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# General Continuity Bounds for Quantum Relative Entropies

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**Abstract**—In this article, we generalize a proof technique by Alicki, Fannes and Winter and introduce a method to prove continuity bounds for entropic quantities derived from different quantum relative entropies. For the Umegaki relative entropy, we mostly recover known almost optimal bounds, whereas, for the Belavkin-Staszewski relative entropy, our bounds are new. Finally, we use these continuity bounds to derive a new entropic uncertainty relation. This is a short version of [1].

## I. INTRODUCTION

Entropic quantities are essential in order to understand processes both in classical and quantum information theory. Examples include (conditional) entropy, (conditional) mutual information, and many others. In applications, one often needs to bound these quantities in terms of a metric distance to nearby states that can be evaluated more easily. Therefore, continuity bounds for entropic quantities have become indispensable tools in quantum information theory. They take the form of a bound on

$$\sup\{|g(\rho) - g(\sigma)| : \rho, \sigma \in \mathcal{S}_0, d(\rho, \sigma) \leq \epsilon\}.$$

Here,  $g$  is the entropic quantity of interest,  $\mathcal{S}_0$  is a suitable subset of the set of quantum states  $\mathcal{S}(\mathcal{H})$  in a finite-dimensional Hilbert space, and  $d$  is a metric on  $\mathcal{S}(\mathcal{H})$ .

Bounds of this form have a long history. In 1973, Fannes [2] proved uniform continuity bounds for the von Neumann entropy, which were sharpened in [3], [4]. Later Alicki and Fannes proved an inequality for the conditional entropy [5], which Winter improved in [6] to an almost tight version.

As realized by Shirokov [7], [8], the proof by Winter and related versions [9], [10] do not only work for the conditional entropy but can be generalized and made applicable to a great variety of entropic quantities. Shirokov coined it the *Alicki-Fannes-Winter (AFW) method*. The present article continues along this line of work, further generalising the method. Our aim is to go beyond entropic quantities derived from the Umegaki relative entropy [11], defined as

$$D(\rho\|\sigma) := \text{Tr}[\rho(\log \rho - \log \sigma)]$$

for quantum states  $\rho, \sigma$  such that  $\ker \sigma \subseteq \ker \rho$  and  $+\infty$  otherwise, and to be able to prove continuity bounds for entropic quantities derived from the Belavkin-Staszewski relative entropy (BS-entropy) [12] as well. The *BS-entropy* is defined as

$$\widehat{D}(\rho\|\sigma) := \text{Tr}[\rho \log(\rho^{1/2} \sigma^{-1} \rho^{1/2})]$$

for quantum states  $\rho, \sigma$  such that  $\ker \sigma \subseteq \ker \rho$  and  $+\infty$  otherwise.

Both Umegaki's relative entropy and BS-entropy reduce to the classical Kullback-Leibler relative entropy [13] if the quantum states commute, but the BS-entropy is much less studied [14]–[16]. Recently, there has been renewed interest in the BS-entropy and its properties [17], [18] because of its applications to channel capacities [19] and quantum many-body systems [20]–[22]. In the latter applications, the authors make use of the *BS-mutual information*

$$\widehat{I}_\rho(A : B) := \widehat{D}(\rho_{AB} \| \rho_A \otimes \rho_B),$$

which is defined as the mutual information, but replacing the Umegaki relative entropy by the BS-entropy. Other entropic quantities derived from the BS-entropy have been considered in [23], [24].

In Section II, we introduce our method. First, we choose a quantum relative entropy  $\mathbb{D}(\rho\|\sigma)$ , for example, the Umegaki relative entropy. Then, we prove that it is *almost concave*, i.e.,

$$-f(p) \leq \mathbb{D}(\rho_p \| \sigma_p) - p\mathbb{D}(\rho_0 \| \sigma_0) - (1-p)\mathbb{D}(\rho_1 \| \sigma_1) \leq 0$$

for  $\rho_p = p\rho_0 + (1-p)\rho_1$ ,  $\sigma_p = p\sigma_0 + (1-p)\sigma_1$ , with  $\rho_0, \rho_1, \sigma_0, \sigma_1 \in \mathcal{S}(\mathcal{H})$ ,  $p \in [0, 1]$  and a proper function  $f$ .

For suitably chosen subsets of the set of quantum states which we call *perturbed  $\Delta$ -invariant subsets*, the almost concavity of the quantum relative entropy translates into the *almost local affinity* (c.f. Definition 2.1) of derived entropic quantities  $g$ , i.e.,

$$-a_g(p) \leq g(p\rho + (1-p)\sigma) - pg(\rho) - (1-p)g(\sigma) \leq b_g(p)$$

for functions  $a_g, b_g$ . That means that the entropic quantity is close to being convex and concave. Our method allows us to convert the almost local affinity into uniform continuity bounds on the entropic quantities  $g$ , where the continuity bounds are given in terms of how close the entropic quantity is to being convex and concave:

$$\sup_{\substack{\rho, \sigma \in \mathcal{S}_0 \\ d(\rho, \sigma) \leq \epsilon}} |g(\rho) - g(\sigma)| \leq C(\epsilon; a_g, b_g).$$

This is illustrated in Figure 1. Restriction to perturbed  $\Delta$ -invariant subsets is vital to the proof of the AFW method and the well-known discontinuities of the Umegaki as well as the Belavkin-Staszewski relative entropies further make it

necessary to use the more general perturbed property instead of the original  $\Delta$ -invariance presented in [8].

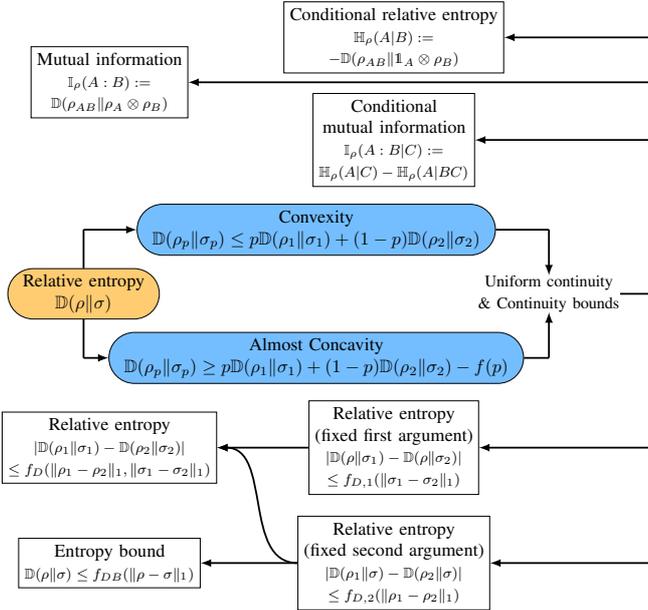


Fig. 1. The proof of convexity and concavity of a divergence gives, through our method, a variety of continuity bounds for entropic quantities derived from that divergence. It further allows concluding continuity bounds for the divergence itself. In the above figure, we depicted the scheme and listed quantities that our approach produces results for.

While the primary focus lies on the introduction of a method to derive uniform continuity bounds in a unified manner, we showcase its usefulness by first applying it to the Umegaki relative entropy in Section III-A and then to the BS-entropy in Section III-B. We begin by establishing almost concavity of the relative entropies and then present the various continuity bounds that follow from our method for each entropy separately.

We continue with Section IV, where we use the obtained bounds to derive results in various contexts within quantum information theory, namely new entropic uncertainty relations, continuity of optimized quantities and robustness of the set of approximate quantum Markov chains. Finally, we conclude in Section V with a short discussion and an outlook.

## II. METHOD

Our method derives continuity bounds for functions from the quantum states to the reals, such as quantities based on a relative entropy (e.g. conditional entropy, mutual information, and  $\rho \mapsto \mathbb{D}(\rho||\sigma)$  with fixed  $\sigma$ ). However, it should be noted that the method is not limited to this specific setting.

The method is based on two main ingredients. First, the boundedness of the function on a set of quantum states, and second a property called almost local affinity (as defined in Definition 2.1). We mentioned earlier that for quantities related to a relative entropy, almost local affinity can be obtained from the convexity and almost concavity of this relative entropy, provided the almost concave remainder is "well-behaved". This means that under the reduction from the relative entropy to the

entropic quantity of choice (possibly also restricting the set of input states), the almost concave bound of the divergence becomes independent of the input state.

Before giving the main theorem, we will introduce two definitions that are necessary for its formalisation. The first is the concept of almost local affinity.

*Definition 2.1 (Almost locally affine):* Let  $f$  be a real-valued function on the convex set  $\mathcal{S}_0 \subseteq \mathcal{S}(\mathcal{H})$ . We say that  $f$  is *almost locally affine*, if there exist continuous functions  $a_f, b_f : [0, 1] \rightarrow \mathbb{R}$ , which are non decreasing on  $[0, \frac{1}{2}]$ , vanish as  $p \rightarrow 0^+$  and satisfy

$$-a_f(p) \leq f(p\rho + (1-p)\sigma) - pf(\rho) - (1-p)f(\sigma) \leq b_f(p) \quad (1)$$

for all  $p \in [0, 1]$  and  $\rho, \sigma \in \mathcal{S}_0$ .

In the introduction, we mentioned the perturbed  $\Delta$ -invariant subsets. This property, together with convexity, constitutes a constraint to the domain of the entropic quantity investigated. The property is needed, as it guarantees that for two elements of a set, two interpolation states, or "intermediate states" are as well contained in the set.

*Definition 2.2 (Perturbed  $\Delta$ -invariant):* A set  $\mathcal{S}_0 \subseteq \mathcal{S}(\mathcal{H})$  is called perturbed  $\Delta$ -invariant with perturbation parameter  $t \in [0, 1)$ , if for all  $\rho, \sigma \in \mathcal{S}_0$  with  $\rho \neq \sigma$ , there exists a state  $\tau$  such that both states

$$\Delta^\pm(\rho, \sigma, \tau) = t\tau + (1-t)\varepsilon^{-1}[\rho - \sigma]_\pm,$$

lie in  $\mathcal{S}_0$ . In the above  $\varepsilon = \frac{1}{2}\|\rho - \sigma\|_1$ , and  $[\cdot]_\pm$  denote the positive and negative parts of a self-adjoint operator, respectively. For  $t = 0$ , this definition reduces to that in [8].

Now that we have introduced all the necessary definitions, we can formulate the method.

*Theorem 2.1:* Let  $\mathcal{S}_0 \subseteq \mathcal{S}(\mathcal{H})$  be a perturbed  $\Delta$ -invariant, convex subset of  $\mathcal{S}(\mathcal{H})$  containing more than one element, and let  $f$  be an almost locally affine (ALAFF) function. If the function  $f$  satisfies

$$C_f^t := \sup_{\substack{\rho, \sigma \in \mathcal{S}_0 \\ \frac{1}{2}\|\rho - \sigma\|_1 = 1-t}} |f(\rho) - f(\sigma)| < +\infty. \quad (2)$$

then it is uniformly continuous, and we have for all  $\varepsilon \in (0, 1]$

$$\begin{aligned} & \sup_{\substack{\rho, \sigma \in \mathcal{S}_0 \\ \frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon}} |f(\rho) - f(\sigma)| \\ & \leq C_f^t \frac{\varepsilon}{1-t} + \frac{1-t+\varepsilon}{1-t} E_f^{\max} \left( \frac{\varepsilon}{1-t+\varepsilon} \right), \end{aligned} \quad (3)$$

where  $E_f^{\max} : [0, 1) \rightarrow \mathbb{R}$ ,

$$p \mapsto E_f^{\max}(p) = (1-p) \max \left\{ \frac{E_f(s)}{1-s} : 0 \leq s \leq p \right\}.$$

Here we defined  $E_f = a_f + b_f$  (c.f. Definition 2.1). Note that for  $\varepsilon \in (0, 1-t]$   $E_f$  and  $E_f^{\max}$  coincide.

### III. RELATIVE ENTROPIES

The natural method to derive the almost local affinity property of entropic quantities relies on having "well-behaved" almost concave bounds for the relative entropy  $\mathbb{D}(\rho\|\sigma)$  they are subordinate to. A divergence  $\mathbb{D}(\rho\|\sigma)$  is called *almost (jointly) concave* on a convex subset  $\mathcal{S}_0 \subset \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H})$  if for every  $(\rho_j, \sigma_j) \in \mathcal{S}_0$  ( $j = 0, 1$ ) there is a function  $f(p)$  such that for every  $p \in [0, 1]$

$$-f(p) \leq \mathbb{D}(\rho_p\|\sigma_p) - p\mathbb{D}(\rho_0\|\sigma_0) - (1-p)\mathbb{D}(\rho_1\|\sigma_1) \leq 0,$$

where  $\rho_p := p\rho_0 + (1-p)\rho_1$  and  $\sigma_p := p\sigma_0 + (1-p)\sigma_1$ . The well-known (joint) convexity of the Umegaki [25] and BS [15], [26] relative entropies ensures that the inequality on the right-hand-side holds. Our main results provide a remainder  $f(p)$  involving the function

$$f_{\alpha_1, \alpha_2}(p) = p \log(p + (1-p)\alpha_1) + (1-p) \log((1-p) + p\alpha_2).$$

Here,  $\alpha_1$  and  $\alpha_2$  are scalar constants defined in terms of the states  $\rho_j, \sigma_j$  via the map

$$\alpha(O, P, Q) := \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[ OP^{\frac{1+it}{2}} QP^{\frac{1-it}{2}} \right].$$

In the above  $O, P, Q$  are positive semi-definite operators (exponentiation and inverses are restricted to the support), and  $\beta_0(t)dt$  is a suitable probability measure on  $\mathbb{R}$ , see (18).

*Theorem 3.1:* The Umegaki relative entropy  $D(\cdot\|\cdot)$  is almost concave on the subset  $\mathcal{S}_0 \subset \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H})$  of all pairs  $(\rho, \sigma)$  such that  $\ker \rho \subset \ker \sigma$  with remainder function

$$f(p) = h(p) \frac{1}{2} \|\rho_1 - \rho_2\|_1 + f_{c_1, c_2}(p), \quad (4)$$

where  $c_1 = \alpha(\rho_1, \sigma_1^{-1}, \sigma_2)$  and  $c_2 = \alpha(\rho_2, \sigma_2^{-1}, \sigma_1)$ .

As for the BS-relative entropy, we require that the second input state belongs to the set  $\mathcal{S}_+(\mathcal{H})$  of full-rank states.

*Theorem 3.2:* The BS-relative entropy  $\widehat{D}(\cdot\|\cdot)$  is almost concave on  $\mathcal{S}_0 = \mathcal{S}(\mathcal{H}) \times \mathcal{S}_+(\mathcal{H})$  with remainder function

$$\widehat{f}(p) = h(p)(1 - \delta_{\rho_1, \rho_2})\widehat{c}_0 + \widehat{f}_{\widehat{c}_1, \widehat{c}_2}(p),$$

where  $\delta_{\rho_1, \rho_2} \in \{0, 1\}$  is one if and only if  $\rho_1 = \rho_2$ , and

$$\begin{aligned} \widehat{c}_0 &= \max \{ \|\sigma_1^{-1}\|_\infty, \|\sigma_2^{-1}\|_\infty \}, \\ \widehat{c}_1 &= \alpha(\rho_1, \rho_1^{1/2} \sigma_1^{-1} \rho_1^{1/2}, \rho_1^{-1/2} \sigma_2 \rho_1^{-1/2}), \\ \widehat{c}_2 &= \alpha(\rho_2, \rho_2^{1/2} \sigma_2^{-1} \rho_2^{1/2}, \rho_2^{-1/2} \sigma_1 \rho_2^{-1/2}). \end{aligned}$$

The expressions for  $f(p)$  and  $\widehat{f}(p)$  simplify in some particular cases.

Special cases	Umegaki	BS-entropy
$p = 0, 1$	$f(p) = 0$	$f(p) = 0$
$\sigma = \sigma_1 = \sigma_2$	$f(p) = h(p)$	$\widehat{f}(p) = \widehat{c}_0 h(p)$
$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ $\sigma_i = (\rho_i)_A \otimes \mathbb{1}_B$ $i = 1, 2$	$f(p) = h(p)$	$\widehat{f}(p) = \widehat{c}_0 h(p) + f_{\widehat{c}_0, \widehat{c}_0}(p)$

#### A. Umegaki Relative Entropy

The estimate given in Theorem 3.1 is tight: if  $\dim(\mathcal{H}) \geq 2$ , and we fix two orthonormal pure states  $|0\rangle, |1\rangle$  in  $\mathcal{H}$  and any  $t \in (0, 1)$ , then the states  $\rho_0 := |0\rangle\langle 0|$ ,  $\rho_1 := |1\rangle\langle 1|$ ,  $\sigma_0 := t|0\rangle\langle 0| + (1-t)|1\rangle\langle 1|$  and  $\sigma_1 := (1-t)|0\rangle\langle 0| + t|1\rangle\langle 1|$  satisfy that for every  $p \in [0, 1]$

$$f(p) = pD(\rho_0\|\sigma_0) + (1-p)D(\rho_1\|\sigma_1) - D(\rho_p\|\sigma_p),$$

where  $f(p)$  is given in (4).

Combining Theorems 3.1 and 2.1, we can provide estimates for the uniform continuity of  $D(\cdot\|\sigma)$  on the set of states  $\rho$  with  $\ker \sigma \subset \ker \rho$ . As a consequence, we obtain that for  $\rho, \sigma$  as before, denoting  $\varepsilon := \frac{1}{2} \|\rho - \sigma\|_1 \leq 1$  and  $r(\varepsilon) := (1 + \varepsilon)h(\varepsilon/(1 + \varepsilon))$  it holds that

$$D(\rho\|\sigma) \leq \varepsilon \log \widetilde{m}_\sigma^{-1} + r(\varepsilon) \leq \left(1 + \frac{\log \widetilde{m}_\sigma^{-1}}{\sqrt{2}}\right) \sqrt{2\varepsilon}, \quad (5)$$

where  $\widetilde{m}_\sigma$  is the minimal non-zero eigenvalue of  $\sigma$ . Let us remark that alternative estimates of  $D(\rho\|\sigma)$  have been provided before [27]–[29], all requiring that  $\sigma$  has full rank, in contrast to the above result. Moreover, the logarithmic scaling with  $\widetilde{m}_\sigma^{-1}$  in (5) improves the linear dependence that appears in the results by Bratteli-Robinson [28] and Vershynina [29]. Numerical simulations show, however, a slight advantage of the estimate by Audenaert-Eisert [27].

Combining Theorems 3.2 and 2.1 we also deduce continuity estimates for well-known entropic quantities derived from the Umegaki relative entropy. Let us recall the definitions and describe the corresponding bounds. For a state on a bipartite system  $\rho \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$  the *conditional entropy* (CE) is given by

$$\begin{aligned} \widehat{H}_\rho(A|B) &:= -D(\rho_{AB} \|\mathbb{1}_A \otimes \rho_B) \\ &= \max_{\sigma_B \in \mathcal{S}(\mathcal{H}_B)} -D(\rho_{AB} \|\mathbb{1}_A \otimes \sigma_B), \end{aligned}$$

and the *mutual information* (MI) by

$$I_\rho(A : B) := D(\rho_{AB} \|\rho_A \otimes \rho_B) = S(\rho_A) - H_\rho(A|B).$$

For a state on a tripartite system  $\rho \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$  the *conditional mutual information* (CMI) is defined as

$$I_\rho(A : B|C) = H_\rho(A|C) - H_\rho(A|BC).$$

Fixed  $\rho, \sigma \in \mathcal{S}(\mathcal{H})$  and  $\varepsilon$  with  $\|\rho - \sigma\|_1/2 \leq \varepsilon \leq 1$  and  $r(\varepsilon)$  as above, we have the following estimates:

Entropic Quantity	Upper Bound
CE: $ H_\rho(A B) - H_\sigma(A B) $	$2\varepsilon \log d_A + r(\varepsilon)$
MI: $ I_\rho(A : B) - I_\sigma(A : B) $	$2\varepsilon \log \min\{d_A, d_B\}$
CMI: $ I_\rho(A : B C) - I_\sigma(A : B C) $	$+2r(\varepsilon)$

Notice that the estimate on the conditional entropy coincides with the result of Winter [6] which he proved to be almost tight. As for the MI and the CMI, the estimates coincide with the tightest previously known results, see e.g. [7] and [30, Lemma 4], respectively.

Our method also allows us to derive continuity bounds for  $D(\rho|\cdot)$  on the set of states  $\sigma$  with  $\ker \sigma \subset \ker \rho$  and  $\tilde{m}\rho \leq \sigma$  for  $1 > \tilde{m} > 0$ . For such a case, we need the full scope of the method and get a rather involved continuity bound, which can with  $\varepsilon = \frac{1}{2}\|\sigma_1 - \sigma_2\|$  be simplified to the following expression

$$|D(\rho|\sigma_1) - D(\rho|\sigma_2)| \leq \frac{2\varepsilon}{l_{\tilde{m}}} \log \tilde{m}^{-1} + \log \left( 1 + \frac{\varepsilon}{l_{\tilde{m}} + \varepsilon \tilde{m}} \right). \quad (6)$$

This, jointly with the aforementioned continuity bound for  $D(\cdot|\sigma)$ , provides continuity bounds for the relative entropy in the same regimes mentioned above. We can derive, with  $\varepsilon = \frac{1}{2}\|\rho_1 - \rho_2\|$ ,  $\delta = \frac{1}{2}\|\sigma_1 - \sigma_2\|$  the following

$$\begin{aligned} |D(\rho_1|\sigma_1) - D(\rho_2|\sigma_2)| \\ \leq (\sqrt{2} + \log \tilde{m}^{-1})\varepsilon^{1/2} + \frac{3\delta}{l_{\tilde{m}}} \log \tilde{m}^{-1} \\ + 2 \log \left( 1 + \frac{\delta}{l_{\tilde{m}} + \delta \tilde{m}} \right). \end{aligned}$$

This is, as far as we know, the first continuity bound for the relative entropy in both inputs existing in the literature.

### B. Belavkin-Staszewski Relative Entropy

In contrast to the case of relative entropy, the estimate in Theorem 3.2 is not tight. Indeed, the BS and the Umegaki relative entropies for commuting states coincide. However,  $\hat{f}(p)$  does not reduce to  $f(p)$  in this case, since  $c_j = \hat{c}_j$  for  $j = 1, 2$ , but  $\hat{c}_0$  is larger than one if  $\mathcal{H} \neq \mathbb{C}$ . This is strong evidence for the existence of an improved bound.

Combining Theorems 3.2 and 2.1, we can again prove continuity and divergence bounds for entropic quantities related to the BS-relative entropy. Here again, the full scope of the method is needed, since former quantities are discontinuous for positive semi-definite states. Let us recall the main definitions and some subtle differences with respect to their counterparts. The *BS-conditional entropy* is defined for a state on a bipartite system  $\rho \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$  as

$$\hat{H}_\rho(A|B) := -\hat{D}(\rho_{AB}|\mathbb{1}_A \otimes \rho_B).$$

This quantity is discontinuous on  $\mathcal{S}(\mathcal{H})$  and not only do numerics indicate that it differs from the variational definition

$$\hat{H}_\rho^{var}(A|B) = \sup_{\sigma_B \in \mathcal{S}(\mathcal{H}_B)} -\hat{D}(\rho_{AB}|\mathbb{1}_A \otimes \sigma_B),$$

where  $\hat{H}_\rho(A|B) \leq \hat{H}_\rho^{var}(A|B)$ , but we are further able to prove that  $\hat{H}_\rho^{var}(A|B)$  is continuous on  $\mathcal{S}(\mathcal{H})$  (see Section IV-B). The *BS-mutual information* is defined as

$$\hat{I}_\rho(A : B) := \hat{D}(\rho_{AB}|\rho_A \otimes \rho_B)$$

and again differs from the variational definition

$$\hat{I}_\rho^{var}(A : B) := \inf_{\sigma_A \otimes \sigma_B \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)} \hat{D}(\rho_{AB}|\sigma_A \otimes \sigma_B).$$

Indeed,  $\hat{I}_\rho(A : B)$  may scale with  $\|\rho_A^{-1}\|_\infty$  and  $\|\rho_B^{-1}\|_\infty$ , whereas  $\hat{I}_\rho^{var}(A : B)$  is bounded by  $2 \log \min\{d_A, d_B\}$ , just

like  $I_\rho(A : B)$ . For a tripartite state  $\rho \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$  the *BS-conditional mutual information* is defined by

$$\hat{I}_\rho(A : B|C) = \hat{H}_\rho(A|C) - \hat{H}_\rho(A|BC).$$

Fixed  $0 < m < d_{\mathcal{H}}^{-1}$  and states  $\rho, \sigma$  with minimal eigenvalues  $\geq m$  and  $\varepsilon := \frac{1}{2}\|\rho - \sigma\|_1$ , we have the following estimates:

Entropic Quantity	Upper Bound
BS-CE: $ \hat{H}_\rho(A B) - \hat{H}_\sigma(A B) $	$\frac{C\sqrt{\varepsilon}}{m(d_{\mathcal{H}}^{-1} - m)}$
BS-MI: $ \hat{I}_\rho(A : B) - \hat{I}_\sigma(A : B) $	$(C : \text{absolute constant})$
BS-CMI: $ \hat{I}_\rho(A : B C) - \hat{I}_\sigma(A : B C) $	

Although the BS-conditional entropy is discontinuous on the set of positive semi-definite operators, numerics suggest that a bound independent of the minimal eigenvalues of the (full-rank) states might be feasible, or even an almost convexity estimate for the BS-conditional entropy also independent of these minimal eigenvalues. This would also yield an improved bound for the BS-CMI.

For the BS-mutual information, however, we know that a continuity bound will depend on the minimal eigenvalues of the states, since  $\hat{I}_\rho(A : B)$  itself may scale with  $\|\rho_A^{-1}\|_\infty$ , as we mentioned before.

## IV. APPLICATIONS

The results of almost concavity for the Umegaki and BS relative entropies, as well as the continuity bounds from the previous section, find a plethora of applications in the context of quantum information theory. Here, we show three families of applications: Obtaining new entropic uncertainty relations, deriving continuity of optimized quantities and showing the robustness of the set of approximate quantum Markov chains.

### A. Entropic uncertainty relations

Let us consider a system  $A$  and two POVMs  $\mathbf{X} := \{X_x\}_{x \in \mathcal{X}}$  and  $\mathbf{Y} := \{Y_y\}_{y \in \mathcal{Y}}$ . Let us further assume the presence of side information  $M$  which might help to better predict the outcomes of  $\mathbf{X}$  and  $\mathbf{Y}$ . Then, for any bipartite state  $\rho_{AM} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_M)$ , the following inequality holds [31]:

$$\begin{aligned} H(X|M)_{(\Phi_{\mathbf{X}} \otimes \text{id}_M)(\rho)} + H(Y|M)_{(\Phi_{\mathbf{Y}} \otimes \text{id}_M)(\rho)} \\ \geq -\ln c + H(A|M)_\rho, \end{aligned} \quad (7)$$

where  $c = \max_{x,y} \{\text{tr}(X_x Y_y)\}$  and  $\Phi_{\mathbf{Z}}$  is given for  $\mathbf{Z} \in \{\mathbf{X}, \mathbf{Y}\}$  by

$$\Phi_{\mathbf{Z}}(\rho_A) := \sum_{z \in \mathcal{Z}} \text{tr}(\rho_A Z_z) |z\rangle\langle z|_{\mathcal{Z}}. \quad (8)$$

This is known as an *entropic uncertainty relation*. Some references exploring similar inequalities are [32] for the special case of measurements in two orthonormal bases and [33] for the case without a memory. We also refer the reader to the review [34] as well as to [35] for a more in detail discussion of the setting and related topics.

In this paper, we consider two von Neumann algebras  $\mathcal{N}_X$  and  $\mathcal{N}_Y$  such that  $\mathcal{N}_X$ , resp.  $\mathcal{N}_Y$ , is diagonal in some orthonormal basis  $\{|e_x^{(\mathcal{X})}\rangle\}$ , resp.  $\{|e_y^{(\mathcal{Y})}\rangle\}$ , and we denote by  $E_{\mathcal{Z}}$  the Pinching map onto the diagonal spanned by  $\{|e_z^{(\mathcal{Z})}\rangle\langle e_z^{(\mathcal{Z})}| \}$  for  $\mathcal{Z} \in \{\mathcal{X}, \mathcal{Y}\}$ . We further consider the algebra  $\mathcal{M} = \mathbb{C}\mathbb{1}_\ell \otimes \mathcal{B}(\mathcal{H}_M)$  and the associated map  $E_{\mathcal{M}} \otimes \text{id}_M = \frac{1}{d_A}\mathbb{1}_A \otimes \text{Tr}_A[\cdot]$ , for  $d_A$  the dimension of  $\mathcal{H}_A$ . In this setting, a similar inequality to Eq. (7) was proven in [36], where  $c$  was given by

$$c := d_A \max_{x,y} |\langle e_x^{(\mathcal{X})} | e_y^{(\mathcal{Y})} \rangle|^2. \quad (9)$$

Here, as a consequence of our continuity bound from Eq. (6), we can prove the following result, which can be viewed as a new entropic uncertainty relation.

*Corollary 1:* In the conditions above, the following inequality holds for any  $\rho_{AM} \in \mathcal{S}(\mathcal{H}_{AM})$ :

$$H(X|M)_{(E_{\mathcal{X}} \otimes \text{id}_M)(\rho_{AM})} + H(Y|M)_{(E_{\mathcal{Y}} \otimes \text{id}_M)(\rho_{AM})} \geq -\xi_{\text{RE}} + H(A|M)_{\rho_{AM}}, \quad (10)$$

where

$$\xi_{\text{RE}} := \frac{2 \log m^{-1} + m^{-1}}{1 - m} \left( \sum_x \max_y \left| \frac{1}{d_A} - |\langle e_x^{(\mathcal{X})} | e_y^{(\mathcal{Y})} \rangle|^2 \right| \right)^{1/2}, \quad (11)$$

for  $m$  given by:

$$m := \min \left\{ \frac{1}{(d_A)^2}, \frac{1}{d_A} \min_{x,y} |\langle e_x^{(\mathcal{X})} | e_y^{(\mathcal{Y})} \rangle|^2 \right\}. \quad (12)$$

### B. Continuity of optimized quantities

Given  $\mathcal{C} \in \mathcal{S}(\mathcal{H})$  a compact, convex subset of the set of quantum states with at least one positive definite state, we can define the minimal distance to  $\mathcal{C}$  in terms of a relative entropy  $\mathbb{D}(\cdot||\cdot)$  as follows:

$$\mathbb{D}_{\mathcal{C}}(\rho) := \inf_{\gamma \in \mathcal{C}} \mathbb{D}(\rho||\gamma). \quad (13)$$

The fact that  $\mathcal{C}$  contains a positive definite state guarantees that  $D_{\mathcal{C}}(\rho) < \infty$  for all  $\rho \in \mathcal{S}(\mathcal{H})$  (cf. [6]), and the infimum is attained due to the lower semi-continuity of the relative entropy [37] and Weierstrass' theorem [38, Theorem 2.43]. A prominent example of such a  $\mathcal{C}$  includes  $\text{SEP}_{AB}$ , the set of separable states for systems  $A, B$ .

For  $\mathcal{C}$  as above, we can prove the following results of almost local affinity for the Umegaki relative entropy

$$-h(p) \leq D_{\mathcal{C}}(p\rho + (1-p)\sigma) - pD_{\mathcal{C}}(\rho) - (1-p)D_{\mathcal{C}}(\sigma) \leq 0,$$

and for the BS-entropy

$$-g_d(p) \leq \widehat{D}_{\mathcal{C}}(p\rho + (1-p)\sigma) - p\widehat{D}_{\mathcal{C}}(\rho) - (1-p)\widehat{D}_{\mathcal{C}}(\sigma) \leq 0,$$

with  $g_d(p) := \frac{d}{p^{1/d}}h(p) - \log(1 - p^{1/d})$  for  $p \in (0, 1)$  and  $g_d(0) := 0$ . These inequalities, jointly with Theorem 2.1, allow us to prove uniform continuity and derive continuity bounds for any quantity defined as  $D_{\mathcal{C}}(\rho)$  and  $\widehat{D}_{\mathcal{C}}(\rho)$ , with  $\mathcal{C}$  as above. In particular, we recover the previously known bound for the relative entropy of entanglement from [6, Corollary 8]:

$$|D_{\text{SEP}_{AB}}(\rho) - D_{\text{SEP}_{AB}}(\sigma)| \leq \varepsilon \log \min\{d_A, d_B\} + r(\varepsilon).$$

Moreover, we show that  $\widehat{H}_{\rho}^{\text{var}}(A|B)$  is continuous in  $\mathcal{S}(\mathcal{H})$ , as opposed to  $\widehat{H}_{\rho}(A|B)$ , which is only continuous in  $\mathcal{S}_+(\mathcal{H})$ . Following an analogous procedure, we obtain additional continuity bounds for both the Rains and the BS-Rains information.

### C. Approximate Quantum Markov Chains

Consider a tripartite Hilbert space  $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ ,  $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$  a positive state on it, and the conditional mutual information of  $\rho_{ABC}$  between  $A$  and  $C$  conditioned on  $B$ . This CMI vanishes if, and only if, it coincides with its Petz recovery [39]

$$\rho_{ABC} = \rho_{AB}^{1/2} \rho_B^{-1/2} \rho_{BC} \rho_B^{-1/2} \rho_{AB}^{1/2}, \quad (14)$$

i.e., whenever  $\rho_{ABC}$  is a quantum Markov chain. As a consequence of our continuity bounds, we can obtain a robustness result for the condition of the quantum Markov chain, namely, we can show that a  $\rho_{ABC}$  has a "small" CMI if, and only if, it is close to its Petz recovery. More specifically, combining the continuity bound for the CMI from Section III-A with the findings of [40], we have

$$\begin{aligned} & \left( \frac{\pi}{8} \right)^4 \|\rho_B^{-1}\|_{\infty}^{-2} \|\rho_{ABC}^{-1}\|_{\infty}^{-2} \left\| \rho_{ABC} - \rho_{AB}^{1/2} \rho_B^{-1/2} \rho_{BC} \rho_B^{-1/2} \rho_{AB}^{1/2} \right\|_1^4 \\ & \leq I_{\rho}(A : C|B) \\ & \leq 2(\log \min\{d_A, d_C\} + 1) \left\| \rho_{ABC} - \rho_{AB}^{1/2} \rho_B^{-1/2} \rho_{BC} \rho_B^{-1/2} \rho_{AB}^{1/2} \right\|_1^{1/2}. \end{aligned}$$

## V. CONCLUSION

In this work, we have introduced a new method to derive uniform continuity bounds for various entropic quantities arising from quantum divergences. We have then applied this method to the Umegaki relative entropy and the BS-entropy. Finally, we have applied the new bounds to prove new entropic uncertainty relations, continuity of optimized quantities and robustness of approximate quantum Markov chains.

While for the Umegaki relative entropy, our bounds recover previous results which are known to be almost tight, there is evidence that our bounds for the entropic quantities arising from the BS-entropy can be improved. First, if  $\rho$  and  $\sigma$  commute, Umegaki and BS-relative entropy coincide, and so should their continuity bounds. However, our bounds do not have this property. Moreover, numerics suggest that the bound for the BS-conditional entropy should be independent of the minimal eigenvalues of the state, such that our bounds can likely be improved. On the other hand, the BS-conditional entropy exhibits discontinuities for singular states, such that possible eigenvalue-independent continuity bounds can only hold on the set of full-rank states. This behaviour is different from the behaviour of the usual conditional entropy. It is therefore important to better understand the pathologies arising from the definition of the BS-entropy compared to the Umegaki relative entropy. We leave these improvements for future work.

Our method is not limited to the relative entropies studied in this article but works more generally for all almost concave quantum divergences. Therefore, it is natural to ask whether

this method can work for quantities such as Tsallis entropies, different variants of Rényi entropies (Petz [37], sandwiched [41], geometric [19]), and many more. We are exploring these directions in a subsequent project.

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## APPENDIX

### A. Proof of Theorem 2.1

Let  $t \in [0, 1]$  and  $\varepsilon \in (0, 1]$ . Let further  $\rho, \sigma \in \mathcal{S}_0$  with  $\frac{1}{2}\|\rho - \sigma\|_1 = \varepsilon$ . By the property of perturbed  $\Delta$ -invariance there exists at least one  $\tau \in \mathcal{S}(\mathcal{H})$  such that  $\gamma_{\pm} := \Delta^{\pm}(\rho, \sigma, \tau) \in \mathcal{S}_0$ . Now for every such  $\gamma_{\pm}$  we have that

$$\begin{aligned}\omega &= \frac{1-t}{1-t+\varepsilon}\rho + \frac{\varepsilon}{1-t+\varepsilon}\gamma_{-} \\ &= \frac{1-t}{1-t+\varepsilon}\sigma + \frac{\varepsilon}{1-t+\varepsilon}\gamma_{+},\end{aligned}$$

which can be easily checked by inserting the explicit form of  $\gamma_{\pm}$  and using that  $[\rho - \sigma]_{+} - [\rho - \sigma]_{-} = \rho - \sigma$ . We get that  $\omega \in \mathcal{S}_0$ , since  $\mathcal{S}_0$  is convex, which allows us to evaluate  $f$  at  $\omega$  and use Eq. (1) for both of the representations of  $\omega$ . This gives us

$$\begin{aligned}-a_f(p) &\leq f(\omega) - (1-p)f(\rho) - pf(\gamma_{-}) \leq b_f(p), \\ -a_f(p) &\leq f(\omega) - (1-p)f(\sigma) - pf(\gamma_{+}) \leq b_f(p),\end{aligned}$$

where we set  $p = p(\varepsilon) = \frac{\varepsilon}{1-t+\varepsilon}$  for better readability. Note that  $p \in (0, \frac{1}{2-t}] \subseteq [0, 1]$  as  $\varepsilon \in (0, 1]$  and  $t \in [0, 1]$  and further that  $p(\varepsilon)$  is monotone with respect to  $\varepsilon$ . We recombine the above to get

$$\begin{aligned}(1-p)(f(\rho) - f(\sigma)) &\leq p(f(\gamma_{+}) - f(\gamma_{-})) + a_f(p) + b_f(p), \\ (1-p)(f(\sigma) - f(\rho)) &\leq p(f(\gamma_{-}) - f(\gamma_{+})) + a_f(p) + b_f(p).\end{aligned}$$

Those two inequalities immediately give us

$$(1-p)|f(\rho) - f(\sigma)| \leq p|f(\gamma_{+}) - f(\gamma_{-})| + (a_f + b_f)(p).$$

If we now insert  $E_f = a_f + b_f$ , we obtain

$$|f(\rho) - f(\sigma)| \leq \frac{p}{1-p}|f(\gamma_{+}) - f(\gamma_{-})| + \frac{1}{1-p}E_f(p).$$

In the case that  $C_f^t$  is finite, we have  $|f(\gamma_{+}) - f(\gamma_{-})| \leq C_f^t$  as  $\gamma_{\pm} \in \mathcal{S}_0$  and fulfil  $\frac{1}{2}\|\gamma_{+} - \gamma_{-}\|_1 = 1-t$ . This allows us to take the supremum over all  $\rho, \sigma \in \mathcal{S}_0$  with  $\frac{1}{2}\|\rho - \sigma\|_1 = \varepsilon$  and even extend to  $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$  in two steps. In the first step we bound  $\frac{1}{1-p}E_f(p)$  from above with  $\frac{1}{1-p}E_f^{\max}(p)$ . In the second step, we use that  $\frac{1}{1-p}E_f^{\max}(p)$  is engineered to be non-decreasing on  $[0, 1]$ . Hence for the specific  $p = p(\varepsilon) = \frac{\varepsilon}{1-t+\varepsilon} \in [0, \frac{1}{2-t}] \subset [0, 1]$ , it is non-decreasing in  $\varepsilon$  as well. This gives us the upper bound in Eq. (3) at last. The reduction of  $E_f^{\max}$  to  $E_f$  on  $\varepsilon \in (0, 1-t]$  is due to  $E_f$  being non-decreasing on  $[0, \frac{1}{2}]$ . Hence,  $E_f^{\max}$  inherits the vanishing property as  $p \rightarrow 0^+$ , which translates to  $E_f^{\max}(p(\varepsilon)) \rightarrow 0$  if  $\varepsilon \rightarrow 0^+$ . Thus we conclude uniform continuity. ■

### B. Proof of Theorem 3.1

It is clear that  $\mathcal{S}_0$  is a convex set and that the bound holds trivially for  $p = 0$  and  $p = 1$ . Hence let  $p \in (0, 1)$  and  $(\rho_1, \sigma_1), (\rho_2, \sigma_2) \in \mathcal{S}_0$  in the following. We find that

$$\begin{aligned}pD(\rho_1\|\sigma_1) + (1-p)D(\rho_2\|\sigma_2) - D(\rho\|\sigma) \\ &= -pS(\rho_1) - (1-p)S(\rho_2) + S(\rho) \\ &\quad + (1-p)\text{tr}[\rho_2(\log\sigma - \log\sigma_2)] \\ &\quad + p\text{tr}[\rho_1(\log\sigma - \log\sigma_1)] \\ &\leq h(p)\frac{1}{2}\|\rho_1 - \rho_2\|_1 + f_{c_1, c_2}(p),\end{aligned}\tag{15}$$

where we split the relative entropies and used that the von Neumann entropy fulfils [43, Theorem 14]

$$S(\rho) \leq \frac{1}{2}\|\rho_1 - \rho_2\|_1 h(p) + pS(\rho_1) + (1-p)S(\rho_2).\tag{16}$$

Furthermore, we upper bound the remaining terms by  $f_{c_1, c_2}(p)$ , estimating the two separately. We will only demonstrate the derivation for the second term, as it is completely analogous to the first one. We have

$$\begin{aligned}p\text{tr}[\rho_1(\log(\sigma) - \log(\sigma_1))] \\ &= p\text{tr}[\exp(\log(\rho_1))(\log(\sigma) - \log(\sigma_1))] \\ &\leq p\log\text{tr}[\exp(\log(\rho_1) + \log(\sigma) - \log(\sigma_1))] \\ &\leq p\log\int_{-\infty}^{\infty} dt\beta_0(t)\text{tr}\left[\rho_1(\sigma_1^{-1})^{\frac{1+it}{2}}\sigma(\sigma_1^{-1})^{\frac{1-it}{2}}\right].\end{aligned}\tag{17}$$

The first estimate follows immediately using the well-known Peierls-Bogolubov inequality [44]. The second one involves a generalisation of the Araki-Lieb-Thirring inequality [45], [46] by Sutter et al. [47, Corollary 3.3] with

$$\beta_0(t) = \frac{\pi}{2} \frac{1}{\cosh(\pi t) + 1}\tag{18}$$

a probability density on  $\mathbb{R}$ . In the above steps, i.e. (17), we relied on  $\rho_1, \sigma_1$  and  $\sigma$  to be full rank. If this is not the case one can also rigorously obtain the same result taking into account the supporting subspaces, however, the procedure is more involved. We restrict to the full-rank setting for the sake of clarity. Note here that in the most general case  $\cdot^{-1}$  in the RHS of (17) is the Moore-Penrose pseudoinverse. The trace in the integral can now be estimated for each  $t$  by

$$\begin{aligned}\text{tr}[\rho_1(\sigma_1^{-1})^{\frac{1+it}{2}}\sigma(\sigma_1^{-1})^{\frac{1-it}{2}}] \\ &= p + (1-p)\text{tr}[\rho_1(\sigma_1^{-1})^{\frac{1+it}{2}}\sigma_2(\sigma_1^{-1})^{\frac{1-it}{2}}].\end{aligned}\tag{19}$$

Here, we just split  $\sigma$  and used the cyclicity of the trace to get rid of the unitary. To see that  $c_1 < \infty$ , we upper bound  $\sigma_2$  by  $\mathbb{1}$  and  $\sigma_1^{-1}$  by  $\tilde{m}_{\sigma_1}^{-1}\mathbb{1}$  where  $\tilde{m}_{\sigma_1}$  is the smallest non-zero eigenvalue of  $\sigma_1$ . This can be done, since  $\ker\sigma_1 \subseteq \ker\rho_1$ . We end up with  $c_1 \leq \tilde{m}_{\sigma_1}^{-1} < \infty$ . Inserting (19) into (17), we obtain the first part of  $f_{c_1, c_2}(p)$  and repeating the steps for  $(1-p)\text{tr}[\rho_2(\log(\sigma) - \log(\sigma_2))]$  the second one as well. This concludes the proof. ■

### C. Proof of Theorem 3.2

We next give some auxiliary results that will be needed later. The first of those concerns an operator inequality for the term inside the trace in the definition of the BS-entropy.

*Lemma A.1:* Let  $A_1, A_2 \in \mathcal{B}(\mathcal{H})$  positive semi-definite,  $p \in [0, 1]$  and  $A := pA_1 + (1-p)A_2$ . Then

$$-A \log(A) \leq -pA_1 \log(A_1) - (1-p)A_2 \log(A_2) + h_{A_1, A_2}(p) \mathbb{1}$$

with  $h_{A_1, A_2}(p) = -p \log(p) \operatorname{tr}[A_1] - (1-p) \log(1-p) \operatorname{tr}[A_2]$  a distorted binary entropy.

*Proof:* It holds that

$$\begin{aligned} & -A \log(A) + pA_1 \log(A_1) + (1-p)A_2 \log(A_2) \\ & \leq \| -A \log(A) + pA_1 \log(A_1) + (1-p)A_2 \log(A_2) \|_1 \mathbb{1}. \end{aligned} \quad (20)$$

Now, since  $x \mapsto -x \log(x)$  is operator concave [48, Theorem 2.6], we have

$$-A \log(A) \geq -pA_1 \log(A_1) - (1-p)A_2 \log(A_2),$$

giving us that

$$-A \log(A) + pA_1 \log(A_1) + (1-p)A_2 \log(A_2) \geq 0,$$

and hence

$$\begin{aligned} & \| -A \log(A) + pA_1 \log(A_1) + (1-p)A_2 \log(A_2) \|_1 \\ & = \operatorname{tr}[-A \log(A) + pA_1 \log(A_1) + (1-p)A_2 \log(A_2)]. \end{aligned} \quad (21)$$

We now use the operator monotonicity of the logarithm to find

$$\begin{aligned} & -\operatorname{tr}[A \log(A)] = -p \operatorname{tr}[A_1 \log(A)] - (1-p) \operatorname{tr}[A_2 \log(A)] \\ & \leq -p \operatorname{tr}[A_1 \log(pA_1)] - (1-p) \operatorname{tr}[A_2 \log((1-p)A_2)] \\ & = -p \operatorname{tr}[A_1 \log(A_1)] - (1-p) \operatorname{tr}[A_2 \log(A_2)] + h_{A_1, A_2}(p) \end{aligned}$$

Inserting this into (21) and then into (20) yields the claimed result.  $\blacksquare$

The next auxiliary result concerns an equivalent formulation for the BS-entropy constructed from the function  $x \mapsto x \log x$ .

*Lemma A.2:* [37, Eq. (7.35)] Let  $\rho \in \mathcal{S}(\mathcal{H})$  and  $\sigma \in \mathcal{S}_+(\mathcal{H})$ , then

$$\widehat{D}(\rho \parallel \sigma) = \operatorname{tr}[\sigma(\sigma^{-1/2} \rho \sigma^{-1/2}) \log(\sigma^{-1/2} \rho \sigma^{-1/2})]. \quad (22)$$

Building on the previous results from this section, we proceed to prove the main result, namely the almost concavity for the BS-entropy. Let  $(\rho_1, \sigma_1), (\rho_2, \sigma_2) \in \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H})_+$ ,  $p \in [0, 1]$  and consider  $\rho := p\rho_1 + (1-p)\rho_2$ ,  $\sigma := p\sigma_1 + (1-p)\sigma_2$ . The formula for  $p = 0, 1$  is trivial, hence let  $p \in (0, 1)$ . We find that

$$\begin{aligned} & p\widehat{D}(\rho_1 \parallel \sigma_1) + (1-p)\widehat{D}(\rho_2 \parallel \sigma_2) - \widehat{D}(\rho \parallel \sigma) \\ & \leq p(\widehat{D}(\rho_1 \parallel \sigma_1) - \widehat{D}(\rho_1 \parallel \sigma)) \\ & + (1-p)(\widehat{D}(\rho_2 \parallel \sigma_2) - \widehat{D}(\rho_2 \parallel \sigma)) + \hat{c}_0 h(p). \end{aligned} \quad (23)$$

Indeed, as of Lemmas A.2 and then A.1 with  $A_1 = \sigma^{-1/2} \rho_1 \sigma^{-1/2}$ ,  $A_2 = \sigma^{-1/2} \rho_2 \sigma^{-1/2}$  respectively, we can prove

$$\begin{aligned} & -\widehat{D}(\rho \parallel \sigma) = \operatorname{tr}[\sigma(-\sigma^{-1/2} \rho \sigma^{-1/2} \log(\sigma^{-1/2} \rho \sigma^{-1/2}))] \\ & \leq p \operatorname{tr}[\sigma(-\sigma^{-1/2} \rho_1 \sigma^{-1/2} \log(\sigma^{-1/2} \rho_1 \sigma^{-1/2}))] \\ & + (1-p) \operatorname{tr}[\sigma(-\sigma^{-1/2} \rho_2 \sigma^{-1/2} \log(\sigma^{-1/2} \rho_2 \sigma^{-1/2}))] \\ & + h_{A_1, A_2}(p) \\ & = -p\widehat{D}(\rho_1 \parallel \sigma) - (1-p)\widehat{D}(\rho_2 \parallel \sigma) + h_{A_1, A_2}(p). \end{aligned}$$

At last we can estimate  $\operatorname{tr}[A_j] = \operatorname{tr}[\sigma^{-1} \rho_j] \leq \|\sigma^{-1}\|_\infty \leq \hat{c}_0$  for  $j = 1, 2$  using Hölder's inequality, giving us  $h_{A_1, A_2}(p) \leq \hat{c}_0 h(p)$ .

We now have to estimate terms of the form  $\widehat{D}(\rho_j \parallel \sigma_j) - \widehat{D}(\rho_j \parallel \sigma)$  for  $j = 1, 2$ . This is done using the Peierls-Bogolubov inequality [44] and the multivariate trace inequalities of Sutter et al. [47]:

$$\begin{aligned} & \widehat{D}(\rho_j \parallel \sigma_j) - \widehat{D}(\rho_j \parallel \sigma) \\ & = \operatorname{tr}[\rho_j (\log(\rho_j^{1/2} \sigma_j^{-1} \rho_j^{1/2}) - \log(\rho_j^{1/2} \sigma^{-1} \rho_j^{1/2}))] \\ & \leq \operatorname{tr}[\exp(\log(\rho_j) + \log(\rho_j^{1/2} \sigma_j^{-1} \rho_j^{1/2}) - \log(\rho_j^{1/2} \sigma^{-1} \rho_j^{1/2}))] \\ & \leq \operatorname{tr}[\exp(\log(\rho_j) + \log(\rho_j^{1/2} \sigma_j^{-1} \rho_j^{1/2}) + \log(\rho_j^{-1/2} \sigma \rho_j^{-1/2}))] \\ & \leq \log \alpha(\rho_j, \rho_j^{1/2} \sigma_j^{-1} \rho_j^{1/2}, \rho_j^{-1/2} \sigma \rho_j^{-1/2}) \\ & = \begin{cases} \log(p + (1-p)\hat{c}_1) & j = 1 \\ \log((1-p) + p\hat{c}_2) & j = 2 \end{cases}. \end{aligned} \quad (24)$$

In the third line, we use that

$$-\log(\rho_j^{1/2} \sigma^{-1} \rho_j^{1/2}) \leq \log(\rho_j^{-1/2} \sigma \rho_j^{-1/2})$$

which is true since for  $P_\rho$  the projection on the support of  $\rho$ , we have

$$P_\rho(P_\rho \sigma P_\rho)^{-1} P_\rho \leq P_\rho \sigma^{-1} P_\rho,$$

as  $x \rightarrow x^{-1}$  is operator convex and hence fulfills the Sherman-Davis inequality [48, Theorem 4.19]. Note that  $\sigma$  is invertible and that by  $(P_\rho \sigma P_\rho)^{-1}$  we mean the Moore-Penrose pseudoinverse. We find

$$\begin{aligned} & -\log(\rho_j^{1/2} \sigma^{-1} \rho_j^{1/2}) = -\log(\rho_j^{1/2} P_\rho \sigma^{-1} P_\rho \rho_j^{1/2}) \\ & \leq -\log(\rho_j^{1/2} P_\rho (P_\rho \sigma P_\rho)^{-1} P_\rho \rho_j^{1/2}) \\ & = \log(\rho_j^{-1/2} P_\rho \sigma P_\rho \rho_j^{-1/2}) \\ & = \log(\rho_j^{-1/2} \sigma \rho_j^{-1/2}). \end{aligned}$$

The argument why the inequalities in (24) hold in the case of  $\rho_j$  not being full rank is simpler than in the case of the corresponding inequality for the Umegaki relative entropy. For the BS-entropy, we can already restrict (24) to the support of  $\rho_j$  as all operators involved,  $\rho_j, \rho_j^{1/2} \sigma_j^{-1} \rho_j^{1/2}$  and  $\rho_j^{1/2} \sigma^{-1} \rho_j^{1/2}$ , commute with the projection onto this support. In the last step we split  $\sigma$  and evaluated the first term to  $p$  in case  $j = 1$  or

the second term in case  $j = 2$  to  $(1 - p)$  and left the other one untouched, respectively. This concludes the proof.  $\blacksquare$

#### D. Proof of Corollary 1

First, note that, given  $\rho_{AM} \in \mathcal{S}(\mathcal{H}_{AM})$ , for  $Z \in \{\mathcal{X}, \mathcal{Y}\}$  the following holds:

$$\begin{aligned} & H(Z|M)_{(E_Z \otimes \text{id}_M)(\rho_{AM})} \\ &= -D\left((E_Z \otimes \text{id}_M)(\rho_{AM}) \left\| \frac{\mathbb{1}_A}{d_A} \otimes \rho_M\right.\right) + \log d_A, \end{aligned}$$

and  $(E_M \otimes \text{id}_M)(\rho_{AM}) = \frac{\mathbb{1}_A}{d_A} \otimes \rho_M$ . Moreover, by [49, Lemma 3.4],

$$\begin{aligned} & D\left((E_Z \otimes \text{id}_M)(\rho_{AM}) \left\| (E_M \otimes \text{id}_M)(\rho_{AM})\right.\right) \\ &= D\left(\rho_{AM} \left\| (E_M \otimes \text{id}_M)(\rho_{AM})\right.\right) \\ &\quad - D\left(\rho_{AM} \left\| (E_Z \otimes \text{id}_M)(\rho_{AM})\right.\right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & H(X|M)_{(E_X \otimes \text{id}_M)(\rho_{AM})} + H(Y|M)_{(E_Y \otimes \text{id}_M)(\rho_{AM})} \\ &= D\left(\rho_{AM} \left\| (E_X \otimes \text{id}_M)(\rho_{AM})\right.\right) \\ &\quad + D\left(\rho_{AM} \left\| (E_Y \otimes \text{id}_M)(\rho_{AM})\right.\right) \\ &\quad - 2D\left(\rho_{AM} \left\| (E_M \otimes \text{id}_M)(\rho_{AM})\right.\right) + 2 \log d_A. \end{aligned}$$

Additionally, it is not difficult to see that

$$\begin{aligned} & D\left(\rho_{AM} \left\| ((E_X \circ E_Y) \otimes \text{id}_M)(\rho_{AM})\right.\right) \\ &= D\left(\rho_{AM} \left\| (E_X \otimes \text{id}_M)(\rho_{AM})\right.\right) \\ &\quad + D\left((E_X \otimes \text{id}_M)(\rho_{AM}) \left\| ((E_X \circ E_Y) \otimes \text{id}_M)(\rho_{AM})\right.\right) \\ &\leq D\left(\rho_{AM} \left\| (E_X \otimes \text{id}_M)(\rho_{AM})\right.\right) \\ &\quad + D\left(\rho_{AM} \left\| (E_Y \otimes \text{id}_M)(\rho_{AM})\right.\right), \end{aligned}$$

also as a consequence of [49, Lemma 3.4] for the first equality and the data-processing inequality for the relative entropy. Furthermore, by Eq. (6), we have

$$\begin{aligned} & \left| D\left(\rho_{AM} \left\| \frac{\mathbb{1}_A}{d_A} \otimes \rho_M\right.\right) - D\left(\rho_{AM} \left\| ((E_X \circ E_Y) \otimes \text{id}_M)(\rho_{AM})\right.\right) \right| \\ & \leq \frac{-2 \log m + \frac{1}{m}}{1 - m} \left\| \frac{\mathbb{1}_A}{d_A} \otimes \rho_M - ((E_X \circ E_Y) \otimes \text{id}_M)(\rho_{AM}) \right\|_1^{1/2}, \end{aligned}$$

Note that we can rewrite the last term on the right-hand side as:

$$\begin{aligned} & \left\| \frac{\mathbb{1}_A}{d_A} \otimes \rho_M - ((E_X \circ E_Y) \otimes \text{id}_M)(\rho_{AM}) \right\|_1 \\ &= \|((E_M - E_X \circ E_Y) \otimes \text{id}_M)(\rho_{AM})\|_1, \end{aligned}$$

A similar term was computed e.g. in [36, Section 4.1]. Following analogous ideas, we have:

$$\begin{aligned} & ((E_X \circ E_Y) \otimes \text{id}_M)(\rho_{AM}) \\ &= \sum_{x,y} |e_x^{(\mathcal{X})}\rangle \langle e_x^{(\mathcal{X})}| e_y^{(\mathcal{Y})}\rangle \langle e_y^{(\mathcal{Y})}| \rho_{AM} |e_y^{(\mathcal{Y})}\rangle \langle e_y^{(\mathcal{Y})}| e_x^{(\mathcal{X})}\rangle \langle e_x^{(\mathcal{X})}| \\ &= \sum_{x,y} |\langle e_x^{(\mathcal{X})}| e_y^{(\mathcal{Y})}\rangle|^2 |e_x^{(\mathcal{X})}\rangle \langle e_x^{(\mathcal{X})}| \otimes \langle e_y^{(\mathcal{Y})}| \rho_{AM} |e_y^{(\mathcal{Y})}\rangle, \end{aligned}$$

and

$$(E_M \otimes \text{id}_M)(\rho_{AM}) = \frac{1}{d_A} \sum_{x,y} |e_x^{(\mathcal{X})}\rangle \langle e_x^{(\mathcal{X})}| \otimes \langle e_y^{(\mathcal{Y})}| \rho_{AM} |e_y^{(\mathcal{Y})}\rangle,$$

Then,

$$\begin{aligned} & \|((E_M - E_X \circ E_Y) \otimes \text{id}_M)(\rho_{AM})\|_1 \\ &= \left\| \sum_{x,y} \left( \frac{1}{d_A} - |\langle e_x^{(\mathcal{X})}| e_y^{(\mathcal{Y})}\rangle|^2 \right) |e_x^{(\mathcal{X})}\rangle \langle e_x^{(\mathcal{X})}| \right. \\ &\quad \left. \otimes \langle e_y^{(\mathcal{Y})}| \rho_{AM} |e_y^{(\mathcal{Y})}\rangle \right\|_1 \\ &\leq \sum_{x,y} \left| \frac{1}{d_A} - |\langle e_x^{(\mathcal{X})}| e_y^{(\mathcal{Y})}\rangle|^2 \right| \left\| \langle e_y^{(\mathcal{Y})}| \rho_{AM} |e_y^{(\mathcal{Y})}\rangle \right\|_1 \\ &\leq \sum_x \max_y \left| \frac{1}{d_A} - |\langle e_x^{(\mathcal{X})}| e_y^{(\mathcal{Y})}\rangle|^2 \right|, \end{aligned}$$

Additionally, note that for  $\Phi \in \mathcal{B}(\mathcal{H}_A)$ ,

$$\begin{aligned} & (E_X \circ E_Y)(\Phi) \\ &= \sum_{x,y} |e_x^{(\mathcal{X})}\rangle \langle e_x^{(\mathcal{X})}| e_y^{(\mathcal{Y})}\rangle \langle e_y^{(\mathcal{Y})}| \Phi |e_y^{(\mathcal{Y})}\rangle \langle e_y^{(\mathcal{Y})}| e_x^{(\mathcal{X})}\rangle \langle e_x^{(\mathcal{X})}| \\ &\geq \min_{x,y} |\langle e_x^{(\mathcal{X})}| e_y^{(\mathcal{Y})}\rangle|^2 \text{tr}[\Phi] \mathbb{1}_A. \end{aligned}$$

Moreover,

$$\rho_{AM} \leq e^{D_{\max}(\rho_{AM} \left\| \frac{\mathbb{1}_A}{d_A} \otimes \rho_M\right.)} \frac{\mathbb{1}_A}{d_A} \otimes \rho_M \leq e^{2 \log d_A} \frac{\mathbb{1}_A}{d_A} \otimes \rho_M.$$

Therefore, since  $m$  should satisfy  $m\rho_{AM} \leq ((E_X \circ E_Y) \otimes \text{id}_M)(\rho_{AM})$  and  $m\rho_{AM} \leq \frac{\mathbb{1}_A}{d_A} \otimes \rho_M$ , then it can be taken to be

$$m := \min \left\{ \frac{1}{(d_A)^2}, \frac{1}{d_A} \min_{x,y} |\langle e_x^{(\mathcal{X})}| e_y^{(\mathcal{Y})}\rangle|^2 \right\}. \quad (25)$$

This concludes the proof.  $\blacksquare$

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