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► **To cite this version:**

Kala Agbo Bidi. Feedback stabilization and observer design for sterile insect technique model. 2024. hal-04433395v1

HAL Id: hal-04433395

<https://hal.science/hal-04433395v1>

Preprint submitted on 1 Feb 2024 (v1), last revised 10 Dec 2024 (v2)

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Feedback stabilization and observer design for sterile insect technique model

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Abstract

This paper focuses on the feedback global stabilization and observer construction for a sterile insect technique model. The Sterile Insect Technique (SIT) is one of the most ecological methods for controlling insect pests responsible for worldwide crop destruction and disease transmission.

In this work, we construct a feedback law that globally asymptotically stabilizes a SIT model at extinction equilibrium. Since the application of this type of control requires the measurement of different states of the target insect population, and in practice, some states are more difficult and very expensive to measure than others, it is important to know how to construct a state estimator which from a few measured states, estimates the other ones as the one we build in the second part of our work. In the last part of our work, we show that we can apply the feedback control with estimated states to stabilize the full system.

Keywords: Sterile Insect Technique, Pest control, Feedback control design, Observer design, Lyapunov stability, Mosquito population control, Vector-borne disease.

1 Introduction

The Sterile Insect Technique, or SIT, is presently one of the most ecological methods for controlling insect pests responsible for disease transmission or crop destruction worldwide. This technique consists of releasing sterile males into the insect pest population [8, 20, 29]. This approach aims to reducing fertility and, consequently, reducing the target insect population after a few generations. Classical SIT has been modeled and studied theoretically in a large number of papers to derive results to study the success of these strategies using discrete, continuous, or hybrid modeling approaches (for instance, the recent papers [2–4, 10, 11, 25, 28]).

Despite this extensive research, little has been done concerning the stabilization of the target population near extinction after the decay caused by the massive initial SIT intervention and their still major difficulties due to the complexity of the dependency on climate, landscape and many other parameters which would be difficult to be integrated into the mathematical models studied. Not being able to consider all these parameters in our mathematical models and knowing that these external factors strongly impact the evolution of the density of the target population, we focus our studies on releases that now depend on the target population density measurements since, as we will see below, this makes our control more robust. Indeed, several tools can provide information on the size of the wild population throughout the year and during monitoring. So, a control that considers this information to adapt the size of the releases is possible and useful. This was already the case of [1, 11] in which a state feedback control law gives significant robustness qualities to the mathematical model of SIT. Although this approach provides evidence in terms of robustness because the control is directly adjusted according to the density of the population, its application requires continuously measuring the different states of the model. In practice, traps allow data to be collected to analyze the control's impact. New technology is being developed that may allow us to obtain continuous data in the near future.

However, specific categories of data are still problematic or very expensive to obtain. For example, during a SIT intervention, it is difficult to measure the density of young females that have not yet been fecundated or of females that were fecundated by wild males. We add in this work using another control theory tool, which consists of constructing a state estimator for a dynamic system and using these estimates to apply feedback control. A state observer or state estimator is a system that provides an estimation of the natural state using some partial measurements of the real system. In our case, using traps, wild males, as well as sterile males, can be measured. Using the observer system technique, we have built a system that allows us to estimate all other states. The problem of observer design for linear systems was established and solved by [22] and [26]. While Kalman's Observer [22] was highly successful for linear systems, extending it to nonlinear systems took a lot of work. In several cases, the observer can be obtained from the extended Kalman filter by a particular choice of the matrix gain using Linear Matrix Inequalities (LMIs). The development of the observer in this paper was motivated by its application to the SIT model. A model of this process can be contained in the process structure:

$$\dot{x} = Ax + B(y)x + Du, \quad (1.1)$$

$$y = Cx, \quad (1.2)$$

where $y \in \mathbb{R}^m$ is the output, $x \in \mathbb{R}^n$ is the state vector, and $u \in \mathbb{R}^p$ is the input. The output matrix $B(y)$ is such that the coefficients $b(y)_{ij}$ are bounded for all i, j . Our paper has three parts. A first part where, thanks to the backstepping approach, we build a feedback control law that stabilizes the zero population state for the SIT model for the mosquito population, which considers only the compartments of young females and fertilized females presented in [5]. A second part in which we construct a state estimator for the SIT model, then a final part where we show that the application of this feedback, depending on the measured states and the ones estimated thanks to the state estimator, globally stabilizes the system.

2 Mosquito Population dynamics

The mosquito life cycle has several phases. The aquatic stage comprises eggs, larvae, and pupa, followed by the adult stage, where we consider both wild males and females. After emergence from the pupa, a female mosquito needs to mate and get a blood meal before it starts laying eggs. Then every 4 – 5 days, it will take a blood meal and lay 100 – 150 eggs at different places (10 – 15 per place). For the mathematical description, we will consider the following compartments [5].

- E the density of population in aquatic stage,
- Y the density of young females, not yet laying eggs,
- F the density of fertilized and egg-laying females,
- M the density of males,
- M_s the density of sterile males,
- U the density of females that mate with sterile males.

The Y compartment represents the stage of the young female before the start of her gonotropic cycle, i.e., when she mates and takes her first blood meal, which generally takes 3 to 4 days. The sterile insect technique introduces male mosquitoes to compete with wild males. We denote by M_s the density of sterile mosquitoes and by U the density of females that will mate with them. We assume that a female mating mosquito has probability $\frac{M}{M+M_s}$ to mate with a wild male and probability $\frac{M_s}{M+M_s}$ to mate with a sterile one. Hence, the transfer rate η from the compartment Y splits into transfer rate of $\frac{\eta_1 M}{M+M_s}$ to compartment F and a transfer rate of $\frac{\eta_2 M_s}{M+M_s}$ to compartment U of females

that will be laying sterile (nonhatching) eggs. The mathematical model is the system of ordinary differential equations presented in [17]

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\delta_E + \nu_E)E, \quad (2.1)$$

$$\dot{M} = (1 - \nu)\nu_E E - \delta_M M, \quad (2.2)$$

$$\dot{Y} = \nu\nu_E E - \frac{\eta_1 M}{M + M_s} Y - \frac{\eta_2 M_s}{M + M_s} Y - \delta_Y Y, \quad (2.3)$$

$$\dot{F} = \frac{\eta_1 M}{M + M_s} Y - \delta_F F, \quad (2.4)$$

$$\dot{U} = \frac{\eta_2 M_s}{M + M_s} Y - \delta_U U, \quad (2.5)$$

$$\dot{M}_s = u - \delta_s M_s. \quad (2.6)$$

The parameter δ_Y is the mortality rate, for young females (they can die without mating for diverse reason like predators or other hostile environmental conditions). Male mosquitoes can mate for most of their lives. A female mosquito needs a successful mating to reproduce for the rest of her life. $\beta_E > 0$ is the oviposition rate; $\delta_E, \delta_M, \delta_F, \delta_Y > 0$ are the death rates for eggs, wild adult males, fertilized females and young females, respectively; $\nu_E > 0$ is the hatching rate for eggs; $\nu \in (0, 1)$ the probability that a pupa gives rise to a female, and $(1 - \nu)$ is, therefore, the probability of giving rise to a male. $K > 0$ is the environmental capacity for eggs. It can be interpreted as the maximum density of eggs that females can lay in breeding sites. Since here the larval and pupal compartments are not present, it is as if E represents all the aquatic compartments, in which case, in this term K represents a logistic law's carrying capacity for the aquatic phase, which also includes the effects of competition between larvae. The control function u represents the number of mosquitoes released during the SIT intervention. It is interesting to follow the evolution of the state U because female mosquitoes, once fertilized by sterile males, will continue their gonotrophic cycle normally and, therefore, can still transmit disease. We will assume in this document that

$$\delta_s \geq \delta_M. \quad (2.7)$$

In [5, 17], equilibria and their stability property were studied for the system without control.

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\delta_E + \nu_E)E, \quad (2.8)$$

$$\dot{M} = (1 - \nu)\nu_E E - \delta_M M, \quad (2.9)$$

$$\dot{Y} = \nu\nu_E E - (\eta_1 + \delta_Y)Y, \quad (2.10)$$

$$\dot{F} = \eta_1 Y - \delta_F F. \quad (2.11)$$

$$(2.12)$$

Its basic offspring number is $\mathcal{R}_0 = \frac{\eta_1 \beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E) (\eta_1 + \delta_Y)}$. For the rest of our work, we assume that

$$\mathcal{R}_0 > 1. \quad (2.13)$$

3 Global stabilization by feedback law

We assume that wild males are more likely to fertilize young females because they are born on the same egg-laying site. We define

$$\Delta\eta = \eta_1 - \eta_2 \geq 0. \quad (3.1)$$

Other authors, such as [5], have already studied the stability of this type of model. The difference in our approach lies in the kind of control used initially for global stabilization. Indeed, in most of

the prior studies the controls u studied were independent of system states. Some previous works have considered certain simple applications of feedback control to SIT (see, for instance, [9, 11, 16]). In a previous paper, [1], we used the backstepping method to build a feedback control system that simplifies the SIT model, which is presented in [28], assuming that all females are immediately fertilized. Here we consider the system

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\delta_E + \nu_E)E, \quad (3.2)$$

$$\dot{M} = (1 - \nu)\nu_E E - \delta_M M, \quad (3.3)$$

$$\dot{Y} = \nu\nu_E E - \frac{\Delta\eta M}{M + M_s} Y - (\eta_2 + \delta_Y)Y, \quad (3.4)$$

$$\dot{F} = \frac{\eta_1 M}{M + M_s} Y - \delta_F F, \quad (3.5)$$

$$\dot{U} = \frac{\eta_2 M_s}{M + M_s} Y - \delta_U U, \quad (3.6)$$

$$\dot{M}_s = u - \delta_s M_s. \quad (3.7)$$

Let $\mathcal{N} := [0, +\infty)^6$ and $\mathcal{X} := (E, M, Y, F, U, M_s)^T$. When applying a feedback law $u : \mathcal{N} \rightarrow [0, +\infty)$, the closed-loop system is the system

$$\dot{\mathcal{X}} = H(x, u(\mathcal{X})), \quad (3.8)$$

where H is the right side of the equation (3.2)-(3.7). The construction method remains the same as in our previous paper [1]. In this work, we also consider solutions in the Filippov sense of our discontinuous closed-loop system (see, for instance [7, 13, 15, 18, 19, 21]). Let us define $x := (E, M, Y, F, U)^T$. We must put the target system (3.2)-(3.6) in the following form to apply the backstepping method (see, for instance [14, Theorem 12.24, page 334]).

$$\begin{cases} \dot{x} = f(x, M_s), \\ \dot{M}_s = u - \delta_s M_s, \end{cases} \quad (3.9)$$

where $f : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ represents the right hand side of (3.2)-(3.6). We then consider the control system $\dot{x} = f(x, M_s)$ with the state being $x \in \mathcal{D} := [0, +\infty)^5$ and the control being $M_s \in [0, +\infty)$. We assume that M_s is of the form $M_s = \theta M$ for a constant $\theta > 0$ we define and study the closed-loop system

$$\dot{x} = f(x, \theta M). \quad (3.10)$$

where the offspring number is

$$\mathcal{R}(\theta) := \frac{\beta_E \eta_1 \nu \nu_E}{\delta_F (\nu_E + \delta_E) (\Delta\eta + (1 + \theta)(\eta_2 + \delta_Y))}. \quad (3.11)$$

Note that if $\mathcal{R}(\theta) \leq 1$, $\mathbf{0} \in \mathbb{R}^5$ is the only equilibrium point of the system in \mathcal{D} . Our next proposition shows that the feedback law $M_s = \theta M$ stabilizes our control system $\dot{x} = f(x, M_s)$ if $\mathcal{R}(\theta) < 1$.

Proposition 3.1 *Assume that*

$$\mathcal{R}(\theta) < 1. \quad (3.12)$$

Then $\mathbf{0}$ is globally asymptotically stable in \mathcal{D} for system (3.10).

Proof. We apply Lyapunov's second theorem. To do so, we define $V : [0, +\infty)^5 \rightarrow \mathbb{R}_+$, $x \mapsto V(x)$,

$$V(x) := \frac{(1 + 2\mathcal{R}(\theta))\nu\nu_E}{(\nu_E + \delta_E)(1 - \mathcal{R}(\theta))} E + \nu M + \frac{3\mathcal{R}(\theta)}{(1 - \mathcal{R}(\theta))} Y + \frac{(2 + \mathcal{R}(\theta))\beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E) (1 - \mathcal{R}(\theta))} F + \sigma U. \quad (3.13)$$

As (3.12) holds, V is of class C^1 , $V(x) > V((0,0,0,0,0)^T) = 0$, $\forall x \in [0, +\infty)^5 \setminus \{(0,0,0,0,0)^T\}$, $V(x) \rightarrow +\infty$ when $\|x\| \rightarrow +\infty$ with $x \in \mathcal{D}$. and

$$\begin{aligned} \dot{V}(x) = & -\frac{\beta_E \nu \nu_E}{(\nu_E + \delta_E)} F - \nu \delta_M M - \frac{(1 + 2\mathcal{R}(\theta)) \nu \nu_E}{(\nu_E + \delta_E)(1 - \mathcal{R}(\theta))} \frac{\beta_E}{K} FE \\ & - \nu^2 \nu_E E - \frac{\eta_1 \beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E)(1 + \theta)} Y - \frac{\sigma \eta_2}{1 + \theta} Y + \sigma \eta_2 Y - \sigma \delta_U U. \end{aligned}$$

By choosing

$$\sigma := \frac{\eta_1 \beta_E \nu \nu_E \mathcal{R}(\theta)}{(1 + \theta) \eta_2 (\nu_E + \delta_E) \delta_F} \quad (3.14)$$

we get

$$\begin{aligned} \dot{V}(x) = & -\frac{\beta_E \nu \nu_E}{(\nu_E + \delta_E)} F - \nu \delta_M M - \frac{(1 + 2\mathcal{R}(\theta)) \nu \nu_E}{(\nu_E + \delta_E)(1 - \mathcal{R}(\theta))} \frac{\beta_E}{K} FE \\ & - \nu^2 \nu_E E - \frac{\eta_1 \beta_E \nu \nu_E (1 + \theta (1 - \mathcal{R}(\theta)))}{\delta_F (\nu_E + \delta_E) (1 + \theta)^2} Y - \sigma \delta_U U. \end{aligned}$$

and using once more (3.12), we get

$$\dot{V}(x) \leq -cV(x), \quad \forall x \in [0, +\infty)^5, \quad (3.15)$$

with

$$c := \min \left\{ \frac{\nu(\nu_E + \delta_E)(1 - \mathcal{R}(\theta))}{(1 + 2\mathcal{R}(\theta))}, \delta_M, \frac{\delta_F(1 - \mathcal{R}(\theta))}{2 + \mathcal{R}(\theta)}, \frac{\eta_1 \beta_E \nu \nu_E (1 + \theta (1 - \mathcal{R}(\theta))) (1 - \mathcal{R}(\theta))}{\delta_F (\nu_E + \delta_E) (1 + \theta)^2} \frac{1}{3\mathcal{R}(\theta)}, \delta_U \right\} > 0. \quad (3.16)$$

This concludes the proof of Proposition 3.1. \square

We define

$$\phi := \frac{(2 + \mathcal{R}(\theta)) \eta_1 \beta_E \nu \nu_E - 3\mathcal{R}(\theta) \Delta \eta \delta_F (\nu_E + \delta_E)}{\delta_F (\nu_E + \delta_E) (1 - \mathcal{R}(\theta)) (1 + \theta)} - \frac{\eta_1 \beta_E \nu \nu_E \mathcal{R}(\theta)}{(1 + \theta)^2 (\delta_E + \nu_E) \delta_E}, \quad (3.17)$$

and the map $G : \mathcal{N} := [0, +\infty)^5 \rightarrow \mathbb{R}$, $(x^T, M_s)^T \mapsto G((x^T, M_s)^T)$ by

$$\begin{aligned} G((x^T, M_s)^T) := & \frac{\phi Y (\theta M + M_s)^2}{\alpha (M + M_s) (3\theta M + M_s)} + \frac{((1 - \nu) \nu_E \theta E - \theta \delta_M M) (\theta M + 3M_s)}{3\theta M + M_s} \\ & + \delta_s M_s + \frac{1}{\alpha} (\theta M - M_s) \text{ if } M + M_s \neq 0, \quad (3.18) \end{aligned}$$

$$G((x^T, M_s)^T) := 0 \text{ if } M + M_s = 0. \quad (3.19)$$

Finally, let us define the feedback law $u : \mathcal{N} \rightarrow [0, +\infty)$, $(x^T, M_s)^T \mapsto u((x^T, M_s)^T)$, by

$$u((x^T, M_s)^T) := \max(0, G((x^T, M_s)^T)). \quad (3.20)$$

The global stability result is the following.

Theorem 3.1 *Assume that (3.12) holds. Then $\mathbf{0} \in \mathcal{N}$ is globally asymptotically stable in \mathcal{N} for system (3.2)-(3.6) with the feedback law (3.20).*

Lemma 3.1 *Assume that (2.13) and (3.12) hold, then $\phi > 0$.*

Proof.

Let us define $\phi_1 := \frac{(2+\mathcal{R}(\theta))\eta_1\beta_E\nu\nu_E-3\mathcal{R}(\theta)\Delta\eta\delta_F(\nu_E+\delta_E)}{\delta_F(\nu_E+\delta_E)(1-\mathcal{R}(\theta))(1+\theta)}$. We get from the relation (2.13) that $\eta_1\beta_E\nu\nu_E > \delta_F(\nu_E + \delta_E)(\eta_1 + \delta_Y)$. So

$$\phi_1 > \frac{2\eta_1}{(1+\theta)} + \frac{(2+\mathcal{R}(\theta))\delta_Y + 3\mathcal{R}(\theta)\eta_2}{(1-\mathcal{R}(\theta))(1+\theta)}. \quad (3.21)$$

From relation (3.12) we get $\frac{\beta_E\eta_1\nu\nu_E}{\delta_F(\nu_E+\delta_E)} < \Delta\eta + (1+\theta)(\eta_2 + \delta_Y)$. Thus

$$\begin{aligned} \phi &> \frac{2\eta_1}{(1+\theta)} + \frac{(2+\mathcal{R}(\theta))\delta_Y + 3\mathcal{R}(\theta)\eta_2}{(1-\mathcal{R}(\theta))(1+\theta)} - \frac{\Delta\eta\mathcal{R}(\theta) + (1+\theta)(\eta_2 + \delta_Y)\mathcal{R}(\theta)}{(1+\theta)^2}, \\ &> \frac{2\eta_1\mathcal{R}(\theta)}{(1+\theta)} + \frac{(2+\mathcal{R}(\theta))\delta_Y + 3\mathcal{R}(\theta)\eta_2}{(1-\mathcal{R}(\theta))(1+\theta)} + \frac{\eta_2\mathcal{R}(\theta)}{(1+\theta)^2} - \frac{\eta_1\mathcal{R}(\theta)}{(1+\theta)^2} - \frac{(\eta_2 + \delta_Y)\mathcal{R}(\theta)}{1+\theta}, \\ &> \frac{\eta_1\mathcal{R}(\theta)(1+2\theta)}{(1+\theta)^2} + \frac{2\mathcal{R}(\theta)\eta_2 + 2\delta_Y + \mathcal{R}(\theta)^2(\eta_2 + \delta_Y)}{(1-\mathcal{R}(\theta))(1+\theta)} + \frac{\eta_2\mathcal{R}(\theta)}{(1+\theta)^2}, \\ &> 0. \end{aligned}$$

□

Proof of Theorem 3.1. Let $\alpha > 0$ and define $W : \mathcal{N} \rightarrow \mathbb{R}$ by

$$W((x^T, M_s)^T) := V(x) + \alpha \frac{(\theta M - M_s)^2}{\theta M + M_s} \text{ if } M + M_s \neq 0, \quad (3.22)$$

$$W((x^T, M_s)^T) := V(x) \text{ if } M + M_s = 0. \quad (3.23)$$

We have

$$W \text{ is continuous,} \quad (3.24)$$

$$W \text{ is of class } \mathcal{C}^1 \text{ on } \mathcal{N} \setminus \{(E, M, Y, F, U, M_s)^T \in \mathcal{N}; M + M_s = 0\}, \quad (3.25)$$

$$W((x^T, M_s)^T) \rightarrow +\infty \text{ as } \|x\| + M_s \rightarrow +\infty, \text{ with } x \in \mathcal{D} \text{ and } M_s \in [0, +\infty), \quad (3.26)$$

$$W((x^T, M_s)^T) > W(\mathbf{0}) = 0, \quad \forall (x^T, M_s)^T \in \mathcal{N} \setminus \{\mathbf{0}\}. \quad (3.27)$$

From now on, and until the end of this proof, we assume that $(x^T, M_s)^T$ is in \mathcal{D}' and until (3.37) below we further assume that

$$(M, M_s) \neq (0, 0). \quad (3.28)$$

One has

$$\begin{aligned} \dot{W}((x^T, M_s)^T) &= \nabla V(x) \cdot f(x, M_s) + \alpha(\theta M - M_s) \frac{2(\theta \dot{M} - \dot{M}_s)(\theta M + M_s) - (\theta \dot{M} + \dot{M}_s)(\theta M - M_s)}{(\theta M + M_s)^2}, \\ &= \nabla V(x) \cdot f(x, \theta M) + \nabla V(x) \cdot (f(x, M_s) - f(x, \theta M)) \\ &\quad + \alpha(\theta M - M_s) \frac{\theta \dot{M}(\theta M + 3M_s) - \dot{M}_s(3\theta M + M_s)}{(\theta M + M_s)^2}. \end{aligned}$$

$$\begin{aligned} \nabla V(x) \cdot (f(x, M_s) - f(x, \theta M)) &= \begin{pmatrix} \frac{(1+2\mathcal{R}(\theta))\nu\nu_E}{(\nu_E+\delta_E)(1-\mathcal{R}(\theta))} \\ \nu \\ \frac{3\mathcal{R}(\theta)}{(1-\mathcal{R}(\theta))} \\ \frac{(2+\mathcal{R}(\theta))\beta_E\nu\nu_E}{\delta_F(\nu_E+\delta_E)(1-\mathcal{R}(\theta))} \\ \frac{\eta_1\beta_E\nu\nu_E\mathcal{R}(\theta)}{(1+\theta)\eta_2(\nu_E+\delta_E)\delta_F} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -\frac{\Delta\eta(\theta M - M_s)}{(M+M_s)(1+\theta)} Y \\ \frac{\eta_1(\theta M - M_s)}{(M+M_s)(1+\theta)} Y \\ -\frac{\eta_2(\theta M - M_s)}{(M+M_s)(1+\theta)} Y \end{pmatrix} \\ &= \frac{\phi Y(\theta M - M_s)}{M + M_s}, \end{aligned}$$

$$\begin{aligned}
\dot{W}((x^T, M_s)^T) &= \nabla V(x) \cdot f(x, \theta M) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \\
&\quad \left[\frac{(\nabla V(x) \cdot (f((x^T, M_s)^T) - f(x, \theta M)))(\theta M + M_s)^2}{\alpha(\theta M - M_s)} \right. \\
&\quad \left. + \theta \dot{M}(\theta M + 3M_s) - \dot{M}_s(3\theta M + M_s) \right] \\
&= \dot{V}(x) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \left[\frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)} + ((1 - \nu)\nu_E \theta E - \theta \delta_M M)(\theta M + 3M_s) \right. \\
&\quad \left. - u(3\theta M + M_s) + \delta_s M_s(3\theta M + M_s) \right]. \tag{3.29}
\end{aligned}$$

We take u as given by (3.20). Therefore, in case

$$\begin{aligned}
\frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)} + ((1 - \nu)\nu_E \theta E - \theta \delta_M M)(\theta M + 3M_s) \\
+ \delta_s M_s(3\theta M + M_s) + \frac{1}{\alpha}(\theta M - M_s)(3\theta M + M_s) > 0, \tag{3.30}
\end{aligned}$$

$$\begin{aligned}
u = \frac{1}{3\theta M + M_s} \left[\frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)} + ((1 - \nu)\nu_E \theta E - \theta \delta_M M)(\theta M + 3M_s) \right. \\
\left. + \delta_s M_s(3\theta M + M_s) + \frac{1}{\alpha}(\theta M - M_s)(3\theta M + M_s) \right],
\end{aligned}$$

which, together with (3.29), leads to

$$\dot{W}((x^T, M_s)^T) = \dot{V}(x) - \frac{(\theta M - M_s)^2(3\theta M + M_s)}{(\theta M + M_s)^2}. \tag{3.31}$$

Otherwise, i.e. if (3.30) does not hold,

$$\begin{aligned}
\frac{\phi Y(\theta M + M_s)^2}{\alpha(M + \gamma_s M_s)} + ((1 - \nu)\nu_E \theta E - \theta \delta_M M)(\theta M + 3M_s) \\
+ \delta_s M_s(3\theta M + M_s) + \frac{1}{\alpha}(\theta M - M_s)(3\theta M + M_s) \leq 0, \tag{3.32}
\end{aligned}$$

so, by (3.20),

$$u = 0. \tag{3.33}$$

We consider two cases. If $\theta M > M_s$ using (3.29), (3.32) and (3.33)

$$\begin{aligned}
\dot{W}((x^T, M_s)^T) &\leq \dot{V}(x) - \frac{(\theta M - M_s)^2(3\theta M + M_s)}{(\theta M + M_s)^2}, \\
&\leq -cV(x) - \frac{(\theta M - M_s)^2}{\theta M + M_s}, \\
&\leq c_1 W((x^T, M_s)^T), \tag{3.34}
\end{aligned}$$

with

$$c_1 := \min\{c, 1\} > 0. \tag{3.35}$$

If $\theta M \leq M_s$, using once more (3.29) and (3.33)

$$\begin{aligned}
\dot{W}((x^T, M_s)^T) = \dot{V}(x) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \left[\frac{\phi Y(\theta M + M_s)^2}{\alpha(M + M_s)} \right. \\
\left. + \theta((1 - \nu)\nu_E E - \delta_M M)(\theta M + 3M_s) + \delta_s M_s(3\theta M + M_s) \right]. \tag{3.36}
\end{aligned}$$

Using (2.7) we get $-\delta_M M(\theta M + 3M_s) + \delta_s M_s(3\theta M + M_s) \geq \delta_M(M_s - \theta M)(M_s + \theta M)$, which, together with (3.36) and the lemma 3.1, implies that

$$\begin{aligned} \dot{W}((x^T, M_s)^T) &\leq \dot{V}(x) - \alpha \delta_M \frac{(\theta M - M_s)^2}{(\theta M + M_s)}, \\ &\leq c_2 W((x^T, M_s)^T), \end{aligned} \quad (3.37)$$

with

$$c_2 := \min\{c, \alpha \delta_M\} > 0. \quad (3.38)$$

Let us now deal with the case where (3.28) is not satisfied. Note that, for every $\tau \geq 0$, $M(\tau) + M_s(\tau) > 0$ implies that $M(t) + M_s(t) > 0$ for all $t \geq \tau$. Thus, if $M(0) + M_s(0) = 0$, there exists $t_s \in [0, +\infty]$ such that $M(t) + M_s(t) = 0$ if and only if $t \in [0, t_s] \setminus \{+\infty\}$. Let us study only the case $t_s \in (0, +\infty)$ (the case $t_s = 0$ is obvious and the case $t_s = +\infty$ is a corollary of our study of the case $t_s \in (0, +\infty)$). Let us first point out that, for every $(M, M_s)^T \in [0, +\infty)^2$ such that $M + M_s > 0$,

$$\frac{M}{M + M_s} \in [0, 1], \quad \frac{(\theta M + M_s)^2}{(M + M_s)(3\theta M + M_s)} \in [0, \theta + 1], \quad \text{and} \quad \frac{\theta M + 3M_s}{3\theta M + M_s} \in [0, 3]. \quad (3.39)$$

Let $t \mapsto \mathcal{X}(t) = (E(t), M(t), Y(t), F(t), U(t), M_s(t))^T$ be a solution (in the Filippov sense) of the closed-loop system (3.2)-(3.6) such that, for some $t_s \in (0, +\infty)$

$$M(t) + M_s(t) = 0 \quad \forall t \in [0, t_s]. \quad (3.40)$$

Note that (3.40) implies that

$$M(t) = M_s(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.41)$$

From (3.39), (3.41) and the definition of a Filippov solution, one has on $(0, t_s)$

$$\begin{pmatrix} \dot{E} \\ \dot{M} \\ \dot{Y} \\ \dot{F} \\ \dot{U} \\ \dot{M}_s \end{pmatrix} = \begin{pmatrix} \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E \\ (1 - \nu)\nu_E E - \delta_M M \\ \nu\nu_E E - \kappa(t)\Delta\eta Y - (\eta_2 + \delta_Y)Y \\ \eta_1 Y \kappa(t) - \delta_F F \\ \eta_2(1 - \kappa(t))Y - \delta_U U \\ Y g_1(t) + E g_2(t) - \delta_s M_s \end{pmatrix} \quad (3.42)$$

with

$$\kappa(t) \in [0, 1], \quad g_1(t) \in \frac{\phi}{\alpha}[0, \theta + 1] \quad \text{and} \quad g_2(t) \in (1 - \nu)\nu_E \theta [0, 3]. \quad (3.43)$$

From (3.41) and the second line of (3.42), one has

$$E(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.44)$$

From the first line of (3.42) and (3.44), we get

$$F(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.45)$$

Let us first consider the case where $Y(0) = 0$. Then, from the third line of (3.42) and (3.44), one has

$$Y(t) = 0, \quad \forall t \in [0, t_s]. \quad (3.46)$$

To summarize, from (3.41), the fifth line of (3.42), (3.44), (3.45), and (3.46)

$$E(t) = M(t) = Y(t) = F(t) = M_s(t) = 0 \quad \text{and} \quad \dot{U}(t) = -\delta_U U(t), \quad \forall t \in [0, t_s], \quad (3.47)$$

which, with (3.13), (3.16), and (3.23), gives

$$\dot{W}(t) = -\sigma \delta_U U(t) \leq -\delta_U W(t), \quad \forall t \in [0, t_s]. \quad (3.48)$$

Let us finally consider the case where $Y(0) > 0$. Then, from the third line of (3.42),

$$Y(t) > 0, \forall t \in [0, t_s], \quad (3.49)$$

which, together with the fourth line of (3.42) and (3.45), implies

$$\kappa(t) = 0, \forall t \in [0, t_s]. \quad (3.50)$$

To summarize, from (3.41), the third and the fifth line of (3.42), (3.44), (3.45), and (3.50),

$$E(t) = M(t) = F(t) = M_s(t) = 0, \dot{Y}(t) = -(\eta_2 + \delta_Y)Y(t), \text{ and } \dot{U}(t) = \eta_2 Y - \delta_U U(t), \forall t \in [0, t_s],$$

which, with (3.13), (3.14), (3.16), and (3.23), gives

$$\begin{aligned} \dot{W}(t) &= -(\eta_2 + \delta_Y) \frac{3\mathcal{R}(\theta)}{(1 - \mathcal{R}(\theta))} Y(t) + \eta_2 \sigma Y(t) - \sigma \delta_U U(t) \\ &= -\mathcal{R}(\theta) \left((\eta_2 + \delta_Y) \frac{3}{(1 - \mathcal{R}(\theta))} - \frac{\eta_1 \beta_E \nu \nu_E}{(1 + \theta)(\nu_E + \delta_E) \delta_F} \right) Y(t) - \sigma \delta_U U(t) \\ &= -\mathcal{R}(\theta) \left(\frac{Q}{(1 - \mathcal{R}(\theta))(1 + \theta)(\nu_E + \delta_E) \delta_F} \right) Y(t) - \sigma \delta_U U(t) \end{aligned} \quad (3.51)$$

where

$$Q := 3(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E) \delta_F - (1 - \mathcal{R}(\theta)) \eta_1 \beta_E \nu \nu_E. \quad (3.52)$$

To end the proof we have to prove that $Q > 0$. Using the relation (3.11) and (3.12) we have

$$\beta_E \eta_1 \nu \nu_E < \mathcal{R}(\theta) \delta_F (\nu_E + \delta_E) \Delta \eta + \delta_F (\nu_E + \delta_E) (1 + \theta) (\eta_2 + \delta_Y). \quad (3.53)$$

Recall that $\Delta \eta = \eta_1 - \eta_2$. One has

$$\begin{aligned} Q &= 3(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E) \delta_F - \eta_1 \beta_E \nu \nu_E + \mathcal{R}(\theta) \eta_1 \beta_E \nu \nu_E \\ &> 2(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E) \delta_F - \mathcal{R}(\theta) \Delta \eta (\nu_E + \delta_E) \delta_F + \mathcal{R}(\theta) \eta_1 \beta_E \nu \nu_E \\ &> 2(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E) \delta_F - \mathcal{R}(\theta) \eta_1 (\nu_E + \delta_E) \delta_F + \mathcal{R}(\theta) \eta_1 \beta_E \nu \nu_E + \mathcal{R}(\theta) \eta_2 (\nu_E + \delta_E) \delta_F. \end{aligned}$$

From the relation (2.13), $\eta_1 \beta_E \nu \nu_E > \delta_F (\nu_E + \delta_E) (\eta_1 + \delta_Y)$.

$$\begin{aligned} Q &> 2(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E) \delta_F - \mathcal{R}(\theta) \eta_1 (\nu_E + \delta_E) \delta_F + \mathcal{R}(\theta) \delta_F (\nu_E + \delta_E) (\eta_1 + \delta_Y) + \mathcal{R}(\theta) \eta_2 (\nu_E + \delta_E) \delta_F \\ &> 2(\eta_2 + \delta_Y)(1 + \theta)(\nu_E + \delta_E) \delta_F + \mathcal{R}(\theta) (\eta_2 + \delta_Y) (\nu_E + \delta_E) \delta_F \\ &> 0. \end{aligned}$$

We get

$$\dot{W}(t) \leq -c' W(t), \forall t \in [0, t_s], \quad (3.54)$$

where

$$c' := \min \left\{ \frac{Q}{3(1 + \theta) \delta_F (\nu_E + \delta_E)}, \delta_U \right\}. \quad (3.55)$$

This proves Theorem 3.1 and gives the global exponential stability and provides an estimate on the exponential decay rate. \square

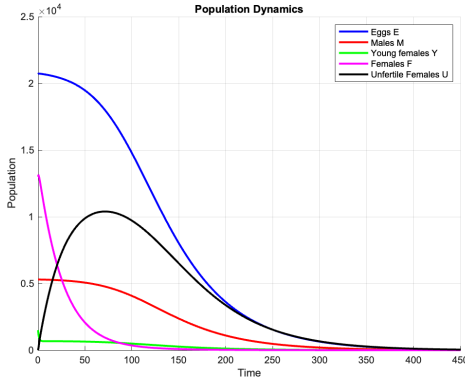
3.1 Numerical simulations

Note that η_1 represents the natural fertility rate in the mosquito population. Wild males have a shorter maturity time in their life cycle than females. Thus, the fertilization phase is essentially around the hatching site. Sterile males are artificially released into the intervention region. We denote by p with $0 \leq p \leq 1$ the proportion of sterile males that are releases in the adequate places. Also, the effective fertilization during the mating could be diminished due to the sterilization, which leads us to assume that the effective mating rate of sterile insects is given by $q\eta_1$ with $0 \leq q \leq 1$. Putting together these assumptions we get that the probability of young female mate with sterile males is $\frac{\eta_2 M}{M+M_s}$ with $\eta_2 = pq\eta_1$. For the numerical simulation we take $\eta_1 = 1$ and $\eta_2 = 0.7$. The numerical simulations of the dynamics when applying the feedback (3.20) gives figure 1. The parameters we use are given in the following table.

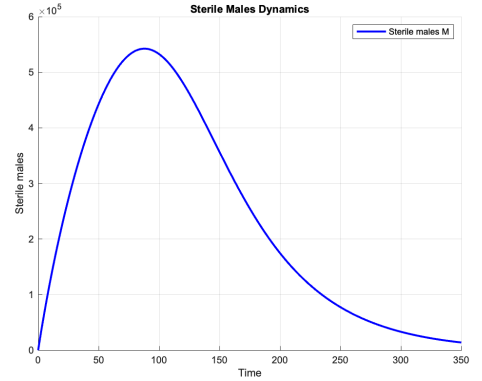
Parameters	Description	Value
β_E	Effective fecundity	10
ν_E	Hatching parameter	0.05
δ_E	Mosquitoes in aquatic phase death rate	0.03
δ_F	Fertilized female death rate	0.04
δ_Y	Young female death rate	0.04
δ_M	Male death rate	0.1
δ_s	Sterilized male death rate	0.12
ν	Probability of emergence	0.49

Table 1: Value for the parameters of system (3.2)-(3.5) (see [5] [28]). Units are days⁻¹ except for ν .

With the parameters given in table 1, condition (3.12) is $\theta > 102,06$. We fix $K = 21000$ and we consider the persistence equilibrium as initial condition. That gives $E^0 = 20700$, $M^0 = 5300$, $Y_0 = 1500$, $F^0 = 13000$ and $U^0 = M_s^0 = 0$. We take $\theta = 290$ and $\alpha = 90$.



(a) Evolution of states E , M , Y and F



(b) Evolution of M_s

Figure 1: (a): Plot of E , M , Y , F and U when applying the feedback (3.20) with the initial condition z_0 . (b): Plot of M_s .

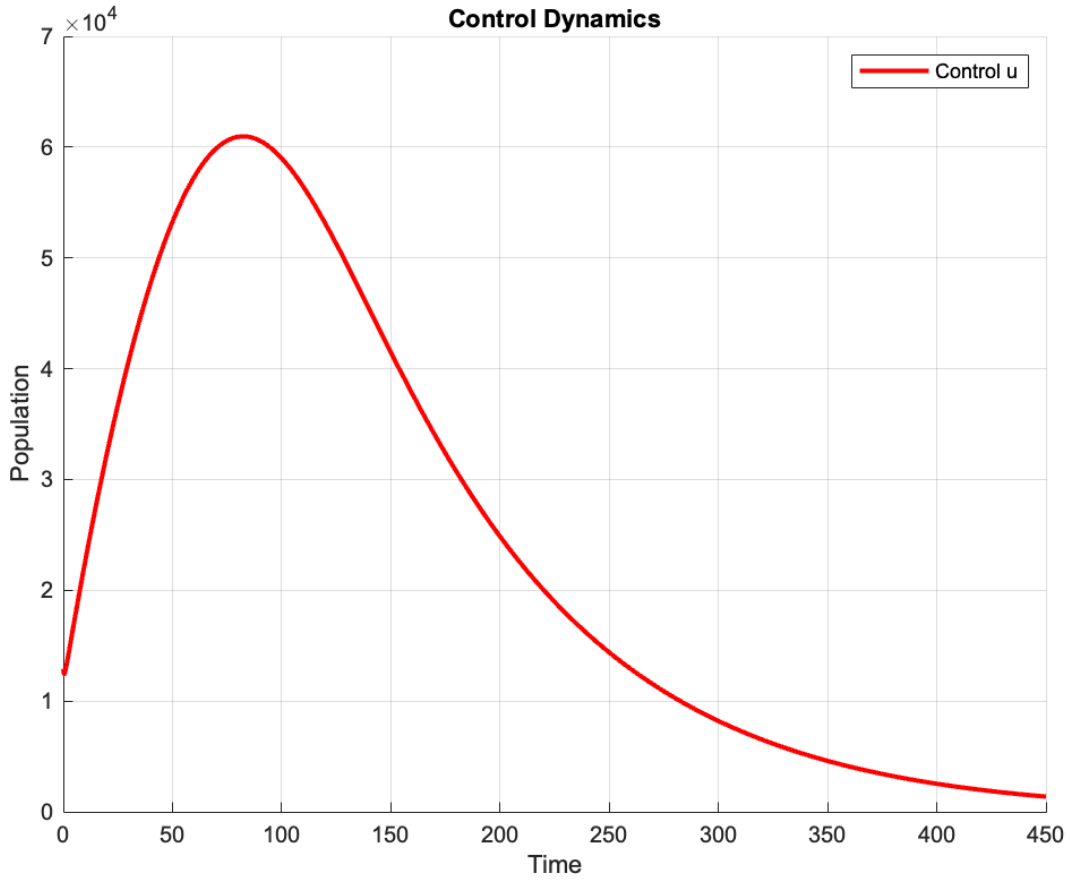


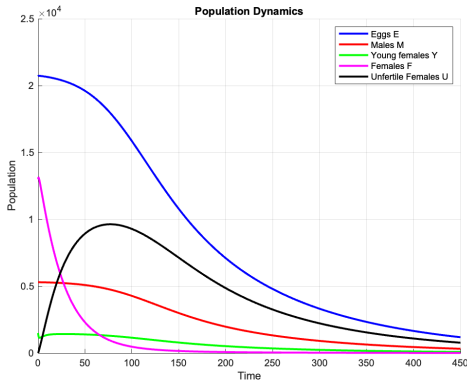
Figure 2: Evolution of the control function u

Remark 3.1 *It is important to note that the backstepping feedback control (3.20) does not depend on the environmental capacity K , which is also an interesting feature for the field applications. Note that $\mathcal{N} := [0, +\infty)^6$ and $\mathcal{X} := (E, M, Y, F, U, M_s)^T$ the feedback satisfies*

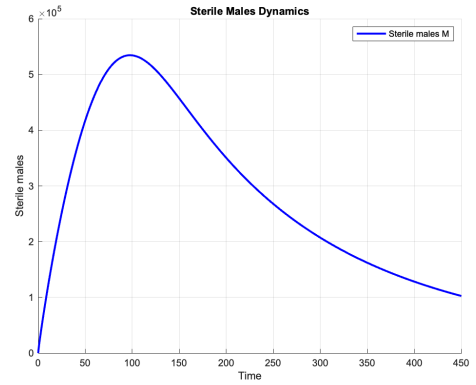
$$\sup_{\varepsilon \rightarrow 0} \{|u(\mathcal{X})| : \mathcal{X} \in \mathcal{N}, \|X\|_1 \in \mathcal{B}(0, \varepsilon)\} \longrightarrow 0. \quad (3.56)$$

The advantage of applying feedback control is that when the density of the target population decreases, the control also decreases.

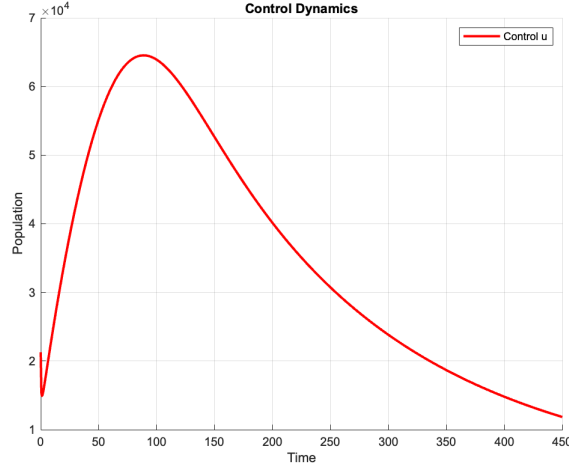
Remark 3.2 *Let us assume that the heterogeneity of the intervention zone strongly impacts the mating of female mosquitoes with sterile males more than we would have estimated. Suppose the estimated mating rate for the control (3.20) is $\eta_2^e = 0.7$ and let the mating rate be $\eta_2^r = 0.4$ for the dynamics. Keeping the other parameters and the same condition, we obtain the following figure.*



(a) Evolution of states E , M , Y and F



(b) Evolution of M_s



(c) Evolution of the control function u

Figure 3: (a): Plot of E, M, Y, F and U when applying the feedback (3.20) with $\eta_2^e = 0.7$ while $\eta_2^r = 0.4$ for the dynamics. (b): Plot of M_s . (c): Plot of the feedback control function u .

This parameter considerably impacts the convergence time of the states of the system. Note that with $e = 3 \times 10^{-1}$ of error difference, we still have convergence. Estimation errors of the order of $e = 10^{-2}$ will have a negligible impact on the convergence time. This is because the control backstepping also depends on the states of the system. Thus, the states make a correction that can compensate for a certain margin of error. Unlike control, which only depends on the parameters, estimation errors have no correction from the dynamics. Therefore, this can be fatal to the success of the intervention. In practice, many external factors impact the life cycle of mosquitoes. These factors modify parameters such as birth, hatching, and fertilization rates. These factors are, for example, rainfall and the topography of the region. A SIT model that can integrate these factors is challenging to study. Success of SIT intervention depends strongly on the robustness. The results of our previous test that is reported in Figure 3, show us the advantage feedback control can provide in terms of robustness.

4 Observer design for SIT model

The application of feedback control requires measuring states such as eggs E and young females Y of the intervention zone over time. In practice, it is always important to estimate the density of adult mosquitoes to intervene in an area. These data are collected using mosquito traps distributed throughout the region. Despite various technological advances to improve these traps, it should be noted that some data are still easier to be measured than others. Measuring mosquito density in the aquatic phase E is difficult, specially in a heterogeneous area. It is also challenging to measure

young females Y because females come in three categories, and we need to distinguish between unfertilized and fertilized females. Males are more easily measured because they are distinguishable. It can be also easy to distinguish wild males from laboratory males by marking processes applied to laboratory males. In this part of our paper, we will assume that the density of wild males and that of sterile males can be measured continuously. Our objective is to estimate the other densities. Observer design for nonlinear dynamic systems is a technique used in control theory to estimate the states of a system when only partial or indirect measurements are available. The difficulties in dealing with observer problems for general nonlinear systems is the proof of global convergence of the estimation error. Much literature exists on state observers and filters for nonlinear systems as they play crucial roles in control theory. In this section we consider the SIT model for a population with high environmental capacity K . So our control system is

$$\dot{E} = \beta_E F - (\delta_E + \nu_E)E, \quad (4.1)$$

$$\dot{M} = (1 - \nu)\nu_E E - \delta_M M, \quad (4.2)$$

$$\dot{Y} = \nu\nu_E E - \frac{\Delta\eta M}{M + M_s} Y - (\eta_1 + \delta_Y)Y, \quad (4.3)$$

$$\dot{F} = \frac{\eta_1 M}{M + M_s} Y - \delta_F F, \quad (4.4)$$

$$\dot{U} = \frac{\eta_2 M_s}{M + M_s} Y - \delta_U U, \quad (4.5)$$

$$\dot{M}_s = u - \mu_s M_s, \quad (4.6)$$

$$y_1 = M, \quad (4.7)$$

$$y_2 = M_s, \quad (4.8)$$

where the states is $X = (E, M, Y, F, U, M_s)^T \in \mathcal{N}$, the control is $u \in [0, +\infty)$ and the output is $y = (M, M_s)^T \in \mathbb{R}_+^2$.

In particular, in this model we are confronted with a difficulty in which most observer construction theories are invalid because of the singularity at the origin. To go around this difficulty, we will use the fact that the main nonlinearity term $\frac{M}{M+M_s}$ is bounded and essentially the most accessible data to measure. This leads us to develop an observer for this type of system.

4.1 Observer design for class of nonlinear systems

The usual observers for linear systems are the Luenberger observer and the Kalman observer. The observer design for a nonlinear system is a complex problem in control theory and has received much attention from many authors yielding a large literature of methods. Among than, the most famous are the change of coordinates to transform the nonlinear system into a linear system [6, 23, 24] and a second approach consists in using the Extended Kalman Filter (EKF) [12, 27]. The state observer is called an exponential state observer if the observer error converges exponentially to zero. In this section we provide an explicit construction of a global observer for the following system.

$$\begin{cases} \dot{x}(t) = Ax(t) + B(y(t))x(t) + Du(t), \\ y(t) = Cx(t), \end{cases} \quad (4.9)$$

where $x(t) \in \mathbb{R}^n$, is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector and $y(t) \in \mathbb{R}^p$ is the output vector. $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times n}$ are the appropriate matrices. The matrice $B(y(t))$ is in the form

$$B(y(t)) = \sum_{i,j=1}^{n,n} b_{ij}(y(t))e_n(i)e_n^T(j). \quad (4.10)$$

We assume that for all $y(t) \in \mathbb{R}^m$ the coefficients b_{ij} are bounded for all $i = 1, \dots, n$ and $j = 1, \dots, n$ and denote

$$\bar{b}_{ij} = \max_t(b_{ij}(y(t))) \text{ and } \underline{b}_{ij} = \min_t(b_{ij}(y(t))). \quad (4.11)$$

Then, the parameter vector $b(t)$ remains in a bounded convex domain $\mathcal{S}_{n,n}$ of which $2^{(n^2)}$ vertices are defined by:

$$\mathcal{V}_{\mathcal{S}_{n,n}} = \{\eta = (\eta_{11}, \dots, \eta_{1n}, \dots, \eta_{nn}) | \eta_{ij} \in \{\underline{b}_{ij}, \bar{b}_{ij}\}\}.$$

A state observer corresponding to (4.9) is given as follows:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + B(y(t))\hat{x} + Du(t) - L(C\hat{x} - y(t)), \\ \hat{y}(t) = C\hat{x}(t), \end{cases} \quad (4.12)$$

where $\hat{x}(t)$ denotes the estimate of the state $x(t)$. The dynamics of the observer error $e(t) := \hat{x}(t) - x(t)$ is $\dot{e}(t) = (A - LC)e(t) + B(y(t))e(t) = (A + B(y(t)) - LC)e(t)$. We define

$$\mathcal{A}(b(t)) = A + \sum_{i,j=1}^{n,n} b_{ij}(y(t))e_q(i)e_n^T(j). \quad (4.13)$$

The dynamics of the observer error becomes

$$\dot{e}(t) = (\mathcal{A}(b(t)) - LC)e(t). \quad (4.14)$$

The observation problem consists of finding a gain L such that (4.14) converges exponentially and asymptotically towards zero. We use the following results in [30].

Theorem 4.1 *The observer error converges exponentially towards zero if there exist matrices $P = P^T > 0$ and R of appropriate dimensions such that following Linear Matrix Inequalities (LMIs) are feasible:*

$$\mathcal{A}^T(\eta)P - C^T R + P\mathcal{A}(\eta) - R^T C + \xi I < 0, \quad (4.15)$$

$$\forall \eta \in \mathcal{V}_{\mathcal{S}_{n,n}}, \quad (4.16)$$

for some constant $\xi > 0$. When these LMIs are feasible, the observer gain L is given by $L = P^{-1}R^T$.

Proof. We follow [30] and consider the following quadratic Lyapunov function

$$\mathcal{V}(e) = e^T P e, \quad (4.17)$$

where $P^T = P > 0$. We have $\dot{\mathcal{V}}(e)(t) = e(t)^T F(b(t))e(t)$, where $F(b(t)) = (\mathcal{A}(b(t)) - LC)^T P + P(\mathcal{A}(b(t)) - LC)$. For $e(t) \neq 0$ the condition $\mathcal{V}(e(t)) > 0$ is satisfied because $P > 0$ and the condition $\dot{\mathcal{V}}(e(t)) < 0$ is satisfied if we have

$$F(b(t)) < 0 \text{ for all } b(t) \in \mathcal{S}_{n,n}. \quad (4.18)$$

Since the matrix function F is affine in $b(t)$, using a convexity argument we deduce that $\forall t \geq 0$

$$\dot{\mathcal{V}}(e(t)) < -\xi \gamma \|e(t)\|_P^2, \quad (4.19)$$

if the following condition is satisfied $F(\eta) < -\varepsilon I \quad \forall \eta \in \mathcal{U}_{\mathcal{V}_{n,q}}$. Thus, if (4.15) holds, this inequality is also satisfied. \square

4.2 Application to the SIT model

We rewrite the output SIT models (4.1) -(4.7) of the study.

$$\begin{cases} \dot{X} = AX + B(y)X + Gu, \\ y = CX, \end{cases} \quad (4.20)$$

where $X = (E, M, Y, F, U, M_s)^T$,

$$A = \begin{pmatrix} -(\delta_E + \nu_E) & 0 & 0 & 0 & \beta_E & 0 \\ (1 - \nu)\nu_E & -\delta_M & 0 & 0 & 0 & 0 \\ \nu\nu_E & 0 & -(\eta_2 + \delta_Y) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_F & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta_U & 0 \\ 0 & 0 & 0 & 0 & 0 & -\delta_s \end{pmatrix}, \quad B(y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\Delta\eta\frac{y_1}{y_1+y_2} & 0 & 0 & 0 \\ 0 & 0 & \eta_1\frac{y_1}{y_1+y_2} & 0 & 0 & 0 \\ 0 & 0 & \eta_2\frac{y_2}{y_1+y_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

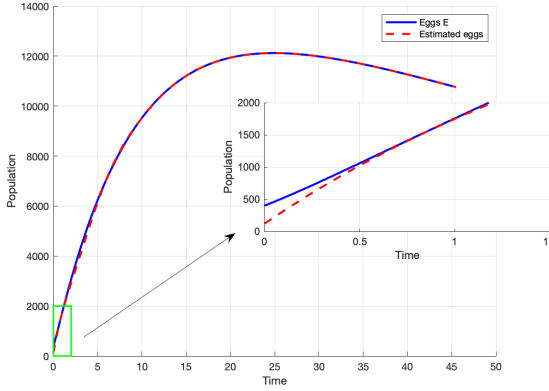
$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As, \mathcal{N} is an invariant set, one has $0 \leq \frac{y_1}{y_1+y_2} \leq 1$. Solving the corresponding equation of (4.15) in MATLAB, we get

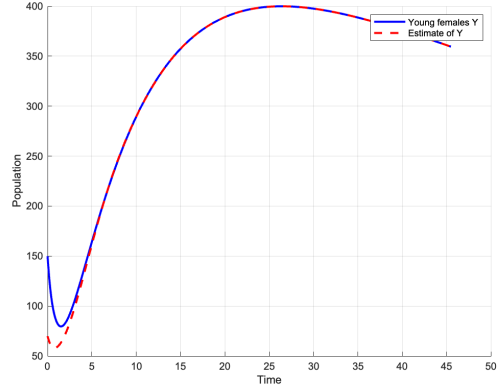
$$P = 10^4 \begin{pmatrix} 0.0219 & -0.1567 & -0.1531 & -0.1703 & -0.0344 & 0 \\ -0.1567 & 8.9301 & -0.8472 & -0.8081 & -0.4929 & 0 \\ -0.1531 & -0.8472 & 4.5716 & 0.9277 & 1.0845 & 0 \\ -0.1703 & -0.8081 & 0.9277 & 4.3088 & -2.3012 & 0 \\ -0.0344 & -0.4929 & 1.0845 & -2.3012 & 4.7413 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.7267 \end{pmatrix}, \quad L = \begin{pmatrix} 50.6342 & 0 \\ 1.4150 & 0 \\ 0.9426 & 0 \\ 2.6547 & 0 \\ 1.6023 & 0 \\ 0 & 0.3800 \end{pmatrix}.$$

$$R = 10^3 \begin{pmatrix} 0.2352 & 0.9704 & -0.4415 & -1.1401 & 0.0690 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.4162 \end{pmatrix},$$

With the parameters given in table 1, we take initially $x_0 = (400, 100, 150, 120, 120, 50)^T$. and $\hat{x}_0 = (120, 70, 70, 50, 60, 0)^T$.

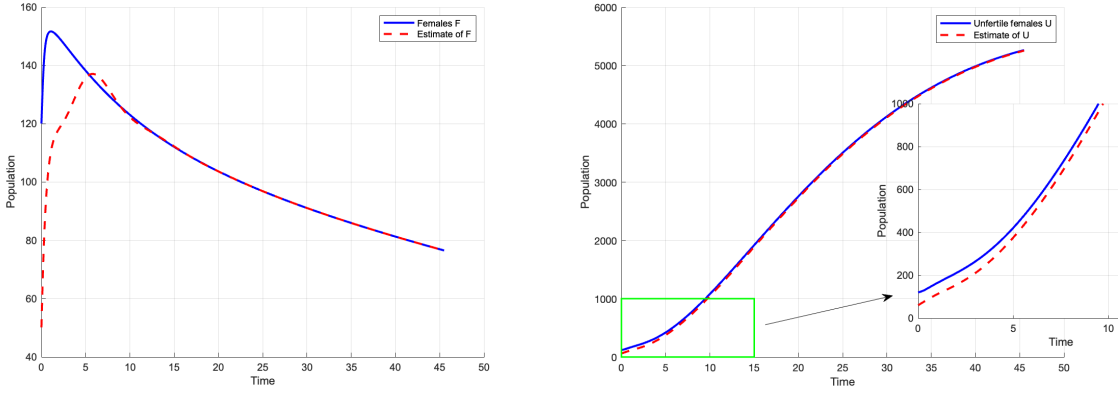


(a) Evolution of states E and corresponding \hat{E}



(b) Evolution of states Y and corresponding \hat{Y}

Figure 4: Simulation of the system and the observer



(a) Evolution of states F and corresponding \hat{F} (b) Evolution of states U and corresponding \hat{U}

Figure 5: Simulation of the system and the observer

4.3 Global stability by an estimate feedback law

The control feedback (3.20) depends on the states E , M , Y and M_s . From the measurement of states M and M_s , an observer system has been built in the previous section. This state observer is used to estimate both eggs E and young females Y . In this section we show that $\mathbf{u}(\hat{X}, y)$ stabilizes the dynamics at the origin. We consider the coupled system

$$\begin{cases} \dot{X} = f(X, \hat{\mathbf{u}}(\hat{X}, y)), \\ \dot{\hat{X}} = f(\hat{X}, \hat{\mathbf{u}}(\hat{X}, y)) - L(C\hat{x} - y), \end{cases} \quad (4.21)$$

with

$$\hat{\mathbf{u}}(\hat{X}, y) = \max\left(0, S(\hat{X}, y)\right). \quad (4.22)$$

where $S : \mathbb{R}^4 \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$, $(\hat{X}, y)^T \mapsto S(\hat{X}, y)$ is defined by

$$S(\hat{X}, y) := G(\hat{E}, M, \hat{F}, M_s) \quad (4.23)$$

The main result of this section is the following theorem.

Theorem 4.2 *Assume that (3.12) holds. Then $\mathbf{0} \in \mathcal{E} = \mathcal{N} \times \mathbb{R}^6$ is globally asymptotically stable in \mathcal{D}' for system (4.21) with the feedback law (3.20).*

Proof. Let $\lambda > 0$ and we define $H : \mathcal{E} \rightarrow \mathbb{R}$ by

$$H(X, \hat{X}) = W(X) + \lambda\sqrt{\mathcal{V}(e)} \quad (4.24)$$

with $e = \hat{X} - X$.

$$H \text{ is continuous on } \mathcal{E} \text{ and } \mathcal{C}^1 \text{ on } \mathcal{E} \setminus \left\{ (X, \hat{X}) \in \mathcal{E}; M + M_s = 0 \right\}, \quad (4.25)$$

$$H(X, \hat{X}) \rightarrow +\infty \text{ as } \|(X, \hat{X})\| \rightarrow +\infty, \quad (4.26)$$

$$H(X, \hat{X}) > H(\mathbf{0}) = 0 \quad \forall (X, \hat{X}) \in \mathcal{E} \setminus \{\mathbf{0}\}. \quad (4.27)$$

In this proof, from now on we assume that $(X, \hat{X})^T$ is in \mathcal{E} . Until (4.33) included, we also assume that

$$(M, M_s) \neq (0, 0). \quad (4.28)$$

One has

$$\begin{aligned} \dot{H}(X, \hat{X}) &= \dot{V}(x) + \alpha \frac{(\theta M - M_s)}{(\theta M + M_s)^2} \left[\frac{\phi Y (\theta M + M_s)^2}{\alpha (M + M_s)} \right. \\ &\quad \left. + ((1 - \nu)\nu_E \theta E - \theta \delta_M M)(\theta M + 3M_s) \right. \\ &\quad \left. - \hat{\mathbf{u}}(\hat{X}, y)(3\theta M + M_s) + \delta_s M_s(3\theta M + M_s) \right] + \lambda \frac{\dot{V}(e)}{2\sqrt{\mathcal{V}(e)}}. \end{aligned}$$

Replacing the term $\hat{\mathbf{u}}(\hat{X}, y)$ by $\hat{\mathbf{u}}(\hat{X}, y) - \hat{\mathbf{u}}(X, y) + \hat{\mathbf{u}}(X, y)$ we get

$$\dot{H}(X, \hat{X}) = \dot{W}(X) + \alpha \frac{(\theta M - M_s)(3\theta M + M_s)}{(\theta M + M_s)^2} (\mathbf{u}(\hat{X}, y) - \mathbf{u}(X, y)) + \lambda \frac{\dot{V}(e)}{2\sqrt{\mathcal{V}(e)}}. \quad (4.29)$$

Lemma 4.1 *There exist $C > 0$ such that, for all $(X, \hat{X}) \in \mathcal{E}$ and for all $y \in \mathbb{R}_+^2$,*

$$\|\hat{\mathbf{u}}(\hat{X}, y) - \hat{\mathbf{u}}(X, y)\| \leq C \|\hat{X} - X\|. \quad (4.30)$$

Note that $\dot{V}(e) \leq -\varepsilon\gamma \|e\|_P^2$. Thanks to this lemma, there exists $C' > 0$ independent of y such that

$$\dot{H}(X, \hat{X}) \leq \dot{W}(X) + C' \|e\| - \varepsilon\gamma\lambda \frac{\|e\|_P^2}{2\sqrt{\mathcal{V}(e)}}. \quad (4.31)$$

Note that there exists a constant $\beta > 0$ such that and $\|e\| \leq \beta \|e\|_P$. So

$$\dot{H}(X, \hat{X}) \leq \dot{W}(X) - \left(\frac{\varepsilon\gamma}{2}\lambda - \beta C'\right) \|e\|_P. \quad (4.32)$$

Hence for $\lambda > 2C'\beta/\varepsilon\gamma$, that there exists a constant $c_0 > 0$ such that

$$\dot{H}(X, \hat{X}) < -c_0 H(X, \hat{X}), \text{ if } M + M_s \neq 0 \quad (4.33)$$

Let us now deal with the case where (3.28) is not satisfied. As we explain previously in the proof of the Theorem 3.1, we study only the case $t_s \in (0, +\infty)$. Let $t \mapsto (E(t), M(t), Y(t), F(t), U(t), M_s(t), \hat{E}(t), \hat{M}(t), \hat{Y}(t), \hat{F}(t), \hat{U}(t), \hat{M}_s(t))^T$ be a solution (in the Filippov sense) of the closed-loop system (4.21) such that, for some $t_s \in (0, +\infty)$

$$M(t) + M_s(t) = 0 \quad \forall t \in [0, t_s] \quad (4.34)$$

Note that (4.34) implies that

$$M(t) = M_s(t) = 0, \quad \forall t \in [0, t_s] \quad (4.35)$$

From (3.39), (3.41) and the definition of a Filippov solution, one has on $(0, t_s)$

$$\begin{pmatrix} \dot{E} \\ \dot{M} \\ \dot{Y} \\ \dot{F} \\ \dot{U} \\ \dot{M}_s \end{pmatrix} = \begin{pmatrix} \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E \\ (1 - \nu)\nu_E E - \delta_M M \\ \nu\nu_E E - \kappa(t)\Delta\eta Y - (\eta_2 + \delta_Y)Y \\ \eta_1 Y \kappa(t) - \delta_F F \\ \eta_2(1 - \kappa(t))Y - \delta_U U \\ \max(0, \hat{Y}g_1 + \hat{E}g_2) - \delta_s M_s \end{pmatrix} \quad (4.36)$$

$$\begin{pmatrix} \dot{\hat{E}} \\ \dot{\hat{M}} \\ \dot{\hat{Y}} \\ \dot{\hat{F}} \\ \dot{\hat{U}} \\ \dot{\hat{M}}_s \end{pmatrix} = \begin{pmatrix} \beta_E \hat{F} - (\nu_E + \delta_E)\hat{E} \\ (1 - \nu)\nu_E \hat{E} - \delta_M \hat{M} \\ \nu\nu_E \hat{E} - \kappa(t)\Delta\eta \hat{Y} - (\eta_2 + \delta_Y)\hat{Y} \\ \eta_1 \hat{Y} \kappa(t) - \delta_F \hat{F} \\ \eta_2(1 - \kappa(t))\hat{Y} - \delta_U \hat{U} \\ \max(0, \hat{Y}g_1 + \hat{E}g_2) - \delta_s \hat{M}_s \end{pmatrix} - LC\hat{X}, \quad (4.37)$$

with

$$\kappa(t) \in [0, 1], g_1(t) \in \frac{\phi}{\alpha}[0, \theta + 1] \text{ and } g_2(t) \in (1 - \nu)\nu_E\theta[0, 3]. \quad (4.38)$$

From (4.35) and the second line of (4.36), one has

$$E(t) = 0, \forall t \in [0, t_s] \quad (4.39)$$

From the first line of (4.36) and (4.39), we get

$$F(t) = 0, \forall t \in [0, t_s]. \quad (4.40)$$

In the case where $Y(0) = 0$, from the third line of (4.36) and (4.39), one has

$$Y(t) = 0, \forall t \in [0, t_s]. \quad (4.41)$$

To summarize, from (4.35), the fifth line of (4.36), (4.39), (4.40), and (4.41)

$$E(t) = M(t) = Y(t) = F(t) = M_s(t) = 0 \text{ and } \dot{U}(t) = -\delta_U U(t), \forall t \in [0, t_s], \quad (4.42)$$

which, with (3.13), (3.16), and (3.23), gives

$$\dot{W}(t) = -\sigma\delta_U U(t) \leq -\delta_U W(t), \forall t \in [0, t_s]. \quad (4.43)$$

In the case where $Y(0) > 0$. Then, from the third line of (4.36),

$$Y(t) > 0, \forall t \in [0, t_s], \quad (4.44)$$

which, together with the fourth line of (4.36) and (4.40), implies

$$\kappa(t) = 0, \forall t \in [0, t_s]. \quad (4.45)$$

Referring to this case already studied in the proof of Theorem 3.1 we get

$$\dot{W}(t) \leq -c'W(t), \forall t \in [0, t_s]. \quad (4.46)$$

$$\kappa(t) \in [0, 1], g_1(t) \in \frac{\phi}{\alpha}[0, \theta + 1] \text{ and } g_2(t) \in (1 - \nu)\nu_E\theta[0, 3], \quad (4.47)$$

$$\dot{M}_s(t) = \max(0, \hat{Y}g_1 + \hat{E}g_2) - \delta_s M_s \quad (4.48)$$

Since $M_s(t) = 0 \forall t \in [0, t_s]$, $\max(0, \hat{Y}g_1 + \hat{E}g_2) = 0$. For all $\kappa(t) \in [0, 1]$, in these two cases, the dynamics of the observation error remains

$$\dot{e} = (\mathcal{A}(\kappa(t)) - LC)e, \quad (4.49)$$

and one has

$$\dot{H}(X, \hat{X}) = -c''W(X) - \frac{\varepsilon\gamma}{2}\lambda\|e\|_{\mathcal{P}}, \quad (4.50)$$

and

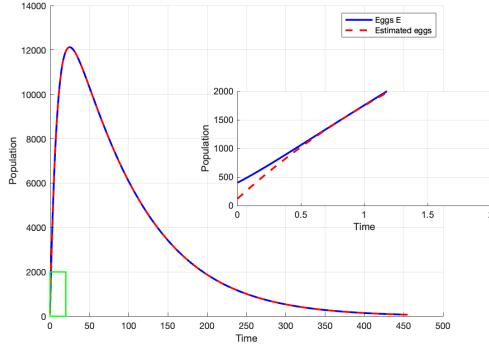
$$\dot{H}(X, \hat{X}) < -\lambda\xi\gamma H(X, \hat{X}). \quad (4.51)$$

□

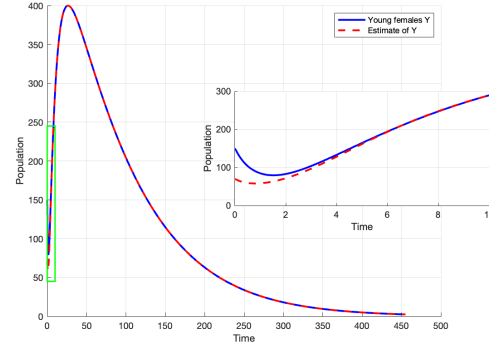
4.4 Numerical simulations

We apply the backstepping control u function of the measured states y and the estimated states \hat{E} and \hat{Y} given by the relation (4.22) in the following condition.

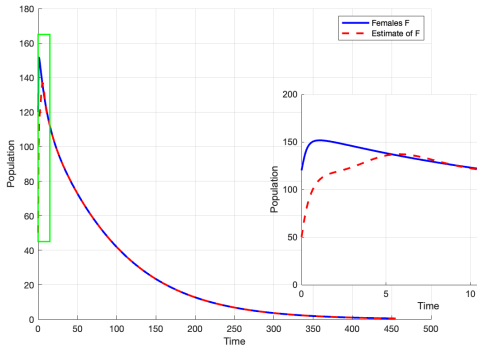
We take initially $x_0 = (20000, 5000, 1500, 12000, 500)$ and $\hat{x}_0 = (2000, 500, 150, 1200, 0)$.



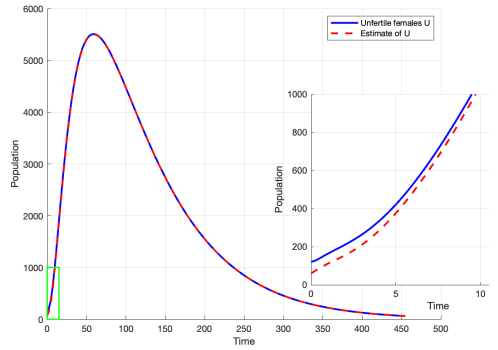
(a) Evolution of states E and estimate \hat{E}



(b) Evolution of states Y and estimate \hat{Y}



(c) Evolution of states F and estimate \hat{F}



(d) Evolution of states U and estimate \hat{U}

Figure 6: Simulation of the SIT model when applying backstepping feedback law of estimate and measure states (4.22) .

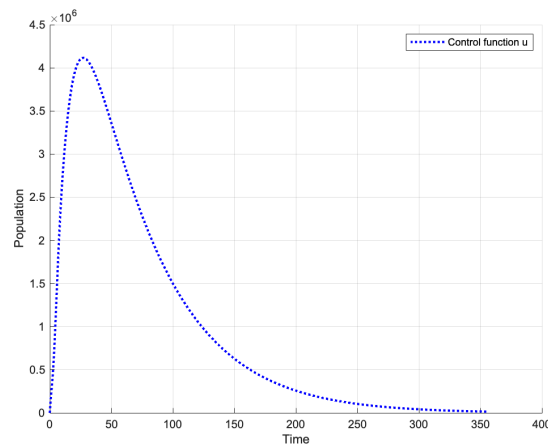


Figure 7: Evolution of control function $u(\hat{X}, y)$.

Conclusion

In this work, we have built a feedback control law to stabilize the SIT model presented in [5, 17] at extinction. Control by state feedback is a type of control rarely proposed in the literature for the overall stabilization of the SIT model. The advantage of this type of control is its robustness to changing dynamics parameters. We have shown in Remark 3.1 that despite the margin of error that can be made in the estimation of the dynamic parameters, this feedback control still make the system converge to extinction.

In section 4 of our work, we build an observer for the SIT model. Using the measurement of male mosquitoes, our state estimator gives us an estimate of the other states of the system. This aspect is rarely studied for this type of dynamics. An accurate estimate of the mosquito population enables resources to be allocated more efficiently. If intervention is effective in some areas but not in others, resources can be reallocated to maximize impact. On the other hand, the data collected during the SIT intervention provides essential information on the impact of the control in the conditions of the intervention area. This will enable informed decisions on future control strategies to be adopted according to conditions in the intervention zone by adding complementary methods or adapting existing approaches.

One of the applications we made was to show in section 4.3 that by using the data estimated via our observer to adjust the feedback control, we globally stabilize the system upon extinction. The Figure 6 shows that the difficulty of estimating eggs and young females during an intervention can be compensated by the application of the observer system. Data collected on the mosquito population is also used in epidemic prevention programs. They help to adapt public health programs for better control of mosquito-borne diseases.

Acknowledgements

The author wishes to thank Luis Almeida and Jean-Michel Coron for having drawn his attention to this problem and for the many enlightening discussions during this work

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