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STIRLING-RAMANUJAN CONSTANTS ARE EXPONENTIAL PERIODS

VICENTE MUÑOZ AND RICARDO PÉREZ-MARCO

ABSTRACT. Ramanujan studied a general class of Stirling constants that are the resummation of some natural divergent series. These constants include the classical Euler-Mascheroni, Stirling and Glaisher-Kinkelin constants. We find natural integral representations for all these constants that appear as exponential periods in the field $\mathbb{Q}(t, e^{-t})$ which reveals their natural transalgebraic nature. We conjecture that all these constants are transcendental numbers. Euler-Mascheroni's and Stirling's integral formula are classical, but the integral formula for Glaisher-Kinkelin appears to be new, as well as the integral formulas for the higher Stirling-Ramanujan constants. The method presented generalizes naturally to prove that many other constants are exponential periods over the field $\mathbb{Q}(t, e^{-t})$.

1. INTRODUCTION

A celebrated result of De Moivre [17] and Stirling [46] is the asymptotic expansion of the logarithm of the factorial function,

$$\log s! = \sum_{k=1}^s \log k = \left(s + \frac{1}{2}\right) \log s - s + S_0 + \mathcal{O}(1/s).$$

The constant term in this asymptotic expansion is the Stirling constant

$$S_0 = \frac{\log 2\pi}{2} = \lim_{s \rightarrow +\infty} \sum_{k=1}^s \log k - \left(\left(s + \frac{1}{2}\right) \log s - s \right).$$

It is well known that the coefficients of the decaying rest $\mathcal{O}(1/s)$ of the asymptotic expansion in the bases of the $(s^{-k})_{k \geq 1}$ are all rational numbers given by Bernoulli numbers. Thus, except for the Stirling constant, all coefficients in the asymptotic

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expansion are rational numbers. It is not known if the Stirling constant is transcendental, and not even known if it is irrational¹.

Euler defined the Gamma function that interpolates the factorial function. The Euler-Mascheroni constant γ appears when looking at the logarithmic derivative of the Gamma function. It can also be defined in a similar way as the Stirling constant by looking at the asymptotic expansion of the divergent harmonic series

$$\sum_{k=1}^s \frac{1}{k} = \log s + \gamma + \mathcal{O}(1/s).$$

So we have

$$\gamma = \lim_{s \rightarrow +\infty} \sum_{k=1}^s \frac{1}{k} - \log s.$$

Again, in this asymptotic expansion all coefficients other than γ are rational numbers, the decaying part in $\mathcal{O}(1/s)$ is given by Bernoulli numbers. The arithmetic nature of the Euler-Mascheroni constant is completely mysterious, but it is conjectured to be transcendental (it is not even known to be irrational). For the history of the Euler-Mascheroni constant we refer to the classical account by Glaisher [22] and the modern one by Lagarias [30].

In general, using the Euler-McLaurin formula, Ramanujan studied the asymptotic expansion of the logarithm of the n -factorial function (see [11, Chapter 9, p. 273])

$$s!^{(n)} = 1^{1^n} . 2^{2^n} . 3^{3^n} . 4^{4^n} \dots s^{s^n}.$$

For example, for $n = 1, 2, 3$ we have,

$$\begin{aligned} \sum_{k=1}^s k \log k &= \left(\frac{1}{2}s^2 + \frac{1}{2}s + \frac{1}{12} \right) \log s - \frac{1}{4}s^2 + S_1 + \mathcal{O}(1/s), \\ \sum_{k=1}^s k^2 \log k &= \left(\frac{1}{3}s^3 + \frac{1}{2}s^2 + \frac{1}{6}s \right) \log s - \frac{1}{9}s^3 + \frac{1}{12}s + S_2 + \mathcal{O}(1/s), \\ \sum_{k=1}^s k^3 \log k &= \left(\frac{1}{4}s^4 + \frac{1}{2}s^3 + \frac{1}{4}s^2 - \frac{1}{120} \right) \log s - \frac{1}{16}s^4 + \frac{1}{12}s^2 + S_3 + \mathcal{O}(1/s). \end{aligned}$$

The Euler-McLaurin formula shows that, except for the constant coefficient S_n , all coefficients are rational numbers, in particular those of the remaining part $\mathcal{O}(1/s)$ are given by Bernoulli numbers.

¹The constants e and π are conjectured to be algebraically independent over the rationals and this easily implies that the Stirling constant is irrational. The algebraic independence of e and π follows from Schanuel's conjecture that also implies that the Stirling constant is transcendental.

The constants S_k , $k \geq 0$, are generalizations of the Stirling constant that we name, for historical reasons, *Stirling-Ramanujan constants*. The Stirling constant appears as S_0 for $n = 0$. The Euler-Mascheroni constant γ appears as $\gamma = S_{-1}$ and corresponds to the divergent harmonic series.

The constant S_1 , more precisely $A = e^{S_1}$, appears in the works of Glaisher [22] and Kinkelin [29]. It is known as the Glaisher-Kinkelin constant that arises in the theory of the Barnes Gamma function [4].

Again, all the coefficients in the above asymptotic expansions other than the constant one are rational numbers. The arithmetic nature of the Stirling-Ramanujan constants is unknown. It is natural to conjecture:

Conjecture 1. *The Stirling-Ramanujan constants are transcendental.*

These generalized Stirling constants had a deep meaning for Ramanujan transalgebraic view. For him, they are the “sum” of the divergent infinite series by the resummation method he discovered based on the Euler-McLaurin formula. As he puts it, these are “the barycenter of the divergent sum” that balances between the divergent series and the divergent integral in the Euler-McLaurin summation formula [23]. Ramanujan even computes these constants using the derivative of the Riemann zeta function at integer values. Ramanujan resummation procedure is described (in an imperfect form) in the classical book of Hardy [24]. The reader can directly consult Berndt volumes [11] for a more faithful description. A rigorous approach is proposed in the monographs by Candelpergher [13], [14]. Another rigorous approach based on the Poisson-Newton formula [35] was found by the authors [34].

The Stirling-Ramanujan constants reappeared after Ramanujan in 1933 when Bendersky [10] defined a new hierarchy of higher Gamma functions different from that of Barnes [4]. The Bendersky Gamma function interpolates the n -factorial function. Contrary to the Barnes Gamma functions, higher Bendersky Gamma functions are not meromorphic since they have ramified singularities at negative integers. Indeed, historically works of Kinkelin [27] and Alekseevsky [2] [3] preceded by many decades those of Barnes and Bendersky. See the recent historical account by Neretin [37].

The Stirling-Ramanujan constants have been rediscovered multiple times by several authors, unaware that they were already studied in Ramanujan notebooks (for example, see constants B and C in [45, p. 56] and the comments therein). In some recent literature these constants are named *Bendersky-Adamchik constants* (in [1] Adamchik computes these constants in terms of values of the zeta function and its derivative at integer values). For example in the recent article by Coppo [16] where he gives remarkable converging series for the constants. Another noteworthy series representation of the Glaisher-Kinkelin constant is given by Pain in [39].

All these higher Gamma functions and the associated Stirling-Ramanujan constants arise in multiple mathematical contexts, as for example in Shintani's work in Algebraic Number theory, in the computation of determinants of Laplacians in Riemannian geometry, and in the zeta-regularization theory in Physics. As a curiosity, in the article [25] in the Physics literature we find the computation of the Stirling-Ramanujan constants up to S_3 (probably the record in the literature).

The main purpose of this article is to give a general whole integral formulas for all the Stirling-Ramanujan constants S_n .

Theorem 2 (General Exponential Period Formula). *For $n \geq 0$ we have*

$$S_n = (-1)^{n+1} n! \int_0^{+\infty} \frac{1}{t^n} \left(\frac{1}{1 - e^{-t}} - \sum_{k=-1}^n b_k t^k - r_n t^{n+1} \right) \frac{e^{-t} dt}{t},$$

where r_n is the rational number

$$r_n = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k!} b_{n-k} H_k,$$

where the (H_k) are the harmonic numbers, $H_k = 1 + \frac{1}{2} + \dots + \frac{1}{k}$, and the coefficients (b_k) have the generating function

$$\frac{1}{1 - e^{-t}} = \sum_{k=-1}^{+\infty} b_k t^k,$$

and they are given by $b_k = (-1)^{k+1} \frac{B_{k+1}}{(k+1)!}$, where B_k is the k -th Bernoulli number.

There are multiple different expressions for the Euler-Mascheroni, Stirling and Glasher-Kinkelin constants in the literature (see [45] for example). But starting from S_1 these integral expressions seem to be new.

For $n = -1$ the formula makes sense and we recover the classical integral formula for $\gamma = S_{-1}$

$$\gamma = \int_0^{+\infty} t \left(\frac{1}{1 - e^{-t}} - \frac{1}{t} \right) \frac{e^{-t}}{t} dt.$$

This integral formula is due to Euler back in 1770 ([19] section 25), but Whittaker-Whatson [47, Example 2, p. 248] attributes this formula to Dirichlet (presumably to [18], 1836). Moreover, Euler also gives this formula in [20] (1785) and devotes a full article to it ([21], 1789).

For $n \leq -2$ the formula makes sense. In that case the series

$$\zeta(n) = \sum_{s=0}^{+\infty} \frac{1}{(s+1)^n}$$

is converging, the asymptotic expansion of the partial sum reduces to a constant (its sum), and the integral formula is nothing else but the classical famous formula for the Riemann zeta function used by Riemann in his investigations of the complex extension of ζ

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^s}{1-e^{-t}} \frac{e^{-t} dt}{t}.$$

For $n = 0$, for the Stirling constant S_0 we recover the classical integral formula due to Pringsheim [42] (see also [47], [39, p. 288])

$$S_0 = \frac{\log(2\pi)}{2} = - \int_0^{+\infty} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} - t \right) \frac{e^{-t}}{t} dt.$$

For $n = 1$, for the Glaisher-Kinkelin constant $\log A$ we get the following integral formula that appears to be new in the literature:

Corollary 3 (Integral formula for Glaisher-Kinkelin constant).

$$S_1 = \log A = \int_0^{+\infty} \frac{1}{t} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} - \frac{t}{12} + \frac{t^2}{4} \right) \frac{e^{-t}}{t} dt.$$

We also get the following new integral formulas for the constants S_2 and S_3 .

Corollary 4. *We have*

$$S_2 = -2 \int_0^{+\infty} \frac{1}{t^2} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} - \frac{t}{12} - \frac{1}{72}t^3 \right) \frac{e^{-t}}{t} dt$$

$$S_3 = 6 \int_0^{+\infty} \frac{1}{t^3} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} - \frac{t}{12} + \frac{t^3}{720} - \frac{1}{288}t^4 \right) \frac{e^{-t}}{t} dt$$

An interesting fact about all these new formulas is that all the Stirling-Ramanujan constants appear as Exponential Periods with base field of functions $\mathbb{Q}(t, e^{-t})$. This reveals their transalgebraic nature. In general, we can consider a class \mathbb{K} of transalgebraic functions, as those appearing in Liouville towers of functions (see for example [26], [44]), obtained by algebraic, logarithmic, integral and exponential extensions. A *Transalgebraic period* over \mathbb{K} is a number of the form

$$\omega = \int_{\eta} f(s) ds$$

where the integral is converging and taken over an open path η where $f \in \mathbb{K}$ is holomorphic and η has end-points that are singularities of f . This notion is interesting particularly when the class of functions \mathbb{K} is a countable class, then most of complex numbers are not periods over \mathbb{K} . For example this occurs for a Liouville type of extension of the differential field of rational functions $\mathbb{Q}(s)$. When we adjoint a

finite number of exponential function $\mathbb{K} = \mathbb{Q}(t, e^{\omega_1 t}, \dots, e^{\omega_n t})$, we obtain *Exponential Periods*.

A different notion of exponential periods is considered by Kontsevich and Zagier [28] by analogy with the more classical algebraic periods, where an algebraic differential form is integrated over a cycle of an algebraic variety. It is easy to see that these are more restrictive than Transalgebraic Periods. Our point of view follows the classical path. Exponential periods were studied by Liouville, Picard, Fuchs,... already in the 19th century. They appear in the natural context of log-Riemann surface theory as the location of the ramifications, and as asymptotic values of transcendental functions associated to log-Riemann surfaces. These log-Riemann surfaces are transalgebraic curves that generalize classical algebraic curves by allowing a finite number of infinite ramification points. The determinant of the matrix of periods which comes from a bases of a vector space of transcendental functions associated to the transalgebraic curve is the Ramificant Determinant. This determinant is a fundamental object of the transalgebraic algebra of transalgebraic curves (see the original manuscript [6], and subsequent articles [7], [8], [9], and the more recent one [41]).

If we remove the combinatorial complexity in the definition and the constants, up to a rational affine combination, the family of Stirling-Ramanujan constants is rationally equivalent to the family of Upsilon constants considered by the authors in a related context:

Definition 5 (Upsilon constants). *For $n \geq -1$ we define*

$$\Upsilon_n = \int_0^{+\infty} \frac{1}{t^n} \left(\frac{1}{1 - e^{-t}} - \sum_{k=-1}^n b_k t^k \right) \frac{e^{-t} dt}{t}.$$

Obviously we have

$$S_n = (-1)^{n+1} n! (\Upsilon_n - r_n),$$

and the transcendence conjecture is equivalent to the transcendence conjecture for the Upsilon constants.

Conjecture 6. *Upsilon constants Υ_n are transcendental.*

It is natural to raise the question of their algebraic independence for $n \geq -1$.

2. HIGHER FRULLANI INTEGRALS

The classical Frullani integral expresses $\log s$ as an exponential period:

$$\log s = \int_0^{+\infty} \frac{1}{t} (e^{-t} - e^{-st}) dt.$$

This formula takes a nicer form when we single out the differential element $\frac{e^{-t} dt}{t}$, that is the natural differential² in \mathbb{C}^* for periods with an exponential singularity at ∞ and a polar singularity at 0. The expressions are also nicer when we work with functions of the variable $s + 1$ instead of s . This is because $\log(s + 1)$, contrary to $\log(s)$, is an LLD function (a Left Located Divisor function has all of its zeros and poles, or in general singularities, in the left half plane $\{\Re s < 0\}$). This is a relevant notion in the theory of Poisson-Newton formula ([34], [35], [36]).

Because of these reasons, the natural way to write down the Frullani integral is

$$\log(s + 1) = \int_0^{+\infty} (1 - e^{-st}) \frac{e^{-t} dt}{t}.$$

Observe that the Frullani integral is obtained by integration on the variable s over the interval $[1, s]$ of the elementary integral

$$\frac{1}{s + 1} = \int_0^{+\infty} t e^{-st} \frac{e^{-t} dt}{t},$$

that can be thought of the primitive generator integral. Starting with the Frullani integral and iterating integrations on the variable s over the interval $[1, s]$, we obtain higher Frullani integrals that express $s^k \log s$ as an exponential integral.

Theorem 7. *We have*

$$(s + 1)^n \log(s + 1) = \sum_{k=1}^n \binom{n}{k} (H_n - H_{n-k}) s^k + (-1)^{n+1} n! \int_0^{+\infty} \frac{1}{t^n} \left(e^{-st} - \sum_{k=0}^n \frac{(-s)^k t^k}{k!} \right) \frac{e^{-t} dt}{t},$$

where $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ are the harmonic numbers for $n \geq 1$, and $H_0 = 0$.

This explicit formula can be found in the references [32] and [33]. These higher Frullani integrals are relevant to explain the most popular BBP formulas as was

²For the reader knowledgeable of the theory of log-Riemann surfaces, the geometry associated to this basic differential $\omega = \frac{e^{-t} dt}{t}$ is the tube-log-Riemann surface that derives from the log-Riemann surface of the logarithm replacing a planar sheet by a tube \mathbb{C}/\mathbb{Z} (see [6]).

unveiled in the recent article by D. Barsky and the authors [5]. A complete proof of Theorem 7 can be found in this reference (Prop. 2.1 and Prop. 2.2 of [5]).

We have a polynomial part, without constant term, and an integral part which is the Hadamard regularization of the Laplace transform of $\frac{1}{t^{n+1}}e^{-t}$ as considered in [35].

3. TECHNICAL PRELIMINARIES

3.1. Technical normalizations. It appears combinatorially simpler to find the integral formulas for the constants appearing in a slight different expansion base, which is closer to the Upsilon constants from the introduction.

In the proof it is natural to compute the expansion on the shifted variable $s + 1$, and this comes down to

$$\sum_{k=1}^s k^n \log k = \tilde{A}_n(s) \log(s+1) + \tilde{B}_n(s) + \tilde{S}_n + \tilde{R}_n(1/s),$$

where $\tilde{A}_n \in \mathbb{Q}[s]$, $\tilde{B}_n \in s\mathbb{Q}[s]$, $\deg \tilde{A}_n = \deg \tilde{B}_n = n + 1$, and $\tilde{R}_n(1/s) \in \mathbb{Q}[[1/s]]$.

From this it follows the expansion in the form used to define the Stirling-Ramanujan constants

$$\sum_{k=1}^s k^n \log k = A_n(s) \log s + B_n(s) + S_n + R_n(1/s),$$

with $A_n \in \mathbb{Q}[s]$, $B_n \in s\mathbb{Q}[s]$, and $R_n(1/s) \in \mathbb{Q}[[1/s]]$.

Indeed, we have

$$\begin{aligned} \sum_{k=1}^s k^n \log k &= \tilde{A}_n(s) \log(s+1) + \tilde{B}_n(s) + \tilde{S}_n + \tilde{R}_n(1/s) \\ &= \tilde{A}_n(s) \log(1 + 1/s) + \tilde{A}_n(s) \log s + \tilde{B}_n(s) + \tilde{S}_n + \tilde{R}_n(1/s) \\ &= \tilde{A}_n(s) \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k s^k} + \tilde{A}_n(s) \log s + \tilde{B}_n(s) + \tilde{S}_n + \tilde{R}_n(1/s). \end{aligned}$$

Therefore, if we write down the polynomial as $\tilde{A}_n(s) = \sum_{j=0}^{n+1} \tilde{a}_j s^j$, then we have

$$S_n = \tilde{S}_n + \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \tilde{a}_k. \quad (1)$$

3.2. Asymptotic decay. We use the following classical and well known fact from Laplace transform theory (see [15] for example).

Proposition 8. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ differentiable, in particular at the right at $t = 0$. We assume f' is bounded. Then the Laplace transform*

$$\mathcal{L}f(s) = \int_0^{+\infty} f(t) e^{-st} dt$$

decays as $\mathcal{O}(1/s)$ when $s \rightarrow +\infty$. Moreover, if we assume that f is infinitely differentiable and the derivatives are bounded, then for every $n \geq 1$, we have the asymptotic expansion

$$\mathcal{L}f(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \dots + \frac{f^{(n-1)}(0)}{s^n} + \mathcal{O}(1/s^{n+1}).$$

Proof. The proof is immediate by integration by parts:

$$\begin{aligned} \mathcal{L}f(s) &= \left[-f(t) \frac{e^{-st}}{s} \right]_0^{+\infty} + \frac{1}{s} \int_0^{+\infty} f'(t) e^{-st} dt \\ &= \frac{f(0)}{s} + \frac{1}{s} \int_0^{+\infty} f'(t) e^{-st} dt \\ &= \frac{f(0)}{s} + \frac{1}{s} \mathcal{L}f'(s). \end{aligned}$$

As $f'(s)$ is bounded, we have that $\mathcal{L}f'(s)$ is bounded for $s \geq 1$. This implies that $\mathcal{L}f(s) = \mathcal{O}(1/s)$, and the same will be true for f' , that is, $\mathcal{L}f'(s) = \mathcal{O}(1/s)$, hence $\mathcal{L}f(s) = \frac{f(0)}{s} + \mathcal{O}(1/s^2)$. Working inductively, we get the result for any $n \geq 1$. \square

4. DERIVATION OF THE INTEGRAL FORMULAS

Consider the n -th higher Frullani integrals given in Theorem 7 and we add from $s = 0$ to $s - 1$ to get (we use the convention $0^0 = 1$)

$$\begin{aligned} \sum_{k=1}^s k^n \log k &= \sum_{k=1}^n \binom{n}{k} (H_n - H_{n-k}) \left(\sum_{j=0}^{s-1} j^k \right) + \\ &+ (-1)^{n+1} n! \int_0^{+\infty} \frac{1}{t^n} \left(\frac{1 - e^{-st}}{1 - e^{-t}} - \sum_{k=0}^n \frac{(-t)^k}{k!} \left(\sum_{j=0}^{s-1} j^k \right) \right) \frac{e^{-t} dt}{t}. \end{aligned}$$

By Faulhaber's formula, for $k \geq 1$,

$$\sum_{j=0}^{s-1} j^k$$

is a polynomial in the variable s without constant term, whose coefficients are given by Bernoulli numbers (we remind this fact in the Appendix).

Therefore, we have that the sum

$$\sum_{k=1}^n \binom{n}{k} (H_n - H_{n-k}) \left(\sum_{j=0}^{s-1} j^k \right)$$

is also a polynomial in the variable s without constant term and we can disregard it when looking for the constant coefficient of the asymptotic expansion for $s \rightarrow +\infty$.

Thus, we reduce the problem at looking for the constant term of the asymptotic expansion of the integral

$$\begin{aligned} I_n &= (-1)^n n! \int_0^{+\infty} \frac{1}{t^n} \left(\frac{e^{-st} - 1}{1 - e^{-t}} + \sum_{k=0}^n \frac{(-t)^k}{k!} \left(\sum_{j=0}^{s-1} j^k \right) \right) \frac{e^{-t} dt}{t} \\ &= (-1)^n n! \int_0^{+\infty} \left(e^{-st} \frac{1}{t^n} \frac{1}{1 - e^{-t}} - \frac{1}{t^n} \frac{1}{1 - e^{-t}} + \frac{1}{t^n} \sum_{k=0}^n \frac{(-t)^k}{k!} \left(\sum_{j=0}^{s-1} j^k \right) \right) \frac{e^{-t} dt}{t}. \end{aligned}$$

We use the expansion

$$\frac{1}{1 - e^{-t}} = \sum_{k=-1}^{+\infty} b_k t^k = \frac{1}{t} + \frac{1}{2} + \frac{1}{12}t - \frac{1}{720}t^3 + \dots$$

Then

$$\begin{aligned} \frac{(-1)^n}{n!} I_n &= \int_0^{+\infty} \left(e^{-st} \sum_{k=-1}^n b_k t^{k-n} + e^{-st} \frac{1}{t^n} \left(\frac{1}{1 - e^{-t}} - \sum_{k=-1}^n b_k t^k \right) \right. \\ &\quad \left. - \frac{1}{t^n} \frac{1}{1 - e^{-t}} + \frac{1}{t^n} \sum_{k=0}^n \frac{(-t)^k}{k!} \left(\sum_{j=0}^{s-1} j^k \right) \right) \frac{e^{-t} dt}{t} \\ &= \sum_{k=-1}^n b_k \int_0^{+\infty} \frac{1}{t^{n-k}} \left(e^{-st} - \sum_{l=0}^{n-k} \frac{(-st)^l}{l!} \right) \frac{e^{-t} dt}{t} + \end{aligned} \quad (2)$$

$$+ \int_0^{+\infty} e^{-st} \frac{1}{t^n} \left(\frac{1}{1 - e^{-t}} - \sum_{k=-1}^n b_k t^k \right) \frac{e^{-t} dt}{t} + \quad (3)$$

$$+ \int_0^{+\infty} \left[\sum_{k=-1}^n \frac{b_k}{t^{n-k}} \sum_{l=0}^{n-k} \frac{(-st)^l}{l!} - \frac{1}{t^n} \frac{1}{1 - e^{-t}} + \frac{1}{t^n} \sum_{k=0}^n \frac{(-t)^k}{k!} \left(\sum_{j=0}^{s-1} j^k \right) \right] \frac{e^{-t} dt}{t}. \quad (4)$$

In line (2) we recognize the first $n + 1$ higher Frullani integrals:

$$\begin{aligned} \sum_{k=-1}^n b_k \int_0^{+\infty} \frac{1}{t^{n-k}} \left(e^{-st} - \sum_{l=0}^{n-k} \frac{(-st)^l}{l!} \right) \frac{e^{-t} dt}{t} \\ = \left(\sum_{k=-1}^n b_k \frac{(-1)^{n-k+1}}{(n-k)!} (s+1)^{n-k} \right) \log(s+1) \pmod{s\mathbb{Q}[s]} \\ = \frac{(-1)^n}{n!} \tilde{A}_n(s) \log(s+1) \pmod{s\mathbb{Q}[s]}, \end{aligned}$$

for some polynomial $\tilde{A}_n(s) \in \mathbb{Q}[s]$. Note that the coefficients \tilde{a}_k of the polynomials \tilde{A}_n can be computed explicitly from the coefficients $(b_k)_{k \geq -1}$. Indeed for $0 \leq j \leq n + 1$, we have

$$\tilde{a}_j = \sum_{k=-1}^{n-j} b_k (-1)^{k+1} \frac{n!}{(n-k-j)! j!}. \quad (5)$$

The second integral (3) is $\mathcal{O}(1/s)$ when $s \rightarrow +\infty$ using Proposition 8.

For the third integral (4), we replace the sum of k -powers of the first $s - 1$ integers by the Faulhaber polynomial in s that is given by the Bernoulli type coefficients (b_k) (use Proposition 11 from the Appendix),

$$\begin{aligned} \int_0^{+\infty} \left[\sum_{k=-1}^n \frac{b_k}{t^{n-k}} \sum_{l=0}^{n-k} \frac{(-st)^l}{l!} - \frac{1}{t^n} \frac{1}{1-e^{-t}} + \frac{1}{t^n} \sum_{k=0}^n \frac{(-t)^k}{k!} \left(\sum_{j=0}^{s-1} j^k \right) \right] \frac{e^{-t} dt}{t} \\ = \int_0^{+\infty} \left[\sum_{h=-1}^n \frac{b_h}{t^{n-h}} \sum_{l=0}^{n-h} \frac{(-st)^l}{l!} - \frac{1}{t^n} \frac{1}{1-e^{-t}} + \frac{1}{t^n} \sum_{k=0}^n \frac{(-t)^k}{k!} \left((-1)^{k+1} k! \sum_{l=1}^{k+1} (-1)^l b_{k-l} \frac{s^l}{l!} \right) \right] \frac{e^{-t} dt}{t} \\ = \int_0^{+\infty} \frac{1}{t^n} \left(\sum_{k=-1}^n b_k t^k - \frac{1}{1-e^{-t}} \right) \frac{e^{-t} dt}{t}. \end{aligned}$$

The simplification to the last result comes from the observation that there is a cancellation of the terms in the second line for the first summation when h equals to $k - l$. In the first sum, the condition is $0 \leq l \leq n - h \neq n + 1$. In the second sum the condition is $1 \leq l \leq k + 1 \leq n + 1$. This means that only the terms for $l = 0$ in the first sum remain. This massive cancellation is to be expected since the coefficients of the monomials on the variable s would give divergent integrals if they wouldn't vanish.

Therefore, this gives the modified Stirling-Ramanujan constant

$$\tilde{S}_n = (-1)^{n+1} n! \int_0^{+\infty} \frac{1}{t^n} \left(\frac{1}{1-e^{-t}} - \sum_{k=-1}^n b_k t^k \right) \frac{e^{-t} dt}{t}.$$

Hence using (1) and (5), we get

$$S_n = \tilde{S}_n + \sum_{j=1}^{n+1} \frac{(-1)^{j+1}}{j} \tilde{a}_j = \tilde{S}_n + \sum_{j=1}^{n+1} \frac{(-1)^{j+1}}{j} \sum_{h=-1}^{n-j} b_h (-1)^{h+1} \frac{n!}{(n-h-j)! j!},$$

and $S_n = \tilde{S}_n + \hat{r}_n$ where \hat{r}_n is the rational number

$$\hat{r}_n = \sum_{j=1}^{n+1} \sum_{h=-1}^{n-j} \frac{(-1)^{j+h}}{j} b_h \frac{n!}{(n-h-j)! j!}.$$

We can insert this rational number inside the integral and we finally get

$$S_n = (-1)^{n+1} n! \int_0^{+\infty} \frac{1}{t^n} \left(\frac{1}{1-e^{-t}} - \sum_{k=-1}^n b_k t^k - r_n t^{n+1} \right) \frac{e^{-t} dt}{t},$$

where the rational number r_n is given by

$$r_n = \frac{(-1)^n}{n!} \hat{r}_n = \sum_{j=1}^{n+1} \sum_{h=-1}^{n-j} \frac{(-1)^{n+j+h}}{j} b_h \frac{1}{(n-h-j)! j!}.$$

We can simplify this expression:

Proposition 9. *We have*

$$r_n = \sum_{h=-1}^{n-1} b_h \frac{(-1)^{n-h+1}}{(n-h)!} H_{n-h}.$$

Proof. We compute

$$\begin{aligned} r_n &= \sum_{j=1}^{n+1} \sum_{h=-1}^{n-j} \frac{(-1)^{n+j+h}}{j} b_h \frac{1}{(n-h-j)! j!} \\ &= \sum_{h=-1}^{n-1} b_h \frac{(-1)^{n-h}}{(n-h)!} \sum_{j=1}^{n-h} \frac{(-1)^j}{j} \binom{n-h}{j} \\ &= \sum_{h=-1}^{n-1} b_h \frac{(-1)^{n-h+1}}{(n-h)!} H_{n-h}, \end{aligned}$$

where we use the identity $\sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} = -H_n$. These type of identities of harmonic numbers are immediate when we write them down as exponential periods:

$$H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} = \sum_{k=1}^n \int_0^{+\infty} e^{-(k-1)t} e^{-t} dt = \int_0^{+\infty} \frac{1 - e^{-nt}}{1 - e^{-t}} e^{-t} dt,$$

and then

$$\begin{aligned} H_n &= \int_0^{+\infty} \frac{1 - (1 - (1 - e^{-t}))^n}{1 - e^{-t}} e^{-t} dt \\ &= - \int_0^{+\infty} \sum_{k=1}^n \binom{n}{k} (-1)^k (1 - e^{-t})^{k-1} e^{-t} dt \\ &= - \sum_{k=1}^n \binom{n}{k} (-1)^k \int_0^{+\infty} (1 - e^{-t})^k e^{-t} dt \\ &= - \sum_{k=1}^n \binom{n}{k} \frac{(-1)^k}{k+1} dt. \end{aligned}$$

□

This completes the proof of Theorem 2.

5. PARTICULAR CASES

Using Proposition 9, we obtain

$$\begin{aligned} r_0 &= b_{-1} = 1, \\ r_1 &= -\frac{H_2 b_{-1}}{2!} + \frac{H_1 b_0}{1!} = -\frac{1}{4}, \\ r_2 &= \frac{H_3 b_{-1}}{3!} - \frac{H_2 b_0}{2!} + \frac{H_1 b_1}{1!} = \frac{1}{72}, \\ r_3 &= -\frac{H_4 b_{-1}}{4!} + \frac{H_3 b_0}{3!} - \frac{H_2 b_1}{2!} = \frac{1}{288}. \end{aligned}$$

Therefore Theorem 2 gives

$$\begin{aligned} S_0 &= \frac{\log(2\pi)}{2} = \int_0^{+\infty} \left(\frac{1}{t} + \frac{1}{2} + t - \frac{1}{1-e^{-t}} \right) \frac{e^{-t}}{t} dt, \\ S_1 &= \log A = \int_0^{+\infty} \frac{1}{t} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} - \frac{t}{12} + \frac{t^2}{4} \right) \frac{e^{-t}}{t} dt, \\ S_2 &= -2 \int_0^{+\infty} \frac{1}{t^2} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} - \frac{t}{12} - \frac{1}{72}t^3 \right) \frac{e^{-t}}{t} dt, \\ S_3 &= 6 \int_0^{+\infty} \frac{1}{t^3} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} - \frac{t}{12} + \frac{t^3}{720} - \frac{1}{288}t^4 \right) \frac{e^{-t}}{t} dt. \end{aligned}$$

The numbers S_n up to $n = 4$ are computed by Bendersky [10, p. 265]. Translating into natural logarithm (Bendersky uses decimal logarithm), we have

$$\begin{aligned} S_0 &= \log(10) \cdot 0,399089\dots = 0.918938\dots \\ S_1 &= \log(10) \cdot 0.108032\dots = 0.248754\dots \\ S_2 &= \log(10) \cdot 0.013223\dots = 0.030448\dots \\ S_3 &= \log(10) \cdot (-1 + 0.991029\dots) = -0.008971\dots \end{aligned}$$

This agrees with the numerical values of the integrals in Theorem 2 given above.

For $n = -1$, we also have that the above formulas give $r_{-1} = 0$ and

$$S_{-1} = \gamma = \int_0^{+\infty} t \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) \frac{e^{-t}}{t} dt.$$

6. GENERAL SCOPE OF THE METHOD

The precedent procedure is very general and allows to find a multitude of integral formulas for many constants defined as the constant term in asymptotic expansions of arithmetic nature. Consider sums (typical divergent) of the type

$$\sum_{k=0}^s f(k+1),$$

where $f(s+1)$ is an LLD function (LLD means Left Located Divisor) so that the general divisor of f are in the left half plane. When f is meromorphic (for example as the Euler Gamma function Γ), then the general divisor is only composed by zeros and poles. In general, we can have ramified singularities (as for the logarithm function $\log s$).

We consider the asymptotic when $s \rightarrow +\infty$ of the above sum. For natural arithmetic functions, these sums are divergent and have a natural asymptotic expansion

on a natural bases of functions and a decaying part in $\mathcal{O}(1/s)$. The constant term in the expansion define generalized Stirling-Ramanujan constants.

A particularly interesting case is when f admits a Malmstén type formula. We recall that Malmstén formula for the Euler Gamma function ([31]), for $\Re s > 0$,

$$\log \Gamma(s+1) = \int_0^{+\infty} \left(s + \frac{e^{-st} - 1}{1 - e^{-t}} \right) \frac{e^{-t} dt}{t}.$$

A general Malmstén formula for $g = e^f$ over the ring $\mathbb{Q}(t, e^{-t})[e^{-st}]$ is, for $\Re s > 0$,

$$\log g(s+1) = f(s+1) = \int_0^{\infty} G(t, e^{-t}, e^{-st}) \frac{e^{-t} dt}{t}.$$

Note that this means that $f(s)$ behaves as an exponential period (with parameter s). The higher Frullani integrals are just Malmstén formulas for the $f(s) = s^n \log(s)$, or $g(s) = s^{s^n}$.

In that case the general procedure presented will also work. The key property is that the sum for which we study the asymptotic is also an an exponential period in the same ring $\mathbb{Q}(t, e^{-t})[e^{-st}]$.

7. APPENDIX: BERNOULLI COMPUTATIONS.

We review in this Appendix the computations with Bernoulli numbers that we use.

The classical Bernoulli polynomials $(B_k(s))_{k \geq 0}$ can be defined via their generating function

$$\frac{te^{st}}{e^t - 1} = \sum_{k=0}^{+\infty} B_k(s) \frac{t^k}{k!}.$$

Making the change of variable $t \rightarrow -t$ this can also be written (in a form that is more appealing to us) as

$$\frac{te^{-st}}{1 - e^{-t}} = \sum_{k=0}^{+\infty} (-1)^k B_k(s) \frac{t^k}{k!}.$$

The Bernoulli numbers are defined as $B_k = B_k(0)$.

We prefer to work with a related sequence of polynomials $(b_k(s))_{k \geq 0}$ defined via their generating function

$$\frac{e^{-st}}{1 - e^{-t}} = \frac{1}{t} + \sum_{k=0}^{+\infty} b_k(s) t^k.$$

Thus we have

$$b_k(s) = (-1)^{k+1} \frac{B_{k+1}(s)}{(k+1)!},$$

and the related Bernoulli like numbers $b_k = b_k(0)$, and defining $b_{-1} = 1$, thus

$$\frac{1}{1 - e^{-t}} = \sum_{k=-1}^{+\infty} b_k t^k.$$

We can compute in two ways the expansion of $\frac{1 - e^{-st}}{1 - e^{-t}}$,

$$\frac{1 - e^{-st}}{1 - e^{-t}} = \frac{1}{1 - e^{-t}} - \frac{e^{-st}}{1 - e^{-t}} = \sum_{k=0}^{+\infty} (b_k - b_k(s)) t^k$$

and, for a positive integer s ,

$$\frac{1 - e^{-st}}{1 - e^{-t}} = \sum_{j=0}^{s-1} e^{-jt} = \sum_{k=0}^{+\infty} \frac{(-t)^k}{k!} \left(\sum_{j=0}^{s-1} j^k \right).$$

The sum abote can be taken from $j = 1$ to $s - 1$, except in the case $k = 0$ (in which case, we set $0^0 = 1$).

It follows the form of Faulhaber's formula for the sum of k -powers of the first $s - 1$ integers.

Proposition 10. *We have*

$$\sum_{j=0}^{s-1} j^k = (-1)^k k! (b_k - b_k(s)).$$

As it is well known we can compute the Bernoulli polynomials from the sequence of Bernoulli numbers:

$$B_k(s) = \sum_{j=0}^k \binom{k}{j} B_{k-j} s^j.$$

Hence we have

$$b_k(s) = \sum_{j=0}^{k+1} (-1)^j b_{k-j} \frac{s^j}{j!}$$

and we conclude:

Proposition 11. *We have*

$$\sum_{j=0}^{s-1} j^k = (-1)^{k+1} k! \sum_{j=1}^{k+1} (-1)^j b_{k-j} \frac{s^j}{j!}.$$

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