{L}_K) \rightarrow \Gamma^{n+1}(\kappa_v, \omega_v \otimes \mathcal{L}_v).$$

On the quotient ring, we get a canonical untwisted residue map:  $\partial_v^{\bar{I}} : \bar{\Gamma}^n(K) \rightarrow \bar{\Gamma}^{n+1}(\kappa_v)$  (because of 2.4.10). It is now a routine check to prove that all the maps of the square of Proposition 2.4.11 are compatible with the corresponding residue maps.

The following computation is an analogue of the Gersten exact sequence for Milnor  $K$ -theory (see [Ker09]):

**Theorem 2.5.9.** *Let  $(K, v)$  be a discretely valued field, and  $\mathcal{L}$  be an invertible  $\mathcal{O}_v$ -module.*

(1) *Then the following sequence (see Remark 2.4.5 for the first term) is exact:*

$$K_n^{MW}(\mathcal{O}_v, \mathcal{L}) \xrightarrow{\nu_*} K_n^{MW}(K, \mathcal{L}_K) \xrightarrow{\partial_v} K_{n-1}^{MW}(\kappa_v, \omega_v \otimes_{\kappa_v} \mathcal{L}_{\kappa_v}) \rightarrow 0$$

where  $\nu : \mathcal{O}_v \rightarrow K$  is the obvious inclusion and  $\nu_*$  is defined as in 2.4.6(2).

(2) *If moreover the ring  $\mathcal{O}_v$  contains an infinite field of characteristic not 2, then the map  $\nu_*$  is injective.*

Idea of proof for (1): the surjectivity of  $\partial_v$  is obvious: given any (abelian) generator  $\sigma = [\eta^r, v_1, \dots, v_{n-1+r}] \otimes \bar{\pi}^*$  of the right hand-side group,  $\pi \in \omega_v^\times$ ,  $v_i \in \kappa_v^\times$  (see Proposition 2.2.13), there exists lifts  $u_i \in \mathcal{O}_v^\times$  of  $v_i$ , along the epimorphism  $\mathcal{O}_v \rightarrow \kappa_v$ . Then formulas (Res1) and (Res2) implies that  $[\eta^n, \pi, v_1, \dots, v_n]$  lifts  $\sigma$ .

Also, (Res2) implies that  $\partial_v \nu_* = 0$ . Therefore, one only needs to prove that the induced map  $\text{Im}(\nu_*) \rightarrow \text{Ker}(\partial_v)$  is an isomorphism. This is the serious part! We refer the reader to the proof of [Mor12, Th. 3.22].

Point (2) is the *Gersten conjecture for Milnor-Witt  $K$ -theory* and for the local ring  $\mathcal{O}_v$ . This is due to Gille, Zhong and Scully: cf. [GSZ16].

### 3. A DETOUR ON CHOW-WITT GROUPS OF DEDEKIND SCHEMES

#### 3.1. Chow-Witt groups, quadratic divisors and rational equivalence.

**3.1.1.** We let  $X$  be a connected 1-dimensional scheme which is assumed to be normal (or equivalently regular). Let  $\mathcal{L}$  be an invertible sheaf over  $X$ . The main examples are smooth algebraic curves over a field and the spectrum of a Dedekind ring.

Let  $\kappa(X)$  be the function field of  $X$  and  $\mathcal{L}_\eta$  be the pullback to  $\text{Spec}(\kappa(X))$  seen as an invertible  $\kappa(X)$ -vector space.<sup>31</sup> We let  $X^{(1)}$  be the set of points  $x \in X$  which are closed (*i.e.* of codimension 1). This amounts to ask that the local ring  $\mathcal{O}_{X,x}$  is 1-dimensional, and therefore a discrete valuation ring. In particular,  $x$  uniquely

<sup>31</sup>We will also use the notation  $\mathcal{L}_{\kappa(X)} = \mathcal{L}_\eta$  later.

corresponds to a valuation  $v_x$  on  $\kappa(X)$  and we can consider the associated residue map (Theorem 2.5.2)

$$\partial_x : K_*^{MW}(\kappa(X), \mathcal{L}_\eta) \rightarrow K_{n-1}^{MW}(\kappa_x, \omega_{x/X} \otimes \mathcal{L}_x)$$

where  $\mathcal{L}_x$  is the restriction of the invertible  $\mathcal{O}_X$ -module  $\mathcal{L}$  to  $\kappa(x)$  and  $\omega_{x/X}$  is the normal sheaf of  $(\kappa(X), v_x)$ .<sup>32</sup> Explicitly:

$$\omega_{x/X} := (\mathcal{M}_{X,x}/\mathcal{M}_{X,x}^2)^\vee.$$

Given an element  $f \in K_*^{MW}(\kappa(X))$ , we will interpret  $\partial_x(f)$  as the  $K^{MW}$ -order of  $f$  at  $x$ .

**Lemma 3.1.2.** *With the above notations, for any  $f \in K_n^{MW}(\kappa(X))$ , the set:*

$$\{x \in X \mid \partial_x(f) \neq 0\}$$

*is finite.*

Given the definition of the residue map, and Proposition 2.2.13, this directly follows from the (more classical) fact:

**Lemma 3.1.3.** *Let  $u \in \kappa(X)^\times$  be a unit. Then the set  $\{x \in X \mid v_x(u) \neq 0\}$  is finite.*

Even in our generality, the finiteness is very classical. The alert reader will have recognized the support of the divisor associated with the rational function  $u$  of  $X$  appearing in the previous lemma!

*Remark 3.1.4.* The fact the scheme  $X$  is noetherian is essential here. However, in case one withdraw this assumption, everything would still be fine as we will obtain a locally finite subset of  $X$ . The theory of cycles, and quadratic cycles, would be fine as we will consider locally finite sums. This fits particularly well with the fact Chow groups (as well as Chow-Witt groups) are a kind of *Borel-Moore homology* in topology, and the latter is represented by the complex of locally finite singular chains (for suitable topological spaces).

The following definition is a slight generalization of the known definition of the classical definition of Chow-Witt groups. We refer the reader to [DFJ] for further developments.

**Definition 3.1.5.** Consider the previous notation. We define the *quadratic divisor class* map as the following sum:

$$\widetilde{\text{div}}_X = \sum_x \partial_x : K_*^{MW}(\kappa(X), \mathcal{L}_\eta) \rightarrow \bigoplus_{x \in X^{(1)}} K_{*-1}^{MW}(\kappa_x, \omega_{x/X} \otimes \mathcal{L}_x)$$

which is well-defined according to Lemma 3.1.2. This is a homogeneous morphism of  $\mathbb{Z}$ -graded abelian groups of degree  $-1$ .

<sup>32</sup>The notation  $\omega_{x/X}$  will take all its meaning in Definition 4.1.5. See also Example 4.1.6.

We then define the group  $\widetilde{\mathcal{C}}^p(X, \mathcal{L})_q$  for  $p = 0$  (resp.  $p = 1$ ) as respectively the source (resp. target) of  $\text{div}_X$  with  $* = q$  (resp.  $* = q + 1$ ), and as 0 otherwise. Therefore we have obtained a complex  $\widetilde{\mathcal{C}}^*(X, \mathcal{L})_q$ , concentrated in cohomological degree 0 and 1. We call it the *(cohomological) Rost-Schmid complex* of  $X$ .

We define the *Chow-Witt group*  $\widetilde{\text{CH}}^p(X, \mathcal{L})_q$  of codimension  $p$  and  $\mathbb{G}_m$ -degree  $q$  as the cohomology in degree  $p$  of this complex. When  $q = 0$ , we call it simply the Chow-Witt group, written  $\widetilde{\text{CH}}^p(X, \mathcal{L})$ .

*Remark 3.1.6.* (1) Beware that the differentials of the Rost-Schmid complex are homogeneous of degree  $-1$  with respect to the  $\mathbb{G}_m$ -grading. There are other possible conventions for the bigrading of  $\widetilde{\mathcal{C}}^*(X, \mathcal{L})_*$  but we will not use them here.

- (2) Even if we are mainly interested in the Chow-Witt groups, the other  $\mathbb{G}_m$ -degrees for  $q \neq 0$  will be crucial for computations. See Section 3.3.
- (3) The groups  $\widetilde{\text{CH}}^p(X, \mathcal{L})_q$  are analogues of the higher Chow groups. However, they do not deserve the name higher Chow-Witt groups as they only contribute to some part of the latter (that one can interpret as the Milnor-Witt motivic Borel-Moore homology; see [BY20, BCD<sup>+</sup>22]). In fact, while the latter are represented by a full ring spectrum  $\text{H}_{\text{MW}}\mathbb{Z}$ , the former are represented by a truncation of  $\text{H}_{\text{MW}}\mathbb{Z}$ . On the other hand, the groups just defined satisfy the same formalism than higher Chow groups.
- (4) If one replaces Milnor-Witt K-theory by the usual K-theory, one obtains Rost's ( $\mathbb{G}_m$ -)graded Chow groups  $\text{CH}^p(X)_q$  defined in [Ros96]. This was in fact the model for the previous definition. We refer the reader to 3.1.14 for more discussion.

**Example 3.1.7.** In codimension 0,  $\widetilde{\text{CH}}^0(X, \mathcal{L})$  is the kernel of the map in degree 0:

$$\text{GW}(\kappa(X), \mathcal{L}_\eta) \rightarrow \bigoplus_{x \in X^{(1)}} \text{W}(\kappa_x, \omega_{x/X} \otimes \mathcal{L}_x).$$

A virtual inner  $\mathcal{L}_\eta$ -space over the function field  $\kappa(X)$  which is in the kernel of this map is said to be *unramified* (with respect to the curve  $X$ ).

**3.1.8. Quadratic divisors.** Let us explicit the above definition when  $q = 0$ . The abelian group  $\widetilde{\text{CH}}^1(X, \mathcal{L})$  is the cokernel of the map in degree 1:

$$\widetilde{\text{div}} : \text{K}_1^{\text{MW}}(\kappa(X), \mathcal{L}_\eta) \rightarrow \bigoplus_{x \in X^{(1)}} \text{GW}(\kappa_x, \omega_{x/X} \otimes \mathcal{L}_x).$$

The abelian group at the target will be called the group of *quadratic divisors* (or 1-codimensional cycles) of  $(X, \mathcal{L})$ . These are formal sums of the form

$$(3.1.8.a) \quad \sum_{i \in I} (\sigma_i \otimes \bar{\pi}_i^* \otimes l_i).x_i$$

where:

- $x_i \in X$  is a closed point,
- $\sigma_i \in \text{GW}(\kappa_{x_i})$  is the class of an inner space over  $\kappa_{x_i}$ ,
- $\pi_i$  is a uniformizing parameter of the valuation ring  $\mathcal{O}_{X,x_i}$ , (equivalently a *local parameter* of the closed subscheme  $\{x_i\} \subset X$ ),<sup>33</sup>
- $l_i \in \mathcal{L}$  is a non-zero element.

In practice, one can also view the coefficient  $(\sigma_i \otimes \bar{\pi}_i^* \otimes l_i)$  as a virtual inner  $(\omega_{x_i/X} \otimes \mathcal{L}_{x_i})$ -space over  $\kappa(x_i)$ . Recall also from 2.4.2 the interpretation of this latter element as a  $\kappa(x_i)$ -equivariant map from the space of non zero elements  $(\omega_{x_i/X} \otimes \mathcal{L}_{x_i})^\times$  to the Grothendieck-Witt group  $\text{GW}(\kappa_{x_i})$ .

As in the classical case, quadratic divisors which are in the image of  $\widetilde{\text{div}}$  are said to be *principal*. Two quadratic divisors are *rationally equivalent* if their difference is principal.

**Example 3.1.9.** In the case  $X$  is in addition local, thus the spectrum of a discrete valuation ring  $\mathcal{O}_v$ , Theorem 2.5.9 implies in particular:

$$\widetilde{\text{CH}}^p(\mathcal{O}_v) = \begin{cases} \text{GW}(\mathcal{O}_v) & p = 0, \mathcal{O}_v \supset k_0 \\ 0 & p = 1. \end{cases}$$

where  $k_0$  is an infinite field of characteristic not 2. The vanishing of  $\widetilde{\text{CH}}^1(\mathcal{O}_v)$  can be interpreted by saying that every quadratic divisor of  $X$  is *principal*.

**3.1.10. Quadratic order of vanishing.** One can also make explicit the definition of the quadratic divisor class map. Let us fix a point  $x \in X^{(1)}$ , and  $v_x$  the corresponding discrete valuation  $v_x$  on  $\kappa(X)$ . We know that the abelian group  $\text{K}_1^{MW}(\kappa(X))$  is generated by elements  $[f]$  for a unit  $f \in \kappa(X)^\times$  (see Corollary 2.2.14 and Remark 2.2.15). Given a rational function  $f \in \kappa(X)^\times$  on  $X$ , we get with the notation of the above definition:

$$(3.1.10.a) \quad \partial_x([f]) = m_\epsilon \cdot \langle \bar{u} \rangle \otimes \bar{\pi}_x \in \text{GW}(\kappa_x, \omega_{x/X})$$

where we have chosen a local parameter  $\pi_x$  of  $x$  in  $X$  (*i.e.* a uniformizing parameter of the valuation ring  $\mathcal{O}_{X,x}$ ),  $m = v_x(f)$  is the classical order of vanishing of  $f$  at  $x$ , and  $u = f \cdot \pi_x^{-m}$  and  $\bar{u}$  is its class in  $\kappa_x = \mathcal{O}_{X,x}/\mathcal{M}_{X,x}$ . The formula, as well as the fact this element does not depend on the particular choice of  $\pi_x$ , directly follows from Theorem 2.5.2.

**Definition 3.1.11.** Consider the above assumptions. We define the quadratic order of vanishing of a rational function  $f \in \kappa(X)$  as the element  $\widetilde{\text{ord}}_x(f) = \partial_x([f])$  in  $\text{GW}(\kappa_x, \omega_{x/X})$ .

<sup>33</sup>the notation  $\bar{\pi}_i^*$  reminds the reader that we consider the element in  $\omega_{x/X} = (\mathcal{M}_{X,x_i}/\mathcal{M}_{X,x_i}^2)^\vee$  corresponding to  $\pi_i$



One can rewrite the definition of the divisor class map when  $q = 0$  in more classical terms:

$$\widetilde{\text{div}}([f]) = \sum_{x \in X^{(1)}} \widetilde{\text{ord}}_x(f).x.$$

*Remark 3.1.12.* One should be careful that the quadratic order of vanishing, as well as the quadratic divisor class map, is only additive in  $f$  with respect to the addition of  $K_1^{MW}(\kappa(X))$ , which in general differs from the group law of  $K^\times$  (see Remark 2.2.15).

**3.1.13.** Let us consider the previous definitions modulo  $\eta$ . Then we get in degree 0, 1 a map, independent of  $\mathcal{L}$ :

$$\kappa(X)^\times = K_1^M(\kappa(X), \mathcal{L}_\eta) \xrightarrow{\widetilde{\text{div}}_X \text{ mod } \eta} \bigoplus_{x \in X^{(1)}} K_0^M(\kappa_x, \omega_{x/X} \otimes \mathcal{L}_x) = Z^1(X)$$

where the right hand-side is the groupe of (ordinary!) 0-cycles of  $X$ . This is precisely the divisor class map: in fact, one obviously have the formula

$$\widetilde{\text{ord}}_x([f]) = \text{ord}_x(f) \text{ mod } \eta,$$

which amounts to say that the rank of the underlying inner form of  $\widetilde{\text{ord}}_x([f])$  is the classical order of vanishing  $\text{ord}_x(f)$  of  $f$  at  $x$  (use Formula Equation (3.1.10.a)). In particular, we get:

$$\widetilde{\text{CH}}^p(X, \mathcal{L})/(\eta) = \begin{cases} \mathbb{Z}^{\pi_0(X)} & p = 0 \\ \text{Pic}(X) & p = 1. \end{cases}$$

Moreover, one can describe explicitly the image of the map:

$$\widetilde{\text{CH}}^p(X, \mathcal{L}) \rightarrow \widetilde{\text{CH}}^p(X, \mathcal{L})/(\eta) \simeq \text{CH}^p(X)$$

It is just induced by the rank map: in degree 0, it sends an unramified inner  $\mathcal{L}$ -space  $\sigma$  over  $\kappa(X)$  to its rank  $\text{rk}(\sigma)$ . In degree 1, it sends a quadratic 0-cycle

$$\sigma : \sum_{i \in I} \sigma_i \otimes \bar{\pi}_i^* \otimes l_i.x_i$$

to the 0-cycle:

$$\text{rk}(\sigma) = \sum_{i \in I} \text{rk}(\sigma_i).x_i.$$

**3.1.14.** We can be more precise about the relation between Chow and Chow-Witt groups, using the definitions of Definition 2.4.9. Indeed, Remark 2.5.8 implies that the following diagram is commutative:

$$\begin{array}{ccccc} K_*^M(\kappa(X)) & \xrightarrow{H_\eta} & K_*^{MW}(\kappa(X), \mathcal{L}_\eta) & \xrightarrow{f_\eta} & K_*^M(\kappa(X)) \\ \text{div}_X \downarrow & & \downarrow \widetilde{\text{div}}_X & & \text{div}_X \downarrow \\ Z^1(X) & \xrightarrow{\sum_x H_x} & \widetilde{\text{C}}^1(X, \mathcal{L}) & \xrightarrow{\sum_x f_x} & Z^1(X) \end{array}$$

where  $Z^1(X)$  denotes the group of codimension 1 algebraic cycles (*i.e.* the Weil divisors) of  $X$ . Taking cokernel, one gets well defined maps:

$$\mathrm{CH}^1(X) \xrightarrow{\mathrm{H}} \widetilde{\mathrm{CH}}^1(X, \mathcal{L}) \xrightarrow{\mathrm{F}} \mathrm{CH}^1(X)$$

whose composite is multiplication by 2. This relation obviously extends to We still call them respectively the hyperbolic and forgetful maps.

**3.2. Homotopy invariance over a field.** Our next result was first proved for Milnor K-theory by Milnor: see [Mil70, Th. 2.3] (and also [BT73, 5.2]). It was generalized by Morel in [Mor12, Th. 3.24].

**Theorem 3.2.1** (Morel). *Let  $k$  be an arbitrary field,  $X = \mathbb{A}_k^1$  with function field  $k(t) = \kappa(X)$ . Let  $\varphi : k \rightarrow k(t)$  be the obvious inclusion.*

*Then the quadratic divisor class map fits of  $X$  into the following sequence*

$$0 \rightarrow \mathrm{K}_q^{MW}(k) \xrightarrow{\varphi_*} \mathrm{K}_q^{MW}(k(t)) \xrightarrow{\widetilde{\mathrm{div}}_X} \bigoplus_{x \in X^{(1)}} \mathrm{K}_{q-1}^{MW}(\kappa_x, \omega_{x/X}) \rightarrow 0$$

which is split exact.

In particular,

$$\widetilde{\mathrm{CH}}^p(\mathbb{A}_k^1)_q = \begin{cases} \mathrm{K}_q^{MW}(k) & p = 0 \\ 0 & p = 1. \end{cases}$$

Note that a splitting is easy to get: considering the valuation  $v = \deg$  on  $k(t)$ , the specialization map  $s_v^t$  (Definition 2.5.3) gives a splitting. More generally, any valuation  $v$  on  $k(t)$  trivial on  $k$  with uniformizing parameter  $\pi$  will give a splitting  $s_v^\pi$ .

The proof of this proposition uses the same trick as in Milnor's proof, and argue inductively on the degree in  $t$ . The principle is to filter  $\mathrm{K}_*^{MW}(k(t))$  by the subring  $L_d$  generated by  $\eta$  and symbols of the form  $[P(t)]$  where  $P(t)$  is a polynomial of degree less than  $t$ . We can then argue inductively on the  $\mathbb{Z}$ -graded  $\mathrm{K}_*^{MW}(k)$ -rings  $L_d$  using an explicit presentation of the  $\mathbb{Z}$ -graded  $\mathrm{K}_*^{MW}(k)$ -module  $L_d/L_{d-1}$ .

As an example, the reader is encouraged to work out for himself the case of  $L_1$ . The hint is to use the (obvious!) exact sequence:

$$0 \rightarrow k^\times \xrightarrow{\varphi_*} k(t)^\times \xrightarrow{\sum_x v_x} Z^1(\mathbb{A}_k^1) \rightarrow 0$$

Given that invertible sheaves on  $\mathbb{A}_k^1$  are trivializable, one immediately deduces the twisted version of the previous theorem.

**Corollary 3.2.2.** *Consider the notation of the previous theorem, and let  $\mathcal{L}$  be an invertible sheaf on  $\mathbb{A}_k^1$ . Then the following sequence of abelian groups is exact:*

$$0 \rightarrow \mathrm{K}_q^{MW}(k, \mathcal{L}_0) \xrightarrow{\varphi_*} \mathrm{K}_q^{MW}(k(t), \mathcal{L}_{k(t)}) \xrightarrow{\widetilde{\mathrm{div}}_X} \bigoplus_{x \in X^{(1)}} \mathrm{K}_{q-1}^{MW}(\kappa_x, \omega_{x/X} \otimes \mathcal{L}_x) \rightarrow 0$$

where  $\mathcal{L}_0$  (resp.  $\mathcal{L}_x$ ) is the fiber of  $\mathcal{L}$  over the point 0 (resp. a closed point  $x$ ). In particular,  $\mathcal{L}_0 = \mathcal{L} \otimes_{k[t]} k$  and the morphism  $\varphi_*$  is defined on twist by the canonical isomorphism:

$$\mathcal{L} \rightarrow \mathcal{L}_0 \otimes_k k[t], l \mapsto (l \otimes_{k[t]} 1) \otimes_k 1.$$

**3.3. Localization exact sequences.** In this section, we will illustrate the usefulness of considering the  $\mathbb{G}_m$ -graduation of the Rost-Schmid complex (Definition 3.1.5). The aim is to compute the Chow-Witt groups of the projective line.

**3.3.1.** Let again  $X$  be a normal connected 1-dimensional scheme,  $\mathcal{L}$  an invertible sheaf on  $X$ .

Consider in addition a finite subset  $Z \subset X$  of closed points of  $X$ , seen as reduced closed subscheme,  $i : Z \rightarrow X$ . Let  $\omega_{Z/X} = (\mathcal{I}(Z)/\mathcal{I}(Z)^2)^\vee$  be the normal sheaf of  $i$ , where  $\mathcal{I}(Z) \subset \mathcal{O}_X$  is the ideal sheaf. Let  $U = X - Z$ , and  $j : U \rightarrow X$  the open immersion.

There is an obvious split epimorphism:

$$j^* : \widetilde{C}^1(X, \mathcal{L})_q \rightarrow \widetilde{C}^1(U, \mathcal{L})_q$$

whose kernel is the finite sum:

$$\widetilde{C}H^0(Z, \omega_{Z/X} \otimes \mathcal{L}_Z)_q := \bigoplus_{z \in Z} K_q^{MW}(\kappa_z, \omega_{z/X} \otimes \mathcal{L}_z).$$

Remark that this notation fits in with the previous considerations as for any point  $z \in Z$ , we have a *canonical* isomorphism (this can be checked directly, or see (4.1.7.a)) of invertible  $\kappa_z$ -vector spaces:

$$\omega_{z/X} \otimes \mathcal{L}_z \simeq (\omega_{z/Z} \otimes \omega_{Z/X}|_z) \otimes \mathcal{L}_z \simeq \omega_{z/Z} \otimes (\omega_{Z/X} \otimes \mathcal{L}_z)_z.$$

Assembling all this, we get a commutative diagram whose lines are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widetilde{C}^0(X, \mathcal{L})_{q+1} & \xlongequal{\quad} & \widetilde{C}^0(U, \mathcal{L})_{q+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \text{div}_X & & \downarrow \text{div}_U \\ 0 & \longrightarrow & \widetilde{C}H^0(Z, \omega_{Z/X} \otimes \mathcal{L}_Z)_q & \xrightarrow{i_*} & \widetilde{C}^1(X, \mathcal{L})_q & \xrightarrow{j^*} & \widetilde{C}^1(U, \mathcal{L})_q \longrightarrow 0 \end{array}$$

**Definition 3.3.2.** Consider the previous notation. The exact sequence obtained by applying the snake lemma to the preceding commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & \widetilde{C}H^0(X, \mathcal{L})_{q+1} & \xrightarrow{j^*} & \widetilde{C}H^0(U, \mathcal{L}_U)_{q+1} & \xrightarrow{\partial_{Z/X}} & \widetilde{C}H^0(Z, \omega_{Z/X} \otimes \mathcal{L}_Z)_q \\ & & & & & & \xrightarrow{i_*} \widetilde{C}H^1(X, \mathcal{L})_q \xrightarrow{j^*} \widetilde{C}H^1(U, \mathcal{L}_U)_q \rightarrow 0 \end{array}$$

is called the *localization exact sequence* associated with  $i$ .

The connecting map  $\partial_{Z/X}$  is called the *residue map* associated with  $i$ . It is induced by the following restriction and corestriction of the quadratic divisor class map  $d_X$ :

$$\sum_{z \in Z} \partial_z : K_{q+1}^{MW}(\kappa(X)) \longrightarrow \bigoplus_{z \in Z} K_q^{MW}(\kappa_z, \omega_z \otimes \mathcal{L}_z).$$

### 3.4. Twisted Chow-Witt groups of the projective line.

**3.4.1.** We now illustrate the usage of the localization exact sequence devined in the previous section. Let  $k$  be an arbitrary field.

Let  $\mathbb{P}_k^1 = \text{Proj}(k[x, y])$  be the projective line,  $\infty = [1 : 0]$  be point at infinity with complementary open subscheme  $\mathbb{A}_k^1 = \text{Spec}(k[x])$ . We let  $i^\infty : \{\infty\} \rightarrow \mathbb{P}_k^1$  be the natural closed immersion, and  $j : \mathbb{A}_k^1 \rightarrow \mathbb{P}_k^1$  the complementary open immersion. We fix a line bundle  $\mathcal{L}$  over  $\mathbb{P}_k^1$ , which is therefore determined up to isomorphism by its degree,  $\deg(\mathcal{L})$ . We let  $\mathcal{L}'$  be the restriction of  $\mathcal{L}$  to  $\mathbb{A}_k^1$ .

Then the localization exact sequence of  $i^\infty$  together with Morel's homotopy invariance theorem (see Corollary 3.2.2) gives us the following exact sequence:

$$\begin{aligned} 0 \rightarrow \widetilde{\text{CH}}^0(\mathbb{P}_k^1, \mathcal{L})_{q+1} &\xrightarrow{j^*} \text{K}_{q+1}^{MW}(k, \mathcal{L}_0) \xrightarrow{\partial_{Z/X}} \text{K}_q^{MW}(k, \omega_\infty \otimes \mathcal{L}_\infty) \\ &\xrightarrow{i_*^\infty} \widetilde{\text{CH}}^1(\mathbb{P}_k^1, \mathcal{L})_q \rightarrow 0 \end{aligned}$$

where we have denoted by  $\mathcal{L}_0$  the restriction of  $\mathcal{L}'$  to the point 0 in  $\mathbb{A}_k^1$ .

The main problem is to determine the residue map  $\partial_{Z/X}$ .

**Lemma 3.4.2.** *Consider the above assumptions and notations.*

*Then if  $\deg(\mathcal{L})$  is even,  $\partial_{Z/X} = 0$ . If  $d = \deg(\mathcal{L})$  is odd, after choosing an isomorphism  $\mathcal{L} \simeq \mathcal{O}(d)$ , and using the isomorphism  $\omega_\infty \simeq k$  given by the uniformizing parameter  $1/x$ , we get the following commutative diagram:*

$$\begin{array}{ccc} \text{K}_{q+1}^{MW}(k, \mathcal{L}_0) & \xrightarrow{\partial_{Z/X}} & \text{K}_q^{MW}(k, \omega_\infty \otimes \mathcal{L}_\infty) \\ \sim \downarrow & & \downarrow \sim \\ \text{K}_{q+1}^{MW}(k) & \xrightarrow{\gamma_\eta} & \text{K}_q^{MW}(k). \end{array}$$

*Proof.* One reduces to the case  $\mathcal{L} = \mathcal{O}(d)$ . We consider  $U = U_\infty$  (resp.  $U_0$ ) the open complement of  $\infty$  (resp. 0) in  $\mathbb{P}_k^1$ , so that  $U_\infty = \text{Spec}(k[x])$  and  $U_0 = \text{Spec}(k[y])$ . The glueing map  $U_0 \cap U_\infty \rightarrow U_\infty \cap U_0$  is given by mapping  $x$  to  $y^{-1}$ . Then the line bundle  $\mathcal{L} = \mathcal{O}(d)$  is given on  $U_\infty$  (resp.  $U_0$ ) by a free module  $\mathcal{L}'_\infty = k[x].u$  (resp.  $\mathcal{L}'_0 = k[y].v$ ) with a glueing map  $u \mapsto y^{-d}.v$ .

Note therefore that one has preferred chosen isomorphisms:  $\mathcal{L}_0 \simeq_u k$  and  $\mathcal{L}_\infty \simeq_v k$ . Therefore we deduce a canonical map

$$\text{K}_{q+1}^{MW}(k) \simeq_{u_*^{-1}} \text{K}_{q+1}^{MW}(k, \mathcal{L}_0) \xrightarrow{\partial_{Z/X}} \text{K}_q^{MW}(k, \omega_\infty \otimes \mathcal{L}_\infty) \simeq_{y_* \otimes v_*} \text{K}_q^{MW}(k)$$

denoted by  $\partial'_{Z/X}$ .

We compute the image of  $\sigma \in \text{K}_q^{MW}(k)$  under  $\partial'_{Z/X}$ . First,  $u_*^{-1}(\sigma) = \sigma \otimes u$ . Then we need to use the map  $\varphi_*$  of Corollary 3.2.2, which sends the latter to

$$\sigma \otimes (u \otimes 1) \in \text{K}_q^{MW}(k(t), \mathcal{L}_0 \otimes_k k(t)).$$

In order to compute its residue at  $\infty$ , one needs to write it as an element of  $K_q^{MW}(k(t), \mathcal{L}_\infty \otimes_k k(t))$ . Therefore, one uses the above change of variables:

$$\sigma \otimes (y^{-d}v \otimes 1) = (\langle y^{-d} \rangle \sigma) \otimes (v \otimes 1).$$

Now if  $d$  is even,  $\langle y^{-d} \rangle = 1$  and we get:  $\partial_\infty^y(\langle y^{-d} \rangle \sigma) = 0$  as  $\sigma$  comes from  $K_*^{MW}(k)$ . Thus  $\partial_{X/Z}(\sigma) = 0$ .

If on the contrary,  $d$  is odd,  $\langle y^{-d} \rangle = \langle y \rangle$ . Therefore

$$\partial_\infty^y(\langle y^{-d} \rangle \sigma) = \partial_\infty^y(\langle y \rangle \sigma) = \eta \cdot \sigma$$

and one deduces that  $\partial'_{Z/X}(\sigma) = \eta \cdot \sigma$  as expected.  $\square$

**3.4.3.** Let  $d = \deg(\mathcal{L})$ . The lemma and the localization exact sequence gives the following possibilities:

(1) if  $d$  is even, one gets isomorphisms:

$$\begin{aligned} j^* : \widetilde{\text{CH}}^0(\mathbb{P}_k^1, \mathcal{L})_q &\xrightarrow{\sim} \widetilde{\text{CH}}^0(\mathbb{A}_k^1, \mathcal{L})_q \simeq K_q^{MW}(k, \mathcal{L}_0) \\ i_*^\infty : K_q^{MW}(k, \omega_\infty \otimes \mathcal{L}_\infty) &\xrightarrow{\sim} \widetilde{\text{CH}}^1(\mathbb{P}_k^1, \mathcal{L})_q. \end{aligned}$$

(2) If  $d$  is odd, and after the choices indicated in the above lemma, we get an exact sequence:

$$0 \rightarrow \widetilde{\text{CH}}^0(\mathbb{P}_k^1, \mathcal{L})_{q+1} \xrightarrow{j^*} K_{q+1}^{MW}(k) \xrightarrow{\gamma_\eta} K_q^{MW}(k) \xrightarrow{i_*^\infty} \widetilde{\text{CH}}^1(\mathbb{P}_k^1, \mathcal{L})_q \rightarrow 0$$

Recall that the cokernel of  $\gamma_\eta$  is  $K_q^M(K)$  (see 2.2.9), and its kernel is  $2 \cdot K_q^M(K)$ , that is the  $q$ -th graded part of the ideal generated by 2 in the ring  $K_*^M(K)$  (see Corollary 2.3.10). To summarize, we have obtained the following computation of (graded) Chow-Witt groups, first proved by Jean Fasel for a perfect base field of characteristic not 2 (see [Fas13]):

**Theorem 3.4.4.** *Consider the above assumption:  $k$  is an arbitrary field,  $\mathcal{L}$  an invertible sheaf over  $\mathbb{P}_k^1$  of degree  $d$ . Then*

$$\begin{aligned} \widetilde{\text{CH}}^0(\mathbb{P}_k^1, \mathcal{L})_q &\simeq \begin{cases} K_q^{MW}(k, \mathcal{L}_0) & d \text{ even} \\ 2 \cdot K_q^M(K) & d \text{ odd} \end{cases} \\ \widetilde{\text{CH}}^1(\mathbb{P}_k^1, \mathcal{L})_q &\simeq \begin{cases} K_q^{MW}(k, \omega_\infty \otimes \mathcal{L}_\infty) & d \text{ even} \\ K_q^M(k) & d \text{ odd} \end{cases} \end{aligned}$$

Recall finally from Remark 2.2.19 that:  $\text{Ker}(\eta) = 2_e \cdot K_q^M(K)$  when  $\text{char}(k) \neq 2$ .

Let us draw the picture for Chow-Witt groups:

$$\widetilde{\text{CH}}^p(\mathbb{P}_k^1, \mathcal{L}) = \begin{cases} \text{GW}(k, \mathcal{L}_0) & p = 0, \deg(\mathcal{L}) = 2r, \\ \text{GW}(k, \omega_\infty \otimes \mathcal{L}_\infty) & p = 1, \deg(\mathcal{L}) = 2r, \\ \mathbb{Z} & p = 0, 1, \deg(\mathcal{L}) = 2r + 1. \end{cases}$$

In particular, these groups depend on the twist  $\mathcal{L}$  when  $\text{GW}(k)$  is non-trivial !

4. TRANSFERS

**4.1. Cotangent complexes and canonical sheaves.** Recall for convenience (and completeness) the following definition.

**Definition 4.1.1.** Let  $f : X \rightarrow S$  be a morphism of schemes.

- (1)  $f$  is *smoothable* if there exists a factorization

$$f : X \xrightarrow{i} P \xrightarrow{p} S$$

such that  $p$  is smooth and  $i$  is a closed immersion.

- (2)  $f$  is a *complete intersection* if there exists a factorization

$$f : X \xrightarrow{i} P \xrightarrow{p} S$$

such that  $p$  is smooth and  $i$  is a regular closed immersion.

- (3)  $f$  is a *local complete intersection* if any point  $x \in X$  admits an open neighborhood  $V$  such that the restriction  $f|_V$  is a complete intersection.

Following a classical abuse, we will simply say that  $f$  is lci.

*Remark 4.1.2.* (1) The second definition first appeared in [BGI71, VIII, 1.1] (see also [mDJ, 37.59.2], especially in the non-noetherian case).

(2) A morphism  $f$  is a complete intersection if and only if it is smoothable and lci (see [BGI71, VIII, 1.2]).

**4.1.3. Cotangent Complexes.** For a scheme  $X$ , we let  $D(\mathcal{O}_X)$  be the derived category of  $\mathcal{O}_X$ -modules.

Let  $f : X \rightarrow S$  be a morphism of schemes. Recall that one can associate to  $f$  its cotangent complex  $\mathcal{L}_{X/S}$  (see [Ill71, III, 1.2.3]), a canonically defined object of  $D(\mathcal{O}_X)$  — it is the derived functor of the Kähler differential functor evaluated at  $\mathcal{O}_X/f^{-1}\mathcal{O}_S$ .

If  $f$  is a complete intersection, choosing a factorization as in Definition 4.1.1(2), one can explicitly compute its cotangent complex  $\mathcal{L}_{X/S}$ . It is quasi-isomorphic to a complex concentrated in two degrees

$$\mathcal{C}_{X/P} \rightarrow \Omega_{P/S}|_X$$

where  $\mathcal{C}_{X/P} = \mathcal{I}_i/\mathcal{I}_i^2$  is the *conormal sheaf* associated with  $i$ , placed in homological degree +1, and  $\Omega_{P/S}$  is the *cotangent sheaf* of  $P/S$  (the conormal sheaf of the diagonal of  $P/S$ ) placed in degree 0 (see [Ill71, VIII, 3.2.7]). This obviously implies that if  $f : X \rightarrow S$  is only assumed to be lci, then its cotangent complex is Zariski locally in  $X$  quasi-isomorphic to a complex concentrated in degree 0 and 1 and whose terms are free.<sup>34</sup> In particular,  $\mathcal{L}_{X/S}$  is perfect.<sup>35</sup>

<sup>34</sup>One says that  $\mathcal{L}_{X/Z}$  has perfect homological amplitude in  $[0, 1]$ .

<sup>35</sup>A complex  $\mathcal{K}$  of  $\mathcal{O}_X$ -module is perfect if any point of  $X$  admits an open neighborhood  $U$  such that  $\mathcal{K}|_U$  is quasi-isomorphic to a bounded complex  $\mathcal{L}$  such that for all integers  $n$  the coherent sheaf  $\mathcal{L}^n$  is a direct factor of a finite free  $\mathcal{O}_U$ -module. See [mDJ, Def. 20.44.1/08C4].

The interest of the cotangent complex is to be compatible with composition in the following sense. Consider a commutative diagram

$$(4.1.3.a) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \swarrow \\ & S & \end{array} \begin{array}{c} \Delta \\ \swarrow \quad \searrow \\ \end{array} \begin{array}{c} \\ \\ g \end{array}$$

of morphisms of schemes. Then one has a canonical homotopy exact sequence of  $D(\mathcal{O}_X)$ :

$$(4.1.3.b) \quad (f^* \mathcal{L}_{Y/S}) \rightarrow \mathcal{L}_{X/S} \rightarrow \mathcal{L}_{X/Y}.$$

**4.1.4.** Recall from [Del87, Ex. 4.13] that one associates to a perfect complex  $C$  of  $\mathcal{O}_X$ -modules its *rank*  $\mathrm{rk}(C)$  which is a locally constant function  $X \rightarrow \mathbb{Z}$  and its determinant  $\det(C)$  which is a well-defined invertible sheaf over  $X$ .<sup>36</sup>

**Definition 4.1.5.** Let  $f : X \rightarrow S$  be a morphism whose cotangent complex is perfect (eg: lci). One associates to  $f$  its *canonical sheaf*:

$$\omega_{X/S} = \det(\mathcal{L}_{X/S}).$$

We will also say that  $f$  is of (virtual) relative dimension  $d = \mathrm{rk}(\mathcal{L}_{X/S})$ .

When  $X/S$  is the spectrum of a ring extension  $B/A$ , the canonical sheaf  $\omega_{X/S}$  is determined by its global sections. We will denote by  $\omega_{B/A}$  the  $B$ -module of its global sections, and call it the *canonical module* associated with  $B/A$ .

**Example 4.1.6.** (1) If  $f : X \rightarrow S$  is smooth, the above definition coincides with the classical definition of the canonical sheaf: the cotangent sheaf of  $f$  is locally free  $\Omega_{X/S}$ , and  $\omega_{X/S}$  is the maximal exterior power of  $\Omega_{X/S}$  as a  $\mathcal{O}_X$ -module.

Note that in particular that when  $f$  is étale, one canonical has:  $\omega_{X/S} = \mathcal{O}_X$ . This is really an identity, and not just an isomorphism.

(2) if  $f = i : Z \rightarrow X$  is a regular closed immersion of pure codimension 1, then  $\omega_{Z/S} = \mathcal{C}_{Z/X}^\vee$ , the dual of the conormal sheaf.

(3) A morphism  $f : X \rightarrow S$  of schemes which is flat, of finite presentation and lci is called *syntomic* after Fontaine and Messing. Syntomic morphisms are stable under base change and composition ([mDJ, 29.30.3, 29.30.4]). In this case the virtual relative dimension of  $X/S$  equals the dimension of fibers functions, which to a point  $s \in S$  associates the dimension of  $X_s = f^{-1}(\{s\})$ . This can be seen by reducing to the case where  $S$  is the spectrum of a field as the cotangent complex of  $f$  is stable under (naive) pullbacks.

<sup>36</sup>The couple  $(\det, \mathrm{rk})$  is actually the left Kan extension, as an  $\infty$ -functor, from the  $\infty$ -category of perfect complexes to the  $\infty$ -groupoid of graded line bundles,

$$\mathrm{Perf}(X) \rightarrow \mathcal{P}\mathrm{ic}^{\mathbb{Z}}(X)$$

of the functor sending a locally free  $\mathcal{O}_X$ -module to its rank and its maximal exterior power (see [LO21, §5]). It is also obtained by restriction of the canonical functor from the Thomason-Trobaugh  $K$ -theory space  $K(X)$  to  $\mathcal{P}\mathrm{ic}^{\mathbb{Z}}(X)$  (see [BS17] in the affine case).

**4.1.7.** Let us consider a commutative diagram (4.1.3.a) such that the cotangent complexes of all three morphisms are perfect (for example,  $f$  and  $g$  are lci). Then the above homotopy exact sequence translates into a canonical isomorphism of invertible sheaves over  $X$ :

$$(4.1.7.a) \quad \psi_{\Delta} : \omega_{X/S} \simeq \omega_{X/Y} \otimes (f^* \omega_{Y/S})$$

*Remark 4.1.8.* It is also useful to consider commutative squares:

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ q \downarrow & \ominus & \downarrow p \\ T & \xrightarrow{f} & S. \end{array}$$

Dividing the square into two commutative triangles, and applying the preceding isomorphism for both triangles, one gets a canonical isomorphism:

$$\psi_{\Theta} : \omega_{Y/X} \otimes (g^* \omega_{X/S}) \simeq \omega_{Y/T} \otimes (q^* \omega_{T/S}).$$

When the preceding square is affine corresponding to a commutative square of rings:

$$\begin{array}{ccc} D & \longleftarrow & C \\ \uparrow & \ominus & \uparrow \\ B & \longleftarrow & A \end{array}$$

one gets the following simpler form, an isomorphism of invertible  $D$ -modules:

$$\psi_{\Theta} : \omega_{D/C} \otimes_C \omega_{C/A} \simeq \omega_{D/B} \otimes_B \omega_{B/A}$$

where the tensor product on the left (resp. right) is taken with respect to the induced structure of  $C$ -module on  $\omega_{D/C}$  (resp.  $B$ -module on  $\omega_{D/B}$ ).

**Example 4.1.9.** Let us consider a finitely generated lci  $A$ -algebra  $B$ . We assume that there exists a smooth  $A$ -algebra  $R$  and a regular ideal  $I \subset R$  such that  $B \simeq R/I$  as an  $A$ -algebra so that we get a surjection  $\varphi : B \rightarrow R$ .<sup>37</sup>

Assume  $\text{Spec}(A)$  and  $\text{Spec}(R)$  are irreducible and let  $n$  be the rank of the  $A$ -algebra  $R$ ,  $m$  be the height of  $I$ . Then one can compute the canonical module of  $B/A$  as:

$$\Theta : \omega_{B/A} \simeq \omega_{B/R} \otimes_B (\omega_{R/A} \otimes_R B) \simeq \Lambda_B^m(I/I^2)^\vee \otimes_R \Omega_{R/A}^n.$$

Indeed, as  $I$  is regular,  $I/I^2$  is a locally free  $B$ -module of constant rank  $m$ .

In general, we have an exact sequence of  $B$ -modules:

$$(4.1.9.a) \quad 0 \rightarrow N \rightarrow I/I^2 \xrightarrow{\psi} \Omega_{R/A} \otimes_R B \xrightarrow{\varphi^*} \Omega_{B/A} \rightarrow 0$$

where  $N = \text{Ker}(\psi)$ ,  $\psi$  is induced by the composition

$$I \hookrightarrow R \xrightarrow{d_R} \Omega_{R/A} \rightarrow \Omega_{R/A} \otimes_R B$$

---

<sup>37</sup>By Noether normalization lemma, this will automatically be the case if  $A$  is a field; moreover we can choose  $R$  to be a polynomial  $k$ -algebra.



and the last map is induced by  $\varphi : R \rightarrow B$ . As recalled in 4.1.3, the cotangent complex  $\mathcal{L}_{B/A}$ , that we view as a complex of  $B$ -modules, is concentrated in homological degree  $[0, 1]$  and one deduces from the above exact sequence and the homotopy exact sequence (4.1.3.b) the following isomorphisms:

$$\begin{aligned} H_0(\mathcal{L}_{B/A}) &\simeq \Omega_{B/A}, \\ H_1(\mathcal{L}_{B/A}) &\simeq N. \end{aligned}$$

When  $B/A$  is étale, one gets  $\Omega_{B/A} = N = 0$ , and  $n = m$ . So  $\omega_{B/A} = B$  (this is really an identity), and the isomorphism  $\Theta : B \simeq \Lambda^n(I/I^2)^\vee \otimes_B \Lambda^n(\Omega_{R/A} \otimes_P B)$  is obtained by transposing the isomorphism  $\psi$ .

**Example 4.1.10.** We consider a particular case of the preceding example, that of a finite field extension  $L/K$ . We can choose a set of generators  $(\alpha_1, \dots, \alpha_n)$ ,  $L = K[\alpha_1, \dots, \alpha_n]$ . If we consider the polynomial  $K$ -algebra  $R = K[t_1, \dots, t_n]$ , then one can write  $L = R/I$ , and  $I = (f_1, \dots, f_n)$  where  $f_i$  is a polynomial in the variables  $t_1, \dots, t_i$ , monogenic in  $t_i$ , which is a lift of the characteristic polynomial of the algebraic element  $\alpha_i$  of  $L/K[\alpha_1, \dots, \alpha_{i-1}]$ . Thus,  $I$  is regular. Then we get from the previous example a canonical isomorphism:

$$(4.1.10.a) \quad \Theta : \omega_{L/K} \simeq \Lambda_L^n(I/I^2)^\vee \otimes_R \Omega_{R/K}^n.$$

We then get an explicit basis of the invertible  $B$ -module  $\omega_{B/A}$ , given by the element:

$$(4.1.10.b) \quad (\bar{f}_1 \wedge \dots \wedge \bar{f}_n)^* \otimes (dt_1 \wedge \dots \wedge dt_n).$$

If  $L/K$  is separable, as explained in the end of the previous example,  $\Omega_{L/K}^1 = 0$ , so  $\omega_{L/K} = L$ . According to the description of  $\psi$ , one obtains that the element (4.1.10.b) goes under  $\Theta^{-1}$  to the unit:

$$(\pi'_1(\alpha_1)\pi'_2(\alpha_1, \alpha_2) \dots \pi'_n(\alpha_1, \dots, \alpha_n))^{-1} \in L^\times.$$

We have to take the inverse of the obvious element as  $\Theta$  is obtained after transposition, as seen in the end of the previous example.

Let us assume on the contrary that  $L/K$  is totally inseparable. Let  $p > 0$  be the characteristic of  $K$ . Then  $\alpha_i = (a_i)^{1/q_i}$ ,  $a_i \in K - K^p$ . Moreover, in the sequence (4.1.9.a) with  $B/A = L/K$ , one obtains that  $\psi = 0$ . In other words, one gets isomorphisms:

$$\begin{aligned} \varphi_* : \Omega_{R/K} \otimes_R L &\xrightarrow{\sim} \Omega_{L/K}, \\ N &\simeq I/I^2. \end{aligned}$$

In particular,  $(d\alpha_1, \dots, d\alpha_n)$ , which is the image of  $(dt_1, \dots, dt_n)$  by the isomorphism  $\varphi_*$ , is an  $L$ -basis of  $\Omega_{L/K}$ , which can be identified to  $H_0(\mathcal{L}_{L/K})$ . Similarly,  $N \simeq I/I^2$  can be identified with  $H_1(\mathcal{L}_{L/K})$ , and an  $L$ -basis is given by  $(\bar{f}_1, \dots, \bar{f}_n)$  — each  $\bar{f}_i$  goes to zero in  $\Omega_{R/K} \otimes_R L$ .

*Remark 4.1.11.* In the case of a totally inseparable extension  $L/K$ ,  $\mathbb{F}_p \subset K$ , one defines the *imperfection module*  $\Gamma_{L/K}$  of  $L/K$  by the following exact sequence:

$$0 \rightarrow \Gamma_{L/K} \rightarrow \Omega_{K/\mathbb{F}_p} \otimes_K L \rightarrow \Omega_{L/\mathbb{F}_p} \rightarrow \Omega_{K/\mathbb{F}_p} \rightarrow 0.$$

One deduces that  $H_1(\mathcal{L}_{L/K}) \simeq \Gamma_{L/K}$ . In particular, with the notations of the previous paragraph, the imperfection module  $\Gamma_{L/K}$  is an  $n$ -dimensional  $L$ -vector space which is isomorphic to  $I/I^2$ .

## 4.2. The quadratic degree map.

**4.2.1.** We will now come back to Theorem 3.4.4 and give its fundamental application to build transfers on Milnor-Witt K-theory.

Let  $k$  be an arbitrary field, and  $\omega = \omega_{\mathbb{P}_k^1/k}$  be the canonical sheaf on  $\mathbb{P}_k^1$  (Definition 4.1.5), and let  $\infty$  (resp.  $\eta$ ) be the point at infinity (resp. generic point) of  $\mathbb{P}_k^1$ . We first rewrite the quadratic divisor class map in *homological* conventions (see [Dég23] more generally). Consider a point  $x \in \mathbb{P}_k^1$  with residue field  $\kappa_x$ . Note that  $\kappa_x/k$  is not necessarily separable so the canonical sheaf  $\omega_{\kappa_x/k}$  can be non-trivial. Nevertheless, the commutative diagram

$$\begin{array}{ccc} \text{Spec}(\kappa_x) & \xrightarrow{x} & \mathbb{P}_k^1 \\ & \searrow & \swarrow \\ & \text{Spec}(k) & \end{array}$$

gives a canonical isomorphism  $\psi^x : \omega_{L/k} \simeq \omega_{x/X} \otimes \omega|_x$  — see (4.1.7.a).

In particular, the quadratic divisor class map for  $\mathbb{P}_k^1/k$  in  $\mathbb{G}_m$ -degree  $q \in \mathbb{Z}$  and with twists  $\omega$  can be rewritten as:

$$\widetilde{\text{div}} : K_{q+1}^{MW}(k(t), \omega_{k(t)/k}) \longrightarrow \bigoplus_{x \in \mathbb{P}_{k,(0)}^1} K_q^{MW}(\kappa_x, \omega_{\kappa_x/k}) =: \widetilde{C}_0(\mathbb{P}_k^1)_q.$$

Recall that  $\widetilde{\text{div}}$  is the sum of the residue maps  $\partial_x : K_{q+1}^{MW}(k(t), \omega_{k(t)/k}) \rightarrow K_q^{MW}(\kappa_x, \omega_{\kappa_x/k})$  for  $x$  a closed point in  $\mathbb{P}_k^1$ , corresponding to a valuation  $v_x$  on  $k(t)$  with residue field  $\kappa_x$ .

The cokernel of  $\widetilde{\text{div}}$  equals the Chow-Witt group  $\widetilde{\text{CH}}^1(\mathbb{P}_k^1, \omega)$  and, as  $\omega$  is even, Theorem 3.4.4 and paragraph 3.4.3 tells us that the pushforward map

$$i_*^\infty : K_q^{MW}(k) \rightarrow \widetilde{\text{CH}}^1(\mathbb{P}_k^1, \omega)_q$$

is an isomorphism. Let us introduce the following definition.<sup>38</sup>

**Definition 4.2.2.** Using the above notation, we denote by

$$\widetilde{\text{deg}}_q : \widetilde{\text{CH}}^1(\mathbb{P}_k^1, \omega)_q \rightarrow K_q^{MW}(k)$$

the inverse of the isomorphism  $i_*^\infty$  and call it the *quadratic degree map* in  $\mathbb{G}_m$ -degree  $q$  (associated with  $\mathbb{P}_k^1$ ).

<sup>38</sup>This is the mother case of the degree map on Chow-Witt groups. See [Dég23].

In degree  $q = 0$ , we therefore get a map:

$$\widetilde{\text{deg}} : \widetilde{\text{CH}}^1(\mathbb{P}_k^1, \omega) \rightarrow \text{GW}(k).$$

Following Bass and Tate ([BT73, I.5.4]) and Morel ([Mor12, §4.2]), we can be more precise about this notion of quadratic degree.

**Proposition 4.2.3.** *Consider the above assumptions and notation. Then there exists a unique family of maps*

$$\text{Tr}_{\kappa_x/k}^{MW} : K_q^{MW}(\kappa_x, \omega_{\kappa_x/k}) \rightarrow K_q^{MW}(k), x \in \mathbb{P}_{k,(0)}^1$$

which fits into the following commutative diagram

$$\begin{array}{ccccc} & & K_q^{MW}(k) & & \\ & & \downarrow i_*^\infty & \searrow & \\ K_{q+1}^{MW}(k(t), \omega_{k(t)/k}) & \xrightarrow{\quad \widetilde{\text{div}} \quad} & \tilde{C}_0(\mathbb{P}_k^1)_q & \xrightarrow{\quad \sum_x \text{Tr}_{\kappa_x/k}^{MW} \quad} & K_q^{MW}(k) \end{array}$$

in such a way that composition of the horizontal maps is zero.

In particular, the quadratic degree map is defined at the level of cycles:

$$\widetilde{\text{deg}}_q := \sum_{x \in \mathbb{P}_{k,(0)}^1} \text{Tr}_{\kappa_x/k}^{MW} : \tilde{C}_0(\mathbb{P}_k^1)_q \rightarrow K_q^{MW}(k).$$

The last condition in the above statement can be translated by saying that the quadratic 0-cycles of degree 0 on  $\mathbb{P}_k^1$  are exactly the principal (*i.e.* rationally trivial) quadratic divisors (using the terminology of 3.1.8).

*Remark 4.2.4.* (1) Note that the commutative triangle corresponds to the normalization property:  $\text{Tr}_{\kappa_\infty/k}^{MW} = \text{Id}$ .

(2) The formula  $\widetilde{\text{deg}} \circ \widetilde{\text{div}} = 0$  is the quadratic analogue of the *Weil reciprocity formula*. Given the preceding normalization property, it can be restated as the following equation:

$$\sum_{x \in (\mathbb{A}_k^1)_{(0)}} \text{Tr}_{\kappa_x/k}^{MW} \circ \partial_x = -\partial_\infty.$$

**4.2.5. Quadratic and classical degree.** As the pushforward morphism  $i_*^\infty$  is compatible with the forgetful and hyperbolic maps of 3.1.14, we get by definition the following commutative diagram:

$$\begin{array}{ccccc} \text{CH}^1(\mathbb{P}_k^1) & \xrightarrow{\quad \text{H} \quad} & \widetilde{\text{CH}}^1(\mathbb{P}_k^1, \omega) & \xrightarrow{\quad \text{F} \quad} & \text{CH}^1(\mathbb{P}_k^1) \\ \text{deg} \downarrow & & \downarrow \widetilde{\text{deg}} & & \downarrow \text{deg} \\ \mathbb{Z} & \xrightarrow{\quad \text{H} \quad} & \text{GW}(k) & \xrightarrow{\quad \text{rk} \quad} & \mathbb{Z}. \end{array}$$

Specializing at a point  $x \in \mathbb{P}_k^1$  with residue field  $\kappa_x$  as before, one gets the following computation:

$$(4.2.5.a) \quad \forall n \in \mathbb{Z}, \mathrm{Tr}_{\kappa_x/k}^{MW}(n.h) = (d_x n).h,$$

$$(4.2.5.b) \quad \forall \sigma \in \mathrm{GW}(\kappa_x, \omega_{\kappa_x/k}), \mathrm{rk}(\mathrm{Tr}_{\kappa_x/k}^{MW}(\sigma)) = d_x \cdot \mathrm{rk}(\sigma).$$

where  $d_x = [\kappa_x : k]$ .

*Remark 4.2.6.* Notice in particular that every quadratic cycle which comes from the hyperbolic map will have a degree of the form  $n.h$  for  $n \in \mathbb{Z}$  and  $h$  the hyperbolic form.

**4.2.7. Transfers in the monogenic case.** Let  $E/k$  be a monogenic finite extension field. Giving a generator  $\alpha \in E$  is equivalent to give a closed embedding  $x : \mathrm{Spec}(E) \rightarrow \mathbb{A}_k^1$ , corresponding to the (monogenic) minimal polynomial of  $\alpha$  in  $E$ . Therefore, the preceding proposition gives for any integer  $q \in \mathbb{Z}$  a well-defined transfer map:

$$\mathrm{Tr}_{E/k}^{MW,\alpha} : K_q^{MW}(E, \omega_{E/k}) \rightarrow K_q^{MW}(k),$$

which a priori depends on the chosen parameter  $\alpha$ .

We also define an  $\mathcal{L}$ -twisted version, for an invertible  $k$ -vector space  $\mathcal{L}$ , as follows:

$$\begin{aligned} \mathrm{Tr}_{E/k}^{MW,\alpha} : K_q^{MW}(L, \omega_{E/k} \otimes \mathcal{L}_E) &\rightarrow K_q^{MW}(k, \mathcal{L}), \\ \sigma \otimes w \otimes a &\mapsto \mathrm{Tr}_{E/k}^{MW,\alpha}((u)\sigma \otimes w) \otimes a' \end{aligned}$$

where  $\sigma \in K_q^{MW}(E)$ ,  $w \in \omega_{E/k}^\times$ ,  $l \in \mathcal{L}_E^\times$  and we have written:  $l = l' \otimes u$  for  $l' \in \mathcal{L}^\times$ ,  $u \in E^\times$  (as according to our notation  $\mathcal{L}_E = \mathcal{L} \otimes_k E$ ).

*Remark 4.2.8.* (1) We will see in Proposition 4.4.13 that the above transfers are independent of the generator  $\alpha$  (and extend its definition to the non necessarily monogenic case).

(2) Our construction is a variation on Morel's one, as done in [Mor12, §4.2, 5.1]. The main difference is that one uses appropriate twists (by canonical sheaves) which allows us to work over an arbitrary base field  $k$ , in particular allowing inseparable extensions from the start.

**4.2.9.** Bass and Tate method, already mentioned, was applied to Milnor K-theory ([BT73, I.5.4]). They constructed the transfer map on Milnor K-theory for monogenic finite extensions, and later, Kato proved that these transfers extend to arbitrary finite extensions  $E/k$  ([Kat80, §1.7, Prop. 5]), giving a transfer map<sup>39</sup>:

$$\mathrm{Tr}_{E/k}^M : K_*^M(E) \rightarrow K_*^M(k).$$

In particular, when  $E/k$  is monogenic, this map coincides with Bass-Tate morphism for any choice of generator  $\alpha$  of  $E/k$ . As the (twisted) hyperbolic and forgetful maps (Definition 2.4.9) are compatible with residues (Remark 2.5.8) we

<sup>39</sup>Kato called this the norm homomorphism

easily derive from the above construction the following compatibility lemma (extending 4.2.5).

**Lemma 4.2.10.** *Let  $E/k$  be a monogenic finite extension with generator  $\alpha \in E$ . Then the following diagram is commutative:*

$$\begin{array}{ccccc} K_*^M(E) & \xrightarrow{H} & K_*^{MW}(E, \omega_{E/k}) & \xrightarrow{F} & K_*^M(E) \\ \text{Tr}_{E/k}^M \downarrow & & \downarrow \text{Tr}_{E/k, \alpha}^{MW} & & \downarrow \text{Tr}_{E/k}^M \\ K_*^M(k) & \xrightarrow{H} & K_*^{MW}(k) & \xrightarrow{F} & K_*^M(k). \end{array}$$

### 4.3. A variation on Scharlau's quadratic reciprocity property.

**4.3.1.** Let  $E/k$  be a finite extension field. Recall that Scharlau has defined in [Sch72] (Definition p. 79) a notion of transfer maps for Witt groups, depending on the choice of a  $k$ -linear map  $s : E \rightarrow k$ .<sup>40</sup>

Using the differential trace map  $\text{Tr}_{E/k}^\omega : \omega_{E/k} \rightarrow k$  (see Definition 6.2.4), it is possible to give a uniform definition, which does not depend on such a choice. Moreover, we will see that it coincides with the trace maps  $\text{Tr}_{E/k}^{MW, \alpha}$  just defined in degree  $q \leq 0$ .

The definition is very similar to Scharlau's definition, but motivated by the form of MW-transfers (see 4.2.7), we use  $\mathcal{L}$ -valued inner product spaces (see 2.1.14). Given an arbitrary  $\omega_{E/k}$ -valued inner product space  $\Phi : V \otimes_E V \rightarrow \omega_{E/k}$ , one can consider the composite map

$$\text{Tr}_{E/k}^\omega \circ \Phi : V \otimes_k V \rightarrow V \otimes_E V \xrightarrow{\Phi} \omega_{E/k} \xrightarrow{\text{Tr}_{E/k}^\omega} k,$$

which is again a non-degenerate symmetric bilinear  $k$ -form. It is compatible with isomorphisms and orthogonal sums, therefore it induces a well-defined map:

$$\begin{aligned} \text{Tr}_{E/k}^\omega : \text{GW}(E, \omega_{E/k}) &\rightarrow \text{GW}(k) \\ [\Phi] &\mapsto [\text{Tr}_{E/k}^\omega \circ \Phi]. \end{aligned}$$

As, by definition, the map  $\text{Tr}_{E/k}^\omega : \omega_{E/k} \rightarrow k$  is  $k$ -linear, one deduces that  $\text{Tr}_{E/k}^\omega$  is  $\text{GW}(k)$ -linear (recall the  $\text{GW}(k)$ -action on both sides from 2.1.14).

**Definition 4.3.2.** Let  $E/k$  be an arbitrary finite extension fields. We call the  $\text{GW}(k)$ -linear morphism  $\text{Tr}_{E/k}^\omega : \text{GW}(E, \omega_{E/k}) \rightarrow \text{GW}(k)$  just defined the (*differential*) *GW-transfer map*.

Modding out by the ideal  $(h)$ , one gets a (differential) W-transfer map that we still denote:  $\text{Tr}_{E/k}^\omega : \text{W}(E, \omega_{E/k}) \rightarrow \text{W}(k)$ .

<sup>40</sup>These maps are denoted by  $s^* : W(E) \rightarrow W(k)$  in *loc. cit.*, but we will prefer the notation  $s_*$  (for obvious reasons). Scharlau originally considered fields of characteristic not 2 but the definition makes sense in arbitrary characteristic. Moreover, one can replace non-degenerate quadratic forms by non-degenerate symmetric bilinear forms.

**Example 4.3.3.** If  $E/k$  is separable, then  $\omega_{E/k} = E$  and  $\mathrm{Tr}_{E/k}^\omega$  is just the usual trace map:  $\mathrm{Tr}_{E/k} : E \rightarrow k$  (see Corollary 6.3.14). In particular,  $\mathrm{Tr}_{E/k^*}^\omega = \mathrm{Tr}_{E/k^*} : \mathrm{GW}(E) \rightarrow \mathrm{GW}(k), \mathrm{W}(E) \rightarrow \mathrm{W}(k)$  is the usual Scharlau transfer associated with the trace “form”  $\mathrm{Tr}_{E/k}$ .

In the inseparable case on the contrary,  $\mathrm{Tr}_{E/k} = 0$ . The link with Scharlau traces will be explained in Remark 4.3.5.

**Example 4.3.4.** One can compute the GW-differential transfer maps more explicitly.

Consider a monogenic field extension  $E/k$  of degree  $d$ , written as  $E = k[\alpha]$ . Let  $f$  be the minimal polynomial of  $\alpha$ , so that for  $I = (f)$ ,  $E = k[t]/I$ . Then, as explain in Example 4.1.10,  $\omega_{E/k} \simeq (I/I^2)^\vee \otimes \omega_{k[t]/k}$ . In particular, the invertible  $k$ -vector space  $\omega_{E/k}$  admits an explicit base given by the element  $\bar{f}^* \otimes dt$ . In particular, any  $\omega_{E/k}$ -valued inner product space

$$\Phi : V \otimes_E V \rightarrow \omega_{E/k}$$

can be written as  $(x, y) \mapsto \phi(x, y) \otimes_k (\bar{f}^* \otimes dt)$  where  $\phi : V \otimes_E V \rightarrow E$  is an inner product space.

With this notation, Corollary 6.3.13 gives the following computation:

$$\mathrm{Tr}_{E/k}^\omega(\phi \otimes (\bar{f}^* \otimes dt)) = \tau_{E/k}^\alpha \circ \phi =: \tau_{E/k^*}^\alpha(\phi),$$

where we recall that the *Tate trace map*  $\tau_{E/k}^\alpha : E \rightarrow k$  is the  $k$ -linear form associated with the element  $\alpha^{d-1}$  of the  $k$ -base  $(1, \alpha, \dots, \alpha^{d-1})$  of  $E$ .

When  $E/k$  is non monogenic, one writes  $E = k[\alpha_1, \dots, \alpha_n] = k[t_1, \dots, t_n]/(f)$ , where  $f = (f_1, \dots, f_n)$  for monic polynomials  $f_i \in k[t_1, \dots, t_n]$ . Then combining the notation Example 4.1.10 and Proposition 6.3.12, one gets the formula:

$$\mathrm{Tr}_{E/k}^\omega\left(\phi \otimes ((\bar{f}_1 \wedge \dots \wedge \bar{f}_n)^* \otimes (dt_1 \wedge \dots \wedge dt_n))\right) = \tau_f \circ \phi,$$

where  $\tau_f$  is the Sheja-Storch trace map (Definition 6.3.8) associated with the presentation  $f$  of  $E/k$ .

*Remark 4.3.5. Comparison with Scharlau transfer.* A particular case of Grothendieck duality (see 6.2.7) gives the following isomorphism:

$$\omega_{E/k} \xrightarrow{\sim} \mathrm{Hom}_k(E, k), w \mapsto s_w := \mathrm{Tr}_{E/k}^\omega(-.w).$$

According to Example 2.4.4, one has a canonical identification  $\mathrm{GW}(E, \omega_{E/k}) \simeq \mathrm{GW}(E) \otimes_{\mathbb{Z}[E^\times]} \mathbb{Z}[\omega_{E/k}^\times]$ . With this notation, one can see that the above transfers incorporate all Scharlau’s transfer maps at once: for  $\sigma \in \mathrm{GW}(E)$  and a non-zero  $w \in \omega_{E/k}$ , one gets:

$$\mathrm{Tr}_{E/k^*}^\omega(\sigma \otimes w) = s_{w^*}(\sigma).$$

**4.3.6.** One easily derives from the previous definition the following basic properties of the differential GW-trace map, for a finite extension  $\varphi : k \rightarrow E$  of degree  $d$ :

- (1) For any  $\sigma \in \text{GW}(E, \omega_{E/k})$ , one has  $\text{rk}(\text{Tr}_{E/k}^\omega(\sigma)) = d \cdot \text{rk}(\sigma)$ .
- (2) If  $L/E$  and  $E/k$  are finite extensions,  $\text{Tr}_{L/E}^\omega \circ \text{Tr}_{E/k}^\omega = \text{Tr}_{L/k}^\omega$  where we have hidden the canonical isomorphism  $\omega_{L/k} \simeq \omega_{L/E} \otimes_L \omega_{E/k}$  (4.1.7).<sup>41</sup>
- (3) For  $\sigma \in \text{GW}(E, \omega_{E/k})$ ,  $\sigma' \in \text{GW}(k)$ , one has:  $\text{Tr}_{E/k}^\omega(\sigma \cdot \varphi_*(\sigma')) = \text{Tr}_{E/k}^\omega(\sigma) \cdot \sigma'$ .<sup>42</sup>

The main result for the GW-differential transfer map is the following *quadratic reciprocity formula* which extends to the Milnor-Witt case a formula due to Scharlau first proved in [Sch72, Th. 4.1], with a similar proof.

**Theorem 4.3.7.** *Let  $k$  be an arbitrary field. Then the following formula holds:*

$$(4.3.7.a) \quad \sum_{x \in (\mathbb{P}_k^1)_{(0)}} \text{Tr}_{\kappa_x/k}^\omega \circ \partial_x = 0,$$

as maps  $K_1^{MW}(k(t), \omega_{k(t)/t}) \rightarrow \text{GW}(k)$ . Here, the map

$$\partial_x : K_1^{MW}(k(t), \omega_{k(t)/k}) \rightarrow \text{GW}(\kappa_x, \omega_{\kappa_x/k})$$

stands for the residue map associated with the discrete valuation on  $k(t)$  associated to the closed point  $x \in \mathbb{P}_k^1$  (see 4.2.1).

*Proof.* The abelian group  $K_1^{MW}(k(t), \omega_{k(t)/k})$  is generated by elements of the form  $[f] \otimes dt$  where  $f \in k(t)^\times$  is a rational function on  $\mathbb{P}_k^1$ . So we need only to check the vanishing on these particular elements.

Consider the prime decomposition of  $f$ :

$$(4.3.7.b) \quad f = u \cdot \pi_1^{m_1} \cdots \pi_r^{m_r},$$

where  $u \in k^\times$ ,  $m_i \in \mathbb{Z}$  and  $\pi_i$  is an irreducible monic polynomial in  $k[t]$ . Each polynomial  $\pi_i$  corresponds to a closed point  $x_i$  in  $\mathbb{A}_k^1 \subset \mathbb{P}_k^1$ , with residue field  $\kappa_i = \kappa(x_i) = k[t]/(\pi_i)$ . With this notation, we will write  $\alpha_i \in \kappa_i$  for the obvious generator of  $\alpha_i$  (*i.e.* corresponding to  $t$ ).

We first remark that, computing the quadratic order of vanishing of at  $\infty$  using the uniformizer  $1/t$ , we find in  $\text{GW}(k)$ :<sup>43</sup>

$$\partial_\infty([f] \otimes dt) = -d_\epsilon \langle u \rangle \in \text{GW}(k).$$

Let us write  $f_i = \prod_{j \neq i} f_j^{m_j}$ , so that  $f = u f_i \pi_i^{m_i}$ . Applying formula (3.1.10.a) (with the added twist  $dt$ ), one gets:

$$\sum_{x \in (\mathbb{P}_k^1)_{(0)}} \text{Tr}_{\kappa_x/k}^\omega \circ \partial_x([f] \otimes dt) = \sum_{i=1}^r (m_i)_\epsilon \langle u \rangle \text{Tr}_{\kappa_i/k}^\omega(\langle f_i(\alpha_i) \rangle \otimes dt \otimes \bar{\pi}_i^*) - d_\epsilon \langle u \rangle.$$

<sup>41</sup>This follows from the functoriality of the differential trace map.

<sup>42</sup>This is generically called the projection formula, and more specifically *Frobenius reciprocity* in the theory of quadratic forms ([Sch72, p. 80]).

<sup>43</sup>Note that  $\omega_{\kappa_\infty/k} = k$  so that we identify  $\text{GW}(\kappa_\infty, \omega_{\kappa_\infty/k})$  with  $\text{GW}(k)$ . With this identification, the GW-differential trace map  $\text{Tr}_{\kappa_\infty/k}^\omega$  is just the identity.

Let us denote by  $(*)$  the right-hand side, so that we need to show that  $(*)$  is 0 in  $\text{GW}(k)$ . The (virtual) rank of  $(*)$  is

$$\sum_{i=1}^r m_i \cdot \deg(\pi_i) - d$$

which is obviously zero — according to relation (4.3.7.b).

Therefore, one needs only to show that the class of  $(*)$  is zero in  $W(k)$ . Obviously, one can assume that  $u = 1$ . Moreover, as  $n_\epsilon = 0$  for  $n$  even in  $W(k)$ , one can assume that  $m_i = 1$  for all  $i$ .

Let us consider the monogenic  $k$ -algebra  $A = k[t]/(f)$ , and write  $\alpha$  its generator. Recall that  $A$  is a finite  $k$ -vector space with basis  $\mathcal{B} = (1, \alpha, \dots, \alpha^{d-1})$ . The Chinese remainder lemma gives an isomorphism of  $k$ -algebras:

$$\Theta : A \xrightarrow{\sim} \prod_{i=1}^r \kappa_i, g \mapsto (f_i(\alpha_i)g(\alpha_i))_{1 \leq i \leq r}.$$

Applying Remark 6.2.5, one deduces that

$$\sum_{i=1}^r \text{Tr}_{\kappa_i/k^*}^\omega (\langle f_i(\alpha_i) \rangle \otimes dt \otimes \bar{\pi}_i^*) = \text{Tr}_{A/k^*}^\omega (\langle 1 \rangle \otimes dt \otimes \bar{f}^*).$$

We are now reduced to show the following equality in  $W(k)$ :

$$(4.3.7.c) \quad \text{Tr}_{A/k^*}^\omega (\langle 1 \rangle \otimes (dt \otimes \bar{f}^*)) = d_\epsilon.$$

One can apply Corollary 6.3.13 (see Example 4.3.4) to compute the left-hand side: if one denotes by  $\tau_{A/k}^\alpha : A \rightarrow k$  the Tate trace map associated with  $A/k$  and its generator  $\alpha$ , — that is the linear form associated with  $\alpha^{d-1}$  in the basis  $\mathcal{B}$  — this inner product on the  $k$ -vector space  $A$  is given by the formula:

$$A \otimes_k A \rightarrow k, (g, g') \mapsto \tau_{A/k}^\alpha(gg').$$

One easily computes the form of the symmetric  $(d \times d)$ -matrix of this symmetric bilinear in the basis  $\mathcal{B}$  as:

$$\begin{pmatrix} & & & & 1 \\ & & & & / \\ & 0 & & & \\ & / & & & \\ 1 & & * & & \end{pmatrix}.$$

But the class of the corresponding inner product space in  $\text{GW}(k)$  is  $d_\epsilon = \langle 1, -1, \dots \rangle$  as it has a totally isotropic subspace of rank  $n$  spanned by  $(1, \dots, \alpha^{n-1})$  if  $d = 2n$  or  $d = 2n + 1$ , and its determinant is  $(-1)^{d-1}$ . This proves (4.3.7.c).  $\square$

*Remark 4.3.8.* It is interesting to note that the end of the previous proof also shows the following degree formula, for any finite degree  $d$  extension  $E/k$ :

$$\text{Tr}_{E/k^*}^\omega (\langle 1 \rangle \otimes dt) = d_\epsilon.$$



*Remark 4.3.9.* Multiplying by  $\eta$ , and looking modulo the hyperbolic form  $h$  (granted the  $\mathrm{GW}(k)$ -linearity of each involved operator), the equation (4.3.7.a) gives a twisted version of Scharlau's quadratic reciprocity formula: for any class  $\sigma \in \mathrm{W}(k(t), \omega_{k(t)/k})$  of a  $\omega_{k(t)/k}$ -valued inner product space over  $k(t)$ , one has:

$$\sum_{x \in \mathbb{P}_{k, (0)}^1} \mathrm{Tr}_{kx/k^*}^\omega(\partial_x(\sigma)) = 0.$$

In fact, using Example 4.3.4 and applying this equality to  $\sigma = \sigma_0 \otimes dt$ , one gets back precisely Scharlau's formula (see also [GHKS70, §2, Satz] in the characteristic 2 case).

The main application of the previous theorem, taking into account the uniqueness statement of Proposition 4.2.3 is the following comparison result between the two transfer maps we have introduced.

**Corollary 4.3.10.** *Let  $E/k$  be a monogenic finite extension field. Then for any generator  $\alpha$  of  $E/k$ , and any  $q < 0$ , one has commutative diagrams*

$$\begin{array}{ccc} \mathrm{K}_0^{MW}(E, \omega_{E/k}) & \xrightarrow{\mathrm{Tr}_{E/k}^{MW, \alpha}} & \mathrm{K}_0^{MW}(k) \\ \sim \uparrow & & \uparrow \sim \\ \mathrm{GW}(E, \omega_{E/k}) & \xrightarrow{\mathrm{Tr}_{E/k^*}^\omega} & \mathrm{GW}(k) \end{array} \quad \begin{array}{ccc} \mathrm{K}_q^{MW}(E, \omega_{E/k}) & \xrightarrow{\mathrm{Tr}_{E/k}^{MW, \alpha}} & \mathrm{K}_q^{MW}(k) \\ \sim \uparrow & & \uparrow \sim \\ \mathrm{W}(E, \omega_{E/k}) & \xrightarrow{\mathrm{Tr}_{E/k^*}^\omega} & \mathrm{W}(k) \end{array}$$

where the vertical isomorphisms come from Example 2.4.4.

#### 4.4. General trace maps.

**4.4.1.** Let  $E/k$  be a finite extension with canonical module  $\omega_{E/k}$ .

We have already seen (Definition 4.3.2) how the Grothendieck differential trace map induces a transfer map on twisted Grothendieck-Witt and Witt groups. We now show how to extend these transfers to Milnor-Witt K-theory using Morel's fundamental square from Corollary 2.3.7. We first need a lemma.

**Lemma 4.4.2.** *Consider the above notation. For any integer  $n \in \mathbb{Z}$ , one has:*

$$\mathrm{Tr}_{E/k^*}^\omega(\mathrm{I}^n(E, \omega_{E/k})) \subset \mathrm{I}^n(k)$$

where we have used notation 2.4.10 for  $\mathrm{I}^n$  and the transfer map on Witt groups was defined in Definition 4.3.2.

Using Remark 4.3.5, the lemma follows from [Ara75, Satz 3.3]. At this point, one can easily deduce it from our earlier computations so we give a proof for completeness.

*Proof.* The case  $n \geq 0$  is trivial. We note the case  $n = 1$  is easy (use 4.3.6(1)). For the other cases, using the functoriality of GW-transfers 4.3.6(2), one reduces to the case where  $E/k$  is monogenic, with say a fixed generator  $\alpha$ . This case now

follows from Corollary 4.3.10, Theorem 2.3.5 and the fact  $\mathrm{Tr}_{E/k}^{MW,\alpha}$  (defined in 4.2.7) commutes with multiplication by  $\eta$  and  $h$ .  $\square$

In particular we get well-defined transfer maps on the algebra functor  $I^*$ . As an intermediate step, we show that these transfers are compatible with the monogenic transfers obtained so far on the Milnor-Witt K-theory functor (4.2.7).

**Lemma 4.4.3.** *Let  $E/k$  be a monogenic finite extension, with a generator  $\alpha \in E$ . Then the following diagram is commutative:*

$$\begin{array}{ccc} K_*^{MW}(E, \omega_{E/k}) & \xrightarrow{\mathrm{Tr}_{E/k}^{MW,\alpha}} & K_*^M(k) \\ \mu'_E \downarrow & & \downarrow \mu'_k \\ I^*(E, \omega_{E/k}) & \xrightarrow{\mathrm{Tr}_{E/k^*}^\omega} & I^*(k). \end{array}$$

Given the previous lemma, and the construction of the morphism  $\mu'$  (see 2.3.6) this statement reduces to Corollary 4.3.10.

**4.4.4.** According to Proposition 2.4.11,  $K_n^{MW}(E, \omega_{E/k})$  can be identified with the abelian group made of pairs  $(\sigma, \tau) \in I^n(E) \times K_n(E)$  such that  $\pi(\sigma) = \mu(\tau)$ . The following lemma is the last step needed to define the transfer map associated with  $E/k$  on Milnor-Witt K-theory.

**Lemma 4.4.5.** *Consider the above notation. Then one has the following equality in  $\bar{I}^n(k)$ :*

$$\pi(\mathrm{Tr}_{E/k^*}^\omega(\sigma)) = \mu(\mathrm{Tr}_{E/k}^M(\tau)).$$

*Proof.* By functoriality of the differential trace map (Remark 6.2.6) and of Kato's transfer map on Milnor K-theory, one reduces to the case of finite monogenic extensions  $E = k[\alpha]$ . Then the result follows from the existence of the trace map  $\mathrm{Tr}_{E/k}^{MW,\alpha}$ , and its compatibility with both Kato's transfer (Lemma 4.2.10) and the differential transfer on  $I^*$  (Lemma 4.4.3).  $\square$

We finally obtain the main definition of this section.

**Definition 4.4.6.** Let  $E/k$  be a finite extension with canonical module  $\omega_{E/k}$ . One defines the transfer map on Milnor-Witt K-theory by the following formula:

$$\begin{array}{ccc} K_*^{MW}(E, \omega_{E/k}) & \xrightarrow{\mathrm{Tr}_{E/k}^{MW}} & K_*^{MW}(k) \\ (\mu'_E, \mathbf{F}_E) \downarrow \sim & & \sim \downarrow (\mu'_k, \mathbf{F}_k) \\ I^*(E, \omega_{E/k}) \times_{\bar{I}^*(E)} K_*^M(E) & & I^*(k) \times_{\bar{I}^*(k)} K_*^M(k) \\ (\sigma, \tau) & \xrightarrow{\quad\quad\quad} & (\mathrm{Tr}_{E/k^*}^\omega(\sigma), \mathrm{Tr}_{E/k}^M(\tau)) \end{array}$$

well-defined according to the previous lemma. The vertical isomorphisms come from Proposition 2.4.11.

As in the end of 4.2.7, one also defines for an invertible  $k$ -vector space  $\mathcal{L}$ , an  $\mathcal{L}$ -twisted transfers:

$$\mathrm{Tr}_{E/k}^{MW} : K_q^{MW}(E, \omega_{E/k} \otimes \mathcal{L}_E) \rightarrow K_q^{MW}(k, \mathcal{L}).$$

When we denote by  $\varphi : k \rightarrow E$  the structural map of the extension  $E/k$ , it is customary to use the notation  $\varphi^* = \mathrm{Tr}_{E/k}^{MW}$ . We also call it occasionally the trace map.<sup>44</sup>

*Remark 4.4.7.* This trace map has all the good properties of its analog on Milnor K-theory. It is compatible with composition (as this is the case for  $\mathrm{Tr}_{E/k}^\omega$  and  $\mathrm{Tr}_{E/k}^M$ ). It satisfies the so-called projection formula: for  $(\sigma, \beta) \in K_*^{MW}(E, \omega_{E/k} \otimes \mathcal{L}_E) \times K_*^{MW}(k, \mathcal{M})$ , one has in  $K_*^{MW}(k, \mathcal{L}_k \otimes \mathcal{M})$ :

$$\varphi^*(\sigma \cdot \varphi_*(\beta)) = \varphi^*(\sigma) \cdot \beta.$$

This follows from 4.3.6(3) and the corresponding formula for Milnor K-theory (see [BT73, formula (5), p. 378]).

Finally, we note that from a geometric point of view, if one denotes by  $f : \mathrm{Spec}(E) \rightarrow \mathrm{Spec}(k)$  the induced morphism, one can also denote:  $\varphi^* = f_*$  and  $\varphi_* = f^*$ . In this way, the previous formula looks like the "classical" projection formula (for Chow groups, cohomology,...)

As an immediate corollary of the previous definition, we obtain the following explicit description of transfers on Milnor-Witt K-theory.

**Corollary 4.4.8.** *Let  $E/k$  be a finite extension with canonical module  $\omega_{E/k}$ , and  $n$  an integer.*

(1) *If  $n = 0$  (resp.  $n < 0$ ) then through the identification*

$$K_0^{MW}(E, \omega_{E/k}) = \mathrm{GW}(E, \omega_{E/k}) \quad (\text{resp. } K_n^{MW}(E, \omega_{E/k}) = \mathrm{W}(E, \omega_{E/k}))$$

*of Proposition 2.3.1, one has  $\mathrm{Tr}_{E/k}^{MW} = \mathrm{Tr}_{E/k^*}^\omega$  where  $\mathrm{Tr}_{E/k}^\omega$  is the differential trace map (see Definition 4.3.2).*

(2) *If  $n > 0$ , any element  $\sigma \in K_n^{MW}(E, \omega_{E/k})$  can be written as a sum of elements of the form  $([\phi] \otimes w, \sigma')$  where:*

- $(V, \phi : V \otimes_E V \rightarrow E)$  is an inner product space over  $E$ ,  $[\phi]$  is its class in  $\mathrm{W}(E)$  and  $[\phi] \in \Gamma^n(E)$ ,
- $w \in \omega_{E/k}$  is a non-zero differential  $k$ -form on  $E$  of maximal degree if  $E/k$  is not separable, and just a unit of  $E$  if  $E/k$  is separable,
- $\sigma' = \{u_1, \dots, u_n\}$  is a symbol in  $K_n^M(E)$ , for certain units  $u_i \in E^\times$ .

*For such an element, one has:*

$$\mathrm{Tr}_{E/k}^{MW}([\phi] \otimes w, \sigma') = ([\mathrm{Tr}_{E/k}^\omega \circ (\phi \cdot w)], \mathrm{Tr}_{E/k}^M(\sigma'))$$

<sup>44</sup>Other terminologies that we prefer to avoid are the norm (Kato) and corestriction (Rost).

where  $\mathrm{Tr}_{E/k}^\omega \circ (\phi.w)$  is the class in  $W(k)$  (and in fact  $\Gamma^n(k)$ ) of the inner product space on  $V$  over  $k$  with bilinear form

$$(x, y) \mapsto \mathrm{Tr}_{E/k}^\omega (\phi(x, y).w),$$

and  $\mathrm{Tr}_{E/k}^M$  is the transfer map on Milnor  $K$ -theory.

**Example 4.4.9.** In general, we refer the reader to Example 4.3.4 and Remark 4.3.5 for the computation of the differential trace map  $\mathrm{Tr}_{E/k^*}^\omega$  on the Grothendieck-Witt or Witt group. One can single out the following explicit computations.

- (1) If  $E/k$  is separable, then  $\omega_{E/k} = E$  and  $\mathrm{Tr}_{E/k}^\omega = \mathrm{Tr}_{E/k}$  is the usual trace map (Corollary 6.3.14). In particular,  $K_0^{MW}(E, \omega_{E/k}) \simeq \mathrm{GW}(E)$  and for any unit  $u \in E^\times$ ,  $\mathrm{Tr}^{MW}(\langle u \rangle) = [\mathrm{Tr}_{E/k}(u.-)]$  the GW-class of the *scaled trace form*,  $(x, y) \mapsto \mathrm{Tr}_{E/k}(uxy)$ .
- (2) Let  $E/k$  be a finite monogeneous extension of degree  $d$ , with generator  $\alpha$ . According to Remark 4.3.5, one has an isomorphism

$$\omega_{E/k} \simeq \mathrm{Hom}_k(E, k), w \mapsto s_w.$$

In particular, there exists a unique non-zero form  $w \in \omega_{E/k}$  such that  $s_w$  is the  $k$ -linear form which maps  $\alpha^0$  to 1 and  $\alpha^i$  to 0 for  $0 < i < d$ .

Then for any unit  $u \in E^\times$ , and for the particular choice of  $w$  made above, one has:

$$\mathrm{Tr}_{E/k}^{MW}([u] \otimes w) = [N_{E/k}(u)] \in K_1^{MW}(k)$$

where  $N_{E/k} : E^\times \rightarrow k^\times$  is the usual norm of the finite extension  $E/k$ . According to the previous corollary, this follows from the [Lam05, VII, Cor. 2.4] for the Witt part<sup>45</sup> and [BT73, I.§5, Th. 5.6] for the Milnor part.

This formula generalizes to arbitrary finite extension provided one chooses the correct differential form  $w$ .

- (3) Let  $k$  be a field of characteristic  $p > 0$ ,  $a \in k$  be an element which is not a  $p$ -th root and  $E = k[\sqrt[q]{a}] = k[t]/(t^q - a)$ . Put  $\alpha = \sqrt[q]{a} \in E$ . There exists a canonical non-zero element  $w = dt \otimes (t^q - a)^*$  of  $\omega_{E/k}$  (see Example 4.1.10 with  $n = 1$ ).

Then for any unit  $u \in E^\times$ , and again for the particular choice of  $w$  made above, one has:

$$\mathrm{Tr}^{MW}(\langle u \rangle \otimes w) = [\tau_{E/k}^\alpha(u.-)]$$

where  $\tau_{E/k}^\alpha$  is the *Tate trace map* associated with the  $q$ -th root  $\alpha$  (see Remark 6.3.10), and  $[\tau_{E/k}^\alpha(u.-)]$  is the GW-class of the scaled (Tate) trace form of the  $k$ -vector space  $E$ :

$$E \otimes_k E \rightarrow k, (x, y) \mapsto \tau_{E/k}^\alpha(uxy).$$

<sup>45</sup>the computation of *loc. cit.* extends in characteristic 2 as well

Note in particular that one gets the following degree formula:

$$\mathrm{Tr}^{MW}(\langle 1 \rangle \otimes w) = [\tau_{E/k}^\alpha]$$

In comparison with the last example, one gets the following more general *degree formula* in Milnor-Witt K-theory.

**Corollary 4.4.10.** *Let  $E/k$  be a finite extension of degree  $d$ . We consider a minimal family of generators  $(\alpha_1, \dots, \alpha_n)$  and the associated presentation  $E = k[t_1, \dots, t_n]/(f_1, \dots, f_n)$  as in Example 4.1.10. Let  $w = (\bar{f}_1 \wedge \dots \wedge \bar{f}_n)^* \otimes (dt_1 \wedge \dots \wedge dt_n)$  be the canonical element of  $\omega_{E/k}$  as in loc. cit.*

*Then one has in  $\mathbf{K}_0^{MW}(k) = \mathrm{GW}(k)$ :*

$$\mathrm{Tr}_{E/k}^{MW}(\langle 1 \rangle \otimes w) = d_\epsilon$$

where we have used the notation of 2.2.18.

*Proof.* By multiplicativity of  $d_\epsilon$  (2.2.18), and the functoriality of the MW-trace map, one reduces to the monogenic case. Then, it follows from Remark 4.3.8.  $\square$

*Remark 4.4.11.* In general, any element of  $w' = \omega_{E/k}$  can be written as  $w' = u.w$ . One should be careful however that if one replaces  $w$  in the above corollary by  $w'$ , this completely changes the above result. For example, in the case of Example 4.4.9(3), one gets

$$\mathrm{Tr}^{MW}(\langle 1 \rangle \otimes w') = [\tau_{E/k}^\alpha(u.-)].$$

**4.4.12.** Finally, we want to compare the previous definition of transfers on Milnor-Witt K-theory with the original one due to Morel for finitely generated extensions of some perfect field: [Mor12, Rem. 4.32].

Recall the construction of Morel, for a finite extension  $E/k$ .<sup>46</sup> We fix a finite generating family  $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$  of  $E/k$ , to which we associate a tower of finite monogenic extensions  $\kappa_i = k[\alpha_1, \dots, \alpha_i]$ :

$$k \subset \kappa_1 \subset \dots \subset \kappa_n = E.$$

Then we can define the following composite map, denoted by  $\mathrm{Tr}_{E/k}^{MW, \underline{\alpha}}$ :

$$\begin{aligned} \mathbf{K}_q^{MW}(E, \omega_{E/k}) &\simeq \mathbf{K}_q^{MW}(E, \omega_{E/\kappa_{n-1}} \otimes \omega_{\kappa_{n-1}/k}|_E) \xrightarrow{\mathrm{Tr}_{E/\kappa_{n-1}}^{MW, \alpha_n}} \mathbf{K}_q^{MW}(\kappa_{n-1}, \omega_{\kappa_{n-1}/k}) \\ &\simeq \mathbf{K}_q^{MW}(\kappa_{n-1}, \omega_{\kappa_{n-1}/\kappa_{n-2}} \otimes \omega_{\kappa_{n-2}/k}|_{\kappa_{n-1}}) \xrightarrow{\mathrm{Tr}_{\kappa_{n-1}/\kappa_{n-2}}^{MW, \alpha_{n-1}}} \dots \\ &\dots \mathbf{K}_q^{MW}(\kappa_1, \omega_{\kappa_1/k}) \xrightarrow{\mathrm{Tr}_{\kappa_1/k}^{MW, \alpha_1}} \mathbf{K}_q^{MW}(k) \end{aligned}$$

where the morphism  $\mathrm{Tr}_{\kappa_i/\kappa_{i-1}}^{MW, \alpha_i}$  is the  $(\omega_{\kappa_{i-1}/k})$ -twisted MW-transfer associated with  $(\kappa_i/\kappa_{i-1}, \alpha_i)$ , as defined in 4.2.7.

<sup>46</sup>Note that contrary to the previously known constructions, we do not to assume that  $E$  and  $k$  are finitely generated extensions over some base perfect field.

The main result of [Mor12, §4.2] (see Th. 4.27), is that this composite map, at least for finitely generated extensions of some perfect base field, is independent of the chosen family of generators. This result has also been proved later in [Fel20b] by direct transport of the proof of Kato (again under the same assumptions). Actually, given the method we have chosen, we get another proof of this theorem (without any restriction on the fields considered).

**Proposition 4.4.13.** *Consider the above notation. Then one has an equality:*

$$\mathrm{Tr}_{E/k}^{MW} = \mathrm{Tr}_{E/k}^{MW,\alpha}$$

where the left hand-side was defined in Definition 4.4.6.

In particular, the computations given precedingly apply to the already known (geometric) transfer map on Milnor-Witt K-theory.

*Proof.* As the transfers of Definition 4.4.6 are compatible with composition (Remark 4.4.7), one reduces to the monogenic case. This is then a consequence (already observed) of the definition, and lemmas 4.2.10, 4.4.3.  $\square$

## 5. THE MILNOR-WITT PREMODULE STRUCTURE

We now summarize the properties of Milnor-Witt's K-theory that we have obtained in the previous sections. They correspond to the axioms of Feld's theory of Milnor-Witt premodules [Fel20a].<sup>47</sup>

The axioms are difficult to grasp at first sight. A general rule is that Milnor-Witt K-theory should be viewed as a bigraded cohomology theory on (spectrum of) fields equipped with a natural functoriality, an exceptional functoriality (called transfers), a structure of bigraded algebra (product) and residue maps (corresponding to boundary morphisms of localization exact sequences).

**5.1. Functorialities.** For any triple  $(E, \mathcal{L}, n)$ , where  $E$  is a field,  $\mathcal{L}$  an invertible  $E$ -vector space, and  $n \in \mathbb{Z}$  an integer, one has an abelian group  $K_n^{MW}(E, \mathcal{L})$ . It satisfies the following functorial behavior (as in [Fel20a, Def. 3.1]):

**D1** (see 2.4.6(2)): Given any morphism  $\varphi : E \rightarrow F$  of fields, one has a morphism of abelian groups:

$$\varphi_* : K_n^{MW}(E, \mathcal{L}) \rightarrow K_n^{MW}(F, \mathcal{L} \otimes_E F).$$

**D2** (see Definition 4.4.6): Given a finite morphism  $\varphi : E \rightarrow F$  of fields, one has a *transfer map*:

$$\psi^* = \mathrm{Tr}_{F/E}^{MW} : K_n^{MW}(F, \omega_{F/E} \otimes_E \mathcal{L}) \rightarrow K_n^{MW}(E, \mathcal{L})$$

where  $\omega_{F/E}$  is the canonical invertible  $F$ -vector space associated with the finite field extension  $F/E$  (see Definition 4.1.5).

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<sup>47</sup>This theory follows the model of Rost cycle modules [Ros96], and its developments made in Schmid's thesis [Sch97] for application to the Witt groups.

**D3** (see 2.4.6(1)): It has a structure of bigraded algebras. Given triples  $(E, \mathcal{L}, n)$  and  $(E, \mathcal{M}, m)$ , one has a *product*:

$$K_n^{MW}(E, \mathcal{L}) \otimes K_m^{MW}(E, \mathcal{M}) \rightarrow K_{n+m}^{MW}(E, \mathcal{L} \otimes_E \mathcal{M}).$$

In other words,  $K_*^{MW}(E, *)$  is a  $K_*^{MW}(E, *)$ -module, graded with respect to  $\mathbb{Z}$  and to the set of isomorphism classes of invertible  $E$ -vector spaces.

**D4** (see Theorem 2.5.2 and 2.5.5): Let  $(E, v)$  be a discretely valued field with ring of integers  $\mathcal{O}_v$ ,  $\mathcal{L}$  be an invertible  $\mathcal{O}_v$ -modules and  $n \in \mathbb{Z}$  an integer. We let  $\kappa_v$  be the residue field,  $\mathcal{L}_E = \mathcal{L} \otimes_{\mathcal{O}_v} E$ . One has a morphism of abelian groups, called the *residue map*:

$$\partial_v : K_n^{MW}(E, \mathcal{L}_E) \rightarrow K_n^{MW}(E, \omega_v \otimes_{\mathcal{O}_v} \mathcal{L})$$

where  $\omega_v = (\mathcal{M}_v/\mathcal{M}_v^2)^\vee$  — the normal sheaf of  $\text{Spec}(\kappa_v) \rightarrow \text{Spec}(\mathcal{O}_v)$ .

There is a further functorial property hidden in the axioms of [Fel20a] that we now state explicitly:

**D1'** (see 2.4.6(3)) Given an isomorphism  $\theta : \mathcal{L} \rightarrow \mathcal{M}$  of  $E$ -vector spaces, one has a morphism of abelian groups:

$$\theta_* : K_n^{MW}(E, \mathcal{L}) \rightarrow K_n^{MW}(E, \mathcal{M}).$$

*Remark 5.1.1.* Recall from Remark 2.4.7 that, using the category of twisted fields, one can unify the functoriality datas of (D1) and (D1').

**5.1.2.** There are many similar examples that we have seen in this text.

(1) The periodized Witt ring  $W[t, t^{-1}]$ , which therefore becomes a  $\mathbb{Z}$ -graded algebra: this comes from the isomorphism  $\phi$  of Corollary 2.3.3 and the fact multiplication by  $\eta$  is compatible with datas  $D^*$  on  $K_*^{MW}$ .

In particular, the canonical map

$$K_*^{MW} \rightarrow W[t, t^{-1}]$$

(essentially obtained by inverting  $\eta$ ,) is compatible with datas  $D^*$ .

(2) The graded algebra  $I^*$  associated with the fundamental ideal  $I$  of the Witt group (2.2.4): a quick way of seeing that is to use the isomorphism  $\psi$  of Theorem 2.3.5 and the fact multiplication by  $h$  is compatible with all datas  $D^*$  on  $K_*^{MW}$ .

In particular, the inclusion  $I^* \subset W[t, t^{-1}]$ , as well as the canonical map  $\mu' : K_*^{MW} \rightarrow I^*$  (Corollary 2.3.7) are compatible with all datas  $D^*$ .

(3) The Milnor K-theory  $K_*^M$ : a first way of seeing that is the isomorphism from 2.2.9, and again the fact that multiplication by  $\eta$  is compatible with datas  $D^*$  on  $K_*^{MW}$ .

On the other hand, recall that in this example, one has a canonical isomorphism  $K_*^M(E, \mathcal{L}) \simeq K_*^M(E)$  (Example 2.4.8). Or in other words, the data  $D1^*$  is trivial: for any automorphism  $\theta : \mathcal{L} \rightarrow \mathcal{L}$  of invertible  $E$ -vector spaces, the map  $\theta_* : K_*^M(E, \mathcal{L}) \rightarrow K_*^M(E, \mathcal{L})$  is equal to the identity. We will say that  $K_*^M$  is *orientable*.

In this case, the functoriality  $D^*$  actually corresponds to the functoriality of Rost cycle premodules [Ros96, Def. 1.1]. Moreover, the hyperbolic and forgetful maps

$$K_*^M \xrightarrow{H} K_*^{MW} \xrightarrow{F} K_*^{MW}$$

of Definition 2.2.10 are compatible with data  $D^*$ :  $D1$  and  $D1'$  are obvious,  $D2$  comes from Definition 4.4.6,  $D3$  comes from the fact both maps are morphisms of rings,  $D4$  was observed in Remark 2.5.8.

(4) The graded algebra  $\bar{\Gamma}^*$  (see again 2.2.4): a quick way of seeing that is to use the Milnor conjecture Theorem 2.2.3 and to use the preceding point. One can also use the fact  $\Gamma^*$  is a sub-algebra of  $W[t, t^{-1}]$  and therefore, all datas  $D^*$  descend to the quotient  $\bar{\Gamma}^*$ , as  $t$  on the right hand-side is compatible with all datas  $D^*$ .

Note that  $\bar{\Gamma}^*$  is also orientable as in the previous point. According to the previous remarks, one sees that the canonical maps (Corollary 2.3.7)

$$K_*^M \xrightarrow{\mu} \bar{\Gamma}^*, \quad \Gamma^* \xrightarrow{\pi} \bar{\Gamma}^*$$

are compatible with the datas  $D^*$ .

**5.2. Rules.** The preceding functoriality satisfy a very precise list of rules, again taken from [Fel20a, Def. 3.1]. Let  $\varphi : E \rightarrow F$ ,  $\psi : F \rightarrow L$  be morphisms of fields, and respectively  $\Phi : E \rightarrow F$ ,  $\Psi : F \rightarrow L$  be finite morphisms of fields,

**R1a** (obvious)  $(\psi \circ \varphi)_* = \psi_* \varphi_*$

**R1b** (see Remark 4.4.7)  $(\Psi \circ \Phi)^* = \Phi^* \Psi^*$

**R1c** (see Theorem 5.3.9 more generally) Assume that  $\Phi$  or  $\psi$  are separable. Then:

$$\psi_* \Phi^* = \sum_{x \subset F \otimes_E L} \Phi_x^* \psi_{x*}$$

where  $x$  is a prime ideal,  $\kappa(x) = (F \otimes_E L)/x$  its residue field,  $\Phi_x : E \rightarrow \kappa(x)$  and  $\psi_x : F \rightarrow \kappa(x)$  the induced map, and we have used the fact  $\omega_{F/E} = F$ ,  $\omega_{\kappa(x)/E} = \kappa(x)$ .

*Remark 5.2.1.* The statement (R1c) will be reinforced in *loc. cit.*, following an idea of [Fel20b]. Note however that this formula is enough to develop the theory of Chow-Witt groups (especially pullbacks). However, one can give a direct proof of (R1c) from our definition of Milnor-Witt transfers (Definition 4.4.6): one reduces to the corresponding formulas for Milnor K-theory (see [BT73, (5.8)]) and for the differential trace map (see [Con00], last statement of 3.4.1, or use the explicit computation of Proposition 6.3.12).

Next, one assumes  $\sigma, \sigma', \beta$  are elements ("Milnor-Witt symbols") of the relevant Milnor-Witt K-group:

**R2a** (obvious)  $\varphi_*(\sigma \cdot \sigma') = \varphi_*(\sigma) \cdot \varphi_*(\sigma')$

**R2b** (see Remark 4.4.7)  $\Phi^*(\Phi_*(\sigma) \cdot \beta) = \sigma \cdot \Phi^*(\beta)$

**R2c** (see Remark 4.4.7)  $\Phi^*(\sigma \cdot \Phi_*(\beta)) = \Phi^*(\sigma) \cdot \beta$



Finally, one considers discretely valued fields  $(E, v)$ ,  $(F, w)$ ,  $\mathcal{O}_v$ ,  $\mathcal{O}_w$ , (resp.  $\mathcal{M}_v, \mathcal{M}_w$ ) the corresponding valuation ring (resp. maximal ideal). In (R3a,c,d) (resp. R3b), we consider in addition a morphism  $\varphi : E \rightarrow F$  (resp.  $\Phi : E \rightarrow F$ ).

**R3a** (formula (Res2) of Theorem 2.5.2) Assume that  $w \circ \varphi = v$ . Thus, one has an induced morphism  $\varphi : \kappa(v) \rightarrow \kappa(w)$  and an induced isomorphism of invertible  $\kappa(v)$ -vector spaces:

$$\begin{aligned} \theta : \omega_v = (\mathcal{M}_v/\mathcal{M}_v)^\vee \otimes_{\kappa(v)} \kappa(w) &\rightarrow (\mathcal{M}_w/\mathcal{M}_w)^\vee = \omega_w \\ \bar{\pi}^* &\mapsto \overline{\varphi(\pi)}^*. \end{aligned}$$

Then:  $\partial_w \circ \varphi_* = \theta_* \circ \bar{\varphi} \circ \partial_v$ .

**R3b** (see [Dég23]) Let  $B$  be the integral closure of  $\mathcal{O}_v$  in  $F$ . Then a prime ideal of  $B$  is the same thing as a discrete valuation  $w$  of  $F$  extending  $v$  (with eventually a ramification index). For such a valuation  $w$ , one considers the commutative diagram

$$\begin{array}{ccc} \kappa_w & \leftarrow & \mathcal{O}_w \\ \Phi_w \uparrow & \Theta & \uparrow \Phi \\ \kappa_v & \leftarrow & \mathcal{O}_v \end{array}$$

and the canonical isomorphism of invertible  $\kappa_w$ -vector spaces (apply Remark 4.1.8 with  $\Theta$ ):

$$\theta : \omega_w \otimes_{\mathcal{O}_w} \omega_{\mathcal{O}_w/\mathcal{O}_v} \xrightarrow{\sim} \omega_{\kappa_w/\kappa_v} \otimes_{\kappa_v} \omega_v$$

Then one has:

$$\partial_v \circ \Phi^* = \sum_{w/v} \Phi_w^* \circ \theta_* \circ \partial_w.$$

**R3c** (formula (Res2) of Theorem 2.5.2) Assume  $w \circ \varphi = 0$ . Then  $\partial_w \circ \varphi_* = 0$ .

**R3d** (formula (Res2) of Theorem 2.5.2) Assume  $w \circ \varphi = 0$  and let  $\varphi : E \rightarrow \kappa(w)$  be the morphism induced by  $\varphi : F \rightarrow E$ . Let  $\pi$  be a prime of  $w$ , and consider the resulting trivialization (sending  $\bar{\pi}^*$  to 1):

$$\theta^\pi : \omega_v \rightarrow E$$

so that  $\partial_w^\pi = \theta_*^\pi \circ \partial_w$ , and:  $s_w^\pi(\sigma) = \theta_*^\pi \circ \partial_w([\pi] \cdot \sigma)$ .

Then  $s_w^\pi \circ \varphi_* = \bar{\varphi}_*$ .

**R3e** (see 2.5.4) The following formulas hold:

$$\begin{aligned} \partial_v([u] \cdot \sigma) &= \epsilon[\bar{u}] \cdot \partial_v(\sigma), \\ \partial_v(\eta \cdot \sigma) &= \eta \cdot \partial_v(\sigma) \end{aligned}$$

where  $u \in \mathcal{O}_v^\times$  is a unit of  $v$ .

*Remark 5.2.2.* Axiom (R3b) is certainly the most difficult. It was first proved in [Fas08, Cor. 10.4.5] with different conventions than ours, using the Gersten-Witt complex and in characteristic not 2. It is also proved when  $\mathcal{O}_v$  is an essentially

smooth  $k$ -algebra by Morel in [Mor12, Th. 5.26]. We will give a proof in [Dég23], using the formalism of Rost-Schmid complexes in dimension greater than 1.

The last axiom is tautological for Milnor-Witt K-theory, but useful for computations. Let us first remark that, given an invertible  $E$ -vector space  $\mathcal{L}$ , one has an isomorphism:

$$E^\times \rightarrow \text{Aut}_E(\mathcal{L}), u \mapsto (l \mapsto u.l).$$

Given an  $E$ -automorphism  $\Theta$  of  $\mathcal{L}$ , one denotes by  $\delta_\Theta \in E^\times$  the corresponding unit.

**R4a** For any automorphism  $\Theta$  of an invertible  $E$ -vector space  $\mathcal{L}$ , and any  $\sigma \in K_*^{MW}(E, \mathcal{L})$ , one has  $\Theta_*(\sigma) = \langle \delta_\Theta \rangle . \sigma$ .

As already mentioned, datas  $D^*$  and relation  $R^*$  on  $K_*^{MW}$ , once restricted to finitely generated extensions of a perfect field  $k$  and valuations trivial on  $k$  in D4, defines of MW-premodule in the sense of Feld ([Fel20a]). In fact, we can forget about the existence of a base field  $k$ , so we introduce the following obvious extension of Feld’s axioms, inspired by [DFJ].

**Definition 5.2.3.** An application  $M$  from fields to  $\mathbb{Z}$ -graded abelian groups equipped with datas  $D^*$  and satisfying the relations  $R^*$  stated above will be called an *absolute (cohomological) MW-premodule*.

Morphisms of absolute MW-premodules are given by transformations natural with respect to datas D1, D1’, D2 and commuting with datas D3 and D4.

Thus  $K_*^{MW}$  is an absolute MW-premodule.

**5.2.4.** We give a more theoretical view on the structure of a MW-premodule  $M$ . First,  $M$  is a  $\mathbb{Z}$ -graded ”enriched” functor from (twisted) fields to  $\mathbb{Z}$ -graded abelian groups  $\mathcal{A}b^{\mathbb{Z}}$ :

- (1) covariant with respect to morphisms of fields (D1+R1a), and more precisely with the category of *twisted fields* (+D1’+R4, see Remark 2.4.7),
- (2) contravariant with respect to finite field extensions up to a twists (D2+R1b).

These functorialities correspond to the fact  $K_*^{MW}$  is a *twisted homology theory* with respect to fields (*i.e.* a twisted cohomology with respect integral 0-dimensional schemes). In particular, the second functoriality correspond to the *exceptional functoriality* aka Gysin morphisms.

One can define a monoidal structure on such bi-functors in such a way that  $K_*^{MW}$  is a monoid. Because of the exceptional functoriality, this is non-trivial and one must use the equivalence of [Fel21, Th 4.0.1] to get the right formulas. Then data D3 as well as relations R2a, R2, R2c, can be stated by saying that  $M$  is a left  $K_*^{MW}$ -module.<sup>48</sup>

<sup>48</sup>Beware also that that if one wants to a symmetric monoidal structure on the both the category and on  $K_*^{MW}$ , one has to work with graded invertible vector spaces over fields instead of just invertible vector spaces (see [Fel20a, §3.2] and [Fas20, Def. 1.18] and the discussion thereafter). This issue will be discussed when considering products on Chow-Witt groups ([Dég23]).

Finally, data D4 is the trace of a localization exact sequence. In fact, it should be interpreted as a boundary map in such an exact sequence (see 2.5.9 for  $K_*^{MW}$ ). Indeed, this is exactly the case when one considers the localization long exact sequence of Section 3.3 applied to the spectrum of the valuation ring  $\mathcal{O}_v$ , and the immersion of the closed point.

**Example 5.2.5.** Further examples of absolute MW-premodules comes from the setting of 5.1.2. Indeed, one deduces from the construction of datas  $D^*$  on  $W[t, t^{-1}]$ ,  $I^*$ ,  $K_*^M$ ,  $\bar{I}^*$  that they all satisfy the above relations. In particular, they all define absolute MW-premodules.

Moreover, one has the following commutative diagram of MW-premodules:

$$\begin{array}{ccccc} K_*^{MW} & \longrightarrow & K_*^{MW}[\eta^{-1}] & \xrightarrow{\sim} & W[t, t^{-1}] \\ \downarrow & & & & \uparrow \\ K_*^{MW}/(h) & \xlongequal{\quad} & K_*^W & \xrightarrow[\sim]{\psi} & I^* \end{array}$$

from Theorem 2.3.5 and

$$\begin{array}{ccccc} & & \text{F} & & \\ & \searrow & \text{---} & \nearrow & \\ K_*^{MW} & \longrightarrow & K_*^{MW}/(\eta) & \simeq & K_*^M \\ \mu' \downarrow & & & \swarrow \mu & \downarrow \\ I^* & \longrightarrow & \bar{I}^* & \xleftarrow[\sim]{} & K_*^M/(2) \end{array}$$

from Corollary 2.3.7 and Proposition 2.4.11. The last diagram is a cartesian square, and  $\mu$  is the *Milnor map* (Theorem 2.2.3).

One has a last morphism, the hyperbolic map:

$$K_*^M \xrightarrow{H} K_*^{MW}$$

and we recall that  $F \circ H = 2 \cdot \text{Id}$  (see 2.2.11).

**5.3. Stronger axioms and quadratic multiplicities.** In this section, we formulate following Feld [Fel20a, Fel20b], stronger forms of properties (R1c) and (R3a) involving multiplicities, as in the theory of cycle modules [Ros96]. Note that the main difficulty compared to Rost's theory is the necessity to describe what happens on twists. Compare to the formula given by Feld, we make explicit the isomorphisms needed to get a coherent formula.

**5.3.1.** We start with the stronger form of (R3a). We consider a ramified extension  $\varphi : E \rightarrow F$  of valued field  $(E, v)$ ,  $(F, w)$  with ramification index  $e > 0$ :  $w \circ \varphi = e \cdot v$ . We still denote by  $\varphi : \mathcal{O}_v \rightarrow \mathcal{O}_w$  the induced morphism on ring of integers, and by  $\bar{\varphi} : \kappa_v \rightarrow \kappa_w$  the induced map on residue field.

Let us choose prime  $\pi_v \in \mathcal{O}_E$ ,  $\pi_w \in \mathcal{O}_F$  respectively for  $v$  and  $w$ . This defines a unique isomorphism of  $\kappa_w$ -vector spaces:

$$\theta : \omega_v \otimes_{\kappa_v} \kappa_w \rightarrow \omega_w, \bar{\pi}_v^* \otimes 1 \mapsto \bar{\pi}_w^*$$

where  $\omega_v = (\mathcal{M}_v/\mathcal{M}_v^2)^\vee$  (resp.  $\omega_w = (\mathcal{M}_w/\mathcal{M}_w^2)^\vee$ ) are the respective normal sheaves.

Note that there exists a uniquely defined unit  $u \in \mathcal{O}_F^\times$  such that  $\varphi(\pi_v) = u \cdot \pi_w^e$ .

**Proposition 5.3.2** (Property R3a+). *Consider the above hypothesis and notation. Then the following diagram commutes:*

$$\begin{array}{ccc} K_*^{MW}(E) & \xrightarrow{\partial_v} & K_*^{MW}(\kappa(v), \omega_v) \\ \varphi_* \downarrow & & \downarrow \langle \bar{u} \rangle \cdot e_\epsilon \cdot \bar{\varphi}_* \otimes \theta_* \\ K_*^{MW}(F) & \xrightarrow{\partial_w} & K_*^{MW}(\kappa(w), \omega_w). \end{array}$$

Moreover, the right vertical map does not depend on the choice of primes  $\pi_v$  and  $\pi_w$ .

*Proof.* Consider an element  $\sigma \in K_*^{MW}(E)$ . As all maps commute with multiplication by  $\eta$ , one reduces to consider a symbol of the form  $\sigma = [u_1, \dots, u_n]$ . By using relation (MW2) of Milnor-Witt K-theory, the fact  $w(\varphi(\pi_E)) > 0$ , and the properties of the residue map, one reduces to the case where  $\sigma = [\pi_F, u_2, \dots, u_n]$ , with  $u_i \in \mathcal{O}_E^\times$ . We compute the composite of the maps through the left-down right corner:

$$\partial_w(\varphi_*(\sigma)) = \partial_w([u \cdot \pi_F^e, \varphi(u_2), \dots, \varphi(u_n)]) = \langle \bar{u} \rangle \cdot e_\epsilon \cdot [\bar{\varphi}(\bar{u}_2), \dots, \bar{\varphi}(\bar{u}_n)] \otimes \bar{\pi}_F^*.$$

where the last equality follows from Theorem 2.5.2(Res2). Another application of *loc. cit.* gives  $\partial_v(\sigma) = [\bar{u}_2, \dots, \bar{u}_n] \otimes \bar{\pi}_E^*$ , and so the first assertion follows.

For the second assertion, we write  $\pi'_E = u_E \pi_E$ ,  $\pi'_F = u_F \pi_F$ , with  $u_E, u_F$  units. Then a straightforward computation reduces to show the equality in  $\text{GW}(\kappa_w)$ :

$$(5.3.2.a) \quad \langle \bar{u}_F \rangle \cdot e_\epsilon = \langle \bar{u}_F^e \rangle \cdot e_\epsilon.$$

If  $e$  is odd, one gets  $\langle \bar{u}_F^e \rangle = \langle \bar{u}_F \rangle$  and therefore (5.3.2.a) is true. If  $e = 2n$  is even,  $e_\epsilon = n \cdot h$ . But for any unit  $a \in \kappa_w^\times$ , one has:  $\langle a \rangle \cdot h = h$  (Theorem 2.1.11(GW3)). Thus (5.3.2.a) holds true in that latter case.  $\square$

*Remark 5.3.3.* In the preceding proposition, one cannot avoid in general the presence of the correcting unit  $\bar{u}$  in the formula of the right vertical map. Using property R4a, it is possible to give a more compact definition of this map. Indeed, working in the abelian group

$$\mathbb{Z}[\text{Hom}_{\kappa_w}(\omega_v \otimes_{\kappa_v} \kappa_w, \omega_w)] = \mathbb{Z}[\text{Hom}_{\kappa_v}(\omega_v, \omega_w)],$$

one can define the element:

$$\theta_u^e = \sum_{i=0}^{e-1} \delta_{\bar{u}(-1)^i} \circ \theta$$

with the notation of (R4a). With that definition, the formula of the preceding proposition reads:

$$\partial_w \circ \varphi_* = (\bar{\varphi}_* \otimes \theta_u^e) \circ \partial_v.$$

This last formula agrees with the computation of the  $\mathbb{A}^1$ -homotopical defect of the purity isomorphism done in [Fel21, Th. 2.2.2].

**5.3.4.** The preceding formula has interesting corollaries. Let us set up the notation before stating the first one.

We let  $\varphi : E \rightarrow F$  be an arbitrary field extension,  $\varphi' : E(t) \rightarrow F(t)$  the induced extension. A closed point  $x \in \mathbb{A}_{E,(0)}^1$  corresponds to a monic irreducible polynomial  $\pi_x \in E[t]$  and we denote by  $v_x$  the corresponding  $\pi_x$ -adic valuation on  $E(t)$ . One can consider the prime decomposition in  $F[t]$ :

$$\varphi'(\pi_x) = \prod_{y/x} \pi_y^{e_{y/x}}.$$

The product runs over a finite family of closed points  $y \in \mathbb{A}_{F,(0)}^1$ , corresponding to the irreducible polynomial  $\pi_y \in F[t]$ , and the integers  $e_{y/x}$  are some multiplicities. Equivalently, the  $\pi_y$ -adic valuations  $v_y$  on  $F(t)$  runs over the extensions of the valuation  $v_x$ , such that  $v_y \circ \varphi'_* = e_{y/x} \cdot v_x$ . As for the preceding proposition, we consider  $\omega_x$  and  $\omega_y$  the respective normal sheaves associated with  $v_x$  and  $v_y$  respectively. Then one considers the isomorphism  $\theta_y : \omega_x \otimes_{\kappa_x} \kappa_y \rightarrow \omega_y$ , sending  $\bar{\pi}_x^* \otimes 1$  to  $\bar{\pi}_y^*$ . We let  $\varphi_y : \kappa(x) \rightarrow \kappa(y)$  be the induced morphism.

**Corollary 5.3.5.** *Consider the above notation. Then the following diagram commutes:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_*^{MW}(E) & \longrightarrow & K_*^{MW}(E(t)) & \xrightarrow{d_E} & \bigoplus_{x \in \mathbb{A}_{E,(0)}^1} K_*^{MW}(\kappa(x), \omega_x) \longrightarrow 0 \\ & & \varphi_* \downarrow & & \varphi'_* \downarrow & (2) & \downarrow \sum_{y/x} e_{y/x, \epsilon} \cdot \varphi_{y*} \otimes \theta_{y*} \\ 0 & \longrightarrow & K_*^{MW}(F) & \longrightarrow & K_*^{MW}(F(t)) & \xrightarrow{d_F} & \bigoplus_{y \in \mathbb{A}_{F,(0)}^1} K_*^{MW}(\kappa(y), \omega_y) \longrightarrow 0 \end{array}$$

where the two horizontal sequences are the split short exact sequences deduced from Theorem 3.2.1.

*Proof.* The commutativity of the left-hand square is trivial. For square (2), we consider an element  $\sigma \in K_*^{MW}(E(t))$ . As all maps involved commute with  $\eta$ , one can assume  $\sigma = [f_1, \dots, f_n]$ ,  $f_i \in E(t)$ . Let  $S \subset \mathbb{A}_{E,(0)}^1$  be the finite set of points such that the family  $(\pi_x)_{x \in S}$  is exactly made of the irreducible polynomials appearing in the prime decomposition of the  $f_i$ . Thus,  $d_E(\sigma) = \sum_{x \in S} \partial_{v_x}(\sigma)$ .

Similarly, let  $T \subset \mathbb{A}_{F,(0)}^1$  be the finite set such that the family  $(\pi_x)_{x \in T}$  is made of the irreducible polynomials appearing in the prime decomposition of the  $\varphi'(f_i)$ . Then  $d_F(\varphi'(\sigma)) = \sum_{y \in T} \partial_{v_y}(\varphi'(\sigma))$ .

With this notation, the conclusion comes from applying Proposition 5.3.2 to each point  $x \in S$  and then take the sum of the resulting formulas.  $\square$

*Remark 5.3.6.* Considering the graded Chow-Witt groups as defined in Definition 3.1.5, the right vertical map of the diagram can be seen as the definition of a

pullback map

$$f^* : C^1(\mathbb{A}_E^1)_* \rightarrow C^1(\mathbb{A}_F^1)_*$$

associated to the flat morphism  $f : \mathbb{A}_F^1 \rightarrow \mathbb{A}_E^1$  (note that in this particular case, though  $f$  is not of finite type, it is quasi-finite). In fact the commutativity of square (2) gives (after adding twists with a line bundle  $\mathcal{L}$  over  $\mathbb{A}_E^1$ ) a well-defined morphism of complexes, called the *flat pullback*:

$$f^* : C^*(\mathbb{A}_E^1, \mathcal{L}) \rightarrow C^*(\mathbb{A}_F^1, f^{-1}\mathcal{L})_*$$

The definition of pullbacks on Chow-Witt groups associated with any flat morphism is still missing, and we intend to come back to that point after the work [DFJ].

Using the Bass-Tate approach to transfers in the monogenic case, and especially the characterization obtained in Proposition 4.2.3, one deduces from the commutativity of square (2) in the previous theorem the following result, which we state as a lemma for the next statement.

**Lemma 5.3.7.** *Let  $\varphi : E \rightarrow F$  be an arbitrary field extension, and consider the notation of the previous corollary. Then the following square is commutative:*

$$\begin{array}{ccc} \bigoplus_{x \in \mathbb{P}_{E,(0)}^1} K_*^{MW}(\kappa(x), \omega_{\kappa(x)/E}) & \xrightarrow{\sum_x \text{Tr}_{\kappa(x)/E}^{MW}} & K_*^{MW}(E) \\ \downarrow \sum_{y/x} e_{y/x, \epsilon} \cdot \varphi_{y^*} \otimes \theta_{y^*} & & \downarrow \varphi_* \\ \bigoplus_{y \in \mathbb{P}_{F,(0)}^1} K_*^{MW}(\kappa(y), \omega_{\kappa(y)/F}) & \xrightarrow{\sum_y \text{Tr}_{\kappa(y)/F}^{MW}} & K_*^{MW}(F) \end{array}$$

where the sum on the vertical left hand-side runs over the point  $y \in \mathbb{P}_F^1$  which lies above  $x \in \mathbb{P}_E^1$ ,  $e_{y/x}$  is defined as in the previous corollary and  $e_{\infty/\infty} = 1$ . We have abused notation by denoted  $\theta_y$  the isomorphism induced by the one of the previous corollary; explicitly:

$$\begin{aligned} \theta_y : \omega_{\kappa(x)/E} &\simeq \omega_x \otimes \omega_{\mathbb{A}_{\kappa(x)}^1/\kappa(x)} \rightarrow \omega_y \otimes \omega_{\mathbb{A}_{\kappa(y)}^1/\kappa(y)} \simeq \omega_{\kappa(y)/F} \\ &\bar{\pi}_x^* \otimes dt \mapsto \bar{\pi}_y^* \otimes dt. \end{aligned}$$

Indeed, it suffices to apply the preceding corollary, Proposition 4.2.3 together with (R3a) for the case of the valuation at infinity on  $E(t)$ .

**5.3.8.** We are now ready to state the stronger form of axiom (R1c), without any assumption of separability. Namely, we consider a commutative square of rings:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & L \\ \varphi \downarrow & & \downarrow \\ F & \rightarrow & R \end{array}$$

where  $E, F, L$  are fields,<sup>49</sup>  $\Phi$  is finite, and we have put:  $R = F \otimes_E L$ . Then  $R$  is a non necessarily reduced ring.

We choose a presentation  $L = F[t_1, \dots, t_n / (\pi_1, \dots, \pi_n)]$  as in Example 4.1.10. In other words, letting  $\alpha_i$  be the image of  $t_i$  in the the above quotient, one has  $L = F[\alpha_1, \dots, \alpha_n]$ . Moreover,  $\pi_i$  is a polynomial with coefficients in  $F$  involving only the variables  $t_1, \dots, t_i$ , and  $\pi_i(\alpha_1, \dots, \alpha_{i-1}, t_i)$  is the minimal polynomial of  $\alpha_i$  over  $F[\alpha_1, \dots, \alpha_{i-1}]$ .

Then,  $\pi_1$  seen as a polynomial in  $t_1$  can be uniquely factored in  $F$  as:

$$\pi_1 = \prod_{j \in J_1} \pi_{1,j}^{e_{1,j}}$$

where  $\pi_{1,j}$  is an irreducible polynomial in  $F[t_1]$ , and  $e_{1,j}$  a positive integer. Arguing inductively, one obtains the following presentation of  $R$ :

$$R = \prod_{x \in X} F[t_1, \dots, t_n] / (\pi_{1,x}^{e_{1,x}}, \dots, \pi_{n,x}^{e_{n,x}})$$

such that for any  $(i, x) \in [1, n] \times I$ ,  $\pi_{i,x}$  is a polynomial in  $(t_1, \dots, t_i)$  such that  $\pi_{i,x}(\alpha_1, \dots, \alpha_{i-1}, t_i)$  is a prime divisor of  $\pi_i(\alpha_1, \dots, \alpha_{i-1}, t_i)$  in  $F[\alpha_1, \dots, \alpha_{i-1}, t_i]$ . Moreover, the indexing set is  $X = \text{Spec}(R)$ , the set of prime ideals of  $R$ .

Let us fix a point  $x \in X$ . As a prime ideal of  $R$ , one can write  $x = (\pi_{1,x}, \dots, \pi_{n,x})$ . The local Artinian ring  $R_x$  is of length:

$$e_x := \text{lg}(R_x) = e_{x,1} \dots e_{x,n}.$$

Moreover, the residue field  $\kappa(x) := R/x$  is finite over  $F$  and one can define an isomorphism of  $F$ -vector spaces:

$$\begin{aligned} \theta_x : \omega_{L/E} \otimes_L F &\xrightarrow{\sim} \omega_{\kappa(x)/F}, \\ (\bar{\pi}_1 \wedge \dots \wedge \bar{\pi}_n)^* \otimes (dt_1 \wedge \dots \wedge dt_n) &\longmapsto (\bar{\pi}_{1,x} \wedge \dots \wedge \bar{\pi}_{n,x})^* \otimes (dt_1 \wedge \dots \wedge dt_n) \end{aligned}$$

The following result gives a more precise form of [Fel20b, Th. 3.8].

**Theorem 5.3.9** (Property R1c+). *Consider the above notation. Then the following diagram is commutative:*

$$\begin{array}{ccc} K_*^{MW}(L, \omega_{L/E}) & \xrightarrow{\Phi^*} & K_*^{MW}(E) \\ \downarrow \sum_{\mathfrak{p}} e_{\mathfrak{p}, \epsilon} \cdot \varphi_{\mathfrak{p}^*} \otimes \theta_{\mathfrak{p}^*} & & \downarrow \varphi_* \\ \bigoplus_{\mathfrak{p} \subset R} K_*^{MW}(\kappa(\mathfrak{p}), \omega_{\kappa(\mathfrak{p})/F}) & \xrightarrow{\sum_{\mathfrak{p}} \Phi_{\mathfrak{p}}^*} & K_*^{MW}(F) \end{array}$$

where  $\mathfrak{p}$  runs over the prime ideals of  $R = L \otimes_E F$ , and the map  $\Phi_{\mathfrak{p}} : F \rightarrow \kappa(\mathfrak{p})$ ,  $\varphi_{\mathfrak{p}} : L \rightarrow \kappa(\mathfrak{p})$  are induced respectively by  $\Phi, \varphi$ .

Moreover, the left-hand vertical map is independent on the chosen parametrization of  $L/E$ .

<sup>49</sup>only the positive characteristics case is relevant

*Proof.* The proof follows that of *loc. cit.* By multiplicativity of the symbol  $?_\epsilon$  (see the end of 2.2.18), and compatibility of the isomorphism  $\theta_x$  with the number of variables  $n$ , one reduces to the case where  $L/E$  is monogenic, *i.e.*  $n = 1$  with our previous notation. To simplify the notation, we write  $t, \alpha, \pi_x, \dots$  for  $t_1, \alpha_1, \pi_1, \dots$ , etc...

Then the first statement to be proved is a particular case of Lemma 5.3.7, obtained by considering only the point  $x \in \mathbb{A}_{E,(0)}^1$  such that  $L = \kappa(x)$ .

Finally, the second statement follows from (the last statement of) Proposition 5.3.2.  $\square$

*Remark 5.3.10.* This theorem is in fact the projection formula  $f^*p_* = q_*g^*$  in (graded) Chow-Witt groups for the cartesian square:

$$\begin{array}{ccc} \mathbb{P}_F^1 & \xrightarrow{q} & \text{Spec}(F) \\ g \downarrow & & \downarrow f \\ \mathbb{P}_E^1 & \xrightarrow{p} & \text{Spec}(E). \end{array}$$

This is in fact one of the main properties needed for a flat pullback. Compare to [Fel20b], we have avoided the need of a (perfect) base field, and we have described the isomorphisms needed to change the twists (*i.e.* the map  $\theta_{p^*}$ ).

Surprisingly,  $f$  is induced by an arbitrary field extension, not necessarily finitely generated.

## 6. APPENDIX: COHERENT DUALITY, TRACES AND RESIDUES

### 6.1. Categorical duality and traces.

**6.1.1.** We recall the classical categorical framework for expressing duality. We refer the reader to [Mac71], Section XI.1 (resp. VII.7), for references on symmetric monoidal categories (resp. closed symmetric monoidal categories). To simplify the exposition, we will apply MacLane’s coherence theorem (*loc. cit.*, XI.1, Th. 1) and consider that any composite of coherence isomorphisms (*i.e.* expressing unity, associativity, commutativity of the symmetric monoidal structure) is an identity.

The historical references for the next two definitions are [DP80] and [SR72].

**Definition 6.1.2.** Let  $(\mathcal{C}, \otimes, \mathbb{1})$  be a symmetric monoidal category and  $M$  an object of  $\mathcal{C}$ . We say that  $M$  is *dualizable*<sup>50</sup> if there exists an object  $M^\vee$  and morphisms:

$$\pi : M \otimes M^\vee \rightarrow \mathbb{1}, \delta : \mathbb{1} \rightarrow M^\vee \otimes M,$$

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<sup>50</sup>Other possible terminologies are *strongly dualizable* or *rigid*.



respectively called the duality *pairing* and *co-pairing*, such that the following composite maps

$$\begin{aligned} M &\xrightarrow{\text{Id} \otimes \delta} M \otimes M^\vee \otimes M \xrightarrow{\pi \otimes \text{Id}} M \\ M^\vee &\xrightarrow{\delta \otimes \text{Id}} M^\vee \otimes M \otimes M^\vee \xrightarrow{\text{Id} \otimes \pi} M^\vee \end{aligned}$$

are the identity. One says that  $M^\vee$  is the *dual* of  $M$ .

The triple  $(M^\vee, \pi, \delta)$  uniquely determines  $M^\vee$  as the dual of  $M$ . It follows from the definition that the functor  $\tau_M = (M \otimes -)$  is both left and right adjoint to the functor  $\tau_{M^\vee}(M^\vee \otimes -)$ . In particular,  $\tau_{M^\vee}$  is the internal Hom functor with source  $M$ , and one gets a canonical isomorphism, bifunctorial in  $M$  and  $N$ :

$$M^\vee \otimes N \simeq \underline{\text{Hom}}(M, N).$$

When the monoidal category  $\mathcal{C}$  is closed, there exists a canonical isomorphism  $M^\vee \simeq \underline{\text{Hom}}(M, \mathbb{1})$ , characterized as being the evaluation at  $\mathbb{1}$  of the unique isomorphism  $\tau_{M^\vee} \simeq \underline{\text{Hom}}(M, -)$ .

**Example 6.1.3** (exercice). Let  $A$  be a (commutative) ring, and  $A\text{-mod}$  the closed symmetric monoidal category of  $A$ -modules. Then the following conditions are equivalent:

- (i)  $M$  is dualizable in  $A\text{-mod}$ ;
- (ii)  $M$  is a finitely generated and projective  $A$ -module.

**Example 6.1.4.** The preceding example generalizes to a quasi-compact quasi-separated scheme  $X$ . Let  $\text{D}(\mathcal{O}_X)$  be the derived category of  $\mathcal{O}_X$ -modules, endowed with its closed symmetric monoidal structure via the derived tensor product. Let  $K$  be an object of  $\text{D}(\mathcal{O}_X)$ . Then the following conditions are equivalent (see e.g. [mDJ, Ex. 0FPC, Lem. 0FPD, Prop. 09M1]):

- (i)  $K$  is dualizable in  $\text{D}(\mathcal{O}_X)$ ;
- (ii)  $K$  is a perfect complex of  $\mathcal{O}_X$ -modules;
- (iii)  $K$  is compact.

**Definition 6.1.5.** Consider the above setting and let  $M$  be a dualizable object with dual  $(M^\vee, \pi, \delta)$ . We define the *trace* of an endomorphism  $f : M \rightarrow M$  as the following element of the ring  $\text{End}_{\mathcal{C}}(\mathbb{1})$ :

$$\text{tr}_M(f) : \mathbb{1} \xrightarrow{f'} M \otimes M^\vee \xrightarrow{\pi} \mathbb{1}$$

where  $f'$  is obtained by adjunction from  $f$ . This defines a map:

$$\text{tr}_M : \text{End}_{\mathcal{C}}(M) \rightarrow \text{End}_{\mathcal{C}}(\mathbb{1}).$$

*Remark 6.1.6* (exercice). One can derive the following formulas for traces:

- $\text{tr}_M(f \circ g) = \text{tr}_M(g \circ f)$ .
- $\text{tr}_{M \otimes N}(f \otimes g) = \text{tr}_M(f) \otimes \text{tr}_N(g)$ .
- $\text{tr}_M(\lambda \cdot f) = \lambda \cdot \text{tr}_M(f)$ ,  $\lambda \in \text{End}_{\mathcal{C}}(\mathbb{1})$ .

**Example 6.1.7.** Consider the setting of Example 6.1.3. Obviously,  $\mathbb{1} = A$  and  $\text{End}_A(A) = A$ , as a ring.

Given a dualizable  $A$ -module  $M$ , the trace map  $\text{Tr}_M : \text{End}_A(M) \rightarrow A$  defined above coincide with the classical notion for number theory. In particular, when  $M$  admits a (global)  $A$ -basis  $(f_1, \dots, f_n)$ , through the induced isomorphism  $\text{End}_A(M) \simeq \mathcal{M}_n(A)$ , the map  $\text{Tr}_M$  is the usual trace map of matrices.

**6.1.8.** Consider again the abstract situation of a symmetric monoidal category  $(\mathcal{C}, \otimes, \mathbb{1})$ , and a dualizable object  $M$  of  $\mathcal{C}$  with dual  $(M^\vee, \pi, \delta)$ .

We remark that the trace map  $\text{tr}_M$  is induced by an internal trace map:

$$\underline{\text{tr}}_M : \underline{\text{Hom}}(M, M) \simeq M^\vee \otimes M \simeq M \otimes M^\vee \xrightarrow{\pi} \mathbb{1}.$$

This means that  $\text{tr}_M = \underline{\text{Hom}}(\underline{\text{tr}}_M, \mathbb{1})$ .

Assume now that  $M$  admits a product map  $\mu : M \otimes M \rightarrow M$  (for example,  $M$  is a monoid, [Mac71, VII.3]). Then one gets a  $\mu$ -trace morphism:

$$\text{Tr}_M^\mu : M \xrightarrow{\mu'} \underline{\text{Hom}}(M, M) \xrightarrow{\underline{\text{tr}}_M} \mathbb{1}$$

As a particular case, one gets back the following classical definition from algebra:

**Definition 6.1.9.** Let  $A$  be a ring and  $B$  be a commutative  $A$ -algebra which is projective and finitely generated as an  $A$ -module.

Then  $B$  is a dualizable  $A$ -module and we define the *trace morphism*

$$\text{Tr}_{B/A} : B \rightarrow A$$

as the  $A$ -linear map associated above with respect to the multiplication map  $B \otimes_A B \rightarrow B$ .

Concretely, the trace of an element  $b \in B$  is the trace of the endomorphism  $\gamma_b$  such that  $\gamma_b(x) = b.x$ . It can be computed locally by choosing bases of the  $A$ -module  $B$  and using the trace of matrices. The local definitions then glue using faithfully flat descent.

Let us recall the following classical result.

**Proposition 6.1.10.** *Let  $B/A$  be a finitely generated projective ring extension. Then the following conditions are equivalent:*

- (i)  $B/A$  is étale.
- (ii) For every prime ideal  $\mathfrak{q}$  in  $B$ ,  $\mathfrak{p}$  being its inverse image in  $A$ ,  $L = B/\mathfrak{q}$ ,  $K = A/\mathfrak{p}$ , one has:  $\text{Tr}_{L/K} \neq 0$ .
- (iii) The bilinear form  $B \otimes_A B \rightarrow A, x \otimes y \mapsto \text{Tr}_{B/A}(xy)$  is non degenerate - i.e. induces by adjunction an isomorphism  $B \rightarrow B^\vee$  of  $A$ -modules.

In particular, a finite field extension  $L/K$  is separable if and only if  $\text{Tr}_{L/K} \neq 0$ .

In particular, the trace map of an inseparable extension is not interesting. This justifies the use of a finer duality theory, which was introduced by Grothendieck. We recall the abstract setting to end-up this section.

**Definition 6.1.11.** Let  $(\mathcal{C}, \otimes, \mathbb{1})$  be a closed symmetric monoidal category. Let  $K$  be an object of  $\mathcal{C}$ , and write  $D_K(M) = \underline{\text{Hom}}(M, K)$ . The evaluation map  $M \otimes \underline{\text{Hom}}(M, K)$  induces by adjunction a canonical map  $\omega_M : M \rightarrow D_K \circ D_K(M)$ .

One says that  $K$  is *dualizing* if the natural transformation  $\omega : \text{Id}_{\mathcal{C}} \rightarrow D_K \circ D_K$  is an isomorphism.

In words,  $D_K(M)$  is called the weak  $K$ -dual of  $M$ , and the definition asks that any object  $M$  is isomorphic to its double weak  $K$ -dual, by the canonical map  $\omega_M$ .

*Remark 6.1.12.* In the original definition, that of a dualizing complex ([Har66, Definition, p. 258]), one had additional assumptions (finite injective dimension and lower boundedness). One has progressively dismissed this kind of assumptions, in order to extend Grothendieck's theory to other context (torsion étale sheaves [ILO14, VII, 6.1.1], constructible pro-étale sheaves [BS15, 6.7.20],  $D$ -modules, motivic homotopy [Ayo07, 2.3.73] and motivic complexes [CD19, 4.4.24]).

**6.1.13.** Consider a dualizing object  $K$  of  $\mathcal{C}$  as in the above definition. Then:

- (1) The map  $\mathbb{1} \rightarrow \underline{\text{Hom}}(K, K)$ , deduced from  $\text{Id}_K$  by adjunction, is an isomorphism.
- (2) For any object  $M, N$  in  $\mathcal{C}$ , one has an isomorphism:

$$D_K(M \otimes D_K(N)) \simeq \underline{\text{Hom}}(M, N).$$

- (3) An object  $K'$  of  $\mathcal{C}$  is dualizing if and only if there exists a  $\otimes$ -invertible object  $L$  such that  $K' = K \otimes L$ .  
Moreover, in this case, one has  $L \simeq D_{K'}(K) = \underline{\text{Hom}}(K, K')$ .
- (4) If  $M$  is a dualizable object in  $\mathcal{C}$  with dual  $M^\vee$ , then  $D_K(M) \simeq M^\vee \otimes K$ .
- (5) The dualizing object  $K$  is rigid if and only if it is invertible.

We give the arguments for completeness:

- (1) use the isomorphism  $\mathbb{1} \simeq D_K(D_K(\mathbb{1}))$  and  $\underline{\text{Hom}}(\mathbb{1}, -) \simeq \text{Id}_{\mathcal{C}}$  (as a right adjoint to the functor  $(\mathbb{1} \otimes -)$ ).
- (2) Use the sequence of isomorphisms:

$$D_K(M \otimes D_K(N)) = \underline{\text{Hom}}(M \otimes D_K(N), K) \simeq \underline{\text{Hom}}(M, D_K D_K(N)) \simeq \underline{\text{Hom}}(M, N).$$

- (3)  $\Leftarrow$ : use  $\underline{\text{Hom}}(M, N \otimes L) \simeq \underline{\text{Hom}}(M \otimes L^{-1}, N) \simeq \underline{\text{Hom}}(M, N) \otimes L$ .  
 $\Rightarrow$ : one proves  $M \mapsto M \otimes D_K(K')$  is an equivalence. It suffices to apply the equivalence  $D_K$ , point (2) to reduce to the fact  $D_{K'}$  is an equivalence.
- (4) Follows from the definition.
- (5) Follows from point (4).

**Example 6.1.14.** In the category of locally compact abelian groups, the unit circle  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  is a dualizing object. We will see more examples in the next section.

## 6.2. Grothendieck differential trace map and duality.

**6.2.1.** Let  $f : X \rightarrow S$  be a morphism of quasi-compact and quasi-separated schemes. We have an adjoint pair:

$$\mathbf{L}f^* : D(\mathcal{O}_S) \rightleftarrows D(\mathcal{O}_X) : \mathbf{R}f_*$$

We say that a complex  $K$  of  $D(\mathcal{O}_X)$  is quasi-coherent if its cohomology sheaves are quasi-coherent. We let  $D_{\text{qc}}(X)$  be the full sub-category of  $D(\mathcal{O}_X)$  made of quasi-coherent complexes.<sup>51</sup> Both functors  $\mathbf{L}f^*$  and  $\mathbf{R}f_*$  preserves quasi-coherent complexes (see [mDJ, 36.3.8, 36.4.1]). The following theorem is one of the essential part of Grothendieck's duality theory.

**Theorem 6.2.2.** *Consider the above assumptions.*

- (1) *The functor  $\mathbf{R}f_* : D_{\text{qc}}(X) \rightarrow D_{\text{qc}}(S)$  admits a left adjoint. When  $f$  is in addition proper, we denote this left adjoint by  $f^! : D_{\text{qc}}(S) \rightarrow D_{\text{qc}}(X)$ .*
- (2) *If  $f$  is proper smoothable lci with canonical sheaf  $\omega_{X/S}$  and relative dimension  $f$  (see Definition 4.1.5), there exists a canonical isomorphism:*

$$\mathfrak{p}_f : \omega_{X/S}[d] \rightarrow f^!(\mathcal{O}_S)$$

*with the notation of the preceding point. It is called the purity isomorphism associated to  $f$ .*

*Proof.* The first statement is Neeman's theorem (see [Nee96], [mDJ, 48.3.1]).

We could not find an appropriate reference for point (2). However, it follows from the results of [Har66], with some complements brought by many years of improvement. Let us summarize the arguments from the litterature. We now erase the symbols  $\mathbf{L}$  and  $\mathbf{R}$  for readability.

First, we fix a factorisation of  $f$  as  $X \xrightarrow{i} P \xrightarrow{p} S$  such that  $i$  is a regular closed immersion and  $p$  is a smooth morphism. We will consider the functors  $i^!$  and  $p^!$  restricted to  $D_{\text{qc}}^+(X)$ , which is legitimate thanks to [mDJ, 48.3.5].

According to [mDJ, 48.9.2], there exists a canonical isomorphism (uniquely characterized by the adjoint property) of functors  $i^! \simeq i^{\flat}$  where  $i^{\flat} : D_{\text{qc}}^+(P) \rightarrow D_{\text{qc}}^+(X)$  is the functor defined in [Har66, III. §6]. As  $i$  is a regular closed immersion, there exists a canonical isomorphism of functors by [Har66, III. Cor. 7.3]:

$$\mathfrak{p}_i : \omega_{X/P}[-m] = \omega_{X/P}[-m] \otimes i^*(\mathcal{O}_P) \simeq i^{\flat}(\mathcal{O}_P) \simeq i^!(\mathcal{O}_P)$$

where  $m$  is the codimension of  $i$ .

As  $p$  is proper and smooth, there exists a canonical isomorphism of functors, as defined in [Nee20, 4.1.6]:

$$\mathfrak{p}_p : \omega_{P/S}[n] = (\omega_{P/S}[n] \otimes p^*(\mathcal{O}_S)) \simeq p^!(\mathcal{O}_S)$$

where  $n$  is the dimension of  $p$ . In particular, the complex  $p^!(\mathcal{O}_S)$  is perfect.

<sup>51</sup>For Noetherian schemes, this category is equivalent to the derived category of the abelian category of quasi-coherent sheaves: see [mDJ, Prop. 36.8.3].

We now build the desired map as the following composition:

$$\begin{aligned} \omega_{X/S}[d] \simeq \omega_{X/P}[-m] \otimes i^*(\omega_{P/S}[n]) &\xrightarrow{\mathbf{p}_i \otimes i^*(\mathbf{p}_p)} i^!(\mathcal{O}_P) \otimes i^*(p^!(\mathcal{O}_S)) \\ &\stackrel{(*)}{\simeq} i^!(\mathcal{O}_P \otimes p^!(\mathcal{O}_S)) \simeq f^!(\mathcal{O}_S) \end{aligned}$$

where the isomorphism  $(*)$  exists as  $i$  is lci ([mDJ, 48.8.1]).

To justify the word “canonical”, one needs to prove that the above isomorphism does not depend on the choice of the factorisation. The steps for this fact are well-known. The main points may be found in [Har66, §III]: 2.2, 6.2, 6.4, and most of all 8.1 (see also the proof of Th. 3.3.2 [DJK21]).  $\square$

*Remark 6.2.3.* In fact, the purity isomorphism can be generalized in the coherent context as follows. For any bounded quasi-coherent complex  $K$ , one get an isomorphism by the following composite maps:

$$f^!(K) \stackrel{(*)}{\xrightarrow{\quad}} f^*(K) \otimes f^!(\mathcal{O}_S) \xrightarrow{\mathbf{p}_f} f^*(K) \otimes \omega_{X/S}[d].$$

The isomorphism  $(*)$  follows from [Har66, III, 8.8]. This is specific to the coherent case. The analogue isomorphism does not hold in other six functors formalism such as the étale  $\ell$ -adic or motivic one, unless further restrictions to  $f$  are assumed (e.g.  $f$  is smooth, or a nil-immersion).

**Definition 6.2.4.** Assuming  $f$  is proper smoothable lci of relative dimension  $d$ , we will denote by

$$\mathrm{Tr}_f^\omega : \mathbf{R}f_*(\omega_{X/S}[d]) \rightarrow \mathcal{O}_S$$

the map obtained by adjunction from  $\mathbf{p}_f$  and call it the *differential trace map* associated with  $f$ .

When  $f$  is finite lci, taking global sections, we will also consider the induced trace map:

$$\mathrm{Tr}_{X/S}^\omega : \Gamma(X, \omega_{X/S}) \rightarrow \Gamma(S, \mathcal{O}_S).$$

Finally, if  $X/S$  is the spectrum of a finite lci ring extension  $B/A$  the above map will be denoted by:

$$\mathrm{Tr}_{B/A}^\omega : \omega_{B/A} \rightarrow A.$$

*Remark 6.2.5.* The above definition makes clear several properties of the differential trace map. It is functorial with respect to flat base change in  $X$ , and compatible with disjoint sum in  $X$ .

This mean in particular that if we have an isomorphism of finite lci  $A$ -algebras:

$$\Theta : B \xrightarrow{\sim} \prod_{i \in I} B_i,$$

the following diagram is commutative:

$$\begin{array}{ccc} \omega_{B/A} & \xrightarrow[\sim]{\Theta_*} & \bigotimes_{i \in I} \omega_{B_i/A} \\ & \searrow \text{Tr}_{B/A}^\omega & \swarrow \bigotimes_i \text{Tr}_{B_i/A}^\omega \\ & & A \end{array}$$

*Remark 6.2.6.* The compatibility with composition of the trace map is more involved. Consider a factorisation  $X \xrightarrow{g} Y \xrightarrow{h} S$  of  $f$  by proper smoothable and lci morphisms, of respective dimension  $n$  and  $m$ . First recall that there exists a canonical isomorphism (see 4.1.7):

$$\psi : \omega_{X/S} \simeq \omega_{X/Y} \otimes (f^* \omega_{Y/S}).$$

The compatibility with composition of the differential trace map is expressed by the following commutative diagram (again we cancel the symbols  $\mathbf{R}$  and  $\mathbf{L}$  for readability):

$$\begin{array}{ccc} f_*(\omega_{X/S})[d] & \xrightarrow{\text{Tr}_{X/S}^\omega} & \mathcal{O}_S \\ \psi \downarrow \sim & & \parallel \\ f_*(\omega_{X/Y} \otimes (g^* \omega_{Y/S}))[d] & & \mathcal{O}_S \\ \sim \downarrow & \xrightarrow{h_* (\text{Tr}_{X/Y}^\omega \otimes \text{Id})} & \\ h_*(g_*(\omega_{X/Y})[n] \otimes \omega_{Y/S})[m] & \xrightarrow{\text{Tr}_{Y/S}^\omega} & \mathcal{O}_S \end{array}$$

The second vertical map is obtained by the so-called projection formula, which holds here either because  $g$  is proper or even simply as  $\omega_{Y/S}$  is an invertible sheaf. This statement follows from [Har66, III, 10.5] (see also [Con00, Th. 3.4.1]).

In the affine case,  $X = \text{Spec}(C)$ ,  $Y = \text{Spec}(B)$ ,  $S = \text{Spec}(A)$ ,  $f$ ,  $g$  and  $h$  being finite, the diagram takes the following simpler form:

$$\begin{array}{ccc} \omega_{C/A} & \xrightarrow{\text{Tr}_{C/A}^\omega} & A \\ \psi \downarrow \sim & \text{Tr}_{C/B}^\omega \otimes \text{Id} & \parallel \\ \omega_{C/B} \otimes_B \omega_{B/A} & \xrightarrow{\text{Tr}_{B/A}^\omega} & A \end{array}$$

**6.2.7. A particular case of duality.** For any quasi-coherent complex  $K$  over  $X$ , and any proper smoothable lci morphism  $f : X \rightarrow S$ , the adjunction property of the pair  $(\mathbf{R}f_*, f^!)$  gives an isomorphism:

$$\begin{aligned} \text{Hom}_{\mathbf{D}(\mathcal{O}_X)}(K, \omega_{X/S}[d]) &\xrightarrow{\sim} \text{Hom}_{\mathbf{D}(\mathcal{O}_S)}(\mathbf{R}f_*(K), \mathcal{O}_S), \\ (u : K \rightarrow \omega_{X/S}[d]) &\mapsto (\text{Tr}_{X/S}^\omega \circ \mathbf{R}f_*(u)). \end{aligned}$$

In the case of a finite lci ring extension  $B/A$ , and for a  $B$ -module  $M$ , this boils down to an isomorphism:

$$\begin{aligned} \text{Hom}_B(M, \omega_{B/A}) &\xrightarrow{\sim} \text{Hom}_A(M, A), \\ (u : M \rightarrow \omega_{B/A}) &\mapsto (\text{Tr}_{B/A}^\omega \circ u). \end{aligned}$$

Finally for  $M = B$ , we get an isomorphism between  $A$ -linear forms on  $B$  and elements of  $\omega_{B/A}$ :

$$\begin{aligned} \omega_{B/A} &\xrightarrow{\sim} \text{Hom}_A(B, A), \\ w &\mapsto (\psi_w : \lambda \mapsto \text{Tr}_{B/A}^\omega(\lambda.w)). \end{aligned}$$

**Example 6.2.8.** We end-up this section with a few classical examples of duality in the case of coherent sheaves.

(1) A concrete case of duality is obtained when  $S$  is the spectrum of any field  $k$ ,  $X$  a proper smoothable lci  $k$ -scheme. In that case, the first isomorphis of 6.2.7 applied to  $K[n]$  where  $K$  is a bounded complex  $K$  with coherent cohomology, gives an isomorphism of  $k$ -vector spaces:

$$\text{Ext}_{\mathcal{O}_X}^{d-n}(K, \omega_{X/k}) \xrightarrow{\sim} H^n(X, K)^*$$

The trace map  $\text{Tr}_{X/k}^\omega$  induces what I will call the *Gysin map* associated with  $f$ :

$$f_! : H^d(X, \omega_{X/S}) \rightarrow k$$

and the above duality isomorphism corresponds to the *Poincaré duality* (perfect) pairing:

$$\begin{aligned} \text{Ext}_{\mathcal{O}_X}^{d-n}(K, \omega_{X/k}) \otimes H^n(X, K) &\rightarrow k \\ (x, y) &\mapsto f_!(a.b). \end{aligned}$$

(2) In the case of a proper smoothable lci morphism  $f : X \rightarrow S$ , one can interpret Grothendieck duality, for  $K = \mathcal{O}_X$ , by saying that  $\mathbf{R}f_*(\mathcal{O}_X)$  is dualizable (Definition 6.1.2, Example 6.1.4) with dual given by  $\mathbf{R}f_*(\omega_{X/S})[d]$ . One of the pairing coming from this duality is a relative version of the Poincaré duality pairing:

$$\mathbf{R}f_*(\mathcal{O}_X) \otimes \mathbf{R}f_*(\omega_{X/S})[d] \rightarrow \mathbf{R}f_*(\omega_{X/S})[d] \xrightarrow{\text{Tr}_{X/S}^\omega} \mathcal{O}_S$$

where the first map comes from the fact  $\mathbf{R}f_*$  is weakly monoidal (as the right adjoint of a monoidal functor).

(3) Of course, the theory can be considerably generalized - but we will only need the case of finite field extensions! Indeed, Grothendieck's main objective was to obtain duality for any proper morphism  $f : X \rightarrow \text{Spec}(k)$ . He achieved this by constructing a dualizing complex  $K_X = \omega_{X/k}$ , which is no longer an invertible sheaf in general (except if  $X$  is *Gorenstein*, see [Har66, V, 9.3]). We refer the reader to [Har66, Con00] or [LH09, Chap. 1].

### 6.3. Grothendieck and Sheja-Storch Residues.

**6.3.1.** We consider a commutative diagram of schemes:

$$\begin{array}{ccc} & P & \\ i \nearrow & & \searrow p \\ X & \xrightarrow{f} & S \end{array}$$

such that  $f$  is finite lci,  $p$  is smooth of relative dimension  $n$ , and  $i$  is a closed immersion with ideal sheaf  $\mathcal{I} \subset \mathcal{O}_P$ . The hypothesis implies that  $i$  is regular of codimension  $n$ .

Recall that we can associate to the above commutative diagram a canonical isomorphism (4.1.7):

$$\Theta : \omega_{X/S} \simeq \omega_{X/P} \otimes_{\mathcal{O}_X} i^* \omega_{P/S} \simeq (\Lambda^n \mathcal{I}/\mathcal{I}^2)^\vee \otimes_{\mathcal{O}_P} \Omega_{P/S}^n.$$

**Definition 6.3.2.** Consider a global differential  $n$ -form  $w \in \Gamma(P, \Omega_{P/S}^n)$  and a global regular parametrization  $(f_1, \dots, f_n)$  of  $\mathcal{I}$ . We define the Grothendieck residue (symbol) of  $w$  at  $(f_1, \dots, f_n)$  as the element of  $\Gamma(S, \mathcal{O}_S)$ :

$$\text{Res}_{P/S} \left[ \begin{array}{c} w \\ f_1 \dots f_n \end{array} \right] = \text{Tr}_{X/S}^\omega \left( (\bar{f}_1 \wedge \dots \wedge \bar{f}_n)^* \otimes i^*(w) \right)$$

where we have used the differential trace map of Definition 6.2.4 and we have considered the element  $(\bar{f}_1 \wedge \dots \wedge \bar{f}_n)^* \otimes i^*(w)$  as an element of  $\omega_{B/A}$  via the isomorphism  $\Theta$ .

This definition agrees with that of [Har66, III, §9] and that of [Con00, Appendix A, (A.1.4)].<sup>52</sup>

**6.3.3.** We now explain a method of Sheja and Storch to compute the above residue, and therefore the differential trace map. Our reference is [Kun08, §8].

We will restrict to the affine case. Let  $B$  be a finite projective  $A$ -algebra: in other words,  $B$  is dualizable as an  $A$ -module, see Example 6.1.3.

In what follows, we will use a set of indeterminates  $\underline{t} = (t_1, \dots, t_n)$ , and denote by  $A[\underline{t}] = A[t_1, \dots, t_n]$  to simplify notations. We assume that  $B$  is a complete intersection  $A$ -algebra: there exists elements  $\alpha = (\alpha_1, \dots, \alpha_n) \in B^n$  which generate  $B$  as an  $A$ -algebra and the kernel of the surjective map

$$A[\underline{t}] \rightarrow B, t_i \mapsto \alpha_i$$

admits a regular parametrization  $I = (f_1, \dots, f_n)$  for polynomials  $f_i \in A[\underline{t}]$ . We say that  $f = (f_1, \dots, f_n)$  is a *presentation* of the lci  $A$ -algebra  $B$ .

Putting  $S = \text{Spec}(A)$ ,  $X = \text{Spec}(B)$ ,  $P = \mathbb{A}_S^n = \text{Spec}(A[\underline{t}])$ , we are therefore in the conditions of 6.3.1. Consider the composite map<sup>53</sup>:

$$\sigma : B[\underline{t}] \rightarrow B \otimes_A B \xrightarrow{\mu} B$$

<sup>52</sup>The sign in the latter can be explained as:

$$(\bar{f}_1 \wedge \dots \wedge \bar{f}_n)^* = \bar{f}_n^* \wedge \dots \wedge \bar{f}_1^* = (-1)^{n(n-1)/2} \bar{f}_1^* \wedge \dots \wedge \bar{f}_n^*.$$

<sup>53</sup>Geometrically, this map corresponds to the graph  $\gamma_i : X \rightarrow X \times_S P$  of the closed immersion  $i : X \rightarrow P$ . As  $i$  is regular and  $P/S$  is smooth,  $\gamma_i$  is regular. Algebraically, it is just the map evaluating  $t_i$  at  $\alpha_i$ .



where  $\mu$  is the multiplication map, and the first arrow is the natural surjection coming from the identification  $B \otimes_A B = B \otimes_A A[t]/I = B[t]/I$ . We consider the ideals:

$$\begin{aligned} J &= \text{Ker}(\mu) \subset B \otimes_A B, \\ K &= \text{Ker}(\sigma) \subset B[t]. \end{aligned}$$

Then  $K$  admits a regular parametrization,  $K = (t_1 - \alpha_1, \dots, t_n - \alpha_n)$  and one gets  $J = K/I$  as obviously  $I \subset K$  as ideals of  $B[t]$ . Therefore, there exists polynomials  $c_{ij} \in B[t]$  such that

$$\forall i \in [1, n], f_i = \sum_{j=1}^n c_{ij} \cdot (t_j - \alpha_j).$$

Then the element

$$\Delta_f = \det(c_{ij}(\alpha)_{1 \leq i, j \leq n}) \in B \otimes_A B$$

is independent of the chosen polynomials  $c_{ij}$  (see [Kun08, Lemma 4.10]).

**Definition 6.3.4.** Consider the above notation:  $B/A$  is a finite projective complete intersection and  $f = (f_1, \dots, f_n) \in A[t]^n$  is a fixed presentation of  $B/A$ . Then the element  $\Delta_f \in B \otimes_A B$  is called the *Bezoutian* associated with the presentation  $f$  of  $B/A$ .

*Remark 6.3.5.* This definition is of course an extension of the classical Bezoutian (or rather the determinant of the Bezout matrix) arising from Euler and Bezout elimination theory, which corresponds to the case where  $A = k$  is a field and  $n = 2$ .

**6.3.6.** Consider again the setting of 6.3.3. We put  $B^* = \text{Hom}_A(B, A)$ , which is the (canonical) dual of the dualizable  $A$ -module  $B$  (see Definition 6.1.2 and what follow).<sup>54</sup> As  $B$  is dualizable, the canonical map:

$$\begin{aligned} \Phi : B \otimes_A B &\rightarrow \text{Hom}_A(B^*, B), \\ b \otimes b' &\mapsto (\varphi \mapsto \varphi(b) \cdot b') \end{aligned}$$

is an isomorphism. The following lemma is now a formality (see [Kun08, 8.13]; beware to translate the notation:  $I$  (resp.  $\omega_{B/A}$ ) in *loc. cit.* is what we denote by  $J$  (resp.  $B^*$ ) here.)

**Lemma 6.3.7.** *Consider the above notations. Recall that  $J = \text{Ker}(B \otimes_A B \xrightarrow{\mu} B)$ , seen as an ideal of  $B \otimes_A B$ . Then  $\Phi(\text{Ann}(J)) \subset \text{Hom}_A(B^*, B)$  and  $\Phi$  induces an isomorphism:*

$$\text{Ann}(J) \xrightarrow{\sim} \text{Hom}_A(B^*, B).$$

---

<sup>54</sup>Following the usage, we identify the set of morphisms  $\text{Hom}_A$  with the internal Hom-functor in the category of  $A$ -modules.

With this lemma in hands, we see that there exists a unique  $A$ -linear map  $\tau_f : B \rightarrow A$ , equivalently  $\tau_f \in B^*$  such that:

$$(6.3.7.a) \quad \Phi(\Delta_f)(\tau_f) = 1$$

**Definition 6.3.8.** Consider the above notation, as in Definition 6.3.4. We call the  $A$ -linear map  $\tau_f : B \rightarrow A$  the *Sheja-Storch trace map* associated with the presentation  $f$  of  $B/A$ .

**Example 6.3.9.** We consider the *monogenic* case:

$$B = A[\alpha] = A[t]/(f)$$

where  $f$  is a monic polynomial in one variable  $t$ :

$$f(t) = a_0 + \dots + a_{n-1} \cdot t^{n-1} + t^n.$$

Thus  $B$  is a free  $A$ -module with basis  $1, \alpha, \dots, \alpha^{n-1}$ . Then one can compute  $\Delta_f$  explicitly and one finds that

$$\tau_f = (\alpha^{n-1})^*, \alpha^i \mapsto \delta_{n-1}^i.$$

*Remark 6.3.10.* The map  $\tau_f$  does depend on the chosen generator  $\alpha$  of  $B/A$ , or more explicitly on the chosen presentation of  $B/A$ . Therefore, it is sometime customary to put:

$$\tau_{B/A}^\alpha = \tau_f.$$

In view of [Tat52, §1, (2)], corresponding to the case where  $B/A$  is an inseparable extension fields, the map  $\tau_f$  is sometimes called the *Tate trace map* (cf. [Kun08]).

**Theorem 6.3.11.** *Consider the assumption of the above definition.*

*Then the  $A$ -linear map  $\tau_f : B \rightarrow A$  is not  $B$ -torsion, and in fact is a  $B$ -basis of  $\text{Hom}_A(B, A)$ .*

*In other words, the symmetric bilinear form*

$$\varphi_f : B \otimes_A B \rightarrow A, b \otimes b' \mapsto \tau_f(bb')$$

*is non-degenerate: the associated map*

$$B \rightarrow B^* = \text{Hom}_A(B, A), \lambda \mapsto \lambda \cdot \tau_f = \varphi_f(b, -)$$

*is an isomorphism.*

*Proof.* Using the definitions of 6.3.3, one obtains that  $\text{Ann}(J)$  is a principal ideal domain generated by the Bezoutian  $\Delta_f$  (a result attributed to Wiebe, see [Kun08, Cor. 4.12]). According to the previous lemma,  $\text{Ann}(J) \simeq \text{Hom}_A(B^*, B)$  is also an invertible  $B$ -module. So  $\Delta_f$  is a  $B$ -basis of the  $\text{Ann}(J)$ . Relation (6.3.7.a) then implies that  $\tau_f$  is a  $B$ -basis of  $B^*$  as expected. The other assertions are formal consequences of this fact.  $\square$

We are now ready to state the link between the concrete construction of Sheja and Storch and the theory of Grothendieck residue symbols (Definition 6.3.2).

**Proposition 6.3.12.** *Recall the situation of the previous theorem and definition:*

- $B$  is a complete intersection, finite and projective  $A$ -algebra
- $f$  is a presentation of  $B/A$ :  $f = (f_1, \dots, f_n)$  is regular sequence of elements of  $R = A[t_1, \dots, t_n]$ ,  $I = (f_1, \dots, f_n)$  and  $B = R/I$ .

Recall that we have a canonical isomorphism (see Example 4.1.9)

$$\Theta : \omega_{B/A} \simeq \Lambda^n(I/I^2)^\vee \otimes_R \Omega_{R/A}^n.$$

Then, for any  $\lambda \in R$ , with image  $\bar{\lambda}$  in  $B = R/I$ , we get:

$$\text{Res}_{R/A} \begin{bmatrix} \lambda \cdot dt_1 \wedge \dots \wedge dt_n \\ f_1 \dots f_n \end{bmatrix} = \tau_f(\bar{\lambda}).$$

In other words, if we let  $w = (\bar{f}_1 \wedge \dots \wedge \bar{f}_n)^* \otimes i^*(dt_1 \wedge \dots \wedge dt_n)$  seen as an element of  $\omega_{B/A}$  via the isomorphism  $\Theta$ , for any  $b \in B$ , one gets:

$$(6.3.12.a) \quad \text{Tr}_{B/A}^\omega(b.w) = \tau_f(b).$$

Or equivalently, with the notation of 6.2.7:  $\psi_w = \tau_f$ .

*Proof.* In the case where  $A = k$  is a field (the only case we will need!), this is [Kun08, Prop. 8.32]. In general, one can reduce to this case by base change: we need to compare two trace maps associated with  $f : X = \text{Spec}(B) \rightarrow \text{Spec}(A) = S$  which is finite and syntomic. Both traces  $\text{Tr}_{B/A}^\omega$  and  $\tau_f$  are compatible under arbitrary base change (as syntomic finite morphisms are stable under base change) so that we can reduce to residue fields of  $S$ .

Alternatively, both definition of residues, via respectively Grothendieck and Sheja-Storch methods, satisfy the properties (R1)-(R10) of [Har66, III, §9] (see respectively [Con00, Appendix A] and [HL79, Hop83]). This uniquely characterizes the residue symbol.  $\square$

In view of Example 6.3.9, we deduce the following comparison of the Grothendieck differential trace map and the Tate trace map (Remark 6.3.10).

**Corollary 6.3.13.** *Suppose  $B/A$  is a monogenic extension ring, of the form  $B = A[t]/I$  where  $I = (f)$  for a monic polynomial  $f \in A[t]$ . We identify  $\omega_{B/A}$  with the  $B$ -module  $(I/I^2)^* \otimes_{A[t]} \Omega_{A[t]/A}$  (via the isomorphism  $\Theta$  of Example 4.1.9).*

Then for any  $b \in B$ , one gets:

$$(6.3.13.a) \quad \text{Tr}_{B/A}^\omega(b \cdot \bar{f}^* \otimes dt) = \tau_{B/A}^\alpha(b)$$

with the notation of Remark 6.3.10.

**Corollary 6.3.14.** *Let  $B$  be a finite étale  $A$ -algebra. Then  $\omega_{B/A} = B$  and the following diagram commutes:*

$$\begin{array}{ccc} \omega_{B/A} & \xrightarrow{\text{Tr}_{B/A}^\omega} & A \\ \parallel & \searrow & \\ B & \xrightarrow{\text{Tr}_{B/A}} & A \end{array}$$

where  $\mathrm{Tr}_{B/A}$  is the "usual" trace map (Definition 6.1.9).

*Proof.* This is asserted without proof in [Har66, Remark p. 187]. As both trace maps are compatible with composition, the proof reduces to the case where  $B/A$  is monogenic,  $B = A[\alpha] = A[t]/(f)$ ,  $f$  being a monic polynomial in one variable  $t$  such that  $f'(\alpha) \in B^\times$ . Note that under the identification

$$\Theta : B = \omega_{B/A} \simeq (I/I^2)^* \otimes \Omega_{A[t]/A}^1$$

one has  $\Theta^{-1}(\bar{f} \otimes dt) = f'(\alpha)^{-1}$  (as explained in Example 4.1.10, separable case). Example 6.3.9 shows that  $\tau_f = (\alpha^{n-1})^*$  where  $n$  is the degree of  $f$  in  $t$ .

Therefore, the relation of the corollary follows from the previous corollary and the "Euler formula" (see e.g. [NS03, Prop. 1]):

$$\mathrm{Tr}_{B/A}(f'(\alpha)^{-1}\lambda) = (\alpha^{n-1})^*(\lambda).$$

□

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