Large-time optimal observation domain for linear parabolic systems

Idriss Mazari, Yannick Privat, Emmanuel Trélat

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Abstract

Given a well-posed linear evolution system settled on a domain Ω of $\mathbb{R}^d$, an observation subset $\omega \subset \Omega$ and a time horizon T, the observability constant is defined as the largest possible nonnegative constant such that the observability inequality holds for the pair (ω, T). In this article we investigate the large-time behavior of the observation domain that maximizes the observability constant over all possible measurable subsets of a given Lebesgue measure. We prove that it converges exponentially, as the time horizon goes to infinity, to a limit set that we characterize. The mathematical technique is new and relies on a quantitative version of the bathtub principle.

Keywords: parabolic equations, observability, shape optimization, quantitative bathtub principle.

AMS classification: 93B07, 35L05, 49K20, 42B37.

1 Introduction

Let $d \in \mathbb{N}^*$ and let Ω be an open bounded smooth connected subset of $\mathbb{R}^d$. Let $q \in \mathbb{N}^*$. We consider the evolution system

$$\partial_t y + A_0 y = 0 \tag{1}$$

where $-A_0 : D(A_0) \to L^2(\Omega, \mathbb{C}^q)$ is a densely defined operator generating a strongly continuous semigroup on $L^2(\Omega, \mathbb{C}^q)$. Given any $y^0 \in D(A_0)$, there exists a unique solution $y \in C^0([0, +\infty), D(A_0)) \cap C^1((0, +\infty), L^2(\Omega, \mathbb{C}^q))$ of (1) such that $y(0, \cdot) = y^0(\cdot)$.

For an evolution system like (1), it is often required in practice, for instance in engineering problems, to reconstruct initial or final data of solutions, based on partial measurements performed on a subset ω of Ω over a a time horizon T. This inverse problem is feasible as soon as the system is observable on ω in time T, which is mathematically modelled by an *observability inequality*, as follows. For any measurable subset ω of Ω and any $T > 0$, the system (1) is said to be *observable* on ω in time T if there exists $C > 0$ such that

$$C\|y(T, \cdot)\|^2_{L^2(\Omega, \mathbb{C}^q)} \leqslant \int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt \tag{2}$$

for every solution of (1) such that $y(0, \cdot) \in D(A_0)$. In (2), $|y|$ is the Hermitian norm of $y \in \mathbb{C}^q$.

Many results have been established in the literature regarding observability properties for some classes of partial differential equations. Since, in this article, we are going to focus on the case of *parabolic* equations

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(see Section 2.1 for assumptions on A_0), here we simply quote few of them. For the heat equation, that is, when $-A_0$ is the Dirichlet Laplacian, observability holds true for any $T > 0$ and any subset of positive Lebesgue measure (see [18, 25] for open subsets and see [3] for the extension to measurable subsets). The same holds true for anomalous diffusion equations, that is, when A_0 is some power $\alpha > 1/2$ of the Dirichlet Laplacian (see [32]), and for the Stokes equation (see [16, Lemma 1] and [9]). We also refer to [45, Chapter 9] for some general results establishing observability for parabolic systems.

The *observability constant* is defined as the largest possible nonnegative constant for which the observability inequality (2) holds, i.e., by

$$C_T(\mathbb{1}_\omega) = \inf \left\{ \frac{\int_0^T \int_\Omega \mathbb{1}_\omega(x) |y(t,x)|^2 \, dx \, dt}{\|y(T,\cdot)\|^2_{L^2(\Omega,\mathbb{C}^q)}} \;\Big|\; y^0 \in D(A_0) \setminus \{0\} \right\}, \tag{3}$$

where $\mathbb{1}_\omega$ denotes the characteristic function of ω. Note that (1) is observable on ω in time T if and only if $C_T(\mathbb{1}_\omega) > 0$. The observability constant defined by (3) provides an account, in some sense, for the well-posedness of the above-mentioned inverse problem: the larger the constant, the more favorable it is to solve the inverse problem. This is why it is important, in practice, to try to choose the observation subset ω in an optimal way and, when it is possible, the horizon of time T over which observations are performed. Of course, in practice, there are limitations on the choice of the subset ω, starting with its Lebesgue measure. This is why, in what follows, we will consider measurable subsets of a given measure.

Let $L \in (0,1)$ be fixed. Given any $T > 0$, we consider the optimal design problem

$$\boxed{\delta_T = \sup_{\mathbb{1}_\omega \in \mathcal{U}_L} C_T(\mathbb{1}_\omega)} \tag{4}$$

consisting of maximizing the observability constant over the set

$$\mathcal{U}_L = \{\mathbb{1}_\omega \in L^\infty(\Omega; \{0,1\}) \mid \omega \text{ is a measurable subset of } \Omega \text{ of Lebesgue measure } |\omega| = L|\Omega|\}. \tag{5}$$

This shape optimization problem models the best shape and placement of sensors for the evolution system (1) in time T. If an optimal set $\omega_T^\star$ exists, then it represents the best possible place to install some (adequately shaped) sensors.

Unfortunately, the shape optimization problem (5) is very difficult to handle, not only because of the complexity of the definition of the observability constant (3), which is an infimum over all possible solutions of (1), but also because of the set of unknowns $\mathcal{U}_L$, which is very large and does not enjoy good compactness properties. Actually, it is not even known if a maximizer $\omega_T^\star$ exists for (4).

To remedy this lack of compactness, some constraints may be added. The search of an optimal set may be restricted to a compact finite-dimensional set, like in [24, 33, 46, 47], or by adding constraints on the BV norm, on the number of connected components, or anything yielding compactness (see, e.g., [20]), or by fixing the initial data, like in [17, 35, 38]. However, we want to keep the maximal freedom on the choice of the observation subset and this is why we consider the general set $\mathcal{U}_L$ defined by (5).

This choice comes at a price: the set $\mathcal{U}_L$ is not even closed for the weak-star topology of $L^\infty(\Omega)$, thus making even the basic question of existence of an optimizer challenging. Such a question is well known in shape optimization, and this issue is often tackled by considering a relaxed version of the problem. Here, the relaxation procedure consists of extending the definition of C_T to the L^∞ weak-star closure of $\mathcal{U}_L$, which is the set

$$\overline{\mathcal{U}}_L = \left\{ a \in L^\infty(\Omega; [0,1]) \mid \int_\Omega a(x) \, dx = L|\Omega| \right\}$$

and to define, in accordance with (3),

$$C_T(a) = \inf \left\{ \frac{\int_0^T \int_\Omega a(x)|y(t,x)|^2 \, dx \, dt}{\|y(T,\cdot)\|^2_{L^2(\Omega,\mathbb{C}^q)}} \;\middle|\; y^0 \in D(A_0) \setminus \{0\} \right\} \qquad \forall a \in \overline{\mathcal{U}}_L. \tag{6}$$

Then, we consider the relaxed shape optimization problem

$$\boxed{\bar{\delta}_T = \sup_{a \in \overline{\mathcal{U}}_L} C_T(a)} \tag{$\mathscr{P}_T$}$$

Now, since $\overline{\mathcal{U}}_L$ is compact for the L^∞ weak-star topology (see [20, Prop. 7.2.17]) and C_T is upper semi-continuous (as an infimum of bounded linear functionals), it follows that Problem ($\mathscr{P}_T$) has at least one solution $a_T^\star$. Characterizing $a_T^\star$ is more intricate. In this article, we will study $a_T^\star$ as $T \to \infty$ and also give some results on its small-time asymptotics as $T \to 0$.

Of course, we have $\delta_T \leqslant \bar{\delta}_T = C_T(a_T^\star)$ but we do not know, in general, whether the inequality may be strict or not. Anyhow, observe the following *a priori* counter-intuitive result: in some cases, the constant density $a \equiv L$ is not a maximizer (see [37, Prop. 2]).

To tackle the difficulty of optimizing the observability constant (3), or its relaxed version (6), the point of view adopted in the series of papers [36, 37, 39, 40, 41, 42, 43] consisted in considering a *randomized* version $C_{T,\mathrm{rand}}(a)$ of the observability constant, defined by taking the infimum in (6) not on all but on *almost all* initial data in an appropriate sense, thus yielding a more tractable expression of the functional, namely,

$$C_{T,\mathrm{rand}}(a) = \inf_{j \in \mathbb{N}^*} \gamma_j(T) \int_\Omega a(x)|\phi_j(x)|^2 \, dx, \tag{7}$$

where $(\phi_j)_{j\in\mathbb{N}^*}$ is a Hilbert basis of eigenfunctions of A_0 and $(\gamma_j(T))_{j\in\mathbb{N}^*}$ is a sequence of positive coefficients depending on the operator. We have $C_{T,\mathrm{rand}}(a) \geqslant C_T(a)$ for every $a \in \overline{\mathcal{U}}_L$ and the inequality may be strict [39, Theorem 1]. The randomized observability constant $C_{T,\mathrm{rand}}(a)$ is in some sense "less pessimistic" than the deterministic observability constant $C_T(a)$ which provides an account for the worst observation cases. From the harmonic analysis point of view, the constant $C_{T,\mathrm{rand}}(a)$ reflects the independent observations on each mode, with no interaction between them, while the constant $C_T(a)$ also takes into account all interactions between modes, through crossed terms in a spectral expansion of solutions.

In the above-mentioned series of papers, the problem of maximizing the randomized (relaxed or not) observability constant has been solved for various classes of evolution equations, showing strong differences between parabolic and hyperbolic cases. The results have revealed a close relationship with asymptotic properties of high frequency eigenfunctions (quantum ergodicity properties). Roughly speaking, it has been shown that, in hyperbolic cases, all modes count and a maximizer does not exist in general, while in parabolic cases, there exists a unique maximizing domain which is moreover characterized and computable from a finite number of modes only.

The question of maximizing the deterministic (*i.e.* without randomization) observability constant $C_T(a)$ has remained unsolved theoretically, although it has been investigated numerically in 1D (see [4, 34]).

Although we do not solve this difficult problem, the present article takes us one step further by studying the large-time behavior of the optimal observability constant for parabolic evolution systems and of the associated maximizers. It is interesting to note that the authors of [1] studied the relaxed problem for two-phase heat equations and proved that the relaxed maximizers converge, as $T \to +\infty$, to relaxed maximizers of the corresponding stationary problem.

In Section 2, we state our main results concerning the large-time asymptotics of the relaxed maximizers. We prove in Theorems 1 and 2 that, in large time T, maximizing $C_T(a)$ over $\overline{\mathcal{U}}_L$ is approximately equivalent

to maximizing the lowest eigenvalue $\sigma_1(a)$ of a matrix – when A_0 is selfadjoint and its lowest eigenvalue λ_1 is simple then $\sigma_1(a) = \int_\Omega a|\phi_1|^2$ where ϕ_1 is a normalized eigenfunction of $-A_0$ associated to λ_1. The latter limit problem has a unique solution $a_1 = \mathbb{1}_{\omega^*}$, with ω^* enjoying nice regularity properties. We prove that any maximizer $a_T^\star \in \overline{\mathcal{U}}_L$ of C_T converges to $\mathbb{1}_{\omega^*}$ in L^1 norm, exponentially as $T \to +\infty$ at a rate given by the spectral gap of the operator A_0. To determine the speed of convergence, we use an improved, quantitative version of the so-called *bathtub principle*, inspired by [31] and [8, Prop. 2.7]. The use of this technique is one of the main novelties of the present paper.

We provide in Section 2.2 several examples and we comment on the assumptions under which our main results are established. Following a relevant comment by Sylvain Ervedoza during a presentation of one of us, we comment in Section 2.3 on the strategy of proof that we develop, showing that, although the main result is perfectly intuitive, a naive approach of proof is bound to fail, and we have to resort to a more elaborate technique using kinds of quasi-maximizers.

The proofs of the main results are done in Section 3.

In the last Section 4, we comment on related problems: first, the optimal controllability domain problem, which is dual to the one investigated in this paper; second, in contrast to what is developed here, the small-time asymptotics of the optimal observability problems. Finally we give some open problems.

2 Large-time behavior of maximizers

2.1 Main results

Throughout the paper, given $y = (y_1, \dots, y_q)$ and $z = (z_1, \dots, z_q)$ in $\mathbb{C}^q$, we denote by $|y| = (\sum_{i=1}^q |y_i|^2)^{1/2}$ the Hermitian norm and by $y \cdot \bar{z} = \sum_{i=1}^q y_i \bar{z}_i$ the Hermitian inner product.

We consider the following assumptions:

($\mathbf{H_1}$) There exists a Hilbert basis $(\phi_j)_{j\in\mathbb{N}^*}$ of $L^2(\Omega, \mathbb{C}^q)$ consisting of (complex-valued) eigenfunctions of A_0, associated with eigenvalues $(\lambda_j)_{j\in\mathbb{N}^*}$ of finite multiplicity such that

$$\mathrm{Re}(\lambda_1) \leqslant \cdots \leqslant \mathrm{Re}(\lambda_j) \leqslant \cdots \to +\infty \quad \text{as } j \to +\infty.$$

($\mathbf{H_2}$) For every $j \in \mathbb{N}^*$, the eigenfunction ϕ_j is analytic in Ω.

We define

$$J_1 = \{j \in \mathbb{N}^* \mid \mathrm{Re}\,\lambda_j = \mathrm{Re}\,\lambda_1\},$$

$$p_0 = \min\{p \in \mathbb{N}^* \mid \mathrm{Re}\,\lambda_p > \mathrm{Re}\,\lambda_1\} = \min(\mathbb{N}^* \backslash J_1) \geqslant 2.$$

The positive real number $\mathrm{Re}\,\lambda_{p_0} - \mathrm{Re}\,\lambda_1$ is the *spectral gap* of the operator A_0.

($\mathbf{H_L}$) (*Bathtub property at level L*) For every $j \in J_1$, the function ϕ_j belongs to $C^1(\overline{\Omega})$. Given any real-valued function Φ on Ω written as

$$\Phi = \sum_{p\in J_1} \eta_p \left| \sum_{j\in J_1} \beta_{j,p}\phi_j \right|^2$$

for some $(\beta_{j,p})_{j,p\in J_1} \in \mathbb{R}^{N_1\times P}\backslash\{0\}$, $\{\eta_p\}_{p\in J_1} \in [0,1]^P$ such that $\sum_{p\in J_1} \eta_p = 1$, the set $\omega = \{\Phi \geqslant \nu\}$ where $\nu > 0$ is such that $|\omega| = L|\Omega|$, satisfies

$$\inf_{\partial\omega\backslash\partial\Omega} |\nabla\Phi| > 0.$$

In Section 2.2, we will discuss these assumptions, give various examples, and show that Assumption ($\mathbf{H_L}$) cannot be weakened.

This is because of Assumptions ($\mathbf{H_1}$) and ($\mathbf{H_2}$) that we have used, in this paper and in particular in its title, the word "parabolic". However, the word may be a bit of a misnomer because it often means that the semigroup generated by $-A_0$ is analytic. However, there exist some analytic hypoelliptic operators satisfying all above assumptions, while the semigroup they generate is not analytic [11, Remark 4.1].

Under ($\mathbf{H_1}$), for any $y_0 \in D(A_0)$, the solution y of (1) such that $y(0,\cdot) = y_0$ can be expanded as

$$y(t,x) = \sum_{j=1}^{+\infty} a_j e^{-\lambda_j t}\phi_j(x),$$

where

$$a_j = \int_\Omega y^0(x)\overline{\phi_j}(x)\,dx \qquad \forall j \in \mathbb{N}^*.$$

Using the change of variable $b_j = a_j e^{-\lambda_j T}$ and a homogeneity argument, we have

$$C_T(a) = \inf_{\sum_{j=1}^{+\infty}|b_j|^2=1} \int_0^T\int_\Omega a(x)\left|\sum_{j=1}^{+\infty} b_j e^{\lambda_j t}\phi_j(x)\right|^2 dx\,dt. \tag{8}$$

Before stating our main results, let us give some notations. For any $j \in \mathbb{N}^*$, set

$$\gamma_j(T) = \begin{cases} \dfrac{e^{2\,\mathrm{Re}(\lambda_j)T}-1}{2\,\mathrm{Re}(\lambda_j)} & \text{if} \quad \mathrm{Re}(\lambda_j) \neq 0, \\ T & \text{if} \quad \mathrm{Re}(\lambda_j) = 0. \end{cases}$$

We define the Hermitian matrix

$$M_1(a) = \left(\int_\Omega a(x)\,\phi_i(x)\cdot\overline{\phi_j(x)}\,ds\right)_{i,j\in J_1}$$

and we denote by $\sigma_1(a)$ the lowest eigenvalue of $M_1(a)$. When $\#J_1 = 1$, i.e., $J_1 = \{1\}$ (this is the case if A_0 is selfadjoint and λ_1 is simple), then $\sigma_1(a) = M_1(a) = \int_\Omega a|\phi_1|^2$.

Theorem 1. *Under* ($\mathbf{H_1}$), *we have*

$$\bar\delta_T \sim \gamma_1(T)\max_{\mathbb{1}_\omega\in\mathcal{U}_L}\sigma_1(\mathbb{1}_\omega) \quad \textit{as} \quad T\to+\infty. \tag{9}$$

Under ($\mathbf{H_1}$), ($\mathbf{H_2}$) *and* ($\mathbf{H_L}$), *any solution* $a_T^\star \in \overline{\mathcal{U}}_L$ *of Problem* ($\mathscr{P}_T$) *satisfies*

$$\left\|a_T^\star - \operatorname*{argmax}_{a\in\overline{\mathcal{U}}_L}\sigma_1(a)\right\|_{L^1(\Omega)} = \mathrm{O}\left(e^{-(1-\xi)\,\mathrm{Re}(\lambda_{p_0}-\lambda_1)T/2}\right) \qquad \forall \xi > 0. \tag{10}$$

Remark 1. Note that (9) makes sense since

$$\max_{\mathbb{1}_\omega\in\mathcal{U}_L}\sigma_1(\mathbb{1}_\omega) = \max_{a\in\overline{\mathcal{U}}_L}\sigma_1(a) \geqslant \sigma_1(L) = L > 0.$$

Theorem 1 highlights the interest of the shape optimization problem

$$\max_{a\in\overline{\mathcal{U}}_L}\sigma_1(a). \tag{$\mathscr{P}_{\sigma_1}$}$$

Theorem 2. *Under* $(\mathbf{H_1})$ *and* $(\mathbf{H_L})$*, the problem* $(\mathscr{P}_{\sigma_1})$ *has a unique*[1] *solution* $a_1 = \mathbb{1}_{\omega^*} \in \mathcal{U}_L$ *satisfying the following properties:*

- *There exists* $\mu^* > 0$ *such that*

$$\omega^* = \left\{ x \in \Omega \ \middle| \ \sum_{k \in J_1} \alpha_k \left| \sum_{j \in J_1} b_j^k \phi_j(x) \right|^2 > \mu^* \right\}, \tag{11}$$

  for some $(\alpha_k)_{k \in J_1} \in [0,1]^{\#J_1}$ *and* $(b_j^k)_{j \in J_1} \in \mathbb{C}^{\#J_1}$ *such that*

$$\sum_{j \in J_1} \alpha_j = \sum_{j \in J_1} |b_j^k|^2 = 1 \qquad \forall k \in J_1.$$

- *There exists* $K > 0$ *such that*

$$\sigma_1(a_1) \geqslant \sigma_1(a) + K \|a - a_1\|^2_{L^1(\Omega)} \qquad \forall a \in \overline{\mathcal{U}}_L. \tag{12}$$

- *If in addition* $(\mathbf{H_2})$ *holds, then* ω^* *is semi-analytic*[2]*. In particular, it has a finite number of connected components.*

Remark 2. Concerning the randomized observability constant defined by (7), it follows from some results in [39] that, for T is large enough,

$$\sup_{a \in \overline{\mathcal{U}}_L} C_{T,\mathrm{rand}}(a) = \gamma_1(T) \max_{\mathbb{1}_\omega \in \mathcal{U}_L} \sigma_1(\mathbb{1}_\omega).$$

Therefore, Theorem 1 shows that, in large time, maximizing the deterministic observability constant is almost equivalent to maximizing the randomized observability constant.

Remark 3. The exponential rate of convergence (10) in Theorem 1 is established thanks to a *quantitative* version of the *bathtub principle*. The bathtub principle is an elementary inequality yielding the solutions of the shape optimization problem

$$\sup_{\omega \in \mathcal{U}_L} \int_\omega f$$

where f is a given real-valued integrable function on Ω. In recent years, quantified versions of this principle were derived and used in optimal control problems. We will build on the version stated in [31, Proposition 24]. We refer to Section 4.1 for more background on this inequality but we merely state here that, while it is now a known tool for optimal control theory, it is to the best of our knowledge the first time that such quantified inequalities are used in the study of observability constants.

Remark 4. In the theory of rearrangements, the quantitative estimate (12) can be seen as a strengthened Hardy-Littlewood inequality; to the best of our knowledge, the first paper to investigate such strengthening was [10]. However, the results of [10] do not apply here, as they are given in a different functional setting

[1] Here and in the sequel, it is understood that the optimal set is unique within the class of all measurable subsets of Ω quotiented by the set of all measurable subsets of Ω of zero measure.

[2] A subset ω of a real analytic finite dimensional manifold M is said to be semi-analytic if it can be written in terms of equalities and inequalities of analytic functions. Recall that semi-analytic subsets are Whitney stratifiable (see [19, 21]) and enjoy local finiteness properties, such that: local finite perimeter, locally finite number of connected components, etc.

(roughly speaking, with $L^p - L^q$ $(1 < p < \infty, q = p/(p-1))$ constraints rather than $L^\infty - L^1$). The quantitative estimate (12) can take different forms. We refer to [6, Lemma 7.73], [31, Proposition 24], for similar spectral quantitative inequalities in a different context and to [27, Theorem 1] where such inequalities are used to study the stability of Vlasov-Poisson systems. The proofs of the inequality in the three latter references revolve around the same idea, that of partial Schwarz rearrangements. Finally, we mention [8, Proposition 2.7] where similar inequalities are obtained in the context of stability of bang-bang optimal controls, with a more direct proof that we will follow.

2.2 Examples and comments on the assumptions

The Dirichlet-Laplacian operator. Take $q = 1$ and $-A_0$ to be the Dirichlet-Laplacian operator defined on its domain $D(A_0) = H_0^1(\Omega) \cap H^2(\Omega)$. It follows from the spectral theorem [7, Section 6.4] that $(\mathbf{H_1})$ is satisfied. By hypoelliptic analyticity of the Laplacian, the eigenfunctions of A_0 are analytic in Ω (and we take them real-valued) and $(\mathbf{H_2})$ is satisfied. Regarding $(\mathbf{H_L})$, note that the first eigenvalue $\lambda_1(\Omega)$ of A_0 is simple (hence, $J_1 = \{1\}$) according to the Krein-Rutman theorem and the eigenfunction ϕ_1 is positive in Ω. By (11), we have $\omega^* = \{\phi_1^2 > \mu^*\}$ and, by analyticity of ϕ_1, ω^* satisfies the interior sphere property (see [20]), $\partial\omega$ is at a positive distance of $\partial\Omega$ and, according to the Hopf maximum principle,

$$\inf_{\partial\omega} \|\nabla\phi_1\| > 0.$$

Hence $(\mathbf{H_L})$ is satisfied.

The Robin-Laplacian operator. Take $q = 1$, $\beta > 0$ and $-A_0$ to be the Robin-Laplacian operator with boundary condition $\partial_n y + \beta y = 0$ on $\partial\Omega$. By similar arguments, $(\mathbf{H_1})$ and $(\mathbf{H_L})$ are satisfied. The analyticity property $(\mathbf{H_2})$ follows for instance from [15, Section V.4].

The Dirichlet Stokes operator in the 2D unit disk. Consider the Stokes equation

$$\partial_t y - \triangle y + \nabla p = 0, \quad \operatorname{div} y = 0$$

in $\Omega = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$, the Euclidean unit disk of $\mathbb{R}^2$, with Dirichlet boundary conditions. The Stokes operator $A_0 : D(A_0) \to H$ is defined by $A_0 = -\mathcal{P}\triangle$, with $D(A_0) = \{y \in V \mid A_0 y \in H\}$, with $V = \{y \in (H_0^1(\Omega))^2 \mid \operatorname{div} y = 0\}$, $H = \{y \in (L^2(\Omega))^2 \mid \operatorname{div} y = 0,\ y_{|\partial\Omega}.n = 0\}$ and $\mathcal{P} : (L^2(\Omega))^2 = H \overset{\perp}{\oplus} H^\perp \to H$ the Leray projection (see [5]). The first eigenfunction is given by

$$\phi_{0,1}(r,\theta) = \frac{-J_0'(\sqrt{\lambda_{0,1}}r)}{\sqrt{\pi}\sqrt{\lambda_{0,1}}|J_0(\sqrt{\lambda_{0,1}})|} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

in polar coordinates (r,θ) (see [23, 26]), where J_0 is the Bessel function of the first kind of order 0 and $\lambda_{0,1} = z_{1,1}^2$ with $z_{1,1} > 0$ is the first positive zero of J_0. Using that $\phi_{0,1}$ is radially-symmetric, it follows from properties of the Bessel functions that $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_L})$ are satisfied.

System of coupled heat equations in the cube. Setting $Y = (y_1, y_2, y_3)^\top$, consider the system of coupled 1D heat equations

$$\partial_t Y - \triangle Y + AY = 0,$$

in $\Omega = (0,1)^3$, with Dirichlet boundary conditions, where $\triangle Y = (\triangle y_1, \triangle y_2, \triangle y_3)^\top$, where $A \in M_3(\mathbb{C})$ is a 3×3 matrix with three distinct complex eigenvalues μ_1, μ_2, μ_3 and such that $-\pi^2 < \operatorname{Re}\mu_1 = \operatorname{Re}\mu_2 < \operatorname{Re}\mu_3$. Let $E_i = \operatorname{Span}(u_i)$, with $u_i \in \mathbb{C}^3$, be the eigenspace of A associated to μ_i, for $i \in \{1,2,3\}$. Set $\rho = \operatorname{Re}\mu_1$.

The above system is known to be observable in any time $T > 0$ on any nonempty open subset ω of Ω. We refer for instance to [2, Theorem 1.1] or [30, Proof of the first point of Theorem 1].

Here, we have $q = 3$, $A_0 = \triangle + A\,\mathrm{Id}$ on $D(A_0) = (H^2(\Omega)\cap H_0^1(\Omega))^3$, $J_1 = \{1,2\}$, $p_0 = 3$ and $\gamma_1(T) = (e^{2\rho T}-1)/(2\rho)$. The assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$ are satisfied, and $\phi_1(x) = \sin(\pi x_1)\sin(\pi x_2)u_1$ and $\phi_2(x) = \sin(\pi x_1)\sin(\pi x_2)u_2$. We define

$$\Phi(x) = \eta \sin^2(\pi x_1)\sin^2(\pi x_2)\left|\beta_{11}u_1 + \beta_{12}u_2\right|^2 + (1-\eta)\sin^2(\pi x_1)\sin^2(\pi x_2)\left|\beta_{21}u_1 + \beta_{22}u_2\right|^2,$$

where $\eta \in [0,1]$ and $\beta_{ij} \neq 0$. Reasoning as before, and using that $\beta_{11}u_1 + \beta_{12}u_2 \neq 0$ and $\beta_{21}u_1 + \beta_{22}u_2 \neq 0$ (by linear independence of u_1 and u_2), we get that $|\nabla\Phi|$ is uniformly bounded below on any nontrivial level set of Φ, and thus $(\mathbf{H_L})$ is satisfied.

On the sharpness of $(\mathbf{H_L})$: periodic Laplacian in the 1D torus. Take $\Omega = (0,2\pi)$, $q = 1$ and $A_0 = -d^2/dx^2$ with periodic boundary conditions $y(0) = y(2\pi)$ and $y'(0) = y'(2\pi)$. The first eigenvalue is $\lambda_1 = 1$ with eigenspace spanned by $x \mapsto \cos x$ and $x \mapsto \sin x$. For every $a \in \overline{U}_L$, we have

$$M_1(a) = \begin{pmatrix} \int_0^{2\pi} a(x)\cos^2 x\,dx & \int_0^{2\pi} a(x)\cos x\sin x\,dx \\ \int_0^{2\pi} a(x)\cos x\sin x\,dx & \int_0^{2\pi} a(x)\sin^2 x\,dx \end{pmatrix},$$

and thus

$$\sigma_1(a) = \frac{1}{2}\int_0^{2\pi} a(x)\,dx - \frac{1}{2}\sqrt{\left(\int_0^{2\pi} a(x)\cos(2x)\,dx\right)^2 + \left(\int_0^{2\pi} a(x)\sin(2x)\,dx\right)^2}.$$

Therefore, the optimal design problem $(\mathscr{P}_{\sigma_1})$ (maximizing σ_1 over $\overline{\mathcal{U}}_L$) is equivalent to

$$\max_{a\in\overline{\mathcal{U}}_L} \left(\int_0^{2\pi} a(x)\cos(2x)\,dx\right)^2 + \left(\int_0^{2\pi} a(x)\sin(2x)\,dx\right)^2.$$

It is easy to see that the optimal value for this problem is 0, meaning that the maximal value of σ_1 is $\frac{L\pi}{2}$. Moreover, the maximal value is reached for any function a given by

$$a(x) = \frac{L}{2} + \eta\left(a_1^c\cos x + a_1^s\sin x + \sum_{k=3}^{m}(a_k^c\cos(kx) + a_k^s\sin(kx))\right)$$

for any $m \in \overline{\mathbb{N}}\setminus\{0,1,2\}$ and any families of real numbers $(a_k^s)_{1\leqslant k\leqslant m}$ and $(a_k^c)_{1\leqslant k\leqslant m}$, with η chosen small enough such that $a \in \overline{\mathcal{U}}_L$. Hence, there exist maximizers in $\overline{\mathcal{U}}_L\setminus\mathcal{U}_L$ and the conclusion of Theorem 2 is not true in that case. The main reason is that $(\mathbf{H_L})$ is not satisfied here. Indeed, one has $\#J_1 = 2$ and it is possible to choose adequately the coefficients defining Φ so that $\Phi(x) = \cos^2 x + \sin^2 x = 1$, and its normal derivative vanishes on $\partial\omega$.

2.3 Comments on the strategy of proof

In this section, we describe our strategy of proof, highlighting the difficulties that will be encountered, on the example of the Dirichlet heat equations. We take $q = 1$ and $-A_0$ equal to the Dirichlet-Laplacian on

$D(A_0) = H_0^1(\Omega) \cap H^2(\Omega)$. The operator A_0 is selfadjoint, of compact resolvent and then its spectrum is real and the first eigenvalue $\lambda_1(\Omega)$ is simple, and we take real-valued eigenfunctions.

Recalling that $\bar\delta_T$ is the optimal value for Problem ($\mathscr{P}_T$), defined by ($\mathscr{P}_T$), considering particular choices of coefficients b_j in (8), we have

$$\bar\delta_T \;\leqslant\; \gamma_1(T) \max_{\mathbb{1}_\omega \in \mathcal{U}_L} \int_\Omega a(x)\phi_1(x)^2\,dx \qquad \forall a \in \overline{\mathcal{U}}_L \tag{13}$$

(see the beginning of Section 3.2 for details).

Let us now establish a lower bound for $\bar\delta_T$. Let $a_1^T \in \overline{\mathcal{U}}_L$ be the unique solution to the shape optimization problem on the right-hand side of (13). We define

$$E_{T,a}(b) = \int_0^T \int_\Omega a(x) \Big(\sum_{j=1}^{+\infty} b_j e^{\lambda_j t} \phi_j(x) \Big)^2 dx\,dt$$

and we set $\varepsilon = |b_1|^2$ and $c_j = b_j/\sqrt{1-\varepsilon}$ for $j \geqslant 2$, so that $\sum_{j=2}^{+\infty} |c_j|^2 = 1$. A straightforward computation gives

$$E_{T,a}(b) = \varepsilon A_{T,a}(c) + (1-\varepsilon) B_{T,a}(c) + 2\sqrt{\varepsilon(1-\varepsilon)} D_{T,a}(c)$$

where

$$\begin{aligned}
A_{T,a}(c) &= \gamma_1(T) \int_\Omega a\phi_1^2, \\
B_{T,a}(c) &= \int_0^T \int_\Omega a(x) \Big(\sum_{j=2}^{+\infty} c_j e^{\lambda_j t} \phi_j(x) \Big)^2 dx dt, \\
D_{T,a}(c) &= \int_0^T \int_\Omega a(x) e^{\lambda_1 t} \phi_1(x) \sum_{j=2}^{+\infty} c_j e^{\lambda_j t} \phi_j(x)\,dx\,dt = \sum_{j=2}^{+\infty} c_j \frac{e^{(\lambda_j+\lambda_1)T}-1}{\lambda_1+\lambda_j} \int_\Omega a\phi_1\phi_j.
\end{aligned}$$

From the maximality of $\bar\delta_T$, we have

$$\bar\delta_T \geqslant \inf_{\sum_{j=2}^{+\infty} c_j^2 = 1} \min_{\varepsilon \in [0,1]} \varepsilon A_{T,a}(c) + (1-\varepsilon) B_{T,a}(c) + 2\sqrt{\varepsilon(1-\varepsilon)} D_{T,a}(c)$$

Due to the presence of exponential terms, we could expect to be able to exploit this decomposition to prove that the first mode dominates the others when T is sufficiently large. This does not work however, essentially because the modes $j \geqslant 2$ in the decomposition can modify the observability constant in large time. To understand this difficulty, note that, by the Cauchy-Schwarz inequality,

$$|D_{T,a_1^T}(c)| \leqslant \sqrt{A_{T,a_1^T}(c) B_{T,a_1^T}(c)}.$$

Therefore

$$\bar\delta_T \geqslant \min_{\varepsilon \in [0,1]} \Big(\varepsilon A_{T,a_1^T}(c) + (1-\varepsilon) B_{T,a_1^T}(c) - 2\sqrt{\varepsilon(1-\varepsilon)} \sqrt{A_{T,a_1^T}(c) B_{T,a_1^T}(c)} \Big) = 0.$$

The latter equality follows from the fact that the minimum is equal to the first eigenvalue of the matrix

$$\begin{pmatrix} A_{T,a_1^T}(c) & \sqrt{A_{T,a_1^T}(c) B_{T,a_1^T}(c)} \\ \sqrt{A_{T,a_1^T}(c) B_{T,a_1^T}(c)} & B_{T,a_1^T}(c) \end{pmatrix}.$$

Unfortunately, this estimate is not useful. The difficulty here is due to the presence of crossed terms that are difficult to handle. To overcome it, we will consider a particular path of "quasi-maximizers", replacing a_1^T by a convex combination of the expected maximizer in large time and the constant function equal to L. We will choose appropriate convexity weights, to give an increasing importance to the term a_1^T as $T \to +\infty$.

3 Proofs

3.1 Proof of Theorem 2

First of all, noting that, for every $a \in \overline{\mathcal{U}}_L$,

$$\sigma_1(a) = \min_{\sum_{j\in J_1}|b_j|^2=1} \int_\Omega a(x) \left| \sum_{j\in J_1} b_j \phi_j(x) \right|^2 dx$$

is as minimum of linear continuous functionals, σ_1 is upper semicontinuous for the weak star topology of L^∞. By compactness of $\overline{\mathcal{U}}_L$ for this topology, it follows that ($\mathscr{P}_{\sigma_1}$) has at least one solution a_1.

We are going to prove that a_1 is unique and that $a_1 = \mathbb{1}_{\omega^*}$ for some subset ω^* ("bang-bang property").

Bang-bang property. In order to prove that a_1 is actually the characteristic function of some measurable subset, we exploit first-order optimality conditions. From [13, Theorem 1], since $\sigma_1(a)$ has a finite multiplicity, the mapping $\overline{\mathcal{U}}_L \ni a \mapsto \sigma_1(a)$ is sub-differentiable at a_1. Let $(b^k)_{k\in\widetilde{J}_1}$ be an orthonormal basis of the eigenspace associated to $\sigma_1(a_1)$, with $\widetilde{J}_1 \subset J_1$. We have

$$\partial\sigma_1(a_1) = \mathrm{co}\left\{ \partial j_k(a_1) \;\middle|\; k \in \widetilde{J}_1 \right\} \quad \text{with} \quad j_k(a) = \int_\Omega a(x) \left| \sum_{j\in\widetilde{J}_1} b_j^k \phi_j(x) \right|^2 dx,$$

where "co" means "convex hull". We consider the tangent cone[3] $\mathcal{T}_{a_1}$ to the set $\overline{\mathcal{U}}_L$ at a_1, and the indicator function $\iota_{[0,1]}$ given by

$$\iota_{\overline{\mathcal{U}}_L}(u) = \begin{cases} 0 & \text{if } u \in \overline{\mathcal{U}}_L, \\ +\infty & \text{otherwise.} \end{cases}$$

Finally, we define the linear functional $j_0 : \overline{\mathcal{U}}_L \ni a \mapsto \int_\Omega a$. The optimization problem ($\mathscr{P}_{\sigma_1}$) can be recast as

$$\min_{\substack{a\in L^\infty(\Omega) \\ j_0(a)=L|\Omega|}} \left(-\sigma_1(a) + \iota_{\overline{\mathcal{U}}_L}(a) \right).$$

If a_1 is an optimal solution, according to the Lagrange multiplier rule, there must exist $\mu^* \in \mathbb{R}$ such that

$$0 \in \partial(-\sigma_1)(a_1) + \mu^* \partial j_0(a_1) + \partial \iota_{\overline{\mathcal{U}}_L}(a_1).$$

Equivalently, there exists $(\alpha_k)_{k\in\widetilde{J}_1} \in \mathbb{R}_+^{\#\widetilde{J}_1}$ such that $\sum_{k\in\widetilde{J}_1} \alpha_k = 1$ and for every $h \in L^\infty(\Omega)$ such that $0 \leqslant a_1 + \eta h \leqslant 1$ for $\eta > 0$ small enough,

$$-\sum_{k\in\widetilde{J}_1} \alpha_k \langle dj_k(a_1), h\rangle + \langle dj_0(a_1), h\rangle = 0 \iff \int_\Omega h(x)(\Psi(x) - \mu^*)\, dx \geqslant 0, \tag{14}$$

[3] The tangent cone $\mathcal{T}_{a_1}$ is the set of functions $h \in L^\infty(\Omega)$ such that, for any sequence of positive real numbers ε_n decreasing to 0, there exists a sequence of functions $h_n \in L^\infty(\Omega)$ converging to h as $n \to +\infty$, and $a_1 + \varepsilon_n h_n \in \overline{\mathcal{U}}_l$ for every $n \in \mathbb{N}$ (see [12] for example).

where the function $\Psi : \Omega \to \mathbb{R}_+$ is defined by

$$\Psi(x) = \sum_{k \in \widetilde{J}_1} \alpha_k \left| \sum_{j \in \widetilde{J}_1} b_j^k \phi_j(x) \right|^2.$$

Let us now prove that a_1 is necessarily *bang-bang*, i.e., that $a_1 = \mathbb{1}_{\omega^*}$ for some Lebesgue measurable $\omega^* \subset \Omega$. By contradiction, assume that $|\{0 < a_1 < 1\}| > 0$. Let x_0 be a Lebesgue point of $\{0 < a_1 < 1\}$. There exists a sequence $(G_n)_{n \in \mathbb{N}}$ of measurable subsets with $G_n \subset \{0 < a_1 < 1\}$ such that

$$\lim_{\varepsilon \to 0} \frac{|G_n \cap B(x_0, \varepsilon)|}{|B(x_0, \varepsilon|} = 1, \tag{15}$$

where $B(x_0, \varepsilon)$ denotes the ball centered at x_0 with radius ε in $\mathbb{R}^d$. Setting $h = \mathbb{1}_{G_n}$ and noting that $0 \leqslant a_1 \pm \eta h \leqslant 1$ if η is small enough, we infer from (14) that

$$\pm \int_{G_n} (\Psi(x) - \mu^*)\, dx \geqslant 0.$$

Dividing this inequality by $|G_n|$ and letting G_n shrink to $\{x_0\}$ as $n \to +\infty$ we infer that $\Psi(x) = \mu^*$ a.e. in $\{0 < a_1 < 1\}$. Since $0 < |\{0 < a_1 < 1\}| \leqslant |\{\Psi = \mu^*\}|$ and Ψ is analytic, we must have $\Psi = \mu^*$ and $\nabla \Psi = 0$ everywhere in Ω, which contradicts ($\mathbf{H_L}$).

We thus infer that $|\{0 < a_1 < 1\}| = 0$ and hence $a_1 = \mathbb{1}_{\omega^*}$ for some Lebesgue measurable subset ω^* of Ω such that $|\omega^*| = L|\Omega|$.

Characterization of ω^*. Consider a Lebesgue point x_0 of ω^*. Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of measurable subsets such that $G_n \subset \omega^*$ and (15) holds. Setting $h = -\mathbb{1}_{G_n}$ and noting that $0 \leqslant a_1 + \eta h \leqslant 1$ if η is small enough, we infer from (14) that

$$\int_{G_n} (\Psi(x) - \mu^*)\, dx \geqslant 0.$$

Dividing this inequality by $|G_n|$ and letting G_n shrink to $\{x_0\}$ as $n \to +\infty$ yields $\Psi \geqslant \mu^*$ a.e. in ω^*. Similarly, $\Psi \leqslant \mu^*$ in $(\omega^*)^c$. We conclude that $\omega^* = \{\Psi > \mu^*\}$ by noting that $\{\Psi = \mu^*\}$ has zero Lebesgue measure.

Uniqueness and regularity of ω^*. Assume by contradiction that Problem ($\mathscr{P}_{\sigma_1}$) has two distinct maximizers a_1 and a_2. As a consequence of our previous analysis, there exist $\omega_1, \omega_2 \subset \Omega$ such that $a_i = \mathbb{1}_{\omega_i}$ $(i = 1, 2)$. By concavity of σ_1, $(a_1 + a_2)/2$ is also a solution of Problem ($\mathscr{P}_{\sigma_1}$), which contradicts the bang-bang property of maximizers.

Finally, the regularity property of ω^* follows from ($\mathbf{H_2}$), which implies the analyticity of Ψ. Therefore, ω^* is an open semi-analytic set.

Quantitative estimate. At last, we prove the quantitative estimate (12); the argument we follow to establish it is inspired and adapted from [8]. Observe that with the same function Ψ we have, for any $a \in \overline{\mathcal{U}}_L$,

$$\sigma_1(a) - \sigma_1(a_1) \leqslant \int_\Omega (a - a_1)\Psi.$$

Let us prove that there exists $C > 0$ such that

$$\forall a \in \overline{\mathcal{U}}_L, \quad \int_\Omega (a - a_1)\Psi \leqslant -C\|a - a_1\|^2_{L^1(\Omega)},$$

which can be seen as a quantitative version of the bathtub principle.

We have proved that $a_1 = \mathbb{1}_{\omega^*}$ for some $\omega^* \subset \Omega$ such that $|\omega^*| = L|\Omega|$, and moreover that $\Psi \geqslant \mu^*$ in ω^* and $\Psi \leqslant \mu^*$ in $\Omega\backslash\omega^*$. Noting that for every $a \in \overline{\mathcal{U}}_L$, $\int_\Omega (a - a_1)\Psi = \int_\Omega (a - a_1)(\Psi - \mu^*)$, that $a - a_1 \leqslant 0$ on ω^* and $a - a_1 \geqslant 0$ on ω^*, we get that

$$\int_\Omega (a - a_1)\Psi = -\int_\Omega |a - a_1|.|\Psi - \mu^*| \leqslant -\int_{\Omega_\delta} |a - a_1|.|\Psi - \mu^*|$$

where, for a given $\delta > 0$, the set $\Omega_\delta = \{|\Psi - \mu^*| \geqslant \delta\}$ is the complement of a tubular neighborhood of the levet set $\{\Psi = \mu^*\}$. By definition of Ω_δ, it follows that

$$\begin{aligned}\int_\Omega (a - a_1)\Psi \leqslant -\delta\|a - a_1\|_{L^1(\Omega_\delta)} = -\delta(\|a - a_1\|_{L^1(\Omega)} - \|a - a_1\|_{L^1(\Omega\backslash\Omega_\delta)}) \\ \leqslant -\delta(\|a - a_1\|_{L^1(\Omega)} - |\Omega\backslash\Omega_\delta|) \quad (16)\end{aligned}$$

Lemma 1. *There exists $M > 0$ such that $|\Omega\backslash\Omega_\delta| \leqslant M\delta$ for every $\delta > 0$.*

Proof. It suffices to prove this estimate for $\delta > 0$ small enough. We establish the existence of $K > 0$ such that if $s > 0$ is small enough then $\{\Psi = \mu^* + s\} \subset \Sigma_{\mu^*} + K\delta\mathbb{B}$ where $\Sigma_{\mu^*} = \{\Psi = \mu^*\}$ and $\mathbb{B}$ is the centered unit ball. Assuming such a K exists, we infer that

$$\Omega\backslash\Omega_\delta = \bigcup_{s=-\delta}^{\delta} \{\Psi = \mu^* + s\} \subset \Sigma_{\mu^*} + K\delta\mathbb{B}.$$

Now, using the definition of the perimeter of rectifiable curves via the Minkowski content, we obtain $|\Omega\backslash\Omega_\delta| \leqslant 2\delta \operatorname{Per}(\Sigma_{\mu^*})$ for δ small enough, whence the conclusion with $M = 2K \operatorname{Per}(\Sigma_{\mu^*})$. It thus remains to show that

$$\operatorname{dist}_H(\Sigma_{\mu^*}, \{\Psi = \mu^* + s\}) \leqslant Ms$$

for $s > 0$ small enough, where dist_H is the Hausdorff distance between two compact sets.

By contradiction, assume that there exist two sequences $(x_k)_{k\in\mathbb{N}}$ in Σ_{μ^*} and $(y_k)_{k\in\mathbb{N}}$ in $\Sigma_{\mu^*+s_k}$, where $(s_k)_{k\in\mathbb{N}}$ denotes a sequence converging to 0, such that

$$|x_k - y_k| = \operatorname{dist}_H(\Sigma_{\mu^*+s_k}, \Sigma_{\mu^*}) \quad \text{and} \quad \lim_{k\to+\infty} \frac{|\Psi(x_k) - \Psi(y_k)|}{|x_k - y_k|} = 0.$$

By the mean value theorem, there exists z_k belonging to the segment $[x_k, y_k]$ such that

$$\frac{|\Psi(x_k) - \Psi(y_k)|}{|x_k - y_k|} = \left\langle \nabla\Psi(z_k), \frac{y_k - x_k}{|y_k - x_k|} \right\rangle.$$

Since Σ_{μ^*} is $\mathscr{C}^2$, it satisfies the uniform interior ball property, and therefore $y_k = x_k + t_k\nu(x_k) + \mathrm{o}(t_k)$ for k large enough, with $t_k \to 0$, where $\nu(x_k)$ denotes the outward unit normal vector on $\Sigma_{\mu^*} = \partial\{\Psi \leqslant \mu^*\}$. We infer that $\langle \nabla\Psi(z_k), \nu(x_k) + \mathrm{o}(1)\rangle \to 0$ as $k \to +\infty$. This contradicts $(\mathbf{H_L})$. □

Given any $a \in \overline{\mathcal{U}}_L$, we set $\delta = \frac{1}{2M}\|a - a_1\|_{L^1(\Omega)}$. By (16) and by Lemma 1, we have

$$\int_\Omega (a - a_1)\Psi \leqslant -\frac{1}{2M}\|a - a_1\|_{L^1(\Omega)} \left(\|a - a_1\|_{L^1(\Omega)} - \frac{M}{2M}\|a - a_1\|_{L^1(\Omega)} \right) = -\frac{1}{4M}\|a - a_1\|^2_{L^1(\Omega)}.$$

The conclusion follows.

3.2 Proof of Theorem 1

Considering particular choices of coefficients b_j equal to 0 in $\mathbb{N}^*\backslash J_1$ in (8), we have

$$\begin{aligned}\bar{\delta}_T &\leqslant \gamma_1(T) \sup_{a\in\overline{\mathcal{U}}_L} \min_{\sum_{j\in J_1}|b_j|^2=1} \int_\Omega a(x)\left|\sum_{j\in J_1} b_j\phi_j(x)\right|^2 dx \\ &= \gamma_1(T) \max_{\mathbb{1}_\omega\in\mathcal{U}_L} \min_{\sum_{j\in J_1}|b_j|^2=1} \int_\Omega a(x)\left|\sum_{j\in J_1} b_j\phi_j(x)\right|^2 dx \\ &= \gamma_1(T) \max_{\mathbb{1}_\omega\in\mathcal{U}_L} \sigma_1(\mathbb{1}_\omega). \end{aligned} \tag{17}$$

To go from the first to the second line, we used Theorem 2. The rest of the proof is devoted to establishing a lower bound on $\bar{\delta}_T$. We define

$$E_{T,a}(b) = \int_0^T \int_\Omega a(x)\left|\sum_{j=1}^{+\infty} b_j e^{\lambda_j t}\phi_j(x)\right|^2 dx\, dt.$$

and we set $\varepsilon = \sum_{j\in J_1}|b_j|^2$ (so that $1-\varepsilon = \sum_{j\in\mathbb{N}^*\backslash J_1}|b_j|^2$). We also define sequence $(c_j)_{\in\mathbb{N}^*}$ in $\ell^2(\mathbb{R})$ by

$$c_j = \begin{cases} b_j/\sqrt{\varepsilon} & \text{if } j\in J_1 \\ b_j/\sqrt{1-\varepsilon} & \text{otherwise} \end{cases}$$

so that

$$\sum_{j\in J_1}|c_j|^2 = \sum_{j\in\mathbb{N}^*\backslash J_1}|c_j|^2 = 1.$$

We define the two subsets of $\ell^2(\mathbb{C})$

$$\Lambda_{J_1} = \left\{(c_j)\in\mathbb{C}^{\#J_1},\ \sum_{j\in J_1}|c_j|^2 = 1\right\} \quad\text{and}\quad \Lambda_{\mathbb{N}^*\backslash J_1} = \left\{(c_j)\in\ell^2(\mathbb{N}^*\setminus J_1),\ \sum_{j\in\mathbb{N}^*\backslash J_1}|c_j|^2 = 1\right\}.$$

A straightforward computation shows that

$$E_{T,a}(b) = \varepsilon A_{T,a}(c) + (1-\varepsilon)B_{T,a}(c) + 2\sqrt{\varepsilon(1-\varepsilon)}D_{T,a}(c),$$

where

$$\begin{aligned} A_{T,a}(c) &= \gamma_1(T)\int_\Omega a(x)\left|\sum_{j\in J_1} c_j\phi_j(x)\right|^2 dx \\ B_{T,a}(c) &= \int_0^T\int_\Omega a(x)\left|\sum_{j\in\mathbb{N}^*\backslash J_1} c_j e^{\lambda_j t}\phi_j(x)\right|^2 dxdt \\ D_{T,a}(c) &= \mathrm{Re}\int_\Omega\int_0^T a(x)\sum_{k\in J_1} c_k e^{\lambda_1 t}\phi_k(x)\overline{\sum_{j\in\mathbb{N}^*\backslash J_1} c_j e^{\lambda_j t}\phi_j(x)}\, dtdx \end{aligned}$$

Consequently,

$$\begin{aligned} C_T(a) &= \inf_{\sum_{j\in\mathbb{N}^*}|b_j|^2=1} E_{T,a}(b) \\ &= \inf_{\varepsilon\in[0,1]}\inf_{(c_j)\in\Lambda_{J_1}}\left(\varepsilon A_{T,a}(c) + \inf_{(c_j)\in\Lambda_{\mathbb{N}^*\backslash J_1}}(1-\varepsilon)B_{T,a}(c) + 2\sqrt{\varepsilon(1-\varepsilon)}D_{T,a}(c)\right) \end{aligned}$$

Moreover, note that

$$\inf_{(c_j)\in\Lambda_{J_1}} A_{T,a}(c) = \gamma_1(T)\sigma_1(a)$$

Let us consider the unique solution $a_1 = \mathbb{1}_{\omega^*}$ of $(\mathscr{P}_{\sigma_1})$. From now on, with a slight abuse of notation, we will use the notation $A_{T,a_1} = \gamma_1(T)\sigma_1(a_1)$.

Let $\nu \in (0,1)$ to be chosen later, and define $a_\nu = \nu a_1 + (1-\nu)L \in \overline{\mathcal{U}}_L$ (by convexity). By maximality of $\bar{\delta}_T$, we have

$$\begin{aligned}
\bar{\delta}_T &\geqslant \inf_{\varepsilon\in[0,1]} \inf_{(c_j)\in\Lambda_{\mathbb{N}^*\setminus J_1}} (\varepsilon A_{T,a_\nu}(c) + (1-\varepsilon)B_{T,a_\nu}(c) + 2\sqrt{\varepsilon(1-\varepsilon)}D_{T,a_\nu}(c)) \\
&\geqslant \inf_{\varepsilon\in[0,1]} \inf_{(c_j)\in\Lambda_{\mathbb{N}^*\setminus J_1}} (\nu\varepsilon A_{T,a_1} + \nu(1-\varepsilon)B_{T,a_1}(c) + (1-\nu)(1-\varepsilon)B_{T,L}(c) + 2\nu\sqrt{\varepsilon(1-\varepsilon)}D_{T,a_1}(c)).
\end{aligned}$$

To obtain this estimate, we have used that $D_{T,a_\nu}(c) = \nu D_{T,a_1}(c)$, since

$$D_{T,L}(c) = L\,\mathrm{Re}\sum_{\substack{j\in\mathbb{N}^*\setminus J_1\\ k\in J_1}} \overline{c_j}c_k \frac{e^{(\overline{\lambda_j}+\lambda_1)T}-1}{\lambda_1+\overline{\lambda_j}} \int_\Omega \phi_k(x)\cdot\overline{\phi_j(x)}\,dx = 0$$

by orthogonality of the functions ϕ_j in $L^2(\Omega,\mathbb{C}^q)$.

We have

$$\begin{aligned}
&\inf_{(c_j)\in\Lambda_{\mathbb{N}^*\setminus J_1}} (\nu\varepsilon A_{T,a_1} + \nu(1-\varepsilon)B_{T,a_1}(c) + (1-\nu)(1-\varepsilon)B_{T,L}(c) + 2\nu\sqrt{\varepsilon(1-\varepsilon)}D_{T,a_1}(c)) \\
&\quad= \nu\varepsilon A_{T,a_1} + \inf_{(c_j)\in\Lambda_{\mathbb{N}^*\setminus J_1}} \nu(1-\varepsilon)B_{T,a_1}(c) + (1-\nu)(1-\varepsilon)B_{T,L}(c) + 2\nu\sqrt{\varepsilon(1-\varepsilon)}D_{T,a_1}(c) \\
&\quad\geqslant \nu\varepsilon A_{T,a_1} + \inf_{(c_j)\in\Lambda_{\mathbb{N}^*\setminus J_1}} \Big((1-\nu)(1-\varepsilon)B_{T,L}(c) + 2\nu\sqrt{\varepsilon(1-\varepsilon)}D_{T,a_1}(c)\Big) \\
&\quad\geqslant \nu\varepsilon A_{T,a_1} + (1-\nu)(1-\varepsilon)\frac{L\gamma_{p_0}(T)}{2} \\
&\qquad+ \inf_{(c_j)\in\Lambda_{\mathbb{N}^*\setminus J_1}} \left(\frac{(1-\nu)(1-\varepsilon)}{2}B_{T,L}(c) + 2\nu\sqrt{\varepsilon(1-\varepsilon)}D_{T,a_1}(c)\right)
\end{aligned} \tag{18}$$

by using that

$$B_{T,L}(c) = L\sum_{j\in\mathbb{N}^*\setminus J_1} |c_j|^2 \int_0^T e^{2\,\mathrm{Re}\,\lambda_j t}\,dt = L\sum_{j=p_0}^{+\infty} \gamma_j(T)|c_j|^2$$

and that $\sum_{j=p_0}^{+\infty}\gamma_j(T)|c_j|^2 \geqslant \gamma_{p_0}(T)\sum_{j=p_0}^{+\infty}|c_j|^2 = \gamma_{p_0}(T)$.

Lemma 2. *Let $T>0$ and $\mu\in(0,1)$ be given. We define*

$$\varepsilon_{\nu,T,L} = \frac{L^2(1-\nu)^2\gamma_{p_0}(T)}{16\nu^2\gamma_1(T) + L^2(1-\nu)^2\gamma_{p_0}(T)} \in (0,1). \tag{19}$$

For every $\varepsilon\in[0,\varepsilon_{\nu,T,L}]$, we have

$$\inf_{(c_j)\in\Lambda_{\mathbb{N}^*\setminus J_1}} \Big(\frac{(1-\nu)(1-\varepsilon)}{2}B_{T,L}(c) + 2\nu\sqrt{\varepsilon(1-\varepsilon)}D_{T,a_1}(c)\Big) \geqslant 0.$$

Proof of Lemma 2. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|D_{T,a_1}(c)|^2 &= \left| \mathrm{Re} \int_\Omega \int_0^T a_1(x) \sum_{k=1}^{p_0-1} c_k e^{\lambda_1 t} \phi_k(x) \overline{\sum_{j=p_0}^{+\infty} c_j e^{\lambda_j t} \phi_j(x)}\, dt\, dx \right|^2 \\
&\leqslant \int_\Omega \int_0^T a_1(x)^2 \left| \sum_{k=1}^{p_0-1} c_k e^{\lambda_1 t} \phi_k(x) \right|^2 dt\, dx \int_\Omega \int_0^T \left| \sum_{j=p_0}^{+\infty} c_j e^{\lambda_j t} \phi_j(x) \right|^2 dt\, dx \\
&\leqslant \sum_{k=1}^{p_0-1} \gamma_1(T) |c_k|^2 \sum_{j=p_0}^{+\infty} \gamma_j(T) |c_j|^2 = \gamma_1(T) \sum_{j=p_0}^{+\infty} \gamma_j(T) |c_j|^2.
\end{aligned}$$

To go from the second to the third line we used the fact that $0 \leqslant a_1(\cdot) \leqslant 1$. Therefore,

$$\frac{(1-\nu)(1-\varepsilon)}{2} B_{T,L}(c) + 2\nu\sqrt{\varepsilon(1-\varepsilon)} D_{T,a_1}(c) \geqslant \frac{(1-\nu)(1-\varepsilon)}{2} L X_T(c)^2 - 2\nu\sqrt{\varepsilon(1-\varepsilon)}\sqrt{\gamma_1(T)} X_T(c)$$

where

$$X_T(c) = \left(\sum_{j=p_0}^{+\infty} \gamma_j(T) |c_j|^2 \right)^{1/2}.$$

Since $X_T(c) \geqslant \sqrt{\gamma_{p_0}(T)}$, the expected conclusion follows if

$$\sqrt{\gamma_{p_0}(T)} \geqslant \frac{4\nu\sqrt{\varepsilon}\sqrt{\gamma_1(T)}}{L(1-\nu)\sqrt{1-\varepsilon}},$$

which is satisfied for any $\varepsilon \in [0, \varepsilon_{\nu,T,L}]$ where $\varepsilon_{\nu,T,L}$ is defined by (19). □

Lemma 3. *One has*

$$\bar{\delta}_T \geqslant \inf_{\varepsilon \in [0,1]} \psi_{T,\nu,L}(\varepsilon), \tag{20}$$

where

$$\psi_{T,\nu,L}(\varepsilon) = \begin{cases} \min\left(\nu A_{T,a_1}, (1-\nu)\frac{L\gamma_{p_0}(T)}{2} \right) & \text{if } \varepsilon \in [0, \varepsilon_{\nu,T,L}] \\ (1-\nu)(1-\varepsilon)\gamma_{p_0}(T) & \text{if } \varepsilon \in (\varepsilon_{\nu,T,L}, 1]. \end{cases} \tag{21}$$

Proof of Lemma 3. We combine Lemma 2 and (18): first, observe that for any $\varepsilon \in [0,1]$

$$\nu\varepsilon A_{T,a_1} + (1-\nu)(1-\varepsilon)\frac{L\gamma_{p_0}(T)}{2} \geqslant \min\left(\nu A_{T,a_1}, (1-\nu)\frac{L\gamma_{p_0}(T)}{2} \right).$$

Second, the expression of $\psi_{T,\nu,L}$ on $(\varepsilon_{\nu,T,L}, 1]$ follows from (18), using that $E_{T,a_1}(b) \geqslant 0$ for all $b \in \ell^2(\mathbb{R})$ and the fact that

$$\nu\varepsilon A_{T,a_1} + \nu(1-\varepsilon) B_{T,a_1}(c) + (1-\nu)(1-\varepsilon) B_{T,L}(c) + 2\nu\sqrt{\varepsilon(1-\varepsilon)} D_{T,a_1}(c) \geqslant (1-\nu)(1-\varepsilon) B_{T,L}(c)$$

for all $c \in \ell^2(\mathbb{R})$. □

The end of the proof consists in choosing ν and ε adequately to compute $\inf_{\varepsilon \in [0;1]} \psi_{T,\nu,L}(\varepsilon)$.

Lemma 4. *There exists a family $(\nu_T)_{T>0}$ converging to 1 as $T \to +\infty$ such that $\nu_T \in (0,1)$ for every $T > 0$, and*

$$\lim_{T\to+\infty} \varepsilon_{\nu_T,T,L} = 1, \qquad \lim_{T\to+\infty} \frac{(1-\nu_T)\gamma_{p_0}(T)}{\gamma_1(T)} = +\infty, \qquad \lim_{T\to+\infty} \frac{(1-\nu_T)(1-\varepsilon_{\nu_T,T,L})\gamma_{p_0}(T)}{\gamma_1(T)} = +\infty.$$

Proof. Let $(\nu_T)_{T>0}$ be a family of elements in $[0,1]$ converging to 1 as $T \to +\infty$. Let $\theta_T = 1-\nu_T$. Since $\gamma_{p_0}(T)/\gamma_1(T) \sim e^{2T\,\mathrm{Re}(\lambda_{p_0}-\lambda_1)}$ as $T \to +\infty$, let us set $\theta_T = \gamma_1(T))e^{\eta T}/\gamma_{p_0}(T)$, where $\eta \in \mathbb{R}_+^*$ will be chosen appropriately. Note first that

$$1-\nu_T = \theta_T \sim e^{(\eta+2\,\mathrm{Re}(\lambda_1-\lambda_{p_0}))T} \quad \text{as } T\to+\infty \quad \text{and} \quad \frac{(1-\nu_T)\gamma_{p_0}(T)}{\gamma_1(T)} = e^{\eta T}.$$

In particular, this indicates that we should choose $\eta \in (0, 2\,\mathrm{Re}(\lambda_{p_0}-\lambda_1))$.

Regarding the last equality of the lemma, note that

$$1-\varepsilon_{\nu_T,T,L} = \frac{16\nu_T^2\gamma_1(T)}{16\nu^2\gamma_1(T) + L^2(1-\nu_T)^2\gamma_{p_0}(T)}$$

and thus

$$(1-\nu_T)(1-\varepsilon_{\nu_T,T,L})\frac{\gamma_{p_0}(T)}{\gamma_1(T)} = \frac{16\nu_T^2(1-\nu_T)\gamma_{p_0}(T)}{16\nu_T^2\gamma_1(T) + L^2(1-\nu_T)^2\gamma_{p_0}(T)} = \frac{e^{\eta T}}{1+\frac{L^2e^{2\eta T}}{16\nu_T^2}\frac{\gamma_1(T)}{\gamma_{p_0}(T)}}.$$

Since

$$e^{2\eta T}\frac{\gamma_1(T)}{\gamma_{p_0}(T)} = \mathrm{O}\left(e^{2T(\eta+\lambda_1-\lambda_{p_0})}\right),$$

choosing any η such that $\eta \in (\mathrm{Re}(\lambda_{p_0}-\lambda_1), 2\,\mathrm{Re}(\lambda_{p_0}-\lambda_1))$ yields the desired conclusion. Observe moreover that, with this choice, we have

$$\varepsilon_{\nu_T,T,L} = \frac{L^2(1-\nu_T)^2\gamma_{p_0}(T)}{L^2(1-\nu_T)^2\gamma_{p_0}(T) + 16\nu_T^2\gamma_1(T)} = \frac{L^2e^{2\eta T}\gamma_1(T)/\gamma_{p_0}(T)}{16\nu_T^2 + L^2e^{2\eta T}\gamma_1(T)/\gamma_{p_0}(T)} = 1 + \mathrm{O}\left(e^{-2T(\eta-\mathrm{Re}(\lambda_{p_0}-\lambda_1))}\right)$$

since $\eta + \mathrm{Re}(\lambda_1-\lambda_{p_0}) > 0$. □

Conclusion: asymptotics of $\bar\delta_T$. Let us consider $\nu = \nu_T$ in (20), where $(\nu_T)_{T>0}$ denotes the family constructed in Lemma 4. One has

$$\bar\delta_T \geqslant \inf_{\varepsilon\in[0,1]} \psi_{T,\nu_T,L}(\varepsilon)$$

where $\psi_{T,\nu_T,L}$ is given by (21). From Lemma 4, there holds

$$\lim_{T\to+\infty} \frac{(1-\nu_T)\gamma_{p_0}(T)L}{2\gamma_1(T)} = +\infty \quad \text{and} \quad \lim_{T\to+\infty} \frac{(1-\nu_T)(1-\varepsilon_{\nu_T,T,L})\gamma_{p_0}(T)}{\gamma_1(T)} = +\infty.$$

From (20) we infer that

$$\frac{\bar\delta_T}{\gamma_1(T)} \geqslant \min\left(\nu_T\sigma_1(a_1), \frac{(1-\nu_T)\gamma_{p_0}(T)L}{2\gamma_1(T)}, \frac{(1-\nu_T)(1-\varepsilon_{\nu_T,T,L})\gamma_{p_0}(T)}{\gamma_1(T)}\right). \tag{22}$$

Letting T go to $+\infty$ yields

$$\liminf_{T\to+\infty} \frac{\bar\delta_T}{\gamma_1(T)} \geqslant \sigma_1(a_1).$$

Combining this inequality with (17) finally gives (9).

Quantitative estimate on $\mathbf{dist}_{L^1(\Omega)}\left(a_T^\star, \underset{a\in\overline{\mathcal{U}}_L}{\mathbf{argmax}}\ \sigma_1(a)\right)$. Since $\bar\delta_T = C_T(a_T^\star)$ and since $C_T(a_T^\star)$ is defined as an infimum, considering particular choices of coefficients b_j equal to 0 in $\mathbb{N}^*\backslash J_1$ gives

$$C_T(a_T^\star) \leqslant \gamma_1(T) \min_{\sum_{j\in J_1}|b_j|^2} \int_\Omega a(x)\left|\sum_{j\in J_1} b_j\phi_j(x)\right|^2 dx = \gamma_1(T)\sigma_1(a_T^\star).$$

Now, according to (22), with the particular choice of parameter ν_T given by Lemma 4, we also have $C_T(a_T^\star) \geqslant \gamma_1(T)\nu_T\sigma_1(a_1)$ if T is large enough, and therefore

$$\sigma_1(a_T^\star) \geqslant \frac{C_T(a_T^\star)}{\gamma_1(T)} \geqslant \nu_T\sigma_1(a_1).$$

In particular, this implies that $0 \leqslant \sigma_1(a_1) - \sigma_1(a_T) \leqslant (1-\nu_T)\sigma_1(a_1)$. Using the constant K given by Theorem 2 we deduce that

$$K\|a_T^\star - a_1\|^2_{L^1(\Omega)} \leqslant (1-\nu_T)\sigma_1(a_1).$$

By construction of ν_T we have

$$\nu_T \sim e^{(\eta+2\,\mathrm{Re}(\lambda_1-\lambda_{p_0}))T} \quad \text{as } T\to+\infty$$

for some fixed $\eta \in (\mathrm{Re}(\lambda_{p_0}-\lambda_1), 2\,\mathrm{Re}(\lambda_{p_0}-\lambda_1))$, whence the conclusion.

4 Conclusion and further comments

In this paper, we have described the large-time behavior of the maximizers of the observability constant associated with measurements of the solution of parabolic equations on a subdomain.

Hereafter, we show by duality that our optimal observability results can be applied as well to the optimal actuator (controllability) shape and location problem. We then give a partial result regarding the small-time asymptotics, i.e., as $T\to 0$, of the optimal observability problem. We conclude with some open questions.

4.1 Controllability

The control system

$$\partial_t y + A_0 y = u\mathbb{1}_\omega \tag{23}$$

is said to be exactly null controllable in time T if every initial datum $y(0,\cdot)\in L^2(\Omega)$ can be steered to 0 in time T by means of an appropriate control function $u\in L^2((0,T)\times\Omega)$. It is well known that controllability and observability are dual notions (see *e.g.* [14, 44, 45, 48]), in the sense that the latter exact null controllability property is equivalent to the observability property of (1) on ω in time T, and the observability constant $C_T(\mathbb{1}_\omega)$ coincides with the inverse of the minimal L^2 control cost for the controllability problem for (23). The Hilbert Uniqueness Method (HUM, see [28, 29, 44, 48]) provides a characterization unique control solving the exact null controllability problem with a minimal L^2 norm. This control is referred to as the HUM control and is characterized as follows. Define the HUM functional $\mathcal{J}_\omega$ by

$$\mathcal{J}_\omega(\phi^T) = \frac{1}{2}\int_0^T\int_\omega |\phi(t,x)|^2\,dx\,dt + \langle \phi^T, y^0\rangle_{L^2},$$

where ϕ is the solution of $-\partial_t\phi + A_0^*\phi = 0$ such that $\phi(T,\cdot)=\phi^T$. Under the observability assumption, the functional $\mathcal{J}_\omega$ has a unique minimizer and the HUM control u_ω steering y^0 to 0 in time T is $u_\omega(t,x) =$

$\mathbb{1}_\omega(x)\phi(t,x)$. The HUM operator $\Gamma_\omega : L^2(\Omega) \to L^2((0,T)\times\Omega)$ is defined by $\Gamma_\omega(y^0) = u_\omega$ and its operator norm is given by

$$\|\Gamma_\omega\| = \sup\left\{ \frac{\|u_\omega\|_{L^2((0,T)\times\Omega)}}{\|y^0\|_{L^2(\Omega)}} \mid y^0 \in L^2(\Omega)\setminus\{0)\} \right\}.$$

Adopting the convention $1/0 = +\infty$, we have

$$\|\Gamma_\omega\| = \frac{1}{C_T(\mathbb{1}_\omega)}.$$

We refer to [36] for details. Thus

$$\inf_{\mathbb{1}_\omega\in\mathcal{U}_L} \|\Gamma_\omega\| = \Big(\sup_{\mathbb{1}_\omega\in\mathcal{U}_L} C_T(\mathbb{1}_\omega)\Big)^{-1}.$$

Hence, the problem of maximizing the observability constant is equivalent to the problem of minimizing the operator norm of Γ_ω, which models the determination of a best control domain in some sense.

4.2 Small-time asymptotics

In this section, ,we mention an issue similar to the one investigated in this article, which cannot be treated using the same tools as in this paper: the limit of maximizers in small time, as $T \to 0$. We have the following partial result. It would be relevant to study how to obtain quantitative estimates on the convergence of $\bar\delta_T/T$, as well as on the convergence of the maximizers.

Theorem 3. *Assume that $q = 1$ and that A_0 is the second-order elliptic differential operator*

$$A_0 = -\sum_{j=1}^d \partial_{x_j}(g_{jk}\partial_{x_k}) + \sum_{j=1}^d b_j\partial_{x_j} + c$$

where the functions g_{jk}, b_j and c are smooth on $\overline{\Omega}$. We assume that the matrix $G = (g_{jk})_{1\leqslant j,k\leqslant d}$ is real and positive definite in $\overline{\Omega}$, that $\det(G) = 1$ on Ω and that A_0 is symmetric for the Lebesgue measure on Ω. Then

$$\bar\delta_T \sim LT \quad as \quad T\to 0.$$

Proof. Using (8) and the fact that $(\phi_j)_{j\geqslant 1}$ is a Hilbert basis, we first note that, taking $a \equiv L$,

$$C_T(L) = \inf_{\sum_{j=1}^{+\infty}|b_j|^2=1} \int_0^T\int_\Omega L\left|\sum_{j=1}^{+\infty} b_j e^{\lambda_j t}\phi_j(x)\right|^2 dx\,dt = L\inf_{j\in\mathbb{N}^*}\gamma_j(T) = L\gamma_1(T).$$

Since $a_T^\star$ is a maximizer, we have $\bar\delta_T = C_T(a_T^\star) \geqslant C_T(L)$ and thus

$$\liminf_{T\to 0}\frac{\bar\delta_T}{T} \geqslant L\liminf_{T\to 0}\frac{\gamma_1(T)}{T} = L.$$

Let us prove the converse inequality. Actually, let us prove the stronger fact that

$$\limsup_{T\to 0}\frac{C_T(a)}{T} \leqslant L \qquad \forall a\in\overline{\mathcal{U}}_L.$$

Using that $C_T(a)$ is defined by (8) as an infimum, taking $b_j = 1$ and all other b_k equal to 0, we have in particular

$$C_T(a) \leqslant \int_0^T \int_\Omega a(x) e^{\mathrm{Re}\,\lambda_j t} |\phi_j(x)|^2 \, dx \, dt = \gamma_j(T) \int_\Omega a |\phi_j|^2 \qquad \forall j \in \mathbb{N}^*$$

and thus, for every $N \in \mathbb{N}^*$,

$$C_T(a) \leqslant \inf \left\{ \sum_{j=1}^N \alpha_j \gamma_j(T) \int_\Omega a |\phi_j|^2 \;\middle|\; \alpha_1, \dots, \alpha_N \geqslant 0, \; \sum_{j=1}^N \alpha_j = 1 \right\}.$$

Choosing in particular $\alpha_j = \gamma_j(T)^{-1} / \sum_{k=1}^N \gamma_k(T)^{-1}$ for every $j \in \{1, \dots, N\}$, we infer that

$$\frac{1}{N} \sum_{j=1}^N \frac{1}{\gamma_j(T)} C_T(a) \leqslant \int_\Omega a \frac{1}{N} \sum_{j=1}^N |\phi_j|^2 \tag{24}$$

The inequality (24) is valid for every $a \in \overline{\mathcal{U}}_L$, every $N \in \mathbb{N}^*$ and every $T > 0$. Fixing N and letting T tend to 0, using that $\gamma_j(T) \to T$, we obtain

$$\limsup_{T \to 0} \frac{C_T(a)}{T} \leqslant \int_\Omega a \frac{1}{N} \sum_{j=1}^N |\phi_j|^2 \qquad \forall N \in \mathbb{N}^*.$$

To conclude, we use the fact that the Cesàro mean $\frac{1}{N} \sum_{j=1}^N |\phi_j|^2$ converges to the constant function $\frac{1}{|\Omega|}$ as $N \to +\infty$, uniformly on any compact subset of the open set Ω for the C^0 topology and thus weakly in $L^2(\Omega)$ (this follows from [22, Section 17.5, Theorem 17.5.7 and Corollary 17.5.8]). The expected result follows. Note that a similar argument was used in [40, Lemma 1]. □

Theorem 3 gives the asymptotic behavior of the optimal value $\bar{\delta}_T$, but we do not know ho to study the behavior of the maximizers $a_T^\star$ as $T \to 0$, as we did in the case $T \to +\infty$, in particular due to the fact that we are not able to identify a limit problem as $T \to 0$. We let this question as an open problem.

4.3 Other open questions

Other open questions are in order:

(i) A first one is to understand whether optimal profiles $a_T^\star$ are characteristic functions for any $T > 0$. This *bang-bang* property is usually a first step in deriving finer estimates and geometric information.

(ii) A second one is to establish quantitative estimates for the optimal observability constant: can we obtain an estimate akin to $C_T(a) - C_T(a_T^\star) \leqslant -K \|a - a_T^\star\|_{L^1(\Omega)}^2$? To answer this question, we believe that a positive answer should first be obtained regarding the bang-bang property.

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References

[1] G. Allaire, A. Münch, and F. Periago. Long time behavior of a two-phase optimal design for the heat equation. *SIAM J. Control Optim.*, 48(8):5333–5356, 2010.

[2] F. Ammar-Khodja, A. Benabdallah, C. Dupaix, and M. González-Burgos. A kalman rank condition for the localized distributed controllability of a class of linear parabolic systems. *Journal of Evolution Equations*, 9:267–291, 2009.

[3] J. Apraiz, L. Escauriaza, G. Wang, and C. Zhang. Observability inequalities and measurable sets. *Journal of the European Mathematical Society*, 16(11):2433–2475, 2014.

[4] A. Bottois, N. Cîndea, and A. Münch. Optimization of non-cylindrical domains for the exact null controllability of the 1d wave equation. *ESAIM: Control, Optimisation and Calculus of Variations*, 27:13, 2021.

[5] F. Boyer and P. Fabrie. *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, volume 183 of *Applied Mathematical Sciences*. Springer, New York, 2013.

[6] L. Brasco and G. D. Philippis. *7 Spectral inequalities in quantitative form*, pages 201–281. De Gruyter Open Poland, Warsaw, Poland, 2017.

[7] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[8] E. Casas, D. Wachsmuth, and G. Wachsmuth. Sufficient second-order conditions for bang-bang control problems. *SIAM Journal on Control and Optimization*, 55(5):3066–3090, 2017.

[9] F. W. Chaves-Silva, D. A. Souza, and C. Zhang. Observability inequalities on measurable sets for the Stokes system and applications. *SIAM J. Control Optim.*, 58(4):2188–2205, 2020.

[10] A. Cianchi and A. Ferone. A strengthened version of the Hardy-Littlewood inequality. *Journal of the London Mathematical Society*, 77(3), 2008.

[11] Y. Colin de Verdière, L. Hillairet, and E. Trélat. Small-time asymptotics of hypoelliptic heat kernels near the diagonal, nilpotentization and related results. *Ann. H. Lebesgue*, 4:897–971, 2021.

[12] R. Cominetti and J.-P. Penot. Tangent sets to unilateral convex sets. *C. R. Acad. Sci. Paris Sér. I Math.*, 321(12):1631–1636, 1995.

[13] S. J. Cox. The generalized gradient at a multiple eigenvalue. *J. Funct. Anal.*, 133(1):30–40, 1995.

[14] R. F. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1995.

[15] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 2*. Springer-Verlag, Berlin, 1988.

[16] E. Fernández-Cara, S. Guerrero, O. Y. Imanuvilov, and J.-P. Puel. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 83(12):1501–1542, 2004.

[17] E. Fernández-Cara and A. Münch. Numerical null controllability of semi-linear 1-D heat equations: fixed point, least squares and Newton methods. *Math. Control Relat. Fields*, 2(3):217–246, 2012.

[18] A. V. Fursikov and O. Y. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[19] R. M. Hardt. Stratification of real analytic mappings and images. *Invent. Math.*, 28:193–208, 1975.

[20] A. Henrot and M. Pierre. *Shape variation and optimization*, volume 28 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2018. A geometrical analysis, English version of the French publication [MR2512810] with additions and updates.

[21] H. Hironaka. Subanalytic sets. In *Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki*, pages 453–493. 1973.

[22] L. Hörmander. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985. Pseudodifferential operators.

[23] J. P. Kelliher. On the vanishing viscosity limit in a disk. *Math. Ann.*, 343(3):701–726, 2009.

[24] S. Kumar and J. Seinfeld. Optimal location of measurements for distributed parameter estimation. *IEEE Trans. Automat. Control*, 23(4):690–698, 1978.

[25] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2):335–356, 1995.

[26] D.-S. Lee and B. Rummler. The eigenfunctions of the Stokes operator in special domains. III. *ZAMM Z. Angew. Math. Mech.*, 82(6):399–407, 2002.

[27] M. Lemou. Extended rearrangement inequalities and applications to some quantitative stability results. *Communications in Mathematical Physics*, 348:695–727, 2016.

[28] J.-L. Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, volume 8 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1988. Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

[29] J.-L. Lions. Exact controllability, stabilization and perturbations for distributed systems. *SIAM Rev.*, 30(1):1–68, 1988.

[30] P. Lissy and E. Zuazua. Internal observability for coupled systems of linear partial differential equations. *SIAM Journal on Control and Optimization*, 57(2):832–853, 2019.

[31] I. Mazari and D. Ruiz-Balet. Quantitative stability for eigenvalues of schrödinger operator, quantitative bathtub principle, and application to the turnpike property for a bilinear optimal control problem. *SIAM Journal on Mathematical Analysis*, 54(3):3848–3883, 2022.

[32] L. Miller. On the controllability of anomalous diffusions generated by the fractional Laplacian. *Math. Control Signals Systems*, 18(3):260–271, 2006.

[33] K. Morris. Linear-quadratic optimal actuator location. *IEEE Trans. Automat. Control*, 56(1):113–124, 2011.

[34] A. Münch. Optimal location of the support of the control for the 1-d wave equation: numerical investigations. *Computational Optimization and Applications*, 42(3):443–470, 2009.

[35] A. Münch and F. Periago. Optimal distribution of the internal null control for the one-dimensional heat equation. *J. Differential Equations*, 250(1):95–111, 2011.

[36] Y. Privat, E. Trélat, and E. Zuazua. Optimal location of controllers for the one-dimensional wave equation. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 30(6):1097–1126, 2013.

[37] Y. Privat, E. Trélat, and E. Zuazua. Optimal observation of the one-dimensional wave equation. *J. Fourier Anal. Appl.*, 19(3):514–544, 2013.

[38] Y. Privat, E. Trélat, and E. Zuazua. Complexity and regularity of maximal energy domains for the wave equation with fixed initial data. *Discrete Contin. Dyn. Syst.*, 35(12):6133–6153, 2015.

[39] Y. Privat, E. Trélat, and E. Zuazua. Optimal shape and location of sensors for parabolic equations with random initial data. *Arch. Ration. Mech. Anal.*, 216(3):921–981, 2015.

[40] Y. Privat, E. Trélat, and E. Zuazua. Optimal observability of the multi-dimensional wave and Schrödinger equations in quantum ergodic domains. *J. Eur. Math. Soc. (JEMS)*, 18(5):1043–1111, 2016.

[41] Y. Privat, E. Trélat, and E. Zuazua. Randomised observation, control and stabilization of waves [Based on the plenary lecture presented at the 86th Annual GAMM Conference, Lecce, Italy, March 24, 2015]. *ZAMM Z. Angew. Math. Mech.*, 96(5):538–549, 2016.

[42] Y. Privat, E. Trélat, and E. Zuazua. Actuator design for parabolic distributed parameter systems with the moment method. *SIAM J. Control Optim.*, 55(2):1128–1152, 2017.

[43] Y. Privat, E. Trélat, and E. Zuazua. Spectral shape optimization for the Neumann traces of the Dirichlet-Laplacian eigenfunctions. *Calc. Var. Partial Differential Equations*, 58(2):Paper No. 64, 45, 2019.

[44] E. Trélat. *Control in finite and infinite dimension*. Springerbriefs on PDEs and Data Science. Springer, 2024.

[45] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.

[46] D. Uciński and M. Patan. Sensor network design for the estimation of spatially distributed processes. *Int. J. Appl. Math. Comput. Sci.*, 20(3):459–481, 2010.

[47] M. van de Wal and B. de Jager. A review of methods for input/output selection. *Automatica J. IFAC*, 37(4):487–510, 2001.

[48] J. Zabczyk. *Mathematical control theory*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. An introduction, Reprint of the 1995 edition.

[49] E. Zuazua. Stable observation of additive superpositions of partial differential equations. *Systems & Control Letters*, 93:21–29, 2016.