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# DYNAMIC PROGRAMMING FOR THE STOCHASTIC MATCHING MODEL ON GENERAL GRAPHS: THE CASE OF THE ‘N-GRAPH’

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**ABSTRACT.** In this paper, we address the optimal control of stochastic matching models on general graphs and single arrivals having fixed arrival rates, as introduced in [12]. On the ‘N-shaped’ graph, by following the dynamic programming approach of [5], we show that a ‘Threshold’-type policy on the diagonal edge, with priority to the extreme edges, is optimal for the discounted cost problem and linear holding costs.

## 1. INTRODUCTION

In this work, we consider a general stochastic matching (GM) model, as defined in [12]. It is roughly defined as follows: items of different classes enter a system one by one, and are of different classes. We let  $V$  be the set of classes, and also define a compatibility graph  $G = (V, E)$  on the set of classes. If two nodes (i.e., classes)  $i$  and  $j$  in  $V$  share an edge in  $G$ , we consider that any couple of items of respective classes  $i$  and  $j$  are *compatible*. Then, upon arrival into the system, any element can either be stored in a buffer, or matched with a compatible item that is already present in the buffer. It is then the role of the matching policy to determine the match of a given item, in case of multiple choices. We say that the matching policy is *greedy* if any incoming item that finds compatible items in line is necessarily matched upon arrival. Otherwise, in the *non-greedy* case, one might want to keep two compatible items together in line, to wait for a more profitable future match.

The GM model is a variant of the the so-called Bipartite stochastic matching (BM) model introduced in [6] for the *First Come, First matched* (FCFM) policy, and generalized in [3]. In this class of models that typically represent skill-based queueing systems, the compatibility graph  $G$  is bipartite (there are classes of *customers* and classes of *servers*) and arrivals occur by pairs customer/server - an assumption that might appear as unpractical, in many applications. The ‘general’ stochastic matching model (GM) we consider in this work, stands for a system in which the compatibility graph is general (i.e., non necessarily bipartite), and arrivals occur by single items rather than couples, entailing a widely different dynamics, and a different analysis. For the GM model, an important line of literature first considered the stability problem. Necessary conditions for stability were shown in [12], entailing in particular that no GM model on a bipartite graph can be stabilizable. Then, [14, 13, 10] obtained various stability results for greedy policies and non-bipartite graphs: [14] shows that greedy class-uniform policies never maximize the stability region, and that neither do greedy strict priority policies in general; By applying the dynamic reversibility argument in [1] to general graphs and single arrivals, [13] proves that the greedy *First Come, First Matched* policy maximizes the stability region, and construct explicitly the invariant measure and the matching rates between classes (a result that is completed in [7] by an insightful comparison

to order-independent queues). Then, [10] proved that greedy matching policies of the ‘Max-Weight’ type also maximize the stability region.

The performance optimization of this class of systems have recently been considered from the point of view of access control, by providing explicit procedures to *construct* the arrival rates in order to achieve given matching rates, for a *fixed* matching policy: see [8, 2]. Another approach has recently received an increasing interest: designing (greedy or non-greedy) matching policies able to maximize a given reward or to minimize a given cost, in the long run. In [15], for a vast class of matching structures, a variant of the greedy primal-dual algorithm was shown to be optimal for the long-term average matching reward, where the rewards are put on the matchings. Regarding the minimization of cumulative holding costs, [9] derives a lower bound for the long-run cumulative holding cost, and shows that this bound can be asymptotically approached under some conditions. Then, [4] constructs a policy that is approximately optimal with bounded regret, in the heavy-traffic regime. It is also shown in [11] that greedy policies are hindsight optimal (i.e., nearly maximize the total value function simultaneously at all times), and that a *static* hindsight optimal greedy policy can be prescribed, whenever  $G$  is a tree. For fixed arrival rates, [5] show that, under the stability conditions prescribed in [3], a matching policy of the ‘threshold’ type is optimal for holding costs, for a BM model, whenever  $G$  is the ‘N’-graph of Figure 1 below, for the discounted cost as well as the average cost problem.

In this work, we show that the result of [5] concerning a BM model on the ‘N’-graph, also holds for the GM model. Specifically, by using the tools of dynamic programming, we show that a ‘threshold’-type policy on the diagonal edge of  $G$  is optimal, for the discounted cost problem. The analysis follows the main steps of the main proof in [5]; However, the technical arguments change significantly, due to the fact that arrivals are single rather than pairwise. In particular, it is remarkable that, due to Theorem 2 in [12], the system is not stabilizable by any matching policy (greedy or not). However, optimality is intended for the discounted cost problem (and for linear holding costs), guaranteeing the well-posedness of the optimization scheme, despite the instability of the process at hand. By doing so, for this simple graph, we show what appears as the first optimal control result for general stochastic models having fixed arrival rates.

This paper is organized as follows. In Section 2 we start with some preliminary, and the definition of the general stochastic matching model at hand. Then, in Section 3 we introduce our dynamic programming problem and our main result, stating the optimality of a threshold-type policy on the diagonal edge with priority on the other edges. The proof of our main result is left to Section 4.

## 2. PRELIMINARY

**2.1. Notation.** Let  $\mathbb{R}$ ,  $\mathbb{N}$  and  $\mathbb{N}^*$  denote the sets of real numbers, non-negative and positive integers, respectively. For any  $d \in \mathbb{N}^*$ , we let  $\llbracket 0, d - 1 \rrbracket$  be the integer interval  $\{0, 1, 2, \dots, d - 1\}$ . For any  $i \in \llbracket 0, d - 1 \rrbracket$ , we let  $\mathbf{e}_i$  be the  $i$ -th vector of the canonical basis of  $\mathbb{R}^d$ . Vectors of  $\mathbb{N}^d$  will in general be denoted in bold, that is, we write e.g.

$$\mathbf{x} = (x_0, \dots, x_{d-1}).$$

For an un-directed graph  $G = (V, E)$ , we write  $i - j$  if  $\{i, j\} \in E$ . For any  $i \in V$ , we let

$$E(i) = \{j \in E : i - j\}.$$

Throughout, all random variables are defined on a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

**2.2. A general stochastic matching model.** We consider a stochastic matching model, as defined in [12]: Consider the ‘N-graph’, represented in Figure 1. It is denoted by

$$G = (V, E) := (\{0, 1, 2, 3\}, \{\ell_1, \ell_2, \ell_3\}) := (\{0, 1, 2, 3\}, \{\{0, 1\}, \{1, 2\}, \{2, 3\}\}).$$

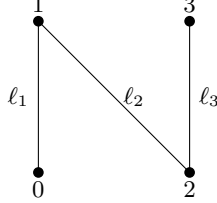


FIGURE 1. The ‘N-graph’.

The dynamics of the system can be described as follows: Items enter one by one, in discrete time, and are of different *classes*. Upon arrival, the class  $i \in V$  of the incoming item is drawn, independently of everything else, from a fixed probability distribution  $\mu$  on  $V$ , and we say in that case that the item is a *i-item*. In other words we see the arrival process as an infinite  $\mathbb{N}^4$ -valued IID sequence  $\{A_n; n \in \mathbb{N}\}$  such that for all  $n \in \mathbb{N}$  and  $i \in V := \llbracket 0, 3 \rrbracket$ ,

$$\mathbb{P}(\text{The incoming item at step } n \text{ is a } i\text{-item}) := \mathbb{P}(A_n = \mathbf{e}_i) = \mu(i),$$

where  $\mu$  is a probability measure over  $V$ . For a node  $j \in E(i)$ , any *j-item* is said to be *compatible* with the *i-item*.

Upon arrival, any *i-item* faces the following situation:

- (1) Either it does not find a compatible item in line, in which case it is stored in line;
- (2) Or, there exists at least one compatible item in line at that time, in which case it is the role of the *decision rule*, to be defined hereafter, to determine a *match* for the *i-item*. Following the latter policy, we have the following sub-cases:
  - 2a) Either the *i-item* is matched right away with a *j-item* for a given  $j \in E(i)$ , in which case both items leave the system right away.
  - 2b) Or, the *i-item* is put in line to wait for a potential better match, despite the presence of one (or more) compatible item(s) in line.

At any time  $n \in \mathbb{N}$ , we denote by  $X_n \in \mathbb{N}^4$ , the state of the buffer just after time  $n$ . Namely, for any  $i \in \llbracket 0, 3 \rrbracket$  we denote by  $X_n(i)$ , the number of stored *i-items* in line just after time  $n$ , and set

$$X_n = (X_n(0), \dots, X_n(3)).$$

The decision rule is to be *Markovian*, if the choice of match at any time  $n$  is prescribed by an operator  $u_n: \mathbb{N}^4 \rightarrow \mathbb{N}^4$  that is deterministic, or independent of the filtration generated by the  $X_p$ ,  $p \leq n$ , and such that

$$X_{n+1} = u_n(X_n) + A_{n+1}.$$

The decision rule is said to be *greedy* if sub-case 2b) above is not allowed, namely, whenever an incoming item finds compatible items in line, it must be matched with one of those right away. An *admissible policy*  $\pi$  is a sequence  $\{u_n; n \in \mathbb{N}\}$  of Markovian decision rules. We then say that the policy  $\pi$  is *structured* by the decision rules  $u_n$ ,  $n \in \mathbb{N}$ . We denote by ADM, the set of admissible policies. Notice that a decision rule may be constant, namely  $u_n = u$  for all  $n$ , in which case we say

that then the policy  $\pi$  is *stationary*. The state space of  $\{X_n; n \in \mathbb{N}\}$  is a subset  $\mathbb{X}_\pi$  of  $\mathbb{N}^4$  that depends on the policy  $\pi$ . For instance, if  $\pi$  is greedy, then

$$\mathbb{X}_\pi = \{\mathbf{x} = (x_0, \dots, x_3) \in \mathbb{N}^4 : x_i x_j = 0 \text{ for any } \{i, j\} \in E\}.$$

Under a stationary policy  $\pi$ , the process  $\{X_n; n \in \mathbb{N}\}$  is thus a  $\mathbb{X}_\pi$ -valued homogeneous Markov chain, and for all state  $\mathbf{x} \in \mathbb{X}_\pi$  we have for all  $v \in V$ ,

$$\mathbb{P}(u(\mathbf{x}) + \mathbf{e}_v \mid \mathbf{x}) := \mathbb{P}(X_{n+1} = u(\mathbf{x}) + \mathbf{e}_v \mid X_n = \mathbf{x}) = \mu(v).$$

**Definition 1.** Any state  $\mathbf{x} = (x_0, \dots, x_3) \in \mathbb{X}_\pi$  corresponds uniquely to a block model graph  $\mathbf{G}_\mathbf{x} = (\mathbf{V}_\mathbf{x}, \mathbf{E}_\mathbf{x})$  having  $|\mathbf{x}|$  nodes, in which there are  $x_i$  nodes of type  $i$  for all  $i \in \llbracket 0, 3 \rrbracket$  and, for any  $i \neq j \in \llbracket 0, 3 \rrbracket$ , and any pair of nodes  $u$  and  $v$  of respective types  $i$  and  $j$ , there exists an edge between  $u$  and  $v$ , that is,  $\{u, v\} \in \mathbf{E}_\mathbf{x}$ , if and only if  $\{i, j\} \in E$ . Note that  $\mathbf{G}_\mathbf{x}$  has no edge if  $\pi$  is greedy.

**Definition 2.** For any  $\mathbf{x} \in \mathbb{X}_\pi$ , we denote by  $\mathbf{M}_\mathbf{x}$ , the set of all possible matches from state  $\mathbf{x}$ , that is, the set of all matchings on  $\mathbf{G}_\mathbf{x}$  or in other words, of all sub-graphs of  $\mathbf{G}_\mathbf{x}$  such that each node has degree 0 or 1.

For any pair of states  $\mathbf{x}, \mathbf{y} \in \mathbb{X}_\pi$ , we write  $\mathbf{x} \rightarrow \mathbf{y}$  whenever  $\mathbb{P}(\mathbf{y} \mid \mathbf{x}) > 0$ . In the sequel, we denote by  $\mathbb{E}_\mathbf{x}^\pi$  the expected value of r.v.'s, under the policy  $\pi$ , given that the initial state is  $\mathbf{x} \in \mathbb{X}_\pi$ . In the discussion that follows, we identify:

- Any edge  $e = \{i, j\} \in E$  with the vector  $\mathbf{e}_i + \mathbf{e}_j \in \mathbb{N}^4$ ;
- For any  $\mathbf{x} \in \mathbb{N}^4$ , any matching  $\mathbf{m} \in \mathbf{M}_\mathbf{x}$  with the vector  $\mathbf{z} \in \mathbb{N}^4$  such that for all  $i \in V$ ,  $z_i$  is the number of items of class  $i$  matched in  $\mathbf{m}$ . In other words,

$$\mathbf{m} \equiv \sum_{\substack{v \in \mathbf{m}: \\ v \text{ is of type } i}} \mathbf{e}_i,$$

in a way that  $\mathbf{x} - \mathbf{m}$  indeed represents the state of a system after executing all matchings in  $\mathbf{m}$  in a system in state  $\mathbf{x}$ .

### 3. OPTIMIZATION OF THE 'N-GRAPH' MODEL

**3.1. Dynamic programming.** We now introduce our optimization problem.

**Definition 3.** An admissible linear cost function is a linear mapping  $c$  of the form

$$c : \begin{cases} \mathbb{N}^4 & \longrightarrow & \mathbb{R}_+ \\ \mathbf{x} := (x_0, x_1, x_2, x_3) & \longmapsto & c_0 x_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 \end{cases},$$

whose coefficients  $c_0, c_1, c_2, c_3$  are non-negative, and such that

$$c_2 \leq c_0 \quad \text{and} \quad c_1 \leq c_3.$$

Let  $c : \mathbb{N}^4 \rightarrow \mathbb{R}_+$  be an admissible linear cost function. For a fixed discount factor  $\gamma \in (0, 1)$  and a fixed initial state  $\mathbf{x} \in \mathbb{X}_\pi$ , our aim is to minimize over  $\pi$ , the so-called *discounted cost*

$$(1) \quad v_\gamma^\pi(\mathbf{x}) = \lim_{N \rightarrow +\infty} \sum_{n=0}^N \gamma^n \mathbb{E}_\mathbf{x}^\pi [c(X_n)].$$

**Definition 4.** Let  $v$  be a real function on  $\mathbb{N}^4$ , and  $A$  be a r.v. of law  $\mu$ . For any  $\mathbf{x} \in \mathbb{N}^4$ , denote

$$L_\mathbf{m}^\gamma v(\mathbf{x}) = c(\mathbf{x}) + \gamma \mathbb{E}[v(\mathbf{x} - \mathbf{m} + A)], \quad \text{for all } \mathbf{m} \in \mathbf{M}_\mathbf{x}$$

and

$$\begin{aligned} L^\gamma v(\mathbf{x}) &= \min_{\mathbf{m} \in \mathbf{M}_\mathbf{x}} L_\mathbf{m}^\gamma v(\mathbf{x}) \\ &= c(\mathbf{x}) + \gamma \min_{\mathbf{m} \in \mathbf{M}_\mathbf{x}} \mathbb{E}[v(\mathbf{x} - \mathbf{m} + A)]. \end{aligned}$$

Then we use the principle of dynamic programming, and obtain that for any state  $\mathbf{x} \in \mathbb{N}^4$  that is admissible for some policy,

$$\begin{aligned} v(\mathbf{x}) &:= \inf_{\pi \in \text{ADM}} v_\gamma^\pi(\mathbf{x}) \\ &= \inf_{\pi \in \text{ADM}} \mathbb{E}_\mathbf{x}^\pi \left[ \sum_{n=0}^{+\infty} \gamma^n c(X_n) \right] \\ &= \inf_{\pi \in \text{ADM}} \mathbb{E}_\mathbf{x}^\pi \left[ c(\mathbf{x}) + \sum_{n=1}^{+\infty} \gamma^n c(X_n) \right] \\ &= c(\mathbf{x}) + \gamma \inf_{\pi \in \text{ADM}} \mathbb{E}_\mathbf{x}^\pi \left[ \sum_{n=0}^{+\infty} \gamma^n c(X_{n+1}) \right] \\ &= c(\mathbf{x}) + \gamma \inf_{\substack{\mathbf{y} \in \mathbb{N}^4: \\ \mathbf{x} \rightarrow \mathbf{y} \text{ for some } \pi \in \text{ADM}}} \mathbb{E}[v(\mathbf{y})] \\ &= c(\mathbf{x}) + \gamma \min_{\mathbf{m} \in \mathbf{M}_\mathbf{x}} \mathbb{E}[v(\mathbf{x} - \mathbf{m} + A)] = L^\gamma v(\mathbf{x}). \end{aligned}$$

The mapping  $v$  is then called the *value function* of our optimization problem, and the argument above shows that  $v$  solves a fixed point equation of the Bellman type: for all  $\mathbf{x}$  that is admissible for some policy,

$$(2) \quad v(\mathbf{x}) = L^\gamma v(\mathbf{x}).$$

In this paper, we put in evidence a simple policy that is able to achieve a solution to (2) in the case of the ‘N’-graph.

**3.2. Threshold-type decision rules.** A decision rule will be of threshold type in  $\ell_2$ , with priority for  $\ell_1$  and  $\ell_3$ , if it matches all possible  $\ell_1$  and  $\ell_3$  edges, before matching a certain amount of  $\ell_2$  edges according to a threshold that depends on the difference between the number of remaining items 0 and 2. Formally,

**Definition 5.** Let  $(t_i)_{i \in \mathbb{Z}}$  be a family of elements of  $\mathbb{N} \cup \{+\infty\}$ . The map  $u: \mathbb{N}^4 \rightarrow \mathbb{N}^4$  is said to be a threshold type decision rule with priority for  $\ell_1$  and  $\ell_3$  and thresholds  $(t_i)_{i \in \mathbb{Z}}$  in  $\ell_2$ , if it can be written as

$$u(\mathbf{x}) = \mathbf{x} - m(\mathbf{x}), \quad \mathbf{x} \in \mathbb{N}^4,$$

where

$$m: \begin{cases} \mathbb{N}^4 & \longrightarrow & \mathbb{N}^4 \\ \mathbf{x} & \longmapsto & (x_0 \wedge x_1)\ell_1 + (x_2 \wedge x_3)\ell_3 + k_{t_{i(\mathbf{x})}}(\mathbf{x})\ell_2 \end{cases},$$

for

$$\begin{cases} k_t(\mathbf{x}) & = ((x_1 - x_0) \wedge (x_2 - x_3) - t)^+; \\ i(\mathbf{x}) & = (x_1 - x_0) - (x_2 - x_3). \end{cases}$$

We denote by  $\mathcal{T}$ , the set of threshold decision rules. A policy  $\pi$  is said to be of the threshold type, if it is structured by decision rules of  $\mathcal{T}$ .

**3.3. Main result.** The following result shows that we can minimize the cost in the long run, by applying at any time, a matching policy of the threshold type, conveying the natural idea that it is preferable to match edges  $\ell_3$  and  $\ell_1$  over edges  $\ell_2$ , up to a certain state of congestion of the buffer. We have the following,

**Theorem 1.** *For the discounted cost problem, under the assumptions of Definition 3 there exists an optimal stationary policy of the threshold type.*

Theorem 1 is proved in Section 4.

**Remark 1.** It is significant that the above result holds regardless of the fact that the model at hand is instable - a fact that is implied by the first assertion of Theorem 2 in [12], as the ‘N’-graph is bipartite. It is easy to observe that the quantity in (1) is well-defined for all  $\pi$  and  $\mathbf{x}$ , since for all  $n$ , the  $n$ -th term of the summation is upper-bounded by  $\gamma^n (\max\{c_0, \dots, c_3\}(|\mathbf{x}| + n))$ .

#### 4. PROOF OF THEOREM 1

We will crucially use the following result,

**Theorem 2** ([16], 6.11.3). *Suppose that there exists a function  $w: \mathbb{N}^4 \rightarrow \mathbb{R}_+$  such that*

(i) *For any  $\mathbf{x} \in \mathbb{N}^4$ ,*

$$\sup_{\mathbf{z} \in \mathbb{N}^4: \mathbf{x}-\mathbf{z} \in \mathbb{N}^4} \frac{c(\mathbf{x}-\mathbf{z})}{w(\mathbf{x})} < +\infty \quad \text{and} \quad \sup_{\mathbf{z} \in \mathbb{N}^4: \mathbf{x}-\mathbf{z} \in \mathbb{N}^4} \frac{1}{w(\mathbf{x})} \sum_{\mathbf{y}} \mathbb{P}(\mathbf{y}|\mathbf{x}-\mathbf{z})w(\mathbf{y}) < +\infty.$$

(ii) *For all  $\varepsilon \in [0 : 1)$ , there exists  $\eta \in [0 : 1)$  and  $p \in \mathbb{N}^*$  such that for all  $p$ -tuple  $\pi = (u_1, u_2, \dots, u_p)$  of deterministic Markovian decision rules  $u_i: \mathbb{N}^4 \rightarrow \mathbb{N}^4$ ,*

$$\varepsilon^p \mathbb{E}_{\mathbf{x}}^{\pi} [w(X_p)] < \eta w(\mathbf{x}).$$

(iii) *If we denote*

$$V_w = \left\{ v: \mathbb{N}^4 \rightarrow \mathbb{R}_+ : \sup_{x \in \mathbb{N}^4} \frac{v(\mathbf{x})}{w(\mathbf{x})} < \infty \right\},$$

*then for all  $v \in V_w$  there exists a deterministic Markovian decision rule  $u$  such that for all  $\mathbf{x} \in \mathbb{N}^4$ ,*

$$L^\gamma v(\mathbf{x}) = L_{u(\mathbf{x})}^\gamma v(\mathbf{x}).$$

*Suppose also that there exist two sets  $\mathcal{V}$  and  $\mathcal{D}$  such that*

- (1)  *$\mathcal{V}$  is stable under  $L^\gamma$  and under pointwise convergence;*
- (2) *If  $v \in \mathcal{V}$ , then there exists a deterministic markovian decision rule  $u \in \mathcal{D}$  such that for all  $\mathbf{x} \in \mathbb{N}^4$ ,*

$$L^\gamma v(\mathbf{x}) = L_{u(\mathbf{x})}^\gamma v(\mathbf{x}).$$

*Then, there exists an optimal stationary policy  $\pi^*$  structured by a single Markovian decision rule  $u^* \in \mathcal{D}$ .*

The strategy is as follows. We start by defining a set of test functions  $\mathcal{V}$  that is stable under the action of  $L^\gamma$ , and such that if  $v \in \mathcal{V}$ , we can always find a threshold type decision rule  $m \in \mathcal{T}$  such that (2) holds for all  $\mathbf{x} \in \mathbb{N}^4$ .

#### 4.1. Properties of value functions.

**Definition 6.** A mapping  $v: \mathbb{N}^4 \rightarrow \mathbb{R}_+$  is said to be increasing in  $\ell_1$ , and we write  $v \in \mathcal{I}_1$  if, for any  $\mathbf{x} \in \mathbb{N}^4$  we have

$$v(\mathbf{x}) \leq v(\mathbf{x} + \ell_1).$$

Likewise, a mapping  $v: \mathbb{N}^4 \rightarrow \mathbb{R}_+$  is said to be increasing in  $\ell_3$ , and we write  $v \in \mathcal{I}_3$  if, for any  $\mathbf{x} \in \mathbb{N}^4$  we have

$$v(\mathbf{x}) \leq v(\mathbf{x} + \ell_3).$$

It is reasonable to think that if we have the choice between matching an  $\ell_2$  edge and both an  $\ell_1$  and an  $\ell_3$  edge, we will rather do the latter. Hence the following definition,

**Definition 7.** We say that  $v: \mathbb{N}^4 \rightarrow \mathbb{R}_+$  belongs to  $\mathcal{I}'$  if, for all  $\mathbf{x} \in \mathbb{N}^4$  such that  $x_1 x_2 > 0$ ,

$$\begin{cases} v(\mathbf{x}) \leq v(\mathbf{x} + \ell_1 - \ell_2) \\ v(\mathbf{x}) \leq v(\mathbf{x} + \ell_3 - \ell_2). \end{cases}$$

Moreover, we want to highlight that it is less and less interesting to keep items of classes 1 or 2 in line. Hence the following definition,

**Definition 8.** We say that  $v: \mathbb{N}^4 \rightarrow \mathbb{R}_+$  is convex in  $\ell_2$  and we write  $v \in \mathcal{C}_2$  if, for all state  $\mathbf{x} \in \mathbb{N}^4$  such that  $x_2 \geq x_3 - 1$  and  $x_1 \geq x_0 - 1$ ,

$$v(\mathbf{x} + 2\ell_2) - v(\mathbf{x} + \ell_2) \geq v(\mathbf{x} + \ell_2) - v(\mathbf{x}).$$

**4.2. Optimal policies.** We now show that these properties are already enough to satisfy the required property, namely, that a value function solving (2) corresponds to a policy structured by a fixed threshold decision rule. We have the following result,

**Lemma 1.** Let  $\gamma, v \in \mathcal{I}_1 \cap \mathcal{I}_3 \cap \mathcal{I}'$  and  $\mathbf{x} \in \mathbb{N}^4$ . Then, there exists a matching  $\mathbf{m} \in \mathbf{M}_{\mathbf{x}}$  such that

$$L_{\mathbf{m}}^{\gamma} v(\mathbf{x}) = L^{\gamma} v(\mathbf{x}),$$

and such that  $\mathbf{m}$  matches all possible  $\ell_1$  and  $\ell_3$  edges in a system in state  $\mathbf{x}$ .

*Proof.* Let  $\mathbf{x} \in \mathbb{N}^4$ , and let  $\mathbf{m}^0 \in \mathbf{M}_{\mathbf{x}}$ . We let  $\mathbf{m}^1$  be the matching we obtain by executing all matches of edges  $\ell_1$  and  $\ell_3$  that are still possible in the state  $\mathbf{x} - \mathbf{m}$ , that is,

$$\mathbf{m}^1 = \mathbf{m}^0 + (x_0 - m_0^0) \wedge (x_1 - m_1^0) \ell_1 + (x_2 - m_2^0) \wedge (x_3 - m_3^0) \ell_3.$$

Because  $v \in \mathcal{I}_1 \cap \mathcal{I}_3$  and because  $\mathbf{m}^1$  can be obtained from  $\mathbf{m}^0$  by successively adding only  $\ell_1$  and  $\ell_3$  edges, we have

$$L_{\mathbf{m}^1}^{\gamma} v(\mathbf{x}) \leq L_{\mathbf{m}^0}^{\gamma} v(\mathbf{x}).$$

Let  $n_2^1$  denotes the number of  $\ell_2$  edges in  $\mathbf{m}^1$ . We are now going to find a matching that is at least as good and which has  $\min(x_0, x_1)$   $\ell_1$  edges. To do so, we define  $\mathbf{m}^2$  to be the matching obtained from  $\mathbf{m}^1$  by transforming as many  $\ell_2$  edges in  $\ell_1$  edges as possible, that is,

$$\mathbf{m}^2 = \mathbf{m}^1 + (n_2^1 \wedge (x_0 - m_0^1)) (\ell_1 - \ell_2).$$

As  $\mathbf{m}^2$  can be obtained from  $\mathbf{m}^1$  by adding  $\ell_1 - \ell_2$  a finite number of times, we get that

$$L_{\mathbf{m}^2}^{\gamma} v(\mathbf{x}) \leq L_{\mathbf{m}^1}^{\gamma} v(\mathbf{x}).$$

Notice that the matching  $\mathbf{m}^2$  has exactly  $x_0 \wedge x_1$   $\ell_1$  edges. Indeed, if it was not so, then we would be in the following alternative:



- (i) Either in  $\mathbf{x} - \mathbf{m}^2$  there would be a positive number of unmatched 0-items and a positive number of unmatched 1-items;
- (ii) Or, in  $\mathbf{m}^2$  there would be a positive number of 1-items matched with 2-items.

But (i) is in contradiction with the definition of  $\mathbf{m}^1$ , and (ii) contradicts the definition of  $\mathbf{m}^2$ , an absurdity.

Now  $n_2^2$  denotes the number of  $\ell_2$  edges in  $\mathbf{m}^2$  (and notice that  $n_2^2 = 0$  if  $x_0 - m_0^1 \geq n_2^1$ ). We now define  $\mathbf{m}^3$  to be the matching obtained from  $\mathbf{m}^2$  by transforming as many  $\ell_2$  edges in  $\ell_3$  edges as possible, that is,

$$\mathbf{m}^3 = \mathbf{m}^2 + (n_2^2 \wedge (x_3 - m_3^2))(\ell_3 - \ell_2).$$

Again, as  $\mathbf{m}^2$  can be obtained from  $\mathbf{m}^1$  by adding  $\ell_3 - \ell_2$  a finite number of times, we get that

$$L_{m^3}^\gamma v(\mathbf{x}) \leq L_{m^2}^\gamma v(\mathbf{x}).$$

Notice that the matching  $\mathbf{m}^3$  also has exactly  $x_3 \wedge x_2$   $\ell_3$  edges. Indeed, if it was not so, we would, again, either find in  $\mathbf{x} - \mathbf{m}^3$  a positive number of unmatched 3-items and a positive number of unmatched 2-items, or find in  $\mathbf{m}^2$ , a positive number of 2-items matched with 1-items, another absurdity given the definitions of  $\mathbf{m}^1$  and  $\mathbf{m}^3$ .

As a conclusion, we have thus shown that for every matching  $\mathbf{m}^0$  we can find a matching  $\mathbf{m}^3$  which has exactly  $\min(x_0, x_1)$   $\ell_1$  edges and  $\min(x_2, x_3)$   $\ell_3$  edges, and such that

$$L_{m^3}^\gamma v(\mathbf{x}) \leq L_{m^0}^\gamma v(\mathbf{x}).$$

Since there is a finite number of possible matchings, the proof is complete.  $\square$

Observe the following result,

**Corollary 1.** *For any  $0 < \gamma < 1$ , the function set  $\mathcal{I}_1 \cap \mathcal{I}_3 \cap \mathcal{I}'$  is stable by the operator  $L^\gamma$ .*

*Proof.* First, it is easily checked that any linear cost function  $c$  satisfying to definition 3 is in particular, an element of  $\mathcal{I}_1 \cap \mathcal{I}_3 \cap \mathcal{I}'$ . For instance, for any  $\mathbf{x}$  such that  $x_1 x_2 > 0$ ,

$$c(\mathbf{x} + \ell_1 - \ell_2) - c(\mathbf{x}) = c(\ell_1) - c(\ell_2) = c_0 - c_2 \geq 0,$$

and the other arguments are similar. Let  $v \in \mathcal{I}_1 \cap \mathcal{I}_3 \cap \mathcal{I}'$ . First let  $\mathbf{x} \in \mathbb{N}^4$ , and  $\bar{\mathbf{x}} = \mathbf{x} + \ell_1$ . According to Lemma 1, we can choose a matching  $\bar{\mathbf{m}} = (\bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3) \in \mathbf{M}_{\bar{\mathbf{x}}}$ , which matches all possible  $\ell_1$  and  $\ell_3$  edges, and such that  $L_{\bar{\mathbf{m}}}^\gamma v(\bar{\mathbf{x}}) = L^\gamma v(\bar{\mathbf{x}})$ . In particular, we have that  $\bar{m}_0 \geq 1$  and  $\bar{m}_1 \geq 1$ , and so we can define a matching  $\mathbf{m} = \bar{\mathbf{m}} - \ell_1 \in \mathbf{M}_{\mathbf{x}}$  such that  $\bar{\mathbf{x}} - \bar{\mathbf{m}} = \mathbf{x} - \mathbf{m}$ . It follows that

$$\begin{aligned} L^\gamma v(\mathbf{x}) &\leq L_{\mathbf{m}}^\gamma v(\mathbf{x}) \\ &= c(\mathbf{x}) + \gamma \mathbb{E}[v(\mathbf{x} - \mathbf{m} + A)] \\ &\leq c(\bar{\mathbf{x}}) + \gamma \mathbb{E}[v(\bar{\mathbf{x}} - \bar{\mathbf{m}} + A)] \\ &= L_{\bar{\mathbf{m}}}^\gamma v(\bar{\mathbf{x}}) = L^\gamma v(\bar{\mathbf{x}}). \end{aligned}$$

As the above holds for all  $\mathbf{x}$  we conclude that  $L^\gamma v \in \mathcal{I}_1$ , and the same argument shows that  $L^\gamma v \in \mathcal{I}_3$ .

We now prove that  $L^\gamma v \in \mathcal{I}'$ . Let  $\mathbf{x} \in \mathbb{N}^4$  be such that  $x_1 x_2 > 0$ , and  $\tilde{\mathbf{x}} = x + \ell_1 - \ell_2$ . According to Lemma 1, we can choose a matching  $\tilde{\mathbf{m}} = (\tilde{m}_0, \tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \in \mathbf{M}_{\tilde{\mathbf{x}}}$ , which matches all possible  $\ell_1$  and  $\ell_3$  edges, and such that  $L_{\tilde{\mathbf{m}}}^\gamma v(\tilde{\mathbf{x}}) = L^\gamma v(\tilde{\mathbf{x}})$ . Again,

we have that  $\tilde{m}_0 \geq 1$  and  $\tilde{m}_1 \geq 1$ , and so the matching  $\mathbf{m}' := \tilde{\mathbf{m}} + \ell_2 - \ell_1$  is an element of  $\mathbf{M}_{\mathbf{x}}$  such that  $\tilde{\mathbf{x}} - \tilde{\mathbf{m}} = \mathbf{x} - \mathbf{m}'$ . Thus we have again

$$\begin{aligned} L^\gamma v(\mathbf{x}) &\leq L_{\mathbf{m}'}^\gamma v(\mathbf{x}) \\ &= c(\mathbf{x}) + \gamma \mathbb{E}[v(\mathbf{x} - \mathbf{m}' + A)] \\ &\leq c(\tilde{\mathbf{x}}) + \gamma \mathbb{E}[v(\tilde{\mathbf{x}} - \tilde{\mathbf{m}} + A)] \\ &= L_{\tilde{\mathbf{m}}}^\gamma v(\tilde{\mathbf{x}}) = L^\gamma v(\tilde{\mathbf{x}}). \end{aligned}$$

We can prove similarly that the same holds true by replacing  $\tilde{\mathbf{x}}$  by  $\mathbf{x} + \ell_3 - \ell_2$ . This shows that  $L^\gamma v \in \mathcal{I}'$ , which concludes the proof.  $\square$

**Proposition 1.**

For any  $\gamma$  and  $v \in \mathcal{I}_1 \cap \mathcal{I}_3 \cap \mathcal{C}_2 \cap \mathcal{I}'$ , there exists a decision rule  $m^*$  with threshold  $(t_i)_{i \in \mathbb{Z}}$ , such that for all  $\mathbf{x} \in \mathbb{N}^4$ ,

$$L_{m^*(\mathbf{x})}^\gamma v(\mathbf{x}) = L^\gamma v(\mathbf{x})$$

*Proof.* First, observe that if  $i \leq 0$ , by the convexity of  $v$  there exists  $t_i \in \mathbb{N} \cup \{+\infty\}$  such that the sequence  $(\mathbb{E}[v(A + |i|\mathbf{e}_2 + j\ell_2)])_{j \in \mathbb{N}}$  is decreasing on  $\llbracket 0, t_i \rrbracket$  and increasing on  $\llbracket t_i, +\infty \rrbracket$ . All the same, if  $i \geq 0$ , then by convexity of  $v$  there exists  $t_i \in \mathbb{N} \cup \{+\infty\}$  such that the sequence  $(\mathbb{E}[v(A + i\mathbf{e}_1 + j\ell_2)])_{j \in \mathbb{N}}$  is decreasing on  $\llbracket 0, t_i \rrbracket$  and increasing on  $\llbracket t_i, +\infty \rrbracket$ . We call  $m^*$  the decision rule with thresholds  $(t_i)_{i \in \mathbb{Z}}$ .

Now let  $\mathbf{x} \in \mathbb{N}^4$ . According to Lemma 1, there exists  $\mathbf{m} \in \mathbf{M}_{\mathbf{x}}$  that realizes the minimum  $L^\gamma v(\mathbf{x})$ , and matches all the possible  $\ell_1$  and  $\ell_3$  edges of  $\mathbf{x}$ . We call  $k_{\mathbf{x}}$  the number of  $\ell_2$  edges in  $\mathbf{m}$ . We are in the following alternative:

- (1) If  $x_0 \leq x_1$  or  $x_2 \leq x_3$ , then  $k_{\mathbf{x}} = k_{t_{i(\mathbf{x})}}(\mathbf{x}) = 0$  and  $\mathbf{m} = m^*(\mathbf{x})$ . We thus have

$$L_{m^*(\mathbf{x})}^\gamma v(\mathbf{x}) = L_{\mathbf{m}}^\gamma v(\mathbf{x}) = L^\gamma v(\mathbf{x}).$$

- (2) Otherwise, denoting  $j(\mathbf{x}) = (x_1 - x_0) \wedge (x_2 - x_3)$ , we have that

$$\begin{aligned} L_{m^*(\mathbf{x})}^\gamma v(\mathbf{x}) &= c(\mathbf{x}) + \mathbb{E}[v(A + (x_1 - x_0)\mathbf{e}_1 + (x_2 - x_3)\mathbf{e}_2 - k_{t_{i(\mathbf{x})}}(\mathbf{x})\ell_2)] \\ &= c(\mathbf{x}) + \mathbb{E}\left[v\left(A + (x_1 - x_0 - j(\mathbf{x}))^+ \mathbf{e}_1 + (x_2 - x_3 - j(\mathbf{x}))^+ \mathbf{e}_2 + (j(\mathbf{x}) - k_{t_{i(\mathbf{x})}}(\mathbf{x}))\ell_2\right)\right]. \end{aligned}$$

We are in the following sub-alternative:

- 2a) If  $i(\mathbf{x}) \leq 0$ , then we get that

$$L_{m^*(\mathbf{x})}^\gamma v(\mathbf{x}) = c(\mathbf{x}) + \mathbb{E}[v(A + |i(\mathbf{x})|\mathbf{e}_2 + (j(\mathbf{x}) - k_{t_{i(\mathbf{x})}}(\mathbf{x}))\ell_2)].$$

and we also have

$$L_{\mathbf{m}}^\gamma v(\mathbf{x}) = c(\mathbf{x}) + \mathbb{E}[v(A + i(\mathbf{x})\mathbf{e}_2 + (j(\mathbf{x}) - k_{\mathbf{x}})\ell_2)].$$

We show that, in all cases,

$$(3) \quad L_{m^*(\mathbf{x})}^\gamma v(\mathbf{x}) \leq L_{\mathbf{m}}^\gamma v(\mathbf{x}).$$

- If  $t_{i(\mathbf{x})} = +\infty$ , we get that  $k_{t_{i(\mathbf{x})}}(\mathbf{x}) = 0 \leq k_{\mathbf{x}}$ . So  $j(\mathbf{x}) - k_{t_{i(\mathbf{x})}}(\mathbf{x}) \geq j(\mathbf{x}) - k_{\mathbf{x}}$ . But in that case, the sequence  $(\mathbb{E}[v(A + |i(x)|\mathbf{e}_2 + j\ell_2)])_j$  is decreasing, which implies (3).
- If  $t_{i(\mathbf{x})} = 0$ , then the sequence  $(\mathbb{E}[v(A + |i(\mathbf{x})|\mathbf{e}_2 + j\ell_2)])_j$  is increasing and

$$j(\mathbf{x}) - k_{t_{i(\mathbf{x})}}(\mathbf{x}) = 0 \leq j(\mathbf{x}) - k_{\mathbf{x}},$$

implying again (3).

- Otherwise, the sequence  $(\mathbb{E}[v(A + |i(\mathbf{x})|\mathbf{e}_2 + j\ell_2)])_j$  is decreasing on  $\llbracket 0, t_{i(\mathbf{x})} \rrbracket$  and increasing on  $\llbracket t_{i(\mathbf{x})}, +\infty \rrbracket$ . In that case,

- \* If  $j(\mathbf{x}) \leq t_{i(\mathbf{x})}$ , then  $k_{t_{i(\mathbf{x})}}(\mathbf{x}) = 0$  and thus  $t_{i(\mathbf{x})} \geq j(\mathbf{x}) - k_{t_{i(\mathbf{x})}}(\mathbf{x}) \geq j(\mathbf{x}) - k_{\mathbf{x}}$ . Again, the sequence  $(\mathbb{E}[v(A + |i(\mathbf{x})\mathbf{e}_2 + j\ell_2)])_j$  is decreasing on  $\llbracket 0, t_{i(\mathbf{x})} \rrbracket$ , implying (3).
- \* If  $j(\mathbf{x}) \geq t_{i(\mathbf{x})}$ , then  $j(\mathbf{x}) - k_{t_{i(\mathbf{x})}}(\mathbf{x}) = t_{i(\mathbf{x})}$  and (3) follows from the fact that the sequence  $(\mathbb{E}[v(A + |i(\mathbf{x})\mathbf{e}_2 + j\ell_2)])_j$  reaches its minimum at  $j = t_{i(\mathbf{x})}$ .

2b) If  $i(\mathbf{x}) > 0$ , then  $j(\mathbf{x}) = x_2 - x_3$  and we have

$$\begin{aligned} L_{m^*}^\gamma v(\mathbf{x}) &= c(\mathbf{x}) + \mathbb{E} [v(A + (x_1 - x_0 - j(\mathbf{x}))\mathbf{e}_1 + (j(\mathbf{x}) - k_{t_{i(\mathbf{x})}}(\mathbf{x}))\ell_2)] \\ &= c(\mathbf{x}) + \mathbb{E} [v(A + i(\mathbf{x})\mathbf{e}_1 + (j(\mathbf{x}) - k_{t_{i(\mathbf{x})}}(\mathbf{x}))\ell_2)]; \end{aligned}$$

$$L_{\mathbf{m}}^\gamma v(\mathbf{x}) = c(\mathbf{x}) + \mathbb{E} [v(A + i(\mathbf{x})\mathbf{e}_1 + (j(\mathbf{x}) - k_{\mathbf{x}})\ell_2)].$$

By the symmetry between  $\ell_1$  and  $\ell_2$ , we then conclude that (3) holds, just like in sub-case 2a). □

**4.3. Value function property conservation.** In this section, we aim at finding a subset of  $\mathcal{I}_1 \cap \mathcal{I}_3 \cap \mathcal{C}_2 \cap \mathcal{I}'$  that is stable under  $L^\gamma$  for well chosen values of  $\gamma$ . For this we introduce the following two functional spaces,

**Definition 9.** We denote by  $\mathcal{B}$ , the set of mappings  $v: \mathbb{N}^4 \rightarrow \mathbb{R}_+$  such that, for all  $\mathbf{b} \in \mathbb{N}^4$ ,

(i) If  $\mathbf{b} = (1, 0, \beta, 0) + \mathbf{e}_i$  for  $i \in \{0, 2, 3\}$  and  $\beta \geq 0$ ,

$$(4) \quad v(\mathbf{b} + 2\ell_2 - \ell_1) - v(\mathbf{b} + \ell_2 - \ell_1) \geq v(\mathbf{b} + \ell_2 - \ell_1) - v(\mathbf{b});$$

(ii) If  $\mathbf{b} = (0, \beta, 0, 1) + \mathbf{e}_i$  for  $i \in \{0, 1, 3\}$  and  $\beta \geq 0$ ,

$$(5) \quad v(\mathbf{b} + 2\ell_2 - \ell_3) - v(\mathbf{b} + \ell_2 - \ell_3) \geq v(\mathbf{b} + \ell_2 - \ell_3) - v(\mathbf{b});$$

(iii) If  $\mathbf{b} = (1, 0, 0, 1) + \mathbf{e}_i$  for  $i \in \llbracket 0, 3 \rrbracket$ ,

$$v(\mathbf{b} + 2\ell_2 - \ell_1 - \ell_3) - v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3) \geq v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3) - v(\mathbf{b}).$$

Observe the following,

**Lemma 2.** For any  $\beta \geq 0$ ,

(i) (4) holds for any  $v \in \mathcal{I}_1 \cap \mathcal{C}_2 \cap \mathcal{B}$  and  $\mathbf{b} = (1, 0, \beta, 0) + \mathbf{e}_i$ ,  $i \in \llbracket 0, 3 \rrbracket$ ;

(ii) (5) holds for any  $v \in \mathcal{I}_3 \cap \mathcal{C}_2 \cap \mathcal{B}$  and  $\mathbf{b} = (0, \beta, 0, 1) + \mathbf{e}_i$ ,  $i \in \llbracket 0, 3 \rrbracket$ .

*Proof.* (i) As  $v \in \mathcal{B}$ , by the very assertion (i) in definition 9 it is sufficient to check that (4) holds for  $\mathbf{b}$  of the form  $\mathbf{b} = (1, 1, \beta, 0)$ . But this is true since, as  $v \in \mathcal{I}_1 \cap \mathcal{C}_2$ , we then have

$$v(\mathbf{b} + \ell_2 - \ell_1) - v(\mathbf{b}) \leq v(\mathbf{b} + \ell_2 - \ell_1) - v(\mathbf{b} - \ell_1) \leq v(\mathbf{b} + 2\ell_2 - \ell_1) - v(\mathbf{b} + \ell_2 - \ell_1).$$

(ii) Likewise, it is enough to show that (5) holds for  $\mathbf{b}$  of the form  $\mathbf{b} = (0, \beta, 1, 1)$ .

But then, as  $v \in \mathcal{I}_3 \cap \mathcal{C}_2$  we get

$$v(\mathbf{b} + \ell_2 - \ell_3) - v(\mathbf{b}) \leq v(\mathbf{b} + \ell_2 - \ell_3) - v(\mathbf{b} - \ell_3) \leq v(\mathbf{b} + 2\ell_2 - \ell_3) - v(\mathbf{b} + \ell_2 - \ell_3). \quad \square$$

**Definition 10.** We denote by  $\mathcal{C}'$ , the set of mappings  $v: \mathbb{N}^4 \rightarrow \mathbb{R}_+$  such that, for all  $\mathbf{c} \in \mathbb{N}^4$ ,

(i) If  $\mathbf{c} = (\alpha, 0, \beta, 0) + \mathbf{e}_i$  for  $\alpha \geq 2$ ,  $\beta \geq 0$  and  $i \in \llbracket 0, 3 \rrbracket$ ,

$$v(\mathbf{c} + 2(\ell_2 - \ell_1)) - v(\mathbf{c} + \ell_2 - \ell_1) \geq v(\mathbf{c} + \ell_2 - \ell_1) - v(\mathbf{c});$$

- (ii) If  $\mathbf{c} = (\alpha, 0, 0, 1) + \mathbf{e}_i$  for  $\alpha \geq 2$  and  $i \in \llbracket 0, 3 \rrbracket$ ,  
 $v(\mathbf{c} + 2(\ell_2 - \ell_1) - \ell_3) - v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3) \geq v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3) - v(\mathbf{c});$
- (iii) If  $\mathbf{c} = (0, \beta, 0, \alpha) + \mathbf{e}_i$  for  $\alpha \geq 2$ ,  $\beta \geq 0$  and  $i \in \llbracket 0, 3 \rrbracket$ ,  
 $v(\mathbf{c} + 2(\ell_2 - \ell_3)) - v(\mathbf{c} + \ell_2 - \ell_3) \geq v(\mathbf{c} + \ell_2 - \ell_3) - v(\mathbf{c});$
- (iv) If  $\mathbf{c} = (1, 0, 0, \alpha) + \mathbf{e}_i$  for  $\alpha \geq 2$  and  $i \in \llbracket 0, 3 \rrbracket$ ,  
 $v(\mathbf{c} + 2(\ell_2 - \ell_3) - \ell_1) - v(\mathbf{c} + \ell_2 - \ell_3 - \ell_1) \geq v(\mathbf{c} + \ell_2 - \ell_3 - \ell_1) - v(\mathbf{c});$
- (v) If  $\mathbf{c} = (\alpha, 0, 0, \beta) + \mathbf{e}_i$  for  $\alpha, \beta \geq 2$  and  $i \in \llbracket 0, 3 \rrbracket$ ,  
 $v(\mathbf{c} + 2(\ell_2 - \ell_1 - \ell_3)) - v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3) \geq v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3) - v(\mathbf{c});$

We proceed with our key technical result, namely, the study of the stability under  $L^\gamma$ .

**Proposition 2.** *For any  $0 < \gamma < 1$ , the set*

$$\mathcal{V} := \mathcal{I}_1 \cap \mathcal{I}_3 \cap \mathcal{I}' \cap \mathcal{C}_2 \cap \mathcal{B} \cap \mathcal{C}',$$

*is stable by the operator  $L^\gamma$ .*

*Proof.* Again, it is immediate to observe that any admissible linear cost function is itself an element of  $\mathcal{V}$ . Let  $v \in \mathcal{V}$ . We know from Corollary 1 that  $L^\gamma v$  belongs to  $\mathcal{I}_1 \cap \mathcal{I}_3 \cap \mathcal{I}'$ . We also let  $m^*$  be a decision rule of threshold  $(t_i)_{i \in \mathbb{Z}}$  for  $v$ , satisfying the properties of Proposition 1. We then show successively that  $L^\gamma$  also is an element of  $\mathcal{C}_2$ ,  $\mathcal{B}$  and  $\mathcal{C}'$ .

Step I:  $L^\gamma v \in \mathcal{C}_2$ . Let  $\mathbf{x} \in \mathbb{N}^4$  be such that  $x_1 \geq x_0 - 1$  and  $x_2 \geq x_3 - 1$ . Denote  $\bar{\mathbf{x}} = \mathbf{x} + \ell_2$ ,  $\bar{\bar{\mathbf{x}}} = \mathbf{x} + 2\ell_2$  and the r.v.  $\mathbf{b} = \mathbf{x} - m^*(\mathbf{x}) + A$ . Then, observing that  $i(\mathbf{x}) = i(\bar{\mathbf{x}}) = i(\bar{\bar{\mathbf{x}}})$  by definition, denote

$$t^* = t_{i(\mathbf{x})} = t_{i(\bar{\mathbf{x}})} = t_{i(\bar{\bar{\mathbf{x}}})}.$$

First case:  $x_1 \geq x_0$  and  $x_2 \geq x_3$ . We are in the following alternative:

- (1a) If  $k_{t^*}(\bar{\mathbf{x}}) > 0$ , then  $k_{t^*}(\bar{\bar{\mathbf{x}}}) > 0$  and in that case we get

$$k_{t^*}(\bar{\bar{\mathbf{x}}}) = k_{t^*}(\bar{\mathbf{x}}) + 1 = k_{t^*}(\mathbf{x}) + 2$$

and so

$$\mathbf{x} - m^*(\mathbf{x}) = \bar{\mathbf{x}} - m^*(\bar{\mathbf{x}}) = \bar{\bar{\mathbf{x}}} - m^*(\bar{\bar{\mathbf{x}}}).$$

Therefore, as  $c \in \mathcal{C}_2$  we get

$$\begin{aligned} L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\mathbf{x}) &= L_{m^*(\bar{\bar{\mathbf{x}}})}^\gamma v(\bar{\bar{\mathbf{x}}}) - L_{m^*(\mathbf{x})}^\gamma v(\mathbf{x}) = c(\bar{\bar{\mathbf{x}}}) - c(\mathbf{x}) + \gamma \mathbb{E}[v(\mathbf{b}) - v(\mathbf{b})] \\ &\leq c(\bar{\bar{\mathbf{x}}}) - c(\bar{\mathbf{x}}) + \gamma \mathbb{E}[v(\mathbf{b}) - v(\mathbf{b})] \\ &= L_{m^*(\bar{\bar{\mathbf{x}}})}^\gamma v(\bar{\bar{\mathbf{x}}}) - L_{m^*(\bar{\mathbf{x}})}^\gamma v(\bar{\mathbf{x}}) = L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\bar{\mathbf{x}}). \end{aligned}$$

- (1b) If  $k_{t^*}(\bar{\mathbf{x}}) = 0$  and  $k_{t^*}(\bar{\bar{\mathbf{x}}}) > 0$ , then

$$k_{t^*}(\bar{\bar{\mathbf{x}}}) = k_{t^*}(\bar{\mathbf{x}}) + 1 = k_{t^*}(\mathbf{x}) + 1.$$

In that case, we have

$$\bar{\mathbf{x}} - m^*(\bar{\mathbf{x}}) = \mathbf{x} - m^*(\mathbf{x}) + \ell_2 \quad \text{and} \quad \bar{\bar{\mathbf{x}}} - m^*(\bar{\bar{\mathbf{x}}}) = \bar{\mathbf{x}} - m^*(\bar{\mathbf{x}}).$$

Consequently, as  $m^*(\bar{\mathbf{x}}) + \ell_2 \in \mathbf{M}_{\bar{\mathbf{x}}}$  we get that

$$\begin{aligned} L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\mathbf{x}) &= c(\bar{\bar{\mathbf{x}}}) - c(\mathbf{x}) + L_{m^*(\bar{\mathbf{x}})}^\gamma v(\bar{\bar{\mathbf{x}}}) - L_{m^*(\bar{\mathbf{x}}) + \ell_2}^\gamma v(\bar{\bar{\mathbf{x}}}) \\ &\leq c(\bar{\bar{\mathbf{x}}}) - c(\mathbf{x}) \\ &\leq c(\bar{\bar{\mathbf{x}}}) - c(\bar{\mathbf{x}}) = L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\bar{\mathbf{x}}). \end{aligned}$$

(1c) If  $k_{t^*}(\bar{\mathbf{x}}) = k_{t_2}(\bar{\mathbf{x}}) = 0$ , reasoning as for the previous case we get that

$$\bar{\bar{\mathbf{x}}} - m^*(\bar{\bar{\mathbf{x}}}) = \bar{\mathbf{x}} - m^*(\bar{\mathbf{x}}) + \ell_2 = \mathbf{x} - m^*(\mathbf{x}) + 2\ell_2,$$

implying, as  $v$  belongs to  $\mathcal{C}_2$ , that

$$\begin{aligned} L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\mathbf{x}) &= c(\bar{\bar{\mathbf{x}}}) - c(\mathbf{x}) + \gamma \mathbb{E}[v(\mathbf{b} + \ell_2) - v(\mathbf{b})] \\ &\leq c(\bar{\bar{\mathbf{x}}}) - c(\bar{\mathbf{x}}) + \gamma \mathbb{E}[v(\mathbf{b} + 2\ell_2) - v(\mathbf{b} + \ell_2)] = L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\bar{\mathbf{x}}). \end{aligned}$$

Second case:  $x_1 = x_0 - 1$  and  $x_2 \geq x_3$ . In that case we have  $\mathbf{b} = (1, 0, \beta, 0) + \mathbf{e}_i$  with  $i \in \llbracket 0, 3 \rrbracket$ , and  $m^*(\bar{\mathbf{x}}) = m^*(\mathbf{x}) + \ell_1$ , so that

$$(6) \quad L^\gamma v(\bar{\mathbf{x}}) - L^\gamma v(\mathbf{x}) = c(\bar{\mathbf{x}}) - c(\mathbf{x}) + \gamma \mathbb{E}[v(\mathbf{b} + \ell_2 - \ell_1) - v(\mathbf{b})].$$

Then,

(2a) If  $k_{t^*}(\bar{\mathbf{x}}) > 0$ , we also have  $m^*(\bar{\bar{\mathbf{x}}}) = m^*(\bar{\mathbf{x}}) + \ell_2$ , hence  $\bar{\bar{\mathbf{x}}} - m^*(\bar{\bar{\mathbf{x}}}) = \bar{\mathbf{x}} - m^*(\bar{\mathbf{x}})$ . Thus, as  $v \in \mathcal{I}'$  it follows from (6) that

$$L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\mathbf{x}) \leq c(\bar{\bar{\mathbf{x}}}) - c(\mathbf{x}) \leq c(\bar{\bar{\mathbf{x}}}) - c(\bar{\mathbf{x}}) = L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\bar{\mathbf{x}}).$$

(2b) If  $k_{t^*}(\bar{\mathbf{x}}) = 0$ , then  $m^*(\bar{\bar{\mathbf{x}}}) = m^*(\bar{\mathbf{x}})$  and, as  $c \in \mathcal{C}_2$  and  $v \in \mathcal{B}$ , (6) entails

$$L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\mathbf{x}) \leq c(\bar{\bar{\mathbf{x}}}) - c(\bar{\mathbf{x}}) + \gamma \mathbb{E}[v(\mathbf{b} + 2\ell_2 - \ell_1) - v(\mathbf{b} + \ell_2 - \ell_1)] = L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\bar{\mathbf{x}}).$$

Third case:  $x_1 \geq x_0$  and  $x_2 = x_3 - 1$ . This case is symmetrical to the second case, replacing  $\ell_1$  by  $\ell_3$ , and using now assertion (ii) of definition 9).

Fourth case:  $x_1 = x_0 - 1$  and  $x_2 = x_3 - 1$ . Then  $\mathbf{b}$  is of the form  $(1, 0, 0, 1) + \mathbf{e}_i$  for some  $i \in \llbracket 0, 3 \rrbracket$ . Then,  $m^*(\bar{\mathbf{x}}) = m^*(\mathbf{x}) + \ell_1 + \ell_3$ , and as  $c \in \mathcal{C}_2$  we have that

$$(7) \quad \begin{aligned} L^\gamma v(\bar{\mathbf{x}}) - L^\gamma v(\mathbf{x}) &= c(\bar{\mathbf{x}}) - c(\mathbf{x}) + \gamma \mathbb{E}[v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3) - v(\mathbf{b})] \\ &\leq c(\bar{\bar{\mathbf{x}}}) - c(\bar{\mathbf{x}}) + \gamma \mathbb{E}[v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3) - v(\mathbf{b})]. \end{aligned}$$

We have the following alternative:

(4a) If  $k_{t^*}(\bar{\mathbf{x}}) > 0$ , then we have  $m^*(\bar{\bar{\mathbf{x}}}) = m^*(\bar{\mathbf{x}}) + \ell_2$ , hence  $\bar{\bar{\mathbf{x}}} - m^*(\bar{\bar{\mathbf{x}}}) = \bar{\mathbf{x}} - m^*(\bar{\mathbf{x}})$ . Thus it follows from (7) and the fact that  $v \in \mathcal{I}' \cap \mathcal{I}_1$ , that

$$L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\mathbf{x}) \leq c(\bar{\bar{\mathbf{x}}}) - c(\bar{\mathbf{x}}) + \gamma \mathbb{E}[v(\mathbf{b}) - v(\mathbf{b})] = L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\bar{\mathbf{x}}).$$

(4b) If  $k_{t^*}(\bar{\mathbf{x}}) = 0$ , then  $m^*(\bar{\bar{\mathbf{x}}}) = m^*(\bar{\mathbf{x}})$  and hence, as  $v \in \mathcal{B}$ , (7) implies that

$$\begin{aligned} L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\mathbf{x}) &\leq c(\bar{\bar{\mathbf{x}}}) - c(\bar{\mathbf{x}}) + \gamma \mathbb{E}[v(\mathbf{b} + 2\ell_2 - \ell_1 - \ell_3) - v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3)] \\ &= L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\bar{\mathbf{x}}), \end{aligned}$$

which concludes the proof in the fourth case.

To summarize, in all cases we get that

$$L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\mathbf{x}) \leq L^\gamma v(\bar{\bar{\mathbf{x}}}) - L^\gamma v(\bar{\mathbf{x}}).$$

As this is true for all  $\mathbf{x} \in \mathbb{N}^4$  such that  $x_1 \geq x_0 - 1$  and  $x_2 \geq x_3 - 1$ , we conclude that  $L^\gamma v \in \mathcal{C}_2$ .

Step II:  $L^\gamma v \in \mathcal{B}$ .

Assertion (i). We first check assertion (i) of definition 9. So we suppose that  $\mathbf{b} = (1, 0, \beta, 0) + \mathbf{e}_i$  for  $\beta \geq 0$  and  $i \in \{0, 2, 3\}$ . Throughout this part of the proof we denote  $\bar{\mathbf{b}} = \mathbf{b} + \ell_2 - \ell_1$  and  $\bar{\bar{\mathbf{b}}} = \mathbf{b} + 2\ell_2 - \ell_1$ , and let  $t^* = t_{i(\bar{\mathbf{b}})} = t_{i(\bar{\bar{\mathbf{b}}})}$ . We show

that  $L^\gamma(\bar{\mathbf{b}}) - L^\gamma(\mathbf{b}) \leq L^\gamma(\bar{\bar{\mathbf{b}}}) - L^\gamma(\bar{\mathbf{b}})$  for each  $i \in \{0, 2, 3\}$ :

- (ia) If  $i = 0$ , then  $\mathbf{b} = (2, 0, \beta, 0)$ ,  $\bar{\mathbf{b}} = (1, 0, \beta + 1, 0)$  and  $\bar{\bar{\mathbf{b}}} = (1, 1, \beta + 2, 0)$ , implying that  $m^*(\mathbf{b}) = m^*(\bar{\mathbf{b}}) = \mathbf{0}$  and  $m^*(\bar{\bar{\mathbf{b}}}) = \ell_1$ . So as  $c \in \mathcal{B}$ , it follows from (i) of definition 10 that

$$\begin{aligned} L^\gamma(\bar{\mathbf{b}}) - L^\gamma(\mathbf{b}) &= c(\bar{\mathbf{b}}) - c(\mathbf{b}) + \gamma\mathbb{E}[v(\mathbf{b} + A + \ell_2 - \ell_1) - v(\mathbf{b} + A)] \\ &\leq c(\bar{\bar{\mathbf{b}}}) - c(\bar{\mathbf{b}}) + \gamma\mathbb{E}[v(\mathbf{b} + A + 2(\ell_2 - \ell_1)) - v(\mathbf{b} + \ell_2 - \ell_1 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{b}}}) - L^\gamma(\bar{\mathbf{b}}). \end{aligned}$$

- (ib) If  $i = 2$ , then  $\mathbf{b} = (1, 0, \beta + 1, 0)$ ,  $\bar{\mathbf{b}} = (0, 0, \beta + 2, 0)$  and  $\bar{\bar{\mathbf{b}}} = (0, 1, \beta + 3, 0)$ , in a way that  $m^*(\mathbf{b}) = m^*(\bar{\mathbf{b}}) = \mathbf{0}$  and  $m^*(\bar{\bar{\mathbf{b}}}) = k_{t^*}(\bar{\bar{\mathbf{b}}})\ell_2$ . Then,  
 - If  $k_{t^*}(\bar{\bar{\mathbf{b}}}) = 0$ , then in view of (i) of Lemma 2 we have

$$\begin{aligned} L^\gamma(\bar{\mathbf{b}}) - L^\gamma(\mathbf{b}) &= c(\bar{\mathbf{b}}) - c(\mathbf{b}) + \gamma\mathbb{E}[v(\mathbf{b} + A + \ell_2 - \ell_1) - v(\mathbf{b} + A)] \\ &\leq c(\bar{\bar{\mathbf{b}}}) - c(\bar{\mathbf{b}}) + \gamma\mathbb{E}[v(\mathbf{b} + A + 2\ell_2 - \ell_1) - v(\mathbf{b} + \ell_2 - \ell_1 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{b}}}) - L^\gamma(\bar{\mathbf{b}}). \end{aligned}$$

- If  $k_{t^*}(\bar{\bar{\mathbf{b}}}) = 1$ , then, as  $v \in \mathcal{I}'$ ,

$$\begin{aligned} L^\gamma(\bar{\mathbf{b}}) - L^\gamma(\mathbf{b}) &= c(\bar{\mathbf{b}}) - c(\mathbf{b}) + \gamma\mathbb{E}[v(\mathbf{b} + A + \ell_2 - \ell_1) - v(\mathbf{b} + A)] \\ &\leq c(\bar{\mathbf{b}}) - c(\mathbf{b}) \\ &\leq c(\bar{\bar{\mathbf{b}}}) - c(\bar{\mathbf{b}}) = L^\gamma(\bar{\bar{\mathbf{b}}}) - L^\gamma(\bar{\mathbf{b}}). \end{aligned}$$

- (ic) If  $i = 3$ , then  $\mathbf{b} = (1, 0, \beta, 1)$ ,  $\bar{\mathbf{b}} = (0, 0, \beta + 1, 1)$  and  $\bar{\bar{\mathbf{b}}} = (0, 1, \beta + 2, 1)$ . Then we distinguish three cases :

- If  $\beta \geq 1$  and  $k_{t^*}(\bar{\bar{\mathbf{b}}}) = 0$ , then applying (i) of Lemma 2 to  $\mathbf{b} - \ell_3 + A$  we obtain

$$\begin{aligned} L^\gamma(\bar{\mathbf{b}}) - L^\gamma(\mathbf{b}) &= c(\bar{\mathbf{b}}) - c(\mathbf{b}) + \gamma\mathbb{E}[v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{b} - \ell_3 + A)] \\ &\leq c(\bar{\bar{\mathbf{b}}}) - c(\bar{\mathbf{b}}) + \gamma\mathbb{E}[v(\mathbf{b} + 2\ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{b}}}) - L^\gamma(\bar{\mathbf{b}}). \end{aligned}$$

- If  $\beta \geq 1$  and  $k_{t^*}(\bar{\bar{\mathbf{b}}}) = 1$ , then as  $v \in \mathcal{I}'$  we get

$$\begin{aligned} L^\gamma(\bar{\mathbf{b}}) - L^\gamma(\mathbf{b}) &= c(\bar{\mathbf{b}}) - c(\mathbf{b}) + \gamma\mathbb{E}[v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{b} - \ell_3 + A)] \\ &\leq c(\bar{\mathbf{b}}) - c(\mathbf{b}) \\ &\leq c(\bar{\bar{\mathbf{b}}}) - c(\bar{\mathbf{b}}) \\ &= c(\bar{\mathbf{b}}) - c(\bar{\mathbf{b}}) + \gamma\mathbb{E}[v(\bar{\bar{\mathbf{b}}} - \ell_2 - \ell_3 + A) - v(\bar{\mathbf{b}} - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{b}}}) - L^\gamma(\bar{\mathbf{b}}). \end{aligned}$$

- If  $\beta = 0$  and  $k_{t^*}(\bar{\mathbf{b}}) = 0$ , then applying (iii) of definition 9 to  $\mathbf{b} + A$  we obtain

$$\begin{aligned} L^\gamma(\bar{\mathbf{b}}) - L^\gamma(\mathbf{b}) &= c(\bar{\mathbf{b}}) - c(\mathbf{b}) + \gamma\mathbb{E}[v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{b} + A)] \\ &\leq c(\bar{\bar{\mathbf{b}}}) - c(\bar{\mathbf{b}}) + \gamma\mathbb{E}[v(\mathbf{b} + 2\ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{b}}}) - L^\gamma(\bar{\mathbf{b}}). \end{aligned}$$

- If  $\beta = 0$  and  $k_{t^*}(\bar{\mathbf{b}}) = 1$  then, using successively the fact that  $v$  is an element of  $\mathcal{I}_3$  and of  $\mathcal{I}'$  we get

$$\begin{aligned} L^\gamma(\bar{\mathbf{b}}) - L^\gamma(\mathbf{b}) &= c(\bar{\mathbf{b}}) - c(\mathbf{b}) + \gamma\mathbb{E}[v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{b} + A)] \\ &\leq c(\bar{\mathbf{b}}) - c(\mathbf{b}) + \gamma\mathbb{E}[v(\mathbf{b} + \ell_2 - \ell_1 + A) - v(\mathbf{b} + A)] \\ &\leq c(\bar{\mathbf{b}}) - c(\mathbf{b}) \\ &\leq c(\bar{\bar{\mathbf{b}}}) - c(\bar{\mathbf{b}}) = L^\gamma(\bar{\bar{\mathbf{b}}}) - L^\gamma(\bar{\mathbf{b}}). \end{aligned}$$

Assertion (ii). By symmetry between  $\ell_1$  and  $\ell_3$ , the proof is similar to that of assertion (i), by using Assertion (v) of definition 10.

Assertion (iii). Set  $\mathbf{b} = (1, 0, 0, 1) + \mathbf{e}_i$ , for  $i \in \llbracket 0, 3 \rrbracket$ , and denote  $\bar{\mathbf{b}} = \mathbf{b} + \ell_2 - \ell_1 - \ell_3$  and  $\bar{\bar{\mathbf{b}}} = \mathbf{b} + 2\ell_2 - \ell_1 - \ell_3$ . We show that  $L^\gamma(\bar{\mathbf{b}}) - L^\gamma(\mathbf{b}) \leq L^\gamma(\bar{\bar{\mathbf{b}}}) - L^\gamma(\bar{\mathbf{b}})$  for each  $i \in \llbracket 0, 3 \rrbracket$ .

- (iiia) If  $i = 0$ , then  $\mathbf{b} = (2, 0, 0, 1)$ ,  $\bar{\mathbf{b}} = (1, 0, 0, 0)$  and  $\bar{\bar{\mathbf{b}}} = (1, 1, 1, 0)$ . Then, from Assertion (ii) of definition 10 we obtain that

$$\begin{aligned} L^\gamma(\bar{\mathbf{b}}) - L^\gamma(\mathbf{b}) &= c(\bar{\mathbf{b}}) - c(\mathbf{b}) + \gamma\mathbb{E}[v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{b} + A)] \\ &\leq c(\bar{\bar{\mathbf{b}}}) - c(\bar{\mathbf{b}}) + \gamma\mathbb{E}[v(\mathbf{b} + 2(\ell_2 - \ell_1) - \ell_3 + A) - v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{b}}}) - L^\gamma(\bar{\mathbf{b}}). \end{aligned}$$

- (iiib) If  $i = 1$ , then  $\mathbf{b} = (1, 1, 0, 1)$ ,  $\bar{\mathbf{b}} = (0, 1, 0, 0)$  and  $\bar{\bar{\mathbf{b}}} = (0, 2, 1, 0)$ . We distinguish two cases:

- If  $k_{t_i}(\bar{\mathbf{b}}) = 0$ , then, applying assertion (ii) of Lemma 2 for  $\beta = 0$  to  $\mathbf{b} - \ell_1 + A$ , we get that

$$\begin{aligned} L^\gamma(\bar{\mathbf{b}}) - L^\gamma(\mathbf{b}) &= c(\bar{\mathbf{b}}) - c(\mathbf{b}) + \gamma\mathbb{E}[v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{b} - \ell_1 + A)] \\ &\leq c(\bar{\bar{\mathbf{b}}}) - c(\bar{\mathbf{b}}) + \gamma\mathbb{E}[v(\mathbf{b} + 2\ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{b}}}) - L^\gamma(\bar{\mathbf{b}}). \end{aligned}$$

- If  $k_{t_i}(\bar{\mathbf{b}}) = 1$ , then, as  $v \in \mathcal{I}'$  we obtain that

$$\begin{aligned} L^\gamma(\bar{\mathbf{b}}) - L^\gamma(\mathbf{b}) &= c(\bar{\mathbf{b}}) - c(\mathbf{b}) + \gamma\mathbb{E}[v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{b} - \ell_1 + A)] \\ &\leq c(\bar{\mathbf{b}}) - c(\mathbf{b}) \\ &\leq c(\bar{\bar{\mathbf{b}}}) - c(\bar{\mathbf{b}}) \\ &= c(\bar{\bar{\mathbf{b}}}) - c(\bar{\mathbf{b}}) + \gamma\mathbb{E}[v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{b} + \ell_2 - \ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{b}}}) - L^\gamma(\bar{\mathbf{b}}). \end{aligned}$$

- (iiic) If  $i = 2$ , then  $\mathbf{b} = (1, 0, 1, 1)$ ,  $\bar{\mathbf{b}} = (0, 0, 1, 0)$  and  $\bar{\bar{\mathbf{b}}} = (0, 1, 2, 0)$ . This case is symmetric to (iiib), using now Assertion (i) of Lemma 2 for  $\beta = 0$  to  $\mathbf{b} - \ell_3 + A$  if  $k_{t_i(\bar{\mathbf{b}})}(\bar{\bar{\mathbf{b}}}) = 0$ .
- (iiid) If  $i = 3$ , then  $\mathbf{b} = (1, 0, 0, 2)$ ,  $\bar{\mathbf{b}} = (0, 0, 0, 1)$  and  $\bar{\bar{\mathbf{b}}} = (0, 1, 1, 1)$  and the argument is symmetric to that of (iiia), applying now Assertion (iii) of definition 10.

Step III:  $L^\gamma v \in \mathcal{C}'$ .

Assertion (i). We first show that  $L^\gamma$  satisfies Assertion (i) of definition 10. For this, let  $\mathbf{c} = (\alpha, 0, \beta, 0) + \mathbf{e}_i$  for  $\alpha \geq 2$ ,  $\beta \geq 0$  and  $i \in \llbracket 0, 3 \rrbracket$ . Let us also set  $\bar{\mathbf{c}} = \mathbf{c} + \ell_2 - \ell_1$  and  $\bar{\bar{\mathbf{c}}} = \mathbf{c} + 2(\ell_2 - \ell_1)$ .

- (ia) If  $i = 0$ , then we get  $\mathbf{c} = (\alpha + 1, 0, \beta, 0)$ ,  $\bar{\mathbf{c}} = (\alpha, 0, \beta + 1, 0)$  and  $\bar{\bar{\mathbf{c}}} = (\alpha - 1, 0, \beta + 2, 0)$ . So as  $c \in \mathcal{C}'$ , applying Assertion (i) of definition 10 to  $v$  we get that

$$\begin{aligned} L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\mathbf{c}) &= c(\bar{\bar{\mathbf{c}}}) - c(\mathbf{c}) + \gamma \mathbb{E}[v(\mathbf{c} + \ell_2 - \ell_1 + A) - v(\mathbf{c} + A)] \\ &\leq c(\bar{\bar{\mathbf{c}}}) - c(\bar{\mathbf{c}}) + \gamma \mathbb{E}[v(\mathbf{c} + 2(\ell_2 - \ell_1) + A) - v(\mathbf{c} + \ell_2 - \ell_1 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

- (ib) If  $i = 1$ , then  $\mathbf{c} = (\alpha, 1, \beta, 0)$ ,  $\bar{\mathbf{c}} = (\alpha - 1, 1, \beta + 1, 0)$  and  $\bar{\bar{\mathbf{c}}} = (\alpha - 2, 1, \beta + 2, 0)$  and thus

$$(8) \quad L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\mathbf{c}) = c(\bar{\bar{\mathbf{c}}}) - c(\mathbf{c}) + \gamma \mathbb{E}[v(\mathbf{c} + \ell_2 - 2\ell_1 + A) - v(\mathbf{c} - \ell_1 + A)].$$

We are in the following alternative:

- If  $\alpha = 2$  and  $k_{t_i(\bar{\mathbf{c}})}(\bar{\bar{\mathbf{c}}}) = 0$ , then applying Assertion (i) of Lemma 2 to  $\mathbf{b} := \mathbf{c} - \ell_1 + A$  we obtain from (8) that

$$\begin{aligned} L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\mathbf{c}) &\leq c(\bar{\bar{\mathbf{c}}}) - c(\bar{\mathbf{c}}) + \gamma \mathbb{E}[v(\mathbf{c} + 2\ell_2 - 2\ell_1 + A) - v(\mathbf{c} + \ell_2 - 2\ell_1 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

- If  $\alpha = 2$  and  $k_{t_i(\bar{\mathbf{c}})}(\bar{\bar{\mathbf{c}}}) = 1$ , then as  $v \in \mathcal{T}'$ , (8) implies

$$\begin{aligned} L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\mathbf{c}) &\leq c(\bar{\bar{\mathbf{c}}}) - c(\mathbf{c}) + \gamma \mathbb{E}[v(\mathbf{c} - \ell_1 + A) - v(\mathbf{c} - \ell_1 + A)] \\ &\leq c(\bar{\bar{\mathbf{c}}}) - c(\bar{\mathbf{c}}) \\ &= c(\bar{\bar{\mathbf{c}}}) - c(\bar{\mathbf{c}}) + \gamma \mathbb{E}[v(\mathbf{c} + \ell_2 - 2\ell_1 + A) - v(\mathbf{c} + \ell_2 - 2\ell_1 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

- If  $\alpha > 2$  then, applying (i) of definition 10 to  $\mathbf{c} - \ell_1 + A$ , (8) entails that

$$\begin{aligned} L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\mathbf{c}) &\leq c(\bar{\bar{\mathbf{c}}}) - c(\bar{\mathbf{c}}) + \gamma \mathbb{E}[v(\mathbf{c} + 2\ell_2 - 3\ell_1 + A) - v(\mathbf{c} + \ell_2 - 2\ell_1 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

- (ic) If  $i = 2$ , then we get  $\mathbf{c} = (\alpha, 0, \beta + 1, 0)$ ,  $\bar{\mathbf{c}} = (\alpha - 1, 0, \beta + 2, 0)$  and  $\bar{\bar{\mathbf{c}}} = (\alpha - 2, 0, \beta + 3, 0)$ . So, as in case (ia), applying Assertion (i) of definition 10 to  $v$  we get that

$$\begin{aligned} L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\mathbf{c}) &= c(\bar{\bar{\mathbf{c}}}) - c(\mathbf{c}) + \gamma \mathbb{E}[v(\mathbf{c} + \ell_2 - \ell_1 + A) - v(\mathbf{c} + A)] \\ &\leq c(\bar{\bar{\mathbf{c}}}) - c(\bar{\mathbf{c}}) + \gamma \mathbb{E}[v(\mathbf{c} + 2(\ell_2 - \ell_1) + A) - v(\mathbf{c} + \ell_2 - \ell_1 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

- (id) If  $i = 3$ , then  $\mathbf{c} = (\alpha, 0, \beta, 1)$ ,  $\bar{\mathbf{c}} = (\alpha - 1, 0, \beta + 1, 1)$  and  $\bar{\bar{\mathbf{c}}} = (\alpha - 2, 0, \beta + 2, 1)$ . We have the following cases:



– If  $\beta > 0$ , then applying Assertion (i) of definition 10 to  $\mathbf{c} - \ell_3$  we get that

$$\begin{aligned} L^\gamma(\bar{\mathbf{c}}) - L^\gamma(\mathbf{c}) &= c(\bar{\mathbf{c}}) - c(\mathbf{c}) + \gamma\mathbb{E}[v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{c} - \ell_3 + A)] \\ &\leq c(\bar{\mathbf{c}}) - c(\bar{\mathbf{c}}) + \gamma\mathbb{E}[v(\mathbf{c} + 2\ell_2 - 2\ell_1 - \ell_3 + A) - v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

– If  $\beta = 0$ , then Assertion (ii) of definition 10 applied to  $\mathbf{c}$ , shows that

$$\begin{aligned} L^\gamma(\bar{\mathbf{c}}) - L^\gamma(\mathbf{c}) &= c(\bar{\mathbf{c}}) - c(\mathbf{c}) + \gamma\mathbb{E}[v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{c} + A)] \\ &\leq c(\bar{\mathbf{c}}) - c(\bar{\mathbf{c}}) + \gamma\mathbb{E}[v(\mathbf{c} + 2\ell_2 - 2\ell_1 - \ell_3 + A) - v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

This shows that  $L^\gamma v$  satisfies Assertion (i) of definition 10.

Assertion (ii). We now show that  $L^\gamma$  satisfies Assertion (ii) of definition 10. Let  $\mathbf{c} = (\alpha, 0, 0, 1) + \mathbf{e}_i$  for  $\alpha \geq 2$  and  $i \in \llbracket 0, 3 \rrbracket$ . Let us also now set  $\bar{\mathbf{c}} = \mathbf{c} + \ell_2 - \ell_1 - \ell_3$  and  $\bar{\bar{\mathbf{c}}} = \mathbf{c} + 2(\ell_2 - \ell_1) - \ell_3$ .

(ia) If  $i = 0$ , then  $\mathbf{c} = (\alpha + 1, 0, 0, 1)$ ,  $\bar{\mathbf{c}} = (\alpha, 0, 0, 0)$  and  $\bar{\bar{\mathbf{c}}} = (\alpha - 1, 0, 1, 0)$ . So, as  $v$  satisfies (ii) of definition 10,

$$\begin{aligned} L^\gamma(\bar{\mathbf{c}}) - L^\gamma(\mathbf{c}) &= c(\bar{\mathbf{c}}) - c(\mathbf{c}) + \gamma\mathbb{E}[v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{c} + A)] \\ &\leq c(\bar{\mathbf{c}}) - c(\bar{\mathbf{c}}) + \gamma\mathbb{E}[v(\mathbf{c} + 2(\ell_2 - \ell_1) - \ell_3 + A) - v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

(iib) If  $i = 1$  we obtain  $\mathbf{c} = (\alpha, 1, 0, 1)$ ,  $\bar{\mathbf{c}} = (\alpha - 1, 1, 0, 0)$  and  $\bar{\bar{\mathbf{c}}} = (\alpha - 2, 1, 1, 0)$ . Then,

$$(9) \quad L^\gamma(\bar{\mathbf{c}}) - L^\gamma(\mathbf{c}) = c(\bar{\mathbf{c}}) - c(\mathbf{c}) + \gamma\mathbb{E}[v(\mathbf{c} + \ell_2 - 2\ell_1 - \ell_3 + A) - v(\mathbf{c} - \ell_1 + A)].$$

Then we are in the following alternative:

– If  $\alpha = 2$  and  $k_{t_i(\bar{\mathbf{c}})}(\bar{\mathbf{c}}) = 0$  then, applying (iii) of definition 9 to  $\mathbf{b} = \mathbf{c} - \ell_1 + A$ , it follows from (9) that

$$\begin{aligned} L^\gamma(\bar{\mathbf{c}}) - L^\gamma(\mathbf{c}) &\leq c(\bar{\mathbf{c}}) - c(\bar{\mathbf{c}}) + \gamma\mathbb{E}[v(\mathbf{c} + 2\ell_2 - 2\ell_1 - \ell_3 + A) - v(\mathbf{c} + \ell_2 - 2\ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

– If  $\alpha = 2$  and  $k_{t_i(\bar{\mathbf{c}})}(\bar{\mathbf{c}}) = 1$ , then as  $v \in \mathcal{I}_3 \cap \mathcal{I}'$ , (9) implies

$$\begin{aligned} L^\gamma(\bar{\mathbf{c}}) - L^\gamma(\mathbf{c}) &\leq c(\bar{\mathbf{c}}) - c(\mathbf{c}) + \gamma\mathbb{E}[v(\mathbf{c} - \ell_1 + \ell_2 - \ell_1 + A) - v(\mathbf{c} - \ell_1 + A)] \\ &\leq c(\bar{\mathbf{c}}) - c(\mathbf{c}) \\ &= c(\bar{\mathbf{c}}) - c(\bar{\mathbf{c}}) + \gamma\mathbb{E}[v(\mathbf{c} + \ell_2 - 2\ell_1 - \ell_3 + A) - v(\mathbf{c} + \ell_2 - 2\ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

– If  $\alpha > 2$ , we apply (ii) of definition 10 to  $\mathbf{c} - \ell_1 + A$ . Then (9) implies that

$$\begin{aligned} L^\gamma(\bar{\mathbf{c}}) - L^\gamma(\mathbf{c}) &= c(\bar{\mathbf{c}}) - c(\mathbf{c}) + \gamma\mathbb{E}[v(\mathbf{c} + \ell_2 - 2\ell_1 - \ell_3 + A) - v(\mathbf{c} - \ell_1 + A)] \\ &\leq c(\bar{\mathbf{c}}) - c(\bar{\mathbf{c}}) + \gamma\mathbb{E}[v(\mathbf{c} + 2\ell_2 - 3\ell_1 - \ell_3 + A) - v(\mathbf{c} + \ell_2 - 2\ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

(iic) If  $i = 2$ , then we get  $\mathbf{c} = (\alpha, 0, 1, 1)$ ,  $\bar{\mathbf{c}} = (\alpha - 1, 0, 1, 0)$  and  $\bar{\bar{\mathbf{c}}} = (\alpha - 2, 0, 2, 0)$ .

So, applying Assertion (i) of definition 10 for  $\mathbf{c} - \ell_3 + A$  yields to

$$\begin{aligned} L^\gamma(\bar{\mathbf{c}}) - L^\gamma(\mathbf{c}) &= c(\bar{\mathbf{c}}) - c(\mathbf{c}) + \gamma\mathbb{E}[v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{c} - \ell_3 + A)] \\ &\leq c(\bar{\mathbf{c}}) - c(\bar{\mathbf{c}}) + \gamma\mathbb{E}[v(\mathbf{c} + 2(\ell_2 - \ell_1) - \ell_3 + A) - v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

(iid) If  $i = 3$ , then  $\mathbf{c} = (\alpha, 0, 0, 2)$ ,  $\bar{\mathbf{c}} = (\alpha - 1, 0, 0, 1)$  and  $\bar{\bar{\mathbf{c}}} = (\alpha - 2, 0, 1, 1)$ . So applying (v) of definition 10 we get that

$$\begin{aligned} L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\mathbf{c}) &= c(\bar{\bar{\mathbf{c}}}) - c(\mathbf{c}) + \gamma\mathbb{E}[v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{c} + A)] \\ &\leq c(\bar{\bar{\mathbf{c}}}) - c(\bar{\mathbf{c}}) + \gamma\mathbb{E}[v(\mathbf{c} + 2\ell_2 - 2\ell_1 - 2\ell_3 + A) - v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

Assertions (iii) and (iv). By the symmetry between  $\ell_1$  and  $\ell_3$ , the proofs are analog to that of assertions (i) and (ii), respectively.

Assertion (v). We conclude by proving that  $L^\gamma v$  satisfies assertion (v) of definition 10. We let  $\mathbf{c} = (\alpha, 0, 0, \beta) + \mathbf{e}_i$  for  $i \in \llbracket 0, 3 \rrbracket$ ,  $\alpha \geq 2$  and  $\beta \geq 2$ , and set  $\bar{\mathbf{c}} = \mathbf{c} + \ell_2 - \ell_1 - \ell_3$  and  $\bar{\bar{\mathbf{c}}} = \mathbf{c} + 2(\ell_2 - \ell_1 - \ell_3)$ .

(va) If  $i = 0$ , then  $\mathbf{c} = (\alpha + 1, 0, 0, \beta)$ ,  $\bar{\mathbf{c}} = (\alpha, 0, 0, \beta - 1)$  and  $\bar{\bar{\mathbf{c}}} = (\alpha - 1, 0, 0, \beta - 2)$ . Then, as  $v$  satisfies assertion (v) of definition 10 we obtain that

$$\begin{aligned} L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\mathbf{c}) &= c(\bar{\bar{\mathbf{c}}}) - c(\mathbf{c}) + \gamma\mathbb{E}[v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3 + A) - v(\mathbf{c} + A)] \\ &\leq c(\bar{\bar{\mathbf{c}}}) - c(\bar{\mathbf{c}}) + \gamma\mathbb{E}[v(\mathbf{c} + 2\ell_2 - 2\ell_1 - 2\ell_3 + A) - v(\mathbf{c} + \ell_2 - \ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

(vb) If  $i = 1$ , then  $\mathbf{c} = (\alpha, 1, 0, \beta)$ ,  $\bar{\mathbf{c}} = (\alpha - 1, 1, 0, \beta - 1)$  and  $\bar{\bar{\mathbf{c}}} = (\alpha - 2, 1, 0, \beta - 2)$ .

Then we are in the following alternative:

– If  $\alpha = 2$ , then, applying assertion (iv) of definition 10 to  $v$  and  $\mathbf{c} - \ell_1 + A$  leads to

$$\begin{aligned} L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\mathbf{c}) &= c(\bar{\bar{\mathbf{c}}}) - c(\mathbf{c}) + \gamma\mathbb{E}[v(\mathbf{c} + \ell_2 - 2\ell_1 - \ell_3 + A) - v(\mathbf{c} - \ell_1 + A)] \\ &\leq c(\bar{\bar{\mathbf{c}}}) - c(\bar{\mathbf{c}}) + \gamma\mathbb{E}[v(\mathbf{c} + 2\ell_2 - 2\ell_1 - 2\ell_3 + A) - v(\mathbf{c} + \ell_2 - 2\ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

– If  $\alpha > 2$ , then we apply assertion (v) of definition 10 to  $v$  and  $\mathbf{c} - \ell_1 + A$ , and get that

$$\begin{aligned} L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\mathbf{c}) &= c(\bar{\bar{\mathbf{c}}}) - c(\mathbf{c}) + \gamma\mathbb{E}[v(\mathbf{c} + \ell_2 - 2\ell_1 - \ell_3 + A) - v(\mathbf{c} - \ell_1 + A)] \\ &\leq c(\bar{\bar{\mathbf{c}}}) - c(\bar{\mathbf{c}}) + \gamma\mathbb{E}[v(\mathbf{c} + 2\ell_2 - 3\ell_1 - 2\ell_3 + A) - v(\mathbf{c} + \ell_2 - 2\ell_1 - \ell_3 + A)] \\ &= L^\gamma(\bar{\bar{\mathbf{c}}}) - L^\gamma(\bar{\mathbf{c}}). \end{aligned}$$

(vc) The case  $i = 2$  is symmetric to (vb).

(vd) The case  $i = 3$  is symmetric to (va).

This completes the proof that  $L^\gamma v \in \mathcal{C}'$ .  $\square$

**4.4. Concluding the proof of Theorem 1.** We check that the assumptions of Theorem 2 are satisfied, for the set  $\mathcal{V}$  which has been shown to be stable under the action of  $L^\gamma$  in Proposition 2. It is also immediate that it is stable under point-wise convergence, which proves assertion 1). Setting  $\mathcal{D} \equiv \mathcal{F}$ , assertion 2) is granted by Proposition 1.

We now check that the cost function  $c$  satisfies assumptions (i)-(iii) of Theorem 2. For this, let

$$w = \begin{cases} \mathbb{N}^4 & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & x_0 + x_1 + x_2 + x_3 + 1 \end{cases} .$$

- (i) For all  $\mathbf{x}, \mathbf{z} \in \mathbb{N}^4$  such that  $\mathbf{x} - \mathbf{z} \in \mathbb{N}^4$ , we have  $c(\mathbf{x})/w(\mathbf{x}) \leq \max(c_0, c_1, c_2, c_3)$ .  
Moreover, there is a single arrival at each time step, implying that

$$\begin{aligned} \frac{1}{w(\mathbf{x})} \sum_{\mathbf{y}} \mathbb{P}(\mathbf{y} | \mathbf{x} - \mathbf{z}) w(\mathbf{y}) &= \mathbb{E} \left[ \frac{w(\mathbf{x} - \mathbf{z} + \mathbf{a})}{w(\mathbf{x})} \middle| A = \mathbf{a} \right] \leq \mathbb{E} \left[ \frac{w(\mathbf{x} + \mathbf{a})}{w(\mathbf{x})} \middle| A = \mathbf{a} \right] \\ &= \frac{\sum_{i=0}^3 x_i + 2}{3} \leq 2. \\ &= \sum_{i=0}^3 x_i + 1 \end{aligned}$$

- (ii) Let  $\pi_0$  be the “no matching” policy. With the same argument, we show that for all  $J$ -steps policy  $\pi$ , for all  $\mathbf{x}$ ,

$$\mathbb{E}_{\mathbf{x}}^{\pi} [w(X_p)] \leq \mathbb{E}_{\mathbf{x}}^{\pi_0} [w(X_p)] \leq w(\mathbf{x}) + p,$$

and so

$$\frac{\varepsilon^p}{w(\mathbf{x})} \mathbb{E}_{\mathbf{x}}^{\pi} [w(X_p)] \leq \frac{\varepsilon^p (w(\mathbf{x}) + p)}{w(\mathbf{x})} \leq \varepsilon^p (p + 1),$$

and it is sufficient to choose  $p \in \mathbb{N}$  such that  $\varepsilon^p (p + 1) < \frac{1}{2} < 1$  to conclude.

- (iii) The last condition is satisfied because in the present case, from Lemma 1 for all  $v$  and  $\mathbf{x}$ ,  $L^\gamma v(\mathbf{x})$  by taking the lower bound over a finite subset.

This completes the proof.

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