



HAL
open science

Non-existing and ill-behaved coequalizers of locally ordered spaces

Emmanuel Haucourt, Pierre-Yves Coursolle

► **To cite this version:**

Emmanuel Haucourt, Pierre-Yves Coursolle. Non-existing and ill-behaved coequalizers of locally ordered spaces. *Journal of Applied and Computational Topology*, 2024, 8 (1), 10.1007/s41468-023-00155-4 . hal-04439755

HAL Id: hal-04439755

<https://hal.science/hal-04439755>

Submitted on 5 Feb 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Non-existing and ill-behaved coequalizers of locally ordered spaces

Pierre-Yves Coursole^{1*} and Emmanuel Haucourt^{1*}

^{1*}Laboratoire d'Informatique de l'École polytechnique (LIX - UMR 7161), École polytechnique, 1 rue Honoré d'Estienne d'Orves, Palaiseau, 91120, France.

*Corresponding author(s). E-mail(s): py.coursolle@yahoo.com;
emmanuel.haucourt@polytechnique.edu;

Abstract

Categories of locally ordered spaces are especially well-adapted to the realization of *most* precubical sets, although their colimits are not so easy to determine (in comparison with colimits of *d-spaces* for example). We use the plural here, as the notion of a locally ordered space varies from an author to another, only differing according to seemingly anodyne technical details. However, these differences have dramatic consequences on colimits: most categories of locally ordered spaces are *not* cocomplete, which answers a question that was neglected so far. We proceed by identifying the image of a directed loop γ on a locally ordered space \mathcal{X} to a single point. If we worked in the category of *d-spaces*, such an identification would be likely to create a vortex, while locally ordered spaces have no vortices. In fact, the existence and the nature of the corresponding coequalizer strongly depends on the topology around the image of γ . Besides the pathologies, we provide an example of well-behaved colimit of locally ordered spaces related to the realization of a *singular* precubical set. Moreover, for a well-chosen notion of locally ordered spaces, the latter induce diagrams of ordered spaces whose colimits are expected to be well-behaved; the category of locally ordered spaces is the smallest extension of the category of ordered spaces satisfying this property. Our locally ordered spaces are compared to streams, *d-spaces*, and to the 'original' *locally partially ordered spaces*.

Keywords: coequalizer, locally ordered space, vortex, directed topology, concurrency, higher dimensional automata, precubical set, realization

1 Introduction

Denote by $[\Sigma]$ the *precubical set* [6, 6.1] whose n -dimensional elements are the n -tuples of elements of a given set Σ ; for $i \in \{1, \dots, n\}$ we have

$$\partial_i^+ \langle a_1 \cdots a_n \rangle = \partial_i^- \langle a_1 \cdots a_n \rangle = \langle a_1 \cdots \hat{a}_i \cdots a_n \rangle.$$

A *higher dimensional automaton* (HDA) is a precubical set morphism λ whose target is of the form $[\Sigma]$. Following the works of Pratt and van Glabbeek [25, 31, 32] HDA's are used to represent concurrent programs: intuitively, the components of $\lambda(x) = \langle a_1 \cdots a_n \rangle$ are the actions that can be simultaneously executed by the tuple of processes $\langle p_1 \cdots p_n \rangle$ associated with x .

The usage of topological methods in concurrency theory was explicitly initiated in [6]; one of its key ingredients is the *realization* of *non-singular precubical sets* in the category of *locally partially ordered spaces* [6, 3.4, 6.23, 6.25, 6.30]. Unfortunately, the precubical set $[\Sigma]$ is singular. There is no proof so far that singular precubical sets cannot be realized in the category of locally partially ordered spaces, the construction from [6] simply does not apply to them.

The *locally ordered spaces* considered in this article (Definition 3.27) are not exactly the *locally partially ordered spaces* from [6, 3.4]. Yet, the former are more tractable, and they form a category in which the category of the latter is fully embedded (§6.3). In practice, a colimit of locally ordered spaces may be very difficult to describe (§4); as we shall see, it may not even exist (Example 4.23). This issue, along with the fact that no algebraic topologist would imagine working in categories with ill-behaved colimits, led to abandoning locally ordered spaces for more convenient frameworks (e.g. d-spaces [20, 21] or streams [12, 13], which are almost equivalent (§6.1) although the former seems to be widely preferred). We advocate for a more temperate approach: d-spaces (resp. streams) are better adapted to directed algebraic topology [7, 13] but only those induced by locally ordered spaces should be allowed to model programs (see §6.5 and compare with [14]). Whatever the topic you are interested in, colimits play a crucial role, this is why we carefully study them in the category of locally ordered spaces.

The article is organized as follows:

Section 2 briefly recalls the basics of *ordered spaces*; which form the category **Ord** (Definition 2.1). Section 3 is a rather detailed survey of *locally ordered spaces* with some new results (Propositions 3.25 and 3.46). Informally speaking, *local orders* are related to *locally ordered spaces* (Proposition 3.25 and Corollary 3.28) as *atlases* are to *manifolds* [18, p.2]. *Ordered bases* (Definition 3.29) are to locally ordered spaces what bases of topology are to topological spaces; they are introduced in view of Section 4. We characterise the category of locally ordered spaces (**LoSp**) as the smallest extension of **Ord** in which every *basic diagram* (Definition 3.39) admits a well-behaved colimit (Proposition 3.46). Section 4 is dedicated to the study of some ill-behaved

coequalizers in certain full subcategories of **LoSp**, see [4, §4]. The idea is quite simple: given a locally ordered subspace \mathcal{S}' of the *directed circle* \mathcal{S} (Examples 3.8 and 3.22), and a locally ordered space \mathcal{X} with a distinguished point $*$, one tries to ‘create’ a vortex by ‘pinching to a point’ the section $\mathcal{S}' \times \{*\}$ of the product $\mathcal{S}' \times \mathcal{X}$. Formally, we suppose that $1 \in \mathcal{S}'$ and study the coequalizer of the maps

$$s \in \mathcal{S}' \mapsto (1, *) \in \mathcal{S}' \times \mathcal{X} \quad \text{and} \quad s \in \mathcal{S}' \mapsto (s, *) \in \mathcal{S}' \times \mathcal{X} \quad .$$

We prove in Section 5 that the locally ordered realization of the 2-truncation of the terminal precubical set (i.e. $[\Sigma]$ when Σ is a singleton) exists and is well-behaved. Our exposition is intentionally detailed and illustrated in order to support the conjecture that beyond the combinatorial intricacies, the idea of the proof should apply to *every* precubical set. To compare, pinching a section of the directed cylinder results in a colimit that is very close to the one required to realize the following *cubical* set:

$$K_2 = \{s\} \quad \text{with} \quad \partial_1^+ s = \partial_1^- s \quad \sigma \partial_0^- \partial_0^- s = \partial_0^- s \quad .$$

Concretely, the two ‘vertical’ edges of the square s are identified, and the lower horizontal edge is reduced to a single point. In Section 6, we compare our locally ordered spaces to streams and to d-spaces, explain why they are more convenient than the locally partially ordered spaces from [6], and provide a computer scientist argument in favor of their use in concurrency theory.

2 Topology and order

Our references for all basic definitions are [23, 33] (general topology), [5, 28] (poset theory), and [26] (category theory). The partial order of a poset P is denoted by \leq_P , its underlying set by P (or $|P|$ when we need to emphasize). Given preorders (X, \preceq_X) and (Y, \preceq_Y) , a map from X to Y is said to be *order-preserving* when $f(x) \preceq_Y f(x')$ for all $x, x' \in X$. A preorder \preceq on X is said to be *chaotic* when $a \preceq b$ holds for all $a, b \in X$.

Definition 2.1. A *preordered space* is a topological space X equipped with a preorder. An *ordered space* is a preordered space whose preorder is antisymmetric. A *Nachbin ordered space*, or *pospace*, is an ordered space whose partial order is closed as a subspace of the product $X \times X$.¹

The following pospaces will be repeatedly used in the sequel:

Example 2.2. A subset A of B is said to be *proper* (in B) when neither A nor $B \setminus A$ is empty. The compact unit circle (with its usual topology) is the

¹Nachbin ordered spaces should not be confused with Nachbin-Hewitt spaces, which are also known as ‘realcompact spaces’ [19, p.166]. The term ‘pospace’ does not appear in [24], it was introduced later – see [9, p.272].

4 Colimits of locally ordered spaces

set of complex numbers of magnitude 1, namely

$$S^1 := \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{it} \mid t \in \mathbb{R}\} .$$

Given $x, y \in S^1$, there is a unique $a \in [0, 2\pi[$ such that $x = e^{ia}$ and a unique $b \in]a, a + 2\pi]$ such that $y = e^{ib}$; the *arc* from x to y is the image of the interval $[a, b]$ under the map $t \in \mathbb{R} \mapsto e^{it}$; it is denoted by $[x, y]$. The *open arc* from x to y is defined the same way with $]a, b[$ instead of $[a, b]$; it is denoted by $]x, y[$ (hence we have $]x, y[= [x, y] \setminus \{x, y\}$, note that $]x, x[= S^1 \setminus \{x\}$).

The *standard order* on a proper subset X of the unit circle is the partial order on X with respect to which x is smaller than y when $[x, y] \subseteq X$. The space X equipped with the standard order is a pospace.

Every arc is a proper subset of S^1 ; the *ordered open arc* from x to y is the set $]x, y[$ equipped with its standard order, we denote it by \vec{xy} .

Definition 2.3. A *concrete* category over \mathbf{B} is a faithful functor $U : \mathbf{C} \rightarrow \mathbf{B}$ [26, 1.6.17, p.42] (or [1, Definition 5.1(1), p. 61] when $\mathbf{B} \neq \mathbf{Set}$, the category of sets). In this article, the objects of \mathbf{C} and \mathbf{B} are sets with additional structure, the morphisms are the structure preserving maps, and the functors U just ‘forget’ some or all of the structure, so we omit them and just write that \mathbf{C} is concrete over \mathbf{B} . The colimit of a diagram in \mathbf{C} is said to be *well-behaved* when it (exists and it) is preserved by U , otherwise it is said to be *ill-behaved*. In the case where \mathbf{B} is (any full subcategory of) \mathbf{Top} , the functor U is denoted by Sp .

Remark 2.4. Given two concrete categories $U : \mathbf{C} \rightarrow \mathbf{B}$ and $U' : \mathbf{C}' \rightarrow \mathbf{B}$, every functor $F : \mathbf{C} \rightarrow \mathbf{C}'$ such that $U = U' \circ F$ is faithful because so is U .

Pospaces (resp. ordered spaces, preordered spaces) and order-preserving continuous maps form a bicomplete category, see [16, Corollary 3.14]. This category is concrete over \mathbf{Top} and \mathbf{PoSet} (the category of partially ordered sets and order-preserving maps).

Remark 2.5. The category of pospaces is actually concrete over \mathbf{Haus} (the category of Hausdorff spaces) because the underlying space of a pospace is Hausdorff [9, Proposition 1.4, p.273] or [24, Proposition 2, p.27].

Unfortunately, colimits in the category of ordered spaces (resp. pospaces) are ill-behaved in general:

Example 2.6. In the category of pospaces, we would expect the coequalizer of the maps

$$t \in \{0, 1\} \mapsto t \in [0, 1] \quad \text{and} \quad t \in \{0, 1\} \mapsto 1 - t \in [0, 1]$$

to be the ‘directed circle’; but it is the single element pospace (being understood that $\{0, 1\}$ is the discrete space equipped with the equality relation, while $[0, 1]$ is equipped with the usual topology and order): it suffices to observe that if a morphism $\gamma : [0, 1] \rightarrow X$ satisfies $\gamma(0) = \gamma(1)$, then it is constant because $\gamma(0) \leq_X \gamma(t) \leq_X \gamma(1)$ holds for every $t \in [0, 1]$ and the relation \leq_X is antisymmetric [16, p.131].

One may try to remedy this problem by weakening the notion of morphism between ordered spaces. To this end, we need to introduce some notation. Given preordered sets P and Q with $|P| \subseteq |Q|$, we write $P \rightarrow Q$ to mean that the inclusion map is order-preserving; and $P \hookrightarrow Q$ to mean that it is an order embedding (i.e. order-preserving and order-reflecting) [5, p.23].

Remark 2.7. The following are equivalent: i) $P \hookrightarrow Q$, ii) for all $x, y \in P$ we have $x \preceq_P y$ iff $x \preceq_Q y$, iii) P is a subpreorder of Q [28, p.15].

Remark 2.8. Given preordered sets U, V , and W satisfying $W \hookrightarrow U$ and $W \hookrightarrow V$, we have $a \preceq_U b \Leftrightarrow a \preceq_V b$ for all $a, b \in W$; this is a consequence of Remark 2.7.

The underlying set of a topological space X is denoted by X or $|X|$. Given another topological space Y , we write $Y \rightarrow X$ to mean that $|Y| \subseteq |X|$ and the inclusion map is continuous. We write $Y \hookrightarrow X$ to mean that Y is a subspace of X , and $Y \hookrightarrow\!\!\!\rightarrow X$ to mean that Y is an open subspace of X , i.e. $Y \hookrightarrow X$ and $|Y|$ is open in X . The following simple tricks are pervasively used in the sequel:

Lemma 2.9. Given topological spaces X and Y such that $|X| \subseteq |Y|$, we have $X \hookrightarrow\!\!\!\rightarrow Y$ if, and only if, the collection of open subsets of X is the collection of open subsets of Y that are included in $|X|$, i.e.

$$\{U \text{ open subset of } X\} = \{U \text{ open subset of } Y \mid U \subseteq |X|\}. \quad (1)$$

Proof Define $\tau_X = \{|X| \cap U \mid U \text{ open in } Y\}$, i.e. the topology induced on $|X|$ by Y . In particular τ_X contains every open subset of Y that is included in $|X|$. Suppose that $X \hookrightarrow\!\!\!\rightarrow Y$, so τ_X is the topology of X . Since $|X|$ is open in Y , every open subset of X (i.e. every element of τ_X) is open in Y . Conversely, assume that (1) is satisfied. In particular $|X|$ is open in Y . We want to prove that τ_X is the topology of X . If V is open in X then it is open in Y so $V = |X| \cap V \in \tau_X$. If U is an open subset of Y , then $U \cap |X|$ is an open subset of Y that is included in $|X|$, so it is an open subset of X by (1). \square

As an immediate consequence of Lemma 2.9 we have:

Corollary 2.10. Let U, V , and W be topological spaces such that $W \hookrightarrow\!\!\!\rightarrow U, V$ (resp. $U \hookrightarrow\!\!\!\rightarrow V \hookrightarrow\!\!\!\rightarrow W$). Every subset of $|W|$ that is open in some of the spaces U, V , and W is open in the others.

Lemma 2.11. Given posets U , V , and W satisfying $W \hookrightarrow U$ and $W \twoheadrightarrow V$, if $a \leq_U b$ with $a, b \in W$, then $a \leq_V b$.

Proof We have $a \leq_W b$ because $a \leq_U b$ and $W \hookrightarrow U$. Then $a \leq_V b$ because $W \twoheadrightarrow V$. \square

Lemma 2.12. Given topological spaces U , V , and W satisfying $W \twoheadrightarrow U$ and $W \hookrightarrow V$, if O and $|W|$ are open in U , then $|W| \cap O$ is open in V .

Proof The intersection $|W| \cap O$ is open in U ; it is also open in W because the inclusion map of $|W|$ in $|U|$ is continuous ($W \twoheadrightarrow U$). We deduce from Lemma 2.9 and $W \hookrightarrow V$ that $|W| \cap O$ is open in V . \square

Definition 2.13. Let X and Y be preordered spaces. Denote their underlying preordered sets by P_X and P_Y . We say that Y is an *ordered subspace* of X , and we write $Y \hookrightarrow X$, when $Sp(Y) \hookrightarrow Sp(X)$ and $P_Y \hookrightarrow P_X$. The ordered subspace Y is said to be *open* when $|Y|$ is an open subset of $Sp(X)$; in this case we write $Y \hookrightarrow X$ and the corresponding morphism is called an *open subspace inclusion*.

Lemma 2.14. Let X , Y , Z , W , and W' be ordered spaces:

- i) If $X \hookrightarrow Y$ and $Y \hookrightarrow Z$, then $X \hookrightarrow Z$.
- ii) If $W \hookrightarrow X$, Y and $W' \hookrightarrow Y$, Z then W and W' induce the same ordered space structure on $|W| \cap |W'|$, we denote it by $W \cap W'$ and we have $W \cap W' \hookrightarrow X$, Z .

Proof Denote by P_X and P_Z the underlying posets of X and Z ; we have $P_X \hookrightarrow P_Z$ by Remark 2.8, and $Sp(X) \hookrightarrow Sp(Z)$ by Lemma 2.9. We have proven the first item; note that it is a special case of the second one with $W = X$ and $W' = Y$.

Since $|W|$ and $|W'|$ are open in Y , so is $|W| \cap |W'|$. Hence $|W| \cap |W'|$ is open in W because $Sp(W) \hookrightarrow Sp(Y)$ (Lemma 2.9). We prove the same way that $|W| \cap |W'|$ is open in W' . Denote by V (resp. V') the ordered space induced by W (resp. W') on $|W| \cap |W'|$; we have $V \hookrightarrow W$ and $V' \hookrightarrow W'$. We deduce from the first item that $V \hookrightarrow Y$, X and that $V' \hookrightarrow Y$, Z . In particular V and V' are ordered subspaces of Y with the same underlying set (namely $|W| \cap |W'|$) so they are equal. \square

Definition 2.15. Given preordered spaces X and Y , a set map $f : |X| \rightarrow |Y|$ is said to be *locally order-preserving* when for all $x \in |X|$ and $V \hookrightarrow Y$ containing $f(x)$ there exists $U \hookrightarrow X$ containing x such that $f(U) \subseteq V$ and the restriction $f : U \rightarrow V$ is order-preserving (this implies that f is continuous). The category of ordered spaces and *locally order-preserving* maps is denoted by **Ord**; it is concrete over **Top** (Definition 2.3). The full subcategory of pospaces is denoted by **PoS**; it is concrete over **Haus** (Remark 2.5).

Example 2.16. If $X \hookrightarrow Y$ then the set inclusion $|X| \subseteq |Y|$ induces a locally order-preserving map from X to Y . Indeed, given $x \in X$ and $V \hookrightarrow Y$ containing x , we have $V \hookrightarrow Y$, V and $X \hookrightarrow Y$, X , so by applying Lemma 2.14 (ii) we obtain $X \cap V \hookrightarrow X$, V with $x \in X \cap V$.

The inclusion functor $\mathbf{PoSp} \hookrightarrow \mathbf{Ord}$ does not preserve colimits:

Example 2.17. Denote by D the diagram $\mathbb{R} \leftarrow]-\infty, 0[\hookrightarrow \mathbb{R}$. Its colimit in \mathbf{PoSp} is isomorphic to the subpospace $X = \mathbb{R} \times \{0\} \cup \{0\} \times [0, \infty[$ of \mathbb{R}^2 . To see this, let f and g be two locally order-preserving maps from \mathbb{R} to Z that coincide on $]-\infty, 0[$. Let t_n be a sequence of negative real numbers converging to 0. Define $x_n = f(t_n)$ for every $n \in \mathbb{N}$. Since f is continuous $f(0)$ is a limit of the sequence $\{x_n \mid n \in \mathbb{N}\}$. Since g is continuous and $g(t_n) = f(t_n)$ for every $n \in \mathbb{N}$, $g(0)$ is also a limit of the sequence $\{x_n \mid n \in \mathbb{N}\}$. Then we have $f(0) = g(0)$ because the underlying space of Z is Hausdorff (Remark 2.5). We define the map h from X to Z by $h(t, 0) = f(t)$ for every $t \in \mathbb{R}$, and by $h(0, t) = g(t)$ for every $[0, \infty[$. Formally speaking, we should specify how we embed the two copies of \mathbb{R} in X (it is tacitly given by the definition of h), and check that h is locally order-preserving. Both tasks are tedious routine.

The underlying set Y of the colimit of D in \mathbf{Ord} is made of one copy of $]-\infty, 0[$ and two copies of $[0, \infty[$, that is to say

$$Y =]-\infty, 0[\cup \{a\} \times [0, \infty[\cup \{b\} \times [0, \infty[.$$

We have the ‘projections’ $p_1 : Y \rightarrow \{n, a, b\}$ (with $p_1(y) = n$ when $y \in]-\infty, 0[$) and $p_2 : Y \rightarrow \mathbb{R}$. By definition we have $x \leq_Y y$ when $p_2(x) \leq p_2(y)$ and ($n \in \{p_1(x), p_1(y)\}$ or $p_1(x) = p_1(y)$). Around every point but a and b , the topology is induced by that of \mathbb{R} . A basis of neighborhoods of $(a, 0)$ is given by the sets $]-\varepsilon, 0[\cup \{a\} \times [0, \varepsilon[$; we replace a by b to obtain a basis of neighborhoods of $(b, 0)$. In particular the resulting space is *not* Hausdorff.

The coequalizer (in \mathbf{Ord} and \mathbf{PoSp}) of the diagram of Example 2.6 is still reduced to a point because of the following fact, which is an immediate consequence of [14, Proposition 5.2, p.1740]:

Proposition 2.18. Every locally order-preserving map defined on an interval of \mathbb{R} is order-preserving.

Nevertheless, the idea that motivated Definition 2.15 is the right one, as we shall see in the next section.

3 Locally ordered spaces

Let \mathbf{C} be a concrete category containing \mathbf{Ord} as a concrete full subcategory. The authors are convinced that if *all* the colimits in \mathbf{C} are well-behaved, then

\mathbf{C} has to contain pathological objects. So let us be parsimonious: the only diagrams of \mathbf{C} that are required to have a well-behaved colimit are the *basic* ones (Example 3.42). The category of locally ordered spaces is, in some sense, the smallest one satisfying the above requirement (Proposition 3.46).

3.1 Local orders

Definition 3.1. Two ordered spaces U and V are said to be *compatible* around $x \in U \cap V$ when there exists an ordered space W containing x such that $W \hookrightarrow U$ and $W \hookrightarrow V$. We say that W is a witness of compatibility (for U and V at x). The ordered spaces U and V are said to be *compatible* when they are compatible around every $x \in U \cap V$.

Remark 3.2. Two local orders W and U such that $W \hookrightarrow U$ are compatible: take W as witness of compatibility of W and U at every $x \in W$.

Lemma 3.3. Being compatible at a given point x is an equivalence relation among the ordered spaces containing x .

Proof An immediate consequence of the second item of Lemma 2.14. □

Definition 3.4. A *local order* is a collection \mathcal{U} of pairwise compatible ordered spaces. We denote by $|\mathcal{U}|$ the underlying set of \mathcal{U} , i.e. the union of the underlying sets of the elements of \mathcal{U} .

Definition 2.15 readily extends to local orders:

Definition 3.5. A *locally order-preserving* map from \mathcal{U} to \mathcal{V} , two local orders, is a map $f : |\mathcal{U}| \rightarrow |\mathcal{V}|$ such that for every $x \in |\mathcal{U}|$, every $V \in \mathcal{V}$, and every $B \hookrightarrow V$ containing $f(x)$ there exist $U \in \mathcal{U}$ and $A \hookrightarrow U$ containing x such that $f(A) \subseteq B$ and the restriction $f : A \rightarrow B$ is order-preserving. Local orders and locally order-preserving maps form the category **LO**.

Given two ordered spaces X and Y , a map $f : |X| \rightarrow |Y|$ is locally order-preserving from X to Y (Definition 2.15) iff it is locally order-preserving from $\{X\}$ to $\{Y\}$ (Definition 3.5). In other words we have:

Proposition 3.6. The map sending every ordered space X to the local order $\{X\}$ induces a full embedding of **Ord** into **LO**.

Example 3.7. Let \mathcal{V} be a local order and $\mathcal{U} \subseteq \mathcal{V}$. Consider $x \in |\mathcal{U}|$, $V \in \mathcal{V}$, and $B \hookrightarrow V$ containing x . Given $U \in \mathcal{U}$ contains x , we have an ordered space $W \hookrightarrow U$, V containing x (Definition 3.4). We also have $B \hookrightarrow V$, B , from which we deduce that $W \cap B \hookrightarrow U$, B (Lemma 2.14 (ii)). Hence the set map induced by the inclusion $|\mathcal{U}| \subseteq |\mathcal{V}|$ is locally order-preserving from \mathcal{U} to \mathcal{V} .

Example 3.8. The collection of all ordered open arcs (Example 2.2) forms a local order which we call the *directed circle*. Given any collection \mathcal{A} of ordered open arcs that cover the whole circle S^1 , the identity map id_{S^1} induces an isomorphism from \mathcal{A} to the directed circle. The local order morphisms from $\{\mathbb{R}\}$ to the directed circle are of the form $t \in \mathbb{R} \mapsto e^{i\theta(t)} \in S^1$ with θ morphism from $\{\mathbb{R}\}$ to $\{\mathbb{R}\}$ (such a morphism is order-preserving, see Proposition 2.18).

Example 3.9. The open subsets of a topological space equipped with their equality relations form a local order. Hence **Top** is a full subcategory of **LO**.

Definition 3.10. A *base of a topology* is a collection of sets \mathfrak{B} such that for all $B, B' \in \mathfrak{B}$, $B \cap B'$ is a union of elements of \mathfrak{B} . Equivalently, the collection of unions of elements of \mathfrak{B} is a topology on the set $\bigcup \mathfrak{B}$; we denote this set by $|\mathfrak{B}|$.

The forgetful functor from **Ord** to **Top** extends to **LO**, although the description of this extension requires some care:

Lemma 3.11. Given a local order \mathcal{U} , the collection

$$\{B \subseteq |\mathcal{U}| \mid \exists U \in \mathcal{U}; B \text{ open in } Sp(U)\}$$

is a base of topology on $|\mathcal{U}|$.

Proof Suppose that $x \in B_1 \cap B_2$ with B_i open in $U_i \in \mathcal{U}$ for $i \in \{1, 2\}$. Since \mathcal{U} is a local order, U_1 and U_2 are compatible around x , in particular we have an ordered space W containing x such that $W \xrightarrow{\hookrightarrow} U_1$ and $W \xrightarrow{\hookrightarrow} U_2$. Now $B_1 \cap |W|$ is open in $Sp(U_1)$, open in $Sp(W)$ because $Sp(W) \xrightarrow{\hookrightarrow} Sp(U_1)$ (Lemma 2.9), and open in $Sp(U_2)$ because $Sp(W) \xrightarrow{\hookrightarrow} Sp(U_2)$ (Lemma 2.9). Then $B_1 \cap |W| \cap B_2$ is open in $Sp(U_2)$, included in $B_1 \cap B_2$, and contains x . \square

Definition 3.12. The *underlying topological space* of the local order \mathcal{U} is generated by the base given by Lemma 3.11, we denote it by $Sp(\mathcal{U})$.

Lemma 3.13. For every element U of a local order \mathcal{U} , we have $Sp(U) \xrightarrow{\hookrightarrow} Sp(\mathcal{U})$.

Proof Following Definition 3.12 any open subset of $Sp(U)$ is open in $Sp(\mathcal{U})$. Conversely, let $O \subseteq |\mathcal{U}|$ be open in $Sp(\mathcal{U})$, and let $x \in O$. By Definition 3.12 we have $V \in \mathcal{U}$ and A open subset of $Sp(V)$ such that $x \in A \subseteq O$. Since U and V are compatible around x , we have an ordered space W containing x such that $W \xrightarrow{\hookrightarrow} U$ and $W \xrightarrow{\hookrightarrow} V$. In particular $A \cap |W|$ is open in $Sp(V)$, therefore it is open in $Sp(U)$ (Corollary 2.10), it contains x , and it is included in O . It follows that O is open in $Sp(U)$. We conclude that $Sp(U) \xrightarrow{\hookrightarrow} Sp(\mathcal{U})$ from Lemma 2.9. \square

Corollary 3.14. Given local orders \mathcal{U} and \mathcal{V} with $Sp(\mathcal{U})$ discrete, any set map $f : |\mathcal{U}| \rightarrow |\mathcal{V}|$ induces a locally order-preserving map from \mathcal{U} to \mathcal{V} .

Proof By Lemma 3.13, the underlying space of every $U \in \mathcal{U}$ is discrete, so we can take $A = \{x\}$ in Definition 3.5. \square

Lemma 3.15. Definition 3.12 extends to a functor $Sp : \mathbf{LO} \rightarrow \mathbf{Top}$.

Proof Let $f : \mathcal{U} \rightarrow \mathcal{V}$ be a local order morphism. Suppose that B is an element of the base of $Sp(\mathcal{V})$ (given by Lemma 3.11) containing $f(x)$ for a given $x \in |\mathcal{U}|$. By Definition 3.5 we have an element A of the base of $Sp(\mathcal{U})$ (given by Lemma 3.11) containing x such that $f(A) \subseteq B$. \square

Definition 3.16. Given $U \in \mathcal{U}$ and Y a subspace of $Sp(\mathcal{U})$ or a subset of $|\mathcal{U}|$, we denote by $Y \cap U$ the ordered space induced by U on $|U| \cap |Y|$. The family of ordered spaces $\{Y \cap U \mid U \in \mathcal{U}\}$ is a local order on Y ; it is called the *sub-local order* of \mathcal{U} on Y .

Example 3.17. The underlying space of the directed circle (Example 3.8) is the unit circle S^1 .

3.2 Locally ordered spaces

Definition 3.18. A local order \mathcal{U} is said to be *maximal* when for every ordered space X if $|X| \subseteq |\mathcal{U}|$ and $\mathcal{U} \cup \{X\}$ is a local order, then $X \in \mathcal{U}$.

Lemma 3.19. Let U be an element of a local order \mathcal{U} ; if W is an ordered space such that $W \hookrightarrow U$, then W is compatible with every element of \mathcal{U} .

Proof Let $V \in \mathcal{U}$ and $x \in |W| \cap |V|$. The ordered spaces W and U are compatible around x (Remark 3.2); and so are U and V because they belong to \mathcal{U} . We deduce from Lemma 3.3 that W and V are compatible around x . Hence W is compatible with every element of \mathcal{U} ; since \mathcal{U} is maximal, it contains W . \square

Lemma 3.20. For every local order \mathcal{U} , there is a *unique* maximal local order $\bar{\mathcal{U}}$ such that $|\mathcal{U}| = |\bar{\mathcal{U}}|$ and $\mathcal{U} \subseteq \bar{\mathcal{U}}$; it is the collection of the ordered spaces V that are compatible with every $U \in \mathcal{U}$ and satisfy $|V| \subseteq |\mathcal{U}|$.

Proof We have $\mathcal{U} \subseteq \bar{\mathcal{U}}$ by Definition 3.4. Let $V_1, V_2 \in \bar{\mathcal{U}}$ and $x \in V_1 \cap V_2$. We have $U \in \mathcal{U}$ containing x and by definition of $\bar{\mathcal{U}}$, the ordered spaces U and V_1 , as well as U and V_2 , are compatible around x , therefore V_1 and V_2 are compatible around x (Lemma 3.3). Hence $\bar{\mathcal{U}}$ is a local order. Let X be an element of a local order \mathcal{V} such that $|\mathcal{U}| = |\mathcal{V}|$ and $\mathcal{U} \subseteq \mathcal{V}$. Then X is compatible with every element of \mathcal{V} , hence with every element of \mathcal{U} . Therefore X belongs to $\bar{\mathcal{U}}$. \square

Definition 3.21. The local order $\bar{\mathcal{U}}$ from Lemma 3.20 is called the *saturation* of the local order \mathcal{U} .

Example 3.22. The saturation \mathcal{S} of the directed circle is the collection of ordered spaces U such that $Sp(U)$ is a proper open subspace of S^1 and \leq_U is an extension of the standard order on U (Example 2.2). The order \leq_U induces a partial order \leq' on the set of connected components of U , which we denote by $\pi_0(U)$. For every $u \in U$, $\pi_0(u)$ is the connected component of U containing u . Then for all $u, v \in U$, we have

$$a \leq_U b \iff (\pi_0(a) <' \pi_0(b) \text{ or } (\pi_0(a) = \pi_0(b) \text{ and } [a, b] \subseteq U)). \quad (2)$$

Conversely, from any partial order \leq' on $\pi_0(U)$, we obtain a partial order \leq_U on U taking the equivalence (2) as a definition for $a \leq_U b$. The resulting partial order \leq_U extends the standard order on U . We insist that if a and b are *not* in the same connected component of the proper open subset X of S^1 , then we have $U, V \in \mathcal{S}$ such that $Sp(U) = Sp(V) = X$, $a <_U b$ and $b <_V a$. By a similar argument, we prove that the saturation of the local order $\{(\mathbb{R}, \leq)\}$ (Proposition 3.6) is made of the ordered spaces (U, \leq_U) such that U is open in \mathbb{R} and for all $u, u' \in U$, if $u \leq u'$ and $[u, u'] \subseteq U$, then $u \leq_U u'$.

Lemma 3.23. The underlying space of a local order \mathcal{U} is discrete if, and only if, (*) the saturation of \mathcal{U} is the family of *all* the ordered spaces whose underlying spaces are open in the underlying space of \mathcal{U} .

Proof Suppose that $Sp(\mathcal{U})$ is discrete and let X be an ordered space in $\bar{\mathcal{U}}$, i.e., $|X| \subseteq |\mathcal{U}|$ and X is compatible with every element of \mathcal{U} (Lemma 3.20). Then $\mathcal{U} \cup \{X\}$ is a local order (Definition 3.4) and $Sp(X)$ is open in $Sp(\mathcal{U} \cup \{X\}) = Sp(\mathcal{U})$, so $Sp(X)$ is discrete. Conversely, if X is an ordered space such that $|X| \subseteq |\mathcal{U}|$ and X is discrete, then for every $U \in \mathcal{U}$ and every $x \in |U| \cap |X|$, the singleton $\{x\}$ is a witness of compatibility of X and U around x (Definition 3.1), so $X \in \bar{\mathcal{U}}$.

Suppose that \mathcal{U} satisfies the property (*) of the statement. Let \leq be a total order on $|\mathcal{U}|$, see [30]. Both $(Sp(\mathcal{U}), \leq)$ and $(Sp(\mathcal{U}), \geq)$ belong to the local order $\bar{\mathcal{U}}$. Given $u \in |\mathcal{U}|$ we have an order space W such that $u \in W \xrightarrow{\hookrightarrow} ((Sp(\mathcal{U}), \leq), ((Sp(\mathcal{U}), \geq))$ (Definition 3.4) therefore $W = \{u\}$; and $Sp(\mathcal{U})$ is discrete. \square

Local orders allow us to give concise descriptions of concrete examples (e.g. the directed circle is isomorphic to any pair of ordered open arcs covering the unit circle – Example 3.8), while their saturations are often more convenient to deal with theoretical problems (the same observation applies to atlases and manifolds in differential topology [8, p.54, 58]). For example we have:

Lemma 3.24. For all local orders \mathcal{U} and \mathcal{V} , the map $f : |\mathcal{U}| \rightarrow |\mathcal{V}|$ is locally order-preserving from \mathcal{U} to \mathcal{V} if, and only if, for all $x \in |\mathcal{U}|$ and $V \in \bar{\mathcal{V}}$ containing $f(x)$, there exists $U \in \bar{\mathcal{U}}$ containing x such that $f(U) \subseteq V$ and f is order preserving from U to V .

Proof Given $V' \in \bar{\mathcal{V}}$ containing x , there exists $V \in \mathcal{V}$ such that V and V' are compatible around x (Lemma 3.20). Since f is locally order-preserving, we have ordered spaces A and U containing x such that $A \hookrightarrow U \in \mathcal{U}$ and $f : A \rightarrow V$ is order-preserving; moreover $A \in \bar{\mathcal{U}}$ (Lemma 3.19).

Conversely, suppose that we have B and V ordered spaces containing x such that $B \hookrightarrow V \in \mathcal{V}$. In particular $B \in \bar{\mathcal{V}}$ (Lemma 3.19). Hence we have $U' \in \bar{\mathcal{U}}$ containing x such that $f : U' \rightarrow B$ is order-preserving. There exists $U \in \mathcal{U}$ compatible with U' around x (Lemma 3.20), so there exists an ordered space A containing x such that $A \hookrightarrow U, U'$. In particular $f : A \rightarrow B$ is order-preserving. \square

Local orders are related to locally ordered spaces in the same way as atlases are related to manifolds:

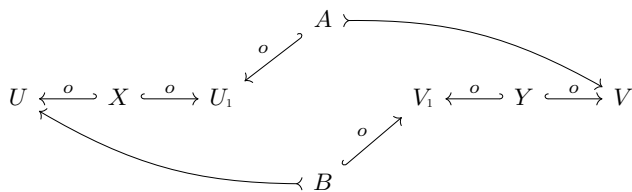
Proposition 3.25. Given two local orders \mathcal{U} and \mathcal{V} with the same underlying set, the following are equivalent:

- i) The local orders \mathcal{U} and \mathcal{V} have the same saturation ($\bar{\mathcal{U}} = \bar{\mathcal{V}}$).
- ii) The identity $\text{id}_{|\mathcal{U}|}$ induces an isomorphism between \mathcal{U} and \mathcal{V} .
- iii) The union $\mathcal{U} \cup \mathcal{V}$ is a local order.
- iv) Every $U \in \mathcal{U}$ is compatible with every $V \in \mathcal{V}$.

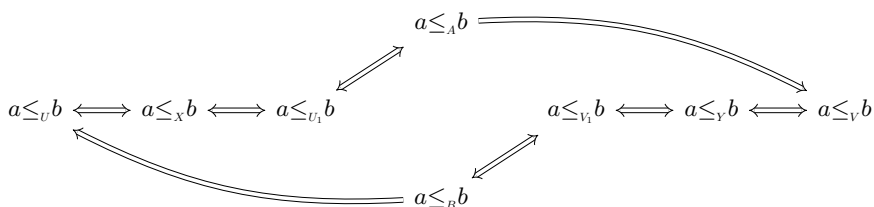
Proof As an immediate consequence of Definition 3.4 the statements iii) and iv) are equivalent. Assuming iii) we have $\mathcal{U} \subseteq \mathcal{U} \cup \mathcal{V} \subseteq \bar{\mathcal{U}}$ by Lemma 3.20. The same way we have $\mathcal{U} \cup \mathcal{V} \subseteq \bar{\mathcal{V}}$. By uniqueness of the maximal local order containing $\mathcal{U} \cup \mathcal{V}$ (Lemma 3.20) we have $\bar{\mathcal{U}} = \bar{\mathcal{V}}$. Given $x \in |\mathcal{U}|$ and $V \in \mathcal{V}$ containing x , there exists $U \in \mathcal{U}$ containing x . Hence we have a witness of compatibility W for U and V around x . In particular we have $W \hookrightarrow V$ so the restriction $\text{id}_{|\mathcal{U}|} : W \rightarrow V$ is order-preserving. Hence iii) implies i) and ii). If i) is satisfied then $\mathcal{U} \cup \mathcal{V} \subseteq \bar{\mathcal{U}}$, so iii) is satisfied too. It remains to prove that ii) implies iv). Let $U \in \mathcal{U}$, $V \in \mathcal{V}$, and $x \in U \cap V$. We have:

- an ordered space $U_1 \in \mathcal{U}$ with $A \hookrightarrow U_1$ containing x such that $|A| \subseteq |V|$ and $(|A|, \leq_A) \rightarrow (|V|, \leq_V)$ because the identity map is a morphism from \mathcal{U} to \mathcal{V} ;
- an ordered space $V_1 \in \mathcal{V}$ with $B \hookrightarrow V_1$ containing x such that $|B| \subseteq |U|$ and $(|B|, \leq_B) \rightarrow (|U|, \leq_U)$ because the identity map is a morphism from \mathcal{V} to \mathcal{U} ;
- an ordered space X containing x such that $X \hookrightarrow U$ and $X \hookrightarrow V_1$ because U and U_1 are compatible around x ;
- an ordered space Y containing x such that $Y \hookrightarrow V$ and $Y \hookrightarrow V_1$ because V and V_1 are compatible around x .

The ordered spaces introduced so far are organised as follows:



By the first point of Lemma 2.14 we have $A \cap X \xrightarrow{o} U$ and $B \cap Y \xrightarrow{o} V$. We know from Lemma 3.13 that $Sp(U)$ and $Sp(V)$ are open subspaces of $Sp(\mathcal{U})$ and $Sp(\mathcal{V})$, which are equal by Lemma 3.15. It follows that the subspace W of $Sp(\mathcal{U})$ whose underlying set is $|A| \cap |B| \cap |X| \cap |Y|$ is an open subspace of $Sp(U)$ and $Sp(V)$. Two elements $a, b \in W$ belong to every ordered space appearing in the preceding diagram, and we have



so the partial orders \le_U and \le_V agree on the set $|W| \times |W|$. The resulting ordered space W is a witness of compatibility of U and V around x . □

Definition 3.26. Two local orders \mathcal{U} and \mathcal{V} with the same underlying set are said to be *equivalent*, which we denote by $\mathcal{U} \sim \mathcal{V}$, when they satisfy one of the items of Proposition 3.25. In particular we denote by $\mathcal{U} \equiv \mathcal{V}$ the (unique) isomorphism from \mathcal{U} to \mathcal{V} induced by the identity map (Proposition 3.25 (ii)).

Definition 3.27. A *locally ordered space* is a maximal local order. The category **LoSp** of locally ordered spaces is full in **LO**.

Corollary 3.28. The inclusion functor $\mathbf{LoSp} \hookrightarrow \mathbf{LO}$ is an equivalence of categories; its quasi-inverse is the functor sending a local order to its saturation.

Proof The saturation $\bar{\mathcal{U}}$ of a local order \mathcal{U} (Definition 3.21) belongs to **LoSp**, and the natural equivalence is given by the collection of isomorphisms $\mathcal{U} \equiv \bar{\mathcal{U}}$ for \mathcal{U} ranging in the class of local orders (Definition 3.26). □

Corollary 3.28 formalizes the idea that the locally ordered spaces are the ‘normal forms’ of local orders.

3.3 Ordered bases

Local orders are made to describe locally ordered spaces; indeed, it is easier to check that a collection of ordered spaces is a local order if it contains fewer

elements (and therefore fewer overlappings). In return, the underlying sets of the elements of a local order \mathcal{U} (which are open in $Sp(\mathcal{U})$ by Lemma 3.13) may not form a base of its topology; this is why every element of \mathcal{U} has to carry a topology. However, if the family $\{|U|; U \in \mathcal{U}\}$ is a base of the topology of $Sp(\mathcal{U})$, which is the case if \mathcal{U} is maximal (Definition 3.18), then we can drop the topology of every $U \in \mathcal{U}$ because it can be recovered from $Sp(\mathcal{U})$ (Lemma 3.13). These observations, along with the fact that \mathcal{U} is equivalent to its saturation (Proposition 3.25), lead to the notion of an ordered base [4, 4.1.1, p.20]; their morphisms being defined in accordance with Lemma 3.24:

Definition 3.29. An *ordered base* is a collection of posets \mathfrak{B} such that for all $B, B' \in \mathfrak{B}$ and $x \in B \cap B'$, there exists $W \in \mathfrak{B}$ containing x that is a subposet of B and B' , i.e. $x \in W \hookrightarrow B, B'$. The collection $\{|B| \mid B \in \mathfrak{B}\}$ is a base of $Sp(\mathfrak{B})$, the underlying space of \mathfrak{B} . By definition, the underlying set of \mathfrak{B} is that of $Sp(\mathfrak{B})$; we denote it by $|\mathfrak{B}|$. The ordered base \mathfrak{B} is said to be *maximal* when for every poset X such that $|X|$ is open in $Sp(\mathfrak{B})$, if $\mathfrak{B} \cup \{X\}$ is still an ordered base, then $X \in \mathfrak{B}$ (compare to Definition 3.18).

Given another ordered base \mathfrak{B}' , a map $f : |\mathfrak{B}| \rightarrow |\mathfrak{B}'|$ is *locally order-preserving* at $x \in |\mathfrak{B}|$ when for all $B' \in \mathfrak{B}'$ containing $f(x)$ there exists $B \in \mathfrak{B}$ containing x such that f induces an order-preserving map from B to B' . A *morphism* from \mathfrak{B} to \mathfrak{B}' is a map $f : |\mathfrak{B}| \rightarrow |\mathfrak{B}'|$ that is locally order-preserving at every point of $|\mathfrak{B}|$. The map f is continuous from $Sp(\mathfrak{B})$ to $Sp(\mathfrak{B}')$. We denote the category of ordered bases by **OB**; it is concrete over **Top**.

Remark 3.30. Given an ordered base \mathfrak{B} , if $B \hookrightarrow B'$ for $B, B' \in \mathfrak{B}$, then $\tilde{B} \hookrightarrow \tilde{B}'$ with \tilde{B} and \tilde{B}' denoting the ordered spaces obtained by equipping the posets B and B' with the topologies induced by $Sp(\mathfrak{B})$.

Every maximal local order induces an ordered base:

Lemma 3.31. Given a maximal local order \mathcal{U} , the collection $OB(\mathcal{U})$ of the underlying posets of the elements of \mathcal{U} is an ordered base whose underlying space is $Sp(\mathcal{U})$.

Proof For every $U \in \mathcal{U}$ denote by P_U the underlying poset of U . Given $U, V \in \mathcal{U}$ and $x \in U \cap V$, since \mathcal{U} is a local order we have an ordered space W such that $x \in W \hookrightarrow U, V$. Since \mathcal{U} is maximal it contains W (Lemma 3.19). In particular we have $P_W \in OB(\mathcal{U})$ and $x \in P_W \hookrightarrow P_U, P_V$. The topological spaces $Sp(\mathcal{U})$ and $Sp(OB(\mathcal{U}))$ are equal because both admit $\{|U| \mid U \in \mathcal{U}\}$ as a base. \square

As an immediate consequence of Lemmas 3.24 and 3.31, we have:

Corollary 3.32. The mapping which sends every maximal local order \mathcal{U} to the ordered base $OB(\mathcal{U})$ induces a fully faithful functor $OB : \mathbf{LoSp} \rightarrow \mathbf{OB}$.

Every ordered base induces a local order:

Lemma 3.33. Given an ordered base \mathfrak{B} , every poset $B \in \mathfrak{B}$ becomes an ordered space \tilde{B} with the topology induced by $Sp(\mathfrak{B})$ (Remark 3.30). The family $LO(\mathfrak{B}) = \{\tilde{B} \mid B \in \mathfrak{B}\}$ is a local order whose underlying space is $Sp(\mathfrak{B})$.

Proof Given $B_1, B_2 \in \mathfrak{B}$ and $x \in B_1 \cap B_2$ we have $W \in \mathfrak{B}$ such that $x \in W \hookrightarrow B_1, B_2$. The set $|W|$ is open in $Sp(\mathfrak{B})$ and by writing $|W| = |W| \cap |B_1|$ we deduce that it is open in $Sp(\tilde{B}_1)$. The same reasoning proves that $|W|$ is open in $Sp(\tilde{B}_2)$. Hence we have $x \in \tilde{W} \xrightarrow{\hookrightarrow} \tilde{B}_1, \tilde{B}_2$ (Remark 3.30). So $LO(\mathfrak{B})$ is a local order, and the topological spaces $Sp(\mathfrak{B})$ and $Sp(LO(\mathfrak{B}))$ both admit $\{|B| \mid B \in \mathfrak{B}\}$ as a base. \square

Lemma 3.34. For all ordered bases \mathfrak{B} and \mathfrak{B}' , the map $f : |\mathfrak{B}| \rightarrow |\mathfrak{B}'|$ is locally order-preserving from \mathfrak{B} to \mathfrak{B}' (Definition 3.29) if, and only if, it is so from $LO(\mathfrak{B})$ to $LO(\mathfrak{B}')$ (Definition 3.5).

Proof Every poset X that belongs to \mathfrak{B} (resp. \mathfrak{B}') becomes an ordered space when it is equipped with the topology induced by $Sp(\mathfrak{B})$ (resp. $Sp(\mathfrak{B}')$), we denote it by \tilde{X} (see the notation from Lemma 3.33).

Suppose that f is locally order-preserving from $LO(\mathfrak{B})$ to $LO(\mathfrak{B}')$. Let $x \in |\mathfrak{B}|$ and $V \in \mathfrak{B}'$ containing $f(x)$. By hypothesis on f , we have $U \in \mathfrak{B}$ and an ordered space A such that $x \in A \xrightarrow{\hookrightarrow} \tilde{U}$, $f(|A|) \subseteq |V|$, and f is order-preserving from (the underlying poset of) A to V . The set $|A|$ is open in $Sp(U)$ and $|U|$ is open in $Sp(LO(\mathfrak{B}))$, hence so is $|A|$ (Lemma 2.9). Since the underlying sets of the elements of \mathfrak{B} form a base of the topology of $Sp(\mathfrak{B}) = Sp(LO(\mathfrak{B}))$, we have $U' \in \mathfrak{B}$ such that $x \in |U'| \subseteq |A|$. Following Definition 3.29, we have $W \in \mathfrak{B}$ such that $x \in W \hookrightarrow U, U'$. Suppose that $w \leq_w w'$ for some w and $w' \in |W|$. We have $w \leq_v w'$ because $W \hookrightarrow U$; therefore $w \leq_A w'$ because $A \xrightarrow{\hookrightarrow} U$. We conclude that $f(w) \leq_v f(w')$.

Conversely, suppose that f is locally order-preserving from \mathfrak{B} to \mathfrak{B}' . Let $x \in |\mathfrak{B}|$, $V \in \mathfrak{B}'$, and an ordered space B such that $f(x) \in B \xrightarrow{\hookrightarrow} \tilde{V}$. The set $|B|$ is open in $Sp(V)$ and $|V|$ is open in $Sp(LO(\mathfrak{B}'))$, hence so is $|B|$ (Lemma 2.9). Since the underlying sets of the elements of \mathfrak{B}' form a base of topology of $Sp(\mathfrak{B}') = Sp(LO(\mathfrak{B}'))$, we have $V' \in \mathfrak{B}'$ such that $f(x) \in |V'| \subseteq |B|$. Following Definition 3.29, we have $W \in \mathfrak{B}'$ such that $f(x) \in W \hookrightarrow V, V'$. By hypothesis on f , we have $U \in \mathfrak{B}$ such that f induces an order-preserving map from U to W . Given $u, u' \in |U|$ such that $u \leq_v u'$, we have $f(u) \leq_w f(u')$, therefore $f(u) \leq_v f(u')$ because $W \hookrightarrow V$, from which we deduce that $f(u) \leq_B f(u')$ because $f(u), f(u') \in |W| \subseteq |V'| \subseteq |B|$ and $B \xrightarrow{\hookrightarrow} \tilde{V}$. \square

Proposition 3.35. The mapping which sends every ordered base \mathfrak{B} to the local order $LO(\mathfrak{B})$ induces an equivalence of categories $LO : \mathbf{OB} \rightarrow \mathbf{LO}$ whose quasi-inverse is the composite $OB \circ Sat$ with Sat denoting the quasi-inverse of the inclusion $\mathbf{LoSp} \hookrightarrow \mathbf{LO}$ (Corollary 3.28). The functor LO induces an isomorphism from the full subcategory of maximal ordered bases to \mathbf{LoSp} .

Proof As an immediate consequence of Lemmas 3.33 and 3.34, LO is a well-defined fully faithful functor. We observe that $\bar{U} = LO(OB(\bar{U}))$; indeed the functor OB ‘drops’ the topologies of the elements of \bar{U} and the functor LO recovers them as a subspaces of $Sp(OB(\bar{U})) = Sp(\bar{U})$ (Lemmas 3.13 and 3.31). Moreover \mathcal{U} and \bar{U} are isomorphic (Proposition 3.25 (ii)). Hence LO is a fully faithful functor whose image meets every isomorphism class of \mathbf{LO} , therefore it is an equivalence of categories [2, 3.4.3 and 3.4.4]. We have actually proven that the composite $LO \circ OB$ is the inclusion functor $\mathbf{LoSp} \hookrightarrow \mathbf{LO}$ whose quasi-inverse is Sat ; we deduce that the quasi-inverse of LO is the expected one. By definition of $LO(\mathfrak{B})$, if it is maximal, then so is \mathfrak{B} . Conversely, suppose that \mathfrak{B} is maximal and let U be an ordered space such that $|U| \subseteq |\mathfrak{B}|$ and $LO(\mathfrak{B}) \cup \{U\}$ is a local order. Denote by P_U the underlying poset of U , the collection $\mathfrak{B} \cup \{P_U\}$ is still an ordered base, hence $P_U \in \mathfrak{B}$. By Lemma 3.13 we know that the underlying topology of U is induced by $Sp(LO(\mathfrak{B}) \cup \{U\}) = Sp(LO(\mathfrak{B}))$ so by definition, U belongs to $LO(\mathfrak{B})$, which is thus maximal. \square

We compare ordered bases the same way as we compare bases of topology:

Definition 3.36. The ordered base \mathfrak{B}' is said to be *coarser* than the ordered base \mathfrak{B} when $|\mathfrak{B}| = |\mathfrak{B}'|$ and for all $B' \in \mathfrak{B}'$ and all $x \in B'$, there is $B \in \mathfrak{B}$ containing x such that $B \subseteq B'$ and the corresponding inclusion map is order-preserving, i.e. $x \in B \mapsto B'$. We say that \mathfrak{B} and \mathfrak{B}' are *equivalent* when each of them is coarser than the other.

Lemma 3.37. The ordered bases \mathfrak{B} and \mathfrak{B}' are equivalent if, and only if, so are their corresponding local orders (in the sense of Definition 3.26).

Proof Note that \mathfrak{B}' is coarser than \mathfrak{B} if, and only if the identity map induces a morphism from \mathfrak{B} to \mathfrak{B}' , and apply Propositions 3.25 (ii) and 3.35. \square

Definition 3.38. We say that $\mathfrak{B} \in \mathbf{OB}$ is an ordered base of the local order \mathcal{U} when $LO(\mathfrak{B}) \sim \mathcal{U}$ (see Definition 3.26 and Lemma 3.33).

3.4 Universal property

We introduce a subclass of local orders from which we deduce the diagrams that are expected to have well-behaved colimits, and prove that the category of local orders is characterised by a universal property (Proposition 3.46).

Definition 3.39. A *basic diagram* is a surjective poset morphism $D : (\mathcal{D}, \leq) \rightarrow (LO(\mathfrak{B}), \overset{\circ}{\rightarrow})$ with \mathfrak{B} an ordered base (Lemma 3.33), such that for all $x, y, y' \in \mathcal{D}$, if $Dx \overset{\circ}{\rightarrow} Dy, Dy'$ then for every $p \in Dx$ there exists $z_p \leq y, y'$ such that $p \in Dz_p \overset{\circ}{\rightarrow} Dx$. We call \mathfrak{B} the underlying base of D .

Remark 3.40. Taking $y' = x$ in the statement of Definition 3.39, we prove that every basic diagram satisfies the following property: (*) for all $x, y \in \mathcal{D}$ such that $Dx \overset{\circ}{\rightarrow} Dy$, for every $p \in Dx$ there exists $z_p \leq x, y$ such that $p \in Dz_p$.

Conversely, if $D : \mathcal{D} \rightarrow (\mathcal{U}, \overset{\circ}{\rightarrow})$ is a poset morphism satisfying $(*)$, then for all $n \in \mathbb{N}$, $x, y_1, \dots, y_n \in \mathcal{D}$, if $Dx \overset{\circ}{\rightarrow} Dy_1, \dots, Dy_n$, then for every $p \in Dx$ there exists $z_p \in \mathcal{D}$ such that $z_p \leq x, y_1, \dots, y_n$ and $p \in Dz_p$. The case $n = 0$ is degenerated yet valid (take $z_p = x$); and the case $n = 1$ is readily given by $(*)$. Let $p \in Dx$ and assume that $Dx \overset{\circ}{\rightarrow} Dy_1, \dots, Dy_{n+1}$. By applying the induction hypothesis, we obtain $z_p \leq x, y_1, \dots, y_n$ such that $p \in Dz_p$. We have $Dz_p \overset{\circ}{\rightarrow} Dy_{n+1}$ so by applying $(*)$ we obtain $z'_p \leq z_p, y_{n+1}$ such that $p \in Dz'_p$. In particular we have $z'_p \leq x, y_1, \dots, y_{n+1}$.

Remark 3.41. We can replace $(LO(\mathfrak{B}), \overset{\circ}{\rightarrow})$ by $(\mathfrak{B}, \hookrightarrow)$ in the statement of Definition 3.39. Indeed, the map sending every ordered space $B \in LO(\mathfrak{B})$ to the underlying poset of B , is a poset isomorphism from $(LO(\mathfrak{B}), \overset{\circ}{\rightarrow})$ to $(\mathfrak{B}, \hookrightarrow)$ (the inverse is given by Lemma 3.33).

Example 3.42. A local order \mathcal{I} is said to be an *ideal* when for all ordered spaces W, I , if $W \overset{\circ}{\rightarrow} I \in \mathcal{I}$ then $W \in \mathcal{I}$. The *idealization* of a local order \mathcal{U} is, by definition, the collection $\mathcal{I}_{\mathcal{U}} = \{W \overset{\circ}{\rightarrow} U \mid U \in \mathcal{U}\}$, it is indeed an ideal. In particular, the underlying posets of the elements of $\mathcal{I}_{\mathcal{U}}$ form an ordered base \mathfrak{B} of \mathcal{U} satisfying $LO(\mathfrak{B}) = \mathcal{I}_{\mathcal{U}}$. Consequently, the identity on $(\mathcal{I}_{\mathcal{U}}, \overset{\circ}{\rightarrow})$ is a basic diagram which we denote by $D_{\mathcal{U}}$.

Definition 3.43. For every local order \mathcal{U} , every $U \in \mathcal{U}$, and every $W \overset{\circ}{\rightarrow} U$, the *canonical inclusion* of W in \mathcal{U} is the locally order-preserving map (Definition 3.5) $\iota_W^{(U)} : \{W\} \rightarrow \mathcal{U}$ induced by the inclusion $|W| \subseteq |\mathcal{U}|$ (the map $\iota_U^{(U)}$ is locally order-preserving by Example 3.7, the map $\iota_W^{(U)}$ is locally order-preserving as the composite of the locally order-preserving map induced by $W \overset{\circ}{\rightarrow} U$, see Example 2.16 and Proposition 3.6, and the map $\iota_U^{(U)}$).

The next example is used in the proofs of Propositions 3.46 and 3.48.

Example 3.44. Given a morphism $f : \mathcal{U} \rightarrow \mathcal{V}$ of **LO** we denote by \mathcal{D}_f the poset of ordered pairs $(U, V) \in \mathcal{I}_{\mathcal{U}} \times \mathcal{I}_{\mathcal{V}}$ such that $f(U) \subseteq V$ with the partial order $\overset{\circ}{\rightarrow} \times \overset{\circ}{\rightarrow}$. In particular the restriction of f to U with values in V induces a morphism of **Ord** (Proposition 3.6). Hence \mathcal{D}_f is a subposet of $(\mathcal{I}_{\mathcal{U}}, \overset{\circ}{\rightarrow}) \times (\mathcal{I}_{\mathcal{V}}, \overset{\circ}{\rightarrow})$; we denote by D_1 and D_2 the restrictions of the first and the second projections to \mathcal{D}_f . We observe that D_2 is a basic diagram (Definition 3.39) because $(\emptyset, V) \in \mathcal{D}_f$ for every $V \in \mathcal{I}_{\mathcal{V}}$. In order to have D_1 surjective, we suppose that its codomain is $\mathcal{J}_f = \{U \in \mathcal{I}_{\mathcal{U}} \mid (U, V) \in \mathcal{D}_f \text{ for some } V \in \mathcal{I}_{\mathcal{V}}\}$. The family \mathcal{J}_f may not be the entire set $\mathcal{I}_{\mathcal{U}}$, but it is an ideal which covers $|\mathcal{U}|$ (because f is locally order-preserving, see Definition 3.5) so it is equivalent to \mathcal{U} . Suppose that we have (U, V) , (U', V') , and $(U'', V'') \in \mathcal{D}_f$, with $U \overset{\circ}{\rightarrow} U'$, U'' and $p \in U$. We have an ordered space $W \in \mathcal{I}_{\mathcal{V}}$ such that $f(p) \in W \overset{\circ}{\rightarrow} V'$, V'' because $\mathcal{I}_{\mathcal{V}}$ is an ideal; and we have an ordered space U''' such that $p \in U''' \overset{\circ}{\rightarrow} U$ (therefore $U''' \overset{\circ}{\rightarrow} U'$, U'' by Lemma 2.14 (i)) and $f(U''') \subseteq W$ because f is locally

order-preserving; hence $(U''', W) \in \mathcal{D}_f$, $(U''', W) \xrightarrow{\circlearrowright} \times \xrightarrow{\circlearrowright} (U', V')$, (U'', V'') with $p \in U'''$. We conclude that D_1 is a basic diagram.

For every (U, V) and (U', V') in \mathcal{D}_f with $U \xrightarrow{\circlearrowright} U'$ and $V \xrightarrow{\circlearrowright} V'$ we have the commutative diagram (in **Ord**)

$$\begin{array}{ccc} U & \xrightarrow{\circlearrowright} & U' \\ f \downarrow & & \downarrow f \\ V & \xrightarrow{\circlearrowright} & V' \end{array}$$

so the family $\{f : U \rightarrow V \mid (U, V) \in \mathcal{D}_f\}$ induces a natural transformation η_f from D_1 to D_2 . The families $\iota_1 = \{\iota_V^{(U)} \mid U \in \mathcal{J}_f\}$ and $\iota_2 = \{\iota_V^{(V)} \mid V \in \mathcal{I}_v\}$ are cones under D_1 and D_2 , and we have

$$f \circ \iota_1 = \iota_2 \circ I\eta_f \quad (3)$$

with I denoting the full embedding of **Ord** in **LO** (Proposition 3.6) because $Sp(f \circ \iota_1) = Sp(\iota_2 \circ I\eta_f)$ and Sp is faithful.

Following [26] we write *cone over* and *cone under* instead of *cone* and *cocone*. The morphisms of a cone under (resp. over) a diagram share the same target (resp. source) which is called the *nadir* (resp. *apex*) of the cone.

The above definitions are motivated by the following result:

Lemma 3.45. Given an ordered base \mathfrak{B} of a local order \mathcal{U} (Definition 3.38), and a basic diagram $D : \mathcal{D} \rightarrow (LO(\mathfrak{B}), \xrightarrow{\circlearrowright})$ (Definition 3.39), the family of canonical inclusions $\{\iota_{Dx}^{(U)} \mid x \in \mathcal{D}\}$ (Definition 3.43) is a universal cone under D .

Proof First we suppose that $\mathcal{U} = LO(\mathfrak{B})$. Let γ be a cone under D with nadir \mathcal{X} . Given $p \in |\mathcal{U}|$, let y and $y' \in \mathcal{D}$ such that $p \in Dy \cap Dy'$. We have $W \in \mathcal{U}$ containing p such that $W \xrightarrow{\circlearrowright} Dy$, $W \xrightarrow{\circlearrowright} Dy'$ because \mathcal{U} is a local order (Lemma 3.33). Since D is a basic diagram (Definition 3.39), we have $W = Dx$ for some $x \in \mathcal{D}$, and we can suppose that $x \leq y, y'$. Since γ is a cone, the mappings γ_y and $\gamma_{y'}$ coincide on Dx . Therefore we can define $f(p) = \gamma_y(p)$ for any $y \in \mathcal{D}$ such that $p \in Dy$ (such an y exists because D is surjective, and we have shown that the choice of y is irrelevant). In particular the restriction of f to Dy is γ_y . Assuming that $f(p) \in U \xrightarrow{\circlearrowright} X \in \mathcal{X}$, we have $B \in \mathcal{U}$ such that $p \in B \xrightarrow{\circlearrowright} Dy$ and f is order-preserving from B to U because γ_x is locally order-preserving. Hence f is locally order-preserving from \mathcal{U} to \mathcal{X} and induces a cone morphism from $\{\iota_{Dx}^{(U)} \mid x \in \mathcal{D}\}$ to γ . Moreover, any locally order-preserving map from \mathcal{U} to \mathcal{X} inducing a cone morphism as above has the same underlying map as f . Therefore f is actually unique and the cone $\{\iota_{Dx}^{(U)} \mid x \in \mathcal{D}\}$ is universal.

Now suppose that \mathfrak{B} is a local order of \mathcal{U} , i.e. $\mathcal{U} \sim LO(\mathfrak{B})$ (Definition 3.38). Let j be the isomorphism $LO(\mathfrak{B}) \equiv \mathcal{U}$ (Definition 3.26). We conclude by observing that

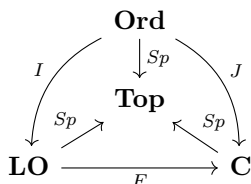
$\iota_{D_x}^{(U)} = j \circ \iota_{D_x}^{(LO(\mathfrak{B}))}$ for all $x \in \mathcal{D}$, so j induces a cone isomorphism from $\{\iota_{D_x}^{(LO(\mathfrak{B}))} \mid x \in \mathcal{D}\}$ to $\{\iota_{D_x}^{(U)} \mid x \in \mathcal{D}\}$, the former being universal by the first part of the proof. \square

We now check that **LoSp** is the smallest concrete category over **Top** in which every basic diagram has a well-behaved colimit:

Proposition 3.46. Let **C** be a concrete category over **Top**, so we have the faithful functor $Sp_C : \mathbf{C} \rightarrow \mathbf{Top}$. We denote by $Sp_{\mathbf{Ord}}$ and $Sp_{\mathbf{LO}}$ the forgetful functors from **Ord** and **LO** to **Top** (Definitions 2.15 and 3.5), and by I the full embedding of **Ord** in **LO** – see Proposition 3.6. Suppose that we have:

- 1) a functor $J : \mathbf{Ord} \hookrightarrow \mathbf{C}$ such that $Sp_{\mathbf{Ord}} = Sp_C \circ J$, and
- 2) a collection $\gamma_V^{(U)}$ of morphisms of **C** with $U \in \mathbf{LO}$ and $V \in \mathcal{I}_U$ the idealization of U (Example 3.42), such that:
 - 2a) for every ordered space X we have $J(id_x) = \gamma_U^{(IX)}$ (recall that $IX = \{X\}$, see Proposition 3.6),
 - 2b) the morphism $\gamma_V^{(U)}$ and the canonical inclusion $\iota_V^{(U)}$ (Definition 3.43) have the same underlying map (i.e. $Sp_C(\gamma_V^{(U)}) = Sp_{\mathbf{LO}}(\iota_V^{(U)})$), and
 - 2c) for every basic diagram $D : \mathcal{D} \rightarrow (LO(\mathfrak{B}), \hookrightarrow)$ (Definition 3.39) with \mathfrak{B} ordered base of U (Definition 3.38), the collection $\{\gamma_{D_x}^{(U)} \mid x \in \mathcal{D}\}$ is a universal cone under JD .

Then we have a unique functor $F : \mathbf{LO} \rightarrow \mathbf{C}$ extending J (i.e. $F \circ I = J$), sending $\iota_V^{(U)}$ to $\gamma_V^{(U)}$ for every $U \in \mathbf{LO}$ and $V \in \mathcal{I}_U$, and such that $Sp_{\mathbf{LO}} = Sp_C \circ F$.



Proof Let \mathcal{I}_U and D_U be the idealization of the local order U and the corresponding basic diagram (Example 3.42). From the hypothesis (2c) we deduce that $\gamma^{(U)} = \{\gamma_V^{(U)} \mid V \in \mathcal{I}_U\}$ is a universal cone under JD_U ; we define $F(U)$ as its nadir.

Let the basic diagrams $D_1^{(f)} : \mathcal{D}_f \rightarrow \mathcal{J}_f$, $D_2^{(f)} : \mathcal{D}_f \rightarrow \mathcal{I}_v$, and the natural transformation η_f from $D_1^{(f)}$ to $D_2^{(f)}$ be as in Example 3.44. In particular $J\eta_f$ is a natural transformation from JD_1 to JD_2 .

As a consequence of the hypothesis (2c) we have the universal cones $\gamma_1 = \{\gamma_U^{(U)} \mid U \in \mathcal{J}_f\}$ and $\gamma_2 = \gamma^{(V)} = \{\gamma_V^{(V)} \mid V \in \mathcal{I}_v\}$ under $JD_1^{(f)}$ and $JD_2^{(f)}$. In particular γ_1 is a subset of $\gamma^{(U)}$ so they have the same nadir, namely $F(U)$; the nadir of γ_2 is $F(V)$ because $\gamma_2 = \gamma^{(V)}$.

A cone under a diagram D can be seen as a natural transformation from D to a constant functor, so the composite $\gamma_2 \circ J\eta_f$ makes sense: it is a cone under JD_1 . Since

the cone γ_1 is universal, we have a unique morphism $F(f) : F(\mathcal{U}) \rightarrow F(\mathcal{V})$ such that

$$F(f) \circ \gamma_1 = \gamma_2 \circ J\eta_f . \quad (4)$$

The morphism $F(f)$ is, as the notation suggests, the image of f under F (the left hand part of the above equality is a composition of natural transformations, provided that we understand $F(f)$ as a natural transformation between two constant functors).

By applying the underlying space functor Sp to Equation (4), we obtain

$$Sp(F(f)) \circ Sp(\gamma_1) = Sp(\gamma_2) \circ (Sp \circ J)\eta_f . \quad (5)$$

By hypothesis (1) we have $(Sp \circ J)\eta_f = Sp\eta_f$, which is a natural transformation from SpD_1 to SpD_2 , so Eq.(5) becomes

$$Sp(F(f)) \circ Sp(\gamma_1) = Sp(\gamma_2) \circ Sp\eta_f . \quad (6)$$

With ι_1 and ι_2 as in Example 3.44 we have $Sp(\gamma_1) = Sp(\iota_1)$ and $Sp(\gamma_2) = Sp(\iota_2)$ by hypothesis (2b). So the cones $Sp(\gamma_1)$ and $Sp(\gamma_2)$ are universal under SpD_1 and SpD_2 (this assertion only depends on general topology: continuous maps that agree on the intersection of their domains of definition, assuming this intersection is a subspace of both domains, can be ‘glued’ to form a continuous map on the ‘union’ of these domains). By applying the functor Sp to Eq.(3) from Example 3.44 we obtain

$$Sp(f) \circ Sp(\gamma_1) = Sp(\gamma_2) \circ Sp\eta_f . \quad (7)$$

The continuous maps $Sp(f)$ and $Sp(F(f))$ thus satisfy the same universal property (see Eq.(6) and Eq.(7)), so they are equal. If $g : \mathcal{V} \rightarrow \mathcal{W}$ is another morphism of \mathbf{C} , then we have

$$Sp(F(gf)) = Sp(gf) = Sp(g)Sp(f) = Sp(F(g))Sp(F(f)) = Sp(F(g)F(f)) .$$

It follows that $F(gf) = F(g)F(f)$ because Sp is faithful. So F is indeed a functor.

We still have to check that $F \circ I = J$ and $F(\iota_v^{(u)}) = \gamma_v^{(u)}$. Given an ordered space X we have

$$id_{F(\{X\})} \circ \gamma_x^{(\{X\})} = F(id_x) \circ \gamma_x^{(\{X\})} = \gamma_x^{(\{X\})} \circ J(id_x) = \gamma_x^{(\{X\})} \circ id_{JX}$$

as a special case of Eq.(4) with $f = id_x = I(id_x)$. Hence $\gamma_x^{(\{X\})}$ is a morphism from JX to $F(\{X\})$; we conclude from the hypothesis (2a) that $F(\{X\}) = JX$. Then given a morphism $g : X \rightarrow Y$ of \mathbf{Ord} , both Jg and Ff , with $f = Ig$, are morphisms from $F(\{X\})$ to $F(\{Y\})$. Moreover we have $Sp_{\mathbf{C}}(Ff) = Sp_{\mathbf{L}\mathbf{O}}(f) = Sp_{\mathbf{Ord}}(g) = Sp_{\mathbf{C}}(Jg)$, therefore $Ff = Jg$ because $Sp_{\mathbf{C}}$ is faithful. Similarly, $F(\iota_v^{(u)})$ and $\gamma_v^{(u)}$ are morphisms from $F(\{V\})$ to $F(\mathcal{U})$ with $Sp(F(\iota_v^{(u)})) = Sp(\iota_v^{(u)}) = Sp(\gamma_v^{(u)})$ (the second equality being given by hypothesis (2b)), so they are equal.

Suppose that $G : \mathbf{LO} \rightarrow \mathbf{C}$ satisfies the properties of F given in the statement. Denoting by $\text{tgt}(\iota_v^{(u)})$ and $\text{tgt}(\gamma_v^{(u)})$ the targets of the morphisms $\iota_v^{(u)}$ and $\gamma_v^{(u)}$, we have

$$G(\mathcal{U}) = G(\text{tgt}(\iota_v^{(u)})) = \text{tgt}(\gamma_v^{(u)}) = F(\mathcal{U}) .$$

Since G extends J we have $GI\eta_f = J\eta_f$, and by functoriality of G we have, for all $U \in \mathcal{I}_u$ and $V \in \mathcal{I}_v$ such that $f(U) \subseteq V$:

$$\begin{aligned} \gamma_v^{(v)} \circ J(\eta_f)_{(U,V)} &= \gamma_v^{(v)} \circ J(f : U \rightarrow V) \\ &= G(\iota_v^{(v)}) \circ GI(f : U \rightarrow V) \\ &= G(\iota_v^{(v)}) \circ I(f : U \rightarrow V) \\ &= G(f \circ \iota_u^{(u)}) \\ &= Gf \circ G(\iota_u^{(u)}) \\ &= Gf \circ \gamma_u^{(u)} \end{aligned}$$

In other word $\gamma_2 \circ J\eta_f = Gf \circ \gamma_1$ from which we deduce that $Ff = Gf$ because they satisfy the same universal property. \square

Remark 3.47. The conclusion of Proposition 3.46 remains valid if (2b) only holds for basic diagrams induced by ideals.

We emphasize that the universal property given by Proposition 3.46 does not guarantee that the order-theoretical information carried by a local order is preserved: consider for example the forgetful functors $J = Sp : \mathbf{Ord} \rightarrow \mathbf{Top}$ and $F = Sp : \mathbf{LO} \rightarrow \mathbf{Top}$.

Proposition 3.48. With the notations and hypotheses from Proposition 3.46, assume that we have two functors $G : \mathbf{LO} \rightarrow \mathbf{E}$ and $H : \mathbf{C} \rightarrow \mathbf{E}$ such that

- i) $GI = HJ$ with $I : \mathbf{Ord} \hookrightarrow \mathbf{LO}$ full embedding (Proposition 3.6),
- ii) $G(\iota_V^{(U)}) = HF(\iota_V^{(U)})$ for every canonical inclusion $\iota_V^{(U)}$ (Definition 3.43),

and for every basic diagram D , with \mathcal{D} the poset on which D is defined, one (at least) of the following properties is satisfied:

- iiia) $\{G(\iota_{D_x}^{(U)}) \mid x \in \mathcal{D}\}$ is a universal cone under GID .
- iiib) $\{H(\gamma_{D_x}^{(U)}) \mid x \in \mathcal{D}\}$ is a universal cone under HJD .

Then $G = HF$.

Proof Let $f : \mathcal{U} \rightarrow \mathcal{V}$ be a morphism of \mathbf{LO} . In order to prove that Gf and HFf are equal, we check that they satisfy the same universal property. Let the basic diagrams $D_1^{(f)} : \mathcal{D}_f \rightarrow \mathcal{J}_f$, $D_2^{(f)} : \mathcal{D}_f \rightarrow \mathcal{I}_V$, the natural transformation η_f from $D_1^{(f)}$ to $D_2^{(f)}$, and the cones ι_1 and ι_2 under $D_1^{(f)}$ and $D_2^{(f)}$ be as in Example 3.44. By applying G to Eq.(3) from Example 3.44 we obtain

$$Gf \circ G\iota_1 = G\iota_2 \circ GI\eta_f . \quad (8)$$

The families $\gamma_1 = \{\gamma_U^{(U)} \mid U \in \mathcal{J}_f\}$ and $\gamma_2 = \{\gamma_V^{(V)} \mid V \in \mathcal{I}_V\}$ are (universal) cones under $JD_1^{(f)}$ and $JD_2^{(f)}$ by hypothesis 2b) of Proposition 3.46. The morphism $F(f)$ satisfies $F(f) \circ \gamma_1 = \gamma_2 \circ J\eta_f$ (Eq.(4) from the proof of Proposition 3.46) to which we apply the functor H to obtain

$$HFf \circ H\gamma_1 = H\gamma_2 \circ HJ\eta_f . \quad (9)$$

We have $GI\eta_f = HJ\eta_f$ by hypothesis i), and given $k \in \{1, 2\}$, we have $G\iota_k = HF\iota_k$ by hypothesis ii), and then $G\iota_k = H\gamma_k$ because $F\iota_1 = \gamma_1$ by Proposition 3.46. So the equations (8) and (9) become

$$Gf \circ H\gamma_1 = H\gamma_2 \circ HJ\eta_f \quad (10)$$

and

$$HFf \circ G\iota_1 = G\iota_2 \circ GI\eta_f . \quad (11)$$

If (iiia) (resp. (iiib)) is satisfied, then we deduce that $Gf = HFf$ from the equations (8) and (11) (resp. (9) and (10)). \square

Proposition 3.48 typically applies when $G : \mathbf{LO} \rightarrow \mathbf{Cat}$ and $H : \mathbf{C} \rightarrow \mathbf{Cat}$ are *fundamental category* functors, see §6.4.

3.5 A categorical approach to cocompleteness

As we will see later (Example 4.23), the category **LO** is not cocomplete. Yet, in order to know ‘how far’ **LO** is from being cocomplete, it is worth paying attention to the fact that a complete well-powered category with a generating family is cocomplete [2, 3.3.8, p.112], and check which are the missing properties.

Denote by \mathcal{G} the local order whose only element is the ordered space $\{0, 1\}$ equipped with the discrete order and topology. Then \mathcal{G} is a *generator* of **LO** in the sense that if $f_1, f_2 : \mathcal{U} \rightarrow \mathcal{V}$ are distinct morphisms of **LO**, then we have $g : \mathcal{G} \rightarrow \mathcal{U}$ such that $f_1 \circ g \neq f_2 \circ g$: given $u, u' \in U$ such that $f_1(u) \neq f_2(u')$, take g such that $g(0) = u$ and $g(1) = u'$, and apply Corollary 3.14.

The category **LO** is also *well-powered* in the sense that given any local order \mathcal{U} the collection of equivalence classes of monomorphisms whose codomain is \mathcal{U} is a set (the monomorphisms $f : \mathcal{X} \rightarrow \mathcal{U}$ and $g : \mathcal{Y} \rightarrow \mathcal{U}$ are equivalent when there exists an isomorphism $\phi : \mathcal{X} \cong \mathcal{Y}$ such that $f = g \circ \phi$).

The local order over the one point space is clearly the terminal object of **LO**, and the equalizer of the morphisms $f, g : \mathcal{U} \rightarrow \mathcal{V}$ is the sub-local order $\{u \in |\mathcal{U}| ; f(u) = g(u)\}$ of \mathcal{U} . The category **LO** even admits *binary products*:

Proposition 3.49. If \mathcal{U} and \mathcal{V} are local orders, then the collection

$$\mathcal{U} \times \mathcal{V} \quad = \quad \{ U \times V \mid U \in \mathcal{U} ; V \in \mathcal{V} \}$$

is (a representative of) the cartesian product of \mathcal{U} and \mathcal{V} in **LO**. Moreover:

- the underlying spaces are preserved, i.e. $Sp(\mathcal{U} \times \mathcal{V}) = Sp(\mathcal{U}) \times Sp(\mathcal{V})$, and
- if $\mathcal{U} \sim \mathcal{U}'$ and $\mathcal{V} \sim \mathcal{V}'$, then

$$\{ U \times V \mid U \in \mathcal{U} ; V \in \mathcal{V} \} \quad \sim \quad \{ U' \times V' \mid U' \in \mathcal{U}' ; V' \in \mathcal{V}' \}.$$

Proof The projections $p : Sp(\mathcal{U}) \times Sp(\mathcal{V}) \rightarrow Sp(\mathcal{U})$ and $p' : Sp(\mathcal{U}) \times Sp(\mathcal{V}) \rightarrow Sp(\mathcal{V})$ are locally order-preserving (Definition 3.5). Assume that $f : \mathcal{X} \rightarrow \mathcal{U}$ and $f' : \mathcal{X} \rightarrow \mathcal{V}$ are locally order-preserving, and denote by $g : Sp(\mathcal{X}) \rightarrow Sp(\mathcal{U}) \times Sp(\mathcal{V})$ the unique map such that $f = g \circ p$ and $f' = g \circ p'$. Given $x \in |\mathcal{X}|$ we have $g(x) = (f(x), f'(x))$. Given $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $g(x) \in U \times V$, we have X and $X' \in \mathcal{X}$ containing x such that f and f' induce order-preserving maps from X to U and from X' to V respectively. Given W a witness of compatibility of X and X' around x , the map g induces an order-preserving map from W to $U \times V$. Hence g is locally order-preserving.

Assume that $\mathcal{U} \sim \mathcal{U}'$ and $\mathcal{V} \sim \mathcal{V}'$, we have the isomorphisms $\mathcal{U} \equiv \mathcal{U}'$ and $\mathcal{V} \equiv \mathcal{V}'$ (Definition 3.26). By the universal property of the cartesian product we have

$$(\mathcal{U} \equiv \mathcal{U}') \times (\mathcal{V} \equiv \mathcal{V}') \quad = \quad (\mathcal{U} \times \mathcal{V}) \equiv (\mathcal{U}' \times \mathcal{V}')$$

and therefore $\mathcal{U} \times \mathcal{V} \sim \mathcal{U}' \times \mathcal{V}'$ (Proposition 3.25 (ii)). □

If **LO** admitted all products, then it would be cocomplete: therefore it does not. Explicitly, one considers the product of infinitely many copies of

the directed circle \mathcal{S} (suppose these copies are indexed by \mathbb{N}). The forgetful functor Sp has a left adjoint (which sends every space X to the local order $\{(X, =)\}$) so it preserves all products. Suppose the product P exists and denote by p_n the n^{th} projection (for $n \in \mathbb{N}$). Then by considering the map $f : [0, 1] \rightarrow P$ such that $p_n \circ f(t) = e^{int}$ for all $t \in [0, 1]$ one proves that every point of P is a vortex, which contradicts the fact that a local order has no vortex [15, 4.3.38, 4.3.40].

By the usual duality argument in category theory, a cocomplete co-well-powered category with a cogenerating family is complete. If we could prove that **LO** is co-well-powered and has a cogenerating family, then we would deduce that it is not cocomplete. The category **Top** is co-well-powered because all its epimorphisms are onto, moreover any non- T_0 -space is a cogenerator of **Top** [1, 7.18(4)]. Yet, as far as we know, these easy arguments about topological spaces cannot be readily adapted to local orders. Moreover, they would certainly no longer hold if we added constraints on the topology of the local orders under consideration. Indeed, the questions about cogenerating families and co-well-poweredness in subcategories of **Top** can be very intricate [10, 27].

4 Ill-behaved diagrams of locally ordered spaces

We describe diagrams of locally ordered spaces whose colimits are not preserved by the functor $Sp : \mathbf{LoSp} \rightarrow \mathbf{Top}$, or do not even exist. Such diagrams are said to be *ill-behaved*. Following Proposition 3.35 we identify any locally ordered space \mathcal{X} with its canonical ordered base. In particular, when we write $U \in \mathcal{X}$, we tacitly refer to the underlying poset of U . We also need to refer to the *points* of \mathcal{X} , namely the points of its underlying space. In order to avoid confusion, we denote this space by X and write $x \in X$ to mean that x is a *point* of \mathcal{X} . More generally, we use the letters $\mathcal{U}, \mathcal{V}, \dots, \mathcal{Z}$ to denote local orders, and the letters U, V, \dots, Z to denote their underlying spaces. This section is taken from the first author's PhD dissertation [4, §4].

4.1 Cylinder

The directed circle is denoted by \mathcal{S} (Example 3.22). The counterexamples we are about to describe are based on products (Proposition 3.49) of the form $\mathcal{S} \times \mathcal{X}$ with \mathcal{X} denoting any locally ordered space (Definition 3.27), see (Fig. 1). We write X for $Sp(\mathcal{X})$, $p_2 : S^1 \times X \rightarrow X$ for the second projection, and we denote the *section* at level $t \in X$ by $i_t : s \in S^1 \mapsto (s, t) \in S^1 \times X$. We fix a point $*$ of X and give a criterion on the lattice of neighborhoods of $*$ for the coequalizer of the pair $(i_*, c_*) : S^1 \rightarrow \mathcal{S} \times \mathcal{X}$ (with $c_* := s \mapsto (1, *)$) to exist, with the unit circle S^1 seen as a locally ordered space (Example 3.9).

Given a locally order-preserving map $f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{Y}$ we denote by

$$K(f) = \{t \in X \mid \forall s, s' \in S^1, f(s, t) = f(s', t)\}. \quad (12)$$

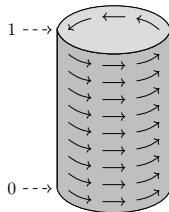


Fig. 1 Cylinder with $X = [0, 1]$

Lemma 4.1. The set $K(f)$ is open in X .

Proof Let $t_0 \in K(f)$ and $U \in \mathcal{Y}$ containing $f(1, t_0)$. For every $s \in S^1$, the map f is locally order-preserving at (s, t_0) . By Definition 3.5 (or 3.29), there exists an ordered arc α_s containing s and $O_s \in \mathcal{X}$ containing t_0 such that $f(\alpha_s \times O_s) \subseteq U$ and the restriction $f_s : \alpha_s \times O_s \rightarrow U$ is order-preserving. Since S^1 is compact, we have a finite subset $J \subseteq S^1$ such that $\{\alpha_s \mid s \in J\}$ covers S^1 . We denote by O the intersection of the elements of $\{O_s \mid s \in J\}$, which is an open neighborhood of t_0 .

We are to show that $O \subset K(f)$. Let $t \in O$, $x, y \in S^1$. There exists a finite sequence $r_0, \dots, r_n \in S^1$ such that $r_0 = x$, $r_n = y$, and for all $k \in \{1, \dots, n\}$ there is $s_k \in J$ such that $r_k, r_{k+1} \in \alpha_{s_k}$ and r_k is less than r_{k+1} in α_{s_k} . In particular (r_k, t) is less than or equal to (r_{k+1}, t) in the product poset $\alpha_{s_k} \times O_{s_k}$; we deduce that

$$f(r_k, t) \leq_U f(r_{k+1}, t)$$

because the restriction f_{s_k} is order-preserving. Hence $f(x, t) \leq_U f(y, t)$ because \leq_U is transitive. By swapping the roles of x and y in the previous reasoning we prove that $f(y, t) \leq_U f(x, t)$. We have $f(x, t) = f(y, t)$ because \leq_U is antisymmetric, so t belongs to $K(f)$, which is therefore open in X . \square

Remark 4.2. Lemma 4.1 remains valid if one replaces the directed circle by a compact locally ordered space that is *strongly connected* in the sense that for every ordered pair of points (a, b) there is a directed path from a to b .

By Lemma 4.1, if $f \circ i_* = f \circ c_*$, then $K(f)$ is an open neighborhood of $*$ in X . We will see that if f collapses the section at level $*$ to the point $*$, then the collapsing spreads around the sections whose levels are close to $*$.

Definition 4.3. Given an open neighborhood O of $*$ in X , the set

$$X_O := O \sqcup \{(s, t) \mid s \in S^1, t \in X \setminus O\} \quad (13)$$

is equipped with the final topology of the map $q_O : S^1 \times X \rightarrow X_O$ defined by

$$q_O(s, t) = \begin{cases} t & \text{if } t \in O \\ (s, t) & \text{if } t \in X \setminus O. \end{cases} \quad (14)$$

Note that $K(q_O) = O$, see (12).

Now we construct a locally ordered space \mathcal{X}_O over X_O such that q_O induces a locally order-preserving map $q_O : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{X}_O$; thanks to Corollary 3.28 and Proposition 3.35, it suffices to provide an ordered base of \mathcal{X}_O :

Lemma 4.4. The topology of X_O is generated by the subsets of the form

$$U_{\alpha,A} := O \cap A \sqcup \{ (s,t) \mid s \in \alpha, t \in (X \setminus O) \cap A \}$$

with A open subset of X and α is a proper open arc.

Proof Those subsets are open in the final topology because $S^1 \times (O \cap A) \cup \alpha \times A$, which is equal to $q_O^{-1}(U_{\alpha,A})$, is an open subset of $S^1 \times X$. Conversely, suppose that $q_O^{-1}(B)$ is open in $S^1 \times X$ with $x \in B \subseteq X_O$. Since q_O is a surjection, there exists $(s,t) \in S^1 \times X$ such that $q_O(s,t) = x$. So we have a proper open arc α and $A \in \mathcal{X}$ such that $(s,t) \in \alpha \times A \subseteq q_O^{-1}(B)$. One readily checks that $U_{\alpha,A} \subseteq B$. \square

An element $u \in U_{\alpha,A}$ is either an element of $O \cap A$ or an ordered pair $(s,t) \in \alpha \times ((X \setminus O) \cap A)$. Depending on the case, the *second component* of u refers to u itself or to t , we denote it by $p_2(u)$. We provide every set $U_{\alpha,A}$ with the image of the partial order $\leq_{\alpha \times A}$ under the mapping q_O , which we denote by $\leq_{\alpha,A}^1$. We insist that $\leq_{\alpha,A}^1$ is a mere relation, not a partial order.

Lemma 4.5. Given $u, u' \in U_{\alpha,A}$ we have $u \leq_{\alpha,A}^1 u'$ if, and only if

$$p_2(u) \leq_A p_2(u') \quad \text{and} \quad \begin{cases} \{u, u'\} \cap O \neq \emptyset \\ \text{or} \\ u = (s,t), u' = (s',t'), \text{ and } s \leq_{\alpha} s'. \end{cases} \quad (15)$$

Proof Let u and u' be elements of $U_{\alpha,A}$ satisfying (15).

- If $u = t \in O$ and $u' = t' \in O$, let $s_0 \in \alpha$, we have $(s_0, t) \leq_{\alpha \times A} (s_0, t')$, $q_O(s_0, t) = u$ and $q_O(s_0, t') = u'$, hence $u \leq_{\alpha,A}^1 u'$.
- If $u = t \in O$ and $u' = (s', t') \in \alpha \times ((X \setminus O) \cap A)$, we have $(s', t) \leq_{\alpha \times A} (s', t')$, $q_O(s', t) = u$ and $q_O(s', t') = u'$, hence $u \leq_{\alpha,A}^1 u'$.
- If $u = (s, t) \in \alpha \times ((X \setminus O) \cap A)$ and $u' = t' \in O$, we have $(s, t) \leq_{\alpha \times A} (s, t')$, $q_O(s, t) = u$ and $q_O(s, t') = u'$, hence $u \leq_{\alpha,A}^1 u'$.
- If $u = (s, t) \in \alpha \times ((X \setminus O) \cap A)$ and $u' = (s', t') \in \alpha \times ((X \setminus O) \cap A)$, we have $(s, t) \leq_{\alpha \times A} (s', t')$, $q_O(s, t) = u$ and $q_O(s', t') = u'$, hence $u \leq_{\alpha,A}^1 u'$.

By definition of q_O , if $u \leq_{\alpha,A}^1 u'$ then (15) is satisfied. \square

Corollary 4.6. If $u_0 \leq_{\alpha,A}^1 \cdots \leq_{\alpha,A}^1 u_i \leq_{\alpha,A}^1 \cdots \leq_{\alpha,A}^1 u_n$ with $i \in \{0, \dots, n\}$ such that $u_i \in O \cap A$, then $u_0 \leq_{\alpha,A}^1 u_i \leq_{\alpha,A}^1 u_n$.

Proof We apply Lemma 4.5 to deduce that $p_2(u_0) \leq_A p_2(u_i) \leq_A p_2(u_n)$ with $u_i \in O \cap A$. We apply it once again (converse implication) to conclude. \square

Lemma 4.7. The transitive closure of the relation $\leq_{\alpha,A}^1$, which we denote by $\leq_{\alpha,A}$, is antisymmetric. Moreover we have $u \leq_{\alpha,A} u'$ if and only if

$$u \leq_{\alpha,A}^1 u' \quad \text{or} \quad \exists u'' \in O \cap A \quad \text{such that} \quad u \leq_{\alpha,A}^1 u'' \leq_{\alpha,A}^1 u'. \quad (16)$$

Proof For the purpose of the current proof, let us write $u \sqsubseteq u'$ to mean that condition (16) is satisfied. The relation \sqsubseteq is an extension of $\leq_{\alpha,A}^1$ and it is transitive by Corollary 4.6, therefore $\leq_{\alpha,A} \subseteq \sqsubseteq$. The converse inclusion is readily deduced from the transitivity of $\leq_{\alpha,A}$. We check that $\leq_{\alpha,A}$ is antisymmetric. Indeed, if $u \leq_{\alpha,A} u'$ $u' \leq_{\alpha,A} u$ then we have $p_2(u) = p_2(u')$. It follows that u, u' , and all u'' such that $u \leq_{\alpha,A} u'' \leq_{\alpha,A} u'$ belong to O or to its complement. The first case is obvious, in the second one we have $s \leq_{\alpha} s' \leq_{\alpha} s$, from which we deduce that $s = s'$. The relation \sqsubseteq is, by definition, an extension of $\leq_{\alpha,A}^1$. \square

Definition 4.8. A set C is said to be *order-convex* in a poset (P, \sqsubseteq) when $C \subseteq P$ and for all $x, y, z \in P$, if $x \sqsubseteq y \sqsubseteq z$ and $x, z \in C$, then $y \in C$.

Corollary 4.9. If $(X \setminus O) \cap A$ is order-convex in (A, \leq_A) then $\leq_{\alpha,A} = \leq_{\alpha,A}^1$.

Proof Suppose we have $u'' \in O \cap A$ such that $u \leq_{\alpha,A}^1 u'' \leq_{\alpha,A}^1 u'$ (Lemma 4.7). In particular $u, u' \in A$. We have $\{u, u'\} \cap O \neq \emptyset$ otherwise $u'' \notin O$ because $(X \setminus O) \cap A$ is order-convex. Then we have $u \leq_{\alpha,A}^1 u'$ by Corollary 4.6. \square

Definition 4.10. Let O be an open subset of X . We say that O is *admissible* for an ordered base \mathfrak{B} of \mathcal{X} (Definition 3.38) when $(X \setminus O) \cap B$ is order-convex in B for every $B \in \mathfrak{B}$. We say that O is *admissible* for \mathcal{X} when it is admissible for some ordered base of \mathcal{X} . The locally ordered space \mathcal{X} is said to be *admissible* when X has a base of admissible open subsets.

Example 4.11. Any locally ordered space \mathcal{X} that is a mere topological space (in the sense of Example 3.9) is admissible; indeed every open subset O of $Sp(\mathcal{X})$ is admissible for the ordered base $\mathfrak{B} = OB(\mathcal{X})$. Given an ordered base \mathfrak{B} of a locally ordered space \mathcal{X} (Definition 3.38), every clopen subset O of X is admissible for the ordered base $\{B \in \mathfrak{B} \mid B \subseteq O \text{ or } B \subseteq X \setminus O\}$ of \mathcal{X} . In particular \mathcal{X} is admissible if $Sp(\mathcal{X})$ is totally disconnected. The directed circle S is admissible: if O is an open arc such that $X \setminus O$ contains more than one

point, then it is admissible for the ordered base

$$\mathfrak{B} = \{B \text{ open arc} \mid \text{length}(B) \leq \text{length}(X \setminus O)\};$$

indeed for every $B \in \mathfrak{B}$, $(X \setminus O) \cap B$ consist of at most one arc. Similarly, the directed real line is admissible: consider open intervals instead of open arcs.

Lemma 4.12. Let O be an open subset of X admissible for an ordered base \mathfrak{B} of \mathcal{X} (Definition 4.10). The collection of posets $(U_{\alpha,B}, \leq_{\alpha,B})$ with $B \in \mathfrak{B}$ and α proper open arc (Lemma 4.7) is an ordered base on X_O (Lemma 4.4). We denote by \mathcal{X}_O the resulting locally ordered space.

Proof Let $u \in X_O$ such that $u \in U_{\alpha_0,B_0} \cap U_{\alpha_1,B_1}$ with α_0, α_1 proper open arcs and $B_0, B_1 \in \mathfrak{B}$. If $u \in O$, then we have $B_2 \in \mathfrak{B}$ such that $B_2 \subseteq O$ and $u \in B_2 \hookrightarrow B_0, B_1$ (Definition 3.29). If $u = (s, t) \in S^1 \times (X \setminus O)$, then we have $B_2 \in \mathfrak{B}$ and a proper open arc α_2 such that $s \in \alpha_2 \subseteq \alpha_0 \cap \alpha_1$ and $t \in B_2 \hookrightarrow B_0, B_1$. In both cases we have $\leq_{\alpha_0,B_2} = \leq_{\alpha_0,B_2}^1$ and $\leq_{\alpha_i,B_i} = \leq_{\alpha_i,B_i}^1$ for every $i \in \{0, 1\}$ because $(X \setminus O) \cap B_i$ (for $i \in \{0, 1, 2\}$) is order-convex in the poset B_i (Corollary 4.9). By Lemma 4.5, the relation \leq_{α_0,B_2}^1 coincide (in both cases) with the restrictions of \leq_{α_0,B_0}^1 and \leq_{α_1,B_1}^1 to B_2 . Therefore we have $u \in U_{\alpha_0,B_2} \hookrightarrow U_{\alpha_0,B_0}, U_{\alpha_1,B_1}$. \square

Lemma 4.13. Let O be an open subset of X admissible for an ordered base \mathfrak{B} of \mathcal{X} . The map q_O is locally order-preserving from \mathcal{X} to \mathcal{X}_O .

Proof Let $x = (s, t) \in S^1 \times X$ and let $U_{\alpha,B}$ such that $q_O(x) \in U_{\alpha,B}$ with $B \in \mathfrak{B}$ and α proper open arc. If $q_O(x) \in O$ then we can suppose that $B \subseteq O$. In that case, the map q_O coincides with the second projection on $\alpha \times O$ so $q_O(x) \in U_{\alpha,B}$ regardless of α ; in particular α may not contain s . Hence we have to choose a proper open arc α' containing s , so that $x \in \alpha' \times B$, $q_O(\alpha' \times B) \subseteq U_{\alpha,B}$ and the restriction of q_O to $\alpha' \times B$ is order-preserving from $\alpha' \times B$ to $U_{\alpha,B}$. If $q_O(x) = (s, t) \in S^1 \times (X \setminus O)$, then $x \in \alpha \times B$, $q_O(\alpha \times B) \subseteq U_{\alpha,B}$ and the restriction of q_O to $\alpha \times B$ is order-preserving from $\alpha \times B$ to $U_{\alpha,B}$. \square

Lemma 4.14. Let O be an open subset of X admissible for an ordered base \mathfrak{B} of \mathcal{X} . Any locally order-preserving map $f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{Y}$ such that $O \subseteq K(f)$ factors through the map q_O in a unique way.

Proof Denote by Y the underlying space of \mathcal{Y} . The map $h : X_O \rightarrow Y$ defined by $h(q_O(s, t)) \hat{=} f(s, t)$ is the only one satisfying $f = h \circ q_O$ because q_O is surjective; and h is continuous because X_O is equipped with the final topology associated with q_O . Let $u \in X_O$ and $W \in \mathcal{Y}$ such that $h(u) \in W$. Let $(s, t) \in S^1 \times X$ such that $u = q_O(s, t)$. Since f is locally order-preserving, we have $B \in \mathfrak{B}$ and a proper open arc α such that $(s, t) \in \alpha \times B$, $f(\alpha \times B) \subseteq W$ and the restriction of f to $\alpha \times B$ is order-preserving from $\alpha \times B$ to W . Then $u \in U_{\alpha,B}$ and $h(U_{\alpha,B}) = h(q_O(\alpha \times B)) = f(\alpha \times B) \subseteq W$. It remains to show that the restriction of h from $U_{\alpha,B}$ to W is order-preserving. Let u' ,

$u'' \in U_{\alpha, B}$ such that $u' \leq_{\alpha, B}^1 u''$. There exist two elements x' and x'' of $\alpha \times B$ such that $x' \leq_{\alpha \times B} x''$, $q_O(x') = u'$ and $q_O(x'') = u''$. Hence $h(u') = f(x') \leq_W f(x'') = h(u'')$. \square

Corollary 4.15. Let \mathcal{X} be an admissible locally ordered space (Definition 4.10). In the category of locally ordered spaces, the morphisms i_* and c_* admit a coequalizer if, and only if, the family of open neighborhoods of $*$ has a smallest element. If O is the smallest open neighborhood of $*$, then the morphism $q_O : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{X}_O$ (Lemma 4.14) is the coequalizer.

Proof Suppose that $e : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{Y}$ is a coequalizer of i_* and c_* . The open set $K(e)$ contains $*$ (Lemma 4.1). Let U be an open neighborhood of $*$ admissible for \mathcal{X} and included in $K(e)$ (such an open set U exists because we made the assumption that \mathcal{X} is admissible). The morphism $q_U : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{X}_U$ (Lemma 4.13) satisfies $q_U \circ i_* = q_U \circ c_*$ so there is a morphism $h : \mathcal{Y} \rightarrow \mathcal{X}_U$ such that $q_U = h \circ f$, therefore $K(e) \subseteq K(q_U) = U$. Hence $K(e)$ is the smallest open neighborhood of $*$.

Conversely, let O be the smallest open neighborhood of $*$. Given $f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{Y}$ such that $f \circ i_* = f \circ c_*$, the set $K(f)$ is an open neighborhood of $*$ (Lemma 4.1) so $O \subseteq K(f)$. By lemma 4.14, there exists a unique factorization of f through q_O . \square

Lemma 4.16. For every continuous map $f : S^1 \times X \rightarrow Y$ with Y locally Hausdorff, the set $K(f)$ is a closed subset of X .

Proof Let $t \in \overline{K(f)}$, we are to show that the continuous map $f \circ i_t : S^1 \rightarrow Y$ is locally constant, which is sufficient to prove that $f \circ i_t$ is constant (i.e. $t \in K(f)$) because S^1 is connected.

Let $s \in S^1$ and $U \in \mathcal{Y}$ such that $f(s, t) \in U$ and U is Hausdorff. Since f is continuous at (s, t) , there exists an open neighborhood $\alpha \times A$ of (s, t) such that $f(\alpha \times A) \subseteq U$ with α denoting a proper open arc and A an open subset of X . From $t \in \overline{K(f)}$ we deduce that there is a net $(t_i)_{i \in I}$ of $A \cap K(f)$ that converges to t . Given $s' \in \alpha$, the nets $(s, t_i)_{i \in I}$ and $(s', t_i)_{i \in I}$ converge to (s, t) and (s', t) respectively. Each t_i belongs to $K(f)$ hence $f(s, t_i) = f(s', t_i)$. The images of the nets $(s, t_i)_{i \in I}$ and $(s', t_i)_{i \in I}$ under f are thus equal and converge, by continuity of f , to $f(s, t)$ and $f(s', t)$. We have $f(s, t) = f(s', t)$ because U is Hausdorff. \square

Remark 4.17. Connectedness is the only property of S^1 that is really used in the above proof. Under the stronger assumption that Y is Hausdorff, the above lemma is valid for any topological space instead of S^1 .

Proposition 4.18. If X is Hausdorff, then for every open subset O of X admissible for an ordered base \mathcal{B} of \mathcal{X} , the following are equivalent:

1. The underlying topology of the locally ordered space \mathcal{X}_O is Hausdorff.
2. The set O is closed in X .

Proof The first point implies the second one by lemma 4.16 (take $f = q_O$). Conversely, assume that O is closed. Let $u, u' \in X_O$ with $u \neq u'$. We have two situations to take into account:

- There are $s, s' \in S^1$ and $t, t' \in X$ with $t \neq t'$ such that $u = q_O(s, t)$ and $u' = q_O(s', t')$. Since X is Hausdorff, there are $B, B' \in \mathfrak{B}$ disjoint and containing t and t' respectively. For any proper open arc α containing s and s' , the subsets $U_{\alpha, B}$ and $U_{\alpha, B'}$ are disjoint open neighborhoods of u and u' respectively.
- There are $t \in X$ and $s, s' \in S^1$ with $s \neq s'$ and $u = q_O(s, t)$ and $u' = q_O(s', t)$. Let α and α' be disjoint proper open arcs containing s and s' respectively, and let $B \in \mathfrak{B}$ containing t and included in $X \setminus O$ (which is an open subset by hypothesis). Then the subsets $U_{\alpha, B}$ and $U_{\alpha', B}$ are disjoint open neighborhoods of u and u' respectively.

□

Definition 4.19. A *locally Nachbin ordered space*, or *local pospace*, is a locally ordered space having an ordered base whose elements are Nachbin ordered spaces (Definition 2.1).

The class of Nachbin ordered spaces is stable under subspaces. In particular, if \mathcal{U} is a local order whose elements are Nachbin ordered spaces, then so is its idealization (Example 3.42). As a consequence, the locally ordered space \mathcal{X} is a locally Nachbin ordered space if, and only if, it is equivalent to a local order whose elements are Nachbin ordered spaces. In addition, if X has a base of admissible open subsets, then we can suppose that the elements of this base are Nachbin ordered spaces.

Proposition 4.20. Assume that \mathcal{X} is a locally Nachbin ordered space. For every open subset O of X , the following propositions are equivalent:

1. The space \mathcal{X}_O is locally Nachbin ordered.
2. The set O is closed in X .

Proof The first point implies the second one by Lemma 4.16. Conversely, assume that O is closed. Let B, α and u be an element of \mathfrak{B} , an ordered arc, and an element of $U_{\alpha, B}$ respectively. We have two situations to deal with. On the one hand, if $u = t \in O \cap B$, there is $B' \in \mathfrak{B}$ included in O such that $t \in B' \hookrightarrow B$, and the partial order $\leq_{B'}$ is closed for the topology induced by X . So the sets $U_{\alpha, B'}$ and B' are equal, the topologies induced on them by X_O and X are the same, and the partial orders $\leq_{\alpha, B'}$ and $\leq_{B'}$ coincide. Moreover, we have $U_{\alpha, B'} \hookrightarrow U_{\alpha, B}$. On the other hand, if $x = (s, t) \in \alpha \times (B \cap (X \setminus O))$ with $X \setminus O$ open, there is $B' \in \mathfrak{B}$ included in $X \setminus O$ such that $t \in B' \hookrightarrow B$ and the partial order $\leq_{B'}$ is closed for the topology induced by X . So the sets $U_{\alpha, B'}$ and $\alpha \times B'$ are equal, the topologies induced on

them by X_O and $S^1 \times X$ are the same, and the partial orders $\leq_{\alpha, B'}$ and $\leq_{\alpha} \times \leq_{B'}$ coincide. Moreover, we have $U_{\alpha, B'} \hookrightarrow U_{\alpha, B}$. \square

Example 4.23 and the next result, require denotations for certain full subcategories of **LoSp** (the category of locally ordered spaces):

Denotation	Objects
H	Locally ordered <i>Hausdorff</i> spaces
N	Locally <i>Nachbin</i> ordered spaces
NH	Locally <i>Nachbin</i> ordered <i>Hausdorff</i> spaces

Corollary 4.21. The coequalizer of i_* and c_* exists in **N** if, and only if, the family of clopen neighborhoods of $*$ has a smallest element. If O is such a neighborhood, then $q_O : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{X}_O$ is the coequalizer. If the underlying topology of X is Hausdorff, the above equivalence also holds in **H** and **NH**.

Proof First we make two remarks ensuring that the choice of the ‘ambient category’, which refers to **N**, **H** or **NH**, does not matter in the rest of the proof. Let $f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of the ambient category such that $f \circ i_* = f \circ c_*$:

i) In each case Y is locally Hausdorff (if the ambient category is **N**, see Remark 2.5) so Lemma 4.16 applies, therefore $K(f)$ is closed. By Lemma 4.1 we conclude that $K(f)$ is a clopen neighborhood of $*$ in X .

ii) Given a clopen subset U of X , we deduce from Propositions 4.18 and 4.20 that if X is an object of **N**, **H** or **NH**, then so is X_U accordingly.

The rest of the proof is a carbon copy of the proof of Corollary 4.15, with clopen subsets instead of open ones. \square

Corollary 4.22. If X is Hausdorff and locally connected, then the coequalizer of i_* and c_* is $q_O : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{X}_O$ with O denoting the connected component of $*$.

Proof Any clopen subset of a locally connected space X is a union of connected components of X . In particular the connected component of $*$ is the smallest clopen subset of X containing $*$. We conclude by applying Corollary 4.21. \square

It is time to provide some concrete examples:

Example 4.23. The spaces \mathbb{R} and \mathbb{Q} are the real line and the space of rational numbers. The coarsest refinement of the topology of \mathbb{R} in which every singleton $\{x\}$ with $x \neq 0$ is open induces a topological space that is denoted by \mathbb{R}_* ; in other words, $X \subseteq \mathbb{R}$ is open in \mathbb{R}_* iff $0 \notin X$ or there is an open subset V of \mathbb{R} such that $0 \in V \subseteq X$. The saturation of $\{(\mathbb{R}, \leq)\}$ is a locally Nachbin ordered Hausdorff space which we denote by $\vec{\mathbb{R}}$. Both $\vec{\mathbb{R}}$ and the directed circle \mathcal{S} are described in Example 3.22. Given an infinite set U with a distinguished element \bar{u} , we denote by \mathcal{U} the topological space on U in which a nonempty subset is open when it contains \bar{u} , i.e. the nonempty open subsets

of \mathcal{U} are the elements of the *principal ultrafilter* on U generated by \bar{u} [5, p.233]. The distinguished elements of \mathbb{R}_* and \mathcal{U} are respectively 0 and \bar{u} . In all the other cases, the distinguished element can be any point of the space. The table below summarizes the cases where the coequalizer of i_* and c_* exists (✓) or not (✗); the space \mathcal{U} does not belong to **H**, **N**, or **NH**, because none of the neighborhoods of \bar{u} is Hausdorff, whence the cases containing ‘n/a’.

category of	X	\mathbb{R}	$\vec{\mathbb{R}}$	\mathcal{S}	\mathbb{R}_*	\mathbb{Q}	\mathcal{U}
locally ordered spaces (LoSp)				✗			✓
locally ordered Hausdorff spaces (H)		✓			✗		n/a
locally Nachbin ordered spaces (N)		✓			✗		n/a
locally Nachbin ordered Hausdorff spaces (NH)		✓			✗		n/a

All the locally ordered spaces appearing on the first row of table above are admissible (Definition 4.10 and Example 4.11); we refer to them as ‘the spaces of the first row’. Apart from \mathcal{U} , their underlying spaces are Hausdorff. The only space of the first row in which the collection of open neighborhoods of $*$ has a least element is \mathcal{U} , so the second row of the table is given by Corollary 4.15; and the coequalizer of \mathcal{U} in **LoSp** is the quotient map q_O with $O = \{\bar{s}\}$, see Eq.(14). Every point of a connected space has a unique clopen neighborhood, namely the entire space (the real line and the unit circle are such spaces). For every $\varepsilon > 0$, the interval $]-\varepsilon, \varepsilon[$ (resp. $]-\varepsilon, \varepsilon[\cap \mathbb{Q}$ with $\varepsilon \in \mathbb{R} \setminus \mathbb{Q}$) is a clopen neighborhood of 0 in \mathbb{R}_* (resp. \mathbb{Q}). Hence the last three rows of the table are given by Corollary 4.21; the coequalizer, when it exists, is the second projection.

The coequalizer of $(i_*, c_*) : \mathcal{S} \rightarrow \mathcal{S} \times \vec{\mathbb{R}}$ (resp. $\mathcal{S} \times \mathbb{R}$) is treated in [17, XIII.4.27, p.256] with a notion of locally ordered space that slightly differs from ours.

4.2 Zebra cylinder

We emphasize that for any point t of $[0, 1]$ the smallest clopen neighborhood of t is $[0, 1]$, so according to Corollary 4.21 the coequalizer of $(i_0, c_0) : \mathcal{S} \rightarrow \mathcal{S} \times [0, 1]$ exists in the category of locally ordered Hausdorff spaces; it is $[0, 1]$ whether it is directed or not.

In this Section, we describe an ordered base \mathfrak{B} on $S^1 \times [0, 1]$ with the property that the coequalizer of the morphisms (i_0, c_0) no longer exists in the category of locally (Nachbin) ordered (Hausdorff) spaces. As before, the strategy consists of setting \mathfrak{B} in a way that:

- every section $S^1 \times \{x\}$ included in a chosen neighborhood V of $S^1 \times \{0\}$ is collapsed, and
- the neighborhood V can be made arbitrarily small.

Let $\overline{\mathbb{N}}$ be the set of extended natural numbers $\mathbb{N} \sqcup \{\infty\}$, and $(d_n)_{n \in \mathbb{N}}$ be a decreasing sequence (i.e. $n < m \Rightarrow d_n > d_m$) with values in the compact unit interval $[0, 1]$. Assume that $\inf\{d_n \mid n \in \mathbb{N}\} = 0$. For every $n \in \mathbb{N}$ we denote by $I(n)$ the interval $[d_{2n+1}, d_{2n}]$ and by $I(\infty)$ the degenerated interval $\{0\}$. The elements of \mathfrak{B} are of the form $\alpha \times O$ where α is an ordered arc and O is an open subset of $[0, 1]$, partially ordered as follows:

$$(s, u) \preceq_{\alpha}^O (s', u') \quad \text{if} \quad \begin{cases} s \leq_{\alpha} s' \text{ and } u = u' \in I(n) \text{ for some } n \in \overline{\mathbb{N}}, \\ \text{or } s = s' \text{ and } u = u' . \end{cases} \quad (17)$$

We denote by \mathcal{Z} the resulting *locally Nachbin ordered Hausdorff space*, see (Fig. 2).

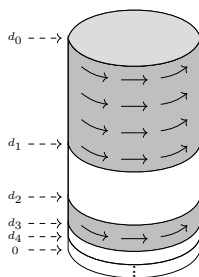


Fig. 2 Zebra cylinder

Let $f : \mathcal{Z} \rightarrow \mathcal{X}$ be a locally order-preserving map such that $f \circ i_0 = f \circ c_0$. We recall that $K(f)$ is the set $\{t \in I \mid \forall s, s' \in S^1, f(s, t) = f(s', t)\}$. We have a result similar to Lemma 4.1:

Lemma 4.24. There exists $n \in \mathbb{N}$ such that for all $m \geq n$, $I(m) \subseteq K(f)$.

Proof The arguments are mostly the same as in the proof of Lemma 4.1. By hypothesis 0 belongs to $K(f)$. Let $U \in \mathcal{X}$ such that $f(1, 0) \in U$. For every $s \in S^1$, f is locally order-preserving at $(s, 0)$, so there exists an open neighborhood $\alpha_s \times O_s$ of $(s, 0)$ such that $f(\alpha_s \times O_s) \subseteq U$ and the restriction $f_s : (\alpha_s \times O_s, \preceq_s) \rightarrow U$ is order-preserving with α_s denoting a proper open arc and \preceq_s the partial order $\preceq_{\alpha_s}^{O_s}$ on $\alpha_s \times O_s$.

Since S^1 is compact, we have a finite subset $J \subset S^1$ such that $\{\alpha_s \mid s \in J\}$ is an open covering of S^1 . We denote by O the intersection of the sets O_s for $s \in J$ which is thus an open neighborhood of 0. Since the sequence $(d_n)_{n \in \mathbb{N}}$ tends to 0, there exists $n \in \mathbb{N}$ such that the intervals $I(m)$ are included in O for every $m \geq n$. Let m be such a natural number. We are to show that $I(m) \subset K(f)$. Given $t \in I(m)$, $x, y \in S^1$, there exists a finite sequence $r_0, \dots, r_l \in S^1$ such that $r_0 = x$, $r_l = y$, and for every $k \in \{1, \dots, l\}$ there is an $s_k \in J$ such that $r_k, r_{k+1} \in \alpha_{s_k}$ and $r_k \preceq_k r_{k+1}$ with \preceq_k denoting the standard order on the proper open arc α_{s_k} . In particular, since $t \in I(m)$, we have

$$(r_k, t) \preceq_{s_k} (r_{k+1}, t)$$

from which we deduce that

$$f(r_k, t) \leq_U f(r_{k+1}, t)$$

because the restriction f_{s_k} is order-preserving. By transitivity of \leq_U , we have $f(x, t) \leq_U f(y, t)$. By swapping the roles of x and y in the previous reasoning we prove that $f(y, t) \leq_U f(x, t)$. By antisymmetry of \leq_U we deduce that $f(x, t) = f(y, t)$, so t belongs to $K(f)$, which therefore contains $I(m)$. \square

Proposition 4.25. The morphisms $i_0, c_0 : \mathcal{S} \rightarrow \mathcal{Z}$ have no coequalizer in the category of locally (Nachbin) ordered (Hausdorff) spaces.

Proof We construct, for each $n \in \mathbb{N}$, a locally ordered space \mathcal{X}_n (which is actually Nachbin and Hausdorff) and a morphism $g_n : \mathcal{Z} \rightarrow \mathcal{X}_n$ such that $g_n \circ i_0 = g_n \circ c_0$ and $K(g_n) = [0, d_{2n}]$. If f were the coequalizer of (i_0, c_0) , then every g_n would factor through f , so we would have $K(f) \subseteq K(g_n)$ for every $n \in \mathbb{N}$; it would follow that

$$0 \in K(f) \subseteq \bigcap_{n \in \mathbb{N}} K(g_n) = \bigcap_{n \in \mathbb{N}} [0, d_{2n}] = \{0\}$$

which would contradict Lemma 4.24. For each $n \in \mathbb{N}$ we define the set (Fig. 3)

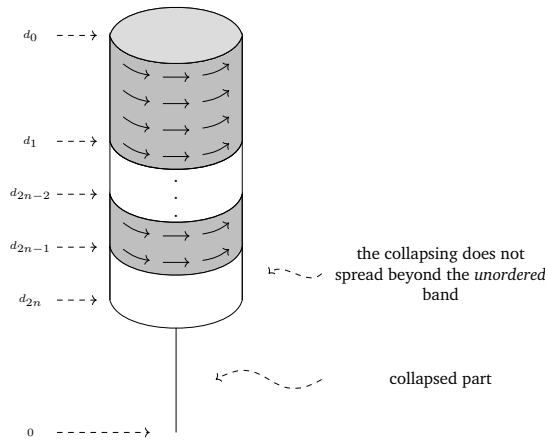


Fig. 3 The locally ordered space X_n

$$X_n := [0, d_{2n}] \sqcup \{(s, t) \mid s \in S^1, t \in]d_{2n}, 1]\}$$

and the (set theoretic) map $g_n : \mathcal{Z} \rightarrow X_n$ by

$$f_n(s, t) = \begin{cases} (s, t) & \text{if } t > d_{2n} \\ t & \text{if } t \leq d_{2n} . \end{cases}$$

We note that $g_n \circ i_0 = g_n \circ c_0$ and $K(g_n) = [0, d_{2n}]$. The final topology of g_n is generated by the subsets of the form

$$O_{\alpha, A} := \begin{cases} B & \text{if } A \subseteq [0, d_{2n}[\\ B \sqcup (S_1 \times C) & \text{if } d_{2n} \in A \\ \alpha \times C & \text{if } A \subseteq]d_{2n}, 1] \end{cases} \quad (18)$$

with α proper open arc, A open interval of $[0, 1]$, $B := A \cap [0, d_{2n}]$ and $C := A \cap]d_{2n}, 1]$. In order to define an ordered base on X_n that makes the map g_n a morphism of locally ordered spaces, we only consider the intervals A whose length is strictly less than $d_{2n-1} - d_{2n}$. By definition, the partial order $\leq_{\alpha, A}$ on $O_{\alpha, A}$ coincides with the equality relation in the first two cases of (18), and with the partial order described at (17) in the third case of (18). In particular, the condition on the length of A guarantees that for any proper open arc α , the partial order on $B \sqcup (S^1 \times C)$ and the partial order on $\alpha \times A$ from the ordered base of \mathcal{Z} agree on $\alpha \times C$. This key observation ensures that g_n is indeed a morphism of locally ordered spaces. \square

4.3 Rational based cylinder

We have seen that the forgetful functor $Sp: \mathbf{LO} \rightarrow \mathbf{Top}$ does not preserve coequalizers: one of the reason is that collapsing a directed loop to a point also collapses the points in a neighborhood of that loop (Lemma 4.1). This phenomenon is mainly due to the strong connectedness of \mathcal{S} (Remark 4.2). In this section we replace \mathcal{S} by some of its dense totally disconnected subspaces; then we exhibit a pair of morphisms whose coequalizer exists, and whose underlying space is the topological coequalizer.

Definition 4.26. The subspace $\{e^{ix} \mid x \in \mathbb{Q}\}$ of S^1 is denoted by $S^1_{\mathbb{Q}}$. The *directed rational circle* $\mathcal{S}_{\mathbb{Q}}$ is the sub-local order of \mathcal{S} on $S^1_{\mathbb{Q}}$ (Definition 3.16).

We overload the denotations i_0 and c_0 which now denote the morphisms

$$s \in \mathcal{S}_{\mathbb{Q}} \mapsto (s, 0) \in \mathcal{S}_{\mathbb{Q}} \times I \quad \text{and} \quad s \in \mathcal{S}_{\mathbb{Q}} \mapsto (1, 0) \in \mathcal{S}_{\mathbb{Q}} \times I$$

As before we identify all the points of the section $S^1_{\mathbb{Q}} \times \{0\}$, the resulting coequalizer in the category of topological spaces is the map

$$q : S^1_{\mathbb{Q}} \times [0, 1] \rightarrow \{0\} \sqcup (S^1_{\mathbb{Q}} \times]0, 1]) \quad (19)$$

with the codomain equipped with the finest topology making q continuous. A base of open neighborhoods of (s, t) with $t > 0$ is given by the traces on $S^1_{\mathbb{Q}}$ of the products $\alpha \times]a, b[$ with $s \in \alpha$ proper open arc and $0 < a < t < b$. The associated partial order is given by the restriction of the product order $\leq_{\alpha} \times =$. The neighborhoods of 0 are a bit harder to describe. To this aim, we consider the set \mathcal{H} of all continuous functions $h : S^1_{\mathbb{Q}} \rightarrow]0, 1]$ such that $\inf h = 0$. For every $h \in \mathcal{H}$ we define the set

$$O_h = \{0\} \sqcup \{ (s, t) \in S^1_{\mathbb{Q}} \times]0, 1] \mid t < h(s) \} \quad .$$

We note that \mathcal{H} is an inf-semilattice in which the greatest lower bound of a pair of elements is computed pointwise. Moreover, the map $h \mapsto O_h$ is a morphism of inf-semilattices. Each set O_h is equipped with the preorder \leq_h defined by $(s, t) \leq_h (s', t')$ when $t = t'$ and there exists a proper open arc α such that $s \leq_{\alpha} s'$ and $(\alpha \cap S^1_{\mathbb{Q}}) \times \{t\} \subseteq O_h$ (and $0 \leq_h 0$). Assume that $(s, t) \leq_h (s', t') \leq_h (s, t)$.

We have $t = t'$. If $s \neq s'$, then we also have two proper open arcs α and β such that $s \leq_\alpha s'$, $s' \leq_\beta s$, and $((\alpha \cup \beta) \cap S_{\mathbb{Q}}^1) \times \{t\} = S_{\mathbb{Q}}^1 \times \{t\} \subseteq O_h$. But then we have $\inf(h) \geq t$, which contradicts the constraint $\inf(h) = 0$. Hence $s = s'$ and \leq_h is antisymmetric.

Lemma 4.27. Let $f : S_{\mathbb{Q}} \times I \rightarrow X$ be a locally order-preserving map such that $f \circ i_0 = f \circ c_0$. Denote by $g : \{0\} \sqcup (S_{\mathbb{Q}}^1 \times]0, 1]) \rightarrow X$ the unique map such that $f = g \circ q$. For every $U \in X$ such that $g(0) \in U$, there exists a family $(\alpha_j, t_j)_{j \in J}$ (with $J \subseteq \mathbb{N}$) such that for every $j \in J$:

- i) α_j is a proper open arc from e^{ia_j} to e^{ib_j} with $a_j, b_j \in \mathbb{R} \setminus \mathbb{Q}$,
- ii) $(\alpha_j \cap S_{\mathbb{Q}}^1) \times]0, t_j[\subseteq f^{-1}(U)$,
- iii) $f(s, t) \leq_U f(s', t)$ for all $s, s' \in \alpha_j \cap S_{\mathbb{Q}}^1$ such that $s \leq_{\alpha_j} s'$ and $0 \leq t \leq t_j$,

and the collection $\{\alpha_j \cap S_{\mathbb{Q}}^1 \mid j \in J\}$ is a partition of $S_{\mathbb{Q}}^1$.

Proof Let $(s_n)_{n \in \mathbb{N}}$ be an enumeration of $S_{\mathbb{Q}}^1$. Since f is locally order-preserving at $(s_0, 0)$, there exists an open neighborhood of $(s_0, 0)$ of the form $A_0 \times]0, t_0[$ such that the restriction of f to $A_0 \times]0, t_0[$ with values in U is order-preserving. Taking A_0 to be the trace on $S_{\mathbb{Q}}^1$ of a proper open arc α_0 from e^{ia_0} to e^{ib_0} with $a_0, b_0 \in \mathbb{R} \setminus \mathbb{Q}$, we obtain a clopen subset of $S_{\mathbb{Q}}^1$. Define $J_0 = \{0\}$. Suppose that we have already defined α_j and t_j for $j \in J_N$, with $N+1$ denoting the cardinality of J_N . If the union U_N of the sets $\alpha_j \cap S_{\mathbb{Q}}^1$ for $j \in J_N$ is $S_{\mathbb{Q}}^1$, then $J := J_N$ and the construction is over. Otherwise, let n be the smallest integer such that s_n does not belong to U_N . We have $n > N$ and we define $J_{N+1} = J_N \cup \{n\}$. We can find a clopen A_n which contains s_n and a number $0 < t_n \leq 1$ so that the restriction of f to $A_n \times]0, t_n[$ with values in U is order-preserving. The union U_N is closed because so is each $\alpha_j \cap S_{\mathbb{Q}}^1$; hence we can suppose that A_n does not meet U_N . Finally, we can assume that A_n is the trace on $S_{\mathbb{Q}}^1$ of a proper open arc α_n from e^{ia_n} to e^{ib_n} with a_n and b_n in $\mathbb{R} \setminus \mathbb{Q}$. \square

Remark 4.28. If the map $f : S_{\mathbb{Q}}^1 \times I \rightarrow X$ from the statement of Lemma 4.27 is only supposed to be continuous, and U is an open subset of X , then items i) and ii) still hold, and the collection $\{\alpha_j \cap S_{\mathbb{Q}}^1 \mid j \in J\}$ is still a partition of $S_{\mathbb{Q}}^1$. It suffices to ignore the order theoretical arguments in the proof.

Lemma 4.29. Let f, g, U , and $(\alpha_j, t_j)_{j \in J}$ be as in Lemma 4.27, and let $(\phi_j)_{j \in J}$ be a family of continuous maps $\phi_j : \alpha_j \rightarrow]0, t_j[$ that tend to 0 at the boundary of α_j (e.g. *bump functions*). The mapping $h : S_{\mathbb{Q}}^1 \rightarrow]0, 1]$ whose restriction to α_j coincides with ϕ_j belongs to \mathcal{H} , and g induces an order-preserving map from (O_h, \leq_h) to (U, \leq_U) (in particular $g(O_h) \subseteq U$).

Proof Suppose that we have $(s, t) \leq_h (s', t)$ in O_h . By definition of the partial order \leq_h there exists an open proper arc α such that $s \leq_\alpha s'$ and $(\alpha \cap S_{\mathbb{Q}}^1) \times \{t\} \subseteq O_h$. Let $j \in J$ be such that $s \in \alpha_j$. Recall that α and α_j are the images of $]a, b[$ and $]a_j, b_j[$ under the complex exponential map $t \in \mathbb{R} \mapsto e^{it} \in S^1$ with $b - a < 2\pi$, $b_j - a_j < 2\pi$. Moreover, once a and b are fixed, one can choose a_j and b_j such that $s = e^{ia}$ for

some $x \in]a, b[\cap]a_j, b_j[$. We also have x' in $]a, b[$ such that $x \leq x'$ and $e^{ix'} = s'$. We cannot have $b_j < x'$ otherwise the interval $]x, b_j[$ would be included in $]a, b[$ so we would have $x'' \in]a, b[\cap \mathbb{Q}$ with $h(e^{ix''})$ arbitrarily small. In particular $(e^{ix''}, t)$ would not belong to O_h . Moreover we have $b_j \neq x'$ because one is rational while the other is not. Since the standard order on \mathbb{R} is total, we have $x' < b_j$. Hence both x and x' belong to $]a_j, b_j[$, and we have $x \leq x'$ so $s \leq_{\alpha_j} s'$. Moreover $t < h(s) \leq t_j$. It follows that $f(s, t) \leq_U f(s', t)$. \square

Remark 4.30. As a continuation of Remark 4.28, if f is continuous and U open in X , then we have $g(O_h) \subseteq U$ with the notation from Lemma 4.29. Once again, it suffices to ignore the order theoretical arguments of the proof.

Lemma 4.31. The collection of posets $\{(O_h, \leq_h) \mid h \in \mathcal{H}\}$ is an ordered base whose underlying set is $\{0\} \sqcup (S_{\mathbb{Q}}^1 \times]0, 1])$, and such that the map q – see Eq.(19), induces the coequalizer of i_0 and c_0 .

Proof In order to prove that O_h is open, we show that $q^{-1}(O_h) = \{(s, t) \in S_{\mathbb{Q}}^1 \times I \mid t < h(s)\}$ is an open subset of $S_{\mathbb{Q}}^1 \times I$. Let $(s, t) \in q^{-1}(O_h)$, since $t < h(s)$ there exist two disjoint open intervals A and A' of $[0, 1]$ such that $A < A'^2$, $t \in A$ and $h(s) \in A'$. Since h is continuous, there is an open subset O of $S_{\mathbb{Q}}^1$ containing s such that $h(O) \subset A'$. Therefore, $O \times A$ is an open neighborhood of (s, t) included in $q^{-1}(O_h)$. We apply Remark 4.30 with $f = q$ to prove that the collection $\{O_h \mid h \in \mathcal{H}\}$ is a base of the topology of the topological coequalizer of i_0 and c_0 . Given $h, h' \in \mathcal{H}$, since $O_{\min(h, h')}$ is the intersection of O_h and $O_{h'}$, the partial order $\leq_{\min(h, h')}$ coincide with the restrictions of both \leq_h and $\leq_{h'}$. So far we have checked that q induces an ordered base morphism. As an immediate consequence of Lemma 4.30, the unique map g such that $f = g \circ q$ is locally order-preserving. Therefore q is the expected coequalizer. \square

5 Locally ordered realization of a singular precubical set

Higher dimensional automata, which play a crucial role in concurrency theory [32], can be defined as precubical set morphisms whose targets are of the form $[\Sigma]$ (see §1). We would like to realize such morphisms in the category of locally ordered spaces, thus providing an interpretation of HDA. Unfortunately, the realization functor described in [6] cannot be applied to $[\Sigma]$ nor to its n -truncations ($n \geq 2$) because they are *singular* [6, 6.25]. It is thus natural to ask whether the locally ordered realization of $[\Sigma]$ exists, and if the colimits involved in its construction are well-behaved. We provide a partial answer.

If the set Σ has a unique element, then the precubical set $[\Sigma]$ is (up to isomorphism) the terminal object Ω . We prove that its 2-truncation, which we denote by Ω_2 , can be realized in the category of locally ordered spaces over its geometric realization. The second author is convinced that the result

²i.e. $a < a'$ holds for all $a \in A$ and $a' \in A'$.

holds for every precubical set; the case of Ω_2 provides a good insight of the problems one meets trying to write a proof, and may provide an intuition of why the colimits occurring in the realization functor construction should not be pathological.

The precubical set Ω_2 is entirely defined by the fact that

$$(\Omega_2)_n = \{n\} \quad (\text{for } n \leq 2) \quad \text{and} \quad (\Omega_2)_n = \emptyset \quad (\text{for } n > 2).$$

The *geometric realization* of Ω_2 is the quotient of the square $[0, 1]^2$ under the least equivalence relation \sim such that $(0, t) \sim (1, t) \sim (t, 1) \sim (t, 0)$ for all $t \in [0, 1]$. The quotient map $q : [0, 1]^2 \rightarrow \{\star\} \cup]0, 1[\cup]0, 1[^2$ coincide with the identity on $]0, 1[^2$; the first projection on $]0, 1[\times \{0, 1\}$; the second projection on $\{0, 1\} \times]0, 1[$; and takes the constant value \star on $\{0, 1\}^2$. The geometric realization of Ω_2 is the codomain of q equipped with the final topology, we denote it by $|\Omega_2|$.

An *elementary directed path* on a subset A of $|\Omega_2|$ is a (continuous) path on A of the form $q \circ \sigma$ with σ an order-preserving path on $[0, 1]^2$. A *directed path* on A is a *finite* concatenation of elementary directed paths on A . The topological space $|\Omega_2|$ and its directed paths form the realization of Ω_2 in the category of d-spaces (see §6.2) which we denote by $|\Omega_2|_{\text{dSPACE}}$. For every directed path on A from x to y we write $x \prec_A y$, and $x \rightarrow_A y$ when the directed path is elementary. Hence \prec_A is the transitive closure of \rightarrow_A . The former is, by definition, the *canonical preorder* on A . The subset A is said to be *antisymmetric* when so is its canonical preorder. In the rest of this section, we describe a base of the topology of $|\Omega_2|$ whose elements are antisymmetric. Then we check that the resulting collection of posets is an ordered base (Definition 3.29) inducing the realization of Ω_2 in the category of locally ordered spaces (Corollary 3.28 and Proposition 3.35).

The base announced in the preceding paragraph contains three kinds of sets, which are actually open balls for a certain metric, depending on whether they are ‘centered’ at a point in $]0, 1[^2$, $]0, 1[$, or $\{\star\}$. Given u and v in $]0, 1[$, and $\varepsilon > 0$, we write

$$U = \{u\} +]-\varepsilon, \varepsilon[, \quad V = \{v\} +]-\varepsilon, \varepsilon[, \quad X =]1 - \varepsilon, 1[, \quad \text{and} \quad Y =]0, \varepsilon[.$$

The parameter $\varepsilon > 0$ is meant to be arbitrarily small, in particular we assume that ε satisfies the following disjunction:

$$(X \cup Y) \cap (U \cup V) = \emptyset.$$

The open subsets centered at (u, v) are of the form $U \times V$, those ‘centered’ at u are of the form

$$B_u = U \cup U \times (X \cup Y) \cup (X \cup Y) \times U, \quad (20)$$

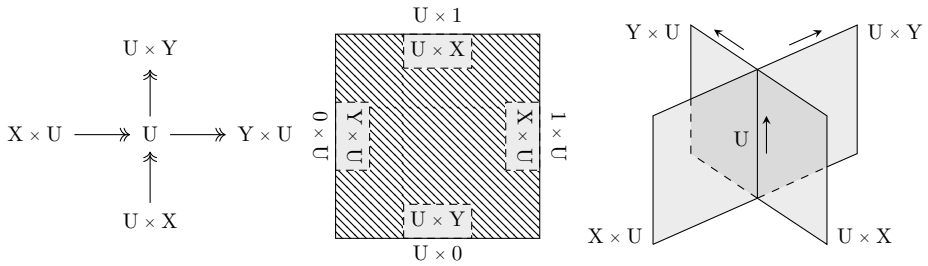


Fig. 4 The canonical preorder of B_u

those ‘centered’ at v , which we denote by B_v , are defined the same way with V instead of U , and those ‘centered’ at the point \star are of the form

$$B_\star = \{\star\} \cup X \cup Y \cup (X \cup Y)^2. \tag{21}$$

The parameter ε , which does not explicitly appear on the denotation, is referred to as the *radius*. In each case, we have to check that the canonical preorder is antisymmetric.

The canonical preorder of $U \times V$ is the relation $\rightarrow_{U \times V}$, namely the standard product order because the map q boils down to the identity on $]0, 1[^2$. The inverse image of U under the quotient map q is

$$q^{-1}U = U \times \{1\} \cup \{1\} \times U \cup U \times \{0\} \cup \{0\} \times U \tag{22}$$

as shown on the middle part of Figure 4. We partition the set B_u , defined at line (20), as suggested by the graph on the left hand part of Figure 4. We refer to any element of this partition as a *tile* of B_u . The graph also describes all the ways of going from a tile to another by means of an elementary directed path. This claim, which requires an easy but rather tedious checking, leads to an extensive description of the relation \rightarrow_{B_u} (we omit the subscript for the sake of readability) :

- the relation \rightarrow is the standard order on each tile of B_u (which is either a subset of \mathbb{R} or a subset of \mathbb{R}^2), and

for all $x \in X, y \in Y$, and $u', u'' \in U$:

- $u' \rightarrow (u'', y)$ and $u' \rightarrow (y, u'')$ when $u' \leq u''$, and
- $(x, u'') \rightarrow u'$ and $(u'', x) \rightarrow u'$ when $u'' \leq u'$.

The graph is acyclic, which means that a directed path cannot go back to a tile that it has already visited (and left). Formally, if $t \leq t'$ and δ is a directed path on B_u such that $\delta(t')$ does not belong to the tile containing $\delta(t)$, then neither does $\delta(t'')$ for $t'' \geq t'$. Hence if we have $a \preceq_{B_u} b$ and $b \preceq_{B_u} a$, then a and b are in the same tile. Since the relation \rightarrow_{B_u} is the standard order on each tile of B_u , we conclude that $a = b$.

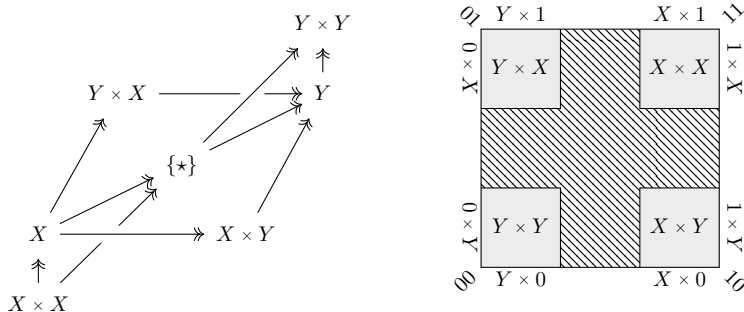


Fig. 5 The canonical preorder of W_\star

More explicitly, the partial order \preceq_{B_u} can be obtained as follows. We equip the set $X \cup \{\star\} \cup Y$ with the total order that makes the following bijection a poset isomorphism:

$$t \in]-\varepsilon, \varepsilon[\mapsto \begin{cases} 1+t & \text{if } t < 0 \\ t & \text{if } t > 0 \\ \star & \text{if } t = 0 \end{cases} \in X \cup \{\star\} \cup Y$$

In particular, the largest elements of the poset $X \cup \{\star\} \cup Y$ are close to ε (dually, the smallest ones are close to $1 - \varepsilon$). From there, the partial order \preceq_{B_u} is isomorphic to the pushout of posets

$$(X \cup \{\star\} \cup Y) \times U \xleftarrow{(\star, z) \mapsto z} U \xrightarrow{z \mapsto (z, \star)} U \times (X \cup \{\star\} \cup Y)$$

This poset is isomorphic to the following subset of \mathbb{R}^3

$$([-\varepsilon, \varepsilon[\times \{0\} \cup \{0\} \times]-\varepsilon, \varepsilon[) \times U,$$

which is represented on the right hand part of Figure 4. The canonical preorder of B_\star is dealt with in a similar fashion. The inverse images of X and Y under the quotient map q are given at line (22) with X and Y instead of U , and the inverse image of the point \star is $q^{-1}\{\star\} = \{0, 1\}^2$. We partition the set B_\star , which is defined at line (21), as suggested by the graph on the left hand part of Figure 5. The elements of this partition are the *tiles* of B_\star . As in the case of B_u , this graph describes all the ways of going from a tile of B_\star to another by means of an elementary directed path. The proof of this claim is still easy, but even more tedious than in the case of B_u . Indeed we have to consider each ordered pair of tiles T, T' and check whether there is an elementary directed

path from T to T' . Once again, this leads to an extensive description of the relation \rightarrow_{B_\bullet} (we omit the subscript for the sake of readability):

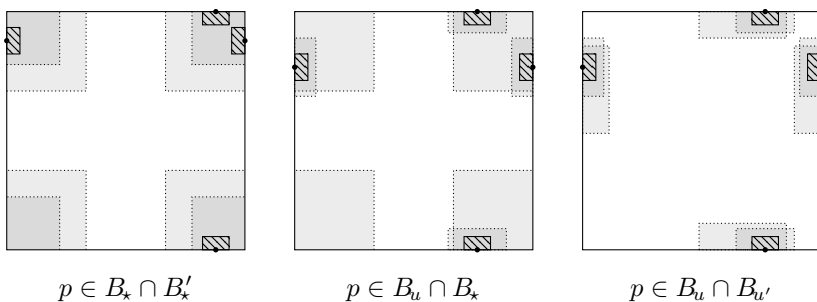
- the relation \rightarrow is the standard order on $X, Y, X \times X, X \times Y, Y \times X$, and $Y \times Y$ (which are subsets of \mathbb{R} and \mathbb{R}^2 respectively),
- the point \star is larger than any element of $X \cup (X \times X)$, and smaller than any element of $Y \cup (Y \times Y)$, which we summarize by writing $X \cup (X \times X) \rightarrow \star \rightarrow Y \cup (Y \times Y)$,

and given x, x' , and x'' in X , and y, y' , and y'' in Y , we have:

- $x' \rightarrow (x, y)$ (resp. $x' \rightarrow (y, x)$) iff $x' \leq x$,
- $(x, y) \rightarrow y'$ (resp. $(y, x) \rightarrow y'$) iff $y \leq y'$,
- $(x, x') \rightarrow x''$ iff $x \leq x''$ or $x' \leq x''$, and
- $y'' \rightarrow (y, y')$ iff $y'' \leq y$ or $y'' \leq y'$.

Finally, the preorder \preceq_{B_\bullet} is antisymmetric because the graph is acyclic and the relation \rightarrow_{B_\bullet} coincides with the standard order on each tile: the arguments are the same as in the case of B_u .

We have described three types of antisymmetric open subsets of $|\Omega_2|$. It remains to check that the resulting collection \mathfrak{B} is an ordered base, which means that given $p \in B \cap B'$ for $B, B' \in \mathfrak{B}$ of radii ε and ε' , we have to find $B_p \in \mathfrak{B}$ containing p such that $B_p \hookrightarrow B$ and $B_p \hookrightarrow B'$. If $p \in]0, 1[$ then we have an open square of radius ε'' centered at p , which is included in both B and B' , and on which both B and B' induce the same ordering (a detailed verification would require that we check 9 cases since both B and B' can be of any of the three type of elements of \mathfrak{B}). If $p \in]0, 1[$ then neither B nor B' are open squares, so it remains three situations to deal with:



On each of the above squares, the dark gray and the hashed regions represent $q^{-1}(B \cap B')$ and $q^{-1}(B_p)$, and the dots represent the inverse image of p . The partial order on B_p clearly coincides with those of B and B' . If p is the point \star then both B and B' are of the third kind, in which case one of them is included in the other.

It remains to check that \mathfrak{B} is the expected colimit in \mathbf{OB} the category of ordered bases (Definition 3.29). Let $f : [0, 1]^2 \rightarrow \mathcal{X}$ be a morphism of ordered bases such that $f(x) = f(y)$ when $q(x) = q(y)$. We have a unique map $g : \{\star\} \cup]0, 1[\cup]0, 1[^2 \rightarrow X$ such that $f = g \circ q$; it is continuous because $|\Omega_2|$ is equipped with the final topology of q . Let $p \in |\Omega_2|$ and $C \in \mathcal{X}$ with $g(p) \in C$. Since \mathfrak{B} induces a base of topology and g is continuous, we have some $B \in \mathfrak{B}$ centered at p such that $g(B) \subseteq C$. Suppose that $x \preccurlyeq_B y$, i.e. there is a directed path δ on B from x to y . Assume that δ is elementary, i.e. $\delta = q \circ \sigma$ with σ some directed path on $[0, 1]^2$. From [14, Corollary 5.1] we deduce that $f(x) \sqsubseteq_C f(y)$. If δ is not elementary, we end up with the same conclusion because the relation \sqsubseteq_C is transitive. Hence the restriction $g : B \rightarrow C$ is order-preserving, and g is a morphism of ordered bases.

6 Other formalisms for directed topology

We compare our locally ordered spaces to streams (§6.1), d-spaces (§6.2), and locally partially ordered spaces from [6] (§6.3); we also check that the *fundamental category*, a well-known invariant in directed algebraic topology, is insensitive to the chosen formalism. We conclude with an example from computer science which supports the idea that, when it comes to models of concurrent programs, locally ordered spaces should be preferred to streams and d-spaces (§6.5).

6.1 Streams

Before the advent of local partial orders in concurrency theory [6], similar mathematical objects were already used in cosmology [29, *finite causal orientation*, p.23], and in Lie theory of semigroups [22, *locally partially ordered manifolds*, p.280]. The latter two are almost identical: they assign a preordered space (N_p, \preccurlyeq_p) to each point p of a Hausdorff space X , such that N_p is an open subset of X and for every $q \in N_p$ there exists a neighborhood of q in which the relations \preccurlyeq_p and \preccurlyeq_q coincide. The concept of a stream (Definition 6.3), which was introduced by S. Krishnan [20, 21] and generalized by the first author [4, §8], is yet another variation of this idea. It is based on the complete lattice of preorders whose underlying sets are open in X , in which least upper bounds are obtained from the reflexive-transitive closure construction:

Definition 6.1. Given a binary relation R on a set X , the *reflexive-transitive closure* of R is the preorder \preccurlyeq on X defined for all $a, b \in X$ by $a \preccurlyeq b$ if, and only if, there exists a chain $a = x_0 R x_1 R \cdots R x_{n-1} R x_n = b$ with $n \in \mathbb{N}$ and $x_i \in X$ for every $i \in \{1, \dots, n\}$.

The underlying set and relation of a preordered set P are denoted by $|P|$ and \preccurlyeq_P . Given another preorder Q , we write $\preccurlyeq_P \subseteq \preccurlyeq_Q$ (and say that \preccurlyeq_Q contains \preccurlyeq_P) to mean that $|P| \subseteq |Q|$ and $a \preccurlyeq_P b \Rightarrow a \preccurlyeq_Q b$ for all $a, b \in |P|$; with these conventions we have:

Lemma 6.2. Given a family \mathcal{F} of preordered sets, define $|\mathcal{F}|$ as the union $\bigcup\{|P| \mid P \in \mathcal{F}\}$ and the relation R on $|\mathcal{F}|$ by $a R b$ if, and only if, we have $a \preceq_P b$ for some $P \in \mathcal{F}$. Then $|\mathcal{F}|$ equipped with \preceq , the transitive-reflexive closure of R , is the *least preorder* on $|\mathcal{F}|$ containing the relation \preceq_P for every $P \in \mathcal{F}$.

Proof The relation \preceq is a preorder (Definition 6.1). By considering chains of length 1 (i.e. $n = 1$) in Definition 6.1, we prove that \preceq contains the relation \preceq_P for every $P \in \mathcal{F}$; and since any preorder is transitive, it is the least such one. \square

Definition 6.3 ([20, 3.1, 3.14]). Let \preceq be a map assigning a *preorder* \preceq_U on U to every open subset U of a topological space X ; the underlying space and set of \preceq are X and $|X|$. Given another such map \preceq' on an open subspace Y of X , we write $\preceq \subseteq \preceq'$ (and say that \preceq' contains \preceq) when $\preceq_U \subseteq \preceq'_U$ holds for every open subset U of X . The map \preceq is said to be a *precirculation* when for all open subsets U, V of X , if $U \subseteq V$ then $\preceq_U \subseteq \preceq_V$. The map \preceq is said to be a *circulation* when it satisfies the *cosheaf condition*: for every open subset U of X and every open cover \mathcal{C} of U (i.e. $U = \bigcup \mathcal{C}$) the relation \preceq_U is the least preorder on U containing \preceq_W for every $W \in \mathcal{C}$. A (*pre*)*stream* is a topological space equipped with a (*pre*)*circulation*.

Example 6.4. Every circulation is a precirculation [20, Lemma 3.8].

Example 6.5. The circulation of the *standard stream* on \mathbb{R} assigns to every open subset U the partial order $\sqsubseteq_U^{(\mathbb{R})}$ defined for all $a, b \in U$ by $a \sqsubseteq_U^{(\mathbb{R})} b$ when $a \leq b$ and $[a, b] \subseteq U$, with \leq denoting the standard order on \mathbb{R} , see [11, Example 7]. The circulation of the *standard stream* on the unit circle S^1 assigns to every *proper* open subset U its standard order (Example 2.2) while the preorder assigned to S^1 is chaotic (see the introduction of §2).

Example 6.6. The least precirculation on a topological space X assigns to each open subset of X the discrete preorder on X : it is a circulation which is called the *discrete circulation* on X .

Example 6.7. The greatest precirculation on a topological space X assigns to each open subset of X the chaotic preorder on it (see the introduction of §2). It is called the *chaotic precirculation* on X . It is not a circulation if X has at least two open connected components.

Definition 6.8. A *prestream morphism* from (X, \preceq) to (Y, \preceq') is a continuous map $f : X \rightarrow Y$ that is order-preserving from $(f^{-1}V, \preceq_{f^{-1}V}^{(X)})$ to $(V, \preceq_V^{(Y)})$ for every open subset V of Y . We denote by **Prestr** the category of prestreams, and by **Str** the full subcategory of streams (Example 6.4). Both **Prestr** and **Str** are concrete over **Top** (Definition 2.3). In line with Definitions 2.15 and 3.5, we say that f is *locally order-preserving* when for every $x \in X$ and every

open subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that f is order-preserving from $(U, \preceq_V^{(X)})$ to $(V, \preceq_V^{(Y)})$.

Remark 6.9. Both **Prestr** and **Str** are *topological* over **Top** [20, 3.22] and [11, Propositions 5 and 11].

Any prestream morphism is locally order-preserving taking $U = f^{-1}V$ in Definition 6.8; the converse is false in general:

Example 6.10. Let \preceq and \preceq' be the discrete circulation and the chaotic precirculation (Examples 6.6 and 6.7) on a *discrete* topological space X , i.e., $v \preceq_v v'$, and $v \preceq'_v v' \Leftrightarrow v = v'$ for every subset V of X and all $v, v' \in V$. Let f be the identity map on X . For *every* subset V of X with (at least) two elements, we have $f^{-1}(V) = V$ and f is not order-preserving from (V, \preceq_v) to (V, \preceq'_v) . Nevertheless f is locally order-preserving (Definition 6.8) because for every $x \in X$, it is order-preserving from (U, \preceq_v) to (V, \preceq'_v) with $U = \{x\}$ and $x \in V$.

Here is a less pathological counterexample:

Example 6.11. The map $\leq^{(\mathbb{R})}$ assigning to every open subset U of \mathbb{R} the total order $\leq_U^{(\mathbb{R})}$ induced by the standard order on \mathbb{R} , is a precirculation. The identity map $id_{\mathbb{R}}$ induces a prestream morphism from $(\mathbb{R}, \sqsubseteq^{(\mathbb{R})})$ (Example 6.5) to $(\mathbb{R}, \leq^{(\mathbb{R})})$, and a locally order-preserving map from $(\mathbb{R}, \leq^{(\mathbb{R})})$ to $(\mathbb{R}, \sqsubseteq^{(\mathbb{R})})$ which is *not* a prestream morphism.

In the category of streams, there is no pathology like those described in Examples 6.10 and 6.11, we have indeed:

Lemma 6.12. For every stream (X, \preceq) and every prestream (Y, \preceq') , a set map $f : |X| \rightarrow |Y|$ induces a prestream morphism from (X, \preceq) to (Y, \preceq') if, and only if, it is locally order-preserving from (X, \preceq) to (Y, \preceq') .

Proof Given an open subset V of Y , put $W = f^{-1}V$. If f is a prestream morphism, then we can take $U = W$ in Definition 6.8. Conversely, if f is locally order-preserving, then for every $x \in W$ there exists $U_x \subseteq W$ open in X containing x such that f is order-preserving from (U_x, \preceq_{U_x}) to (V, \preceq_V) . In particular f is continuous, and $\mathcal{C} = \{U_x \mid x \in |X|\}$ is an open covering of W . Let $a, b \in |W|$. The relation \preceq_W is the least preorder containing \preceq_{U_x} for every $x \in W$ (because \preceq satisfies the cosheaf condition), therefore we have a chain $a = x_0 \preceq_{U_1} x_1 \preceq_{U_2} \cdots \preceq_{U_{n-1}} x_{n-1} \preceq_{U_n} x_n = b$ with $n \in \mathbb{N}$, $U_i \in \mathcal{C}$, and $x_{i-1}, x_i \in U_i$ for every $i \in \{1, \dots, n\}$ (Lemma 6.2). We apply f to all the elements of this chain, and deduce that $f(a) \preceq'_V f(b)$ because \preceq'_V is transitive. \square

Lemma 6.13 ([20, 3.4]). The collection of circulations on X (equipped with the relation \sqsubseteq from Definition 6.3) is a complete lattice in which least upper

bounds are computed pointwise, i.e. the least upper bound of a family \mathcal{F} of circulations on X is the map assigning to every open subset U of X the least preorder on U containing \preceq_U for every element \preceq of \mathcal{F} .

As pointed out in [20], most constructions on streams (in particular products [20, 3.25], [11, Lemma 16], and more generally every limit [11]) are described in **Prestr** and brought back to **Str** via the following construction, which is well-defined by Lemma 6.13:

Definition 6.14 ([20, 3.23]). The *cosheafification* of a precirculation \preceq on X , which we denote by $\preceq^!$, is the greatest circulation on X contained in \preceq .

Lemma 6.15. If the underlying set map of a prestream isomorphism $f : (X, \preceq) \rightarrow (X', \preceq')$ is an identity, then so is f . In particular, for every prestream (X, \preceq) , the identity map id_X induces a morphism from (X, \preceq) to $(X, \preceq^!)$ if, and only if, \preceq is a circulation.

Proof The map f induces a homeomorphism from X to X' with $|X| = |X'|$, therefore $X = X'$. Since f is a prestream morphism, for every open subset U of X and all $a, b \in U = f^{-1}U$, we have $a \preceq_U b \Rightarrow a \preceq'_U b$; the other implication holds because f^{-1} is a prestream morphism. If id_X induces a morphism from (X, \preceq) to $(X, \preceq^!)$, then it is an isomorphism, and then $\preceq = \preceq^!$ is a circulation. Conversely, if \preceq is a circulation, then $\preceq = \preceq^!$ (Definition 6.14). \square

Example 6.16. The circulation $\sqsubseteq^{(\mathbb{R})}$ from Example 6.5 is the cosheafification of the precirculation $\preceq^{(\mathbb{R})}$ from Example 6.11: suppose that \preceq is a circulation containing $\sqsubseteq^{(\mathbb{R})}$ and contained in $\preceq^{(\mathbb{R})}$. For every open interval I of \mathbb{R} we have $\sqsubseteq_I^{(\mathbb{R})} = \preceq_I^{(\mathbb{R})}$, therefore $\sqsubseteq_I^{(\mathbb{R})} = \preceq_I$. If U is the union of a family \mathcal{F} of pairwise disjoint open intervals of \mathbb{R} , then we have

$$\preceq_U = \bigvee \{ \preceq_I \mid I \in \mathcal{F} \} = \bigvee \{ \sqsubseteq_I^{(\mathbb{R})} \mid I \in \mathcal{F} \} = \sqsubseteq_U^{(\mathbb{R})}.$$

Definition 6.17 ([20, 3.16]). Given a set map $f : X \rightarrow Y$ and a preorder \preceq on X , the *pushforward* of \preceq along f is the transitive-reflexive closure (Definition 6.1) of the binary relation R on Y defined by $y R y'$ if, and only if, there exists $x, x' \in X$ with $x \preceq x'$ such that $f(x) = y$ and $f(x') = y'$.

Assuming that $f : X \rightarrow Y$ is a continuous map, for every open subset V of Y , we denote by f_V the restriction of f to $f^{-1}(V)$ with values in V .

The *pushforward* of a precirculation \preceq on X along f is the precirculation \preceq^{f_s} on Y assigning to every open subset V the pushforward of $\preceq_{f^{-1}V}$ along f_V .

Lemma 6.18. Given a continuous map $f : X \rightarrow Y$ and circulations \preceq and \preceq' on X and Y , the map f is a prestream morphism from (X, \preceq) to (Y, \preceq') if,

and only if, for every precirculation \leq contained in \preccurlyeq , the pushforward \leq^{f^*} is contained in \preccurlyeq' .

Proof Let \leq be a prestream contained in \preccurlyeq ; its underlying space is X . Let V be an open subset of Y and $a, b \in V$ such that $a \leq_V^f b$. Let U denotes $f^{-1}V$. By Definition 6.17 we have a chain $x_0 \leq_U x_1 \leq_U \cdots x_{n-1} \leq_U x_n$ with $n \in \mathbb{N}$, $f(x_0) = a$, and $f(x_n) = b$. Since $\leq \subseteq \preccurlyeq$ we have $x_0 \preccurlyeq_V x_1 \preccurlyeq_V \cdots x_{n-1} \preccurlyeq_V x_n$, from which we deduce that $f(x_0) \preccurlyeq'_V f(x_1) \preccurlyeq'_V \cdots f(x_{n-1}) \preccurlyeq'_V f(x_n)$ because f is a prestream morphism. Hence $a \preccurlyeq'_V b$. Taking $\leq = \preccurlyeq$, the converse is immediate from Definition 6.17. \square

Lemma 6.19. With the notations of Definition 6.17, \preccurlyeq^{f^*} is the least prestream \preccurlyeq' on Y such that the map f is a prestream morphism from (X, \preccurlyeq) to (Y, \preccurlyeq') .

Proof The fact that f is a prestream morphism from \preccurlyeq to \preccurlyeq^{f^*} is immediate. The minimality is a consequence of Lemma 6.18. \square

Lemma 6.20 ([20, 3.18]). The pushforward of a circulation is a circulation.

The following result appears in [11, Corollary 3] with a proof based on an alternative description of the cosheafification and some general categorical facts; we provide a more direct proof:

Proposition 6.21. The cosheafification construction extends to a functor $(-)^! : \mathbf{Prestr} \rightarrow \mathbf{Str}$ which is the right adjoint to the inclusion functor $\mathbf{Str} \hookrightarrow \mathbf{Prestr}$, and whose action on morphisms is the identity.

Proof Let $f : (X, \preccurlyeq) \rightarrow (Y, \preccurlyeq')$ be a prestream morphism. Let V be open subset of Y and let $U = f^{-1}V$. Suppose that we have $a \preccurlyeq'_V b$ with $a, b \in U$. We want to prove that $f(a)(\preccurlyeq')^!_V f(b)$. By Definition 6.14 the preorder \preccurlyeq'_V is the least upper bound of the family \mathcal{F} of preorders of the form \leq_U on U with \leq stream contained in \preccurlyeq . By Lemma 6.2 we have a chain $a = x_0 \leq_U^{(1)} x_1 \leq_U^{(2)} \cdots \leq_U^{(n-1)} x_{n-1} \leq_U^{(n)} x_n = b$ with $n \in \mathbb{N}$, $x_k \in U$, and $\leq^{(k)}$ stream contained in \preccurlyeq for every $k \in \{1, \dots, n\}$. Let $k \in \{1, \dots, n\}$. We deduce from Definition 6.17 that $f(x_{k-1}) \leq_V^{(k)} f(x_k)$. Moreover, the pushforward $(\leq^{(k)})^{f^*}$ is a stream (Lemma 6.20) that is contained in \preccurlyeq' (Lemma 6.18), therefore $(\leq^{(k)})^{f^*}$ is contained in the cosheafification $(\preccurlyeq')^!$ (Definition 6.14). It follows that we have $f(x_{k-1})(\preccurlyeq')^!_V f(x_k)$. By transitivity, we conclude that $f(a)(\preccurlyeq')^!_V f(b)$, and therefore that f is a stream morphism from (X, \preccurlyeq) to $(Y, (\preccurlyeq')^!)$. So the cosheafification construction is indeed a functor, and its action on morphisms is the identity.

Let $f : (X, \preccurlyeq) \rightarrow (Y, \preccurlyeq')$ be a prestream morphism with (X, \preccurlyeq) being a stream. In particular $\preccurlyeq = \preccurlyeq'$ and we have already proven that f is a stream morphism from (X, \preccurlyeq) to $(Y, (\preccurlyeq')^!)$. Hence we have

$$\mathbf{Prestr}((X, \preccurlyeq), (Y, \preccurlyeq')) \subseteq \mathbf{Str}((X, \preccurlyeq), (Y, (\preccurlyeq')^!));$$

the converse inclusion is immediate since $(\preceq')^!$ is contained in \preceq' . The counit of the adjunction is made of the prestream morphisms $(X, \preceq') \rightarrow (X, \preceq)$ for (X, \preceq) ranging in the class of prestreams. \square

The material accumulated so far allows us to compare locally ordered spaces with (pre)streams:

Definition 6.22. A prestream \preceq is said to be an *underapproximation* (resp. *overapproximation*) of a locally ordered space \mathcal{X} (Definition 3.27) when for all $U \in \mathcal{X}$, and $a, b \in Sp(U)$, we have $a \preceq_{Sp(U)} b \Rightarrow a \leq_U b$ (resp. $a \leq_U b \Rightarrow a \preceq_{Sp(U)} b$).

Proposition 6.23. Every locally ordered space \mathcal{X} has a greatest underapproximation $\preceq_{(\mathcal{X})}$ and a least overapproximation $\preceq^{(\mathcal{X})}$ in the complete lattice of prestreams on $Sp(\mathcal{X})$.

Proof The discrete stream (Example 6.6) is an underapproximation of \mathcal{X} , and the chaotic prestream (Example 6.7) is an overapproximation of \mathcal{X} . The lattice of prestreams on $Sp(\mathcal{X})$ is complete (Lemma 6.13). \square

Proposition 6.24. Let \mathcal{X} be a locally ordered space. If $Sp(\mathcal{X})$ is Hausdorff, then the least overapproximation of \mathcal{X} is chaotic.

Proof Let \preceq be an overapproximation of \mathcal{X} and let U be an open subset of $Sp(\mathcal{X})$. Given x and y in U we have V_x and V_y in \mathcal{X} such that $x \in V_x \xrightarrow{\circ} U$, $y \in V_y \xrightarrow{\circ} U$, and we can suppose that $|V_x| \cap |V_y| = \emptyset$ because $Sp(\mathcal{X})$ is Hausdorff. We denote by S the subspace of $Sp(\mathcal{X})$ whose underlying set is $|V_x| \cup |V_y|$; it can be equipped with the partial order \leq whose restriction to $|V_x|$ (resp. $|V_y|$) coincide with the partial order of V_x (resp. V_y) and such that $a \leq b$ for all $a \in V_x$ and $b \in V_y$. The space S can also be equipped with the partial order \leq' which is defined as \leq except that $b \leq' a$ for all $a \in V_x$ and $b \in V_y$. Both (S, \leq) and (S, \leq') are compatible with U : given $p \in S$, the ordered space V_x (resp. V_y) is a witness of compatibility of S and U around p if $p \in V_x$ (resp. $p \in V_y$) (Definition 3.1). So both (S, \leq) and (S, \leq') belong to \mathcal{X} (Definition 3.27). As a consequence we have $x \preceq_{|S|} y$ and $y \preceq_{|S|} x$, and therefore $x \preceq_U y$ and $y \preceq_U x$ because $|S| \subseteq U$ and \preceq is a precirculation. \square

As an immediate consequence of Proposition 6.24, the construction assigning to each Hausdorff locally ordered space its least overapproximation is, up to isomorphism, the forgetful functor to **Top** (or **Haus** the full subcategory of Hausdorff spaces). Greatest underapproximations behave better:

Proposition 6.25. Given a locally ordered space \mathcal{X} , the map assigning the greatest lower bound \preceq_U of the family of preorders \leq such that $(U, \leq) \in \mathcal{X}$, to every open subset U of $Sp(\mathcal{X})$, is the greatest underapproximation of \mathcal{X} (if no such preorder exists, then \preceq_U is chaotic).

Proof Let U and V be open subsets of $Sp(\mathcal{X})$ such that $U \subseteq V$. Let $u, u' \in U$ such that $u \not\preceq_v u'$. By definition of \preceq_v there exists a preorder \leq on V such that $(V, \leq) \in \mathcal{X}$ and $u \not\leq u'$. Denote by \leq_U the restriction of \leq to U . We have $(U, \leq_U) \in \mathcal{X}$ because \mathcal{X} is a maximal local order (Definition 3.27), therefore $u \not\preceq_v u'$ because $u \not\leq_U u'$. If \sqsubseteq is an underapproximation of \mathcal{X} , then we have $\sqsubseteq_U \subseteq \leq$ for every preorder \leq such that $(U, \leq) \in \mathcal{X}$, so $\sqsubseteq_U \subseteq \preceq_v$, and \preceq is the greatest underapproximation of \mathcal{X} . \square

With the notations of Proposition 6.25, and even assuming that there exists some preorder \leq such that $(U, \leq) \in \mathcal{X}$ (such a preorder is antisymmetric), the ordered space (U, \preceq_v) may not belong to \mathcal{X} :

Example 6.26. Let \mathcal{Q} be the saturation (Definition 3.21) of the local order induced by the ordered space (\mathbb{Q}, \leq) (Corollary 3.28) with \leq denoting the standard order on \mathbb{R} . In other words \mathcal{Q} is the collection of all ordered spaces (X, \leq) compatible with (\mathbb{Q}, \leq) (Definition 3.26) and such that X is a (necessarily open) subspace of \mathbb{Q} . Let U be an open subset of \mathbb{R} and let $D \subseteq \mathbb{R} \setminus \mathbb{Q}$ be a discrete subspace of U . Given $q, q' \in U \cap \mathbb{Q}$ we write $q \leq q'$ when $q \leq q'$, $[q, q'] \subseteq U$, and $[q, q'] \cap D = \emptyset$. Let $q \in U \cap \mathbb{Q}$, so $q \notin D$ because $D \cap \mathbb{Q} = \emptyset$. Since D is discrete, there is an open interval I such that $q \in I \subseteq U \setminus D$, so we have $(I \cap \mathbb{Q}, \leq) \xrightarrow{\text{cl}} (U \cap \mathbb{Q}, \leq)$, (\mathbb{Q}, \leq) ; hence $(U \cap \mathbb{Q}, \leq) \in \mathcal{Q}$. As a consequence, the discrete order on $U \cap \mathbb{Q}$ is the largest order on $U \cap \mathbb{Q}$ contained in every partial order \leq on $U \cap \mathbb{Q}$ such that $(U \cap \mathbb{Q}, \leq) \in \mathcal{Q}$ (so the greatest underapproximation of \mathcal{Q} is discrete, see Proposition 6.25); yet, the ordered space $(U \cap \mathbb{Q}, =)$ does not belong to \mathcal{Q} .

We were not able to find a locally ordered space with a greatest underapproximation that is not a circulation, or to prove that the greatest underapproximation construction is functorial; yet we have:

Proposition 6.27. Denote by \preceq and \preceq' the greatest underapproximations of the locally ordered spaces \mathcal{X} and \mathcal{Y} , and let the map $f : \mathcal{X} \rightarrow \mathcal{Y}$ be locally order-preserving (in the sense of Definition 3.5). If

1. the greatest underapproximation of \mathcal{X} is a circulation, and
2. the family of open subsets V of $Sp(\mathcal{Y})$ such that $(V, \preceq'_v) \in \mathcal{Y}$ covers $Sp(\mathcal{Y})$,

then f is a prestream morphism from $(Sp(\mathcal{X}), \preceq)$ to $(Sp(\mathcal{Y}), \preceq')$.

Proof Let $x \in Sp(\mathcal{X})$ and V open in $Sp(\mathcal{Y})$ such that $(V, \preceq'_v) \in \mathcal{Y}$. Since f is locally order-preserving, we have $(U, \leq) \in \mathcal{X}$ with $x \in U$ such that f is order-preserving from (U, \leq) to (V, \preceq'_v) . In particular f is order-preserving from (U, \preceq_v) to (V, \preceq'_v) (because $\preceq_v \subseteq \leq$), so f is locally order-preserving from $(Sp(\mathcal{X}), \preceq)$ to $(Sp(\mathcal{Y}), \preceq')$ in the sense of Definition 6.8. Since \preceq is a circulation, we deduce from Lemma 6.12 that f is a prestream morphism from $(Sp(\mathcal{X}), \preceq)$ to $(Sp(\mathcal{Y}), \preceq')$. \square

6.2 D-spaces

The concept of d-space is based on that of path on a topological space; it was introduced by M. Grandis [12, 13], and it is closely related to streams, see [15, §4.4, §4.5] and [16]. A *path* on a topological space X is a continuous map from some compact interval (possibly made of a single element) to X . A *subpath* of $\gamma : [a, b] \rightarrow X$ is a path of the form $\gamma \circ \iota$ with $\iota : [x, y] \hookrightarrow [a, b]$ an inclusion map. A *reparametrization* of $\gamma : [a, b] \rightarrow X$ is a path of the form $\gamma \circ \theta$ with $\theta : [x, y] \rightarrow [a, b]$ continuous, surjective, and order-preserving. Given $\delta : [b, c] \rightarrow X$ such that $\gamma(b) = \delta(b)$, the *concatenation* $\gamma \cdot \delta$ is the map

$$t \in [a, c] \quad \mapsto \quad \begin{cases} \gamma(t) & \text{if } t \in [a, b] \\ \delta(t) & \text{if } t \in [b, c] . \end{cases}$$

Definition 6.28. A *direction* on X is a collection dX of paths on X that contains all constant paths, every subpath and every reparametrization of any of its elements, and that is stable under concatenation. A *d-space* is a topological space together with a direction on it. The *directed paths* of a d-space are the elements of its direction. A *d-subspace* of (X, dX) is a d-space of the form (Y, dY) with Y a subspace of X and dY the collection paths on Y that belong to dX . We use the same denotation for a d-space and its direction. A d-space morphism from dX to dY is a continuous map $f : X \rightarrow Y$ such that $f \circ dX \subseteq dY$. The d-spaces and their morphisms form the category **dTop**. The cartesian product of the d-spaces (X, dX) and (Y, dY) is the d-space $(X \times Y, dX \times dY)$, the first product being taken in **Top**, the second one in **Set**.

The categories **dTop** and **Str** are related by an adjunction. Given a d-space dX and an open subset U of X , we write $p \preceq_v^{(X)} q$ when there exists $\delta \in dX$ from p to q such that $\text{img}(\delta) \subseteq U$. The map sending every open subset U of X to $\preceq_v^{(X)}$ is a circulation which we denote by $S(dX)$ [20, 3.13].

Definition 6.29. A *directed path* on a local order \mathcal{U} (resp. a stream $\preceq^{(X)}$) is a path on the underlying space inducing a local order (resp. stream) morphism; the interval $[a, b]$ on which the path is defined is equipped with the standard local order (resp. standard circulation, Example 6.5).

The collection of directed paths on a stream $(X, \preceq^{(X)})$ is a direction; we denote by $D(X, \preceq^{(X)})$ the resulting d-space. The constructions of S and D extend to a pair of adjoint functors $S \dashv D$ between **Str** and **dTop**.

A path γ on a d-space X is said to be *pseudo-directed* when for every open subset U of X and every subpath $\tilde{\gamma} : [a, b] \rightarrow U$ of γ , there exists $\delta : [x, y] \rightarrow U$ in dX such that $\delta(x) = \tilde{\gamma}(a)$ to $\delta(y) = \tilde{\gamma}(b)$. By merely unfolding the definitions of S and D , one checks that for every d-space dX , the directed paths of

$DS(dX)$ are the pseudo-directed paths of dX , i.e.

$$d(DS(dX)) = \{\text{pseudo-directed paths of } dX\}.$$

Definition 6.30. [15, 4.4.11, 4.5.14, pp. 92, 96] A stream $\preceq^{(X)}$ is said to be *filled* when for every open subset U of X , we have $p \preceq_U^{(X)} q$ if, and only if, there is a preorder-preserving path on $(U, \preceq_U^{(X)})$ from p to q . A d-space dX is said to be *filled* when every pseudo-directed path on dX is directed. The image of D is the reflective subcategory \mathbf{dTop}^* of filled d-spaces, and the image of S is the coreflective subcategory \mathbf{Str}^* of filled streams [16, §5].

The restriction of S to \mathbf{dTop}^* induces an isomorphism $S : \mathbf{dTop}^* \rightarrow \mathbf{Str}^*$ whose inverse $D : \mathbf{Str}^* \rightarrow \mathbf{dTop}^*$ is induced by the restriction of D to \mathbf{Str}^* . We have the commutative diagram [15, p.98]:

$$\begin{array}{ccc} \mathbf{Str} & \begin{array}{c} \xrightarrow{D} \\ \dashv \\ \xleftarrow{S} \end{array} & \mathbf{dTop} \\ \uparrow \text{corefl} \downarrow & & \text{refl} \downarrow \uparrow \\ \mathbf{Str}^* & \begin{array}{c} \xrightarrow{D} \\ \cong \\ \xleftarrow{S} \end{array} & \mathbf{dTop}^* \end{array}$$

Example 6.31. The underlying space of a local order \mathcal{U} equipped with the directed paths on \mathcal{U} is a filled d-space (Definition 6.30); we deduce a functor $F : \mathbf{LO} \rightarrow \mathbf{dTop}^*$. This functor may lose a lot of information: two local orders on the same totally disconnected space have the same image under F . A tedious verification shows that the functor F is induced by the canonical functor $J : \mathbf{Ord} \rightarrow \mathbf{dTop}^*$ via Proposition 3.46; the main point is that every directed path on a local order \mathcal{U} can be written as a finite concatenation of directed paths whose images are entirely contained in some element of \mathcal{U} (by a mere compactness argument).

6.3 Local partial orders

We have written in the introduction that our concept of *local order* (Definition 3.4) is more convenient than the concept of *local partial order* from [6, 3.4]; we explain this claim. A local partial order is a collection \mathcal{U} of ordered spaces whose underlying sets form an open cover of a topological space X , together with a *choice* of open ordered neighborhoods $x \mapsto W_x$ such that for every $U \in \mathcal{U}$ and every $u \in U$, the partial orders of W_u and U agree on $W_u \cap U$. Instead of requiring a witness of compatibility for each $U, V \in \mathcal{U}$ containing x (as in Definition 3.1), the open ordered neighborhood W_x has to be compatible with all the elements of \mathcal{U} containing x : this is an unnecessarily strong assumption with unwanted consequences. Indeed, the equivalence class of a local partial

order may not have a greatest element: given the local partial orders \mathcal{U}_1 , \mathcal{U}_2 , and \mathcal{U}_3 with the choices of ordered neighborhoods W^{12} and W^{23} witnessing that $\mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_2 \cup \mathcal{U}_3$ are local orders, the choice of ordered neighborhoods $x \mapsto W_x^{12} \cap W_x^{23}$ attests that $\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$ is also a local order [6, (5), p.246]. However, this reasoning cannot be adapted to prove the existence of a local partial order containing all the elements of an equivalence class: it would indeed involve possibly infinite intersections of ordered neighborhoods. We provide a counterexample inspired from [14, 5.5]:

Example 6.32. Let \mathcal{C} be a *point-finite* open covering of \mathbb{R} , i.e. every $x \in \mathbb{R}$ belongs to finitely many elements of \mathcal{C} , see [33, Definition 15.9, p. 104]. We obtain a local partial order $\mathcal{U}_{\mathcal{C}}$ by equipping each $U \in \mathcal{U}_{\mathcal{C}}$ with the relation \sqsubseteq_U defined by $u \sqsubseteq_U u'$ if, and only if, $u \leq u'$ and $[u, u'] \subseteq U$ (Example 3.22), and by choosing, for every $x \in \mathbb{R}$, an open interval W_x contained in all the elements of $\mathcal{U}_{\mathcal{C}}$ containing x (e.g. the largest one). Given locally finite open coverings $\mathcal{C}_1, \dots, \mathcal{C}_n$ with $n \in \mathbb{N}$, each local partial order $\mathcal{U}_{\mathcal{C}_k}$ comes with a choice of open ordered neighborhood $x \mapsto W_x^{(k)}$. The union $\mathcal{U}_{\mathcal{C}_1} \cup \dots \cup \mathcal{U}_{\mathcal{C}_n}$ is still a local partial order with the choice of open ordered neighborhoods $x \mapsto W_x^{(1)} \cap \dots \cap W_x^{(n)}$. However, the union of *all* the local partial orders $\mathcal{U}_{\mathcal{C}}$ with \mathcal{C} point-finite open covering of \mathbb{R} is *not* a local partial order: given $x \in \mathbb{R}$ and W_x an open subset of \mathbb{R} containing x , consider $y, z \in W_x$ and $\varepsilon > 0$ such that $y + \varepsilon \leq z - \varepsilon$. We have $y \not\sqsubseteq_U z$ with $U =]-\infty, y + \varepsilon[\cup]z - \varepsilon, +\infty[$ and $y \sqsubseteq_I z$ with $I =]y - \varepsilon, z + \varepsilon[$ whereas $\{I, U\}$ is a point-finite covering of \mathbb{R} .

6.4 Fundamental category functors

A concrete category \mathbf{C} over \mathbf{Top} with a functor $J : \mathbf{Ord} \rightarrow \mathbf{C}$ such that $Sp_{\mathbf{Ord}} = Sp_{\mathbf{C}} \circ J$ is called a *framework for fundamental categories* [16, §2] when:

- i) the forgetful functor $Sp_{\mathbf{C}} : \mathbf{C} \rightarrow \mathbf{Top}$ has a left adjoint,
- ii) the category \mathbf{C} is cartesian (i.e. it has finite products),

and for every tuple $(\iota_1, \dots, \iota_n)$ of nonempty intervals of \mathbb{R} :

- iii) $J(\iota_1 \times \dots \times \iota_n) \cong J(\iota_1) \times \dots \times J(\iota_n)$, and
- iv) the images of the following diagrams (with $r, s \in \mathbb{R}_+$) under J

$$\begin{array}{ccc}
 & [0, r + s] & \\
 \nearrow \iota \rightarrow \iota & & \nwarrow \iota + r \leftarrow \iota \\
 [0, r] & & [0, s] \\
 \nwarrow r \leftarrow 0 & \{0\} & \nearrow 0 \rightarrow 0
 \end{array}$$

are pushout squares in \mathbf{C} , and they are preserved, for every $k \in \{1, \dots, n\}$, by the functors $X \mapsto J(\iota_1) \times \dots \times J(\iota_{k-1}) \times X \times J(\iota_k) \times \dots \times J(\iota_n)$.

The fundamental category functor associated with a given framework is defined as follows. A *directed path* on $X \in \mathbf{C}$ is a morphism $\gamma : J[a, b] \rightarrow X$ with $a \leq b$; its starting and its finishing points are $\gamma(a)$ and $\gamma(b)$. Given two directed

paths $\gamma, \delta : J[a, b] \rightarrow X$, a *directed homotopy* from γ to δ is a morphism $h : J([a, b] \times [c, d]) \rightarrow X$, with $c \leq d$, such that $h(-, c) = \gamma$, $h(-, d) = \delta$, and both $h(a, -)$ and $h(b, -)$ are constant. We say that γ and δ are *d-homotopic* when there exists a zigzag of directed homotopies relating them. A *reparametrization* is a nondecreasing surjective map between two compact intervals. If γ and δ are no longer supposed to be defined on the same interval, we say that γ and δ are *d-homotopic*, and write $\gamma \sim_x \delta$, when there exists two reparametrizations α and β defined on the same interval such that $\gamma \circ J\alpha$ and $\delta \circ J\beta$ are d-homotopic. The item iv) allows us to concatenate paths and directed homotopies. Hence we have a category P_X whose objects and morphisms are the points of X and the directed paths on X defined on intervals of the form $[0, r]$ with $r \geq 0$. The composition of $\gamma : [0, r] \rightarrow X$ and $\delta : [0, r'] \rightarrow X$ is the concatenation of γ and $\delta \circ \tau_r$ with τ_r denoting the translation $t \in [r, r+r'] \mapsto t-r \in [0, r']$; the identities are defined on $\{0\}$. From the vertical composition of directed homotopies one checks that \sim_x induces an equivalence relation on every homset of P_X . From their horizontal composition we deduce that if $\gamma \sim_x \delta$ and $\gamma' \sim_x \delta'$ with γ finishing where γ' starts, then $\gamma' \sim_x \delta' \delta$; so the quotient P_X / \sim_x makes sense: it is by definition the *fundamental category* of X , we denote it by $\bar{\pi}_1 X$. The construction extends to a functor $\bar{\pi}_1 : \mathbf{C} \rightarrow \mathbf{Cat}$ because for every morphism $f : X \rightarrow Y$ and every directed homotopy h from γ to δ on X , the composite $f \circ h$ is a directed homotopy from $f \circ \gamma$ to $f \circ \delta$ on Y .

The categories **LO** (Definition 3.5), **LoSp** (Definition 3.27), **OB** (Definition 3.29), **Prestr**, **Str** (Definition 6.8), **dTop** (Definition 6.28), **Str*** and **dTop*** (Definition 6.30) are typical examples of frameworks for fundamental categories [16], so they all come with a fundamental category functor.

The adjunction $S \dashv D$ from §6.2 allows us to compare the fundamental category functor of streams with that of d-spaces: we have indeed $\bar{\pi}_1(X) = \bar{\pi}_1(DX)$ for all prestreams X , and $\bar{\pi}_1(DX) = \bar{\pi}_1(SDX)$ and $\bar{\pi}_1(X) = \bar{\pi}_1(IX)$ for all streams X with I denoting the full embedding **Str** \hookrightarrow **Prestr** [16, 5.11, 5.12, 5.26, 5.27]. Note that comparing $\bar{\pi}_1(X)$ and $\bar{\pi}_1(SX)$ when X is a d-space remains a tricky problem [16, pp. 145-146].

As long as we only care about the fundamental categories, the category of local order is a kind of common denominator to the categories **LO**, **LoSp**, **OB**, **Prestr**, **Str**, **dTop**, **Str***, and **dTop***; let **C** be one of them and let F be the functor given by Proposition 3.46:

Theorem 6.33. *We have $\bar{\pi}_1^{(\text{LO})} = \bar{\pi}_1^{(\text{C})} \circ F$ with $\bar{\pi}_1^{(\text{LO})}$ and $\bar{\pi}_1^{(\text{C})}$ denoting the fundamental category functors defined on **LO** and **C**.*

Proof Most of the properties required to apply Proposition 3.48 are readily satisfied, the only subtleties are:

1. in the case where **C** = **Prestr**, Lemma 6.12 is required to check that certain locally order preserving maps are indeed prestream morphisms, and

2. the hypotheses iii_a) and ii_b) of Proposition 3.48 are obtained as a variant of the van Kampen theorem [16, 2.8] adapted to infinite open coverings. \square

6.5 A case study from concurrency theory

The use of topological methods in concurrency theory is grounded on the idea that a program Π made of sequential processes running in parallel can be represented by a topological space equipped with a ‘causal structure’; the resulting mathematical object is called the *model* of Π , we denote it by $[\Pi]$. The program Π runs on an abstract machine M whose internal states are 2-tuples (p, μ) with $p \in [\Pi]$ representing the current position of the *instruction pointer*, and μ the current state of the *memory*, see [14] for a detailed exposition. For the purpose of this section, we suppose that $[\Pi]$ is a locally ordered space, a stream, or a d-space, so every execution trace of Π induces a directed path (§6.4) of $[\Pi]$. The model of Π is designed to satisfy the following principles:

- (P1) the number of processes of Π is the dimension of $[\Pi]$, and
(P2) execution traces inducing *d-homotopic* directed paths (in the sense of §6.4) have the same effect on the memory state of M [14, Corollary 6.2].

A *clockwise downward spiral* is a path on the complex plane \mathbb{C} of the form $t \in [a, b] \mapsto \rho(t)e^{i\theta(t)}$ with $\rho : [a, b] \rightarrow [0, +\infty[$ and $\theta : [a, b] \rightarrow \mathbb{R}$ nonincreasing, viz $t < t' \Rightarrow \rho(t) \geq \rho(t')$ and $\theta(t) \geq \theta(t')$. We denote by $d\mathbb{C}$ the collection of clockwise downward spirals, by dS^1 the collection of clockwise paths on S^1 , and by $d\mathbb{R}_+$ the collection of nonincreasing paths on \mathbb{R}_+ . Hence (S^1, dS^1) and $(\mathbb{R}_+, d\mathbb{R}_+)$ are d-subspaces of $(\mathbb{C}, d\mathbb{C})$ (Definition 6.28). The point 0 is a vortex of $(\mathbb{C}, d\mathbb{C})$, i.e. every neighborhood of 0 contains a non-trivial directed loop – see [13, 1.4.7], [16, pp.145-6], [7, 4.4]. Consequently, we have the d-space morphism

$$\pi : (\rho, s) \in d\mathbb{R}_+ \times dS^1 \mapsto \rho s \in d\mathbb{C} .$$

We draw attention to the fact that π is the coequalizer of the morphisms

$$s \in dS^1 \mapsto (0, s) \in d\mathbb{R}_+ \times dS^1 \quad \text{and} \quad s \in dS^1 \mapsto (0, 1) \in d\mathbb{R}_+ \times dS^1$$

in \mathbf{dTop}^* , but not in \mathbf{dTop} (§6.2) because of the downward spirals that turn *infinitely* many times around 0 before reaching it, see [16, p.145].

Consider a program Π made of two processes Π_1 and Π_2 running in parallel. The first one (Π_1) stores finitely many customer requests at the end of a queue; its model is $d\mathbb{R}_+$ in the sense that an execution of Π_1 is represented by a nondecreasing path on \mathbb{R}_+ that ends at 0. The second one (Π_2) consists of a loop at each turn of which one request (at most) is removed from the queue and processed (the memory state of M is updated accordingly); its model is dS^1 in the sense that an execution of Π_2 is represented by a clockwise path on S^1 that starts at 1. Neglecting all the synchronizations between Π_1 and Π_2 , the ‘standard’ model of Π is the d-space product $d\mathbb{R}_+ \times dS^1$ (Definition 6.28);

its directed paths are of the form

$$t \in [0, T] \mapsto (\rho(t), e^{i\theta(t)})$$

with both θ and ρ nonincreasing. An execution trace of Π is thus represented by a path on $\mathbb{R}_+ \times S^1$ whose first component ρ is nonincreasing and stops at 0, and whose second component $e^{i\theta}$ is clockwise. Assume that $\rho^{-1}\{0\} \neq \emptyset$ and let $t_0 = \min \rho^{-1}\{0\}$ (since ρ is nonincreasing, t_0 is the first instant the origin is reached). On such a model, the server Π_2 keeps on running after the last request has been sent (i.e. after t_0) so the processing of requests (which is Π_2 's job) can be postponed until Π_1 has stopped. The directed path $\delta_n : t \in [0, 1] \mapsto e^{2\pi i n t} \in S^1$ (for $n \in \mathbb{N}$) represents an execution trace of Π_2 during which n requests are processed. Let $\gamma : t \in [0, 1] \mapsto (1 - t) \in \mathbb{R}_+$ be the representation of an execution trace of Π_1 , sending requests, no matter how many (note that γ is nonincreasing, so it is indeed a directed path of \mathbb{R}_+ according to our convention). For $n \neq m$ the directed paths $\delta_n \times \gamma$ and $\delta_m \times \gamma$ are not d-homotopic (if they were so, then their underlying paths on $Sp(\mathbb{R}_+ \times S^1)$ would be homotopic in the sense of [3, §6.2, pp. 207-215]).

We wonder whether we can take the d-space $d\mathbb{C}$ as a model of Π instead of $d\mathbb{R}_+ \times dS^1$. We observe that for every $n \in \mathbb{N}$, the map

$$(t, s) \in [0, 1] \times [0, 1] \mapsto (1 - t)e^{i2\pi n s t} \in \mathbb{C}$$

is a directed homotopy from $\gamma_0 = \pi \circ (\delta_0 \times \gamma)$ to $\gamma_n = \pi \circ (\delta_n \times \gamma)$ (§6.4). If the principle (P2) were satisfied, then the execution traces represented by γ_0 and γ_n would have the same effect on the memory state of M . In other words, the program Π would behave as if the last instruction of Π_1 stopped Π_2 and restored the memory of M to its initial state. What such a program would be good for? More concretely, imagine what would happen if Π_2 were actually dealing with bank transactions that, at the end of the day, were all canceled.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

- [1] Adámek, J., Herrlich, H., and Strecker, G. E. *Abstract and Concrete Categories: the joy of cats*. Theory Appl. Categ. reprints, 2006.
- [2] Borceux, F. *Handbook of Categorical Algebra, I. Basic Category Theory*. Cambridge, 1994.
- [3] Brown, R. *Topology and Groupoids*. BookSurge, 2006.
- [4] Coursolle, P.-Y. *A framework for locally structured spaces - application to geometric models of concurrency*, PhD dissert. École polytechnique, 2022.

- [5] Davey, B. A. and Priestley, H. A. *Introduction to Lattices and Order*. Cambridge, 2nd edition, 2002.
- [6] Fajstrup, L., Goubault, É., and Raussen, M. Algebraic Topology and Concurrency. *Theoret. Comput. Sci.*, 357(1):241–278, 2006.
- [7] Fajstrup, L., Goubault, É., Haucourt, E., Mimram, S., and Raussen, M. *Directed Algebraic Topology and Concurrency*. Springer, 2016.
- [8] Gauld, D. B. *Differential Topology: An Introduction*. Marcel Dekker, 1982.
- [9] Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. W., and Scott, D. S. *A Compendium of Continuous Lattices*. Springer, 1980.
- [10] Giuli, E. Bases of topological epi-reflections. *Topol. Appl.* (11):265–273, 1980.
- [11] Goubault-Larrecq, J. Exponentiable Streams and Prestreams. *Appl. Categor. Struct.* 22:514–549, 2014
- [12] Grandis, M., Directed Homotopy Theory, I. The Fundamental Category. *Cah. Topol. Géom. Différ. Catég.*, 44(4):281–316, 2003.
- [13] Grandis, M. *Directed Algebraic Topology*. Cambridge, 2009.
- [14] Haucourt, E. The geometry of conservative programs. *Math. Structures Comput. Sci.*, 28(10):1723–1769, 2018.
- [15] Haucourt, E. *Some Invariants of Directed Topology*. Habilitation. Univ. Paris 7 Denis Diderot / École polytechnique, 2016.
- [16] Haucourt, E. Streams, d-spaces, and their fundamental categories. *Electron. Notes Theor. Comput. Sci.* 283, 2012.
- [17] Haucourt, E. *Topologie Algébrique Dirigée et Concurrence*, PhD dissert. Université Paris 7 Denis Diderot / CEA LIST, 2005. (In french)
- [18] Hicks, N. J. *Notes on Differential Geometry*. Van Nostrand, 1965.
- [19] Johnstone, P. T. *Stone Spaces*. Cambridge, 1982.
- [20] Krishnan, S., A Convenient Category of Locally Preordered Spaces. *Appl. Categ. Struct.*, 17(5):445–466, 2009.
- [21] Krishnan, S. *Directed Algebraic Topology and Concurrency*, PhD dissert. Chicago University, 2006.

- [22] Lawson, J. D. Ordered manifolds, invariant cone fields, and semigroups. *Forum Math.*, 1:273–308, 1989.
- [23] Munkres, J. R. *Topology*. Pearson, 2017.
- [24] Nachbin, L. *Topology and Order*. Van Nostrand, 1965.
- [25] Pratt, V. Modeling Concurrency with Geometry. *Proc. 18th Ann. ACM Symposium on Principles of Programming Languages*, 1991.
- [26] Riehl, E. *Category Theory in Context*. Dover, 2016.
- [27] Schröder, J. The category of Urysohn spaces is not cowellpowered. *Topol. Appl.* (16):237–241, 1982.
- [28] Schröder, B. S. W. *Ordered Sets, An Introduction*. Birkhäuser, 2002.
- [29] Segal, I. E. *Mathematical Cosmology and Extragalactic Astronomy*. Academic Press, 1976.
- [30] Szpilrajn, E. Sur l’extension de l’ordre partiel. *Fundamenta Mathematicae* 16(1):386–389, 1930
- [31] Van Glabbeek, R. J. Bisimulations for Higher Dimensional Automata. Available at <http://theory.stanford.edu/~rvg/hda>, 1991.
- [32] Van Glabbeek, R. J. On the Expressiveness of Higher Dimensional Automata. *Theoret. Comput. Sci.*, 356(3):265–290, 2006.
- [33] Willard, S. *General Topology*. Addison-Wesley, 1970.