

# Coupled queueing and charging game model with power capacity optimization

PGMO days

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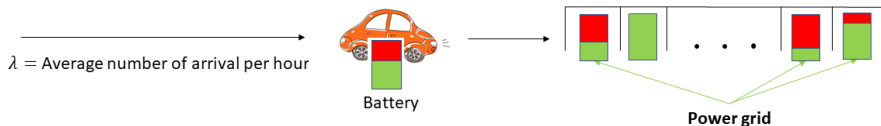
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- 1 Model in the case of 1 CS
- 2 Non-cooperative queueing game
- 3 Optimization of the QoS at equilibrium under constrained power capacity
- 4 Numerical simulations
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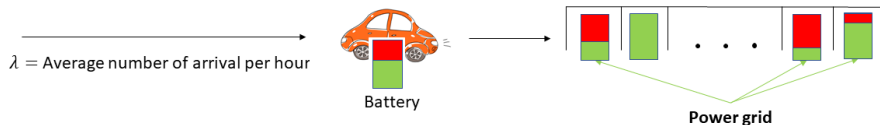
# Model: Mathematical description



## Characteristics of Electric Vehicules (EVs)

- Random time of arrivals: follows a Poisson process of parameter  $\lambda$ .
- Random need in energy when arriving at the station  $B$  to have a fully charged battery: follows an exponential law of parameter  $\mu$ .
  - $Capa$ : Capacity of the battery,
  - $SoC_{arr}$ : State of Charge of the battery when arriving,
  - $B = Capa - SoC_{arr}$ .
- Random parking time  $D$  ( $\perp\!\!\!\perp B$ ): follows an exponential law of parameter  $\nu$ .

# Model: Mathematical description



## Characteristics of a Charging Station (CS)

- $p_{\max}$ : Maximum power delivered for an EV.
- $N_p$ : Number of parking spaces.
- $\alpha$ : Proportion of the number of parking spaces that deliver maximum power  $p_{\max}$  simultaneously.

# Model: Mathematical description

## Notations

- $Z_t$ : Number of charging EVs at the CS at time  $t$ .
- $C_t$ : Number of EVs with full battery, i.e not charging, at the CS at time  $t$ .
- $Q_t := Z_t + C_t$ : total number of EVs at the CS at time  $t$ .

## Power scheduling

- Each arriving EVs immediately starts to charge with power:

$$p(Z_t) = p_{\max} \times \frac{\min(\alpha N_p, Z_t)}{Z_t}.$$

## Remark

Total power  $p_T$  at the CS:  $p_T = p_{\max} \times \alpha N_p$ .

# Model: Number of charging EVs

## Notation

$\bar{z}$  = Average number of charging EVs in stationary regime.

Fixed point equation for  $\bar{z}$  given by [1]

$$\bar{z} = \min(\lambda, \nu N_p) \mathbb{E} \left[ \min \left( D, \frac{B}{\rho_{\max}} \max \left( 1, \frac{\bar{z}}{\alpha N_p} \right) \right) \right]. \quad (1)$$

Proposition [Avg nb of charging EVs]:

In stationary regime, the average number of charging EVs is:

$$\bar{z} = \begin{cases} \frac{\min(\lambda, \nu N_p)}{\nu + \rho_{\max} \mu} & \text{if } \frac{\nu}{\nu + \rho_{\max} \mu} \leq \alpha \text{ or } \frac{\lambda}{\nu + \rho_{\max} \mu} \leq \alpha N_p, \\ \frac{\min(\lambda, \nu N_p) - \rho_{\max} \mu \alpha N_p}{\nu} & \text{else.} \end{cases} \quad (2)$$

Idea of the proof : Resolution of the fixed point equation 1.

# Model: Probability to leave with a fully charged battery

## Proposition [Probability to leave with a fully charged battery]:

In stationary regime, the expected probability  $\bar{P}$  to leave the CS with a fully charged battery is given by:

$$\begin{aligned}\bar{P} &= \frac{\rho_{\max}\mu}{\nu \max\left(1, \frac{\bar{z}}{\alpha N_p}\right) + \rho_{\max}\mu} \\ &= \begin{cases} \frac{\rho_{\max}\mu}{\nu + \rho_{\max}\mu} & \text{if } \frac{\nu}{\nu + \rho_{\max}\mu} \leq \alpha \text{ or } \lambda \leq \alpha N_p(\nu + \rho_{\max}\mu), \\ \frac{\rho_{\max}\mu\alpha N_p}{\min(\theta\lambda, \nu N_p)} & \text{else.} \end{cases} \end{aligned} \quad (3)$$

Ideas of the proof:

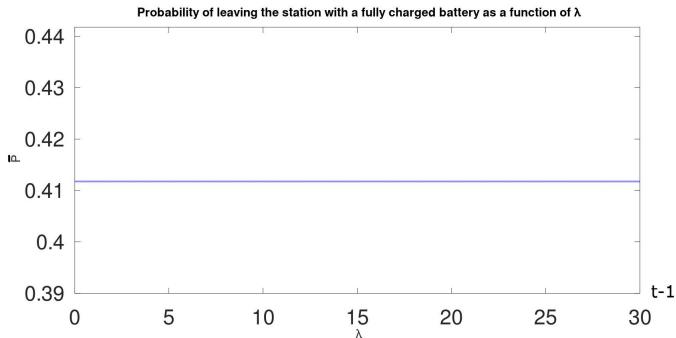
- $\bar{P}$  is the probability that the time to get fully charged is less than the parking time:  $\bar{P} = \mathbf{P}\left(\frac{B}{\rho_{\max}} \max\left(1, \frac{\bar{z}}{\alpha N_p}\right) < D\right)$ .
- $\frac{B}{\rho_{\max}} \max\left(1, \frac{\bar{z}}{\alpha N_p}\right)$  and  $D$  follow exponential distributions.



# Model: Probability to leave with a fully charged battery

- If  $\frac{\nu}{\nu + \rho_{\max}\mu} \leq \alpha$ , then

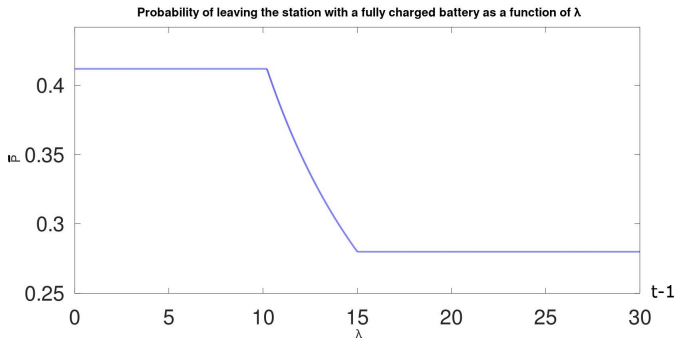
$$\forall \lambda \in \mathbb{R}_+, \bar{P} = \frac{\rho_{\max}\mu}{\nu + \rho_{\max}\mu}.$$



# Model: Probability to leave with a fully charged battery

- If  $\frac{\nu}{\nu + \rho_{\max}\mu} > \alpha$ , then

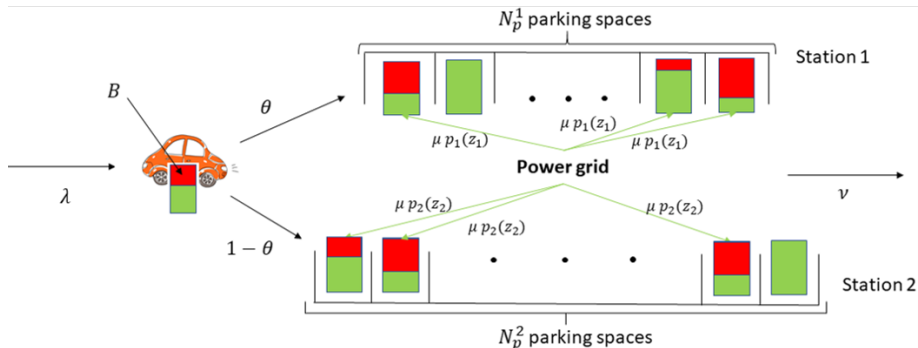
$$\bar{P} = \begin{cases} \frac{\rho_{\max}\mu}{\nu + \rho_{\max}\mu} & \text{if } \lambda \leq \alpha N_p(\nu + \rho_{\max}\mu), \\ \frac{\rho_{\max}\mu\alpha N_p}{\min(\theta\lambda, \nu N_p)} & \text{if } \lambda > \alpha N_p(\nu + \rho_{\max}\mu). \end{cases}$$



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# Game: Definition



## Setting of the game

- EVs choose CS 1 with probability  $\theta$ .
- Criterion of choice: highest probability  $\bar{P}$  of leaving with a fully charged battery.

## Notation

$i \in \{1, 2\}$

- $N_p^i$ : Number of parking spaces at CS  $i$ .
- $\alpha_i$ : Proportion of the number of parking spaces that can deliver maximum power simultaneously at CS  $i$ .
- $p_{\max}$ : Common to the two CSs.
- $\bar{z}_i$ : Stationary average number of charging EVs at CS  $i$ .
- $p_i$ : Power available for a charging EVs at CS  $i$ .
- $\bar{P}_i$ : Probability of leaving CS  $i$  with a fully charged battery.

## Arriving process at station $i$

EVs arrive at station  $i$  according to a Poisson process of parameter

$$\lambda_i := \begin{cases} \theta\lambda & \text{if } i = 1, \\ (1 - \theta)\lambda & \text{if } i = 2. \end{cases}$$

## Definition of the game $\Gamma$

- $\mathcal{N} := [0, 1]$ : Continuum of nonatomic players, which are the EVs;
- $\mathcal{A} := \{CS1, CS2\}$ : Set of pure actions;
- $\mathcal{S} := \Delta[\mathcal{A}] = [0, 1]$ : Set of mixed strategies over the two pure actions;
- Same utility function  $u$  for all the players:

$$u(\theta, \bar{\theta}) := \theta \times \bar{P}_1(\bar{\theta}) + (1 - \theta) \times \bar{P}_2(1 - \bar{\theta}),$$

where  $\bar{\theta}$  is the average strategy used by the players.

# Game: Equilibrium

- In decentralized system, EVs individually choose their best strategy.
- An equilibrium is a situation where no player has an incentive to unilaterally change her strategy.

## Definition [Equilibrium]:

An equilibrium is a set identical mixed-strategies  $\theta_{eq}$  s.t.

$$\forall \theta \in [0, 1], \quad u(\theta_{eq}, \theta_{eq}) \geq u(\theta, \theta_{eq}). \quad (4)$$

## Definition [Set of equilibria]:

The set of equilibria  $S_{eq}$  is:

$$S_{eq} = \{\theta_{eq} \in [0, 1], \quad \theta_{eq} \text{ verify (4)}\}.$$

## Proposition [Characterization]:

The set of equilibrium  $S_{eq}$  is:

$$S_{eq} = \{\theta_{eq} \in [0, 1], \bar{P}_1(\theta_{eq}) = \bar{P}_2(1 - \theta_{eq})\}. \quad (5)$$

## Proposition [Existence and convexity]:

The set of equilibrium  $S_{eq}$  is non-empty and convex.

## Proposition [Unicity]:

There exist a constant  $\bar{P}_{eq}$  s.t.

$$\forall \theta_{eq} \in S_{eq}, \bar{P}_{eq} = \bar{P}_1(\theta_{eq}) = \bar{P}_2(1 - \theta_{eq}).$$



## Proposition [Expression of $\bar{P}_{eq}$ ]:

At equilibrium, the probability  $\bar{P}_{eq}$  for any EV to leave a CS with a full battery is given by:

$$\bar{P}_{eq} = \begin{cases} \frac{\rho_{\max}\mu}{\nu + \rho_{\max}\mu} & \text{if } \begin{cases} \max(\alpha_1, \alpha_2) \geq \frac{\nu}{\nu + \rho_{\max}\mu}, \\ \text{or } \alpha_1 N_p^1 + \alpha_2 N_p^2 \geq \frac{\lambda}{\nu + \rho_{\max}\mu}, \end{cases} \\ \frac{\rho_{\max}\mu}{\lambda} (\alpha_1 N_p^1 + \alpha_2 N_p^2) & \text{if } \begin{cases} \max(\alpha_1, \alpha_2) < \frac{\nu}{\nu + \rho_{\max}\mu}, \\ \alpha_1 N_p^1 + \alpha_2 N_p^2 < \frac{\lambda}{\nu + \rho_{\max}\mu}, \\ N_p^1 + N_p^2 > \frac{\lambda}{\nu}, \\ \max(\alpha_1, \alpha_2) > \frac{\nu}{\lambda} (\alpha_1 N_p^1 + \alpha_2 N_p^2), \end{cases} \\ \frac{\rho_{\max}\mu}{\nu} \max(\alpha_1, \alpha_2) & \text{else.} \end{cases} \quad (6)$$

# Game: PoA

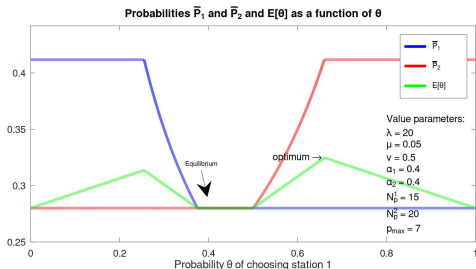
In centralized system, a centralized CS operator chooses the CS for the EVs to optimize their utility:

$$U_{opt} = \max_{\theta \in [0,1]} u(\theta, \theta) = \max_{\theta \in [0,1]} \theta \times \bar{P}_1(\theta) + (1 - \theta) \times \bar{P}_2(1 - \theta).$$

## Price of Anarchy [2] (PoA)

The PoA quantify the inefficiency of the decentralized system in comparison of the centralized system:

$$\begin{aligned} PoA &:= \frac{U_{opt}}{u(\theta_{eq}, \theta_{eq})} \\ &= \frac{U_{opt}}{\bar{P}_{eq}} (\geq 1). \end{aligned} \quad (7)$$



# Game: PoA

Denote, for any couple  $(i, j) \in \{(1, 2), (2, 1)\}$ , the following values:

$$\begin{cases} \gamma_i = \alpha_i N_p^i + \frac{\nu}{\nu + \rho_{\max} \mu} N_p^j, \\ A_i = \frac{\rho_{\max} \mu \alpha_j}{\nu} (1 - \alpha_i N_p^i \frac{\nu + \rho_{\max} \mu}{\lambda}) + \alpha_i N_p^i \frac{\rho_{\max} \mu}{\lambda}. \end{cases}$$

## Proposition [PoA]:

Condition (8) is a necessary and sufficient condition such that  $PoA = 1$ .

$$\max(\alpha_1, \alpha_2) \geq \frac{\nu}{\nu + \rho_{\max} \mu} \text{ or } \min(\gamma_1, \gamma_2) \geq \frac{\lambda}{\nu + \rho_{\max} \mu} \quad (8)$$

## Proposition [Bounds on the PoA]:

Suppose condition (8) does not hold. Then

$$PoA < \max\left(\frac{\nu}{\nu + \rho_{\max} \mu}, \min(\alpha_1, \alpha_2)\right).$$

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# Optimization of the QoS: Problem formulation

## Context

- The two CSs are geographically close,
- they are connected to the same node of the grid,
- they have to share a limited total quantity of power for the two stations.

## Problem formulation

$$\max_{(\alpha_1, \alpha_2) \in [0,1]^2} \bar{P}_{eq}, \quad (9)$$

$$\text{s.t. } \alpha_1 N_p^1 + \alpha_2 N_p^2 \leq C, \quad (10)$$

with  $C \in [1, N_p^1 + N_p^2]$ .

## Proposition [Expression of $\bar{P}_{eq}$ at optimum]:

The optimal value  $\bar{P}_{eq}^{opt}$  of  $\bar{P}_{eq}$  (9) under constraint (10) is:

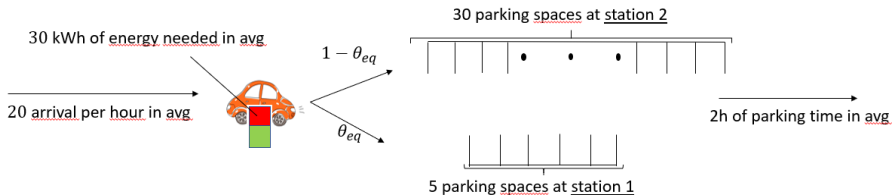
$$\bar{P}_{eq}^{opt} = \begin{cases} \frac{\rho_{\max}\mu}{\nu + \rho_{\max}\mu} & \text{if } C \geq \frac{\min(\lambda, \min(N_p^1, N_p^2)\nu)}{\nu + \rho_{\max}\mu}, & (a) \\ \frac{\rho_{\max}\mu}{\lambda} C & \text{if } \begin{cases} C < \frac{\lambda}{\nu + \rho_{\max}\mu}, \\ \min(N_p^1, N_p^2) > \frac{\lambda}{\nu}, \end{cases} & (b) \\ \frac{\rho_{\max}\mu}{\nu} \frac{C}{\min(N_p^1, N_p^2)} & \text{if } \begin{cases} C < \frac{\min(N_p^1, N_p^2)\nu}{\nu + \rho_{\max}\mu}, \\ \min(N_p^1, N_p^2) \leq \frac{\lambda}{\nu}. \end{cases} & (c) \end{cases} \quad (11)$$

- (a): High resource in power.
- (b): Low resource in power but high number of parking spaces.
- (c): Low resource in power and in quantity of parking spaces.

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# Usecase

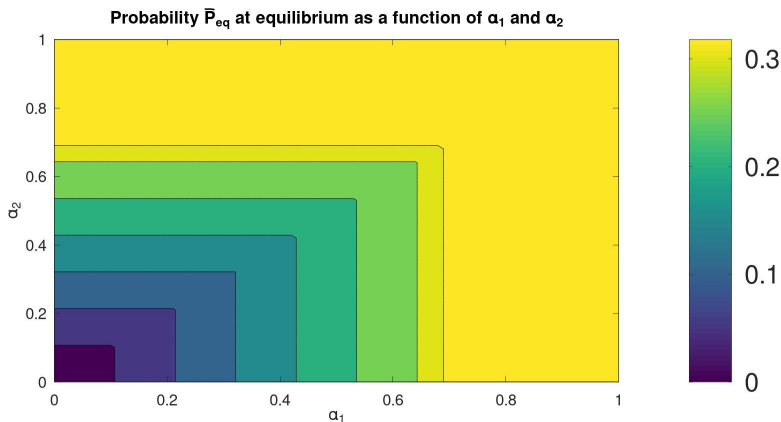


## Value parameters

- $\lambda = 20$ : EVs arrive in average every 3 minutes;
- $\mu = \frac{1}{30}$ : EVs need in average 30kWh to have a fully charged battery;
- $\nu = \frac{1}{2}$ : EVs stay in the CS 2h in average;
- $N_p^1 = 5$ ,  $N_p^2 = 30$ : There are respectively 5 and 30 parking spaces at CS 1 and 2;
- $p_{\max} = 7$ : The maximum charging power for the EVs is 7kW.

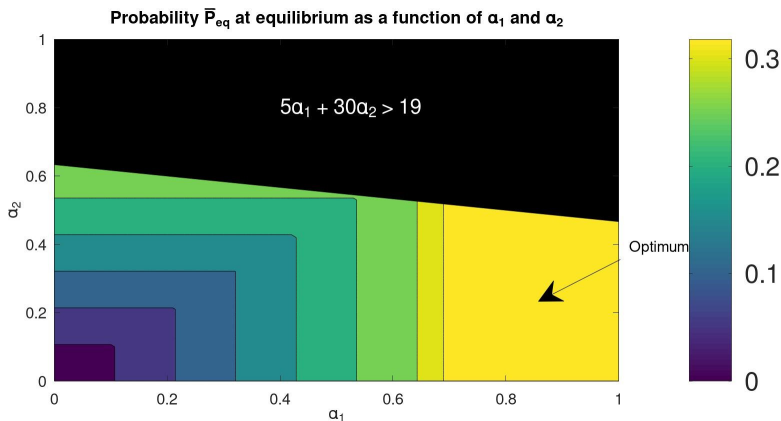


# Usecase: Values of $\bar{P}_{eq}$ at equilibrium



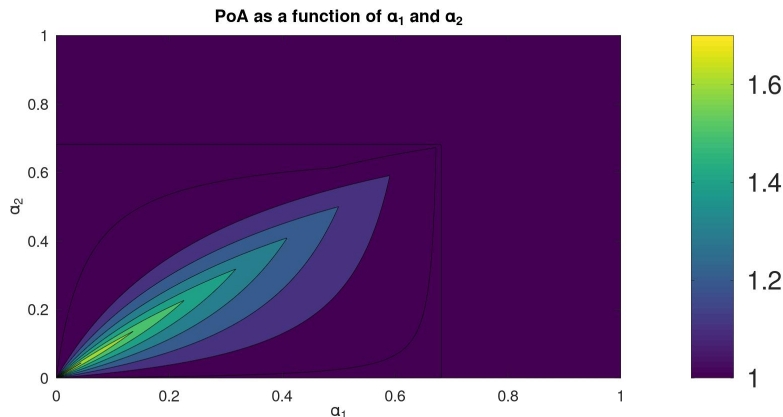
- $\bar{P}_{eq}$  increasing in  $\max(\alpha_1, \alpha_2)$ .

# Usecase: Optimization problem



- $S_{eq}^{opt} \approx \{(\alpha_1, \alpha_2) \in [0, 1]^2, \alpha_1 > 0.68, \alpha_2 \leq \frac{19}{30} - \alpha_1 \frac{1}{6}\}$ .

# Usecase: PoA



- $\max(\alpha_1, \alpha_2) > 0.68 \Rightarrow PoA = 1$ .
- Low values of  $\alpha_1$  and  $\alpha_2$ ,  $\alpha_1 \approx \alpha_2 \Rightarrow$  More power sharing between the EVs  $\Rightarrow$  Decentralized system inefficient.

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# Conclusion

## Main idea

The decentralized system is more efficient in particular when one of the two coefficients  $\alpha_1$  or  $\alpha_2$  is sufficiently high (in comparison of the parking time and the need in energy).

## Perspectives

- Study other types of power scheduling (ex: FIFO (First In First Out));
- Study other optimization criteria;
- Take into account the probability of being rejected;
- Take into account the price (for the EVs as well as for the charging stations);
- Give more information to the EVs (ex: number of charging EVs, current power available) [3];
- Other probability laws for  $B$  or  $D$  (ex: normal or uniform).

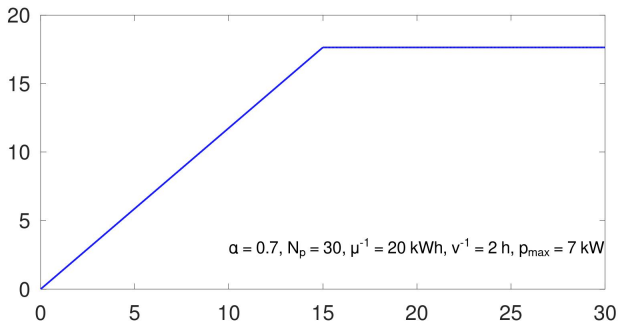
# Thank you for your attention

- [1] A.Aveklouris, M.Vlasiou **and** B.Zwart. “Bounds and limit theorems for a layered queueing model in electric vehicle charging”. *in* *Queueing Systems*: 93 (2019), **pages** 83–137. DOI: <https://doi.org/10.1007/s11134-019-09616-z>.
- [2] T. Roughgarden. “Selfish Routing and the Price of Anarchy”. *in*(2006).
- [3] R. Shone, V. A. Knight **and** J. E. Williams. “Comparisons between observable and unobservable M/M/1 queues with respect to optimal customer behavior”. *in* *European Journal of Operational Research*: 227 (2013), **pages** 133–141.
- [4] R. J. Aumann **and** L. S. Shapley. *Value of Non-atomic Games*. Princeton University Press, 1974, **page** 348. ISBN: 9780691645469.
- [5] C. Wan. *Jeux de congestion dans les réseaux*. **volume** 32. 9-10. Lavoisier, 2013, **pages** 981–1005. DOI: [10.3166/TSI.32.981-1005](https://doi.org/10.3166/TSI.32.981-1005)

# Appendix: Model (Number of charging EVs)

- If  $\frac{\nu}{\nu + \rho_{\max}\mu} \leq \alpha$ , then

$$\forall \lambda \in \mathbb{R}_+, \bar{z} = \frac{\min(\lambda, \nu N_p)}{\nu + \rho_{\max}\mu} \leq \alpha N_p.$$

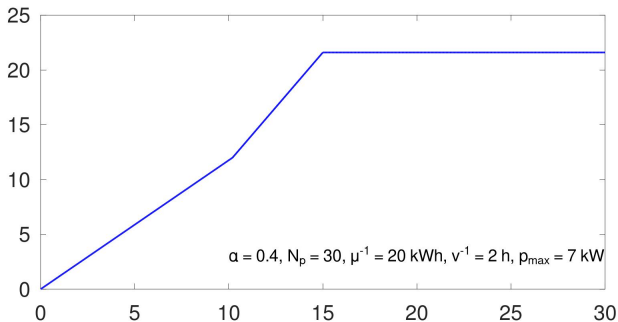




# Appendix: Model (Number of charging EVs)

- If  $\frac{\nu}{\nu + p_{\max}\mu} > \alpha$ , then

$$\bar{z} = \begin{cases} \frac{\min(\lambda, \nu N_p)}{\nu + p_{\max}\mu} \leq \alpha N_p & \text{if } \lambda \leq \alpha N_p(\nu + p_{\max}\mu), \\ \frac{\min(\lambda, \nu N_p) - p_{\max}\mu\alpha N_p}{\nu} > \alpha N_p & \text{if } \lambda > \alpha N_p(\nu + p_{\max}\mu). \end{cases}$$



## Characteristics of the game

- A non-atomic game[4] : The decision of a single EVs have no impact on the decision of the others EVs.
- A congestion game[5]: The decision of an EV to choose station  $i$  depends on the proportion of other EVs which choose station  $i$ . the incentive to choose station  $i$  is decreasing in the proportion of EVs choosing station  $i$ .
- An unobservable game[4]: EVs don't have any information on the number of charging EVs in the stations when arriving.

## Proposition [Characterization]:

The set of equilibrium  $S_{eq}$  is:

$$S_{eq} = \{\theta_{eq} \in [0, 1], \quad \bar{P}_1(\theta_{eq}) = \bar{P}_2(1 - \theta_{eq})\}. \quad (12)$$

Ideas of the proof: Let  $\tilde{S}_{eq} = \{\theta_{eq} \in [0, 1], \quad \bar{P}_1(\theta_{eq}) = \bar{P}_2(1 - \theta_{eq})\}$ .

- $\tilde{S}_{eq} \subseteq S_{eq}$ : Immediate.
- $S_{eq} \subseteq \tilde{S}_{eq}$ :
  - Suppose  $\tilde{\theta}$  s.t.  $\bar{P}_1(\tilde{\theta}) > \bar{P}_2(\tilde{\theta})$ . Then  $\tilde{\theta} = 1$ .
  - $\bar{P}_1(1) \leq \bar{P}_2(0)$ . Contradictory. ■

## Proposition [Existence and convexity]:

The set of equilibrium  $S_{eq}$  is non-empty and convex

Ideas of the proof: Let  $h := \theta \in [0, 1] \mapsto \bar{P}_1(\theta_{eq}) - \bar{P}_2(1 - \theta_{eq}) \in [0, 1]$ .

- $h$  is continuous and decreasing.
- $h(0) \geq 0$  and  $h(1) \leq 0$ . ■

## Proposition [Expression of $\bar{P}_{eq}$ ]:

At equilibrium, the probability  $\bar{P}_{eq}$  for any EV to leave a station with a full battery is given by:

$$\bar{P}_{eq} = \begin{cases} \frac{\rho_{\max}\mu}{\nu + \rho_{\max}\mu} & \text{if } \begin{cases} \max(\alpha_1, \alpha_2) \geq \frac{\nu}{\nu + \rho_{\max}\mu}, \\ \text{or } \alpha_1 N_p^1 + \alpha_2 N_p^2 \geq \frac{\lambda}{\nu + \rho_{\max}\mu}, \end{cases} \\ \frac{\rho_{\max}\mu}{\lambda} (\alpha_1 N_p^1 + \alpha_2 N_p^2) & \text{if } \begin{cases} \max(\alpha_1, \alpha_2) < \frac{\nu}{\nu + \rho_{\max}\mu}, \\ \alpha_1 N_p^1 + \alpha_2 N_p^2 < \frac{\lambda}{\nu + \rho_{\max}\mu}, \\ N_p^1 + N_p^2 > \frac{\lambda}{\nu}, \\ \max(\alpha_1, \alpha_2) \leq \frac{\nu}{\lambda} (\alpha_1 N_p^1 + \alpha_2 N_p^2), \end{cases} \\ \frac{\rho_{\max}\mu}{\nu} \max(\alpha_1, \alpha_2) & \text{else.} \end{cases} \quad (13)$$

Ideas of the proof: Solution of  $\bar{P}_1(\theta) = \bar{P}_2(1 - \theta)$ .

## Appendix: Proof4

Proposition: The optimal value  $\bar{P}_{eq}^{opt}$  of  $\bar{P}_{eq}$  (9) under constraint (10) is:

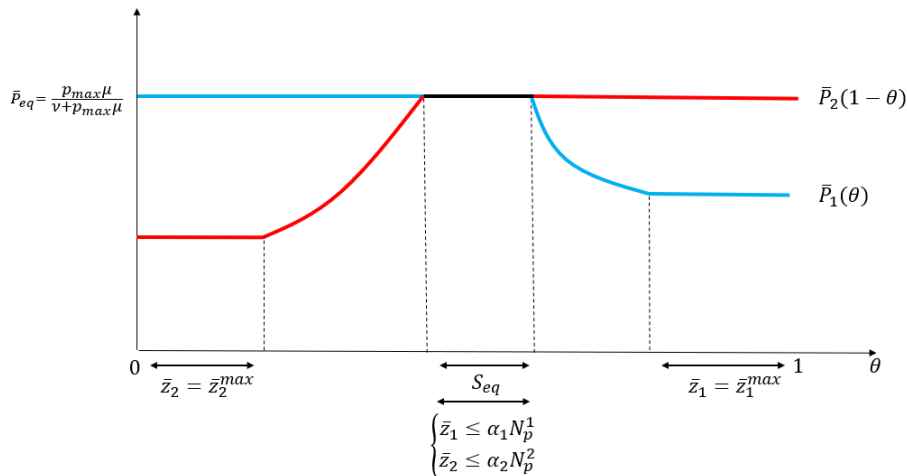
$$\bar{P}_{eq}^{opt} = \begin{cases} \frac{\rho_{\max}\mu}{\nu + \rho_{\max}\mu} & \text{if } C \geq \frac{\min(\lambda, \min(N_p^1, N_p^2)\nu)}{\nu + \rho_{\max}\mu}, & (a) \\ \frac{\rho_{\max}\mu}{\lambda} C & \text{if } \begin{cases} C < \frac{\lambda}{\nu + \rho_{\max}\mu}, \\ \min(N_p^1, N_p^2) > \frac{\lambda}{\nu}, \end{cases} & (b) \\ \frac{\rho_{\max}\mu}{\nu} \frac{C}{\min(N_p^1, N_p^2)} & \text{if } \begin{cases} C < \frac{\min(N_p^1, N_p^2)\nu}{\nu + \rho_{\max}\mu}, \\ \min(N_p^1, N_p^2) \leq \frac{\lambda}{\nu}, \end{cases} & (c) \end{cases} \quad (14)$$

Steps of the proof:

- Replace constraint (10) by constraint  $\alpha_1 N_p^1 + \alpha_2 N_p^2 = C$ .
- Express  $\bar{P}_{eq}$  as a function of  $\alpha_1$  (similarly  $\alpha_2$ ).
- Find the optimal  $\alpha_1$ .
- Notice that  $\bar{P}_{eq}^{opt}$  is increasing in  $C$ , so that the optimal values of (9) under the inequality (10) is the same as under the equality constraint  $\alpha_1 N_p^1 + \alpha_2 N_p^2 = C$ . ■

# Appendix: Equilibrium of the game in situation 1

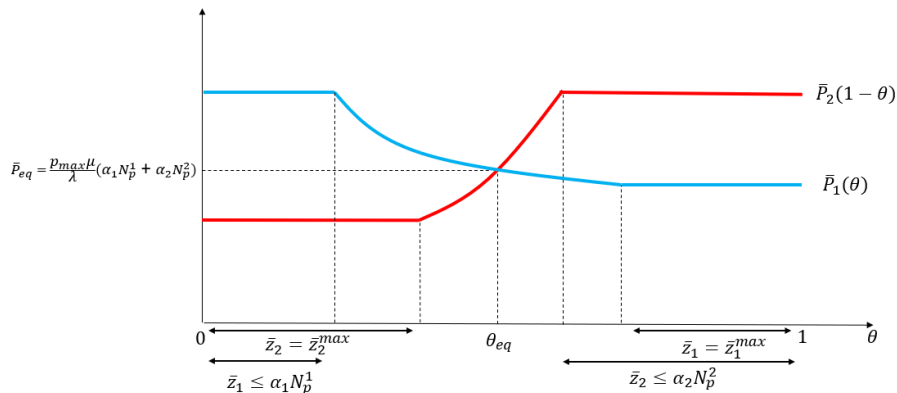
- $\bar{P}_{eq} = \frac{p_{max}\mu}{\nu + p_{max}\mu}$  if  $\max(\alpha_1, \alpha_2) \geq \frac{\nu}{\nu + p_{max}\mu}$  or  $\alpha_1 N_p^1 + \alpha_2 N_p^2 \geq \frac{\lambda}{\nu + p_{max}\mu}$



## Appendix: Equilibrium of the game in situation 2

- Otherwise,

$$\bar{P}_{eq} = \frac{P_{\max}\mu}{\lambda}(\alpha_1 N_p^1 + \alpha_2 N_p^2) \text{ if } \begin{cases} N_p^1 + N_p^2 > \frac{\lambda}{\nu} \\ \max(\alpha_1, \alpha_2) > \frac{\nu}{\lambda}(\alpha_1 N_p^1 + \alpha_2 N_p^2) \end{cases}$$





## Appendix: Equilibrium of the game in situation 3

- Otherwise,  $\bar{P}_{eq} = \frac{p_{max}\mu}{\nu} \max(\alpha_1, \alpha_2)$ .

