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ISOSINGULARITY, DIFFERENTIALITY, AND NON-TRIVIAL ANALYTIC FACTORS OF ALGEBRAIC SINGULARITIES

DAVID BOURQUI AND MARIO MORÁN CAÑÓN

ABSTRACT. We establish a strong connection between the minimal differential closed subset containing a closed singular point of an algebraic variety on the one hand, the isosingular locus of the point and the analytic nature of its singularity on the other hand.

1. INTRODUCTION

A differential ideal of a ring is an ideal that is stable under all the derivations of the ring. If the ring contains the field of rational numbers, then the maximal differential ideal contained in a given ideal is prime, a fact first pointed out by Seidenberg in [Sei67] in the Noetherian case. As a consequence, one can define, for any point x of an algebraic variety defined over a field of characteristic zero, the notion of minimal differential closed subset containing x , corresponding in the affine case $x = \mathfrak{p} \in \text{Spec}(A)$ to the closure of the maximal differential ideal contained in \mathfrak{p} . The main results of the present paper are summed up in the following statement. It provides a geometric interpretation of the minimal differential closed subset containing a given point and of the formal neighborhood of its generic point in terms of the analytic nature of the singularity at x .

Theorem 1.1. *Let x be a closed point of an algebraic variety X defined over a field of characteristic zero, and let F be the minimal differential closed subset containing x .*

Then F is the closure of the unique component through x of the isosingular locus of x . Moreover, the formal neighborhood of the generic point of F is, up to a scalar extension, isomorphic to the non-trivial factor of the formal neighborhood of x .

The notion of isosingular loci of algebraic varieties was defined and studied recently in [CH22] when the base field is algebraically closed. Roughly speaking, they are the subsets where the formal neighborhood, in other words, the analytic nature of the singularities, remains constant (up to isomorphism). The non-trivial factor of a formal neighborhood $\widehat{\mathcal{O}_{X,x}}$ is the complete local ring A such that $\widehat{\mathcal{O}_{X,x}} \xrightarrow{\sim} A[[t_1, \dots, t_r]]$ and A is not isomorphic to $B[[t]]$ for some B . This uniquely determines A (up to isomorphism). In some sense, A contains the interesting information on the analytic nature of the singularity at x .

The second assertion of Theorem 1.1, which is in a certain sense a comparison theorem for formal neighborhoods, was motivated by analogous phenomena observed in the case of arc spaces associated with curves and toric varieties in [BMCS, BMCn23]. However, the general situation for arc spaces seems more involved (see Section 6).

Here is a brief description of the content of the sections of the paper.

In Section 2 we provide a recap, as well as some complements, on the connection between differentiability and cylindricity (*i.e.*, the property of having smooth factors), and the notion of maximal differential ideals.

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In Section 3, we define two notions of analytical isosingularity for closed subsets of an algebraic variety, a strong and a (a priori) weaker one. We use the weak version to extend the notion of isosingular locus discussed in [CH22] over an arbitrary field.

In Section 4, we positively answer Question C of [CH22], showing that over an algebraically closed field of characteristic $p > 0$, informally speaking, the bigger an analytically isosingular subset is, the more cylindrical the formal neighborhoods of its points are. In order to do so, we state and prove a version of the Lipman-Nagata-Zariski simplification lemma valid in arbitrary characteristic in terms of multivariate Hasse-Schmidt derivations. Our arguments also permit us to recover the characteristic zero case.

In Section 5, we prove the results leading to the statement of Theorem 1.1, showing along the way that the two notions of isosingularity defined in Section 3 coincide. One of the key ingredients is the study of the maximal differential ideal with respect to a regular family of derivations.

In Section 6, we discuss the situation in the case of arc spaces.

2. DIFFERENTIALITY AND CYLINDRICITY

In this section, we will provide a recap of classical ideas revolving around Lipman's extension of Zariski's simplification lemma in terms of regular derivations (See [Lip65, Theorem 2] and Subsection 2.15) and the notion of maximal differential ideal first explored in [Sei67], as well as some further results and complements. In view of the upcoming proofs of our main results, we shall in particular focus on the maximal differential ideals with respect to a set of regular derivations (see Definitions 2.2 and 2.5 below, which are motivated by Lipman's result).

2.1. Differentiability.

Definition 2.2. Let A be a ring and \mathfrak{p} be a prime ideal of A . Let r be a positive integer. An r -tuple D_1, \dots, D_r of elements of $\text{Der}(A)$ is said to be *regular* with respect to \mathfrak{p} if there exist elements $x_1, \dots, x_r \in \mathfrak{p}$ such that $\det(D_i(x_j))_{1 \leq i, j \leq r} \notin \mathfrak{p}$. If there exist such x_1, \dots, x_r such that the matrix $(D_i(x_j))$ is moreover diagonal, the r -tuple D_1, \dots, D_r is said to be *normalizable*. In case A is local, the family of derivations D_1, \dots, D_r is said to be *regular* if it is regular with respect to the maximal ideal of A .

Remark 2.3. As we shall see later, there exist regular yet non-normalizable families of derivations (See Lemma 2.39 and Remark 2.40).

Remark 2.4. If D_1, \dots, D_r is regular with respect to \mathfrak{p} and \mathcal{S} is a any set of generators of \mathfrak{p} , then there exist elements $x_1, \dots, x_r \in \mathcal{S}$ such that $\det(D_i(x_j))_{1 \leq i, j \leq r} \notin \mathfrak{p}$. Indeed, take $y_1, \dots, y_r \in \mathfrak{p}$ such that $\det(D_i(y_j))_{1 \leq i, j \leq r} \notin \mathfrak{p}$. Then the multilinearity of the determinant implies that $\det(D_i(y_j))$ is equal modulo \mathfrak{p} to a linear combination of elements of the form $\det(D_i(x_j))_{1 \leq i, j \leq r}$, where $x_1, \dots, x_r \in \mathcal{S}$.

Definition 2.5. Let A be a ring and \mathfrak{p} be a prime ideal of A . Let r be a positive integer. One says that \mathfrak{p} is:

- (1) *r-differential* if every r -tuple D_1, \dots, D_r of elements of $\text{Der}(A)$ is *not* regular with respect to \mathfrak{p} ;
- (2) *locally r-differential* if every r -tuple D_1, \dots, D_r of elements of $\text{Der}(A_{\mathfrak{p}})$ is *not* regular;
- (3) *formally r-differential* if every r -tuple D_1, \dots, D_r of continuous elements of $\text{Der}(\widehat{A_{\mathfrak{p}}})$ is *not* regular.

Remark 2.6. In particular, being 1-differential in the above sense is equivalent to being differential in the usual sense (see also Subsection 2.29).

Remark 2.7. If k is a field and A is endowed with a structure of k -algebra, one defines analogously the notion of being (locally, formally) differential *over* k by considering in the above definition only those derivations of A that are k -linear.

Example 2.8. Let B be a complete local ring (resp. k -algebra) and $A = B[[t_1, \dots, t_r]]$ the ring of formal power series in r indeterminates over B . Then $\partial_{t_1}, \dots, \partial_{t_r}$ is a regular family of elements of $\text{Der}(A)$ (resp. $\text{Der}_k(A)$), thus the maximal ideal of A is not r -differential (resp. not r -differential over k).

Let us recall the following well-known lemma.

Lemma 2.9. *Let A be a ring and $D \in \text{Der}(A)$.*

- (a) *Let S be a multiplicative part of A . Then there exists a unique extension of D to the localization $S^{-1}A$, that is to say: denoting by $\iota: A \rightarrow S^{-1}A$ the localization morphism there exists a unique $D' \in \text{Der}(S^{-1}A)$ such that $D' \circ \iota = \iota \circ D$.*
- (b) *Let \mathcal{I} be an ideal of A . Then there exists a unique continuous (for the limit topology) extension of D to the completion \widehat{A} . That is to say: denoting by $\tau: A \rightarrow \widehat{A}$ the completion morphism, there exists a unique continuous derivation $\widehat{D} \in \text{Der}(\widehat{A})$ such that $\widehat{D} \circ \tau = \tau \circ D$.*

Moreover, if k is a field, A is endowed with a structure of k -algebra, and the derivation D is k -linear, then \widehat{D} and D' are also k -linear.

Remark 2.10. In the notation of lemma 2.9, any derivation on \widehat{A} is automatically continuous for the $\widehat{\mathcal{I}}$ -adic topology, since $D(\widehat{\mathcal{I}}_1^n) \subset \widehat{\mathcal{I}}_1^{n-1}$ for any positive n .

In case \mathcal{I} is finitely generated, the limit topology on \widehat{A} is the $\widehat{\mathcal{I}}$ -adic topology ([Bou85, Ch. III, §2, n° 12, Corollaire 2]). Thus any derivation on \widehat{A} is also continuous for the limit topology.

Proposition 2.11. *Let A be a ring and \mathfrak{p} be a prime ideal of A .*

- (1) *If \mathfrak{p} is formally r -differential, then \mathfrak{p} is locally r -differential.*
- (2) *If \mathfrak{p} is locally r -differential, then \mathfrak{p} is r -differential.*

Assuming that k is a field and A is a k -algebra, the statements whith the analogous notions “over k ” are also true.

Proof. The pull-back by the localization morphism $A \rightarrow A_{\mathfrak{p}}$ (resp. the completion morphism $A \rightarrow \widehat{A}_{\mathfrak{p}}$) of the maximal ideal of $A_{\mathfrak{p}}$ (resp. $\widehat{A}_{\mathfrak{p}}$) is \mathfrak{p} (resp. the maximal ideal of $A_{\mathfrak{p}}$). Thus the result is a consequence of the definitions and Lemma 2.9. The last statement is a consequence of the end of Lemma 2.9. \square

The converse statements of Proposition 2.11 for r -differentiality hold in case A is an algebra of finite type over a field. This easily follows from the arguments in the proofs of [Sei67, Theorems 2 and 12]. Let us state the result and its proof, giving a somewhat simplified version of the aforementioned arguments.

Proposition 2.12. *Let k be a field, A be a k -algebra of finite type and \mathfrak{p} be a prime ideal of A .*

- (1) *For any $D \in \text{Der}_k(A_{\mathfrak{p}})$, there exists $s \notin \mathfrak{p}$ and $D' \in \text{Der}_k(A)$ such that $s \cdot D$ is the extension of D' .*
- (2) *If \mathfrak{p} is r -differential over k , then \mathfrak{p} is locally r -differential over k .*
- (3) *If \mathfrak{p} is locally r -differential over k , then \mathfrak{p} is formally r -differential over k .*

Proof. Let $\varphi: R := k[X_1, \dots, X_n] \rightarrow A$ be a presentation of the k -algebra A , $x_i := \varphi(X_i)$. Let $\mathfrak{n} = \text{Ker}(A \rightarrow A_{\mathfrak{p}}) = \{a \in A, \exists s \notin \mathfrak{p}, sa = 0\}$.

Let $D \in \text{Der}_k(A_{\mathfrak{p}})$. Up to multiplying D by an element $s \notin \mathfrak{p}$ (a common denominator of all the Dx_i , $i = 1, \dots, n$), one may assume that D is the extension of a derivation

$\tilde{D} \in \text{Der}_k(A/\mathfrak{n})$. Let $\mathfrak{n}' = \varphi^{-1}(\mathfrak{n})$ and $\bar{D} \in \text{Der}_k(R)$ such that $\varphi(\bar{D}(X_i)) = \tilde{D}(x_i) \pmod{\mathfrak{n}}$. In particular $\bar{D}\mathfrak{n}' \subset \mathfrak{n}'$.

Now since \mathfrak{n} is finitely generated, one may find $t \notin \mathfrak{p}$ such that $t \cdot \mathfrak{n} = 0$. But then if $T \in R$ is such that $\varphi(T) = t$ then $\text{Ker}(\varphi) \subset \mathfrak{n}'$ is stable under $T\bar{D}$. Thus $T \cdot \bar{D}$ induces $D' \in \text{Der}_k(A)$ whose extension to $A_{\mathfrak{p}}$ is $t \cdot D$.

The second assertion is a consequence of the first: let (D_1, \dots, D_r) be a regular family of k -derivation on $A_{\mathfrak{p}}$ and $(z_1, \dots, z_r) \in \mathfrak{p} \cdot A_{\mathfrak{p}}$ such that $\det(D_i z_j) \notin \mathfrak{p} \cdot A_{\mathfrak{p}}$. Up to replacing z_i by $t_i \cdot z_i$ for a suitable $t_i \notin \mathfrak{p}$ one may assume that the z_i 's are images of elements of A . And by the first assertion, up to replacing D_i by $s_i \cdot D_i$ for a suitable $s_i \notin \mathfrak{p}$ one may assume that D_i is the extension of $D'_i \in \text{Der}_k(A)$. Thus (D'_1, \dots, D'_r) is a regular family of $\text{Der}_k(A)$. Now let $D_1, \dots, D_r \in \text{Der}_k(\widehat{A}_{\mathfrak{p}})$ be a regular r -tuple of continuous derivations.

Let us show that there exists a regular r -tuple of elements of $\text{Der}_k(\widehat{A}_{\mathfrak{p}})$. Quotienting A by the kernel of the localization morphism, one may assume that the localization morphism is injective. By continuity of the D_i 's and the fact that the closure of \mathfrak{p} is $\widehat{\mathfrak{p}}$ (the maximal ideal of $\widehat{A}_{\mathfrak{p}}$), there exist $z_1, \dots, z_r \in \mathfrak{p}$ such that $\det(D_i z_j)_{1 \leq i, j \leq r} \notin \mathfrak{p}$. Therefore, up to changing the D_i 's and the presentation morphism φ , one may assume that $x_1, \dots, x_r \in \mathfrak{p}$ and $(D_i x_j)_{1 \leq i, j \leq r}$ is the identity matrix.

Let f_1, \dots, f_s be a finite family of generators of $\text{Ker}(\varphi)$. Set $y_1 = 1$ and $y_2 = \dots = y_r = 0$. Then any family $(y_j)_{j=r+1, \dots, n}$ of elements of $A_{\mathfrak{p}}$ such that

$$\sum_{1 \leq j \leq n} \partial_{x_j} f_i(x) \cdot y_j = 0, \quad i = 1, \dots, s$$

yields an element of $\text{Der}_k(A_{\mathfrak{p}})$. The above system has a solution in $\widehat{A}_{\mathfrak{p}}$ given by $y_j = D_1 x_j$ for $j \geq r+1$. Since $A_{\mathfrak{p}} \rightarrow \widehat{A}_{\mathfrak{p}}$ is faithfully flat, the system also has a solution in $A_{\mathfrak{p}}$ (see e.g., [Bou61, I.3.7, Proposition 13]). This yields an element $D'_1 \in \text{Der}_k(A_{\mathfrak{p}})$ such that $D'_1 x_1 = 1$, and $D'_1 x_2 = \dots = D'_1 x_r = 0$. Similarly, for any $2 \leq i \leq r$, one may construct D'_i such that $D'_i x_i = 1$ and $D'_i x_j = 0$ for $j \neq i$. \square

2.13. Interlude on families of pairwise commuting derivations. The following will be used to give a more precise version of the Lipman-Nagata-Zariski simplification lemma as stated in [Mat86, Theorem 30.1].

Proposition 2.14. *Let A be a ring, and let E_1, \dots, E_r and D_1, \dots, D_r be elements of $\text{Der}(A)$. Assume that there exists a square matrix $B = [b_{i,j}]_{1 \leq i, j \leq r}$ with coefficients in A such that*

$$D_i = \sum_{j=1}^r b_{i,j} \cdot E_j.$$

Consider the following property, called property (P):

$$\forall 1 \leq i, j, k \leq r, \quad D_i b_{j,k} = D_j b_{i,k}.$$

- (1) Assume that the E_i 's are pairwise commuting.
 - (a) If (P) holds, then the D_i 's are pairwise commuting.
 - (b) If the D_i 's are pairwise commuting and the E_i 's form a free family over A , then (P) holds.
 - (c) If $B \in \text{GL}_r(A)$, then $[D_i, D_j] \in \sum_{k=1}^r A \cdot D_k$.
- (2) Assume that the D_i 's are pairwise commuting.
 - (a) If (P) holds and $B \in \text{GL}_r(A)$, then the E_i 's are pairwise commuting.
 - (b) If E_i 's are pairwise commuting and form a free family over A , then (P) holds.

Proof. For $1 \leq i, j \leq r$, an elementary computation (akin to the one in the proof of [Now86, Lemma 4]) shows the relation

$$[D_i, D_j] = \sum_{1 \leq k \leq r} (D_i b_{j,k} - D_j b_{i,k}) \cdot E_k + \sum_{1 \leq k, \ell \leq r} b_{j,k} b_{i,\ell} [E_k, E_\ell]$$

This immediately shows 1(abc) and 2(b). Moreover if the D_i 's are pairwise commuting and (P) holds, the above relations show that the matrix product $B^2 \cdot (E_k, E_\ell)_{1 \leq k, \ell \leq r}$ is zero, allowing to conclude for 2(a). \square

2.15. Connection with cylindricity.

Definition 2.16. Let k be a field, A be a complete local ring (resp. k -algebra). Let r be a positive integer. Then A is said to be r -cylindrical (resp. r -cylindrical over k) if there exists a complete local ring (resp. k -algebra) B and a ring (resp. k -algebra) isomorphism between A and $B[[t_1, \dots, t_r]]$ (the formal power series ring in r variables with coefficients in B). It is said to be maximally r -cylindrical if it is r -cylindrical and not $(r+1)$ -cylindrical.

Remark 2.17. If A and B are complete local rings (resp. complete local k -algebras) then any ring (resp. k -algebra) isomorphism between A and B is automatically an isomorphism of complete local rings (resp. k -algebras).

Remark 2.18. If A is an r -cylindrical complete local ring (resp. k -algebra), and B_1, B_2 are complete local rings (resp. k -algebras) such that A and $B_1[[t_1, \dots, t_r]]$ on one hand, A and $B_2[[t_1, \dots, t_r]]$ on the other hand, are isomorphic rings (resp. k -algebras) then, by [Ham75, Theorem 4] (and its proof in the case of k -algebras), B_1 and B_2 are isomorphic rings (resp. k -algebras).

For rings containing \mathbf{Q} , the relation between r -differentiality and r -cylindricity is given by the Lipman-Nagata-Zariski simplification lemma, see e.g., [Lip65, Theorem 2] or [Mat86, Theorem 30.1]. Here is a more precise version of [Mat86, Theorem 30.1].

Lemma 2.19. *Let (A, \mathfrak{m}) be a complete local ring containing \mathbf{Q} , and let r be a positive integer.*

- (1) *The ideal \mathfrak{m} is not r -differential if and only if A is r -cylindrical. More precisely, if D_1, \dots, D_r is a regular family of derivations on A and $x_1, \dots, x_r \in \mathfrak{m}$ are such that $\det(D_i x_j)_{1 \leq i, j \leq r} \notin \mathfrak{m}$, then there exists a complete local ring B and an isomorphism $\varphi: B[[t_1, \dots, t_r]] \rightarrow A$ mapping t_i to x_i .*
- (2) *Let D_1, \dots, D_r be a regular family of derivations on A . Then:*
 - (a) *The following are equivalent:*
 - (i) *There exists a complete local ring B and an isomorphism $\varphi: B[[t_1, \dots, t_r]] \rightarrow A$ such that $\varphi(B) = \{a \in A : D_1 a = \dots = D_r a = 0\}$.*
 - (ii) *For any i, j , $[D_i, D_j] \in \sum_{k=1}^r A \cdot D_k$.*
 - (b) *The following are equivalent:*
 - (i) *There exists a complete local ring B and an isomorphism $\varphi: B[[t_1, \dots, t_r]] \rightarrow A$ such that $\varphi \circ \partial_{t_i} = D_i \circ \varphi$ for any i .*
 - (ii) *The D_i 's are pairwise commuting.*

Remark 2.20. Let (A, \mathfrak{m}) be as in the statement of Lemma 2.19 and assume that \mathfrak{m} is not r -differential. Let $B, x_1, \dots, x_r \in \mathfrak{m}$, $\varphi: B[[t_1, \dots, t_r]] \rightarrow A$ be as in the conclusion of Assertion 1. Then the quotient morphism $A \rightarrow A / \sum A \cdot x_i$ induces an isomorphism $\varphi(B) \xrightarrow{\sim} A / \sum A \cdot x_i$. Assume moreover that A is endowed with a structure of k -algebra, where k is a field of characteristic 0. Then $\varphi(B) \xrightarrow{\sim} A / \sum A \cdot x_i$ is a sub- k -algebra of A and φ is a k -algebra isomorphism. Therefore in this case the two equivalent statements of Assertion (1) of the lemma are also equivalent to “ A is r -cylindrical over k ” and “ \mathfrak{m} is not r -differential over k ”.

Proof. (of Lemma 2.19). The “if” in Assertion (1) is a straightforward consequence of Example 2.8.

Assume \mathfrak{m} is not r -differential and let $x_1, \dots, x_r \in \mathfrak{m}$ such that $\det(D_i x_j)_{1 \leq i, j \leq r} \notin \mathfrak{m}$. In particular the matrix $(D_i x_j)$ is invertible and up to changing (D_1, \dots, D_r) by its image by the inverse of the matrix, one may assume that $(D_i x_j)$ is the identity matrix. By the case

$r = 1$ there exists a complete local ring B_1 and an isomorphism $\varphi: B_1[[t_1]] \xrightarrow{\sim} A$ mapping t_1 to x_1 . Identifying A with $B_1[[t_1]]$ via this isomorphism, for $2 \leq i \leq r$, let $\widetilde{D}_i \in \text{Der}(B_1)$ be defined by $\widetilde{D}_i(b) = D_i(b)_{t_i=0}$. Note that $x_2, \dots, x_r \in B_1 = \{a \in A : D_1 a = 0\}$ and that $(\widetilde{D}_i x_j)_{2 \leq i, j \leq r}$ is the identity matrix. Thus one obtains the result by induction.

Let us show Assertion 2(a). $(ii) \Rightarrow (i)$ is part of [Mat86, Theorem 30.1]. Now assume $A = B[[t_1, \dots, t_r]]$ and (D_1, \dots, D_r) is a regular family of derivations on A such that $B = \{a \in A, D_1 a = \dots = D_r a = 0\}$. In particular, for any i one has $D_i = \sum_{j=1}^r D_i t_j \cdot \partial_{t_j}$. Let $x_1, \dots, x_r \in \mathfrak{m}$ such that $\det(D_i x_j) \in A^\times$. Then the equality of matrices

$$(D_i x_j) = (D_i t_j) \cdot (\partial_{t_i} x_j)$$

shows that $\det(D_i t_j) \in A^\times$, and one may apply Assertion 1(c) of Proposition 2.14 in order to conclude.

Let us show Assertion 2(b). $(i) \Rightarrow (ii)$ is straightforward. Now assume that the D_i 's are pairwise commuting and take $x_1, \dots, x_r \in \mathfrak{m}$ such that $\det(D_i x_j) \in A^\times$. As in the proof of Assertion 1, consider the derivations E_1, \dots, E_r determined by the relations

$$D_i = \sum_{j=1}^r D_i x_j \cdot E_j$$

so that $(E_i x_j)$ is the identity matrix. Then by Assertion 2(a) of Proposition 2.14, the E_i 's are pairwise commuting. Thus in the above proof of Assertion 1, since one may chose φ such that $B_1 = \{a \in A, E_1 a = 0\}$, B_1 is then E_i -stable for $i > 1$, and \widetilde{E}_i is simply the restriction of E_i to B_1 . By induction one thus may construct an isomorphism $\varphi: B[[t_1, \dots, t_r]] \rightarrow A$ mapping t_i to x_i and such that $\varphi \circ \partial_{t_i} = E_i \circ \varphi$ for any i and $\varphi(B) = \{a \in A, E_1 a = \dots = E_r a = 0\}$. Note that one also has $\varphi(B) = \{a \in A, D_1 a = \dots = D_r a = 0\}$. In order to conclude, it suffices to find $y_1, \dots, y_r \in \mathfrak{m}$ such that $(D_i y_j)$ is the identity matrix. Identifying A with $B[[t_1, \dots, t_r]]$ and thus R_i with ∂_{t_i} , the existence of such y_i 's is a consequence of [Now86, Theorem 2]. \square

Remark 2.21. One may also recover Assertion (1) by applying Theorem 4.14 below to the formal flow associated with D_1, \dots, D_r (see Example 4.12).

Example 2.22. As an illustration of Assertion 2(a), consider $A = k[[x_1, x_2, x_3]]$, and (D_1, D_2) the regular family given by $D_1 = \partial_{x_1}$ and $D_2 = \partial_{x_2} + x_1 \cdot \partial_{x_3}$. One easily sees that $\{a \in A : D_1 a = D_2 a = 0\} = k$; on the other hand, for any isomorphism $\varphi: B[[t_1, t_2]] \xrightarrow{\sim} A$, one must have $\dim(\varphi(B)) = 1$.

From Propositions 2.11 and 2.12, Lemma 2.19 and Remark 2.20, one deduces the following corollary, that we state in geometric terms.

Corollary 2.23. *Let k be a field and X be a k -scheme of finite type. Let $x \in X$ be a (schematic) point. Then the following assertions are equivalent, and are also equivalent to the same assertions with “over k ” removed:*

- (1) *There exists an open affine subset U of X containing x such that the prime ideal \mathfrak{p}_x of $\mathcal{O}_X(U)$ corresponding to x is r -differential over k .*
- (2) *For any open affine subset U of X containing x , the prime ideal \mathfrak{p}_x of $\mathcal{O}_X(U)$ corresponding to x is r -differential over k .*
- (3) *The maximal ideal of $\widehat{\mathcal{O}}_{X,x}$ is r -differential over k .*
- (4) *The maximal ideal of $\widehat{\mathcal{O}}_{X,x}$ is r -differential over k .*
- (5) *(Requires $\text{char}(k) = 0$) The complete local k -algebra $\widehat{\mathcal{O}}_{X,x}$ is not r -cylindrical over k .*

Definition 2.24. Let k be a field and X be a k -scheme of finite type. Let $x \in X$ be a (schematic) point. One says that x is an r -differential point of X if one of the equivalent assertions of Corollary 2.23 holds for x . If $\text{char } k = 0$, a point that is not r -differential is called an r -cylindrical point.

Example 2.25. Let k be a field of characteristic zero and X be an algebraic k -variety. Then by [Sei67, Theorem 5], the generic point of any component of the singular locus of X is a 1-differential point of X .

Lemma 2.26. *Let k be a field of characteristic zero, A be a complete local Noetherian k -algebra with residue field k -isomorphic to k and r be a positive integer. Assume that there exists an extension L/k such that $A\widehat{\otimes}_k L$ is r -cylindrical. Then A is r -cylindrical.*

Remark 2.27. In case k has positive characteristic and L/k is not separable, the statement does not hold (see Example 4.17). We do not know whether the statement holds in case k has positive characteristic and L/k is separable.

Proof. The case $r = 1$ directly follows from arguments contained in the proofs of [Sei69a, Theorems 2 and 3]. The general case may then be deduced by induction, using Remark 2.18

Alternatively, one can use Lemma 2.19 and the following extension of the aforementioned arguments. Write $A = k[[t_1, \dots, t_s]]/\mathcal{I}$. Let (D_1, \dots, D_r) be a regular family of derivations on $A\widehat{\otimes}_k L = L[[t_1, \dots, t_s]]/\mathcal{I} \cdot L[[t_1, \dots, t_s]]$. By Remark 2.4 up to permuting the t_i 's and changing the D_i 's, one may assume that $(D_i \bar{t}_j)_{1 \leq i, j \leq r}$ is the identity matrix, where \bar{t}_i is the image of t_i in $A\widehat{\otimes}_k L$.

For $1 \leq i \leq r$, one may lift D_i to a derivation D_i on $L[[t_1, \dots, t_s]]$ such that $D_i t_j = \delta_{i,j}$ for $1 \leq j \leq r$ and for any $F \in \mathcal{I}$, $D_i F \in \mathcal{I} \cdot L[[t_1, \dots, t_s]]$. Now let $\{e_\alpha\}$ be a k -basis of L containing 1, and consider the $k[[t_1, \dots, t_s]]$ -linear map $\pi: L[[t_1, \dots, t_s]] \rightarrow k[[t_1, \dots, t_s]]$ induced by the linear projection of L on $k \cdot 1$. Then $D_{k,i} := \pi(D_i)|_{k[[t_1, \dots, t_s]]}$ is a derivation of $k[[t_1, \dots, t_s]]$ such that $D_{k,i} t_j = \delta_{i,j}$, and for any $F \in \mathcal{I}$, one has that $D_{k,i} F \in \pi(\mathcal{I} \cdot L[[t_1, \dots, t_s]]) = \mathcal{I}$. \square

Here is another consequence of Lemma 2.19 (noted in [BS18, §5.6] in the case $r = 1$)

Corollary 2.28. *Let k be a field of characteristic zero and X be a k -scheme of finite type. Let $x \in X$ be an r -cylindrical point. Then there exists an affine k -scheme of finite type Y and $y \in Y$ such that $\kappa(x) \xrightarrow{\sim} \kappa(y)$ with the following property: if $(y, 0)$ denotes the point of $Y \times_k \mathbf{A}_k^r$ corresponding to the prime ideal $\langle \mathfrak{p}_y, t_1, \dots, t_r \rangle$ of $\mathcal{O}(Y)[t_1, \dots, t_r] = \mathcal{O}(Y \times_k \mathbf{A}_k^r)$, there exists an isomorphism of complete local k -algebras*

$$\widehat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{Y \times_k \mathbf{A}_k^r, (y,0)} = \widehat{\mathcal{O}}_{Y,y}[[t_1, \dots, t_r]].$$

Proof. Let D_1, \dots, D_r be a regular family of derivations on $\widehat{\mathcal{O}}_{X,x}$. Since moreover the derivations are continuous (Remark 2.10), there exist elements $x_1, \dots, x_r \in \mathfrak{m} := \mathfrak{m}_{X,x}$ such that $\det(D_i(x_j)) \notin \widehat{\mathfrak{m}}$. By lemma 2.19, there exists a complete local k -algebra B and an isomorphism of k -algebras $\varphi: B[[t_1, \dots, t_r]] \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ mapping t_i to x_i . Thus one has an isomorphism $B \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}/\langle x_1, \dots, x_r \rangle$ and B is isomorphic to the completion of $\frac{\mathcal{O}_{X,x}}{\langle x_1, \dots, x_r \rangle}$, hence the existence of (Y, y) as in the statement \square

2.29. Maximal differential ideals. Let A be a ring and T be a family of derivations of A . Recall that an ideal of A is said to be T -differential (or simply differential in case $T = \text{Der}(A)$) if it is stable under any element $D \in T$. Now let \mathfrak{i} be an ideal of A . The sum of all the T -differential ideals contained in \mathfrak{i} is a T -differential ideal contained in \mathfrak{i} and maximal for this property. We call this ideal the *maximal T -differential ideal of A contained in \mathfrak{i}* . In case A is a local ring and \mathfrak{i} is the maximal ideal, we simply say the maximal T -differential ideal of A .

Lemma 2.30. *Keep the above notation and assume that A is a ring containing \mathbf{Q} and \mathfrak{i} is a proper ideal. Then the maximal T -differential ideal of A contained in \mathfrak{i} is a prime ideal.*

In case A is Noetherian, the above lemma follows from [Sei67, Theorem 1]), which shows that in this case the associated primes of a T -differential ideal are T -differential.

As explained in the proof below, a slight variation of the arguments of [Sei67] allows to drop the Noetherianity hypothesis; this should be useful in the context of the extension of the present work to arc spaces (see Section 6).

Proof. Let $D \in \text{Der}(A)$. Following [Sei67], consider the morphism $\varphi_D: A[[t]] \rightarrow A[[t]]$ mapping t to t and inducing the formal flow $H_D: a \mapsto \sum_{i \geq 0} \frac{D^{(i)}a}{i!} \cdot t^i$ associated with D on A , which is easily seen to be an automorphism, with inverse the morphism ψ_D mapping t to t and inducing $a \mapsto H_D(a)(-t)$ on A . Moreover, for any ideal \mathcal{I} of A , \mathcal{I} is D -stable if and only if $\mathcal{I}[[t]]$ is φ_D -stable.

Now it suffices to show that for any $D \in \text{Der}(A)$ and any proper D -stable ideal \mathcal{I} of A , any minimal prime ideal containing \mathcal{I} is again D -stable. Since A contains \mathbf{Q} and \mathcal{I} is D -stable, the radical $\sqrt{\mathcal{I}}$ is again D -stable (see e.g., [Bui94, Chapter 2, (1.3)]). Thus we may assume that \mathcal{I} is radical.

Let $\{\mathfrak{p}_e\}_{e \in E}$ be the family of minimal prime ideals containing \mathcal{I} . In particular, $\mathcal{I} = \bigcap_{e \in E} \mathfrak{p}_e$. Thus $\mathcal{I}[[t]] = \bigcap_{e \in E} \mathfrak{p}_e[[t]]$. We claim that for any $e \in E$, the prime ideal $\mathfrak{p}_e[[t]]$ is a minimal prime ideal containing $\mathcal{I}[[t]]$. Indeed, if \mathfrak{q} is a prime ideal of $A[[t]]$ such that $\mathcal{I}[[t]] \subset \mathfrak{q} \subset \mathfrak{p}_e[[t]]$, one must have $\mathfrak{q} \cap A = \mathfrak{p}_e$; now for $a = \sum_{i \geq 0} a_i \cdot t^i \in \mathfrak{q}$, one has $a_0 \in \mathfrak{p}_e$, thus $t \cdot \sum_{i \geq 1} a_i \cdot t^{i-1} \in \mathfrak{q}$. Since $\mathfrak{q} \subset \mathfrak{p}_e[[t]]$, one has $t \notin \mathfrak{q}$ and since \mathfrak{q} is prime, one has $\sum_{i \geq 1} a_i \cdot t^{i-1} \in \mathfrak{q}$ thus $a_1 \in \mathfrak{p}_e$ and so on.

Thus $\{\mathfrak{p}_e[[t]]\}_{e \in E}$ is the family of minimal prime ideals containing $\mathcal{I}[[t]]$. Since \mathcal{I} is D -stable, one has $\mathcal{I}[[t]] = \varphi_D(\mathcal{I}[[t]]) = \bigcap_{e \in E} \varphi_D(\mathfrak{p}_e[[t]])$. Thus for $e \in E$ one must have $\varphi_D(\mathfrak{p}_e[[t]]) = \mathfrak{p}_{e'}[[t]]$ for $e' \in E$. Since for $a \in A$, $\varphi_D(a) = a \pmod{\langle t \rangle}$, one infers that $\mathfrak{p}_e \subset \mathfrak{p}_{e'}$ thus $\mathfrak{p}_{e'}[[t]] = \mathfrak{p}_e[[t]]$ and \mathfrak{p}_e is D -stable. \square

Remark 2.31. If k is a field of characteristic $p > 0$, the maximal $\partial_{t_1}, \dots, \partial_{t_r}$ -differential ideal of $k[t_1, \dots, t_r]$ (or $k[[t_1, \dots, t_r]]$) contained in $\langle t_1, \dots, t_r \rangle$ is $\langle t_1^p, \dots, t_r^p \rangle$. In particular the conclusion of Lemma 2.30 does not hold in this case, and the hypothesis that A contains \mathbf{Q} is crucial. See Remark 4.13 below for a characteristic-free version of Lemma 2.30.

Definition 2.32. Let X be a k -scheme of finite type and $x \in X$ a schematic point. The maximal differential generalization of X is the unique differential point $x' \in X$ (see Definition 2.24) whose closure contains x and such that any other differential point of X whose closure contains x also contains x' in its closure.

Remark 2.33. If $\text{Spec}(A)$ is an open affine subset of X containing x and \mathfrak{p} is the prime ideal of A corresponding to x , the prime ideal of A corresponding to the maximal differential generalization of X is the maximal differential ideal of A contained in \mathfrak{p} .

The following lemma shows the functoriality of the notion of maximal T -differential ideal.

Lemma 2.34. Let A and B be rings, $T_A = \{D_i\}_{i \in I}$ (resp. $T_B = \{E_i\}_{i \in I}$) be a family of elements of $\text{Der}(A)$, (resp. $\text{Der}(B)$), and $\varphi: A \rightarrow B$ be a ring morphism such that for any $i \in I$, $E_i \circ \varphi = \varphi \circ D_i$. Let \mathcal{I} be an ideal of B and \mathcal{J} be the maximal T_B -differential ideal of B contained in \mathcal{I} . Then $\varphi^{-1}\mathcal{J}$ is the maximal T_A -differential ideal of A contained in $\varphi^{-1}\mathcal{I}$.

Proof. Due to the relations $E_i \circ \varphi = \varphi \circ D_i$, it is clear that $\varphi^{-1}\mathcal{J}$ is a T_A -differential ideal of A contained in $\varphi^{-1}\mathcal{I}$. Let \mathcal{K} be any T_A -differential ideal of A contained in $\varphi^{-1}\mathcal{I}$. Since $\mathcal{J} \cup \varphi(\mathcal{K})$ is T_B -stable, the ideal $\langle \mathcal{J}, \varphi(\mathcal{K}) \rangle$ is T_B -differential. Since it contains \mathcal{J} and is contained in \mathcal{I} , it coincides with \mathcal{J} , in particular $\varphi(\mathcal{K}) \subset \mathcal{J}$ thus $\mathcal{K} \subset \varphi^{-1}\mathcal{J}$. \square

Corollary 2.35. Let (A, \mathfrak{m}) be a local ring, $D_1, \dots, D_r \in \text{Der}_k(A)$ be a family of derivations on A and \mathfrak{n} be the maximal D_1, \dots, D_r -differential ideal of A . Let $\widehat{D}_1, \dots, \widehat{D}_r$ be the extensions of D_1, \dots, D_r to the completion \widehat{A} and \mathfrak{n}^* be the maximal $(\widehat{D}_1, \dots, \widehat{D}_r)$ -differential ideal of \widehat{A} . Then we have the relation $\mathfrak{n} = \mathfrak{n}^* \cap A$.

2.36. Maximal differential ideals with respect to regular families, 1. We shall pay particular attention, now and later in Section 5, to the description and properties of T -maximal differential ideals in case the family of derivations T is a finite regular family. Lemma 2.37 below is a basic instance of this situation.

Lemma 2.37. *Let A be a ring containing \mathbf{Q} , \mathfrak{p} be a prime ideal of A and r be a positive integer. Then the maximal $(\partial_{t_1}, \dots, \partial_{t_r})$ -differential ideal of $A[t_1, \dots, t_r]$ (resp. $A[[t_1, \dots, t_r]]$) contained in $\langle \mathfrak{p}, t_1, \dots, t_r \rangle$ is $\mathfrak{p}[t_1, \dots, t_r]$ (resp. $\mathfrak{p}[[t_1, \dots, t_r]]$).*

Proof. The prime ideal $\mathfrak{p}[t_1, \dots, t_r]$ of $A[t_1, \dots, t_r]$ is $\partial_{t_1}, \dots, \partial_{t_r}$ -differential and contained in $\langle \mathfrak{p}, t_1, \dots, t_r \rangle$. Now let $\alpha \in A[t_1, \dots, t_r]$ be such that $\alpha \notin \mathfrak{p}[t_1, \dots, t_r]$. Since $\alpha \pmod{\mathfrak{p}[t_1, \dots, t_r]}$ is a non-zero element of $(A/\mathfrak{p})[t_1, \dots, t_r]$ and the latter ring contains \mathbf{Q} , there exist $i_1, \dots, i_r \geq 0$ such that $\partial_{t_1}^{(i_1)} \dots \partial_{t_r}^{(i_r)} \alpha \pmod{\mathfrak{p}[t_1, \dots, t_r]}$ has a non-zero constant term. Thus $\partial_{t_1}^{(i_1)} \dots \partial_{t_r}^{(i_r)} \alpha$ does not belong to $\langle \mathfrak{p}, t_1, \dots, t_r \rangle$. Thus the maximal $(\partial_{t_1}, \dots, \partial_{t_r})$ -differential ideal of $A[t_1, \dots, t_r]$ contained in $\langle \mathfrak{p}, t_1, \dots, t_r \rangle$ is $\mathfrak{p}[t_1, \dots, t_r]$.

A strictly analogous argument works for $A[[t_1, \dots, t_r]]$. \square

Remark 2.38. Let (D_1, \dots, D_r) be a regular family of derivations on a complete local ring (A, \mathfrak{m}) . Then in general the maximal (D_1, \dots, D_r) -differential ideal \mathfrak{p} of A does not coincide, via the isomorphism $A \xrightarrow{\sim} B[[t_1, \dots, t_r]]$ constructed in the proof of Assertion 1 of Lemma 2.19, with the maximal $(\partial_{t_1}, \dots, \partial_{t_r})$ -differential ideal \mathfrak{q} of $B[[t_1, \dots, t_r]]$, that is, the ideal generated by the maximal ideal of B .

Consider e.g., $A = k[[x_1, x_2, x_3]]$, $D_1 = \partial_{x_1}$ and $D_2 = \partial_{x_2} + x_1 \cdot \partial_{x_3}$. Then the derivation \widetilde{D}_2 (remember the notation in the proof of Lemma 2.19) on $k[[x_2, x_3]]$ is ∂_{x_2} , and $\mathfrak{q} = \langle x_3 \rangle$. Yet $D_2 x_3 = x_1$ thus \mathfrak{q} is not stable under D_2 . This shows that $\mathfrak{p} \neq \mathfrak{q}$.

Lemma 2.39. *Let A be a ring containing \mathbf{Q} , \mathfrak{p} be a prime ideal of A , r be a positive integer, and D_1, \dots, D_r be a family of elements of $\text{Der}(A)$ that is regular with respect to \mathfrak{p} . For $0 \leq s \leq r$, denote by \mathfrak{q}_s the maximal D_1, \dots, D_s -differential ideal of A contained in \mathfrak{p} . Then:*

- (1) *One has $\text{codim}_{A/\mathfrak{q}_r} \mathfrak{p} \geq r$.*
- (2) *If D_1, \dots, D_r is moreover normalizable (see Definition 2.2) and $\text{codim}_{A/\mathfrak{q}_r} \mathfrak{p} = r$, one has $\text{codim}_{A/\mathfrak{q}_s} \mathfrak{p} = s$ for any $0 \leq s \leq r$.*

Proof. Localizing and using Lemma 2.34 and 2.9, one may assume that A is local and \mathfrak{p} is its maximal ideal. Let us show the first assertion. Let $(x_1, \dots, x_r) \in \mathfrak{p}$ such that $\det(D_i(x_j))_{1 \leq i, j \leq r} \notin \mathfrak{p}$. Replacing the r -tuple of derivations D_1, \dots, D_r with its product by the inverse of the matrix $(D_i(x_j))$, one may assume that $(D_i(x_j))$ is the identity matrix. Note that \mathfrak{q}_r remains the same by this change of family (however, the \mathfrak{q}_s 's for $s < r$ may have been modified). Recall that \mathfrak{q}_s is prime (Lemma 2.30). Moreover, one has $\mathfrak{q}_{s+1} \subset \mathfrak{q}_s$ and for $0 \leq s \leq r-1$ one has $x_{s+1} \in \mathfrak{q}_s \setminus \mathfrak{q}_{s+1}$ since $D_{s+1} x_{s+1} = 1$ and (in case $s \geq 1$) $D_s x_{s+1} = 0$. The conclusion follows.

As for the second assertion, note that under the given assumptions, the previous argument shows directly (without having to change the family of derivations) that we have a chain of proper inclusions of prime ideals $\mathfrak{q}_r \subsetneq \mathfrak{q}_{r+1} \subsetneq \dots \subsetneq \mathfrak{q}_0 = \mathfrak{p}$. \square

Remark 2.40. In case A is a complete local ring and D_1, \dots, D_r is a finite regular family of derivations that are pairwise commuting, the dimension of the maximal D_1, \dots, D_r -differential ideal equals r . This follows from Lemmas 2.19 and 2.37. The equality holds even under the weaker assumption $[D_i, D_j] \in \sum_{k=1}^r A \cdot D_k$. Indeed, in this case, by Lemma 2.19, one may assume that $A = B[[t_1, \dots, t_r]]$ where (B, \mathfrak{m}_B) is a complete local ring such that $B = \{a \in A : D_1 a = \dots = D_r a = 0\}$. But then $D_i = \sum_{j=1}^r D_i t_j \cdot \partial_{t_j}$ and (see the proof of Assertion 2(a) of Lemma 2.19) $(D_i t_j) \in GL_r(A)$. Thus the maximal D_1, \dots, D_r -differential ideal coincides with the maximal $\partial_{t_1}, \dots, \partial_{t_r}$ -differential ideal, and one may conclude with Lemma 2.37 as before.

In general the inequality in the first assertion of the lemma is strict. Consider again the example $A = k[[x_1, x_2, x_3]]$, $D_1 = \partial_{x_1}$ and $D_2 = \partial_{x_2} + x_1 \cdot \partial_{x_3}$. Then the maximal (D_1, D_2) -differential ideal \mathfrak{q} is zero. Indeed, by the lemma, its dimension is 2 or 3. If it is 2, then one must have $\mathfrak{q} = \langle F \rangle$ with $F \in k[[x_1, x_2, x_3]]$ such that $\partial_{x_1} F \in \langle F \rangle$ and $\partial_{x_2} F + x_1 \partial_{x_3} F \in \langle F \rangle$. The latter conditions are easily seen to imply $F \in k$, a contradiction. Thus $\mathfrak{q} = \{0\}$.

The latter example also shows that the second assertion fails in general. Indeed, in this case, setting $D_3 = \partial_{x_3}$, the maximal (D_1, D_2) -differential ideal coincides with the maximal D_1, D_2, D_3 -differential, both having dimension 3. This shows in particular that D_1, D_2, D_3 is not normalizable (see remark 2.3).

3. ANALYTICALLY ISOSINGULAR SUBSETS AND ISOSINGULAR LOCI OF ALGEBRAIC VARIETIES

Motivated by our comparison theorems for formal neighborhoods in arc schemes ([BMCS, BMCn23]) and Chiu and Hauser's study of isosingular loci of algebraic varieties over algebraically closed field ([CH22]), in this section, we first introduce the notion of strongly and weakly analytically isosingular (in short SAI and WAI) closed subsets of an algebraic variety defined over a perfect field. Then we extend the notion of isosingular loci, as well as some of their properties, to the case of perfect fields.

The definitions of SAI and WAI are such that any SAI subset is trivially WAI, but later we will be able to show that both notions coincide (see Corollary 5.20). We do not know whether a similar statement holds in the case of arc schemes (see Question 6.11).

3.1. Preliminaries : notation and technical lemmas. We will say that a finitely generated field extension K/k is separable if there exists a finite family $\{x_i\}$ of elements of K that are algebraically independent over k and such that any element of K is algebraic over $k(\{x_i\})$ with separable minimal polynomials. An arbitrary extension K/k is said to be separable if any finitely generated subextension L/k is.

Definition 3.2. Let k be a field and A be a local k -algebra with residue field $\kappa(A)$. A k -section of (the local k -algebra) A is a section $s: \kappa(A) \rightarrow A$ of the quotient morphism which is a morphism of k -algebras.

Recall that if A is a complete local k -algebra and k' is sub- k -extension of $\kappa(A)$ such that k'/k is algebraic separable, all the k -sections of A have the same restriction to k' (see e.g., [Bou06, Chapitre 9, §3, Proposition 1]). In particular, if $\kappa(A)/k$ is algebraic separable, there exists a unique k -section of A ; in this case, A will be tacitly assumed to be equipped with the resulting $\kappa(A)$ -algebra structure. More generally, if A is endowed with a k' -algebra structure, every k -section of A is automatically a k' -section.

Notation 3.3. If A is a ring, K is a field and $\iota: K \rightarrow A$ is a morphism, we shall sometimes denote by $A^{(\iota)}$ the ring A equipped with the induced K -algebra structure. This will be relevant in contexts where several K -algebra structures on A are involved.

Remark 3.4. For any extension K/k , two local K -algebras A and B with residue field K -isomorphic to K are isomorphic as k -algebras if and only if there exists a k -section s of B such that the K -algebras A and $B^{(s)}$ are isomorphic. Indeed, if $\varphi: A \rightarrow B$ is a k -algebra isomorphism, one may take $s = \varphi \circ \iota_A$ where $\iota_A: K \rightarrow A$ is the structural morphism.

For the next lemma, see [Sei73, Lemma, p.29].

Lemma 3.5. Let k be a field, $A = k[X_1, \dots, X_N]/\mathcal{I}$ and \mathfrak{m} be a maximal ideal of A such that the residue field $\kappa(\mathfrak{m})$ of \mathfrak{m} is an algebraic separable extension of k . Let α_i be the image of X_i in $\kappa(\mathfrak{m})$. Then the $\kappa(\mathfrak{m})$ -algebras $\widehat{A_{\mathfrak{m}}}$ and $\kappa(\mathfrak{m})[[X_1, \dots, X_N]]/\langle F(X_i + \alpha_i) \rangle_{F \in \mathcal{I}}$ are isomorphic.

Corollary 3.6. Let k be a perfect field, \bar{k} be an algebraic closure of k , X be an algebraic k -variety, x be a closed point of X , $\iota: \kappa(x) \hookrightarrow \bar{k}$ be a k -embedding and x' be the point of $X(\bar{k})$ lying over x induced by ι . Then the \bar{k} -algebras $\widehat{\mathcal{O}_{X_{\bar{k}}, x'}}$ and $\widehat{\mathcal{O}_{X, x} \otimes_{\kappa(x), \iota} \bar{k}}$ are isomorphic.

Lemma 3.7. *Let k be a field.*

- (1) *Let A be a complete local Noetherian k -algebra with residue field k -isomorphic to k , K/k be an extension and $A_K := A \widehat{\otimes}_k K$. Let $s: K \rightarrow A_K$ be a k -section of A_K . Then there exists a k -algebra automorphism φ of A_K inducing $s: K \rightarrow s(K)$.*
- (2) *Let k'/k be an algebraic separable extension of k , A be a complete local Noetherian k' -algebra with residue field k' -isomorphic to k' , K/k' be an extension and B a K -algebra such that the K -algebras $A_K := A \widehat{\otimes}_{k'} K$ and B are isomorphic as k -algebras. Then A_K and B are isomorphic K -algebras.*

Proof. One may write $A_K = K[[X_1, \dots, X_n]]/\langle F_1, \dots, F_m \rangle$ with $F_j \in k[[X_1, \dots, X_n]]$. Denote by x_i the image of X_i in A_K . Consider $\varphi: K[[X_1, \dots, X_n]] \rightarrow A$ mapping $\sum a_I \cdot \mathbf{X}^I$ to $\sum s(a_I) \cdot \mathbf{x}^I$. Since s is a morphism of k -algebras, any $F \in k[[X_1, \dots, X_n]]$ is mapped to $F(x_1, \dots, x_n)$. In particular, for any $1 \leq i \leq m$ one has $\varphi(F_i) = 0$, and therefore φ factors through an endomorphism $\varphi: A \rightarrow A$, mapping $\sum a_I \cdot \mathbf{x}^I$ to $\sum s(a_I) \cdot \mathbf{x}^I$.

Since for any $\alpha \in K$ one has $s(\alpha) - \alpha \in \mathfrak{M}$, one sees that φ induces the identity on $\mathfrak{M}^n / \mathfrak{M}^{n+1}$. Thus φ is a ring automorphism. This shows the first assertion.

As for the second assertion, since A_K and B are isomorphic k -algebras and k'/k is algebraic separable, A_K and B are isomorphic k' -algebras. Thus one reduces to the case $k' = k$. Denote by $\iota_{A_K}: K \rightarrow A_K$ and $\iota_B: K \rightarrow B$ the structural morphisms. Let $\varphi: A_K \rightarrow B$ be a k -algebra isomorphism. Since $\varphi^{-1} \circ \iota_B$ is a k -section of A_K and by the first assertion, there exists a k -algebra automorphism ψ of A such that $\psi \iota_{A_K} = \varphi^{-1} \circ \iota_B$. Then $\varphi \psi: A_K \rightarrow B$ is a ring isomorphism and $(\varphi \psi) \iota_{A,K} = \iota_B$. Thus $\varphi \psi$ is a K -algebra isomorphism between A_K and B . \square

3.8. Strongly and weakly analytically isosingular subsets of an algebraic variety.

Notation 3.9. For any topological space X , one denotes by $X^{(0)}$ the set of its closed points. Note that in case X is an algebraic variety and Y is a locally closed subset of X , one has $Y^{(0)} = Y \cap X^{(0)}$.

Definition 3.10. Let k be a perfect field and X be a k -scheme of finite type.

Let $x_1, x_2 \in X^{(0)}$. One says that x_1 and x_2 are *geometrically analytically equivalent* if there exist an extension K/k and k -embeddings $\iota_1: \kappa(x_1) \hookrightarrow K$ and $\iota_2: \kappa(x_2) \hookrightarrow K$ such that $\widehat{\mathcal{O}_{X,x_1}} \widehat{\otimes}_{\kappa(x_1)} K^{(\iota_1)}$ and $\widehat{\mathcal{O}_{X,x_2}} \widehat{\otimes}_{\kappa(x_2)} K^{(\iota_2)}$ are isomorphic K -algebras.

An irreducible closed subset F of X is said to be *weakly analytically isosingular* (in short WAI) if there exists a non-empty Zariski open subset U of F such that any $x_1, x_2 \in U^{(0)}$ are geometrically analytically equivalent. Such an open subset of F is called an analytical isosingularity (in short AI) open subset of F .

A WAI closed subset F is said to be maximal (in short MWAI) if there does not exist a WAI closed subset G containing properly F and an AI open subset U of G such that $U \cap F \neq \emptyset$.

Remark 3.11. Let \bar{k}/k be an algebraic closure of k . Let $x_1, x_2 \in X^{(0)}$. Then x_1 and x_2 are geometrically analytically equivalent if and only if there exist k -embeddings $\iota_1: \kappa(x_1) \hookrightarrow \bar{k}$ and $\iota_2: \kappa(x_2) \hookrightarrow \bar{k}$ such that $\widehat{\mathcal{O}_{X,x_1}} \widehat{\otimes}_{\kappa(x_1)} \bar{k}^{(\iota_1)}$ and $\widehat{\mathcal{O}_{X,x_2}} \widehat{\otimes}_{\kappa(x_2)} \bar{k}^{(\iota_2)}$ are isomorphic \bar{k} -algebras (see also remark 3.25 below). Indeed, the ‘‘if’’ is clear. Conversely, let K/k , ι_1 , and ι_2 be as in the definition. Upon extending K , one may assume that K/k factors through \bar{k}/k . For $i = 1, 2$, let $\iota'_i: \kappa(x_i) \hookrightarrow \bar{k}$ be the induced embedding and $A_i := \widehat{\mathcal{O}_{X,x_i}} \widehat{\otimes}_{\kappa(x_i)} \bar{k}^{(\iota'_i)}$. Since $A_1 \widehat{\otimes}_{\bar{k}} K$ and $A_2 \widehat{\otimes}_{\bar{k}} K$ are isomorphic K -algebras, by [Sei69b, Theorem 5], A_1 and A_2 are isomorphic \bar{k} -algebras.

In particular, if $k = \bar{k}$, $x_1, x_2 \in X^{(0)}$ are geometrically analytically equivalent if and only if $\widehat{\mathcal{O}_{X,x_1}}$ and $\widehat{\mathcal{O}_{X,x_2}}$ are isomorphic k -algebras.

Example 3.12. Assume that X is an algebraic k -variety.

Then X itself is a MWAI closed subset of X (we call it the trivial MWAI closed subset) with maximal AI open subset the set of smooth points of X .

In case X is smooth, X is the only MWAI closed subset of X .

In case X has only isolated singularities, the non-trivial MWAI closed subsets are the subsets $\{x\}$ where x is a singular point.

Example 3.13. Assume that $\text{char}(k) = 0$ and $X = \text{Spec}(k[X_1, X_2, X_3]/\langle X_1^2 - X_3 X_2^2 \rangle)$ (the Whitney umbrella), whose singular locus is the line $L := \{X_1 = X_2 = 0\}$. Then the non-trivial MWAI closed subsets are $\{(0, 0, 0)\}$ and L (with maximal AI open subset the complement of the origin).

Indeed, for any $x \in L^{(0)} \setminus \{(0, 0, 0)\}$ and any embedding $\iota: \kappa(x) \hookrightarrow \bar{k}$ one has by Lemma 3.5

$$\widehat{\mathcal{O}_{X,x}} \widehat{\mathfrak{O}_{\kappa(x)}} \bar{k}^{(i)} \xrightarrow{\sim} \bar{k}[[X_1, X_2, X_3]] / \langle X_1^2 - X_2^2(X_3 + \alpha) \rangle$$

where $\alpha \in \bar{k}$, $\alpha \neq 0$. Let $\beta \in \bar{k}$ such that $\beta^2 = 0$ and $\sqrt{X_3 + \alpha}$ the unique element of $\bar{k}[[X_3]]$ squaring to $X_3 + \alpha$ and with constant term β . Then the change of coordinates

$$X_1^* = \frac{X_1}{\beta}, \quad X_2^* = \frac{X_2}{\sqrt{X_3 + \alpha}}, \quad X_3^* = X_3$$

defines an isomorphism

$$\frac{\bar{k}[[X_1, X_2, X_3]]}{\langle X_2^2 - X_1^2(X_3 + \alpha) \rangle} \xrightarrow{\sim} \frac{\bar{k}[[X_1, X_2]]}{\langle X_2^2 - \beta^2 \cdot X_1^2 \rangle} [[X_3]].$$

Thus $\widehat{\mathcal{O}_{X,x}} \widehat{\mathfrak{O}_{\kappa(x)}} \bar{k}^{(i)}$ is isomorphic to $\frac{\bar{k}[[X_1, X_2]]}{\langle X_1^2 X_2 \rangle} [[X_3]]$ (in particular x is maximally 1-cylindrical).

On the other hand, if $x = (0, 0, 0)$,

$$\widehat{\mathcal{O}_{X,x}} \xrightarrow{\sim} k[[X_1, X_2, X_3]] / \langle X_1^2 - X_2^2 X_3 \rangle$$

is not cylindrical (see [BS18, Example 5.12]).

Example 3.14. Assume now that $\text{char}(k) = p$ and consider the p -Whitney umbrella $X = \text{Spec}(k[X_1, X_2, X_3]/\langle X_1^p - X_3 X_2^p \rangle)$. Then the only non-trivial MWAI closed subset is the line $L = \{X_1 = X_2 = 0\}$. In fact, for any $x_1, x_2 \in L^{(0)}$, it follows from [CH22, Example 2.6] that $\widehat{\mathcal{O}_{X,x_1}} \widehat{\mathfrak{O}_{\kappa(x_1)}} \bar{k}^{(t_1)} \xrightarrow{\sim} \widehat{\mathcal{O}_{X,x_2}} \widehat{\mathfrak{O}_{\kappa(x_2)}} \bar{k}^{(t_2)}$ for any choice of t_1, t_2 .

Definition 3.15. Let k be a perfect field and X be a k -scheme of finite type. Let F be an irreducible closed subset of X and η_F be its generic point. Then F is said to be *strongly analytically isosingular* (in short SAI) if there exists a non-empty Zariski open subset U of F such that the following holds: there exists a k -section s_F of $\widehat{\mathcal{O}_{X,\eta_F}}$ such that for any $x \in U^{(0)}$, there exists a common extension K of $\kappa(x)$ and $\kappa(\eta_F)$ such that $\widehat{\mathcal{O}_{X,x}} \widehat{\mathfrak{O}_{\kappa(x)}} K$ and $(\widehat{\mathcal{O}_{X,\eta_F}})^{(s_F)} \widehat{\mathfrak{O}_{\kappa(\eta_F)}} K[[t_1, \dots, t_{\dim(F)}]]$ are isomorphic K -algebras.

Remark 3.16. Since k is perfect, by lemma 3.7, one obtains an equivalent notion by requiring that $\widehat{\mathcal{O}_{X,x}} \widehat{\mathfrak{O}_{\kappa(x)}} K$ and $(\widehat{\mathcal{O}_{X,\eta_F}})^{(s_F)} \widehat{\mathfrak{O}_{\kappa(\eta_F)}} K[[t_1, \dots, t_{\dim(F)}]]$ be only isomorphic k -algebras.

The notion of a maximal SAI closed subset is defined similarly to the case of WAI subsets. The following lemma is a direct consequence of the definitions.

Lemma 3.17. *Keep the above notation. If F is SAI, then F is WAI.*

Example 3.18. If X is an algebraic k -variety, any closed subset F meeting the smooth locus of X is SAI, and X itself is MSAI.

Example 3.19. Let us get back to Example 3.13 and show that L is SAI (thus MSAI). Since η_L corresponds to the prime ideal $\langle X_1, X_2 \rangle$, of $\frac{k[[X_1, X_2, X_3]]}{\langle X_1^2 - X_3 X_2^2 \rangle}$, the quotient morphism

$\widehat{\mathcal{O}_{X,\eta_L}} \rightarrow \kappa(\eta_L)$ induces an isomorphism $k(X_3) \xrightarrow{\sim} \kappa(\eta_L)$, thus a natural k -section s of $\widehat{\mathcal{O}_{X,\eta_L}}$ and one has a $k(X_3)$ -algebra isomorphism

$$\widehat{\mathcal{O}_{X,\eta_L}}^{(s)} \xrightarrow{\sim} \frac{k(X_3)[[X_1, X_2]]}{\langle X_1^2 - X_3 X_2^2 \rangle}.$$

Thus for any extension $K/k(X_3)$ such that X_3 is a square in K , one has

$$\widehat{\mathcal{O}_{X,\eta_L}}^{(s)} \widehat{\otimes}_{k(X_3)} K \xrightarrow{\sim} \frac{K[[t_1, t_2]]}{\langle t_1 t_2 \rangle}. \quad (3.1)$$

Therefore, for any $x \in L^{(0)} \setminus \{(0, 0, 0)\}$ and any extension K of $\bar{k}(X_3)[v]/\langle v^2 - X_3 \rangle$, one has by Example 3.13 and for any $\iota: \kappa(x) \hookrightarrow \bar{k}$ an isomorphism

$$(\widehat{\mathcal{O}_{X,\eta_L}}^{(s)} \widehat{\otimes}_{k(X_3)} K)[[v]] \xrightarrow{\sim} \widehat{\mathcal{O}_{X,x}} \widehat{\otimes}_{\kappa(x)} K^{(i)}.$$

Isomorphism (3.1) also shows that η_L is differential; this may also be seen as a consequence of [Sei67, Theorem 5], see Example 2.25). Thus for any $x \in L^{(0)} \setminus \{(0, 0, 0)\}$, η_L is the maximal differential generalization of x .

Theorem 5.18 and corollary 5.20 below will in particular generalize the observations in Example 3.19. Namely, they will show that WAI and SAI turns out to be equivalent notions and give an interpretation of the maximal differential generalization of a closed point x as the generic point of the component through x of its (geometric) isosingular locus (to be discussed in the next subsection), as well as the generic point of the unique MWAI closed subset containing x .

3.20. The geometric isosingular locus of a closed point. We want to use the notion of geometrically analytically equivalent closed points to extend Chiu and Hauser's notion of isosingular locus, defined in [CH22] for closed points of an algebraic variety over an algebraically closed field, to the case where the base field is perfect, keeping in particular the property that such a locus is the set of closed points of a locally closed subset.

First we have to make some reminders on contact groups and their truncations, which play a pivotal role in Chiu and Hauser's argument. See [CH22] for more details and relevant facts.

Let e be a nonnegative integer, N be a positive integer, and $\mathbf{X} = (X_1, \dots, X_N)$ be a set of indeterminates. Let $\mathcal{V}_{N,e}$ be the affine k -space whose L -points, for any extension L/k , are in natural correspondence with the set of coefficients of elements of $L[[\mathbf{X}]]$ modulo $\langle \mathbf{X} \rangle^e$.

For any positive integer n , let $\mathcal{K}_{N,n,e}$ be the associated (truncated) contact group of order e ; this is a linear algebraic group over k , that naturally acts on $\mathcal{V}_{N,e}^n$, in such a way that for any extension L/k and any $(F_{1,1}, \dots, F_{1,n}), (F_{2,1}, \dots, F_{2,n}) \in \mathcal{V}_{N,e}^n(L)$, the L -algebras $L[[\mathbf{X}]]/\langle F_{1,1}(\mathbf{X}), \dots, F_{1,n}(\mathbf{X}) \rangle + \langle \mathbf{X} \rangle^e$ and $L[[\mathbf{X}]]/\langle F_{2,1}(\mathbf{X}), \dots, F_{2,n}(\mathbf{X}) \rangle + \langle \mathbf{X} \rangle^e$ are isomorphic if and only if there exists $g \in \mathcal{K}_{N,n,e}(L)$ such that $g \cdot (F_{1,1}, \dots, F_{1,n}) = (F_{2,1}, \dots, F_{2,n})$.

Let now \mathcal{V}_N be the affine k -space whose L -points, for any extension L/k , are in natural correspondence with the set of coefficients of elements of $L[[\mathbf{X}]]$. For any n -tuple $\mathbf{f} = (f_1, \dots, f_n)$ of elements of $k[[\mathbf{X}]]$, for any k -algebra A and any $\mathbf{a} \in A^n$, set $\text{ev}_{\mathbf{f}}(\mathbf{a}) := (f_i(\mathbf{X} + \mathbf{a}))_{1 \leq i \leq n}$. This defines a morphism $\text{ev}_{\mathbf{f}}: \mathbf{A}_k^n \rightarrow \mathcal{V}_N^n$.

For fixed N and n , note that $(\mathcal{V}_{N,e}^n)_e, (\mathcal{K}_{N,n,e})_e$ are inverse systems such that the action of $\mathcal{K}_{N,n,e}$ on $\mathcal{V}_{N,e}^n$ is compatible with the transition morphisms. In particular the k -group scheme $\mathcal{K}_{N,n} := \varprojlim \mathcal{K}_{N,n,e}$ acts on $\varprojlim \mathcal{V}_{N,e}^n = \mathcal{V}_N$. By [CH22, Lemma 1.5] and Lemma 3.5, this action enjoys the following properties.

Lemma 3.21. (1) *Let K be an extension of k . Let \mathbf{f} and \mathbf{g} be n -tuples of elements of $K[[\mathbf{X}]] = \mathcal{V}_N(K)$. Then $\frac{K[[\mathbf{X}]]}{\langle \mathbf{f} \rangle}$ and $\frac{K[[\mathbf{X}]]}{\langle \mathbf{g} \rangle}$ are isomorphic if and only if there exists $\kappa \in \mathcal{K}_{N,n}(K)$ such that $\kappa \cdot \mathbf{f} = \mathbf{g}$.*

- (2) Let \mathbf{f} be an n -tuple of elements of $k[[\mathbf{X}]]$ and $X := \text{Spec}(k[[\mathbf{X}]]/\langle \mathbf{f} \rangle)$. Let $x_1, x_2 \in X^{(0)}$ such that $\kappa(x_1)$ and $\kappa(x_2)$ are separable extensions of k . Then for any common extension K of $\kappa(x_1)$ and $\kappa(x_2)$ (inducing elements of K^n still denoted by x_1 and x_2), $\widehat{\mathcal{O}_{X,x_1}} \widehat{\otimes}_{\kappa(x_1)} K \xrightarrow{\sim} \widehat{\mathcal{O}_{X,x_2}} \widehat{\otimes}_{\kappa(x_2)} K$ if and only if there exists $\kappa \in \mathcal{K}_{N,n}(K)$ such that $\kappa \cdot \text{ev}_{\mathbf{f}}(x_1) = \text{ev}_{\mathbf{f}}(x_2)$.

Remark 3.22. Let $\mathbf{t} = (t_1, \dots, t_d)$ be a set of indeterminates. Let $f_1, f_2 \in \mathcal{V}_N(k[[\mathbf{t}]])$. Assume that there exists $\kappa \in \mathcal{K}_N(k[[\mathbf{t}]])$ such that $\kappa \cdot f_1 = f_2$. Then the complete local k -algebras $k[[\mathbf{X}, \mathbf{t}]]/\langle f_1 \rangle$ and $k[[\mathbf{X}, \mathbf{t}]]/\langle f_2 \rangle$ are isomorphic. A similar remark holds for $\mathcal{K}_{N,n}$ for any positive integer n .

As pointed out in [CH22, Remark 1.1], the strict extension of the notion of isosingular locus defined there to the case where the base field is not algebraically closed is not well-behaved. Here one adopts the following definition, that coincide with the one in [CH22] in case $\bar{k} = k$ (see Remark 3.11).

Definition 3.23. Let k be a perfect field, X be a k -scheme of finite type and x be a closed point of X . The (geometric) isosingular locus of x , denoted by $\text{Iso}(X, x)$, is the set of elements $x' \in X^{(0)}$ such that x and x' are geometrically analytically equivalent.

Using a Galois descent argument, one obtains the following generalization of [CH22, Proposition 1.9].

Proposition 3.24. Let k be a perfect field, X be a k -scheme of finite type and x a closed point of X . Then there exists a locally closed subset Y of X such that $Y^{(0)} = \text{Iso}(X, x)$.

Proof. We may assume that $X = \text{Spec}(k[[\mathbf{X}]]/\langle \mathbf{f} \rangle)$ is affine, where $\mathbf{X} = (X_1, \dots, X_N)$ and \mathbf{f} is a n -tuple of elements of $k[[\mathbf{X}]]$. Let $\{x'_i\}_{i \in I}$ be the finite set of those elements of $X(\bar{k})$ lying over x . By [CH22, Corollary 1.8], there exists a positive integer e such that for any $y' \in X(\bar{k})$ and any $i \in I$ one has $\widehat{\mathcal{O}_{X(\bar{k}), x'_i}} \xrightarrow{\sim} \widehat{\mathcal{O}_{X(\bar{k}), y'}}$ if and only if

$$\exists \kappa \in \mathcal{K}_{N,n,e}(\bar{k}), \quad \kappa \cdot \text{ev}_{\mathbf{f}}(x'_i) = \text{ev}_{\mathbf{f}}(y'). \quad (3.2)$$

On the other hand, by the definition of $\text{Iso}(X, x)$, for any $y \in X^{(0)}$, one has $y \in \text{Iso}(X, x)$ if and only if there exists $y' \in X(\bar{k})$ lying over y and $i \in I$ such that (3.2) holds. Note that in this case, for any $\gamma \in \text{Gal}(\bar{k}/k)$, (3.2) also holds for $\gamma \cdot x'_i$ and $\gamma \cdot y'$. Thus the set $\text{Iso}(X, x)(\bar{k})$ of elements of $X(\bar{k})$ lying over an element of $\text{Iso}(X, x)$ is the pullback by $\text{ev}_{\mathbf{f}}$ of $\cup_{i \in I} \mathcal{K}_{N,n,e}(\bar{k}) \cdot \text{ev}_{\mathbf{f}}(x'_i)$. Since $\{\text{ev}_{\mathbf{f}}(x'_i)\}_{i \in I}$ is the set of elements of $\mathcal{V}_N^n(\bar{k})$ lying over $\text{ev}_{\mathbf{f}}(x)$, by lemma 3.26 below, the above union of orbits equals $Z(\bar{k})$ where Z is a locally closed subset of $\mathcal{V}_{N,e}^n$. Setting $Y := \text{ev}_{\mathbf{f}}^{-1}(Z)$, one thus has $\text{Iso}(X, x) = Y^{(0)}$. \square

Remark 3.25. The above argument also shows that x_1 and x_2 are geometrically analytically equivalent if and only if for any k -embedding $\iota_1 : \kappa(x_1) \hookrightarrow \bar{k}$, one may find a k -embedding $\iota_2 : \kappa(x_2) \hookrightarrow \bar{k}$ such that $\widehat{\mathcal{O}_{X,x_1}} \widehat{\otimes}_{\kappa(x_1)} \bar{k}^{(\iota_1)}$ and $\widehat{\mathcal{O}_{X,x_2}} \widehat{\otimes}_{\kappa(x_2)} \bar{k}^{(\iota_2)}$ are isomorphic \bar{k} -algebras (see Remark 3.11).

However, in general, such an isomorphism will not exist for any choice (ι_1, ι_2) of such k -embeddings. Indeed, consider e.g., $k = \mathbf{R}$ and the celebrated Whitney example $X = \text{Spec}\left(\frac{k[[X_1, X_2, X_3]]}{X_1 X_2 (X_1 + X_2)(X_1 + X_3 X_2)}\right)$. For $x_1 = x_2$ the closed point given by the maximal ideal $\langle X_1, X_2, X_3^2 + 1 \rangle$, the completions corresponding to the two different \mathbf{R} -embeddings $\kappa(x_1) \rightarrow \mathbf{C}$ are $\frac{\mathbf{C}[[X_1, X_2, X_3]]}{X_1 X_2 (X_1 + X_2)(X_1 + (X_3 + i) X_2)}$ and $\frac{\mathbf{C}[[X_1, X_2, X_3]]}{X_1 X_2 (X_1 + X_2)(X_1 + (X_3 - i) X_2)}$ which are not isomorphic \mathbf{C} -algebras by Whitney's cross-ratio argument.

Lemma 3.26. Let k be a perfect field, G be a smooth connected algebraic group over k acting on an affine k -scheme of finite type X . Let $x \in X^{(0)}$ and $\{x'_i\}_{i \in I}$ be the finite set of the points of $X(\bar{k})$ lying over x .

Then there exists a locally closed subset Y of X such that $\cup_{i \in I} G(\bar{k}) \cdot x'_i = Y(\bar{k})$.

Proof. Let $\Gamma = \text{Gal}(\bar{k}/k)$ be the absolute Galois group of k . Take $J \subset I$ such that the orbits $\{G(\bar{k}).x'_i\}_{i \in J}$ are pairwise disjoint. For $i \in J$, let F_i be the closure of $G(\bar{k}).x'_i$ in $X(\bar{k})$. In particular $U_i := G(\bar{k}).x'_i$ is open in F_i . Since Γ acts transitively on $\{U_i\}_{i \in J}$, $\dim(F_i)$ is constant. Let $i, j \in J$, $i \neq j$. Since $F_i \setminus U_i$ is a union of $G(\bar{k})$ -orbit, if $U_j \cap F_i$ were non empty, one would have $U_j \subset F_i \setminus U_i$. Since $\dim(F_i) = \dim(F_j)$ this is impossible. Thus $U_j \cap F_i = \emptyset$.

It follows that $\mathcal{U} := \cup_{i \in I} U_i$ is open and dense in $\mathcal{F} = \cup_{i \in I} F_i$. Thus since \mathcal{U} is Γ -stable, \mathcal{F} is also Γ -stable, as well as $\mathcal{F} \setminus \mathcal{U}$. This implies that there exist closed subsets F_1 and F_2 of X such that $F_1(\bar{k}) = \mathcal{F}$ and $F_2(\bar{k}) = \mathcal{F} \setminus \mathcal{U}$ (see e.g., [Mil15, Chapter 16]). Therefore $Y := F_1 \setminus F_2$ has the required properties. \square

Notation 3.27. Using a notation strictly similar to [CH22], for $x \in X^{(0)}$ one denotes by $X^{(x)}$ the unique locally closed subset of X such that $(X^{(x)})^{(0)} = \text{Iso}(X, x)$, and one endows it with its reduced subscheme structure.

Remark 3.28. The proof of Lemma 3.26 shows that for a suitable subset $\{x_j\}_{j \in J}$ of the set of elements of $X(\bar{k})$ lying over x , the underlying topological space of $X^{(x)} \otimes_k \bar{k}$ is the disjoint union of the locally closed subsets $X_{\bar{k}}^{(x_j)}$.

One has the following extensions of [CH22, Lemmas 1.13 & 2.5 & Proposition 1.14].

Lemma 3.29. *Let k be a perfect field and X be a k -scheme of finite type.*

- (1) *Let Y be a k -scheme of finite type, $f: X \rightarrow Y$ be an étale morphism and $x \in X^{(0)}$. Then $f^{-1}(\text{Iso}(Y, f(x))) = \text{Iso}(X, x)$. Moreover the induced morphism $X^{(x)} \rightarrow Y^{(f(x))}$ is étale.*
- (2) *Let $x \in X^{(0)}$. If x is r -cylindrical, then $\dim_x(X^{(x)}) \geq r$.*
- (3) *$X^{(x)}$ is smooth over k .*

Remark 3.30. By Corollary 2.28 (in case $\text{char}(k) = 0$) and Remark 4.15 (in case $\text{char}(k) > 0$) the formulation of the second assertion is indeed equivalent to the one of [CH22, Lemma 2.5].

Proof. If $f: X \rightarrow Y$ be an étale morphism and $x \in X^{(0)}$, then by Proposition 5.2, $\widehat{\mathcal{O}_{X,x}}$ and $\widehat{\mathcal{O}_{Y,f(x)}} \widehat{\otimes}_{\kappa(f(x))} \kappa(x)$ are isomorphic $\kappa(x)$ -algebras. That being said, the first and the second assertions can now be proved exactly as the corresponding statements in [CH22].

For the third assertion, one may reduce to the case $k = \bar{k}$ (and thus to [CH22, Proposition 1.14]) using Remark 3.28. \square

Notation 3.31. For any $x \in X^{(0)}$, by Assertion 3 of Lemma 3.29, the closure of $X^{(x)}$ in X has a unique component passing through x . We denote this component by $\mathcal{G}(X, x)$.

Remark 3.32. By Assertion 2 of Lemma 3.29, if x is r -cylindrical, then $\dim(\mathcal{G}(X, x)) \geq r$.

4. ISOSINGULARITY IMPLIES CYLINDRICITY

Our main goal in this section is to show the following result.

Theorem 4.1. *Let k be a perfect field and X be a k -scheme of finite type. Let Z be a locally closed subset of X such that any $x_1, x_2 \in Z^{(0)}$ are geometrically analytically equivalent.*

In case $\text{char}(k) > 0$, assume moreover that $k = \bar{k}$ and there exist $z \in Z(k)$ that is a smooth point of Z (endowed with its reduced scheme structure), an open affine subset U of X containing z and an isomorphism $U \xrightarrow{\sim} \text{Spec}(k[X_1, \dots, X_N]/\langle f_1, \dots, f_n \rangle)$ mapping z to the origin and such that the orbit map of $\mathcal{X}_{N,n}$ through $\mathbf{f} = (f_1, \dots, f_n)$ is separable (see Definition 4.4 below).

Then any $x \in Z^{(0)}$ is $\dim(Z)$ -cylindrical.

Remark 4.2. Taking $Z = X^{(x)}$ and using Lemma 3.29 as well as Corollary 2.28 (in case $\text{char}(k) = 0$) and Remark 4.15 (in case $\text{char}(k) > 0$), the result recovers Theorem B of [CH22] in case $k = \bar{k}$ and $\text{char}(k) = 0$, and answers Question C of *op.cit.*.

Remark 4.3. In case $\text{char}(k) = 0$, the proof reduces to the case $k = \bar{k}$ using Lemma 2.26. If Lemma 2.26 happened to also hold for separable extensions in positive characteristic (see Remark 2.27), one could drop the assumption $\bar{k} = k$ in Theorem 4.1 in case k is a perfect field of positive characteristic.

Definition 4.4. Let k be a field, N a positive integers, $\mathbf{X} = (X_1, \dots, X_N)$ a set of N indeterminates. For any positive integer n and n -tuple $\mathbf{f} = (f_1, \dots, f_n)$ of elements of $k[\mathbf{X}]$, one says that the orbit map of the contact group $\mathcal{K}_{N,n}$ (see Subsection 3.20) through \mathbf{f} is separable if for any e large enough the orbit map $\begin{array}{ccc} \mathcal{K}_{N,n,e} & \longrightarrow & \mathcal{V}_{N,e} \\ \kappa & \longmapsto & \kappa \cdot \mathbf{f} \end{array}$ is separable.

The proof of Theorem 4.1 follows the general line of arguments in [CH22], but one uses a multivariate version of Hasse-Schmidt derivations instead of a mere family of regular derivations. In this framework, the basic idea is that, under the given assumptions, one may lift a formal parametrization of a smooth point z of Z along the action of the contact group, which in turn allows us to construct such a multivariate Hasse-Schmidt derivation locally at z , showing the claimed cylindricity by a generalization of the Lipman-Nagata-Zariski simplification lemma involving multivariate Hasse-Schmidt derivations. In the following, we first state the result allowing us to perform the aforementioned lifting. Then we state and prove an r -cylindricity criterion (Theorem 4.14) in terms of multivariate Hasse-Schmidt derivations, before proving Theorem 4.1. At the end of the section, we explain how very similar arguments allow us to recover a related result of Seidenberg in the context of algebroid geometry.

4.5. Wedge-transitivity of a limit of linear algebraic group actions. The following statement is an abstract version of a result used and proved in [CH22] (see Lemma 2.2 & 2.3 and their proofs therein; the same arguments allow us to show Lemma 4.6).

Lemma 4.6. *Let k be a field. Let $(\{G_n\}_{n \in \mathbf{N}}, \{\pi_{n',n}\}_{n,n' \in \mathbf{N}, n \leq n'})$ be an inverse system of linear algebraic groups over k and $(\{V_n\}_{n \in \mathbf{N}}, \{\tau_{n',n}\}_{n,n' \in \mathbf{N}, n \leq n'})$ be an inverse system of affine algebraic k -varieties. Let G be the affine group scheme over k defined by $G := \varprojlim G_n$ and V be the affine k -scheme defined by $V := \varprojlim V_n$. Assume that for any $n \in \mathbf{N}$ we are given an algebraic action $a_n: G_n \times V_n \rightarrow V_n$ of G_n over V_n compatible with the transition morphisms, thus inducing an algebraic action $a: G \times V \rightarrow V$*

Let $x \in V(k)$ and Y be the closure of the orbit $G \cdot x$, endowed with its reduced subscheme structure. Let $a_x: G \rightarrow Y$ be the morphism $g \mapsto g \cdot x$. For $n \in \mathbf{N}$, let $x_n = \tau_n(x)$, Y_n be the closure of the orbit $G_n \cdot x_n$, endowed with its reduced subscheme structure and $a_{x_n}: G_n \rightarrow Y_n$ be the morphism $g \mapsto g \cdot x_n$.

Assume that either:

- (1) *k is of characteristic zero;*
- (2) *or for n large enough G_n is smooth and the morphism a_{x_n} is separable.*

Then the morphism $\widehat{\mathcal{O}}_{Y,x} \rightarrow \widehat{\mathcal{O}}_{G,1_G}$ admits a retraction.

Remark 4.7. Let $\mathbf{t} = (t_1, \dots, t_r)$ be a r -tuple of indeterminates. For any k -scheme of finite type X , an element w of $X(k[[\mathbf{t}]])$ is usually called an (r) -wedge (an arc in case $r = 1$; see also Subsection 6.1). Its origin is the element of $X(k)$ obtained from w by the specialization $t_1 = \dots = t_r = 0$. Thus a straightforward consequence of the above lemma may be stated as follows: any r -wedge $\gamma \in Y(k[[\mathbf{t}]])$ with origin x may be lifted to a wedge $\tilde{\gamma} \in G(k[[\mathbf{t}]])$ with origin 1_G ; hence the title of the section.

4.8. Hasse-Schmidt multivariate derivations and cylindricity in arbitrary characteristic. As already pointed out, the proof of Theorem 4.1 relies in particular on a version of Lemma 2.19 valid in any characteristic and involving a multivariate version of the notion of Hasse-Schmidt derivation. The latter (which corresponds to the case $r = 1$ in the definition below) is by now very classical, whereas as far as we know, the multivariate version has not been much studied so far in the literature (see [HK16, NM18]).

Definition 4.9. Let A be a ring and r be a positive integer. A *Hasse-Schmidt r -derivation* on A is a morphism $D: A \rightarrow A[[t_1, \dots, t_r]]$ (where t_1, \dots, t_r are indeterminates) such that for any $a \in A$, $D(a)_{t_1=\dots=t_r=0} = a$. We denote by $\text{HS}_r(A)$ the set of Hasse-Schmidt r -derivations on A . If k is a field and A is a k -algebra, we denote by $\text{HS}_{r,k}(A)$ the set of Hasse-Schmidt r -derivations on A that are morphisms of k -algebras.

In case (A, \mathfrak{M}_A) is a local ring, an element $D = \text{Id}_A + \sum_{1 \leq i \leq r} D_i \cdot t_i + \dots$ of $\text{HS}_r(A)$ is said to be *regular* if there exist elements $x_1, \dots, x_r \in \mathfrak{M}_A$ such that

$$\det(D_j x_i) \notin \mathfrak{M}_A.$$

Remark 4.10. If $D = \text{Id}_A + \sum_{1 \leq i \leq r} D_i \cdot t_i + \dots$ is an element of $\text{HS}_r(A)$ (resp. $\text{HS}_{r,k}(A)$) then the D_i 's are elements of $\text{Der}(A)$ (resp. $\text{Der}_k(A)$) and the Hasse-Schmidt r -derivation D is regular if and only if D_1, \dots, D_r is a regular tuple in the sense of Definition 2.2. Moreover, if $x_1, \dots, x_r \in \mathfrak{M}_A$ are such that $\det(D_j x_i) \notin \mathfrak{M}_A$, then there exists $\tilde{D} = \text{Id}_A + \sum_{1 \leq i \leq r} \tilde{D}_i \cdot t_i + \dots \in \text{HS}_r(A)$ such that $(\tilde{D}_j x_i)$ is the identity matrix: just compose D on the target with the inverse of the A -automorphism of $A[[t_1, \dots, t_r]]$ mapping t_i to $D(x_i) - x_i$.

Example 4.11. In case A is a complete local ring, setting $\mathbf{t} = (t_1, \dots, t_r)$ and $\mathbf{u} = (u_1, \dots, u_r)$, the morphism $A[[\mathbf{u}]] \rightarrow A[[\mathbf{u}, \mathbf{t}]]$ mapping $F(\mathbf{u})$ to $F(\mathbf{u} + \mathbf{t})$ is a regular Hasse-Schmidt r -derivation on $A[[\mathbf{u}]]$, which is moreover an element of $\text{HS}_{r,k}(A)$ in case k is a field and A is a k -algebra.

Example 4.12. Let A be a ring containing \mathbf{Q} and D_1, \dots, D_r be a tuple of derivations on A . Then the map $\mathcal{D} = \mathcal{D}_{D_1, \dots, D_r}: A \rightarrow A[[t_1, \dots, t_r]]$ defined by

$$\mathcal{D}(a) = \sum_{(i_1, \dots, i_r) \in \mathbf{N}^r} \frac{(D_1^{(i_1)} \dots D_r^{(i_r)})(a)}{\prod_{1 \leq j \leq r} i_j!} \prod_{1 \leq j \leq r} t_j^{i_j}$$

is a Hasse-Schmidt r -derivation on A , called the formal flow associated with the tuple D_1, \dots, D_r . Indeed, additivity is clear and for $\mathbf{i} \in \mathbf{N}^r$ and $a, b \in A$, one has by the Leibniz rule

$$(D_1^{(i_1)} \dots D_r^{(i_r)})(ab) = D_1^{(i_1)} \dots D_{r-1}^{(i_{r-1})} \left(\sum_{\substack{\alpha_r, \beta_r \in \mathbf{N} \\ \alpha_r + \beta_r = i_r}} \frac{i_r!}{\alpha_r! \beta_r!} D_r^{(\alpha_r)} a D_r^{(\beta_r)} b \right)$$

thus by induction

$$\frac{(D_1^{(i_1)} \dots D_r^{(i_r)})(ab)}{i_1! \dots i_r!} = \sum_{\substack{\alpha, \beta \in \mathbf{N}^r \\ \alpha + \beta = \mathbf{i}}} \frac{1}{\alpha_1! \dots \alpha_r! \beta_1! \dots \beta_r!} (D_1^{(\alpha_1)} \dots D_r^{(\alpha_r)})(a) (D_1^{(\beta_1)} \dots D_r^{(\beta_r)})(b)$$

which shows that $\mathcal{D}(ab) = \mathcal{D}(a)\mathcal{D}(b)$.

Note that in case the derivations D_1, \dots, D_r do not pairwise commute, the associated formal flow will depend on the chosen ordering of the r derivations.

Remark 4.13. Let A be a ring. An ideal \mathcal{I} of A is said to be HS-differential if for any Hasse-Schmidt derivation $D = \sum_{j \geq 0} D_j \cdot t^j$ on A and any j , one has $D_j \mathcal{I} \subset \mathcal{I}$. If this holds, then it also holds for $\sqrt{\mathcal{I}}$ (see e.g., [Tra00, Corollary 2.3]). The version of Lemma 2.30 where differentiability is replaced by HS-differentiability holds in any characteristic. In the Noetherian case, this is a consequence of [BK72, Theorem 1], where it is observed that for any Hasse-Schmidt derivation D on A , the extension $\varphi_D: A[[t]] \rightarrow A[[t]]$ of D to

$A[[t]]$ mapping t to t is an automorphism. The same observation also allows to adapt the argument of the proof of Lemma 2.30 to this context.

Here is a generalization in arbitrary characteristic of the Lipman-Nagata-Zariski simplification lemma, using the notion of regular Hasse-Schmidt r -derivation. Such a generalization is well-known in the case $r = 1$ (see [BK72, Kua74, Ish76, BS19]).

Theorem 4.14. *Let (A, \mathfrak{m}) be a complete local ring.*

Then A is r -cylindrical if and only if there exists a regular Hasse-Schmidt r -derivation on A .

More precisely, assume that there exists a Hasse-Schmidt r -derivation

$$D = \text{Id}_A + \sum_{1 \leq i \leq r} D_i \cdot t_i + \dots : A \rightarrow A[[t_1, \dots, t_r]]$$

on A and $x_1, \dots, x_r \in \mathfrak{m}$ such that $\det(D_i x_j) \in A^\times$.

Then there exists a complete local ring B and an isomorphism of complete local rings $\varphi : B[[u_1, \dots, u_r]] \rightarrow A$ mapping u_i to x_i .

Remark 4.15. By Example 4.11, if A is moreover a k -algebra which is r -cylindrical over k , then there exists a regular element of $\text{HS}_{r,k}(A)$.

On the other hand, in case there exists a regular element of $\text{HS}_r(A)$, and keeping the notation of the statement of the theorem, the quotient morphism $A \rightarrow A/\sum A \cdot x_i$ induces an isomorphism $B \xrightarrow{\sim} A/\sum A \cdot x_i$. Therefore, going back to the case where A is a k -algebra, B is a sub k -algebra of A and φ is an isomorphism of k -algebras.

Proof. By Example 4.11, we only have to show that if there exists a regular Hasse-Schmidt r -derivation on A then A is r -cylindrical.

Let $D = \text{Id}_A + \sum_{1 \leq i \leq r} D_i \cdot t_i + \dots$ be a regular Hasse-Schmidt r -derivation on A and let $x_1, \dots, x_r \in \mathfrak{m}$ such that the matrix $(D_j x_i)$ is the identity matrix (see Remark 4.10)

Consider the ring morphism $\theta : A \rightarrow A$ obtained by composing D with the evaluations $t_1 = -x_1, \dots, t_r = -x_r$. Clearly $x_1, \dots, x_r \in \text{Ker}(\theta)$. On the other hand, by the very definition of θ , for any a , one has $\theta(a) - a \in \sum A \cdot x_i$. One infers that $\text{Ker}(\theta) = \sum A \cdot x_i$ and $A = \text{Ker}(\theta) + \text{Im}(\theta)$. Moreover, since $\theta(a) - a \in \text{Ker}(\theta)$ for any a , one has $\text{Ker}(\theta) \cap \text{Im}(\theta) = \{0\}$.

Thus, setting $B = \text{Im}(\theta)$, the quotient map $A \rightarrow A/\sum A \cdot x_i$ induces an isomorphism $B \xrightarrow{\sim} A/\sum A \cdot x_i$ and one has $A = B + \sum A \cdot x_i$. Thus the morphism of complete local rings $\varphi : B[[u_1, \dots, u_r]] \rightarrow A$ inducing the identity on B and mapping u_i to x_i is onto.

Moreover, for any homogeneous element $F_d \in B[[u_1, \dots, u_r]]$ of degree d , one has $\varphi(F_d) \in (\sum A \cdot x_i)^d$. Therefore, in order to show that φ is injective, it suffices to show the following: let $F_d \in B[[u_1, \dots, u_r]]$ be an homogeneous element of degree d such that $\varphi(F_d) \in (\sum A \cdot x_i)^{d+1}$; then $F_d = 0$.

But since $D(x_i) = x_i + t_i$ for $1 \leq i \leq r$, one has in $A[[t_1, \dots, t_r]]$ the relation

$$D[(\sum A \cdot x_i)^{d+1}] \subset \langle t_1, \dots, t_r \rangle^{d+1} + \langle x_1, \dots, x_r \rangle.$$

On the other hand, for any $F \in B[[u_1, \dots, u_r]]$, one has

$$D\varphi(F) = F(t_1, \dots, t_r) \pmod{\langle x_1, \dots, x_r \rangle}.$$

Hence if $\varphi(F_d) \in (\sum A \cdot x_i)^{d+1}$, one has

$$F_d(t_1, \dots, t_r) \in \langle t_1, \dots, t_r \rangle^{d+1} + \langle x_1, \dots, x_r \rangle.$$

In particular, all the coefficients of $F_d(t_1, \dots, t_r)$ lie in $B \cap \sum A \cdot x_i = \{0\}$, and $F_d = 0$. \square

Corollary 4.16. *Corollary 2.28 also holds in case $\text{char}(k) > 0$.*

Proof. If (A, \mathfrak{m}_A) is any Noetherian local k -algebra and $D = \text{Id}_{\hat{A}} + \sum_{1 \leq j \leq r} D_j \cdot t_j + \dots$ is a regular element of $\text{HS}_{r,k}(A)$, by continuity of the derivations D_i 's, there exist elements $x_1, \dots, x_r \in \mathfrak{m}_A$ that $\det(D_i(x_j)) \notin \hat{\mathfrak{m}}_A$. One may now conclude as in the proof of Corollary 2.28. \square

Example 4.17. Let k be a field of characteristic $p > 0$, s a transcendental element over k and $A = \frac{k(s)[[X_1, X_2]]}{\langle X_1^p - s \cdot X_2^p \rangle}$; note that A is isomorphic to the formal neighborhood of the generic point of the singular locus of the Whitney p -umbrella (see Example 3.14). Then for any extension K/k containing a p -th root of s , a linear change of variables shows that $A \widehat{\otimes}_{k(w)} K$ is isomorphic to $K[[X_1, X_2]]/\langle X_1^p \rangle$ and is thus 1-cylindrical.

Yet A itself is not 1-cylindrical. In fact, denoting by \mathfrak{m}_A the maximal ideal of A , let us show the stronger statement that for any Hasse-Schmidt derivation $D: A \rightarrow A[[t]]$, one has $D\mathfrak{m}_A \subset \mathfrak{m}_A[[t]]$. Let x_1, x_2 be the images of X_1, X_2 in A and $\pi: A \rightarrow k(s)[[t]]$ be the composition of D with the morphism $A[[t]] \rightarrow A/\mathfrak{M}_A[[t]]$ of coefficientwise reduction. Set $x_i(s, t) = \pi(x_i)$. Since $x_1(s, t)^p = s \cdot x_2(s, t)^p$, either $x_1(s, t) = x_2(s, t) = 0$ (and then our statement is proved) or $\text{ord}_t(x_1(s, t)) = \text{ord}_t(x_2(s, t)) =: n$. In the latter case, write $x_i(s, t) = x_{i,n}(s) \cdot t^n + \dots$ where $x_{i,n}(s) \in k(s)^\times$. Then in $k(s)$ one has the relation $x_{1,n}(s)^p = s \cdot x_{2,n}(s)^p$ which is impossible.

4.18. Proof of Theorem 4.1. One may assume that X is affine, say $X = \text{Spec}(k[X_1, \dots, X_N]/\langle f_1, \dots, f_n \rangle)$. Moreover, in case $\text{char}(k) = 0$, we shall reduce to the case where $k = \bar{k}$. Let $X' := X \otimes_k \bar{k}$, $Z' := Z \otimes_k \bar{k}$, and $x' \in Z'(\bar{k})$. Then x' induces an element x_K of $X(K)$ contained in $Z(K)$ (for a certain finite algebraic extension K/k) and an element $x \in X^{(0)}$ contained in Z . Denote by α_i the image of X_i in K by $x_K^\# : k[\mathbf{X}]/\langle f_1, \dots, f_n \rangle \rightarrow K$. Then by [Sei73, Lemma, p.29], one has

$$\widehat{\mathcal{O}_{X,x} \otimes_{k(x)} \bar{k}} \xrightarrow{\sim} \bar{k}[[\mathbf{X}]]/\langle f_j((X_i + \alpha_i))_{1 \leq j \leq n} \rangle \xrightarrow{\sim} \widehat{\mathcal{O}_{X',x'}}.$$

Thus by Lemma 2.26, one may assume that $k = \bar{k}$ in case $\text{char}(k) = 0$.

We moreover may assume that the origin \mathfrak{o} of \mathbf{A}_k^N lies on Z and is a smooth point of Z (equipped with its reduced scheme structure), and, in case $\text{char}(k) > 0$, that the orbit map of $\mathcal{X}_{N,n}$ through $\mathbf{f} = (f_1, \dots, f_n)$ is separable.

Let $d = \dim(Z)$, $\mathbf{t} = (t_1, \dots, t_d)$ a d -tuple of indeterminates and $\mathbf{x}(\mathbf{t}) = (x_i(\mathbf{t}))_{1 \leq i \leq N} \in (\langle \mathbf{t} \rangle \cdot k[[\mathbf{t}]])^N$ be a formal parametrization of Z at the origin, corresponding to a choice of étale coordinates z_1, \dots, z_d of Z at the origin inducing an isomorphism $k[[\mathbf{t}]] \xrightarrow{\sim} \widehat{\mathcal{O}_{Z,\mathfrak{o}}}$. In particular, denoting by $p^\# : \mathcal{O}_X(X) \rightarrow k[[\mathbf{t}]]$ the corresponding morphism, one has

$$\det(\partial_{t_i} p^\#(z_j)) \in k[[\mathbf{t}]]^\times. \quad (4.1)$$

Let $Y \subset (\mathcal{V}_N)^n$ be the closure of the orbit $\mathcal{X}_{N,n} \cdot \mathbf{f}$. By assumption and Lemma 3.21, the set $\text{ev}_{\mathbf{f}}(Z(k))$ is contained in Y , thus, since k is algebraically closed, $\text{ev}_{\mathbf{f}}(Z)$ is contained in Y . Therefore the element $w \in (\mathcal{V}_N)^n(k[[\mathbf{t}]])$ defined by the composition of

$$\widehat{\text{ev}}_{\mathbf{f}} : \widehat{\mathcal{O}_{Z,\mathfrak{o}}} \rightarrow \widehat{\mathcal{O}_{(\mathcal{V}_N)^n, \mathbf{f}}}$$

with the chosen isomorphism $k[[\mathbf{t}]] \xrightarrow{\sim} \widehat{\mathcal{O}_{Z,\mathfrak{o}}}$ is an element of $Y(k[[\mathbf{t}]])$. Note that w corresponds to the element of $(\mathcal{V}_N)^n(k[[\mathbf{t}]])$ defined by $\mathbf{f}(\mathbf{X} + \mathbf{x}(\mathbf{t}))$.

Thus, applying Lemma 4.6 (and Remark 4.7) one obtains the existence of $\tilde{w} \in \mathcal{X}_{N,n}(k[[\mathbf{t}]])$ such that the element of $\mathcal{X}_{N,n}(k)$ induced by mapping \mathbf{t} to 0 is $1_{\mathcal{X}_{N,n}}$ and

$$\mathbf{f}(\mathbf{X} + \mathbf{x}(\mathbf{t})) = \tilde{w} \cdot \mathbf{f}(\mathbf{X}).$$

That means that there exists a $n \times n$ matrix $A(\mathbf{t})$ with coefficient in $k[[\mathbf{X}, \mathbf{t}]]$ and such that $A(\mathbf{t}) = I_n \pmod{\langle \mathbf{t} \rangle \cdot k[[\mathbf{X}, \mathbf{t}]]}$ and a N -tuple $\boldsymbol{\varphi}(\mathbf{t})$ of elements of $\langle \mathbf{X} \rangle \cdot k[[\mathbf{X}, \mathbf{t}]]$ such that $\boldsymbol{\varphi}(\mathbf{t}) = 0 \pmod{\langle \mathbf{t} \rangle \cdot k[[\mathbf{X}, \mathbf{t}]]}$ satisfying

$$A(\mathbf{t})\mathbf{f}(\mathbf{X} + \mathbf{x}(\mathbf{t})) + \boldsymbol{\varphi}(\mathbf{t}) = \mathbf{f}(\mathbf{X})$$

Multiplying by the adjugate matrix $A(\mathbf{t})^{\text{ad}}$ of $A(\mathbf{t})$ yields

$$\mathbf{f}(\mathbf{X} + \mathbf{x}(\mathbf{t})) + \boldsymbol{\varphi}(\mathbf{t}) = \det(A(\mathbf{t}))^{-1} \cdot A(\mathbf{t})^{\text{ad}} \cdot \mathbf{f}(\mathbf{X}) \quad (4.2)$$

Now consider the following Hasse-Schmidt d -derivation on $k[[\mathbf{X}]]$:

$$D: g \mapsto g(\mathbf{X} + \mathbf{x}(\mathbf{t}) + \boldsymbol{\varphi}(\mathbf{t})).$$

By (4.2), for $1 \leq i \leq n$, one has $D(f_i) \in \langle \mathbf{f} \rangle \cdot k[[\mathbf{X}, \mathbf{t}]]$. Thus D defines a Hasse-Schmidt d -derivation on $k[[\mathbf{X}]]/\langle \mathbf{f} \rangle$.

By the Taylor formula and since $\boldsymbol{\varphi}(\mathbf{t}) \in \langle \mathbf{X} \rangle \cdot k[[\mathbf{X}, \mathbf{t}]]$, one has

$$D(g) = p^\#(g) \pmod{\langle \mathbf{X} \rangle \cdot k[[\mathbf{X}, \mathbf{t}]]}.$$

Therefore (4.1) shows that

$$\det(\partial_{t_j} D(z_j)) \in k^\times + \langle \mathbf{X}, \mathbf{t} \rangle \cdot k[[\mathbf{X}, \mathbf{t}]].$$

Thus D is a regular Hasse-Schmidt d -derivation of $k[[\mathbf{X}]]/\langle \mathbf{f} \rangle$. By Theorem 4.14, we are done.

4.19. Isosingularity and cylindricity in an algebroid context. It is interesting to note that the wedge-transitivity of the action of the contact group allows to easily recover the main result of [Sei69a], that is another instance of a situation where isosingularity implies cylindricity (see also [HM89, Remark (b), p. 834]). More precisely, Theorem 4.20 below for $r = d$ recovers [Sei69a, Theorem 3]. Geometrically, it could be interpreted as follows: let X be an algebroid variety and $\pi: X \rightarrow W \times W'$ be a morphism to a product of formal disks of respective dimension r and $d - r$, admitting a section; if the generic fiber over W' is geometrically analytically equivalent to the special fiber, then X is isomorphic to the product of the special fiber with W . For the sake of simplicity, we assume that the base field has characteristic zero but a suitable extra separability assumption on the action of the contact group would allow to also deal with the positive characteristic case.

Theorem 4.20. *Let k be a field of characteristic zero. Let (A, \mathfrak{m}_A) be a complete local Noetherian k -algebra with residue field k -isomorphic to k . Let \mathfrak{n} be a prime ideal of A such that A/\mathfrak{n} is smooth of dimension d , and a_1, \dots, a_d a system of regular parameters at \mathfrak{n} ; in other words, the morphism of complete local k -algebras $k[[t_1, \dots, t_d]] \rightarrow A$ mapping t_i to a_i induces an isomorphism $k[[t_1, \dots, t_d]] \xrightarrow{\sim} A/\mathfrak{n}$. Note that this defines an isomorphism $k((t_1, \dots, t_d)) \xrightarrow{\sim} \kappa_{\mathfrak{n}}$ and a section s of $\kappa_{\mathfrak{n}}$.*

Assume that there exists an extension K of $k((t_1, \dots, t_d))$ such that the K -algebras $A/\langle a_1, \dots, a_r \rangle \widehat{\otimes}_k K$ and $\widehat{A}_{\mathfrak{n}}/\langle a_1 - s(a_1), \dots, a_r - s(a_r) \rangle \widehat{\otimes}_{\kappa_{\mathfrak{n}}} K$ are isomorphic.

Then A is r -cylindrical, more precisely A is isomorphic to $\frac{A}{\langle a_1, \dots, a_r \rangle}[[t_1, \dots, t_r]]$.

Proof. One may write $A = k[[\mathbf{X}, \mathbf{Y}]]/\mathcal{I}$ $\mathbf{X} = (X_1, \dots, X_N)$, $\mathbf{Y} = (Y_1, \dots, Y_r)$, in such a way that there exists an isomorphism $A/\mathfrak{n} \xrightarrow{\sim} k[[t_1, \dots, t_d]]$ mapping a_i to t_i and such that for $1 \leq i \leq r$ one has $a_i = Y_i \pmod{\mathcal{I}}$. The assumption is then translated as: the complete local K -algebras

$$\frac{K[[\mathbf{X}]]}{\langle \mathbf{f}(\mathbf{X}, 0) \rangle_{f \in \mathcal{I}}} \quad \text{and} \quad \frac{K[[\mathbf{X}]]}{\langle \mathbf{f}(\mathbf{X} + \mathbf{x}(\mathbf{t}), t_1, \dots, t_r) \rangle_{f \in \mathcal{I}}}$$

are isomorphic. One may now apply Proposition 4.21 below in order to conclude. \square

Proposition 4.21. *Let k be a field of characteristic zero. Let N, r, d be positive integers, $\mathbf{X} = (X_1, \dots, X_N)$, $\mathbf{Y} = (Y_1, \dots, Y_r)$ be indeterminates, and \mathcal{I} be a proper ideal of $k[[\mathbf{X}, \mathbf{Y}]]$. Let $\mathbf{t} = (t_1, \dots, t_d)$, $\mathbf{x}(\mathbf{t}) \in (\langle \mathbf{t} \rangle \cdot k[[\mathbf{t}]])^N$ and $\mathbf{y}(\mathbf{t}) \in (\langle \mathbf{t} \rangle \cdot k[[\mathbf{t}]])^r$ such that*

$$\forall f \in \mathcal{I}, \quad f(\mathbf{x}(\mathbf{t}), \mathbf{y}(\mathbf{t})) = 0.$$

Assume that there exists an extension K of $k((\mathbf{t}))$ such that the K -algebras $\frac{K[[\mathbf{X}]]}{\langle \mathbf{f}(\mathbf{X}, 0) \rangle_{f \in \mathcal{I}}}$ and

$\frac{K[[\mathbf{X}]]}{\langle \mathbf{f}(\mathbf{X} + \mathbf{x}(\mathbf{t}), \mathbf{y}(\mathbf{t})) \rangle_{f \in \mathcal{I}}}$ are isomorphic.

Then:

- (1) *The k -algebras $\frac{k[[\mathbf{X}, \mathbf{t}]]}{\langle \mathbf{f}(\mathbf{X}, 0) \rangle_{f \in \mathcal{I}}}$ and $\frac{k[[\mathbf{X}, \mathbf{t}]]}{\langle \mathbf{f}(\mathbf{X} + \mathbf{x}(\mathbf{t}), \mathbf{z}(\mathbf{t})) \rangle_{f \in \mathcal{I}}}$ are isomorphic.*

(2) Assuming moreover that $\det(\partial_{t_i} y_i(\mathbf{t}))_{1 \leq i, j \leq r} \in k[[\mathbf{t}]]^\times$, the k -algebra $\frac{k[[\mathbf{X}, \mathbf{Y}]]}{\mathcal{J}}$ is isomorphic to $\frac{k[[\mathbf{X}]]}{\langle f(\mathbf{X}, 0) \rangle_{f \in \mathcal{J}}}[[t_1, \dots, t_r]]$.

Proof. One uses in particular the notation introduced in Subsection 3.20.

Write $\mathcal{J} = \langle f_1, \dots, f_n \rangle$ and let $w = w(\mathbf{t})$ be the element of $\mathcal{V}_{N+r}(k[[\mathbf{t}]])^n$ corresponding to

$$(f_i(\mathbf{X} + \mathbf{x}(\mathbf{t}), \mathbf{y}(\mathbf{t})))_{1 \leq i \leq n}.$$

In particular $w(0) \in (\mathcal{V}_{N+r}(k))^n$ corresponds to $f_i(\mathbf{X}, 0)_{1 \leq i \leq n}$. Denote by Z the closure of the $\mathcal{K}_{N+r,n}$ -orbit of $w(0)$ in \mathcal{V}_{N+r}^n . The assumptions imply that there exists $g \in \mathcal{K}_{N+r,n}(K)$ such that $g \cdot w(0) = w$, where w is seen as an element of $\mathcal{V}_{N+r}(k((\mathbf{t})))^n \subset \mathcal{V}_{N+r}(K)^n$. Thus the generic point of w , that is to say, the schematic point defined by $w \in \mathcal{V}_{N+r}(k((\mathbf{t})))^n$, lies in Z , hence $w \in Z(k[[\mathbf{t}]])$.

By Lemma 4.6 and Remark 4.7, there exists $\tilde{g} \in \mathcal{K}_{N+r}(k[[\mathbf{t}]])$ such that $\tilde{g} \cdot \gamma(0) = \gamma$. By Remark 3.22, this gives the first assertion.

As for the second assertion, the assumption implies that

$$\mathbf{X} \mapsto \mathbf{X} + \mathbf{x}(\mathbf{t}), \quad t_i \mapsto z_i(\mathbf{t}), (1 \leq i \leq r), \quad t_i \mapsto t_i(r+1 \leq i \leq d)$$

is an automorphism of $k[[\mathbf{X}, \mathbf{t}]]$, hence the following isomorphism:

$$\frac{k[[\mathbf{X}, \mathbf{t}]]}{\langle f(\mathbf{X} + \mathbf{x}(\mathbf{t}), \mathbf{y}(\mathbf{t})) \rangle_{f \in \mathcal{J}}} \xrightarrow{\sim} \frac{k[[\mathbf{X}, \mathbf{Y}, t_{r+1}, \dots, t_d]]}{\mathcal{J}} = \frac{k[[\mathbf{X}, \mathbf{Y}]]}{\mathcal{J}}[[t_{r+1}, \dots, t_d]].$$

By the first assertion and the cancellation Theorem for complete local Noetherian k -algebras ([Ham75]; see Remark 2.18), one gets the result. \square

5. CYLINDRICITY IMPLIES ISOSINGULARITY

In this section we show the different assertions of Theorem 1.1, using in particular the properties of the isosingular loci discussed in the two previous sections, as well as the results of Section 2. Key ingredients include Artin's characterization of isomorphic formal neighborhoods in terms of étale neighborhoods and a study of maximal differential ideals with respect to a regular tuple of derivations. A key step is to show that the closure of the associated schematic point is a SAI subset of the expected dimension (hence the name of the section).

5.1. Étaleness and completions.

Proposition 5.2. *Let $f: (R, \mathfrak{M}_R) \rightarrow (S, \mathfrak{M}_S)$ be a local morphism of Noetherian local rings. Choose a section $\sigma: \kappa_R \rightarrow \hat{R}$ of \hat{R} and assume that σ extends to a section of \hat{S} .*

Then the following are equivalent:

- (1) S is R -flat and $\mathfrak{M}_R \cdot S = \mathfrak{M}_S$.
- (2) the map $\hat{R} \otimes_{\kappa_R} \kappa_S \rightarrow \hat{S}$ sending $r \otimes \alpha$ to $\sigma(\alpha) \cdot \hat{f}(r)$ is an isomorphism of κ_S -algebras.

Remark 5.3. By [Bou06, Chapitre 9, §3, Proposition 1], the assumption that a section of \hat{R} extends to a section of \hat{S} holds as soon as κ_S/κ_R is separable.

Remark 5.4. In particular $R \rightarrow S$ is étale if and only if 2) holds and κ_S/κ_R is finite separable. The latter result is stated as an exercise in [Har77, III, exer. 10.4].

Proof. Note that for any $n \geq 1$, one may use σ to endow $\frac{R}{\mathfrak{M}_R^n}$ (resp. $\frac{S}{\mathfrak{M}_S^n}$) with a structure of κ_R -algebra (resp. κ_S -algebra). For $n = 1$, this structure is nothing but the canonical structure of $\kappa_R = R/\mathfrak{M}_R$ -algebra on R/\mathfrak{M}_R^n and ditto for S .

Also note that for any $n \geq 1$, any $r \in \mathfrak{M}_R^n$ and any $\alpha = r' \pmod{\mathfrak{M}_R} \in \kappa_R$, one has $r'r = \sigma(\alpha)r \pmod{\mathfrak{M}_R^{n+1}}$. Thus the natural exact sequence

$$0 \longrightarrow \frac{\mathfrak{M}_R^n}{\mathfrak{M}_R^{n+1}} \xrightarrow{\iota_{R,n}} \frac{R}{\mathfrak{M}_R^{n+1}} \xrightarrow{\pi_{R,n}} \frac{R}{\mathfrak{M}_R^n} \longrightarrow 0$$

is an exact sequence of κ_R -vector spaces, and ditto for S .

For any $n \geq 1$, denote by q_n the group morphism $\frac{\mathfrak{M}_R^n}{\mathfrak{M}_R^{n+1}} \otimes_{\kappa_R} \kappa_S \rightarrow \frac{\mathfrak{M}_S^n}{\mathfrak{M}_S^{n+1}}$ such that

$$q_n(r \pmod{\mathfrak{M}_R^{n+1}} \otimes s \pmod{\mathfrak{M}_S}) = f(r)s \pmod{\mathfrak{M}_S^{n+1}}$$

and by p_n the group morphism $\frac{R}{\mathfrak{M}_R^n} \otimes_{\kappa_R} \kappa_S \rightarrow \frac{S}{\mathfrak{M}_S^n}$ such that

$$p_n(r \pmod{\mathfrak{M}_R^n} \otimes \beta) = f(r)\sigma(\beta) \pmod{\mathfrak{M}_S^n}.$$

Then q_n (resp. p_n) is a morphism of κ_S -vector spaces (resp. κ_S -algebras) and one has a commutative diagram of κ_S -vector spaces:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{\mathfrak{M}_R^n}{\mathfrak{M}_R^{n+1}} \otimes_{\kappa_R} \kappa_S & \xrightarrow{\iota_{R,n} \otimes_{\kappa_R} \kappa_S} & \frac{R}{\mathfrak{M}_R^{n+1}} \otimes_{\kappa_R} \kappa_S & \xrightarrow{\pi_{R,n} \otimes_{\kappa_R} \kappa_S} & \frac{R}{\mathfrak{M}_R^n} \otimes_{\kappa_R} \kappa_S & \longrightarrow & 0 \\ & & \downarrow q_n & & \downarrow p_{n+1} & & \downarrow p_n & & \\ 0 & \longrightarrow & \frac{\mathfrak{M}_S^n}{\mathfrak{M}_S^{n+1}} & \xrightarrow{\iota_{S,n}} & \frac{S}{\mathfrak{M}_S^{n+1}} & \xrightarrow{\pi_{S,n}} & \frac{S}{\mathfrak{M}_S^n} & \longrightarrow & 0 \end{array}$$

Now assume $\mathfrak{M}_R \cdot S = \mathfrak{M}_S$ and S is R -flat. Since $\mathfrak{M}_R \cdot S = \mathfrak{M}_S$, p_1 is an isomorphism. Moreover, since $\mathfrak{M}_R \cdot S = \mathfrak{M}_S$, S is R -flat and using [Mat86, Theorem 22.3], one gets that q_n is an isomorphism for any n . Thus by induction p_n is an isomorphism for any n . Therefore

$$\widehat{R} \widehat{\otimes}_{\kappa_R} \kappa_S = \varprojlim \frac{R}{\mathfrak{M}_R^{n+1}} \otimes_{\kappa_R} \kappa_S = \varprojlim \frac{S}{\mathfrak{M}_S^{n+1}} = \widehat{S}.$$

Conversely, if $\widehat{R} \widehat{\otimes}_{\kappa_R} \kappa_S \rightarrow \widehat{S}$ is an isomorphism, then p_n is an isomorphism for any $n \geq 1$, thus also q_n . Since p_1 is an isomorphism, one has $\mathfrak{M}_R \cdot S = \mathfrak{M}_S$. Now the fact that q_n is an isomorphism for any n and [Mat86, Theorem 22.3] show that S is R -flat. \square

Lemma 5.5. *Let $(R, \mathfrak{M}_R, \kappa_R)$ be a Noetherian local ring, $(S, \mathfrak{M}_S, \kappa_S)$ be the localization of $R[t_1, \dots, t_r]$ at the ideal $\mathfrak{M}_R[t_1, \dots, t_r]$, $f: R \rightarrow S$ be the composition of the canonical injection $R \rightarrow R[t_1, \dots, t_r]$ with the localization, and $\widehat{f}: \widehat{R} \rightarrow \widehat{S}$ be the induced morphism. The following assertions are true:*

- (a) *The natural morphism $R[t_1, \dots, t_r] \rightarrow \kappa_R[t_1, \dots, t_r]$ induces an isomorphism $\kappa_S \xrightarrow{\sim} \kappa_R(t_1, \dots, t_r)$.*
- (b) *Choose a section $\sigma: \kappa_R \rightarrow \widehat{R}$. Let $\sigma': \kappa_R(t_1, \dots, t_r) \rightarrow \widehat{S}$ be the section that extends σ canonically. Then the map*

$$\widehat{R} \widehat{\otimes}_{\kappa_R} \kappa_R(t_1, \dots, t_r) \xrightarrow{\sim} \widehat{S}$$

sending $r \otimes \alpha$ to $\sigma'(\alpha) \cdot \widehat{f}(r)$ is an isomorphism of $\kappa_R(t_1, \dots, t_r)$ -algebras.

Proof. By the exactness of the localization functor, the morphism of the statement induces the following isomorphisms:

$$\left(\frac{R[t_1, \dots, t_r]}{\mathfrak{M}_R[t_1, \dots, t_r]_{(0)}} \right) \xrightarrow{\sim} (\kappa_R[t_1, \dots, t_r])_{(0)} \xrightarrow{\sim} \kappa_R(t_1, \dots, t_r).$$

We now prove part (b). Since $R[t_1, \dots, t_r]$ is a free R -module, $R \rightarrow R[t_1, \dots, t_r]$ is flat. Since any localization morphism is flat, f is also flat, and the result follows from Proposition 5.2. \square

5.6. Maximal differential ideals with respect to regular families, 2.

Proposition 5.7. *Let k be a field of characteristic 0 and X be a k -scheme of finite type. Let r be a positive integer and $x \in X$ be an r -cylindrical point. Then, there exists a regular family of pairwise commuting derivations $D_1, \dots, D_r \in \text{Der}_k(\widehat{\mathcal{O}_{X,x}})$ such that, denoting by \mathfrak{p}^* the maximal (D_1, \dots, D_r) -differential ideal of $\widehat{\mathcal{O}_{X,x}}$, and setting $\mathfrak{p} := \mathfrak{p}^* \cap \mathcal{O}_{X,x}$:*

- (1) *One has $\dim(\mathcal{O}_{X,x}/\mathfrak{p}) = r$.*

- (2) Let $\eta \in X$ be the schematic point corresponding to \mathfrak{p} . Then, there exist a k -section s_x of $\widehat{\mathcal{O}_{X,x}}$ and a common extension K of $\kappa(\eta)$ and $\kappa(x)$ such that for any k -section s_η of $\widehat{\mathcal{O}_{X,\eta}}$, the complete local K -algebras $\widehat{\mathcal{O}_{X,x}}^{(s_x)} \widehat{\otimes}_{\kappa(x)} K$ and $(\widehat{\mathcal{O}_{X,\eta}})^{(s_\eta)} \widehat{\otimes}_{\kappa(\eta)} K[[t_1, \dots, t_r]]$ are isomorphic as k -algebras.

Example 5.8. For an arbitrary regular family of pairwise commuting derivations $D_1, \dots, D_r \in \text{Der}_k(\widehat{\mathcal{O}_{X,x}})$, one always has $\dim(\widehat{\mathcal{O}_{X,x}}/\mathfrak{p}^*) = r$ by Lemma 2.37. However, in general, one may have $\dim(\mathcal{O}_{X,x}/\mathfrak{p}) > r$, even in case $r = 1$ and D is the extension of an element of $\text{Der}_k(\mathcal{O}_{X,x})$, as the following example (taken from [Sin83, Example 2.10], but with a different argument) shows. Consider the derivation D on $k[[t_1, t_2]]$ defined by $D(t_1) = 1$ and $D(t_2) = 1 + t_2$. Since $D(t_1 - \ln(1 + t_2)) = 0$, one has $\mathfrak{p}^* = \langle t_1 - \ln(1 + t_2) \rangle$, whose intersection with $k[[t_1, t_2]]_{\langle t_1, t_2 \rangle}$ is the zero ideal.

Proof. Since $x \in X$ is r -cylindrical, then corollary 2.28 ensures the existence of an affine k -scheme of finite type Y , a point $y \in Y$ and an isomorphism of k -algebras

$$\widehat{\mathcal{O}_{X,x}} \xrightarrow{\sim} \widehat{\mathcal{O}_{Y \times_k \mathbf{A}_k^r, (y, \mathbf{0})}} = \widehat{\mathcal{O}_{Y,y}}[[t_1, \dots, t_r]].$$

Set $(Z, z) := (Y \times_k \mathbf{A}_k^r, (y, \mathbf{0}))$. By [Art69, Corollary 2.6], there exists a k -scheme of finite type W and $w \in W$ and a diagram

$$\begin{array}{ccc} & \widehat{\mathcal{O}_{W,w}} & \\ \theta_X \nearrow & & \nwarrow \theta_Z \\ \widehat{\mathcal{O}_{X,x}} & & \widehat{\mathcal{O}_{Z,z}} \end{array}$$

where θ_Z and θ_X are étale morphisms of local k -algebras, inducing isomorphisms of k -algebras

$$\widehat{\theta}_X: \widehat{\mathcal{O}_{X,x}} \xrightarrow{\sim} \widehat{\mathcal{O}_{W,w}} \quad \text{and} \quad \widehat{\theta}_Z: \widehat{\mathcal{O}_{Z,z}} \xrightarrow{\sim} \widehat{\mathcal{O}_{W,w}}.$$

For the sake of simplicity, we hereby identify $\widehat{\mathcal{O}_{X,x}}$ and $\widehat{\mathcal{O}_{Z,z}} = \widehat{\mathcal{O}_{Y,y}}[[t_1, \dots, t_r]]$ with $\widehat{\mathcal{O}_{W,w}} := A$ via these isomorphisms, and view $\mathcal{O}_{X,x}$, $\mathcal{O}_{Z,z}$ and $\mathcal{O}_{W,w}$ as subrings of A . For $1 \leq i \leq r$, let $D_i \in \text{Der}_k(A)$ be the extension of the the partial derivative ∂_{t_i} on $\mathcal{O}_{Y,y}[[t_1, \dots, t_r]]$; note that the D_i 's are pairwise commuting. Let \mathfrak{p}^* be the maximal D_1, \dots, D_r -differential ideal of A , $\mathfrak{p} := \mathfrak{p}^* \cap \mathcal{O}_{X,x}$, $\mathfrak{p}_Z := \mathfrak{p}^* \cap \mathcal{O}_{Z,z}$, and $\mathfrak{p}_W := \mathfrak{p}^* \cap \mathcal{O}_{W,w}$. In particular, $\theta_X^{-1}(\mathfrak{p}_W) = \mathfrak{p}$ and $\theta_Z^{-1}(\mathfrak{p}_W) = \mathfrak{p}_Z$. We deduce that $\dim(\mathcal{O}_{X,x}/\mathfrak{p}) = \dim(\mathcal{O}_{W,w}/\mathfrak{p}_W) = \dim(\mathcal{O}_{Z,z}/\mathfrak{p}_Z)$, because θ_X and θ_Z are étale morphisms.

Then, for the first part, we only have to show that $\dim(\mathcal{O}_{Z,z}/\mathfrak{p}_Z) = r$. Recall that $(Z, z) = (Y \times_k \mathbf{A}_k^r, (y, \mathbf{0}))$. Let \mathfrak{p}_y be the maximal ideal of $\mathcal{O}_{Y,y}$. Then $\mathcal{O}_{Z,z}$ is the localization of $\mathcal{O}_{Y,y}[[t_1, \dots, t_r]]$ at the maximal ideal $\mathfrak{p}_y + \langle t_1, \dots, t_r \rangle$. From Lemma 2.37, we deduce that the maximal $(\partial_{t_1}, \dots, \partial_{t_r})$ -differential ideal of $\mathcal{O}_{Z,z}$, namely \mathfrak{p}_Z by Corollary 2.35, coincides with $\mathfrak{p}_y[[t_1, \dots, t_r]]$. Thus

$$\frac{\mathcal{O}_{Z,z}}{\mathfrak{q}_Z} = \frac{\mathcal{O}_{Y,y}[[t_1, \dots, t_r]]_{\mathfrak{p}_y + \langle t_1, \dots, t_r \rangle}}{\mathfrak{p}_y[[t_1, \dots, t_r]]} \cong \kappa(y)[t_1, \dots, t_r]_{\langle t_1, \dots, t_r \rangle},$$

and this proves that $\dim(\mathcal{O}_{Z,z}/\mathfrak{q}_Z)$ equals r , the dimension of the latter ring.

Let us now prove the second assertion. We denote by η_W, η_X, η_Z and $\kappa(\eta_W), \kappa(\eta_X), \kappa(\eta_Z)$ respectively the points of W, X, Z corresponding to the prime ideals $\mathfrak{q}_W, \mathfrak{q}_X, \mathfrak{q}_Z$ and their residue fields. Note that $\eta_X = \eta$.

Let $s_y: \kappa(y) \rightarrow \widehat{\mathcal{O}_{Y,y}}$ be an arbitrary k -section. Via the inclusion $\widehat{\mathcal{O}_{Y,y}} \subset \widehat{\mathcal{O}_{Y,y}}[[t_1, \dots, t_r]] = \widehat{\mathcal{O}_{X,x}}$ it may be extended to a k -section $s_x: \kappa(x) = \kappa(y) \rightarrow \widehat{\mathcal{O}_{X,x}}$.

We identify $\kappa(\eta_Z)$ with $\kappa(y)(t_1, \dots, t_r)$ via the isomorphism of Lemma 5.5 and consider the k -section $s_{\eta_Z} : \kappa(y)(t_1, \dots, t_r) \rightarrow \widehat{\mathcal{O}_{Z, \eta_Z}}$ that extends s_y canonically. The same lemma then provides an isomorphism of $\kappa(y)(t_1, \dots, t_r)$ -algebras

$$\widehat{\mathcal{O}_{Z, \eta_Z}}^{(s_{\eta_Z})} \xrightarrow{\sim} \widehat{\mathcal{O}_{Y, y}}^{(s_y)} \widehat{\otimes}_{\kappa(y)} \kappa(y)(t_1, \dots, t_r). \quad (5.1)$$

We now choose an arbitrary k -section s_{η_X} of $\widehat{\mathcal{O}_{X, \eta_X}}$. The étale morphism θ_X induces an étale morphism $\mathcal{O}_{X, \eta_X} \rightarrow \mathcal{O}_{W, \eta_W}$, thus the extension $\kappa(\eta_W)/\kappa(\eta_X)$ is finite separable. Then, by Remark 5.3, s_{η_X} extends to a section $s_1 : \kappa(\eta_W) \rightarrow \widehat{\mathcal{O}_{W, \eta_W}}$. An analogous reasoning with θ_Z shows that s_{η_Z} extends to a section $s_2 : \kappa(\eta_W) \rightarrow \widehat{\mathcal{O}_{W, \eta_W}}$. Hence, Proposition 5.2 provides isomorphisms of complete local $\kappa(\eta_W)$ -algebras

$$\widehat{\mathcal{O}_{W, \eta_W}}^{(s_1)} \xrightarrow{\sim} \widehat{\mathcal{O}_{X, \eta_X}}^{(s_{\eta_X})} \widehat{\otimes}_{\kappa(\eta_X)} \kappa(\eta_W) \quad \text{and} \quad \widehat{\mathcal{O}_{W, \eta_W}}^{(s_2)} \xrightarrow{\sim} \widehat{\mathcal{O}_{Z, \eta_Z}}^{(s_{\eta_Z})} \widehat{\otimes}_{\kappa(\eta_Z)} \kappa(\eta_W). \quad (5.2)$$

On the one hand, from (5.2) we deduce an isomorphism of $\kappa(\eta_W)$ -algebras

$$\widehat{\mathcal{O}_{W, \eta_W}}[[t_1, \dots, t_r]]^{(s_1)} \xrightarrow{\sim} \left(\widehat{\mathcal{O}_{X, \eta_X}}^{(s_{\eta_X})} \widehat{\otimes}_{\kappa(\eta_X)} \kappa(\eta_W) \right) [[t_1, \dots, t_r]]. \quad (5.3)$$

On the other hand, combining (5.2) and (5.1) we obtain an isomorphism of $\kappa(\eta_W)$ -algebras

$$\widehat{\mathcal{O}_{W, \eta_W}}^{(s_2)} \xrightarrow{\sim} \widehat{\mathcal{O}_{Y, y}}^{(s_y)} \widehat{\otimes}_{\kappa(y)} \kappa(\eta_W),$$

that induces an isomorphism of $\kappa(\eta_W)$ -algebras

$$\widehat{\mathcal{O}_{W, \eta_W}}[[t_1, \dots, t_r]]^{(s_2)} \xrightarrow{\sim} \left(\widehat{\mathcal{O}_{Y, y}}^{(s_y)} \widehat{\otimes}_{\kappa(y)} \kappa(\eta_W) \right) [[t_1, \dots, t_r]]. \quad (5.4)$$

Via our previous identification $\widehat{\mathcal{O}_{Y, y}}[[t_1, \dots, t_r]] = \widehat{\mathcal{O}_{X, x}}$ and choice of section s_x , the RHS of the previous isomorphism is nothing but $\widehat{\mathcal{O}_{X, x}}^{(s_x)} \widehat{\otimes}_{\kappa(x)} \kappa(\eta_W)$. Finally, comparing with (5.3), we deduce that the complete local $\kappa(\eta_W)$ -algebras $\widehat{\mathcal{O}_{X, x}}^{(s_x)} \widehat{\otimes}_{\kappa(x)} \kappa(\eta_W)$ and $\left(\widehat{\mathcal{O}_{X, \eta_X}}^{(s_{\eta_X})} \widehat{\otimes}_{\kappa(\eta_X)} \kappa(\eta_W) \right) [[t_1, \dots, t_r]]$ are isomorphic as k -algebras (see Remark 3.4), which proves the second assertion for any common extension K of $\kappa(\eta_W)$ and $\kappa(x)$. \square

Remark 5.9. In Proposition 5.7, if the point x is assumed to be closed, then the field extension $\kappa(x)/k$ is finite. Since $\kappa(\eta_W)/\kappa(\eta)$ is also finite, we may choose K such that $K/\kappa(\eta)$ is a finite field extension (take, e.g., a compositum over k of $\kappa(x)$ and $\kappa(\eta_W)$).

Corollary 5.10. *Let k be a field of characteristic zero, X be a k -scheme of finite type, and $x \in X$ be a maximally r -cylindrical point. Let \mathfrak{p}_x be the maximal differential ideal of $\mathcal{O}_{X, x}$ and $\eta_x \in X$ be its corresponding schematic point. Then*

- (1) *one has $\dim(\mathcal{O}_{X, x}/\mathfrak{p}_x) = r$;*
- (2) *for any regular r -tuple D_1, \dots, D_r of derivations on $\mathcal{O}_{X, x}$, \mathfrak{p}_x is the maximal D_1, \dots, D_r -differential ideal of $\mathcal{O}_{X, x}$;*
- (3) *There exist a k -section s_x of $\widehat{\mathcal{O}_{X, x}}$ and a common extension K of $\kappa(x)$ and $\kappa(\eta_x)$, such that, for any k -section s_{η_x} of $\widehat{\mathcal{O}_{X, \eta_x}}$, $\widehat{\mathcal{O}_{X, x}}^{(s_x)} \widehat{\otimes}_{\kappa(x)} K$ and $\left(\widehat{\mathcal{O}_{X, \eta_x}}^{(s_{\eta_x})} \widehat{\otimes}_{\kappa(\eta_x)} K \right) [[t_1, \dots, t_r]]$ are isomorphic k -algebras.*

Proof. Consider a family of derivations as in Proposition 5.7 and keep the notation of the statement of the proposition.

First, note that \mathfrak{p} is differential. If that were not the case, by the last assertion of Proposition 5.7, $\widehat{\mathcal{O}_{X, x}}^{(s_x)} \widehat{\otimes}_{\kappa(x)} K$ would be $(r+1)$ -cylindrical. By Lemma 2.26, $\widehat{\mathcal{O}_{X, x}}$ would also be $(r+1)$ -cylindrical, a contradiction.

Now consider a regular r -tuple D_1, \dots, D_r of derivations on $\mathcal{O}_{X, x}$ and $\mathfrak{p}' \in \text{Spec}(\mathcal{O}_{X, x})$ the maximal D_1, \dots, D_r -differential ideal of $\mathcal{O}_{X, x}$. Since \mathfrak{p} is differential, one has $\mathfrak{p} \subset \mathfrak{p}'$. On the other hand, by Lemma 2.39, one has $\dim(\mathcal{O}_{X, x}/\mathfrak{p}') \geq r$. Since $\dim(\mathcal{O}_{X, x}/\mathfrak{p}) = r$ by the first assertion of Proposition 5.7, one has $\mathfrak{p}' = \mathfrak{p}$.

Finally, consider \mathfrak{p}_x the maximal differential ideal of $\widehat{\mathcal{O}}_{X,x}$. Since \mathfrak{p} is differential, one has $\mathfrak{p} \subset \mathfrak{p}_x$. Since \mathfrak{p}_x is D_1, \dots, D_r -differential, one has $\mathfrak{p}_x \subset \mathfrak{p}'$. Hence $\mathfrak{p} = \mathfrak{p}_x$ and Proposition 5.7 again allows to conclude. \square

Corollary 5.11. *Let k be a field of characteristic zero, X be a k -scheme of finite type, and $x \in X$ be a maximally r -cylindrical point. For any $s \leq r$ and any regular family D_1, \dots, D_s of derivations on $\widehat{\mathcal{O}}_{X,x}$, if we denote by $\mathfrak{p}_{x,s}$ the maximal D_1, \dots, D_s -differential ideal of $\widehat{\mathcal{O}}_{X,x}$, then one has $s \leq \dim(\widehat{\mathcal{O}}_{X,x}/\mathfrak{p}_{x,s}) \leq r$.*

Proof. The first inequality is given by Lemma 2.39. As for the second one, let \mathfrak{p}_x be the maximal differential ideal of $\widehat{\mathcal{O}}_{X,x}$. In particular, it is D_1, \dots, D_s -differential, then $\mathfrak{p}_x \subset \mathfrak{p}_{x,s}$. Thus $\dim(\widehat{\mathcal{O}}_{X,x}/\mathfrak{p}_{x,s}) \leq \dim(\widehat{\mathcal{O}}_{X,x}/\mathfrak{p}_x)$, and by Corollary 5.10 gives $\dim(\widehat{\mathcal{O}}_{X,x}/\mathfrak{p}_x) = r$. \square

Lemma 5.12. *Let k be a field of characteristic zero and X be a k -scheme of finite type. Then the set of r -cylindrical points of $X^{(0)}$ is open in $X^{(0)}$.*

Notation 5.13. We shall denote by $X_{\text{cyl},r}$ the unique open subset of X such that $X_{\text{cyl},r} \cap X^{(0)}$ is the set of r -cylindrical points of $X^{(0)}$.

Proof. Let $x \in X^{(0)}$ be an r -cylindrical point and D_1, \dots, D_r be a regular family of derivations on $\widehat{\mathcal{O}}_{X,x}$. Up to replacing X by an open affine subset containing x , one may assume that D_i is the extension of $D_i \in \text{Der}_k(\Gamma(X))$, for every $1 \leq i \leq r$. Let \mathfrak{m}_x be the maximal ideal of $\Gamma(X)$ corresponding to x , $f_1, \dots, f_r \in \mathfrak{m}_x$ such that $\det(D_i f_j) \notin \mathfrak{m}_x$ and U_x be the open subset of X defined by $\{\det(D_i(f_j)) \neq 0\}$. Let us show that any point in $x' \in U_x \cap X^{(0)}$ is r -cylindrical.

Let $K = \kappa(x')$. Since x' is a closed point and $\text{char}(k) = 0$, K is a finite separable extension of k . Let us choose $K \subset \widehat{\mathcal{O}}_{X,x'}$ a coefficient field containing k . For any $1 \leq i \leq r$, \widehat{D}_i induces an element E_i of $\text{Der}_k(K, \widehat{\mathcal{O}}_{X,x'})$. Since K/k is a finite separable extension and $E_i(k) = \{0\}$, by [Mat86, Theorem 5.23], one also has $E_i(K) = \{0\}$. Thus $\widehat{D}_i \in \text{Der}_K(\widehat{\mathcal{O}}_{X,x'})$. Let $\pi_{x'}: \widehat{\mathcal{O}}_{X,x'} \rightarrow K$ be the canonical quotient morphism. Then, for any $1 \leq i, j \leq r$, $g_j := f_j - \pi_{x'}(f_j) \in \widehat{\mathfrak{m}}_{X,x'}$ and $\widehat{D}_i(g_j) = \widehat{D}_i(f_j)$. Since $x' \in U_x$, one has $\det(\widehat{D}_i(g_j)) \notin \widehat{\mathfrak{m}}_{X,x'}$. Thus x' is r -cylindrical.

We conclude by taking $X_{\text{cyl},r} = \bigcup_x U_x$, where $x \in X^{(0)}$ is an r -cylindrical point. \square

Notation 5.14. If X is a scheme and $x \in X$ is a schematic point, one denotes by $\dim_X(x)$ the dimension of the closure of x in X .

Corollary 5.15. *Let k be a field of characteristic zero, and X be a k -scheme of finite type. Let $x \in X$ be a closed point that is maximally r -cylindrical, η be the maximal differential generization of x , and F be the Zariski closure of η in X . Let $U := F \cap X_{\text{cyl},r}$ (see Notation 5.13). For any $x' \in U$, there exist a k -section $s_{x'}$ of $\widehat{\mathcal{O}}_{X,x'}$ and $K/\kappa(\eta)$ a finite field extension that is a common extension of $\kappa(x)$ and $\kappa(x')$, such that $\widehat{\mathcal{O}}_{X,x} \widehat{\otimes}_{\kappa(x)} K$ and $(\widehat{\mathcal{O}}_{X,x'}^{(s_{x'})}) \widehat{\otimes}_{\kappa(x')} K[[t_1, \dots, t_{\dim_X(x')}]$ are isomorphic K -algebras. In particular (taking $x' = \eta$), F is SAI. Moreover, U is the maximal AI open subset of F .*

Proof. Let $x' \in U$. First assume that x' is closed (in particular r -cylindrical). Since $x' \in F$, η is a differential generization of x' . On the other hand, since x is closed and by Corollary 5.10, one has $\dim_X(\eta) = r$. Let η' be the maximal differential generization of x' . Then η is a generization of η' , but on the other hand, since x' is r -cylindrical, by Lemma 2.39 one has $\dim_X(\eta') \geq r = \dim_X(\eta)$ thus $\eta' = \eta$ and η is the maximal differential generization of x' . By Corollary 5.10 and Lemma 3.7, x' is maximally r -cylindrical, and there exists a common extension K of $\kappa(\eta)$, $\kappa(x)$ and $\kappa(x')$ such that $K/\kappa(\eta)$ is finite (see Remark 5.9, that also applies when taking the compositum over k of K as in Corollary 5.10 and $\kappa(x')$, because $\kappa(x')/k$ is finite) and, for any k -section s_η of $\widehat{\mathcal{O}}_{X,\eta}$, both the K -algebras $\widehat{\mathcal{O}}_{X,x} \widehat{\otimes}_{\kappa(x)} K$ and $(\widehat{\mathcal{O}}_{X,x'}^{(s_\eta)}) \widehat{\otimes}_{\kappa(x')} K$ are isomorphic to $(\widehat{\mathcal{O}}_{X,\eta}^{(s_\eta)}) \widehat{\otimes}_{\kappa(\eta)} K[[t_1, \dots, t_r]]$, hence the result follows.

Now take any $x' \in U$. Pick a closed point $x'' \in U$ that lies in the closure of x' . By the previous argument, η is the maximal differential generization of x'' , hence also of x' . Assume x' is maximally s -cylindrical, then Corollary 5.10 ensures that $\dim(\widehat{\mathcal{O}}_{X,x'}/\mathfrak{p}_\eta) = s$. Then $s = \dim(\widehat{\mathcal{O}}_{X,x'}/\mathfrak{p}_\eta) = \dim(\widehat{\mathcal{O}}_{X,x}/\mathfrak{p}_\eta) - \dim(x') = \dim(\eta) - \dim(x')$. By Corollary 5.10, there exist a k -section $s_{x'}$ of $\widehat{\mathcal{O}}_{X,x'}$ and a common extension K of $\kappa(\eta)$, $\kappa(x')$ and $\kappa(x'')$ such that $K/\kappa(\eta)$ is finite (Remark 5.9 applies again) and, for any k -section s_η of $\widehat{\mathcal{O}}_{X,\eta}$, there exist k -algebras isomorphisms

$$\widehat{\mathcal{O}}_{X,x''} \widehat{\otimes}_{\kappa(x'')} K \xrightarrow{\sim} (\widehat{\mathcal{O}}_{X,\eta})^{(s_\eta)} \widehat{\otimes}_{\kappa(\eta)} K[[t_1, \dots, t_r]]$$

and

$$\widehat{\mathcal{O}}_{X,x'}^{(s_{x'})} \widehat{\otimes}_{\kappa(x')} K \xrightarrow{\sim} (\widehat{\mathcal{O}}_{X,\eta})^{(s_\eta)} \widehat{\otimes}_{\kappa(\eta)} K[[t_1, \dots, t_{\dim_X(\eta) - \dim_X(x')}]].$$

Therefore, since $\dim_X(\eta) = r$, $\widehat{\mathcal{O}}_{X,x''} \widehat{\otimes}_{\kappa(x'')} K$ and $(\widehat{\mathcal{O}}_{X,x'}^{(s_{x'})} \widehat{\otimes}_{\kappa(x')} K)[[t_1, \dots, t_{\dim_X(x')}]]$ are isomorphic k -algebras. By Lemma 3.7, they are also isomorphic K -algebras. The previously discussed particular case allows to conclude.

Let us show that U is the maximal AI open subset of F . By the very definition of U , any element x' of $(F \setminus U) \cap X^{(0)}$ is not r -cylindrical. In particular, by Lemma 2.26, for any extension $K/\kappa(x')$, $\widehat{\mathcal{O}}_{X,x'} \widehat{\otimes}_{\kappa(x')} K$ is not r -cylindrical, thus not isomorphic to $\widehat{\mathcal{O}}_{X,x} \widehat{\otimes}_{\kappa(x)} K$. This shows that U is the maximal AI open subset of F . \square

Corollary 5.16. *We keep the hypotheses and notation in Corollary 5.15 and assume moreover that the field k is algebraically closed, in particular $x \in X(k)$. Then for any $x' \in U(k)$, the k -algebras $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}}_{X,x'}$ are isomorphic.*

Proof. Corollary 5.15 ensures that there exists an extension K of k such that $\widehat{\mathcal{O}}_{X,x} \widehat{\otimes}_k K$ and $\widehat{\mathcal{O}}_{X,x'} \widehat{\otimes}_k K$ are isomorphic K -algebras. Since k is algebraically closed, by [Sei69b], $\widehat{\mathcal{O}}_{X,x'}$ and $\widehat{\mathcal{O}}_{X,x}$ are isomorphic k -algebras. \square

5.17. Application to the geometric isosingular locus.

Theorem 5.18. *Let k be a field of characteristic zero and X be a k -scheme of finite type. Let $x \in X^{(0)}$ be a maximally r -cylindrical point. Then $\dim(\mathcal{G}(X, x)) = r$ (see Notation 3.31).*

Moreover, for any irreducible closed subset F of X , the following are equivalent:

- (i) *There exists $x \in X^{(0)}$ such that $F = \mathcal{G}(X, x)$.*
- (ii) *There exists $x \in X^{(0)}$ such that F is the closure of the maximal differential generization of x .*
- (iii) *F is a MWAI closed subset of X .*

Proof. For any $x \in X^{(0)}$, by Theorem 4.1, x is $\dim(\mathcal{G}(X, x))$ -cylindrical. Thus if x is maximally r -cylindrical, one has $\dim(\mathcal{G}(X, x)) \leq r$ and by Remark 3.32 one has $\dim(\mathcal{G}(X, x)) = r$.

Moreover, by the definition of $\mathcal{G}(X, x)$, any AI closed subset F whose AI open subset contains x is contained in $\mathcal{G}(X, x)$, and $X^{(x)} \cap F = X^{(x)} \cap \mathcal{G}(X, x) \cap F$ is an AI open subset of F . In particular $\mathcal{G}(X, x)$ is MWAI, and is the only MWAI closed subset of X containing x in its maximal AI open subset.

Now let F be the closure of the maximal differential generization of a maximally r -cylindrical closed point x . Then by Corollary 5.15 F is SAI thus WAI and by Corollary 5.10, one has $\dim(F) = r$. Let U be the maximal AI open subset of F . Thus for any $x' \in U^{(0)}$, one has $\mathcal{G}(X, x') \subset F$ and $\dim(\mathcal{G}(X, x')) = r$ thus $\mathcal{G}(X, x') = F$.

The equivalences in the statement are now clear. \square

Remark 5.19. In particular, the generic point of a MWAI closed subset is always differential. Yet in general the closure of a differential point need not be WAI: [Sei67, Example 3, p.32] is an example of an irreducible integral surface whose singular locus is irreducible of dimension 1 and contains exclusively non-cylindrical points. The generic point of

the singular locus is differential by [Sei67, Theorem 5], yet the singular locus can not be a WAI closed subset. Indeed, since its dimension is positive, by Theorem 4.1, any sufficiently generic k -rational point of the singular locus would be cylindrical.

Corollary 5.20. *Let k be a field of characteristic zero and X be a k -scheme of finite type. Then any WAI closed subset of X is SAI.*

More precisely, let F be a WAI closed subset of X with maximal AI open subset U and generic point η . Let $x' \in U$ and $d' = \dim_X(\eta) - \dim_X(\eta')$. Then there exist a common extension K of $\kappa(\eta)$ and $\kappa(x')$, and a k -section s_η (resp. $s_{x'}$) of $\widehat{\mathcal{O}}_{X,\eta}$ (resp. $\widehat{\mathcal{O}}_{X,x'}$) such that $\widehat{\mathcal{O}}_{X,x'}^{(s_{x'})} \widehat{\otimes}_{\kappa(x')} K$ and $\widehat{\mathcal{O}}_{X,\eta}^{(s_\eta)} \widehat{\otimes}_{\kappa(\eta)} K[[u_1, \dots, u_{d'}]]$ are isomorphic K -algebras.

Proof. In case F is a MWAI closed subset of X , this is a direct consequence of Theorem 5.18, Corollary 5.15 and Remark 2.18. In case F is WAI, let G be a MWAI closed subset containing F , η_G be the generic point of G and U_G be the maximal AI open subset of G . In particular $U_G \cap F = U$ and one may again conclude via Corollary 5.15 applied to G and Remark 2.18. \square

6. ANALYTIC ISOSINGULARITY FOR ARC SPACES

In this concluding section, we initiate a study of the extent to which the concepts and results discussed in the rest of the paper are amenable to generalize to the framework of arc spaces, which are one of the most natural example of so-to-speak infinite-dimensional algebraic varieties. In the next three subsections, we provide brief reminders of the aspects of the topic that are relevant for the discussion to follow (referring the reader to the given references for more details) as well as some remarks and a useful lemma.

6.1. Reminder: arc spaces. (See e.g., [BNS20, CLNS18]) Let k be a field and X be an algebraic k -variety. The arc space associated with X , denoted by $\mathcal{L}_\infty(X)$, is a k -scheme parameterizing the smooth formal germs of curves on X . One stresses that as soon as $\dim(X)$ is positive, $\mathcal{L}_\infty(X)$ is not a k -scheme of finite type. For any extension K/k , there is a functorial bijection between the set of K -points of $\mathcal{L}_\infty(X)$ and the set of $K[[t]]$ -points of X . If $\gamma \in \mathcal{L}_\infty(X)$ is an arc, with residue field $K = \kappa(\gamma)$, it induces a K -point of $\mathcal{L}_\infty(X)$ thus a $K[[t]]$ -points of X , which in turn defines a $K((t))$ -point and a K -point (through the specialization $t = 0$) of X . The corresponding schematic points of X are called respectively the generic point and the center of the arc γ . One says that an arc is *non-degenerate* if its generic point belongs to the smooth locus of X . The set of non-degenerate arcs is a non-empty Zariski open subset of $\mathcal{L}_\infty(X)$.

In case $X = \text{Spec}(\Gamma(X))$ is affine, $\mathcal{L}_\infty(X) = \text{Spec}(\Gamma(X)_\infty)$ is also affine. Assuming moreover that $\text{char}(k) = 0$, one can give an explicit differential description a presentation of $\Gamma(X)_\infty$ in terms of a presentation of $\Gamma(X)$: write $\Gamma(X) = \frac{k[x_1, \dots, x_n]}{\mathcal{I}}$; let $k[x_1, \dots, x_n]_\infty$ be the polynomial k -algebra over the countable set of variables $\{x_{i,j}\}_{\substack{1 \leq i \leq n, \\ j \in \mathbb{N}}}$, and identify $k[x_1, \dots, x_n]$ with a subalgebra of $k[x_1, \dots, x_n]_\infty$ via $x_i \mapsto x_{i,0}$; let Δ be the k -derivation on $k[x_1, \dots, x_n]_\infty$ such that for any i, j one has $\Delta(x_{i,j}) = x_{i,j+1}$; then $\Gamma(X)_\infty$ is isomorphic to $\frac{k[x_1, \dots, x_n]_\infty}{[\mathcal{I}]}$ where \mathcal{I} is the Δ -differential ideal generated by \mathcal{I} . Note that Δ descends to a k -derivation on $\Gamma(X)_\infty$ and there is an inclusion $\Gamma(X) \subset \Gamma(X)_\infty$. Let now \mathfrak{p} be a prime ideal of $\Gamma(X)_\infty$, $\gamma_{\mathfrak{p}} \in \mathcal{L}_\infty(X)$ be the corresponding arc, K be the residue field of \mathfrak{p} and $\theta \in \text{Hom}(\Gamma_\infty(X), K)$ the morphism of k -algebras. Then the induced morphism $\gamma_{\mathfrak{p}}^* : \Gamma(X) \rightarrow K[[t]]$ is given by the composition of the morphism $\Gamma(X) \rightarrow \Gamma(X)_\infty$ mapping φ to $\sum_{j \geq 0} \frac{\Delta^{(j)} \varphi}{j!} t^j$ with $\sum \varphi_j \cdot t^j \mapsto \sum \theta(\varphi_j) \cdot t^j$. The center of $\gamma_{\mathfrak{p}}$ corresponds to the kernel of $(t = 0) \circ \gamma_{\mathfrak{p}}^*$, thus to the prime ideal $\mathfrak{p} \cap \Gamma(X)$, whereas its generic center corresponds to the kernel of $\gamma_{\mathfrak{p}}^*$. That being said, one easily obtains the following lemma, which will be useful below.

Lemma 6.2. *Keep the previous notation. If $\Delta(\mathfrak{p}) \subset \mathfrak{p}$, then the arc $\gamma_{\mathfrak{p}}$ is pseudo-constant, i.e., its center and generic point coincide.*

Proof. For any $\varphi \in \mathfrak{p} \cap \Gamma(X)$ and any $j \geq 0$, one has $\Delta^{(j)}\varphi \in \mathfrak{p}$, thus $\mathfrak{p} \cap \Gamma(X) \subset \text{Ker}(\gamma_{\mathfrak{p}}^*)$. \square

6.3. Reminder: the Drinfeld-Grinberg-Kazhdan theorem and the minimal formal model of a rational non-degenerate arc. (See [Dri20, BS20, BS17]) Let X be an algebraic k -variety of positive dimension and $\gamma \in \mathcal{L}_{\infty}(X)(k)$ be a *non-degenerate* k -rational arc. The Drinfeld-Grinberg-Kazhdan theorem states that there exists a k -scheme of finite type Y and $y \in Y(k)$ such that the formal neighborhood $\widehat{\mathcal{O}_{\mathcal{L}_{\infty}(X), \gamma}}$ is isomorphic (as a topological k -algebra) to $\widehat{\mathcal{O}_{Y, y}[[(u_r)_{r \in \mathbb{N}}]]}$ where u_r are indeterminates. A minimal formal model of γ is such a $\widehat{\mathcal{O}_{Y, y}}$ with the additional property that it is non-cylindrical. By Gabber's cancellation theorem, a minimal formal model of γ is uniquely determined (up to isomorphism).

6.4. Reminder: maximal divisorial sets on arc spaces. (See [Ish08, dFEI08, dF18]). We continue our short reminders by discussing the notion of maximal divisorial sets, which form an important class of irreducible closed sets of the arc space of an algebraic variety, closely related to the well-known Nash problem and birational geometry. We assume that k is a field of characteristic zero. Let ν be a divisorial valuation on an algebraic k -variety X . We associate with it a closed subset $\mathcal{N}_X(\nu)$ of $\mathcal{L}_{\infty}(X)$ as follows: let $W \rightarrow X$ be a resolution of the singularities of X (inducing a morphism $\mathcal{L}_{\infty}(W) \rightarrow \mathcal{L}_{\infty}(X)$ such that there is a prime divisor E on W and a positive integer q such that $\nu = q \cdot \text{ord}_E$). Let $\mathcal{L}_{\infty}(W)^{E, q}$ be the set of arcs on W with contact order $\geq q$ along E . Then $\mathcal{N}_X(\nu)$ is the closure of the image of $\mathcal{L}_{\infty}(W)^{E, q}$ in $\mathcal{L}_{\infty}(X)$. It does not depend on the choice of W and E , and is irreducible. The arc corresponding to the generic point of $\mathcal{N}_X(\nu)$ is a non-degenerate arc, in fact its generic point is the generic point of X (it is a fat arc in the sense of [Ish05]). Moreover its center is the center of the valuation ν on X .

Remark 6.5. By the latter result, the generic point η_{ν} of $\mathcal{N}_X(\nu)$ is not a pseudo-constant arc, thus by Lemma 6.2 is not a differential point. In particular $\widehat{\mathcal{L}_{\infty}(X), \eta_{\nu}}$ (which is Noetherian by [Reg06], see also [dFD20]) is 1-cylindrical. This gives a general explanation of a phenomenon observed in [BMCS] for toric valuations of toric varieties and in [BMCn23, Theorem 1.7] for valuations on curves.

Remark 6.6. Assuming that k is moreover algebraically closed, it is worth remarking that the set of k -rational points of $\mathcal{N}_X(\nu)$ is in $\mathcal{N}_X(\nu)$. Let us roughly explain why (A very similar argument is used to show that $\mathcal{N}_X(\nu)$ is irreducible.). Using suitable étale coordinates and the compatibility of the formation of arc spaces with étale morphism, one reduces to show that $\mathcal{L}_{\infty}(W)^{E, q}(k)$ is dense in $\mathcal{L}_{\infty}(W)^{E, q}$ for $W = \mathbf{A}_k^n$ and $E = \{x_1 = 0\}$; But then $\mathcal{L}_{\infty}(W)^{E, q}$ is the closed subscheme of $\mathcal{L}_{\infty}(\mathbf{A}_k^n) = \text{Spec}(k[[x_{i,j}]]_{\substack{1 \leq i \leq n \\ j \in \mathbb{N}}})$ defined by the ideal $\langle x_{1,j} \rangle_{0 \leq j \leq q-1}$. On the other hand, for any non-zero $f \in k[[x_r]]_{r \in \mathbb{N}}$, one has $f \in k[[x_r]]_{0 \leq r \leq s}$ for a suitable s , hence the existence of $(x_r) \in k^{\mathbb{N}}$ such that $f((x_r)) \neq 0$, and the result.

It is the occasion to recall that in case the algebraically closed field k is countable, there exist closed points of $\text{Spec}(k[[x_r]]_{r \in \mathbb{N}})$ that are not k -rational points: write $k = \{\alpha_r\}_{r \in \mathbb{N} \setminus \{0\}}$ and consider e.g., the kernel of the morphism $k[[x_r]]_{r \in \mathbb{N}} \rightarrow k(t)$ mapping x_0 to t and x_r ($r \geq 1$) to $\frac{1}{t - \alpha_r}$, which is surjective (see also [Ish04, Proposition 2.11])

6.7. SAI and WAI subsets of arc spaces. From now on one assumes that the basefield k is algebraically closed and of characteristic zero. The following result, which is a concatenation of [BMCS, Theorem 6.18] and [BMCn23, Theorem 1.7] was the initial motivation of the present paper.

Theorem 6.8. *Let X be a k -curve or a normal toric k -variety. Let ν be a divisorial valuation on X , assumed to be toric in case X is toric. Then there exists a non-empty Zariski open set U of $\mathcal{N}_X(\nu)$ such that the following holds:*

- (1) *such that for any γ_1, γ_2 in $U(k)$, the formal neighborhoods $\widehat{\mathcal{O}}_{\mathcal{L}_\infty(X), \gamma_1}$ and $\widehat{\mathcal{O}}_{\mathcal{L}_\infty(X), \gamma_2}$ are isomorphic.*
- (2) *Let η_ν be the generic point of $\mathcal{N}_X(\nu)$. Then there exists a k -section s of $\widehat{\mathcal{O}}_{\mathcal{L}_\infty(X), \eta_\nu}$ such that for any $\gamma \in U(k)$, the topological $\kappa(\eta_\nu)$ -algebras $\widehat{\mathcal{O}}_{\mathcal{L}_\infty(X), \gamma} \widehat{\otimes}_k \kappa(\eta_\nu)$ and $\widehat{\mathcal{O}}_{\mathcal{L}_\infty(X), \eta_\nu}^{(s)}[[u_r]]_{r \in \mathbb{N}}$ (where u_r are indeterminates) are isomorphic.*

This motivates the following definition, that features evident analogies with the concepts introduced in Subsection 3.8:

Definition 6.9. Let X be an arbitrary algebraic k -variety and ν be an arbitrary divisorial valuation. One says that ν is weakly analytically isosingular (WAI), resp. strongly analytically isosingular (SAI), if there exists a non-empty Zariski open set U of $\mathcal{N}_X(\nu)$ such that assertion (1) of Theorem 6.8 holds for ν , resp. assertion (2) of Theorem 6.8 holds for ν , possibly only over a suitable extension of K of $\kappa(\eta_\nu)$, that is to say, after applying $-\widehat{\otimes}_{\kappa(\eta_\nu)} K$ to the involved topological $\kappa(\eta_\nu)$ -algebras.

The following proposition supports the chosen terminology.

Proposition 6.10. *If ν is SAI, then ν is WAI.*

Proof. By the very definition, if ν is SAI, there exists an extension K of k such that for any γ_1, γ_2 in $U(k)$, the topological K -algebras $\widehat{\mathcal{O}}_{\mathcal{L}_\infty(X), \gamma_1} \widehat{\otimes}_k K$ and $\widehat{\mathcal{O}}_{\mathcal{L}_\infty(X), \gamma_2} \widehat{\otimes}_k K$ are isomorphic. Since η_ν lies in the open set of non-degenerate arcs, one may assume that any arc in U is non-degenerate. For $i = 1, 2$, let A_i be a minimal finite formal model of γ_i (see 6.3). Thus the topological K -algebras $(A_1 \widehat{\otimes}_k K)[[u_n]]_{n \in \mathbb{N}}$ and $(A_2 \widehat{\otimes}_k K)[[u_n]]_{n \in \mathbb{N}}$ are isomorphic. By Gabber's cancellation theorem ([BS17, Theorem 7.1]), the complete local K -algebras $A_1 \widehat{\otimes}_k K$ and $A_2 \widehat{\otimes}_k K$ are isomorphic. Since k is algebraically closed, by [Sei69b, Theorem 5], the complete local k -algebras A_1 and A_2 are isomorphic. Thus the topological k -algebras $\widehat{\mathcal{O}}_{\mathcal{L}_\infty(X), \gamma_1} \xrightarrow{\sim} A_1[[u_n]]_{n \in \mathbb{N}}$ and $\widehat{\mathcal{O}}_{\mathcal{L}_\infty(X), \gamma_2} \xrightarrow{\sim} A_2[[u_n]]_{n \in \mathbb{N}}$ are isomorphic. Thus ν is WAI. \square

The following question has a positive answer in the cases considered in Theorem 6.8. Recall that the analogous question for subsets of algebraic varieties also has a positive answer, as shown in the present paper (See Corollary 5.20).

Question 6.11. *Let ν be a divisorial valuation on an algebraic variety. If ν is WAI, then is ν SAI?*

The following question is equivalent to one already raised in [BMCS].

Question 6.12. *Is any divisorial valuation SAI (resp. WAI)?*

In case it turns out that there exist in general divisorial valuations that are not SAI (resp. WAI), it would then be interesting to try to characterize them geometrically.

6.13. Concluding observations. It might be possible that a line of thought analogous to the one used in the present paper to show that any WAI closed subset of an algebraic variety is SAI could lead to a positive answer to Question 6.11. This would in particular provide a new unified proof of the main results of [BMCS] and [BMCn23]. It is worth noting that regarding this kind of issues the context of arc spaces may present a priori notorious differences with that of algebraic varieties, as illustrated by the following proposition and Remark 6.15 below.

Proposition 6.14. *Let k be an algebraically closed field of characteristic zero. Let \mathcal{C} be an affine algebraic curve. We assume that \mathcal{C} is singular. For the sake of simplicity, we moreover assume that \mathcal{C} has only one singular point $c \in \mathcal{C}(k)$, and that c is analytically irreducible, i.e., $\widehat{\mathcal{O}}_{\mathcal{C},c}$ is a domain. Choose an isomorphism between the normalization of $\widehat{\mathcal{O}}_{\mathcal{C},c}$ and $k[[t]]$. Let v be the valuation on \mathcal{C} induced by ord_t and U be a non-empty Zariski open subset of $\mathcal{N}_{\mathcal{C}}(v)$ such as in the statement of Theorem 6.8.*

Then for any $\gamma' \in \mathcal{L}_{\infty}(\mathcal{C})(k) \setminus \mathcal{N}_{\mathcal{C}}(v)$ and any $\gamma \in U(k)$, the formal neighborhoods $\widehat{\mathcal{O}}_{\mathcal{L}_{\infty}(\mathcal{C}),\gamma}$ and $\widehat{\mathcal{O}}_{\mathcal{L}_{\infty}(\mathcal{C}),\gamma'}$ are not isomorphic.

Remark 6.15. In particular, $\mathcal{N}_{\mathcal{C}}(v)$ is in some sense a maximal WAI subset, yet, by Remark 6.5, its generic point is not differential, contrarily to what happens in the case of algebraic varieties (See Remark 5.19).

Proof. Under the given assumptions, $\mathcal{L}_{\infty}(\mathcal{C}) \setminus \mathcal{N}_{\mathcal{C}}(v)$ is exactly the set of arcs on \mathcal{C} whose center is smooth (see e.g., [BMCn23, Remark 3.14]). In particular, for any $\gamma' \in \mathcal{L}_{\infty}(\mathcal{C})(k) \setminus \mathcal{N}_{\mathcal{C}}(v)$, $\widehat{\mathcal{O}}_{\mathcal{L}_{\infty}(\mathcal{C}),\gamma'}$ is isomorphic to $k[[u_n]]_{n \in \mathbb{N}}$ (as easily follows from the compatibility of the formation of arc spaces with étale morphisms) and the minimal formal model of γ' is k . On the other hand, by [Bou23, Theorem 1.3], for any $\gamma \in U(k)$, the minimal formal model of γ has embedding dimension the multiplicity of the singularity c minus one. This shows that $\widehat{\mathcal{O}}_{\mathcal{L}_{\infty}(\mathcal{C}),\gamma'}$ and $\widehat{\mathcal{O}}_{\mathcal{L}_{\infty}(\mathcal{C}),\gamma}$ are not isomorphic. \square

As a last observation, let us point out that as a consequence of Corollary 5.15 and Theorem 5.18, in the case X is an algebraic variety and $x \in X(k)$ is a k -rational point, the dimension of the minimal formal model of $\widehat{\mathcal{O}}_{X,x}$ exactly corresponds to the dimension of the formal neighborhood of the generic point η_x of the component through x of its isosingular locus. In case v is a SAI divisorial valuation, it should be sensible that for a generic k -rational arc γ of $\mathcal{N}_X(v)$, a suitable analog of η_x is given by the generic point η_v of $\mathcal{N}_X(v)$ (Proposition 6.14 is a supporting fact in this direction). Both the dimension of the minimal formal model of γ and the dimension of $\widehat{\mathcal{O}}_{\mathcal{L}_{\infty}(X),\eta_v}$ seem to carry subtle yet interesting informations on the nature of the singularities of X (see e.g., [Bou23, MR18, CdFD22]). Yet they differ in general, and it would be interesting to understand the exact discrepancy between them.

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