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Volumetric Aggregation Methods for Scoring Rules with Unknown Weights

Paolo Viappiani

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Abstract Scoring rules are a popular method for aggregating rankings; they are frequently used in many settings, including social choice, information retrieval and sports. Scoring rules are parametrized by a vector of weights (the scoring vectors), one for each position, and declare as winner the candidate that maximizes the score obtained when summing up the weights corresponding to the position of each voter.

It is well known that properly setting the weights is a crucial task, as different candidates can win with different scoring vectors. In this paper, we provide several methods to identify the winner considering all possible weights. We first propose VolumetricTop, a rule that ranks alternatives based on the hyper-polytope representing the set of weights that give the alternative the highest score, and provide a detailed analysis of the rule from the point-of-view of social choice theory. In order to overcome some of its limitations, we then propose two other methods: Volumetric-runoff, a rule that iteratively eliminates the alternative associated with the smallest region until a winner is found, and Volumetric-tournament, where alternatives are matched in pairwise comparisons; we provide several insights about these rules. Finally we provide some test cases of rank aggregation using the proposed methods.

Keywords Scoring rules · rank aggregation · computational social choice · convex sequences · Borda count

1 Introduction

An important issue in group decision-making is how to determine a winner or a social ranking from a set of rankings; the problem of rank aggregation emerges in recommender systems, social choice (aggregation of preferences), information retrieval (aggregation of the output of different search engines), sports (aggregation of the performance in several races into a single score). Among the many aggregation methods, positional scoring rules constitute a particularly popular class. Their appeal lies in their inherent simplicity, as they assign a weight to each position and designate as the winner the candidate that achieves the highest score.

In many contexts, in particular in sport competitions, it is often assumed that weights should constitute a convex sequence [19], meaning that the difference between the weight of the first position and that of the second position is at least as large as the difference between the weight of the second and the third position, and so on.

Properly setting the weights is a crucial task; indeed, it has been noted by several authors [37, 22, 23, 19] that scoring rules with different weights may produce very different outputs. Setting a

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precise vector of weights for a scoring rule can be seen arbitrary; indeed the decision of which weights to use is critical.

Since providing a precise weight vector may be cognitively difficult in many contexts, it is interesting to investigate procedures, based on scoring rules, that discriminate alternatives with imprecise or partially known weights. The goal is to determine a winner without making an arbitrary choice for the weights. Over the years many methods for aggregating rankings without fixed weights have been proposed, starting from the classic work of Cook and Kress [6] that pioneered the use of data-envelope methods for this problem (that opened a whole research area, see e.g. [13, 15, 33]), and the determination of qualitative dominance relations [37]. Decision models with uncertain weights have found applications in (among others) the evaluation of safety of roads according to multiple criteria [28] and the evaluation of human physiological state [42].

However, many of the proposed methods are not convincing, as argued by Llamazares and Peña [22], because they fail to satisfy important social choice properties, or they require the specification of parameters that are hard to set (and their value will determine the winner). Moreover, we claim that the methods that are more satisfactory with respect to their theoretical properties, may not be adapted to all circumstances. Indeed some aggregation methods [23, 24] do provide a reasonable framework, but good properties are obtained by doing some technical choices that could be seen as arbitrary. The recently proposed method based on minimax regret [38] can be a reasonable choice when a robust solution has to be found, but may not be accepted if the decision maker is not pessimistic.

Based on our analysis of the state of the art (that will be carried out in Section 2) we identify some characteristics that an ideal aggregation method should have. We believe that an ideal method for rank aggregation should have the following characteristics:

- The aggregation method should satisfy basic properties identified by social choice theory, such as as anonymity, neutrality, Pareto and monotonicity. We will detail which properties are particularly important and which can be relaxed, bearing in mind that, as we know from the theory, not all properties can be simultaneously fulfilled.
- The aggregation should not depend on arbitrary parameter choices. Indeed, a method that requires specifying parameter values will be subject to the fact that, typically, different winners can emerge with different parameter values; the problem of setting the weights is not solved but just reframed in terms of a different parameter space.
- The aggregation should be able to rank all alternatives; this means that when a full ranking is desired in output, all alternatives can be discriminated (with ties between alternatives being a rather exceptional case).
- The aggregation should not be overly optimistic or pessimistic. While many works insist in picking the alternative that scores better with respect to the most favorable weights, this may not be justified in terms of decision theory [35]. Similarly, while pessimistic approaches can offer robust solutions, they may not be suitable to the typical decision-maker.
- Finally, the aggregation should be easy to explain, so that a decision maker can understand why a particular choice is recommended.

After reviewing the state of the art and showing that current methods fail to satisfy one or more of the previous points, in this article we aim at identifying an approach with the above characteristics.

We assume that a committee (or a decision maker) provides us with general requirements on the weighting vector, such as *“the weights should be decreasing”* and *“the weights should be a convex sequence”*, and more precise ones such as *“the difference between the second and the third position should be more than 5 points”*. We propose an approach that aims at is conceptually simple and that may be easily accepted by decision makers: assuming that we are given constraints on the possible weights, represented by a feasible region, we consider each possible instance of the weights as a signal in favor of the alternative that obtains the highest score with such weights. The winner should be the alternative that is supported by the largest “number” of possible weight vectors (since the space of possible weights is continuous, we reason about volumes). More precisely, the set of feasible weights is partitioned with respect to which alternative has the highest score, and we sort the alternatives based on the volume of each element of the partition. We think that our proposed method better captures the essence of the idea of reasoning about the different possible scoring vector, than many methods previously proposed.

The article is organized as follows:

- In Section 2 we review some background and discuss related works.
- In Section 3 we present our baseline method to discriminate candidates, called *VolumetricTop*, provide some insights and an analysis of its properties.
- In Section 4 we provide an alternative method, *VolumetricTop with runoff* (*Volumetric-runoff*) that proceeds by using *VolumetricTop* repeatedly; it progressively eliminates the worst performing alternative until only one alternative remains, that is declared to be the winner.
- In Section 5 we discuss a variation of *VolumetricTop*, called *VolumetricTop with tournament* (*Volumetric-tournament*), that performs a series of pairwise comparisons between the alternatives. The alternative that prevails in the highest number of comparisons is declared to be the winner.
- We then present a use case with a real dataset in Section 6 considering data from different seasons of the F1 racing championship.
- Then, we conclude with discussions and final remarks (Section 7).
- Appendix A describes an approximate method for computing the volume of the polytope that can be employed by *VolumetricTop* and the other methods (although other algorithms to compute the volume could be used).

2 Background

In this section we review relevant background; we start by presenting the basic decision problem and some notation; then we present some background knowledge on scoring rule and on social choice; finally we discuss related works on scoring rules with uncertain weights, stressing the limitations of previous approaches.

2.1 Preliminaries

We assume n agents, or voters, express preferences in the form of rankings involving a set of m candidates $A = \{a, b, c, \dots\}$, also called items or alternatives; rankings are assumed to be linear orders (complete, transitive, asymmetric and irreflexive binary relation).

Positional scoring rules (see, for an introduction on social choice theory, for example, [32, 43]) discriminate between candidates by fixing a weight to each rank: they assign a score to each alternative based on its *rank distribution* (since all scoring rules in this work are *positional*, we simply call them scoring rules in the rest of the paper). We use v_j^x to denote the number of times alternative x was ranked in the j -th position. Note that $\sum_{j=1}^m v_j^x = n$ for each $x \in A$ and $\sum_{x \in A} v_j^x = n$ for each $j = 1, \dots, m$. We use v^x to denote the vector (v_1^x, \dots, v_m^x) . For example suppose a profile with $m = 3$ alternatives and $n = 2$ voters whose preferences are $a \succ b \succ c$ and $c \succ b \succ a$; then, the rank distributions are as follows: $v^a = (1, 0, 1)$, $v^b = (0, 2, 0)$ and $v^c = (1, 0, 1)$.

A scoring rule specifies the vector of weights w_1, \dots, w_m (also called *scoring vector*) to be assigned to each position. The score obtained by a candidate according to weight vector $w = (w_1, \dots, w_m)$ is

$$s_w(x) = \sum_{j=1}^m w_j v_j^x. \quad (1)$$

The alternative with the highest score is picked as the winner. It is also possible to rank the alternative from the best to the worst. For example *plurality* is obtained by setting $w_1 = 1$ and $w_j = 0$ for all $j \in \{2, \dots, m\}$.

By choosing a particular w , it is possible to specify some preferences on which kind of aggregation is desired, by giving more or less weight to the first positions compared to the positions that came afterwards in the ranking. First of all, we assume that not all weights are null, otherwise the alternatives are not discriminated (degenerated scoring rule). A natural hypothesis is to require that the sequence of weights is non-increasing: $w_i \geq w_{i+1}$ for all $i \in \{1, \dots, m-1\}$; this expresses

the basic intuition that in a ranking an alternative is at least as preferred to the alternatives that come afterwards in the ranking; this hypothesis is in practice always adopted.

A scoring rule is invariant to affine positive transformations of the scoring vector, which means that the ranking obtained with the scoring vector w and the ranking obtained with $w' = \alpha w + \beta$ (with $\alpha > 0$ and arbitrary β) are the same. Therefore, with no loss of generality, we let $w_1 = 1$ and $w_m = 0$ (therefore we have $m - 2$ degrees of freedom). A particular scoring rule is the Borda rule [7] that uses linear weights; given the assumptions, the Borda rule is given by setting $w_j = \frac{m-j}{m-1}$.

Note Borda may not be adequate in decision contexts where greater discrimination between the positions is needed. Indeed it is often assumed that the positional weights constitute a *convex sequence* [37, 19], meaning that the difference between the first and the second weight is not less than the difference between the second and the third, and so on. In such a case the weights need to satisfy the following constraint, for each i between 1 and $m - 2$:

$$w_i - w_{i+1} \geq w_{i+1} - w_{i+2} \iff w_i - 2w_{i+1} + w_{i+2} \geq 0. \quad (2)$$

Note that Borda and plurality are convex; furthermore convexity is often satisfied by the weights used when combining ranks in sports, races and other situations (e.g. formula one world championship, alpine skiing world cup).

We will use W^D to denote the set of scoring vectors with weakly decreasing (i.e. non-increasing) weights

$$W^D = \left\{ (w_1, \dots, w_m) \mid 1 = w_1 \geq w_2 \geq \dots \geq w_{m-1} \geq w_m = 0 \right\}. \quad (3)$$

We use W^C to denote the set of non-increasing scoring vectors whose weights constitute a convex sequence¹ (with our boundary assumptions, $w_1 = 1$ and $w_m = 0$)

$$W^C = \left\{ (w_1, \dots, w_m) \mid w \in W^D \wedge w_i - 2w_{i+1} + w_{i+2} \geq 0 \quad \forall i \in \llbracket m-2 \rrbracket \right\}. \quad (4)$$

While we mainly focus on W^D and W^C , the methods that we propose in this article can as well tackle situations where additional information about the weights is present. In fact, we might formalize the preferences for a certain difference between contiguous weights, that is:

$$w_i - w_{i+1} \geq t_i$$

where t_i represents the minimum difference between contiguous weights.

We also cover situations where only *top- k* positions, with $k < m - 1$, contribute to the score. In this case we have that $w_{k+1} = \dots = w_m = 0$, so that there are only k unknowns. This can be handled in our approach by simply considering, for each x , the ranking distribution $(v_1^x, \dots, v_k^x, v_{\geq k}^x)$ where $v_{\geq k}^x$ is the number of times that alternative x arrives in a position worse than k . This will be the case of the F1 dataset in Section 6. In order to avoid making the presentation overly complex, we assume $k = m - 1$, that is, all but the last position might contribute to the score.

2.2 Social choice theory

Social choice theory deals with the design and the analysis of methods for collective decision making; an important focus of social choice concerns methods for the aggregation of individual preferences into a collective preference. A *social choice function* associates a profile (a vector of linear orders, each one associated with a different voter) with one or more winners in A , according to a voting rule; instead, when a ranking is desired as output, a *social welfare function* is used to associate a profile with a ranking.

Theoreticians have formulated several properties that model good behaviors of a social choice or social welfare function. Ideally, a social choice/welfare function would satisfy all reasonable properties; unfortunately, this is not possible as it was shown by the Arrow's theorem [2]. When proposing a new rule, it is usual to analyze it to see which properties are satisfied and which are

¹ There is some redundancy in the constraints: it is enough to assume convexity and $w_{m-1} \geq 0$ to ensure that the sequence is not increasing.

not, to better appreciate the suitability of the rule in the desired context. Below we provide a short reminder of the traditional properties that are studied in the theory of social choice (see, for instance, [43]); note that these properties can be formulated in slightly different ways depending on whether the output is a set of winners (social choice function) or a ranking (social welfare function).

- *Neutrality* requires the social choice function to have no bias in favor of any alternative.
- For *anonymity* the identity of the voters are irrelevant. This is in contrast to *dictatorship*, where the preferences of a particular voter (the dictator) determines the outcome for the society.
- *Pareto* requires that, if all voters prefer alternative a to b , then the society should also prefers a to b
- A social choice function is *unanimous*, if all voters place an alternative in first position, this alternative is selected as winner.
- According to *Independence to Irrelevant Alternatives (IIA)*, the social preference between two alternatives should depend only on the preferences between these two alternatives.
- To be *monotone*, a social choice function has to satisfy the following: if some voters raises a winning alternative in their ranking without changing the orders of the remaining ones, that alternative continues to be a winner.
- *Homogeneity* requires that replicating the same profile several times lead to the same social choice.
- *Consistency* [40], sometimes also called reinforcement, states that, if an alternative is the winner when considering two different population of voters, then the same alternative should be a winner when the two populations are merged together.

Positional scoring rules, that we introduced in Section 2.1, are a class of social choice function that has been characterized by an important result by Young [40]: a neutral and anonymous social choice function is consistent if and only if it is a scoring rule. Moreover, Fishburn [9] discusses properties of the Borda method that makes it special among the positional scoring rules.

While social choice theory typically assumes that all aspects of the decision problem (including voter preferences) are known, researchers in computational social choice theory (the field dealing with algorithmic and computational aspects of social choice) have considered, in recent years, methods for dealing with partially specified preference profiles [18]. This interest arises from the realization that, in modern applications of social choice (such as group decision making and recommender systems), it is unrealistic to expect full knowledge on the decision context. So far, much of the emphasis has been on uncertainty over the votes (partial orders are provided instead of full rankings), rather than on the rule. A seminal work deals with algorithms (and their computational complexity) for determining possible and necessary winners when aggregating partial orders [39]: an alternative is a possible winner if and only if there exists a full profile (made of full rankings) that extends the given partial profile in which the alternative is a winner (similarly, a necessary winner is an alternative that is winning in all full profiles extending the current partial profile). Other papers have also dealt with uncertainty in votes, but focusing on elicitation [8, 41, 26] (asking additional preferences in order to converge to a winner as soon as possible).

While we consider rules that have weights associated to positions, Baumeister et al. [4] have considered aggregation rules with weights attached to individual voters and provided methods for computing possible winners. Some authors have considered machine learning methods in relation to scoring rules: Procaccia et al. [36] studied how a scoring vector can be learned from examples, while Haghtalab et al. [14] considered, in a setting of repeated social choice, how to learn the weights attached to voters.

One main underlying hypothesis of positional scoring rules is the assumption that all votes contribute equally to the score of an alternative. Indeed, scoring rules use a sum to aggregate the points that an alternative obtains from the different voters. Recently, this assumption has been questioned [5]; in particular, it has been proposed a positional voting system [12, 11] using the Ordered Weighted Average (OWA) operator: the scores are aggregated by taking into account the rank of a score in the ordered list of scores obtained from the votes. This allow to weigh differently the points obtained according to the best rank obtained, the second best rank, etc.; for example, one may want to disregard the best and the worst voter. This is similar to what happens in artistic

sports at the Olympics, that are judged by first removing the high and low scores and averaging the remaining scores achieved.

The work by Narodytska et al. [31] proposes an approach where several voting rules are considered at the same time; their approach works in rounds: the alternatives that are winners for at least one of the rules are kept while the others are discarded from the profile, the voting rules are then applied again to the modified profile and this process continues until a fixed point is found. While their work is relevant somehow, we stress that their work is not focused on scoring rules.

2.3 Scoring rules with uncertain weights

Scoring rules are a very common method for aggregating rankings (that can model preferences, outcomes of races, etc) into a single one and hence determine a winner. It is well known that setting the weights correctly is an important issue, as one can obtain different winners depending on the chosen weights. Motivated by this realization, several authors have proposed approaches to score alternatives without having a precise scoring vector. Similar to works concerning dominance relations in multiattribute domains [34,27,16], it is possible [37] to determine a partial order between the alternatives according to dominance relations that identify alternatives that are less preferred than another one for any feasible scoring vector. An alternative x dominates another alternative y with respect to W , when for all $w \in W$, $s(x;w) \geq s(y;w)$ and, for some w , we have $s(x;w) > s(y;w)$. This means that x will in general be a better choice than y , since x always achieves at the least the score of y and in some cases it achieves a strictly higher score. Note however that dominance only gives us a partial order, so it is usually not enough to unambiguously define a winner.

An important finding is that it is possible to quickly check dominance by considering *cumulative ranks*, also frequently called cumulative standings [37]. These are defined as the cumulative sum of the rank vectors, starting from the first position: for each $x \in A$, the vector $V^x = (V_1^x, \dots, V_{m-1}^x)$ is such that $V_j^x = \sum_{l=1}^j v_l^x$ is the number of times that alternative x has been ranked in position j or better; note that V^x has only $m-1$ components (we do not consider the number of times that an alternative was ranked at least in the last place, as it is always equal to the number of voters n). Cumulative ranks have been used by several authors (see e.g. [37,13,22,25,38]). Let $x, y \in A$; we have that x dominates y with respect to W^D if and only if $V_j^x \geq V_j^y \forall j \in \{1, \dots, m-1\}$, and there is $i \in \{1, \dots, m-1\}$ such that $V_i^x > V_i^y$.

To consider the case of convex weights (W^C), it is needed to consider the *double cumulative rank distribution*, also called cumulative of the cumulative standings [37,19]:

$$\mathcal{V}_j^x = \sum_{l=1}^j V_l^x = \sum_{l=1}^j \sum_{o=1}^l v_o^x = \sum_{l=1}^j (j-l+1)v_l^x \quad \forall j \in \{1, \dots, m-1\}$$

In the case of convex weights (W^C), we can compare the double cumulative ranks of two alternatives componentwise to check if a dominance relation exists. Let $x, y \in A$; x dominates y with respect to W^C if and only if all components of \mathcal{V}^x are higher or equal than the corresponding component of \mathcal{V}^y , and the relation is strict for at least one component.

Example 1 Consider the following numeric example. The first table reports the distribution of the ranks, to be read as follows: a is ranked two times first, two time second, etc., while b is never ranked first but is ranked 6 times second and 2 times third. The second table reports the cumulative ranks (for example, a is ranked 4 times in the second position or better) and the third the double cumulative ranks.

Alternative	v_1	v_2	v_3	v_4	Alternative	V_1	V_2	V_3	Alternative	\mathcal{V}_1	\mathcal{V}_2	\mathcal{V}_3
a	2	2	2	2	a	2	4	6	a	2	6	12
b	0	6	2	0	b	0	6	8	b	0	6	14
c	2	0	4	2	c	2	2	6	c	2	4	10
d	4	0	0	4	d	4	4	4	d	4	8	12

- When considering decreasing weights (set W^D), one can establish dominance by pairwise comparisons of rows in the table of cumulative ranks; for instance alternative a dominates alternative c since $V_1^a = V_1^c$, $V_2^a > V_2^c$, and $V_3^a = V_3^c$. The set of undominated alternatives is then $\{a, b, d\}$.
- When considering convex weights (set W^C), now alternative d dominates alternative a ; moreover d dominates c since $\mathcal{V}_j^d > \mathcal{V}_j^c$ for all $j \in \{1, \dots, m-1\}$. The set of undominated alternatives is $\{b, d\}$.

Dominance relations are useful, but usually are not enough to determine a winner; therefore it is typically necessary to design methods to differentiate the alternatives based on their ranks without knowing the weights precisely.

A method that determines a winner from the rank distribution can be seen as a social choice function and analyzed with respect to the properties studied in social choice. An additional desideratum emerges when considering aggregation with unknown weights: the aggregation method should not be perturbed by changes in the performance of dominated alternatives (alternatives that are always worse off than some others). This intuition is formalized by the property called *Independence from the Rank of Dominated Alternatives (IRDA)*, that is a relaxation of IIA requiring that dominated alternatives do not influence who the winner is; more precisely [38] it is necessary that the set of winners does not change whenever a profile is modified in a way that everything stays the same beside the ranks of dominated alternatives.

Definition 1 A social choice function f satisfies *Independence from the Rank of Dominated Alternatives (IRDA)* if, for any two profiles p, p' , defined on the same alternatives A , we have that $f(p) = f(p')$, the winners are the same in p and p' , whenever

1. the set of undominated alternatives is the same in p and p' , and
2. the rank distributions of all alternatives that are undominated is the same in p and p' .

When discussing methods for aggregation with uncertain weights, we need to verify whether or not this property is verified, in addition to the other properties introduced before. When considering the different methods available in the literature for aggregating rankings with respect to a scoring rule with an unknown or incompletely specified weighting vector, we will be careful to the following points:

- Does the method satisfy reasonable properties in terms of social choice theory?
- Does the method avoid to depend on arbitrary parameter choices? (our preference goes to method that do not introduce extra parameters).
- Is the method able to rank all alternatives? Or it is just capable to recommend a winning alternative? (an ideal method will be able to rank all alternatives, including dominated ones).
- Is the method optimistic or pessimistic? (we prefer methods that avoid rewarding extreme behaviors).
- Is the method easy to explain to a decision maker? (Scoring rules are popular because they are easy to understand; the fact to deal with unknown weights add a layer of complexity, but it would be great if the aggregation is done in a way that the decision maker can understand why an alternative has been chosen instead of another one).

Our analysis will show that none of the methods presented so far is completely convincing with respect to the previous points.

DEA-based models A classic paper is the one by Cook and Kress [6] that proposes, using methods based on *Data Envelopment Analysis (DEA)*, to evaluate each alternative with respect to the most favorable weights (the weight vector that gives the highest score to that alternative), while constraining all aggregate scores to be less than a maximum valid value (typically set to one). The

score of each alternative can be computed using the following linear program:

$$Z^x = \max \sum_{j=1}^m w_j v_j^x \quad (5)$$

$$\text{s.t. } \sum_{j=1}^m w_j v_j^y \leq 1 \quad \forall y \in A \quad (6)$$

$$w_j - w_{j+1} \geq d(j, \epsilon) \quad (7)$$

$$w_k \geq d(k, \epsilon) \quad (8)$$

The main limitation of this procedure is that often several alternatives achieve the maximum attainable score. Moreover, it is not obvious how to choose the functions $d(j, \epsilon)$; this is however critical since, depending on the choice of $d(j, \epsilon)$, a different winner may arise. Cook and Kress propose another optimization method to break ties. However, this second optimization problem is found to be equivalent to scoring rules with specific weights (and not very convincing given the goal of measuring each alternative to weight vector that most favours it against the others).

Several other model, based on modifications of the original idea by Cook and Kress have then been proposed. An excellent review of many of these methods is provided by by Llamazares and Peña [22], underlying the limits of many of these works.

Hashimoto [15] modifies the DEA method of Cook by allowing an alternative x to obtain a score higher than one by constraining the score of all others alternatives to be less than one (this relates to the so-called exclusion method; see [1]) to Cook and Kress's model. This means that the Constraint (6) is substituted with the following:

$$\sum_{j=1}^m w_j v_j^y \leq 1 \quad \forall y \in A \setminus \{x\}.$$

The model also presents a simplification since it fixes $d(j, \epsilon) = \epsilon$, that is, uses a fixed small constant ϵ instead of the discriminative function $d(j, \epsilon)$. The main problem of this method is that, as shown by Llamazares and Peña [22], dominated candidates may change the order of undominated candidates: Hashimoto's model does not satisfy the IRDA property.

A method based on cross-evaluation (each alternative is evaluated not just with respect to the most favouring weight vector but also with respect to vectors favouring the competitors) has been proposed [13] but since it is non monotonic, it is not considered a suitable choice. The models of Obata and Ishii [33] and its extension [10] suffer from the dependency on dominated alternatives (IRDA is not satisfied). Khodabakhshi and Aryavash [17] propose a method that is closely related to Hurwicz's criterion, where the objective function is a combination of an optimistic and of a pessimistic component considering several restrictions on the weight; later Llamazares [20] has provided closed-form expressions for this model.

Llamazares and Peña proposed [23,19] a modification of the DEA model of Cook and Kress where they introduce a new constraint requiring that the sum of the scores of the alternatives not being evaluated is constant. We also note that the idea of letting the weights to be completely flexible has been applied to multiattribute decision making problems [21].

Regret-based method One could argue that the previous methods are "optimistic" in the sense that they associate each alternative with its highest value, under some constraints on the scoring vector. Recently, it has been proposed [38] to adopt the criterion of minimax regret in order to determine the winner in the case of uncertain weights. This approach associates each alternative with its maximum regret and the alternative associated with the minimax value is chosen as the winner. More formally, the *max regret* (MR) of alternative x is defined as:

$$MR(x; W) = \max_{y \in A} \max_{w \in W} \left\{ s_w(y) - s_w(x) \right\} = \max_{y \in A} \max_{w \in W} \left\{ \sum_{j=1}^{m-1} w_j v_j^y - \sum_{j=1}^{m-1} w_j v_j^x \right\}. \quad (9)$$

$MR(x; W)$ is the worst-case loss associated with picking alternative x as a winner when $w \in W$ (max regret is always non-negative). It can be seen as an adversarial selection of the scoring vector w

from W in order to maximize the loss between the chosen alternative x and the alternative with highest score using w as scoring vector.

The *minimax regret* is defined as $\text{MMR}(W) = \min_{x \in A} \text{MR}(x; W)$ and the set S^* of *regret-optimal* alternatives as $S^*(W) = \arg \min_{x \in A} \text{MR}(x; W)$. The set $S^*(W)$ contains the alternatives that achieve minimax regret: for all $x^* \in S^*(W)$, $\text{MR}(x^*; W) = \text{MMR}(W)$. The value of MMR is non-negative. The alternatives in $S^*(W)$ are to be considered as “winners” according to this approach.

The straightforward approach for computing minimax regret involves solving a series of linear optimization problems. The main results of the article by Viappiani is that the winners (the alternatives with lowest max regret) can be easily found without solving an optimization problem using formulas that involve cumulative ranks (in the case of W^D) and double cumulative ranks (in the case of W^C). This method has been then extended [30] to the case of simultaneous uncertainty about the rule and the user votes; the authors propose both a winner determination method based on regret and elicitation strategies; these strategies asks questions specifically picked with the aim of reducing regret.

The method based on regret has the advantage of being able to satisfy reasonable theoretical properties, it does not introduce additional parameters, it can rank all alternatives according to max regret, and it can provide explanations in terms of worst-case loss. However it is a pessimistic approach and as such it does not fit our desiderata.

3 Aggregation using Volumetric Approaches

The idea that we propose in this article is to view all feasible scoring vectors as equally important possible opinions about how the candidates should be ranked; the intuition is that the winner should be the alternative that is supported as winner by the largest number of scoring vectors. Since the space of possible weights is continuous, we associate each alternative with its region of optimality. The method is neither “optimistic” as the model based on data envelope analysis (and does not suffer from the issues associated with several of the approaches based on DEA [22]), nor “pessimistic” as the model based on minimax regret [38].

The first method that we propose, *VolumetricTop*, selects the alternative(s) with the greatest region of optimality. We first formalize the rule in Section 3.1; we then analyze *VolumetricTop* as a social choice function in Section 3.2.

3.1 VolumetricTop rule

We propose a new method to compare the alternatives when the weights are not known. Our method associates each alternative with the volume of the region of the weights that makes it a winner. We think that this method is relatively natural, as it embodies the idea of evaluating alternatives according to the different scoring vectors that are compatible with some given constraints, and not just a single one. More precisely, we define, for each alternative, the notion of optimal region as the set of weights that gives it a score higher than all other alternatives.

Definition 2 Given an alternative a , the *optimal region* of a is the set of feasible weights, denoted $W^{*,a}$, that gives a the highest score among the alternatives in A ; formally:

$$W^{*,a} = \{w \in W : s_w(a) \geq s_w(y) \forall y \in A\}.$$

Recall that the score of x with respect to weight vector w is $s_w(x) = \sum_{j=1}^m w_j v_j^x$. We observe that the region $W^{*,x}$ is a polytope and is encoded by the following linear constraints:

$$\sum_{j=1}^m w_j (v_j^x - v_j^y) \geq 0 \quad \forall y \in A \quad (10)$$

$$w_i - 2w_{i+1} + w_{i+2} \geq 0 \quad \forall i \in \{1, \dots, m-2\} \quad (11)$$

$$w_i \geq w_{i+1} \quad \forall i \in \{1, \dots, m-1\} \quad (12)$$

$$w_1 = 1, w_m = 0 \quad (13)$$

Constraint (10) imposes that the score of x must be higher (or equal to) than the score of all other alternatives. Constraints (11), (12) and (13) encodes the the feasible region of the weights (either W^D or W^C). Constraint (11) imposes the weights to be non decreasing. Constraint (12) imposes the weights to be convex (and should be omitted if this property is not required).

Now, let $\text{Vol}(W^{*,a})$ be the volume of the region $W^{*,a}$. The score Z_a associated to alternative a is the normalized volume of its optimality region $\text{Vol}(W^{*,a})$. The alternative (or, in case of ties, the alternatives) with highest value is (are) chosen as overall winner(s) by the *VolumetricTop* rule.

Definition 3 Let $a \in A$. The value Z_a is the normalized volume of the optimal region $W^{*,a}$ of a :

$$Z_a = \frac{\text{Vol}(W^{*,a})}{\sum_{x \in A} \text{Vol}(W^{*,x})}.$$

The *Volumetric-Top* method declares as winners the alternatives maximizing Z :

$$\arg \max_{a \in A} Z_a = \arg \max_{a \in A} \text{Vol}(W^{*,a}).$$

Obviously, if it is wished to output a ranking, instead of only determining the winners, we can rank the alternatives according to the volume of optimality Z ; i.e. we deem a better than b if $\text{Vol}(W^{*,a}) > \text{Vol}(W^{*,b})$ and we have a tie when two alternatives have the same Z value (therefore, in general, we output a weak order).

With 4 alternatives, the scoring vectors is composed by 4 elements, but since w_1 is set to 1 and w_4 is set to 0, there are only 2 degrees of freedom. As boundary conditions, we have $0 \leq w_2 \leq 1$ and $0 \leq w_3 \leq 1$.

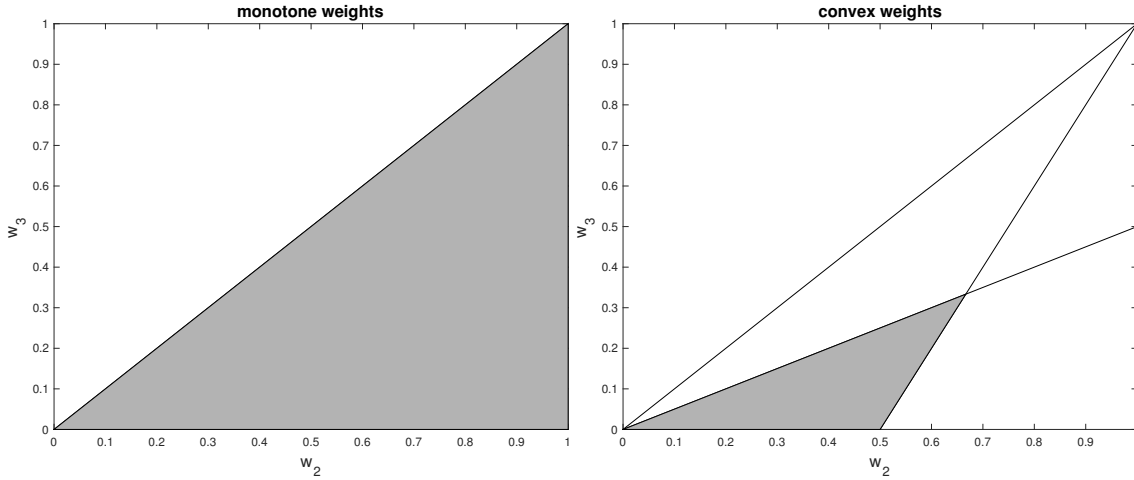


Fig. 1: With 4 alternatives the weight space is fully determined by w_2 and w_3 ; the shaded areas show the feasible monotone weights (on the left) and feasible convex weights (on the right).

For monotone weights, the feasible region is simply determined by the constraint $w_2 \geq w_3$. For convex weights we have as well the following two constraints that derive from Equation 2:

$$\begin{aligned} w_1 - w_2 \geq w_2 - w_3 &\iff w_3 \geq 2w_2 - 1 \\ w_2 - w_3 \geq w_3 - w_4 &\iff w_3 \leq \frac{1}{2}w_2 \end{aligned}$$

The feasible values for the pair (w_2, w_3) are shown in the shaded regions in Figure 1.

Example 2 We assume that the 4 alternatives a, b, c, d have the following rank distribution:

Alternative	v_1	v_2	v_3	v_4
a	3	2	2	2
b	0	7	2	0
c	2	0	5	2
d	4	0	0	5

By looking at the cumulative standings (number of times an alternative has reached at least position i , for all i from 1 to $m - 1$), it is easy to realize [37] that c is dominated by a . Indeed the vector of cumulative standings of a is $(3, 5, 7)$ and that of c is $(2, 2, 7)$; these are compared componentwise and we have that the former has values higher or equal than the latter.

The scores of the alternatives, written in function of the weights (we assume, without loss of generality, that $w_1 = 1$ and $w_4 = 0$) are:

$$s_w(a) = 2w_2 + 2w_3 + 3$$

$$s_w(b) = 7w_2 + 2w_3$$

$$s_w(c) = 5w_3 + 2$$

$$s_w(d) = 4$$

We now analyze the optimality regions of a, b, c, d and their volume, assuming $w \in W^D$. We can see that $W^{*,a}$ is the region satisfying the following constraints:

$$w \in W^D \left\{ \begin{array}{l} w_2 \geq w_3 \\ s_w(a) \geq s_w(x) \quad \forall x \in A \setminus \{a\} \end{array} \right. \left\{ \begin{array}{l} s_w(a) \geq s_w(b) \iff w_2 \leq \frac{3}{5} \\ s_w(a) \geq s_w(c) \iff w_3 \leq \frac{2}{3}w_2 + \frac{1}{3} \\ s_w(a) \geq s_w(d) \iff w_3 \geq -w_2 + \frac{1}{2} \end{array} \right.$$

These constraints encode the polygon with vertices $(0.5, 0), (0.6, 0), (0.6, 0.6), (0.25, 0.25)$ in the (w_2, w_3) plane.

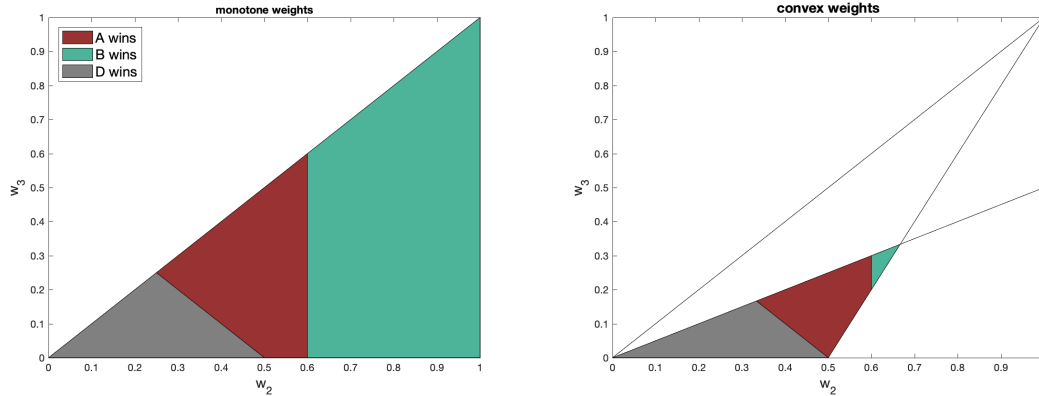


Fig. 2: Optimality regions associated to the alternatives of our example. Monotone weights are displayed on the left; convex weights on the right.

Figure 2 (left) shows the optimality regions for a, b, d (while the optimal region of c is empty) associated with this example when assuming monotone weights. By computing the volumes, which can be done exactly in this simple example, we obtain the following values:

$$Z_a = 0.24, \quad Z_b = 0.64, \quad Z_c = 0, \quad Z_d = 0.12.$$

Therefore, alternative b is the overall winner according to our method. Note that $Z_c = 0$ since c is dominated.

Now assume that the weights are in W^C (Equation 4), considering as well the convexity constraint. The optimal region of a , $W^{*,a}$, is determined by the following set of constraints:

$$w \in W^C \begin{cases} w_1 - w_2 \geq w_2 - w_3 & \iff w_3 \geq 2w_2 - 1 \\ w_2 - w_3 \geq w_3 - w_4 & \iff w_3 \leq \frac{1}{2}w_2 \end{cases}$$

$$s_w(a) \geq s_w(x) \quad \forall x \in A \setminus \{a\} \begin{cases} s_w(a) \geq s_w(b) & \iff w_2 \leq \frac{3}{5} \\ s_w(a) \geq s_w(c) & \iff w_3 \leq \frac{2}{3}w_2 + \frac{1}{3} \\ s_w(a) \geq s_w(d) & \iff w_3 \geq -w_2 + \frac{1}{2} \end{cases}$$

Figure 2 (right) shows the optimality regions in this case. The alternatives are associated with the following values:

$$Z_a = 0.46, \quad Z_b = 0.04, \quad Z_c = 0, \quad Z_d = 0.50.$$

and it is the alternative d being the winner in this case. Note that this time b (the winner assuming monotone weights) is optimal for very few values of w (that is, a very small volume).

Example 3 We consider the following rank distribution table, taken from [23], with 6 alternatives and 20 voters; however it is assumed that only the top 4 positions contribute to the score (and therefore the rank distribution is given only for the first 4 positions).

Alternative	v_1	v_2	v_3	v_4
a	3	3	4	3
b	4	5	5	2
c	6	2	3	2
d	6	2	2	6
e	0	4	3	4
f	1	4	3	3

We note that alternatives a , e and f are dominated. We obtain the following values:

$$Z_a = 0, \quad Z_b = 0.77, \quad Z_c = 0.03, \quad Z_d = 0.20, \quad Z_e = 0 \quad \text{and} \quad Z_f = 0;$$

therefore b is the winner according to our method. Note that, according to the first method of Llamazares [23], the four alternatives a, b, c, d are tied winners; the result is not very satisfying nor very informative (especially since a is dominated). According to Llamazares' second method [23], that is more discriminating, either alternative c or d , are declared winners (depending on some specific assumptions about the minimum difference between consecutive weights). We find choosing c as winner not very convincing: as we can see from our analysis, alternative c can only obtain the highest score for very few values of the weights, so it is perhaps not a very good choice.

We now consider convex weights; we therefore impose Constraint (11) as well. The scores are

$$Z_a = 0, \quad Z_b = 0.37, \quad Z_c = 0.26, \quad Z_d = 0.38, \quad \text{and} \quad Z_e = 0 \quad Z_f = 0;$$

the alternative d is now the winner. The fact that d is now preferred, rather than b , can be explained by the tendency of convex weights to favour more alternatives that are more times in the first positions (d is first six times, while b only four times).

In this article we focus on this article on the use of the volume of the optimal region as a social choice, and not so much on its actual computation and its issues. There are many techniques and software tools that can be used to compute the volume of a polytope. In Appendix A we provide a Monte Carlo approach to provide an approximate computation of Z values.

3.2 Analysis of of VolumetricTop

A probabilistic interpretation We now give a probabilistic interpretation of the volumetric approach. Assume that the decision maker does have a "true" set of weights but (due cognitive and time limitations) cannot state them precisely, but only reports that the weight belongs to W . From the point of view of the computer system tasked with deciding who should win, any weight w that

belongs to W seems plausible. Without any other information, it is reasonable to assume a uniform prior on w (according to the *principle of indifference*). If the “true” weights are sampled uniformly at random, from the space of feasible weights W , then it is easy to see that the probability of being the true winner is proportional to the volume of the optimal region. The VolumetricTop rule then consists in picking the alternative with maximum probability of winning assuming a uniform prior.

Theoretical properties We provide an analysis of the *Volumetric-top* aggregation method with respect to the traditional properties that are studied in the theory of social choice; in particular, we refer to the properties that were listed in Section 2.2. We view the procedure as a *social choice function*, mapping a profile to one or more winners. Note that the information basis is the rank distribution, imposing an equivalence relation between profiles that induces that same rank distribution v^x for each alternative $x \in A$.

The aggregation method is *neutral* and *anonymous*, and *unanimous* (if all voters place an alternative in first position, this alternative is selected as winner: indeed the optimality region of this alternative will cover the entire feasible region of the weights).

The aggregation method is *monotone*: if some voters raise a winning alternative in their ranking without changing the orders of the remaining ones, that alternative continues to be a winner. Observe that, when an alternative improves its position in the rankings, its optimality region cannot decrease and therefore its Z value cannot decrease. Hence, if alternative a is a winner, it continues to be a winner after a has improved with respect to some voters.

With now check *consistency* [40] (sometimes also called reinforcement). We remind (see Section 2.2) that a rule is consistent if an alternative that is the winner considering two different population of voters, then the same alternative should be a winner when the two populations are merged together. Volumetric aggregation is not consistent: this is not surprising, since, according to Young’s axiomatization a rule that is neutral, anonymous and consistent should be a scoring rule with a fixed weight vector (and volumetric aggregation is not a scoring rule). In Example 4 below we present an example showing that aggregation with Volumetric-top does not satisfy the property of consistency: indeed in both profiles 1 and 2, the alternative a is the winner according to Volumetric-top. In the combined profile, however, b , and not a , is the winner.

Example 4 *An example showing that aggregation with VolumetricTop does not satisfy consistency. We assume decreasing weights; alternative a is picked as the optimal alternative in both profile 1 and profile 2. Now consider the merged profile obtained by considering both the voters of profile 1 and of profile 2 (the rank distribution of the merged profile is the result of the sum of the rank distribution of profile 1 and 2, as can be easily checked by looking at the tables below). Since b , and not a , is selected by the rule in the merged profile, this means that the rule is not consistent.*

Profile 1:	Alternative	v_1	v_2	v_3	v_4	Alternative	Z
	a	4	0	0	4	a	0.333
	b	0	6	0	2	b	0.311
	c	3	0	3	2	c	0.056
	d	1	2	5	0	d	0.300
Profile 2:	Alternative	v_1	v_2	v_3	v_4	Alternative	Z
	a	4	0	0	4	a	0.333
	b	0	6	0	2	b	0.311
	c	1	2	5	0	c	0.300
	d	3	0	3	2	d	0.056
Profile 1 \cup Profile 2:	Alternative	v_1	v_2	v_3	v_4	Alternative	Z
	a	8	0	0	8	a	0.277
	b	0	12	0	4	b	0.282
	c	4	2	8	2	c	0.221
	d	4	2	8	2	d	0.221

We now consider *homogeneity*, requiring that replicating the same profile several times lead to the same social choice. In terms of rank distribution, when a profile is replicated k times, the value v_j^x is multiplied by k , for all positions j and for all alternatives x . When the rank distributions change in this way, the optimality region that does not change. Therefore homogeneity is satisfied by our method.

Another property commonly studied in social choice is the *independence from adding symmetric profiles* [29]: adding a symmetric profile (a profile where all permutations are present in the same number) should not change the result of the aggregation. This is actually the case: adding a symmetric profile induces all scores to be incremented by a constant value; the optimality regions are unchanged.

It is quite obvious that our rule does not satisfy *Independence to irrelevant alternatives (IIA)*; however we do not view the lack of satisfaction of IIA as necessarily problematic.

We look for relaxations of IIA that deal only with the rank of the alternatives. Llamazeres and Peña [22] have noticed that in some aggregation methods that have been proposed in the literature, the inefficient candidates may change the order of efficient candidates. Therefore it is desirable that an aggregation method is not perturbed by changes in the performance of dominated alternatives (alternatives that are always worse off than some others). Our method satisfies the property, called *Independence from the Rank of Dominated Alternatives (IRDA)*, that requires that dominated alternatives do not influence who the winner is. IRDA [38] requires that the set of winners does not change whenever a profile is modified in a way that everything stays the same beside the ranks of dominated alternatives.

Volumetric-top satisfies IRDA. We provide an intuition of this fact. If an alternative a dominates b , then it means that, for any feasible weight, the score of a is higher than b ; therefore b cannot be optimal (its region of optimality is void). In the case that the dominance is weak, then b can at most tie with a , and the region of optimality of a includes that of b . Therefore dominated alternatives cannot have an impact on the performance of undominated ones.

To conclude the analysis, we can say that overall *VolumetricTop* displays good characteristics. Its main limitation is that it does not discriminate all alternatives, in particular all dominated alternatives obtain a score of zero. Therefore it is not a good method when it is desired to obtain a full ranking in output.

4 Aggregation with Volumetric-runoff

Algorithm 1: VolumetricTop with Runoff (Volumetric-runoff)

Data: The set of alternatives A and their rank distribution v
Result: Overall winner
 $C \leftarrow A$;
while $|C| > 1$ **do**
 Compute Z_x for all alternatives in C ;
 Let $x^- \in \arg \min_{x \in C} Z_x$;
 Remove x^- from C ;
end
Let x^+ be the remaining item in C ;
return x^+ ;

Runoff methods are popular in social choice; they iteratively discard less promising alternatives until a winner is found. The advantage of runoff methods is to eliminate any impact of less interesting alternatives when comparing the more promising ones. A common case is the plurality rule: an alternative a may be better than b according to plurality (because the former gets more times the first ran) but when another alternative c is eliminated, b may gets more first ranksm, and b may win instead of a . Since c was not going to win anyway, one may want to compare a and b without considering c ; the runoff version does exactly that. An analogous phenomenon may happen with *VolumetricTop*; we will describe it with Example 2 below.

The method proposed in this Section is inspired by Single Transferable Vote (STV) [43] (runoff version of plurality) and Baldwin's method [3] (runoff version of Borda). The *runoff version* of the volumetric approach, whose pseudocode is shown in Algorithm 1, is called *Volumetric-runoff* and maintains a set of alternatives under consideration, initialized as the set A of all alternatives. At each iteration, the algorithm computes the normalized volume of the optimal region associated with each of the alternatives and eliminated the alternative associated with the lowest value. When a single alternative remains, it is returned as the winner.

The algorithm can be used to retrieve a ranking by sorting by the inverse order of removal; that is, by constructing a ranking by placing in the last position the item that is removed from C in the first round, by placing in the second last position the item that is removed from C in the second round, etc. Note, however, that dominated alternatives are not being differentiated, since they all have a score of zero in the initial round (this problem is solved by the method presented in Section 5). An important, subtle point is that the rank distribution is left unchanged, but when removing an alternative from the set C , then the optimality region is computed using the constraint 10 with respect to a smaller set $C \subseteq A$.

Example 2 (continued) *We provide an example of the application of the volumetric runoff variant, considering the example that we used in the previous Section for illustrating VolumetricTop.*

Alternative	v_1	v_2	v_3	v_4
a	3	2	2	2
b	0	7	2	0
c	2	0	5	2
d	4	0	0	5

In the first round, the scores are the following:

$$Z_a = 0.235, \quad Z_b = 0.64, \quad Z_c = 0, \quad Z_d = 0.125.$$

The item c is eliminated. Since c was dominated, its optimal region was empty (and its score was zero); this means that the optimal regions for a , b and d are not impacted from the deletion of c . Therefore, in the second round, the scores for a, b and d , are unchanged.

$$Z_a = 0.235, \quad Z_b = 0.64, \quad Z_d = 0.125.$$

In the third round, only alternatives a and b are left. The optimal regions are shown in Figure 3. We obtain the scores:

$$Z_a = 0.36, \quad Z_b = 0.64;$$

therefore a is eliminated and finally b is returned as winner. If we are interested in a ranking, according to the method the output ranking is $b \succ a \succ d \succ c$.

Now, we suppose that the weights are convex ($w \in W^C$) and we describe how *VolumetricTop-runoff* evaluates the alternatives. In the first round, we have:

$$Z_a = 0.46, \quad Z_b = 0.04, \quad Z_c = 0, \quad Z_d = 0.50.$$

Therefore, item c is eliminated. Subsequently (c was dominated, the optimal regions do not change):

$$Z_a = 0.46, \quad Z_b = 0.04, \quad Z_d = 0.50.$$

Item b is eliminated in the second round. In the third and final round the remaining items a and d obtain the following scores (Figure 4 shows how the optimal regions of a and d partition W^C):

$$Z_a = 0.50, \quad Z_d = 0.50.$$

Finally a and d are tied (note that using the original version of *VolumetricTop* d was the only winner). The final output ranking is $a \sim d \succ b \succ c$.

We conclude that, using the *Volumetric-runoff* method, the winner is b with monotone weights, while the winners are a and d that are tied when considering the convex weights.

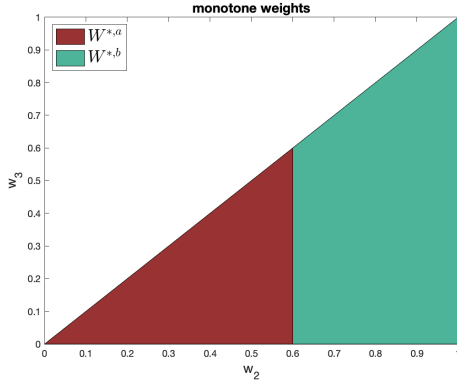


Fig. 3: The optimal regions in the third round of Volumetric-runoff, when only alternatives a and b are considered (Example 2; W^D).

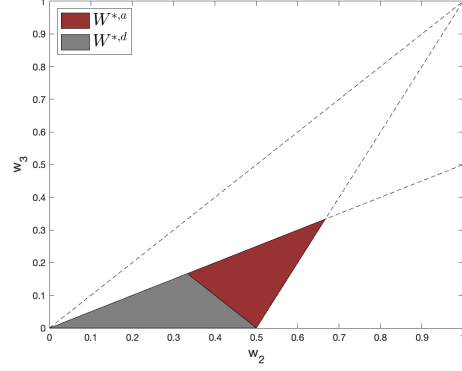


Fig. 4: The optimal regions in the third round of Volumetric-runoff, when only alternatives a and b are considered (Example 2; W^C).

Note that in the previous example, when considering convex weights (W^C), alternatives a and d obtain the same Z value when considered alone (they are tied in the last round of the runoff method); however d obtained a larger Z value than a when b was present (and therefore d was the only winner according to VolumetricTop). Since c is clearly far from winning (although c is not dominated), one may desire that the preference between a and d should not depend on the performance of c . In this sense the result obtained with Volumetric-runoff seems more plausible.

Example 3 (continued) We show how Volumetric-runoff, the runoff variant of VolumetricTop, works in the following scenario.

Alternative	v_1	v_2	v_3	v_4
a	3	3	4	3
b	4	5	5	2
c	6	2	3	2
d	6	2	2	6
e	0	4	3	4
f	1	4	3	3

Let's assume monotone weights. The Z scores obtained in the first round are:

$$Z_a = 0 \quad Z_b = 0.7733 \quad Z_c = 0.0267 \quad Z_d = 0.2000 \quad Z_e = 0.$$

In the first three rounds a , d , and e are eliminated since they have empty optimal regions (and the other Z values do not change after their elimination); c is eliminated next since it has the lowest value. The remaining contestants are b and d ; in the next round we have:

$$Z_b = 0.7778 \quad Z_d = 0.2222$$

Therefore the winner is b .

We now consider convex weights; the Z scores obtained in the first round are:

$$Z_a = 0 \quad Z_b = 0.3719 \quad Z_c = 0.2555 \quad Z_d = 0.3727 \quad Z_e = 0 \quad Z_f = 0.$$

The alternatives a , e and f are eliminated. In the next iteration, we have

$$Z_b = 0.3719 \quad Z_c = 0.2555 \quad Z_d = 0.3727.$$

and c is eliminated. Finally we have:

$$Z_b = 0.39 \quad Z_d = 0.61.$$

Therefore the winner is d . Note that in this example the winners obtained with Volumetric-runoff are the same as the winners obtained with VolumetricTop.

4.1 Theoretical Properties

It is straightforward to see that the runoff variant is: neutral, anonymous, unanimous, homogeneous, independent from adding symmetric profiles, independent from the rank of dominated alternatives. It is not consistent.

The main inconvenience of *Volumetric-runoff* is that it is not monotone. We show this by providing an example where an alternative is the winner in a given profile but does not win in a modified profile obtained by improving its performance.

Example 5 *First, consider the following rank distribution:*

Alternative	v_1	v_2	v_3
a	40	0	40
b	12	52	16
c	28	28	24

We have 3 alternatives and 80 voters; remember that we assume, without loss of generality, that $w_1 = 1$ (the weight of the first position) and $w_3 = 0$. We assume monotone weights. The uncertainty is on the weight w_2 associated with the second position. In this 1-dimensional setting, the score of an alternative can be represented by a line in the Cartesian plane; the scores of a , b , and c as a function of w_2 are shown in the left part of Figure 5.

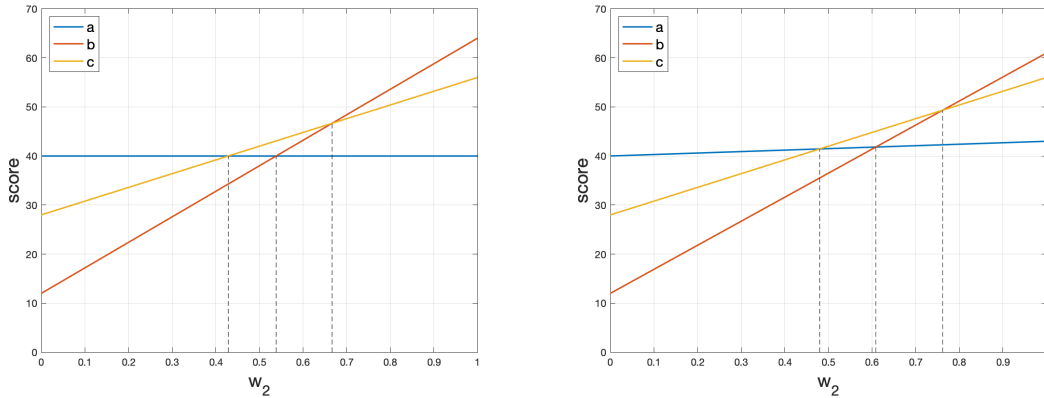


Fig. 5: The score $s_w(a)$, $s_w(b)$ and $s_w(c)$ of the alternatives as a function of w_2 (Example 5). Z values are the length of the interval that makes each alternative the one with the highest score. a wins in the original profile (on the left) but when the profile is modified, then c is the winner (on the right).

We compute the optimal regions:

- The optimal region of a is $0 \leq w_2 \leq 0.4286$, and therefore $Z(a) = 0.4286 - 0 = 0.4286$.
- The optimal region of b is $0.6667 \leq w_2 \leq 1$, and therefore $Z(b) = 1 - 0.6667 = 0.3333$
- The optimal region of c is $0.4286 \leq w_2 \leq 0.6667$, and therefore $Z(c) = 0.6667 - 0.4286 = 0.2381$.

Using the iterative runoff method, a is the winner. Indeed, according to the score just calculated, c obtains the lowest value and is therefore eliminated. Then in the second round:

$$Z_a = 0.5385, \quad Z_b = 0.4615.$$

And a is declared the winner.

Now, suppose that 3 voters change their preferences and they rank a in the second position rather than in the third. The resulting rank distribution is as follows:

Alternative	v_1	v_2	v_3
a	40	3	37
b	12	49	19
c	28	28	24

The right part of Figure 5 shows the scores of a , b , and c as a function of w_2 in the modified profile. Now a has the highest score when w_2 is between 0 and 0.48, b has the highest score when w_2 is between 0.7619 and 1, and c when $0.48 \leq w_2 \leq 0.7619$. Therefore the volumes of the optimal regions (that are also the scores obtained by standard *VolumetricTop*) are :

$$Z_a = 0.4800, \quad Z_b = 0.2381, \quad Z_c = 0.2819;$$

this means that alternative b is now eliminated in the first round. In the second round, the competition is between a and c , and a is eliminated.

$$Z_a = 0.48, \quad Z_c = 0.52.$$

Therefore, in the modified profile, c is the winner. Since the modified profile was obtained from the original one by improving the performance of a (at the expense of b ; with the positions of c unchanged), we can conclude that the method is not monotone.

Note that, in this example, the standard version of *VolumetricTop* would have declared a a winner in both profiles since a is associated with the larger optimal region; indeed standard *VolumetricTop* is monotone. Non-monotonicity emerges from the iterative process of elimination in the runoff version.

The fact that *Volumetric-runoff* is not monotone can be seen as a serious drawback. Moreover, it is not capable of ranking all alternatives; all dominated alternatives are tied. In the next Section we will see a method that can overcome these problems.

5 Aggregation with Volumetric-tournament

In some situations we are interested in providing a full ranking as output. *Volumetric-top* may not be able to differentiate between the alternatives that are inefficient (there is no scoring vector that make them optimal) whose Z score is 0. Still, we might want to differentiate them. Consider again Example 3: both $Z_e = 0$ and $Z_f = 0$, therefore a and e are tied. However it is desirable to rank e before f as the first dominates the latter.

Moreover, it is often possible that the number of alternatives is large but only a very few alternatives have a non empty optimal region; this make impossible to differentiate the other alternatives. This actually happens with the dataset of F1 drivers (see Section 6): each year only about 4-5 are undominated and therefore obtain a Z value higher than zero with *VolumetricTop*; all other drivers are tied for *VolumetricTop*. Reckoning this limitation of the *Volumetric-Top* rule, we design an approach based on *pairwise comparisons* of alternatives, that is able to rank all alternatives. For each pair of alternatives, we consider the region that makes the first preferred to the second, and the region of weights that makes the second preferred to the first.

Definition 4 Let $a, b \in A$ with $a \neq b$. The region *pairwise optimal region* of alternative a against b , denoted as $W^{a \geq b}$ is defined as:

$$W^{a \geq b} = \{w \in W : s_w(a) \geq s_w(b)\}$$

and $Z^{a \geq b}$ is the ratio between its volume and the whole region of weights W :

$$Z^{a \geq b} = \frac{\text{Vol}(W^{a \geq b})}{\text{Vol}(W)}.$$

Then, we compare $Z^{a \geq b}$ and $Z^{b \geq a}$; when the first is higher, or equivalently when $Z^{a \geq b} > 0.5$, we take this as indication that we should prefer a over b (informally, we say that a *beats* b or that a *wins against* b). Following this intuition, if there is an alternative that wins all pairwise comparisons against all others, then this alternative should be declared as winner. This is formalized by the notion of *Volumetric-Condorcet*, that it is analogous to the Condorcet property in standard methods of social choice.

Definition 5 The alternative a is a Volumetric-Condorcet winner if it holds that:

$$|\{b \in A \setminus \{a\} : Z^{a \geq b} > Z^{b \geq a}\}| = m - 1.$$

If there is no Volumetric-Condorcet winner, then we pick the alternative that beats most of the others in pairwise comparisons. We associate to each alternative $a \in A$ the score Z_a^{VT} that is the number of times that a is better than another alternative in pairwise comparisons.

Definition 6 The Volumetric-tournament score of alternative a is:

$$Z_a^{VT} = |\{b \in A \setminus \{a\} : Z^{a \geq b} > Z^{b \geq a}\}|$$

The alternative with highest Z^{VT} value is declared as winner.

It is immediate to realize that Volumetric-tournament will declare as winner the Volumetric-Condorcet winner if it exists.

It is possible to display graphically the outcome of Volumetric-tournament using a directed graph (the *tournament graph*) where the nodes correspond to alternatives and a direct edge connects an alternative with another where the former beats the latter. This will be shown in the examples below.

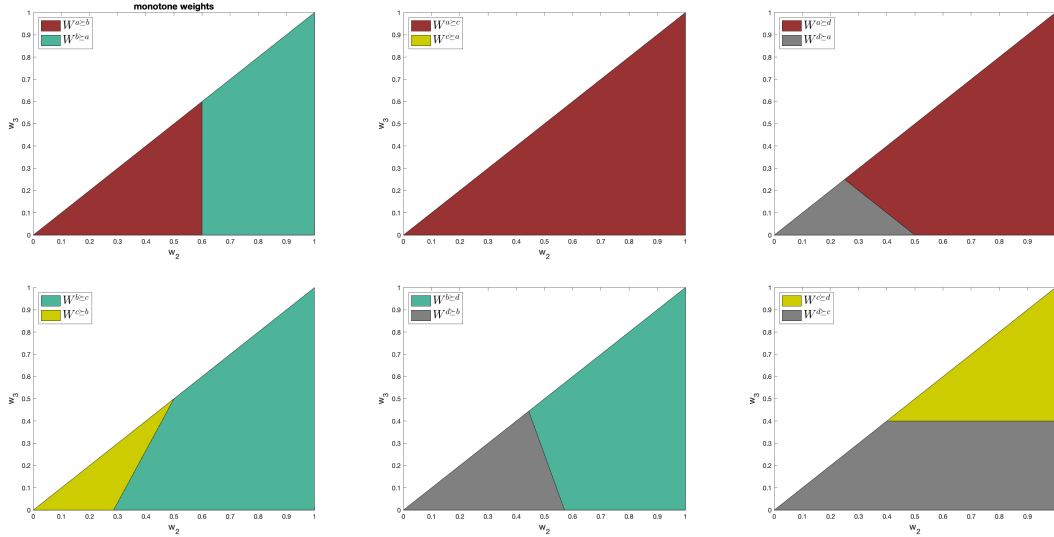


Fig. 6: The pairwise optimal regions for each pair of alternatives in the case of monotone weights.

Example 2 (continued) We compute the scores according to VolumetricTournament; as usual, we first consider the general class of decreasing weights. Remember that the scores of the alternatives in function of the weights (with $w_1 = 1$ and $w_4 = 0$) are:

$$\begin{aligned} s_w(a) &= 2w_2 + 2w_3 + 3, & s_w(b) &= 7w_2 + 2w_3, \\ s_w(c) &= 5w_3 + 2, & s_w(d) &= 4. \end{aligned}$$

The region $W^{a \geq b}$ is given by the constraints:

$$\begin{cases} w \in W^D & \iff w_2 \geq w_3 \\ s_w(a) \geq s_w(b) & \iff w_2 \leq \frac{3}{5} \end{cases}$$

while $W^{b \geq a}$ (note that $W^{a \geq b}$ and $W^{b \geq a}$ partition W^D) by the constraints:

$$\begin{cases} w \in W^D & \iff w_2 \geq w_3 \\ s_w(a) \leq s_w(b) & \iff w_2 \geq \frac{3}{5} \end{cases}$$

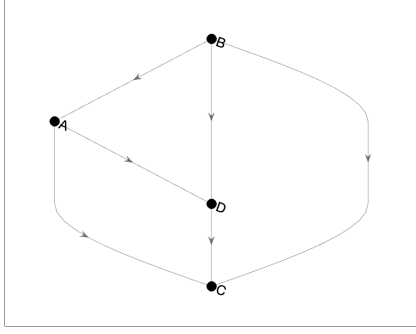


Fig. 7: The tournament graph of Example 2 when using Volumetric-tournament (case W^D).

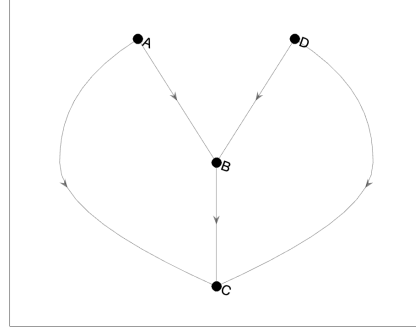


Fig. 8: The tournament graph of Example 2 when using Volumetric-tournament (case W^C).

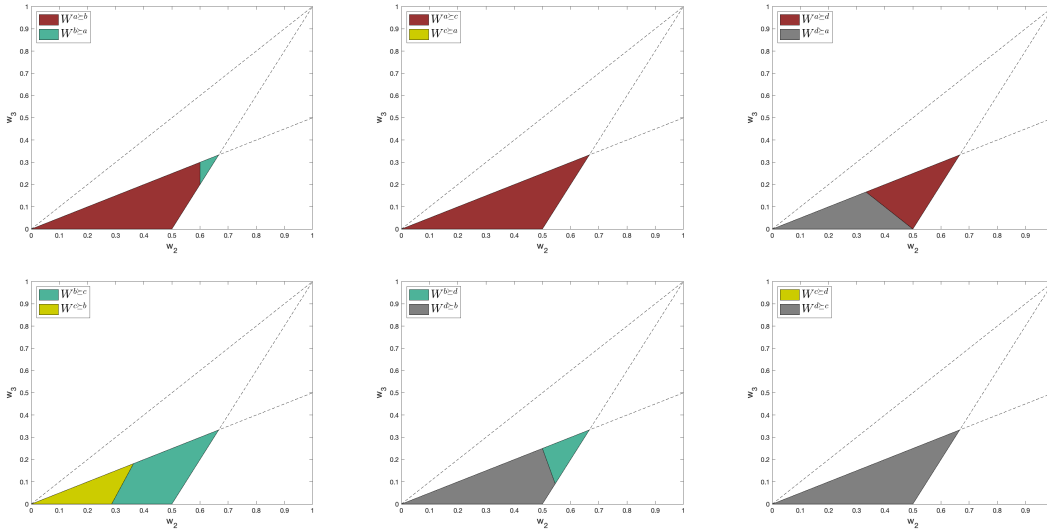


Fig. 9: The pairwise optimal regions for each pair of alternatives in the case of convex weights.

Therefore $W^{a \geq b}$ is the polygon with vertices $(0, 0)$, $(0.6, 0)$, $(0.6, 0.6)$ in the (w_2, w_3) plane, while $W^{b \geq a}$ is the one with vertices $(0.6, 0)$, $(1, 0)$, $(1, 1)$. All other pairwise optimal regions for this problems with respect to W^D can be determined in a similar way; they are displayed in Figure 6 and the (normalized) volumes of the pairwise regions are given in the following table:

Z^{\geq}	a	b	c	d
a	—	0.3600	1	0.8750
b	0.6400	—	0.8571	0.7460
c	0	0.1429	—	0.3600
d	0.1250	0.2540	0.6400	—

The scores can be determined considering, for each row, how many values are greater than 0.5.

- We can see that a beats c and d , but loses against b ; therefore $Z_a^{VT} = 2$.
- Alternative b wins against a , c and d (it is a Volumetric-Condorcet winner): $Z_b^{VT} = 3$.
- Alternative c loses all pairwise comparisons and $Z_c^{VT} = 0$.
- Alternative d wins against c and loses the other comparisons: $Z_d^{VT} = 1$.

Therefore the winner is alternative b (The tournament graph is shown in Figure 7).

Assuming convex weights ($w \in W^C$) we have the following volumes (Figure 9 shows the pairwise optimal regions):

Z^{\geq}	a	b	c	d
a	0	0.96	1	0.5
b	0.04	0	0.69	0.18
c	0	0.31	0	0
d	0.5	0.81	1	0

Therefore, alternatives a and d are both winners (they win against b and c and tie between themselves; c only wins against b):

$$Z^{VT}(a) = 2; \quad Z^{VT}(b) = 1; \quad Z^{VT}(c) = 0; \quad Z^{VT}(d) = 2.$$

The tournament graph is shown in Figure 8. Note that this result is consistent with the result obtained with Volumetric-runoff, but differs from Volumetric-top.

Example 3 (continued) We compute the scores according to the pairwise variant of Volumetric. With monotone weights (W^D) we have the following pairwise values:

Z^{\geq}	a	b	c	d	e	f
a	–	0	0	0	1.0	1.0
b	1.0	–	0.89	0.78	1.0	1.0
c	1.0	0.11	–	0.25	1.0	1.0
d	1.0	0.22	0.75	–	1.0	1.0
e	0	0	0	0	–	0
f	0	0	0	0	1.0	–

Therefore, considering the number of pairwise comparisons won by each alternative, we have the following scores:

$$Z^{VT}(a) = 2; \quad Z^{VT}(b) = 5; \quad Z^{VT}(c) = 3; \quad Z^{VT}(d) = 4; \quad Z^{VT}(e) = 0; \quad Z^{VT}(f) = 1.$$

Thus b is the overall winner (and it is a Volumetric-Condorcet). The final ranking obtained is $b \succ d \succ c \succ a \succ f \succ e$. Note that this version, differently from VolumetricTop, is able to produce a total order as output (as it can discriminated between the dominated alternatives as well).

Let's now consider convex weights; these are the associated volumes of the pairwise regions:

Z^{\geq}	a	b	c	d	e	f
a	0	0	0	0	1	1
b	1	0	0.43	0.40	1	1
c	1	0.57	0	0.40	1	1
d	1	0.60	0.60	0	1	1
e	0	0	0	0	0	0
f	0	0	0	0	1	0

Alternative d obtains a score of 5 and it is a Volumetric-Condorcet winner.

$$Z_a^{VT} = 2; \quad Z_b^{VT} = 3; \quad Z_c^{VT} = 4; \quad Z_d^{VT} = 5; \quad Z_e^{VT} = 0; \quad Z_f^{VT} = 1.$$

The final ranking obtained is $d \succ c \succ b \succ a \succ f \succ e$.

Note that alternative c , that is the winner according to Llamazares [23] (in some of variants of his approach), it is ranked third in both cases: this is because it is rarely a good alternatives according to most weights in either $w \in W^D$ or $w \in W^C$, when compared to the other alternatives (especially b and d).

Example 5 (continued) Reconsider the example that was used to show the nonmonotonicity of the runoff version. Considering pairwise matches (left part of Figure 5) we have that c beats both a and b ; a beats b but loses with c ; b loses with both.

- Considering a and b : $W^{a \geq b} = \{0 \leq w_2 \leq 0.5385\}$ and $W^{b \geq a} = \{0.5385 \leq w_2 \leq 1\}$ (refer to the left part of Figure 5); therefore a beats b since $W^{a \geq b}$ is a longer interval than $W^{b \geq a}$

- Considering a and c : $W^{a \geq c} = \{0 \leq w_2 \leq 0.4286\}$ and $W^{c \geq a} = \{0.4286 \leq w_2 \leq 1\}$ (refer to the left part of Figure 5); therefore c beats a
- Considering b and c : we have that $W^{b \geq c} = \{0.6666 \leq w_2 \leq 1\}$ and $W^{c \geq b} = \{0 \leq w_2 \leq 0.6666\}$; therefore c beats a

The tournament version of *VolumetricTop*, therefore, declares alternative c as the winner as it gets 2 points, alternative a is second with 1 point and finally, alternative b is third with 0 points.

In the modified profile (right part of Figure 5) c continues in being the winner according to *VolumetricTop*-tournament. Therefore there is no violation of monotonicity in this case. In fact, we shall see that the tournament version of *VolumetricTop* is monotone.

One could wonder if the binary relation induced by pairwise comparisons of optimal regions is transitive. We show that this is not true in general by providing a counterexample. Indeed in the following profile (the rank distribution is the same as the combined profile in Example 4) there is a violation of transitivity.

Alternative	v_1	v_2	v_3	v_4	Z^{\geq}	a	b	c	d
a	8	0	0	8	a	–	0.44	0.55	0.55
b	0	12	0	4	b	0.56	–	0.45	0.45
c	4	2	8	2	c	0.45	0.55	–	0.50
d	4	2	8	2	d	0.45	0.55	0.50	–

We have that:

- Alternative b beats a
- Alternative a beats c and d
- However, c beats b and d beats b

The tournament graph is shown in Figure 10. The two cycles (b - a - c - b and b - a - d - b) are reminiscent of Condorcet cycles in classic voting theory [43]. Note that our method, *Volumetric-tournament*, would pick a as a winner in this setting as it is pairwise better than two other alternatives, while b, c, d beat only one (and a tie holds between c and d).

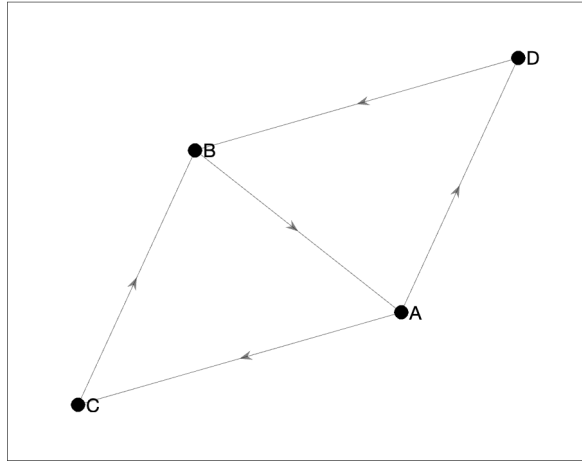


Fig. 10: An example of an intransitive tournament graph: alternative a beats c , c beats b , and b beats c . Moreover d beats b , b beats a , and a beats d .

5.1 Theoretical Properties

The rule is neutral, anonymous, unanimous, homogeneous, independent from adding symmetric profiles, and independent from the rank of dominated alternatives. It is not consistent (but we do not view this as problematic, since consistency is inherently linked to a scoring rule with known weights: see also discussion in Section 2.2). The rule is also monotone; therefore it is more satisfying than the runoff variant. Moreover, it can rank all alternatives (something that the original Volumetric-top cannot make).

Since this method satisfies all the characteristics that we have put forward as desirable, it is the one that look more convincing.

6 Case study

Alternative	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	$v_{\geq 16}$
L. di Grassi	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	17
R. Barrichello	0	0	0	1	1	1	1	1	2	3	0	4	0	3	0	2
M. Schumacher	0	0	0	3	0	3	2	0	3	2	2	0	1	0	1	2
J. Alguersuari	0	0	0	0	0	0	0	0	2	1	4	4	4	0	2	2
N. Heidfeld	0	0	0	0	0	0	0	1	1	0	1	0	0	0	0	2
N. Hulkenberg	0	0	0	0	0	1	1	1	0	4	0	0	2	2	1	7
T. Glock	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	17
J. Trulli	0	0	0	0	0	0	0	0	0	0	0	0	1	0	2	16
V. Liuzzi	0	0	0	0	0	1	1	0	2	2	1	1	2	0	1	8
K. Chandhok	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	7
F. Massa	0	2	3	3	0	1	2	1	1	1	1	0	0	0	3	1
K. Kobayashi	0	0	0	0	0	1	2	2	1	2	1	1	0	1	0	8
B. Senna	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	16
N. Rosberg	0	0	3	1	5	3	0	2	0	1	0	0	1	0	0	3
F. Alonso	5	2	3	2	0	0	2	2	0	0	0	0	1	1	0	1
A. Sutil	0	0	0	0	2	1	1	1	3	1	1	2	1	0	0	6
M. Webber	4	4	2	0	1	2	0	3	1	0	0	0	0	0	0	2
L. Hamilton	3	5	1	2	2	2	0	0	0	0	0	0	0	1	0	3
R. Kubica	0	1	2	1	4	1	3	2	1	0	1	0	0	0	0	3
V. Petrov	0	0	0	0	1	1	1	0	1	1	2	0	3	1	1	7
H. Kovalainen	0	0	0	0	0	0	0	0	0	0	0	1	2	2	1	13
S. Yamamoto	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	6
P. de la Rosa	0	0	0	0	0	0	1	0	0	0	2	2	0	2	0	7
S. Buemi	0	0	0	0	0	0	0	1	1	1	3	3	1	1	1	7
S. Vettel	5	2	3	3	0	1	1	0	0	0	0	0	0	0	1	3
C. Klien	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3
J. Button	2	3	2	3	3	0	1	2	0	0	0	1	0	0	0	2

Table 1: The rank distribution for the race results of the 2010 season of the F1 championships; assuming at most the first 15 positions gives points to the score.

We provide some numerical tests with real data obtained from the PREFLIB data repository², providing the rankings of each race in the F1 championships from 1961 to 2018. In this context, an alternative is a driver and each ranking is the result of a race. The rankings of a given year have to be aggregated in order to determine the winner for that year. The number of drivers (m) changes from year to year, ranging from 22 to 62; and as well the number of races (n).

We suppose that at most $k = 15$ positions contributes to the score; this is consistent with typical point systems used in race competitions. Obviously a driver that does not participate to a race does not get any point. If there are only $q < k$ drivers that finish a particular race, the points associated with positions $q + 1, \dots, k$ are not allocated.

First of all, we consider in details the competition of the year 2010 ($m = 27$ and $n = 19$). In Table 1 we show the rank distributions of the drivers taking part in the 2010 edition of the F1

² <https://www.preflib.org/dataset/00052>

championship. Because finishing a race with position $k + 1$ or worse does not give any point, the rank distribution has $k + 1$ elements, with the last component $v_{\geq k+1}$ being the number of times the racer finished in position $k + 1, k + 1, \dots, m$.

Dominance relations can be checked³ using the methods presented in Section 2.3. For example, assuming $w \in W^D$; we compare the components of the cumulative standings $(v_1, v_1 + v_2, \dots, \sum_{i=1}^k v_i)$ of Button and Schumacher:

$$(2, 5, 7, 10, 13, 13, 14, 16, 16, 16, 16, 17, 17, 17, 17, 19) \succ (0, 0, 0, 3, 3, 6, 8, 8, 11, 13, 15, 15, 16, 16, 17)$$

Since the first vector is component-wise strictly higher than the second, Button is *necessarily better* than Schumacher: for any feasible w , Button obtains a higher score (this situation is called *strong dominance*).

Now consider Button and Kubica; comparing the cumulative standings, we have that

$$(2, 5, 7, 10, 13, 13, 14, 16, 16, 16, 16, 17, 17, 17, 17) \succeq (0, 1, 3, 4, 8, 9, 12, 14, 15, 15, 16, 16, 16, 16, 16)$$

here \succeq means that all components of the first vector are higher or equal to the corresponding components of the second vector. This means that the Kubica cannot have a better score than Button for any $w \in W^D$. Indeed, when using 11-approval (the rule that gives 1 point if the driver is at most in the 11th position) Button and Kubica are tied with the same score (they both get 16 points); however there is no $w \in W^D$ that makes Kubica *strictly* better than Button.

In Table 1 the lines in bold correspond to undominated drivers (the drivers such that there exists no driver that dominated them): Alonso, Webber, Hamilton, Vettel and Button. All other drivers are dominated: there is no possible w that makes them strictly better than these five, and there is at least w for which they are worse.

We now consider how different aggregation methods differentiate the drivers with respect to the races of the year 2010:

- Alonso and Vettel win according to plurality, they both obtain 5 points. Webber gets 4, Hamilton 3 and Button 2.
- According to 3-approval, that is a scoring rule with weight $w = (1, 1, 1, 0, \dots, 0)$, Alonso, Webber and Vettel are tied with 10 points while Hamilton gets 9 points; finally Button gets 7 points.
- According to Borda, that is $w = (10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$, we have that Alonso wins, followed by Webber, Vettel and then Hamilton and Button.
- The official F1 competition made use of the scoring vector $(25, 18, 15, 12, 10, 8, 6, 4, 2, 1)$, when using these weights the winner is Sebastian Vettel.
- Instead, if one uses the weighting vector $(25, 20, 16, 13, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$ then we have that the winner is Fernando Alonso.
- Now, consider our volumetric approaches assuming monotone weights ($w \in W^D$):
 - By applying VolumetricTop, assuming monotone weights, F. Alonso is the winner with 0.54, followed by Webber with 0.38, Vettel is third with 0.05, then Hamilton with 0.02, and Button is fifth (0.01).
 - Using Volumetric-runoff, we also obtain that Alonso is the winner. Indeed, after having eliminated all the dominated alternatives, only Alonso, Webber, Hamilton, Vettel and Button remain. Button is then eliminated in the second round. Then, Hamilton is eliminated and Alonso, Webber and Vettel are the three remaining. After eliminating Vettel, in the last stage we have Alonso and Webber: finally Alonso wins.
 - For Volumetric-tournament, Alonso is the winner with 26 points.
- Considering convex weights ($w \in W^C$) we have that only Alonso, Webber and Vettel are undominated. VolumetricTop picks Vettel as the winner with Z-score of 0.96 (meaning that for the vast majority of compatible scoring weights he is the winner) while Alonso gets 0.04 and Webber a value less than 0.01.

Vettel is also the winner for Volumetric-runoff and Volumetric-tournament.

We now show that the tournament version of VolumetricTop allows the assessment of all race participants, including those dominated. In Table 2 we can see how the different drivers rank according to the tournament method when assuming, respectively, monotone and convex weights.

(a) monotone assumption W^D			(b) convex assumption W^C		
Rank	Driver	Tournament Score	Rank	Driver	Tournament Score
1	F. Alonso	26	1	S. Vettel	26
2	M. Webber	25	2	F. Alonso	25
3	S. Vettel	24	3	M. Webber	24
4	J. Button	23	4	L. Hamilton	23
5	L. Hamilton	22	5	J. Button	22
6	R. Kubica	21	6	F. Massa	21
7	N. Rosberg	20	7	N. Rosberg	20
8	F. Massa	19	8	R. Kubica	19
9	M. Schumacher	18	9	M. Schumacher	18
10	R. Barrichello	17	10	R. Barrichello	17
11	A. Sutil	16	11	A. Sutil	16
12	K. Kobayashi	15	12	K. Kobayashi	15
13	J. Alguersuari	14	13	V. Petrov	14
14	V. Petrov	13	14	N. Hulkenberg	13
15	N. Hulkenberg	12	15	V. Liuzzi	12
16	V. Liuzzi	11	16	J. Alguersuari	11
17	S. Buemi	10	17	S. Buemi	10
18	P. de la Rosa	9	18	P. de la Rosa	9
19	N. Heidfeld	8	19	N. Heidfeld	8
20	H. Kovalainen	7	20	H. Kovalainen	7
21	K. Chandhok	6	21	J. Trulli	6
22	J. Trulli	5	22	K. Chandhok	5
23	L. di Grassi	2	23	L. di Grassi	2
23	T. Glock	2	23	T. Glock	2
23	B. Senna	2	23	B. Senna	2
26	S. Yamamoto	1	26	S. Yamamoto	1
27	C. Klien	0	27	C. Klien	0

Table 2: The scores obtained by the F1 drivers in 2010 according to the tournament version of VolumetricTop.

Note that Di Grassi, Glock and Senna are tied in the 23-th position as they have exactly the same rank distribution.

We now present the results of aggregation for the different seasons in the table below, by comparing our method(s) with classical approaches of social choice. Table 3 shows, for the period 1987–2010, the scores of different drivers according to different methods: plurality (the driver who arrives first in a race takes 1 point, all others that 0), 3-approval (the first three drivers in a race take 1 point), Borda, the point system used by F1 in 2010, and the methods that we propose: VolumetricTop, runoff-Volumetric and Volumetric-tournament (either considering monotone weights, W^D or convex weights, W^C). For Volumetric-runoff the score corresponds to the number of rounds in which the alternative avoided being eliminated (the higher the better); for Volumetric-tournament the value is the number of times in which the alternatives has the larger volume in pairwise comparisons (the maximum being $m - 1$, with m being the number of drivers). Only drivers that are undominated in W^D are shown in the table; moreover, a checkmark in the last column means that the driver is also undominated in W^C . Dominance is checked using the techniques discussed in Section 2.3.

By analyzing the F1 data it is surprising to see that, in this dataset, the three variations of VolumetricTop always pick the same winner. However, it is often the case that a different assumption on the weights (monotone or convex) changes the winner: in 13 seasons the winner according VolumetricTop when assuming W^D is different from the winner according VolumetricTop when assuming W^C . Moreover, the way the three Volumetric methods rank the undominated alternatives changes depending on the method: for example, in 1969, assuming monotone weights ($w \in W^D$) both VolumetricTop and Volumetric-runoff rank Stewart first, then Beltoise as second, and Ickx third; while Volumetric-tournament ranks Ickx as second after Stewart, with Beltoise third.

³ See also [38] for analysis of this dataset with respect to possible winners and with minimax-regret.

Year	m	Alternative	Plur	3-appr	Borda	F1	VolTop W^D	VolTop W^C	runoff W^D	runoff W^C	tourn W^D	tourn W^C
1950	81	fagioli farina	0.00 3.00	5.00 3.00	4.87 4.73	3.48 3.72	0.72 0.28	0.50 0.50	80.00 79.00	80.00 79.00	80.00 79.00	80.00 79.00 ✓
1951	84	ascari fangio	2.00 2.00	4.00 5.00	5.27 5.67	4.12 4.28	0.04 0.96	0.00 1.00	82.00 83.00	0.00 83.00	82.00 83.00	82.00 83.00 ✓
1952	105	ascari farina	6.00 0.00	6.00 4.00	6.00 5.27	6.00 3.68	1.00 0.00	1.00 0.00	104.00 103.00	104.00 0.00	104.00 103.00	104.00 103.00 ✓
1953	108	ascari farina hawthorn	5.00 1.00 1.00	5.00 5.00 3.00	6.33 5.73 6.53	5.64 4.16 4.36	0.30 0.00 0.70	1.00 0.00 0.00	106.00 105.00 107.00	107.00 0.00 0.00	106.00 105.00 107.00	107.00 105.00 106.00 ✓
1954	97	fangio	6.00	7.00	7.67	7.08	1.00	1.00	96.00	96.00	96.00	96.00 ✓
1955	84	moss fangio mieres trintignant	1.00 4.00 0.00 1.00	3.00 5.00 0.00 1.00	3.67 5.00 3.33 3.67	2.52 4.72 1.44 2.00	0.00 1.00 0.00 0.00	0.00 1.00 0.00 0.00	80.00 83.00 81.00 82.00	0.00 83.00 0.00 0.00	81.00 83.00 80.00 82.00	82.00 83.00 ✓ 80.00 81.00
1956	85	fangio collins	2.00 2.00	5.00 4.00	5.87 4.53	4.44 3.52	1.00 0.00	1.00 0.00	84.00 83.00	84.00 0.00	84.00 83.00	84.00 83.00 ✓
1957	76	fangio	4.00	6.00	6.07	5.44	1.00	1.00	75.00	75.00	75.00	75.00 ✓
1958	87	moss hawthorn	4.00 1.00	5.00 7.00	5.47 7.87	4.72 5.60	0.00 1.00	0.01 0.99	85.00 86.00	85.00 86.00	85.00 86.00	85.00 86.00 ✓
1959	88	jack brabham trintignant	2.00 0.00	5.00 2.00	6.07 5.47	4.40 2.92	0.98 0.02	1.00 0.00	87.00 86.00	87.00 0.00	87.00 86.00	87.00 86.00 ✓
1960	91	phil hill trips jack brabham mclaren ireland	1.00 0.00 5.00 1.00 0.00	2.00 0.00 5.00 6.00 3.00	5.33 5.33 6.13 6.33 5.20	2.72 2.20 5.48 4.84 2.92	0.01 0.06 0.21 0.72 0.00	0.00 0.00 0.99 0.01 0.00	87.00 88.00 89.00 90.00 0.00	0.00 0.00 90.00 89.00 0.00	87.00 88.00 89.00 90.00 86.00	87.00 86.00 90.00 ✓ 89.00 88.00
1961	62	gurney phil hill moss	0.00 2.00 2.00	3.00 6.00 2.00	5.80 6.07 3.60	3.40 4.72 2.64	0.26 0.74 0.00	0.00 1.00 0.00	60.00 61.00 0.00	0.00 61.00 0.00	60.00 61.00 59.00	60.00 61.00 ✓ 59.00
1962	61	hill	4.00	6.00	7.80	6.32	1.00	1.00	60.00	60.00	60.00	60.00 ✓
1963	62	clark	7.00	9.00	9.33	8.48	1.00	1.00	61.00	61.00	61.00	61.00 ✓
1964	41	gurney bandini hill clark ginther surtees	2.00 1.00 2.00 3.00 0.00 2.00	2.00 4.00 5.00 3.00 2.00 6.00	4.80 5.67 6.67 5.47 7.00 5.80	2.48 3.32 5.04 4.12 3.84 4.76	0.00 0.00 0.24 0.00 0.76 0.00	0.00 0.00 1.00 0.00 0.00 0.00	0.00 0.00 39.00 37.00 40.00 38.00	0.00 0.00 40.00 38.00 0.00 39.00	35.00 37.00 39.00 36.00 40.00 38.00	35.00 36.00 40.00 ✓ 37.00 ✓ 38.00 ✓ 39.00 ✓
1965	54	hill clark	2.00 6.00	6.00 6.00	8.40 6.47	6.12 6.04	1.00 0.00	0.78 0.22	53.00 52.00	53.00 52.00	53.00 52.00	53.00 52.00 ✓
1966	33	jack brabham rindt	4.00 0.00	5.00 3.00	6.27 5.80	5.20 3.48	0.93 0.07	1.00 0.00	32.00 31.00	32.00 0.00	32.00 31.00	32.00 31.00 ✓
1967	45	clark jack brabham amon hulme	4.00 2.00 0.00 2.00	5.00 6.00 4.00 8.00	6.53 8.20 7.00 8.53	5.00 6.08 3.76 6.44	0.00 0.05 0.00 0.95	0.00 0.00 0.00 1.00	41.00 43.00 42.00 44.00	42.00 43.00 0.00 44.00	41.00 43.00 42.00 44.00	42.00 ✓ 43.00 ✓ 41.00 44.00 ✓
1968	43	siffert hill hulme rodriguez stewart	1.00 3.00 2.00 0.00 3.00	1.00 6.00 3.00 3.00 4.00	5.93 7.27 7.47 6.07 7.20	2.76 5.72 4.84 3.00 5.04	0.00 0.31 0.67 0.00 0.02	0.00 1.00 0.00 0.00 0.00	38.00 41.00 42.00 39.00 40.00	0.00 42.00 0.00 0.00 0.00	38.00 41.00 42.00 39.00 40.00	38.00 42.00 ✓ 40.00 ✓ 39.00 41.00
1969	31	ickx stewart beltoise	2.00 6.00 0.00	5.00 7.00 3.00	7.53 8.60 7.27	5.04 7.28 3.72	0.00 0.99 0.01	0.00 1.00 0.00	28.00 30.00 29.00	0.00 30.00 0.00	29.00 30.00 28.00	29.00 30.00 ✓ 28.00
1970	43	jack brabham amon hulme rodriguez ickx pescarolo rindt	1.00 0.00 0.00 1.00 3.00 0.00 5.00	4.00 3.00 4.00 2.00 5.00 1.00 5.00	6.27 7.20 6.93 6.13 6.53 6.27 5.53	3.36 3.80 4.20 3.44 4.96 2.60 5.00	0.00 0.74 0.19 0.00 0.07 0.00 0.00	0.00 0.00 0.00 0.00 0.66 0.00 0.34	37.00 42.00 41.00 0.00 40.00 38.00 39.00	0.00 42.00 40.00 0.00 42.00 38.00 41.00	39.00 42.00 41.00 37.00 40.00 38.00 36.00	38.00 39.00 ✓ 40.00 ✓ 37.00 42.00 ✓ 36.00 41.00 ✓
1971	50	peterson stewart	0.00 6.00	5.00 7.00	7.07 8.07	4.56 7.12	0.00 1.00	0.00 1.00	48.00 49.00	0.00 49.00	48.00 49.00	48.00 49.00 ✓
1972	42	emerson fittipaldi amon hulme	5.00 0.00 1.00	8.00 1.00 7.00	8.33 4.87 7.73	7.04 2.44 5.48	0.96 0.00 0.04	1.00 0.00 0.00	41.00 0.00 40.00	41.00 0.00 0.00	41.00 39.00 40.00	41.00 ✓ 39.00 40.00
1973	43	hulme stewart	1.00 5.00	3.00 8.00	8.60 11.20	4.68 8.84	0.00 1.00	0.00 1.00	41.00 42.00	0.00 42.00	41.00 42.00	41.00 42.00 ✓
1974	62	emerson fittipaldi regazzoni	3.00 1.00	7.00 7.00	9.93 10.20	7.28 6.92	0.11 0.89	0.93 0.07	60.00 61.00	61.00 60.00	60.00 61.00	61.00 ✓ 60.00 ✓
1975	52	lauda	5.00	8.00	11.00	8.52	1.00	1.00	51.00	51.00	51.00	51.00 ✓
1976	54	hunt lauda scheckter	6.00 5.00 1.00	8.00 9.00 5.00	9.73 9.93 10.47	8.24 8.28 6.84	0.03 0.15 0.82	0.16 0.84 0.00	51.00 52.00 53.00	52.00 53.00 0.00	51.00 52.00 53.00	52.00 ✓ 53.00 ✓ 51.00 ✓
1977	61	reutemann hunt mario andretti lauda scheckter	1.00 3.00 4.00 3.00 3.00	6.00 5.00 5.00 10.00 9.00	10.00 7.67 8.07 11.00 8.80	6.32 5.28 5.92 8.80 6.88	0.03 0.00 0.00 0.97 0.00	0.00 0.00 0.00 1.00 0.00	59.00 0.00 0.00 60.00 0.00	0.00 0.00 0.00 60.00 0.00	59.00 56.00 57.00 60.00 58.00	58.00 56.00 57.00 ✓ 60.00 ✓ 59.00
1978	46	laffite peterson mario andretti	0.00 2.00 6.00	2.00 7.00 7.00	8.53 8.53 9.73	4.04 6.36 7.80	0.04 0.00 0.96	0.00 0.00 1.00	44.00 43.00 45.00	0.00 0.00 45.00	44.00 43.00 45.00	43.00 44.00 45.00 ✓
1979	36	gilles villeneuve jones	3.00 4.00	7.00 5.00	10.47 7.33	7.04 5.24	0.01 0.00	0.00 0.00	34.00 0.00	34.00 0.00	34.00 33.00	34.00 ✓ 33.00 ✓

Year	m	Alternative	Plur	3-appr	Borda	F1	VolTop W^D	VolTop W^C	runoff W^D	runoff W^C	tourn W^D	tourn W^C
		vettel	5.00	10.00	15.07	11.24	0.20	0.40	23.00	23.00	23.00	23.00 ✓
2013	23	vettel	13.00	16.00	17.27	15.88	1.00	1.00	22.00	22.00	22.00	22.00 ✓
2014	24	ricciardo	3.00	9.00	14.00	9.76	0.01	0.00	21.00	0.00	21.00	21.00
		rosberg	5.00	15.00	15.27	12.68	0.26	0.00	22.00	0.00	22.00	22.00
		bottas	0.00	6.00	12.47	6.76	0.00	0.00	20.00	0.00	20.00	20.00
		hamilton	11.00	16.00	15.53	14.36	0.73	1.00	23.00	23.00	23.00	23.00 ✓
2015	22	hamilton	10.00	17.00	17.13	15.24	1.00	1.00	21.00	21.00	21.00	21.00 ✓
2016	24	ricciardo	1.00	8.00	16.27	10.24	0.00	0.00	22.00	0.00	21.00	21.00
		rosberg	9.00	16.00	18.33	15.40	1.00	0.99	23.00	23.00	23.00	23.00 ✓
		hamilton	10.00	17.00	17.60	15.20	0.00	0.01	21.00	22.00	22.00	22.00 ✓
2017	25	bottas	3.00	13.00	16.53	12.20	0.00	0.00	23.00	0.00	23.00	23.00
		hamilton	9.00	13.00	17.73	14.52	1.00	1.00	24.00	24.00	24.00	24.00 ✓
2018	20	hamilton	11.00	17.00	18.73	16.32	1.00	1.00	19.00	19.00	19.00	19.00 ✓
2019	20	hamilton	11.00	17.00	19.07	16.28	1.00	1.00	19.00	19.00	19.00	19.00 ✓
2020	23	verstappen	2.00	11.00	10.87	8.44	0.00	0.00	0.00	0.00	20.00	20.00
		bottas	2.00	10.00	12.33	8.60	0.00	0.00	0.00	0.00	21.00	21.00
		hamilton	11.00	13.00	14.93	13.44	1.00	1.00	22.00	22.00	22.00	22.00 ✓

Table 3: The scores obtained by undominated alternatives in F1 championships (years 1961-2008) with different aggregation methods (Plur=plurality; 3-appr=3-approval voting; VolTop=VolumetricTop; runoff=VolumetricRunoff; tourn=VolumetricTournament).

7 Conclusions

Scoring rules are social choice functions used to rank a set of alternatives. Their main issue is that the choice of the weights associated with the different positions is critical, as different winners can emerge depending on the chosen weights. Indeed the weights used in many contexts (for instance, in sports competitions) can often be seen as arbitrary. In this article, we considered preference aggregation in the context of social choice when the weights are not defined apriori, therefore avoiding the cognitive burden of specifying a scoring vector precisely. We provided a family of methods that evaluate alternatives with respect to all feasible weights belonging to a predefined wide class (monotone weights, convex weights). The proposed methods are analyzed from the point of view of social choice theory. We illustrate the use of our rules with several examples and provide a detailed presentation of its use in a real dataset.

Our approach allows to aggregate different opinions (expressed as rankings) and determine cardinal scores to differentiate the alternatives. The key idea of our work is to consider, for each alternative, the set of weights (the optimal region) that give it the highest score among the alternatives. The first method that we proposed, called VolumetricTop, sorts the alternatives by the volume of their optimal region. We believe that the VolumetricTop rule naturally catches the idea that, roughly speaking, an alternative that scores well with respect to many possible weights should be preferable to one that scores well for only few combinations of weights. Differently from previous works, the proposed rule is neither optimistic nor pessimistic, and it is also relatively easy to explain to a decision maker. Moreover, VolumetricTop has a probabilistic interpretation: it picks the alternative that has the highest probability of winning according to the scoring rule with uncertain w , assuming a uniform distribution on the weights in W .

We then study two modifications of VolumetricTop inspired by classic social choice methods: the first variant is a runoff version that iteratively eliminates the alternative with the smallest volume. Volumetric-runoff returns more plausible results than VolumetricTop in some circumstances since in VolumetricTop less performing alternative may have some undue influence on how the most performing one are ranked; however we show this alternative method to be non-monotonic, so this method is not entirely satisfying. A second variant, called Volumetric-tournament, is based on a tournament between the alternatives, where all possible pairs of alternatives are compared with respect to the volume of the region of the weights that make one better than the other. Volumetric-tournament satisfies the good properties of the original VolumetricTop method, while at the same time returns more reasonable results as Volumetric-runoff, and can also provide a full ranking of the alternatives as output (VolumetricTop only ranks the undominated ones). These observations make Volumetric-runoff the most promising approach.

In future works we are interested in evaluating the behavior of the proposed rules in case of manipulation, in studying additional theoretical properties of VolumetricTop and its variants, and in considering the effect of the presence of missing votes.

Compliance with Ethical Standards

The authors do not have any conflict of interests.

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A Sampling Techniques for Volume Approximation

Algorithm 2: MCMC computation of Z values

```

Data: The set of alternatives  $A$  and their rank distribution  $v$  and  $x \in A$ , number  $b$  of burn-in samples
Result:  $Z$  values
Let  $Z_x = 0 \quad \forall x \in A$ ;
Let  $w = (w_1, \dots, w_m) \in W$ ; // initialized to a feasible weight
for  $n = 2, \dots, N + b$  do
  for  $i = 1, \dots, m - 1$  do
    Compute  $w_i^\perp = \arg \min_{w \in W} w_i$ ;
    Compute  $w_i^\top = \arg \max_{w \in W} w_i$ ;
    Set  $w_i$  to random value sampled uniformly in the interval  $[w_i^\perp, w_i^\top]$ 
  end
  if  $n \leq b$  then // burn-in samples
    continue;
  end
   $x^* = \arg \max_{x \in A} s(x; w)$ ;
   $Z_{x^*} = Z_{x^*} + 1$ ; // increment Z value for winner wrt to vector  $w$ 
end
Let  $Z_x = Z_x / N \quad \forall x \in A$ ; // normalization of Z values
return  $Z$ ;

```

Our method requires to compute the volume of the optimal regions, that are multi-dimensional convex polytopes. This is not a trivial task. To do this we adopt a method based on a Markov chain using Monte Carlo. The approach,

detailed in Algorithm ??, proceeds as follows. Starting from an initial vector w , we repeatedly produce new samples starting by fixing all positions of the weight vector w except one, and sample uniformly at random the new value for such position in a way that the new vector satisfy the constraints of the feasible region. As usual in Monte Carlo methods, several samples at the beginning of the process are discarded (burn-in period).

As we show below, w_i^\perp and w_i^\top can be readily computed when considering monotone or convex weights. Indeed, let w_i be the value of w in the i -th dimension.

- For monotone weights with threshold values, at each step, w_i must be sampled uniformly between $w_{i+1} + t_{i+1}$ and $w_{i-1} - t_{i-1}$.

$$w_i^\perp = w_{i+1} + t_{i+1} \quad (14)$$

$$w_i^\top = w_{i-1} - t_{i-1} \quad (15)$$

- For convex sequences, we need to sample the new value so that the resulting vector is still non-increasing and convex. The constraints (from Equation 2) implicating position i are the following:

$$w_{i-2} - w_{i-1} \geq w_{i-1} - w_i$$

$$w_{i-1} - w_i \geq w_i - w_{i+1}$$

$$w_i - w_{i+1} \geq w_{i+1} - w_{i+2}$$

From the first and the third constraint we derive the lower bound for w_i to be $\max[2w_{i-1} - w_{i-2}, 2w_{i+1} - w_{i+2}]$, while the second constraint gives the upper bound $\frac{w_{i-1} - w_{i+1}}{2}$, from which we sample uniformly. Therefore, we derive:

$$w_i^\perp = \max[2w_{i-1} - w_{i-2}, 2w_{i+1} - w_{i+2}] \quad (16)$$

$$w_i^\top = \frac{w_{i-1} - w_{i+1}}{2} \quad (17)$$

This approach to computing the volume seems effective in practice.