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# Linear realisability on untyped nets

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## ABSTRACT

We present a new realisability model based on orthogonality for Linear Logic in the context of nets – untyped proof structures with generalized axiom. We show that it adequately models second order multiplicative linear logic.

As usual, not all realizers are representations of a proof, but we identify specific types (sets of nets closed under bi-orthogonality) that capture exactly the proofs of a given sequent thus proving a completeness theorem. Furthermore these types are orthogonal’s of finite sets; this ensures the existence of a correctness criterion that runs in finite time.

## CCS CONCEPTS

• **Theory of computation** → **Linear logic**; **Proof theory**; *Program semantics*.

## KEYWORDS

Linear Logic, Realisability, Proof nets, Second order quantification

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## 1 INTRODUCTION

Realisability is a technique that extracts the computational content of proofs [12]. It was first introduced in 1945 by Kleene for Heyting Arithmetic – an Intuitionistic axiomatization of arithmetic – based on the codes of Gödel’s partial recursive functions [9]. Fixing an untyped computational model, the methodology of Realisability is based on two aspects:

- Types are given a computational status: the interpretation of a type<sup>1</sup>  $A$  is a set of programs  $\llbracket A \rrbracket$ , which behave similarly – its elements are called *realizers* of  $A$ . The interpretation process however is not unique and is parametrized by an

interpretation basis  $\mathcal{B}$ , thus a same formula  $A$  may be interpreted by different sets of programs  $\llbracket A \rrbracket_{\mathcal{B}}$  or  $\llbracket A \rrbracket_{\mathcal{B}'}$ .

- A co-compiler, a simple process transforming the proofs of the realized proof system into programs is defined, introducing a non trivial predicate on programs, namely, some programs represent a proof, the *correct* programs, while others do not, the *incorrect* programs.

For instance, whenever the computational model is a freely-generated language equipped with a binary relation capturing program execution, correct programs are well-formed terms.

Two properties are of interest for an interpretation basis  $\mathcal{B}$ : *Completeness* stating for any formula  $A$  any program in the type  $\llbracket A \rrbracket_{\mathcal{B}}$  is a proof of  $A$ . Hence completeness provides a procedure to assert whether a program is correct or not. *Adequacy* which ensure that the co-compiler process and the interpretations are coherent, stating that each proof of  $A$  is a realizer of  $A$ . Whenever both properties hold this means that the elements of the interpretation of a formula  $A$  are exactly its corresponding correct programs; the programs which are correctly typeable by  $A$ .

For most interpretations completeness does not hold; not all realizers are correct programs thus not all realizers represent a proof. In fact, it was revealed by realisability models based on *orthogonality* that the presence of incorrect programs is crucial to give a computational status to correctness.

This can be traced back to the introduction of *classical realisability* by Jean Louis Krivine [10] aiming at extending realisability techniques from intuitionistic to classical logic, proposing a model based on orthogonality. Krivine’s construction is based on an extension of the untyped lambda calculus, but, in order to capture a given context (stack) to potentially restore it later, the syntax is not only extended with the call/cc operator but also with a countably infinite set of *stack constants*. In that context the correct programs are those who do not contain stack constants. The orthogonality relation is understood as a notion of test: a program (here a  $\lambda$ -term) follows a specification if it passes (e.g. is orthogonal to) the adequate tests (here they are stacks). This is how orthogonality is used to define the interpretation of types (as the set of terms passing a given set of tests). From the point of view of orthogonality, the presence of "incorrect" terms is essential, as it introduces in the syntax semantic information [13] that can be used to *test* correct (and incorrect) terms.

In parallel with the work of Krivine, similar realisability constructions have been introduced by Jean-Yves Girard in order to interpret Linear Logic. While the orthogonality construction was clearly put forth in Ludics [7], the ideas and first occurrences can be traced back to the first model of geometry of interaction (GoI) [4], which is restricted to multiplicative linear logic, and interprets proofs as permutations. Furthermore, in a series of recent papers

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<sup>1</sup>Equivalently, having the Curry–Howard correspondence in mind,  $A$  is a formula.

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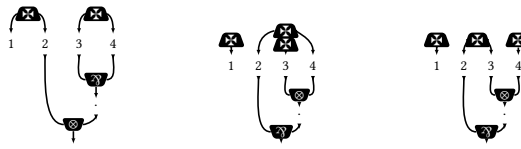
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[15–19], Thomas Seiller proposed a combinatorial approach to the Geometry of Interaction, *interaction graphs*, which specialises to all the previous ‘geometries’ of interaction proposed by Girard, and constructs the types of Linear Logic via a realisability method involving orthogonality within the computational model of interaction graphs. However, proofs are interpreted in these models as abstract objects (generalisations of dynamical systems) and remain far from the general intuition of what a proof is, this makes the co-compiler hard to reverse and so completeness results are lacking.

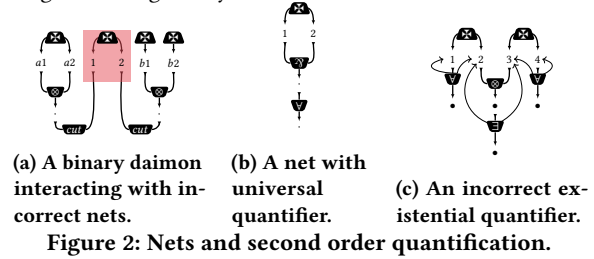
The goal of our work is to explore the question of completeness for realisability models of linear logic (LL). To do so, we work in the context of nets – essentially the proof structures introduced by Girard [5]. Since their introduction, proof structures have been known to represent proofs of LL (meaning a co-compiler is known), and several *correctness criteria*, e.g. procedures to assert whether a proof structure represents a proof, have been provided. The missing piece to perform realisability in this context is a way to interpret formulas as types; this will be based on orthogonality relying on the computational rules on proof structures, rewriting rules generalizing cut-elimination on proof trees. Furthermore this work will relate correctness criteria with completeness results, for instance in this paper the Danos Regnier criterion [3] applied to a would-be proof  $\pi$  yields a set of “tests” whose interaction with  $\pi$  by cut elimination will allow to determine whether or not  $\pi$  is indeed a proof.

It will be necessary to consider generalized axiom (daimon) rules; the correctness of a multiplicative net with binary daimons (which can be identified with axioms) can be tested by the Danos Regnier criterion, this criterion can be performed via orthogonality with some nets that we’ll call *tests* however these nets may have non binary daimons making it necessary to work in the  $\text{MLL}^{\boxtimes}$  fragment (Figure 3). See for instance below the net in figure 1a represents a proof in MLL while its two tests – the nets in figure 1b and 1c – represent proofs of  $\text{MLL}^{\boxtimes}$ . The orthogonality will consist of making the two nets interact by placing a cut between their output, if the resulting reduces to a net made of a single daimon link with no output the nets are orthogonal (essentially this means that no cycles or disconnections have appeared during cut elimination defined in definition 2.132.15). In particular in the figure 1 the orthogonality between the net and its tests will involve only cuts of the form  $(\boxtimes/\boxtimes)$  or  $(\otimes/\boxtimes)$  – later on defined as homogeneous cuts (definition 2.13). A folklore result which we prove in theo-



(a) An MLL proof-net. (b) First test for 1a. (c) Second test for 1a.  
**Figure 1: Daimons are needed to construct tests for MLL nets.** rem 5.7 ensures that any multiplicative test represents proofs of  $\text{MLL}^{\boxtimes}$ . However, to construct the interpretation of multiplicative second-order formulas the use of the incorrect terms is necessary. Realisability naturally provides interpretations for the universal (resp. existential) second order quantifiers as an intersection (resp. union) on the set of all types. For instance we want a binary daimon to behave like an axiom, it can introduce any pair of dual formulas

$A, A^\perp$ . Recall that our objects are however untyped therefore this means that the binary daimon should belong to any interpretation  $\llbracket A, A^\perp \rrbracket_{\mathcal{B}}$ , that is to be orthogonal to any pair of orthogonal nets. In particular orthogonality should hold with incorrect nets and this is necessary to obtain an adequate interpretation of MLL since the daimon behave well with any correct net. In the figure 2 below the binary daimon interacts with a net containing a cycle and a net containing a disconnection – these error will cancel each other making the orthogonality relation hold.



**Figure 2: Nets and second order quantification.**

The presence of incorrect nets enables adequacy for nets with second order quantifiers ( $\text{MLL}_2$ ) – with a dummy version of cut elimination. For instance the net in figure 2b realizes the formula  $\forall X(X, X^\perp)$  but it does not realise the formula  $(\forall XX), X^\perp$ .

Importantly we will identify interpretations basis  $\mathcal{B}$  such that for any formula  $\llbracket A \rrbracket_{\mathcal{B}}$  is the orthogonal of a finite set. This ensures that asserting if a net realizes  $A$  or not is decidable in finite time. Furthermore these basis can be complete and adequate and so this provides a way to test correctness in finite time, in other words a correctness criterion. This will be true for the realisability model of  $\text{MLL}_2$  hinting towards a correctness criterion for that fragment (this is what we explore in the last section). To handle second order in an untyped setting we define nets with pointers and define a contractibility criterion (Danos [2]) which generalizes the formulation provided by curien in [1] for multiplicatives. This new criterion somehow breaks down correctness for second order nets into three independent pieces (1) the existential witness in the net must be coherent (2) the net must be multiplicatively e.g. pass the tests of Danos Regnier (3) the net must be sequentializable in the sense of a contractibility reduction which read backward provides the reconstruction of a proof.

Specifically the first (1) step is required in an untyped setting: an existential quantifier isn’t allowed to abstract two distinct formulas. This is illustrated in figure 2c where two universal quantifiers are binding distinct formulas  $X$  and  $Y$  and the existential quantifier abstract both  $X$  and  $Y$  and creates a conclusion which would be typed by  $\exists Z.Z \otimes Z$ , we forbid this kind of scenario as we want an existential to substitute exactly one formula with a propositional variable, hence the conclusion of the existential quantifier should correspond to a formula  $\exists Z.Z \otimes Y$  or  $\exists Z.X \otimes Z$  (resulting in an incorrect net).

*Ouline of the paper.* In Section 2, we recall the general notion of hypergraph and we adapt it to our framework: a multiplicative net is a (particular) ordered hypergraph that can be understood as an untyped version of the notion of proof-structure ([5]). Indeed, it is possible to perform cut elimination on these nets, and the main novelty here is the presence of daimon links (denoted by  $\boxtimes$  and already present in the literature, see for example [7]) yielding the non homogeneous cut elimination steps  $(\otimes/\boxtimes)$  and  $(\boxtimes/\boxtimes)$ , this

last step being non deterministic (see remark 11). The intuition is that a  $\bowtie$ -link with  $n$  conclusions represents an agent whose logical correctness is not guaranteed, but which is capable to interact correctly with  $n$  other agents.

Section 3 introduces our realisability model, based on the interaction between nets (definition 3.1) and the consequent notion of orthogonality: two nets are orthogonal when their interaction yields a  $\bowtie$ -link without conclusions (definition 3.4). One can thus define the orthogonal of a set of nets and the crucial notion of type, a set of net equals to its bi-orthogonal (definition 3.5). The set of types is stable with respect to constructions allowing to interpret MLL formulas in the very spirit of realisability: a formula is interpreted as a set of nets with a similar behaviour (belonging to the same type interpreting the formula).

Adequacy for MLL and  $\text{MLL}^{\bowtie}$  follows in Section 4: a net coming from a proof of a formula  $A$  in MLL or  $\text{MLL}^{\bowtie}$  always belongs to the interpretation of  $A$  (theorem 4.3).

Let us say some words on our path to completeness of  $\text{MLL}^{\bowtie}$  and MLL proven in Section 5. We are naturally led to introduce the notion of test (definition 5.6) and to prove that a test of a formula  $A$  is a proof of  $A^\perp$  (theorem 5.7) which allows to characterize proofs as those nets with a good interaction with their tests (theorem 5.8), in a Danos-Regnier style. This last result and adequacy yield completeness for  $\text{MLL}^{\bowtie}$  (theorem 5.10). The MLL case (theorem 5.13) follows once the intuition that the conclusion of an MLL axiom is any pair of dual formulas is taken in consideration: the proof relies on the definition of the intersection and the union type (definition 5.11).

With Section 6 we smoothly move to the second order case: we introduce a first version of the quantifier links, that are here only unary links with no pointers. Cut elimination immediately extends to these new nets. We can thus define the interpretation of second order formulas as types (definition 6.1) and prove adequacy for this first version of second order nets (theorem 6.2): a proof of a formula  $A$  in  $\text{MLL}_2$  always belongs to the interpretation of  $A$ .

In Section 7 we introduce the notion of computable type (definition 7.1): the fundamental point is that one can check in finite time whether or not a given net belongs to such a type. We then notice that computability is preserved by the previously defined constructions on types (propositions 7.2 and 7.3), so that one can interpret  $\text{MLL}_2$  formulas with computable types. This paves the way to the possibility of defining a new correctness criterion for  $\text{MLL}_2$  (remark 28).

In Section 8 we present a new correctness criterion for untyped second order multiplicative linear logic, more precisely for  $\text{MLL}_2^{\bowtie}$ . To handle quantifiers (remaining untyped!) we add pointers to hypergraphs. There are pointers both for existential and universal quantifiers but with a different intended meaning: universal pointers indicate that the pointed location contains a formula which uses the quantified variable while the target of existential pointers are necessarily above the corresponding existential link and indicate positions of existential witnesses. The notion of pointed tree (definition 8.1) guarantees the uniformity of existential witnesses, a particularly delicate point in the absence of types: any two targets of the same existential link need to be coherent (definition 8.6). We then define a list of contraction rules (definition 8.7) allowing to characterize  $\text{MLL}_2^{\bowtie}$ -proofs (theorem 8.9).

## 2 UNTYPED NETS

### 2.1 Directed hypergraph

Throughout this document we fix a set  $\text{Pos}$  of elements called *positions*. Given a set  $X$  we will let  $\mathcal{P}_{\leq}(X)$  denote the set of totally ordered finite subsets of  $X$ .

*Definition 2.1.* Suppose given a set  $L$  of *labels*. A *directed ( $L$ -labelled) hypergraph* is a tuple  $(V, E, s, t, \ell)$  where  $V$  is a set of positions, and  $E$  is a set of elements called *links*,  $s : E \rightarrow \mathcal{P}_{\leq}(V)$  is the *source map*,  $t : E \rightarrow \mathcal{P}_{\leq}(V)$  is the *target map*, and  $\ell : E \rightarrow L$  is the *labelling map*.

Given a link  $e \in E$ , since the finite sets  $t(e)$  and  $s(e)$  are ordered we will represent them as sequences: they are respectively called the *target* and the *source* sets of  $e$ . A *source* of  $e$  is an element of its source set  $s(e)$ , a *target* of  $e$  is an element of its target set. The set of targets and sources of  $e$  is the *domain* of the link  $e$ . We will use superscripts to denote sequences of positions  $(\bar{p}, \bar{q}, \bar{u}, \dots)$ . A link is a *loop* when its target set and source set are not disjoint.

*Remark 1.* Along this work we assume all the hypergraphs to be loop-free e.g. containing only links which are not loops.

Given an hypergraph  $\mathcal{H}$  having  $E$  as set of links, we denote  $s(\mathcal{H})$  (resp.  $t(\mathcal{H})$ ) the set of all positions which are source (resp. target) of at least one link:

$$s(\mathcal{H}) = \bigcup_{e \in E} s(e), \quad t(\mathcal{H}) = \bigcup_{e \in E} t(e).$$

A *conclusion/output* (resp. a *premise/input*) of a directed hypergraph  $\mathcal{H}$  is a position which is the source of no link in  $\mathcal{H}$ , e.g. an element of  $V \setminus s(\mathcal{H})$  (resp. in  $V \setminus t(\mathcal{H})$ ). The set of conclusions (resp. premises) of an hypergraph  $\mathcal{H}$  is denoted  $\text{out}(\mathcal{H})$  (resp.  $\text{in}(\mathcal{H})$ ).

The *size* of a directed hypergraph is the number of its link. An hypergraph is *finite* when it contains a finite number of links and positions.

A position  $p$  is *isolated* in an hypergraph  $\mathcal{H}$  if  $p$  is both an output and an input of  $\mathcal{H}$ , e.g. it belongs to no link as an input nor as an output.

*Notation 1.* A link  $e$  will be written as  $\langle u \triangleright_l v \rangle$  where  $s(e) = u$ ,  $t(e) = v$ , and  $\ell(e) = l$ . With this notation, a directed labelled hypergraph with no isolated position can always be written as a formal sum

$$\sum_{e \in E} \langle s(e) \triangleright_{\ell(e)} t(e) \rangle.$$

*Notation 2.* We write  $u \cdot v$  the concatenation of sequences.

Given  $u = (u_1, \dots, u_n)$  a sequence of element in a set  $X$  and an integer  $i \in \{1, \dots, n\}$ , we denote by  $u_{<i}$  (resp.  $u_{>i}$ ) the sequence  $(u_1, \dots, u_{i-1})$  (resp.  $(u_{i+1}, \dots, u_n)$ ).

Moreover, given two – potentially empty – sequences  $u$  and  $v$  we denote by  $u[i \leftarrow v]$  the sequence  $u_{<i} \cdot v \cdot u_{>i}$ . We can generalize this notation as follow

$$u[i_1 \leftarrow v_1, \dots, i_n \leftarrow v_n] = (\dots (u[i_1 \leftarrow v_1]) \dots) [i_n \leftarrow v_n].$$

*Definition 2.2 (Initial and final links).* A link is *initial* (resp. *final*) when it has no input (resp. no output). A position is *initial* (resp. *final*) when it belongs to the domain of an initial (resp. final) link.

$$A, B := X, X^\perp \mid A \otimes B \mid A \wp B \mid \forall X A \mid \exists X A. \quad \frac{}{A, A^\perp} \text{ax} \quad \frac{}{\Gamma} \wp \quad \frac{\Gamma, A \quad \Delta, A^\perp}{\Gamma, \Delta} \text{cut} \quad \frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B} \otimes \quad \frac{\Gamma, A, B}{\Gamma, A \wp B} \wp \quad \frac{\Gamma, A, B, \Delta}{\Gamma, B, A, \Delta} \text{ex} \quad \frac{\Gamma, A}{\Gamma, \exists X A} \exists \quad \frac{\Gamma, A}{\Gamma, \forall X A} \forall$$

**Figure 3: The rules ( $\wp$ ,  $\otimes$ ,  $\wp$ , cut, ex) define the  $\text{MLL}^{\wp}$  proof system. Adding to  $\text{MLL}^{\wp}$  the quantifier rules ( $\forall$ ,  $\exists$ ) constitutes  $\text{MLL}_2^{\wp}$ . Substituting the daimon rule ( $\wp$ ) with the axiom rule (ax) in  $\text{MLL}^{\wp}$  (resp.  $\text{MLL}_2^{\wp}$ ) results in  $\text{MLL}$  (resp.  $\text{MLL}_2$ ).**

*Example 2.3.* For instance a link  $\langle \triangleright a, b, c \rangle$  is an initial link. And in the hypergraph  $\langle \triangleright a, b, c \rangle + \langle a \triangleright d \rangle + \langle \triangleright e \rangle + \langle e \triangleright \rangle$  the set of initial positions are  $\{a, b, c, e\}$ , while  $e$  is the only final position belonging to the final link  $\langle e \triangleright \rangle$ .

A link is *terminal* when its outputs are conclusions – thus a final link is a terminal link.

## 2.2 Sum of hypergraphs

Hypergraphs enjoy a natural notion of sum based on the disjoint union of the set of links.

*Notation 3.* Given a function  $f : X \rightarrow E$  and  $X_0 \subseteq X$  we denote the restriction of  $f$  to  $X_0$  by  $f \upharpoonright_{X_0}$  or simply  $f \upharpoonright X_0$ .

Given two disjoint sets  $X_0$  and  $X_1$  and two functions  $f : X_0 \rightarrow E$  and  $g : X_1 \rightarrow E$  we denote  $f \uplus g$  the function which takes an element  $x$  of  $X_0 \uplus X_1$  and returns  $f(x)$  if  $x \in X_0$  or  $g(x)$  if  $x \in X_1$ .

*Definition 2.4.* Given two hypergraphs  $\mathcal{H}_1 = (V_1, E_1, t_1, s_1)$  and  $\mathcal{H}_2 = (V_2, E_2, t_2, s_2)$ . The sum of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  is defined as:

$$\mathcal{H}_1 + \mathcal{H}_2 = (V_1 \cup V_2, E_1 \uplus E_2, t_1 \uplus t_2, s_1 \uplus s_2).$$

*Remark 2.* Whenever  $\mathcal{H}_1 = (V_1, E_1, t_1, s_1)$  and  $\mathcal{H}_2 = (V_2, E_2, t_2, s_2)$  are such that  $E_1 \cap E_2 \neq \emptyset$ , we will abusively write their sum as  $\mathcal{H}_1 + \mathcal{H}_2 = (V_1 \cup V_2, E_1 \uplus E_2, t_1 \uplus t_2, s_1 \uplus s_2)$ , using implicitly the isomorphism between  $E_1 \uplus E_2$  and  $E_1 \cup E_2$  which exists in this case, since up to renaming the set of links of two hypergraphs can always be considered to be disjoint.

*Remark 3.* Vertices may overlap in a sum (as we take the union of vertex sets rather than the disjoint union). As a consequence, a position may be input (or output) of several distinct links.

As mentioned above, we can therefore describe hypergraphs as sums of simple hypergraphs; namely those that contain only one link. Indeed, an hypergraph consisting of two links  $\langle a_1, \dots, a_i \triangleright_\ell b_1, \dots, b_j \rangle$  and  $\langle a'_1, \dots, a'_i \triangleright_{\ell'} b'_1, \dots, b'_j \rangle$  is in fact equal to the sum of the single-link hypergraphs  $\langle a_1, \dots, a_i \triangleright_\ell b_1, \dots, b_j \rangle$  and  $\langle a'_1, \dots, a'_i \triangleright_{\ell'} b'_1, \dots, b'_j \rangle$ . By induction on the number of links, this formally shows that  $\mathcal{H} = \sum_{e \in E} \langle s(e) \triangleright_{\ell(e)} t(e) \rangle$  when considering an hypergraphs with no isolated positions.

*Remark 4.* The sum of hypergraphs enjoys the properties of an abelian monoid; associativity, commutativity, and a neutral element which is the empty hypergraph.

We will also use extensively the notion of *parallel composition* or *parallel sum* of hypergraphs, an analogue of the *union-graph* of two simple graphs.

*Definition 2.5 (Parallel sum).* Given  $\mathcal{H}_1 = (V_1, E_1, t_1, s_1)$  and  $\mathcal{H}_2 = (V_2, E_2, t_2, s_2)$  two hypergraphs we define their *parallel sum* as:  $\mathcal{H}_1 \parallel \mathcal{H}_2 = (V_1 \uplus V_2, E_1 \uplus E_2, t_1 \uplus t_2, s_1 \uplus s_2)$ .

*Remark 5.* The parallel sum of two hypergraphs  $\mathcal{H}_1$  and  $\mathcal{H}_2$  corresponds to a regular sum whenever the sets of vertices are disjoint. Just like the sum, parallel composition can always be performed between two hypergraphs up to renaming their vertex sets.

## 2.3 Ordered Hypergraphs

An *arrangement* of a directed hypergraph  $\mathcal{H}$  is a total order  $<_{\mathbf{a}}$  on its set of conclusions, equivalently, the order may be identified as a bijection  $\mathbf{a} : \{1, \dots, \text{card}(\text{out}(\mathcal{H}))\} \rightarrow \text{out}(\mathcal{H})$ .

*Definition 2.6 (Ordered hypergraph).* An *ordered hypergraph* is a pair  $(\mathcal{H}, \mathbf{a})$  of an hypergraph together with an arrangement. The arrangement  $\mathbf{a}$  is denoted  $\mathbf{a}(\mathcal{H})$ .

*Notation 4.* Given a finite and ordered hypergraph  $(\mathcal{H}, \mathbf{a})$  with  $n$  conclusions for an integer  $1 \leq i \leq n$ , we denote  $\mathbf{a}(i)$  by  $\mathcal{H}(i)$  whenever there is no ambiguity.

*Notation 5.* A segment of  $\mathbb{N}$  is denoted  $[n; n+k]$  and corresponds to set of integers  $i$  such that  $n \leq i \leq n+k$ . For any integer  $k$  we denote  $\text{tr}_k : \mathbb{N} \rightarrow \mathbb{N}, n \mapsto n+k$ .

Given a partial function  $f : \mathbb{N} \rightarrow E$  with a finite domain of cardinality  $n$  and ordered as  $i_1 < i_2 < \dots < i_n$ , we denote  $f \parallel$  the total function of type  $[1; n] \rightarrow E$  such that  $f(m) = f(i_m)$  for any integer  $1 \leq m \leq n$ .

In particular given a restricted function  $f \upharpoonright X_0$  we denote  $f \upharpoonright X_0 \parallel$  by  $f \parallel X_0$ . Furthermore, we denote  $f \parallel X_0^c$  by  $f \parallel X_0$  meaning the function  $f$  to which we have removed the element of  $X_0$  from its domain and rearrange the domain of  $f$  to be a segment of  $\mathbb{N}$ .

Given  $f : [1; n] \rightarrow E$  and  $g : [1; m] \rightarrow E$  two functions defined on segments and with a common domain, we denote by  $f \uplus g$  their sum with translation, namely the disjoint functional sum of  $f \uplus (g \circ \text{tr}_{-n} \upharpoonright [n+1; n+m])$ . The sum with translation is not a commutative operation.

*Definition 2.7.* Two hypergraphs  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are *summable* whenever their target sets and source sets are disjoint, e.g.

$$t(\mathcal{H}_1) \cap t(\mathcal{H}_2) = \emptyset \quad \text{and} \quad s(\mathcal{H}_1) \cap s(\mathcal{H}_2) = \emptyset$$

*Remark 6.* A position  $p$  can occur in two summable hypergraphs only if  $p$  is a target in  $\mathcal{H}_1$  and a source in  $\mathcal{H}_2$  – or the converse – or whenever  $p$  is isolated in (at least) one of the two hypergraphs.

Since the conclusions of an hypergraphs with no isolated points are targets, summable hypergraphs have arrangements  $\mathbf{a}_1$  and  $\mathbf{a}_2$  with disjoint codomains. As a consequence, their disjoint sum function  $\mathbf{a}_1 \uplus \mathbf{a}_2$  remains a bijection.

*Definition 2.8 (Arrangement of the sum of ordered hypergraphs).* Given two summable ordered hypergraphs  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . The arrangement  $\mathbf{a}(\mathcal{H}_1 + \mathcal{H}_2)$  of there sum is;

$$\mathbf{a}(\mathcal{H}_1 + \mathcal{H}_2) = \mathbf{a}(\mathcal{H}_1) \parallel s(\mathcal{H}_2) \uplus \mathbf{a}(\mathcal{H}_2) \parallel s(\mathcal{H}_1)$$

*Remark 7.* Given two summable hypergraphs since  $s(\mathcal{H}_1)$  and  $s(\mathcal{H}_2)$  are disjoint the arrangement  $\mathbf{a}(\mathcal{H}_1 + \mathcal{H}_2)$  is equal to

$$\mathbf{a}(\mathcal{H}_1) \uplus \mathbf{a}(\mathcal{H}_2) \parallel s(\mathcal{H}_1) \cup s(\mathcal{H}_2).$$

*Remark 8.* The ordered sum of ordered hypergraphs  $\mathcal{H}_1$  and  $\mathcal{H}_2$  given as  $(\mathcal{H}_1 + \mathcal{H}_2, \mathbf{a}(\mathcal{H}_1) + \mathbf{a}(\mathcal{H}_2))$  is – unlike the standard sum of hypergraphs – *not commutative*.

However an hypergraph with no conclusions commutes with any other ordered hypergraph under the ordered sum. Furthermore

internal sub-hypergraphs commute for the ordered sum, an hypergraph  $\mathcal{H}_0 \subset \mathcal{H}_1 + \mathcal{H}_2$  included in a sum and such that none of the conclusion of  $\mathcal{H}_0$  is a conclusion of  $\mathcal{H}_1 + \mathcal{H}_2$  commutes with any link in the sum  $\mathcal{H}_1 + \mathcal{H}_2$ . More precisely, in an hypergraph  $\mathcal{H}$  given as a sum  $\sum_{e \in E} \langle se \triangleright_\ell te \rangle$ , each link  $e$  commutes with any link of  $E$  if  $e$  is not terminal or  $e$  has no conclusion.

## 2.4 Properties of hypergraphs

*Definition 2.9.* We say a hypergraph  $\mathcal{H} = (V, E, t, s)$  is:

- *surjective* whenever  $t(\mathcal{H}) = V$ .
- *source-disjoint* if the sets  $s(e)$  for  $e \in E$  are pairwise disjoint.
- *target-disjoint* if the sets  $t(e)$  for  $e \in E$  are pairwise disjoint.

If  $\mathcal{H}$  is both surjective and target-disjoint, we will say it is full. In this case, the sets  $t(e)$  for  $e \in E$  define a partition of  $V$ .

Up to this point we have allowed any kind of link to occur in an hypergraph. We now restrict the discussion to untyped multiplicative nets in which will only considered specific sets of links:

- *Generalized axioms:* links of the form  $e = \langle \triangleright p_1, p_2 \dots, p_n \rangle$ ;
- *Cuts:* links of the form  $e = \langle p_1, p_2 \triangleright \rangle$ .
- *Connectives:* links of the form  $e = \langle p_1, p_2 \dots, p_n \triangleright a \rangle$ .

In particular, we will restrict in the next section to binary connectives only, with the additional restriction that those binary connectives should be labelled either by  $\otimes$  or  $\wp$ . As a consequence, the hypergraphs considered will closely resemble multiplicative linear logic proof nets, with two important points of divergence: the absence of typing, and the presence of generalised axioms.

Note that these restrictions leave a number of questions open for future investigations. For instance, one can wonder how a generalised cut of the form  $\langle a, b, c \triangleright \rangle$  would behave.

## 2.5 Multiplicative nets

We now fix the set of labels as the set made of the *daimon* ( $\boxtimes$ ) the tensor ( $\otimes$ ) the parr ( $\wp$ ) and the (cut) symbol.

*Definition 2.10.* A *multiplicative module* is an ordered hypergraph  $(\mathcal{H}, a)$  where  $\mathcal{H}$  is source-disjoint and target-disjoint, and such that  $\boxtimes$ -labelled links have no inputs, cut-labelled links have exactly two inputs and no outputs,  $\otimes$ - and  $\wp$ -labelled links have exactly two inputs and one output.

In other words,  $\mathcal{H}$  is a sum of links of the form:

$$\langle \triangleright_{\boxtimes} p_1, \dots, p_n \rangle, \langle p_1, p_2 \triangleright_{\otimes} p \rangle, \langle p_1, p_2 \triangleright_{\wp} p \rangle, \langle p_1, p_2 \triangleright_{\text{cut}} \rangle.$$

Whenever there is no ambiguity we might relax the notation:  $\langle \triangleright p_1, \dots, p_n \rangle$  will denote  $\langle \triangleright_{\boxtimes} p_1, \dots, p_n \rangle$ , and  $\langle p_1, p_2 \triangleright \rangle$  will denote  $\langle p_1, p_2 \triangleright_{\text{cut}} \rangle$ .

*Definition 2.11.* A *multiplicative net* is a surjective multiplicative module.

*Notation 6.* We will denote  $\text{Mod}$  the set of multiplicative modules and  $\text{Net}$  the set of multiplicative nets.

*Notation 7.* Given an integer  $n$  we denote by  $\boxtimes_n$  any multiplicative net consisting of a single daimon link with  $n$  outputs, e.g. isomorphic<sup>2</sup> to  $\langle \triangleright_{\boxtimes} p_1, \dots, p_n \rangle$ .

<sup>2</sup>The notion of isomorphism between hypergraphs is recalled in the appendix, see definition B.1.

## 2.6 Homogeneous cut elimination

The set of untyped nets we have defined comes with its very own notion of computation, which corresponds to some graph rewriting. The computation is called the *cut elimination* and to describe it roughly consists in removing the cut links. How to remove a cut link will depend on what this link is attached to, this defines the *type* of the cut, we give it a clear definition in the following.

*Definition 2.12 (Types of cut).* Given a multiplicative net  $S$  the *type* of cut link  $c = \langle p, q \triangleright \rangle$  occurring in  $S$  is the pair of labels of the links of output  $p$  and  $q$ . Thus there are 6 *types* of cuts (up to symmetry). More precisely, we distinguish:

- *multiplicative* cuts, of type  $(\otimes/\wp)$ ;
- *clash* cuts, of type  $(\otimes/\otimes)$  or  $(\wp/\wp)$ ;
- *glueing* cuts, of type  $(\boxtimes/\boxtimes)$ ;
- *non-homogeneous* cuts, of type  $(\otimes/\boxtimes)$  or  $(\wp/\boxtimes)$ , they are respectively called *reversible* and *irreversible* cuts.

*Remark 9.* A cut to which we can assign a type forms a *redex*, whenever a redex occurs in an hypergraph  $\mathcal{H}$  the links which define the redex commute with any link contained in  $\mathcal{H}$  under the ordered sum – as in remark 8– this allows us to define cut-elimination as a local rewriting of redexes.

*Definition 2.13 (homogeneous cut elimination).* The relation of *homogeneous cut elimination* on multiplicative nets is the rewriting relation defined as the contextual closure of the following:

$$\langle \triangleright_{\boxtimes} \bar{p} \rangle + \langle \triangleright_{\boxtimes} \bar{q} \rangle + \langle p_i, q_j \triangleright \rangle \rightarrow \langle \triangleright_{\boxtimes} \bar{p}[i \leftarrow \epsilon] \cdot \bar{q}[j \leftarrow \epsilon] \rangle$$

$$\langle p_1, p_2 \triangleright_{\otimes} p \rangle + \langle q_1, q_2 \triangleright_{\wp} q \rangle + \langle p, q \triangleright \rangle \rightarrow \langle p_1, q_1 \triangleright \rangle + \langle p_2, q_2 \triangleright \rangle$$

*Remark 10.* The commutation of a cut link  $\langle p, q \triangleright_{\text{cut}} \rangle$  is the link  $\langle q, p \triangleright_{\text{cut}} \rangle$ . The input of a cut link  $\langle p, q \triangleright_{\text{cut}} \rangle$  are ordered, however this plays no role during cut elimination, the redex of a cut link  $\langle p, q \triangleright_{\text{cut}} \rangle$  or  $\langle q, p \triangleright_{\text{cut}} \rangle$  are the same upto the commutation of cutlinks.

The following result is easily established, in particular using that the number of links strictly decreases during homogeneous cut elimination.

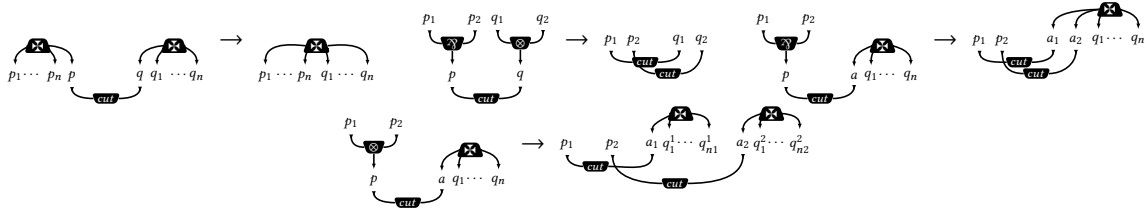
**PROPOSITION 2.14.** *Homogeneous cut elimination is confluent and strongly normalizing.*

## 2.7 Non homogeneous cut elimination

*Definition 2.15 (Non-homogeneous cut elimination).* The non homogeneous reduction is denoted  $\rightarrow_{nh}$ , and defined as the extension of the homogeneous cut-elimination with the following rules:

- A redex  $\langle \triangleright_{\boxtimes} \bar{p} \rangle + \langle p_i, q \triangleright_{\text{cut}} \rangle + \langle q_1, q_2 \triangleright_{\wp} q \rangle$  reduces to  $\langle \triangleright_{\boxtimes} \bar{u}, r_1 \rangle + \langle \triangleright_{\boxtimes} \bar{v}, r_2 \rangle + \langle r_1, q_1 \triangleright_{\text{cut}} \rangle + \langle r_2, q_2 \triangleright_{\text{cut}} \rangle$ , where  $\bar{u}, \bar{v}$  are such that  $\bar{u} \cdot \bar{v}$  is a reordering of  $\bar{p}[i \leftarrow \epsilon]$ , and  $r_1, r_2$  are fresh positions.
- A redex  $\langle \triangleright_{\boxtimes} \bar{p} \rangle + \langle p_i, q \triangleright_{\text{cut}} \rangle + \langle p_1, p_2 \triangleright_{\otimes} p \rangle$  reduces to  $\langle \triangleright_{\boxtimes} u \rangle + \langle r_1, q_1 \triangleright_{\text{cut}} \rangle + \langle r_2, q \triangleright_{\text{cut}} \rangle$ , where  $r_1, r_2$  are fresh positions and  $u = \bar{p}_{<i} \cdot r_1 \cdot r_2 \cdot \bar{p}_{>i}$ .

*Remark 11.* From the interactive viewpoint daimon links behave like proof nets ([5]) and they are able to interact with any net. As a consequence, the non homogeneous cut-eliminations simulates proof search in the sequent calculus.



**Figure 4: Rules for the homogeneous cut elimination and non homogeneous cut-elimination.**

The non-homogeneous cut elimination of a daimon against a  $\mathfrak{Y}$ -link is non deterministic,  $\{q_1^1, \dots, q_{n1}^1\}, \{q_1^2, \dots, q_{n2}^2\}$  is a partition of  $\{q_1, \dots, q_n\}$ . On the left hand side of each reduction rule one can find the different types of cuts of definition 2.12.

$$\frac{}{\Gamma, A^\perp \otimes B^\perp} \mathfrak{X} \rightarrow \frac{}{\Delta_1, A^\perp} \mathfrak{X} \otimes \frac{}{\Delta_2, B^\perp} \mathfrak{X} \quad \frac{}{\Gamma, A^\perp \mathfrak{Y} B^\perp} \mathfrak{X} \rightarrow \frac{}{\Gamma, A^\perp, B^\perp} \mathfrak{X} \mathfrak{Y}$$

This is where the non determinism comes from, in particular the rule eliminating the  $(\mathfrak{X}/\mathfrak{Y})$  cuts corresponds to proof search on a formula of the form  $A \otimes B$ , going from bottom to top the  $\otimes$ -introduction rules splits the context  $\Gamma$ , which isn't a deterministic process. Simultaneously, some of the different possible reductions will, play the role of the switchings used in correctness criteria [3]. These intuitions will be important in understanding further results.

A consequence of non determinism is the loss of confluence for cut-elimination; since splitting the context is irreversible, see for instance the reduction of the net in figure 2.

### 3 REALISABILITY MODEL

#### 3.1 Interaction of nets

*Definition 3.1 (Interaction between nets).* The *interface* of two nets  $S$  and  $R$  is an injective and functional relation on  $\text{out}(S) \times \text{out}(R)$ . An interface is *total* whenever it is defined for all conclusions of  $S$ .

An interface between two nets  $(S, p_1 < \dots < p_n)$  and  $(R, q_1 < \dots < q_m)$ , is *regular* whenever it contains only pair of the form  $(p_i, q_i)$ . The *identity interface* is the relation, denoted  $\text{id}_n$ , defined when  $n = m$  by  $\text{id}_n = \{(p_i, q_i) \mid 1 \leq i \leq n\}$ . The *projective interface* of size  $k \leq n$  is the regular interface between  $S$  and  $R$  that is defined exactly for the elements  $\{p_1, \dots, p_k\}$  we denote it  $\text{pr}_k$ .

The *interaction* of the nets  $S$  and  $T$  along the interface  $\sigma$ , is denoted  $S ::_\sigma T$  and corresponds to

$$S ::_\sigma T \triangleq S + T + \sum_{(p,q) \in \sigma} \langle p, q \triangleright_{\text{cut}} \rangle.$$

*Notation 8.* For the purpose of readability given two nets  $S$  and  $T$  with respective conclusions  $p_1, \dots, p_n$  and  $q_1, \dots, q_k$ , say  $m = \min(n, k)$ , we will denote the interaction  $S ::_{\text{pr}_m} T$  by  $S :: T$ , whenever there is no ambiguity.

For instance if  $S$  and  $T$  have the same number of conclusion,  $S :: T$  denotes  $S ::_{\text{id}} T$ .

*Notation 9.* Given  $\sigma \subseteq \text{out}(S) \times \text{out}(T + R)$  an interface between a net  $S$  and a sum of nets  $T \parallel R$ . The notation  $\sigma \mid T$  will denote the interface which relates conclusion of  $S$  to conclusion of  $T$  given as  $\sigma \cap \text{out}(S) \times \text{out}(T)$ . Obviously this also defines  $\sigma \mid R$ .

**PROPOSITION 3.2 (INTERACTION AS AN ACTION).** *Given  $S, T$  and  $R$  three multiplicative nets, and  $\sigma$  an interface between  $S$  and  $T \parallel R$ .  $S ::_\sigma (T \parallel R)$  is equal to  $(S ::_{\sigma \mid T} T) ::_{\sigma \mid R} R$ .*

*Definition 3.3.* A net  $S$  fails cut elimination when it has no reduction to  $\langle \triangleright_{\mathfrak{X}} \rangle$ .

*Remark 12.* Failure of cut elimination is preserved by anti-reduction. Moreover, clashing cuts are preserved during cut elimination, thus a net containing such a cut fails cut elimination.

#### 3.2 Orthogonality

*Definition 3.4 (Orthogonality).* Two nets  $S_1$  and  $S_2$  are *orthogonal* if there exists an interface  $\sigma$  such that  $S_1 ::_\sigma S_2 \rightarrow^* \mathfrak{X}_0$ .<sup>3</sup> In that case we write  $S_1 \perp S_2$ .

*Remark 13.* Since cuts are asymmetric, namely  $\langle p, q \triangleright_{\text{cut}} \rangle$  and  $\langle q, p \triangleright_{\text{cut}} \rangle$  are distinct the interaction  $S :: T$  and  $T :: S$  are not the same nets. However this has no consequence on cut elimination, because the reduction steps are defined up to commutation of the input of a cut link thus  $S :: T$  reduces to  $\mathfrak{X}_0$  if and only if  $T :: S$  does. As expected the relation of orthogonality is symmetric.

*Remark 14.* Since cut elimination preserves the conclusions of a net, two nets may only be orthogonal when they have the same number of conclusion. Furthermore if two nets are orthogonal, there exists a reordering of the conclusions of  $S_1$  such that the orthogonality is witnessed by the identity interface.

*Definition 3.5 (Type).* Given a set  $A$  of multiplicative nets, we define the *orthogonal* of  $A$  as  $A^\perp = \{P \mid \forall R \in A, P \perp R\}$ . A *type*  $\mathbf{A}$  is a set of multiplicative nets such that  $\mathbf{A}^{\perp\perp} = \mathbf{A}$ .<sup>4</sup>

*Remark 15.* Two nets which are orthogonal have the same number of output. As a consequence two orthogonal types which are non empty, must contain nets with the same number of outputs.

*Notation 10.* We will denote by  $\#S$  the number of outputs of  $S$ . For a type  $\mathbf{A}$  we let  $\#\mathbf{A}$  denote the number of outputs of the nets in  $\mathbf{A}$ .

*Remark 16.* There cannot be simultaneously a net with a terminal  $\mathfrak{Y}$  (or  $\otimes$ ) link in its  $i$ -th position in  $\mathbf{A}$  and  $\mathbf{A}^\perp$ : their interaction contains a clashing cut and by remark 12 it would fail cut elimination, meaning the nets cannot be orthogonal.

**PROPOSITION 3.6 (SYMMETRY OF TYPES).** *For any type  $\mathbf{A}$  with one conclusion; (1) there exists a net  $S$  with a terminal  $\mathfrak{Y}$ -link that belongs to  $\mathbf{A}$  or  $\mathbf{A}^\perp$ , and (2) there exists a net  $S$  with a terminal  $\otimes$ -link that belongs to  $\mathbf{A}$  or  $\mathbf{A}^\perp$ .*

<sup>3</sup>Note that, we require the *existence* of such a reduction, not all reductions need to behave this way.

<sup>4</sup>Equivalently, a type is a set  $\mathbf{A}$  such that  $\mathbf{A} = B^\perp$  for some set  $B$ , see, for instance, [8].

### 3.3 Construction on Types

*Definition 3.7 (Constructions on types).* Given two sets of nets  $\mathbf{A}$  and  $\mathbf{B}$  their *functional composition* denoted  $\mathbf{A} \succ \mathbf{B}$ , and their *parallel composition* denoted  $\mathbf{A} \parallel \mathbf{B}$  are defined as follows:

$$\begin{aligned} \mathbf{A} \succ \mathbf{B} &\triangleq \{S \mid \text{for any } \bar{a} \in \mathbf{A}^\perp, S :: \bar{a} \in \mathbf{B} \text{ and } \#S \geq \#\mathbf{A}\}, \\ \mathbf{A} \parallel \mathbf{B} &\triangleq \{a \parallel b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp}. \end{aligned}$$

*Remark 17.* The side condition that  $\#S \geq \#\mathbf{A}$  is to avoid ambiguous situation where potentially  $\mathbf{A}$  contains a net  $T$  which has more conclusions than  $S$ . In that case in  $S :: T$  the remaining conclusions are conclusions of  $T$  hence for some other net  $U$  the interaction  $(S :: T) :: U$  isn't  $S :: (T \parallel U)$  but  $T :: (S \parallel U)$  (upto commutation of the input of cuts between  $S$  and  $T$ ). With this condition the interaction  $(S :: T) :: U$  when  $T \in \mathbf{A}^\perp$  and  $U \in \mathbf{B}^\perp$  is the net  $S :: (T \parallel U)$ .

**PROPOSITION 3.8 (DUALITY).** *Given two types  $\mathbf{A}$  and  $\mathbf{B}$ :  $(\mathbf{A} \parallel \mathbf{B})^\perp = \mathbf{A}^\perp \succ \mathbf{B}^\perp$  and  $(\mathbf{A} \succ \mathbf{B})^\perp = \mathbf{A}^\perp \parallel \mathbf{B}^\perp$ .*

*Remark 18.* The duality of the constructions – proposition 3.8 – ensures that the set of types is closed under the  $\parallel$  and  $\succ$  operations. Furthermore the set of types is also closed under any intersection however it isn't closed under union, this will influence the definition of intersection and union type 5.11.

**PROPOSITION 3.9 (ASSOCIATIVITY).** *Given  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  three types we have the following commutations:*

- $(\mathbf{A} \parallel \mathbf{B}) \parallel \mathbf{C}$  is equal to  $\mathbf{A} \parallel (\mathbf{B} \parallel \mathbf{C})$ .
- $(\mathbf{A} \succ \mathbf{B}) \succ \mathbf{C}$  is equal to  $\mathbf{A} \succ (\mathbf{B} \succ \mathbf{C})$ .

**PROOF.** It is clear that the parallel construction is associative. Hence its dual construction, the composition, is associative too.  $\square$

*Definition 3.10 (Sequential constructions on types).* Given  $\mathbf{A}$  and  $\mathbf{B}$  two types with one conclusion, we define their *tensor product* (denoted  $\otimes$ ) and their *compositional product* (denoted  $\wp$ ):

$$\begin{aligned} \mathbf{A} \otimes \mathbf{B} &\triangleq \{S + \langle S(1), S(2) \triangleright_\otimes p \rangle \mid S \in \mathbf{A} \parallel \mathbf{B}\}^{\perp\perp}, \\ \mathbf{A} \wp \mathbf{B} &\triangleq \{S + \langle S(1), S(2) \triangleright_\wp p \rangle \mid S \in \mathbf{A} \succ \mathbf{B}\}^{\perp\perp}, \end{aligned}$$

where  $p$  denotes a fresh position.

**PROPOSITION 3.11 (DUALITY).** *Given  $\mathbf{A}$  and  $\mathbf{B}$  two types with one conclusion,  $(\mathbf{A} \otimes \mathbf{B})^\perp = \mathbf{A}^\perp \wp \mathbf{B}^\perp$  and  $(\mathbf{A} \wp \mathbf{B})^\perp = \mathbf{A}^\perp \otimes \mathbf{B}^\perp$ .*

### 3.4 Interpretation of formulas

We interpret formulas and hypersequent of multiplicative and second order multiplicative linear logic. We will respectively denote MLL and MLL<sub>2</sub> the multiplicative and second order multiplicative fragments of Linear Logic. On the other MLL<sup>⊗</sup> will denote the multiplicative fragment with generalized axiom, as found in [1].

The definition of formulas and proofs of MLL<sub>2</sub> (and MLL) is standard and can be found in [11]. The notion of hypersequent we use is similar to the tree-hypersequents used in some modal logics, for instance in [14]. Namely our hypersequent are constructed by a simple induction; they are either a formula  $A$ , or  $\mathcal{H}_1, \mathcal{H}_2$ , or  $\mathcal{H}_1 \parallel \mathcal{H}_2$  where  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are hypersequents.

*Definition 3.12 (Interpretation Basis).* An *interpretation basis*  $\mathcal{B}$  is a function that associate with each atomic proposition  $X$  a type  $\llbracket X \rrbracket_{\mathcal{B}}$ , the *interpretation* of  $X$ , such that:

- Each net in  $\llbracket X \rrbracket_{\mathcal{B}}$  has one conclusion.
- For any atomic proposition  $X$ , we have  $\llbracket X^\perp \rrbracket_{\mathcal{B}} \subseteq \llbracket X \rrbracket_{\mathcal{B}}^\perp$ .

*Definition 3.13 (Interpretation of an hyper-sequent).* Given an interpretation basis  $\mathcal{B}$ , the *interpretation* of a formula is lifted from atomic formulas to any formula of MLL and any hypersequent of MLL formulas by induction;

$$\begin{aligned} \llbracket \mathbf{A} \otimes \mathbf{B} \rrbracket_{\mathcal{B}} &\triangleq \llbracket \mathbf{A} \rrbracket_{\mathcal{B}} \otimes \llbracket \mathbf{B} \rrbracket_{\mathcal{B}}, & \llbracket \mathcal{H}_1, \mathcal{H}_2 \rrbracket_{\mathcal{B}} &\triangleq \llbracket \mathcal{H}_1 \rrbracket_{\mathcal{B}} \succ \llbracket \mathcal{H}_2 \rrbracket_{\mathcal{B}}, \\ \llbracket \mathbf{A} \wp \mathbf{B} \rrbracket_{\mathcal{B}} &\triangleq \llbracket \mathbf{A} \rrbracket_{\mathcal{B}} \wp \llbracket \mathbf{B} \rrbracket_{\mathcal{B}}, & \llbracket \mathcal{H}_1 \parallel \mathcal{H}_2 \rrbracket_{\mathcal{B}} &\triangleq \llbracket \mathcal{H}_1 \rrbracket_{\mathcal{B}} \parallel \llbracket \mathcal{H}_2 \rrbracket_{\mathcal{B}}. \end{aligned}$$

*Definition 3.14 (Realizer).* A multiplicative net *realizes* – with respect to an interpretation basis  $\mathcal{B}$  – an hypersequent  $\mathcal{H}$  of MLL formulas whenever it belongs to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}$ .

**Notation 11.** Given a hypersequent  $\mathcal{H}$ ,  $S \Vdash_{\mathcal{B}} \mathcal{H}$  denotes the assertion  $S \in \llbracket \mathcal{H} \rrbracket_{\mathcal{B}}$ . Whenever there is no ambiguity the notation  $S \Vdash_{\mathcal{B}} \mathcal{H}$  (resp.  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}$ ) may be relaxed to  $S \Vdash \mathcal{H}$  (resp.  $\llbracket \mathcal{H} \rrbracket$ ).

## 4 ADEQUACY

As we mentioned in the introduction realisability models comes with a simple process transforming sequent calculus proofs into programs (in our case nets). In the case of Proof structures (e.g. nets) that process is well-known. Proof structures have been introduced by Jean Yves Girard in his seminal paper [5] and provide a natural way to represent the proofs of linear logic up to "irrelevant" permutations of their deduction rules. The nets which represent a proof, called *proof nets*, or *correct nets*, are inductively defined on the size of the sequent calculus proof they represent. For second order multiplicative linear logic the definition can for instance be found in [5, 6] and easily adapts to our framework.

**Notation 12.** Let  $\mathfrak{P}$  denote one of the proof system under consideration, namely MLL, MLL<sup>⊗</sup> or later on MLL<sub>2</sub>, MLL<sub>2</sub><sup>⊗</sup>. Let  $S$  be a multiplicative net, we denote  $S \vdash_{\mathfrak{P}} \Gamma$  whenever there exists a proof  $\pi$  in  $\mathfrak{P}$  such that  $S$  is the representation of  $\pi$  in Net.

Furthermore we let  $\{\Gamma : \mathfrak{P}\}$  denote the set of all the nets  $S \vdash_{\mathfrak{P}} \Gamma$ .

*Definition 4.1 (Approximable type and basis).* A type with one conclusion is *approximable* whenever it contains the net  $\mathfrak{X}_1$ . Let  $X$  be a propositional variable, an interpretation basis is; (1) *standard* in  $X$  whenever  $\llbracket X \rrbracket_{\mathcal{B}}^\perp = \llbracket X^\perp \rrbracket_{\mathcal{B}}$ ; (2) *approximable* in  $X$  whenever both  $\llbracket X \rrbracket_{\mathcal{B}}$  and  $\llbracket X^\perp \rrbracket_{\mathcal{B}}$  are approximable.

An interpretation basis is: (1) *regular* whenever it is standard or approximable in any propositional variable; (2) *approximable* when it maps atomic formulas to approximable types.

*Remark 19.* A type  $\mathbf{A}$  is approximable if and only if  $\{\mathfrak{X}_1\}^{\perp\perp} \subseteq \mathbf{A}$  which is also equivalent to  $\mathbf{A}^\perp \subseteq \{\mathfrak{X}_1\}^\perp$ .

*Remark 20.* An approximable interpretation basis is regular.

**PROPOSITION 4.2.** *For any types with one output  $\mathbf{A} \subseteq \mathbf{B}^\perp$ ;  $\mathfrak{X}_2$  belongs to  $\mathbf{A}^\perp \succ \mathbf{B}^\perp$ .*

**THEOREM 4.3.** *Let  $S$  be a multiplicative net,  $\mathcal{B}$  an interpretation basis, and  $\Gamma$  a sequent:*

- if  $\mathcal{B}$  is regular, then  $S \vdash_{\text{MLL}} \Gamma \Rightarrow S \Vdash_{\mathcal{B}} \Gamma$ ,
- if  $\mathcal{B}$  is approximable, then  $S \vdash_{\text{MLL}^\otimes} \Gamma \Rightarrow S \Vdash_{\mathcal{B}} \Gamma$ .



## 5 COMPLETENESS

### 5.1 Test of a formula

*Definition 5.1 (Support of a daimon).* For a daimon link  $d = \langle \triangleright_{\mathfrak{X}} p_1, \dots, p_n \rangle$  we define its *support* as  $s(d) = \{p_1, \dots, p_n\}$ . For a sum of daimon links  $D = d_1 + \dots + d_k$ , we define its support  $s(D) = \{s(d_1), \dots, s(d_k)\}$ .

*Definition 5.2 (switching).* The *switching* rewriting, denoted  $\rightarrow_{\mathfrak{X}}$ , is the non-deterministic rewriting given by

$$\begin{aligned} \langle \bar{p} \triangleright_l \bar{q}_1, p_1, \bar{q}_2 \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle &\rightarrow_{\mathfrak{X}}^l \langle \bar{p} \triangleright_l \bar{q}_1, p, \bar{q}_2 \rangle \\ \langle \bar{p} \triangleright_l \bar{q}_1, p_2, \bar{q}_2 \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle &\rightarrow_{\mathfrak{X}}^r \langle \bar{p} \triangleright_l \bar{q}_1, p, \bar{q}_2 \rangle \end{aligned}$$

The *detached position* of a switching rewriting is  $p_2$  (resp.  $p_1$ ) in the  $\rightarrow_{\mathfrak{X}}^l$  (resp.  $\rightarrow_{\mathfrak{X}}^r$ ) case of the rewriting. Each switching of a net  $S$  equipped with an order  $u = (p_1, \dots, p_n)$  on its conclusions, produces a net  $S'$  with conclusions ordered as  $u \cdot q$  where  $q$  is the detached position of the switching step.

A *switching* of a net  $S$  is a normal form of  $S$  with respect to the switching rewriting.

*Remark 21.* A normal form for the switching rewriting is a net which contains no  $\mathfrak{X}$ -link.

*Definition 5.3.* A position  $p$  is *above* a position  $q$  in a net  $S$  if there exists a directed path from  $p$  to  $q$ .

Given a position  $q$  we denote  $q \uparrow^i S$  the set of *initial* positions which are above  $q$  in  $S$ .

*Remark 22.* Given a net  $S$  with conclusions  $p_1, \dots, p_n$  the sets  $p_1 \uparrow^i S, \dots, p_n \uparrow^i S$  form a partition of the initial positions of  $S$ . We denote this partition  $\uparrow^i S$ .

*Notation 13.* Given a net  $S$  we write  $S^{\mathfrak{X}}$  the net made of its  $\mathfrak{X}$  links.

*Notation 14.* Given  $\Gamma = (A_1, \dots, A_n)$  the representant of a sequent we denote by  $\Gamma(i)$  the formula  $A_i$ . We denote a net  $S$  by  $S[p_1, \dots, p_n]$  whenever the conclusion of  $S$  are ordered as  $p_1, \dots, p_n$ .

*Definition 5.4 (Witness of a sequent).* A net  $S$  is a *witness* of a sequent  $\Gamma$ , denoted  $S \vdash \Gamma$ , when:

- If  $\Gamma$  is a sequent of  $n$  atomic formulas,  $S = \mathfrak{X}_n$ ;
- If  $S \vdash A, B, \Gamma$ , then  $S + \langle S(1), S(2) \triangleright_{\otimes} p \rangle \vdash A \otimes B, \Gamma$ .
- If  $S \vdash A, B, \Gamma$ , then  $S + \langle S(1), S(2) \triangleright_{\mathfrak{X}} p \rangle \vdash A \mathfrak{X} B, \Gamma$ .
- If  $S[\bar{p}_1, q_1, q_2, \bar{p}_2] \vdash \Gamma, A, B, \Delta$ , then  $S[\bar{p}_1, q_2, q_1, \bar{p}_2] \vdash \Gamma, B, A, \Delta$ .

*Remark 23.* A cut-free proof net  $S \vdash \Gamma$  is in particular a witness  $S \vdash \Delta$  of a sequent  $\Delta$  such that  $\theta\Delta = \Gamma$  for some substitution  $\theta$ . However a net  $S \vdash \Gamma$  may not be a proof net and could contain cycles or disconnections: the witness condition only provide information on multiplicative connectives constituting the net  $S$ . This information is crucial in order to apply the test of a formula  $A$ : orthogonality with the tests of  $A$  coincides with correctness for any nets  $S \vdash A$ .

*Definition 5.5 (Inversion).* Given a net  $S$  the *inversion* of  $S$  denoted  $\bar{S}$  corresponds to  $S$  in which the  $\mathfrak{X}$  and  $\otimes$  labels have been inverted.

*Remark 24.* Whenever  $S$  witnesses a sequent  $A_1, \dots, A_n$  then  $\bar{S}$  witnesses  $A_1^\perp, \dots, A_n^\perp$ .

*Definition 5.6 (Test of a formula).* A cut-free net  $T$  is a test of a formula  $A$  if (1)  $T$  is a witness of  $A^\perp$  and (2) there exists a switching  $\sigma\bar{T}$  of the net  $\bar{T}$  such that  $\uparrow^i \sigma\bar{T} = \text{supp}(T^{\mathfrak{X}})$ .

We denote by  $\text{tests}(A)$  the set of tests of  $A$ .

**THEOREM 5.7 (CORRECTNESS OF TESTS).** *Any test of a formula  $A$  is a proof in  $\text{MLL}^{\mathfrak{X}}$  of the negation  $A^\perp$ .*

**PROOF SKETCH.** Consider a test of the formula  $A$ , meaning a net  $T$  which witness  $A^\perp$  and such that  $\text{supp}(T^{\mathfrak{X}}) = \uparrow^i \sigma\bar{T}$ . We will assume that  $T$  is not correct and show it implies that  $\mathfrak{X}_2$  isn't correct, which is absurd. To conclude we perform a series of transformation preserving correctness on  $\mathfrak{X}_2$  which results in  $T$ :

- (1) First, perform some  $\eta$ -expansions from  $\mathfrak{X}_2$  to obtain a witness of  $A, A^\perp$  that we denote  $W$ .
- (2) Perform the switching on the syntactical part  $A$  in  $W$  obtaining a net  $W'$  with  $n+1$  conclusions.
- (3) Make the  $n$  conclusions of the switching interact with  $\mathfrak{X}_1$ .
- (4) The switching of  $A$  is made only of tensor links thus the cut-elimination results in some daimons  $D$  which have for support the partition of the switching  $\text{supp}(D) = \uparrow^i \sigma A$ .
- (5) After eliminating the remaining glueing cuts the partitions goes above  $A^\perp$ , we obtain a net  $T$  witnessing  $A^\perp$  and such that  $\text{supp}(T^{\mathfrak{X}}) = \text{supp}(D) = \uparrow^i \sigma A$ .  $\square$

**THEOREM 5.8 (DANOS REGNIER TESTS, [3]).** *Given a cut-free net  $S \vdash A_1, \dots, A_n$ , the two following assertions are equivalent:*

- $S \vdash_{\text{MLL}^{\mathfrak{X}}} A_1, \dots, A_n$ ;
- $S$  is orthogonal to  $\text{tests}(A_1) \parallel \dots \parallel \text{tests}(A_n)$ .

### 5.2 Intersection Types and Completeness

*Definition 5.9 (self dual type).* A type  $A$  is *self-dual* whenever  $A \subseteq A^\perp$ . An interpretation basis is self-dual when it maps any propositional variable to a self dual type.

*Notation 15.* A substitution  $\theta : \mathcal{V} \rightarrow \mathcal{F}_{\text{MLL}}$  can be lifted to formulas by the induction  $\theta A \square B = \theta A \square \theta B$  where  $\square$  is a symbol  $\mathfrak{X}$  or  $\otimes$ . Furthermore substitutions can be lifted to hypersequent by  $\theta \mathcal{H}_1, \mathcal{H}_2 = \theta \mathcal{H}_1, \theta \mathcal{H}_2$  and  $\theta \mathcal{H}_1 \parallel \mathcal{H}_2 = \theta \mathcal{H}_1 \parallel \theta \mathcal{H}_2$ .

Given two hypersequents we denote  $\Delta \leq \Gamma$  whenever there exists a substitution  $\theta$  such that  $\theta\Delta = \Gamma$ .

*Remark 25.* Given a net  $S$  since daimon rules may introduce any sequent, whenever  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Delta$  and  $\Delta \leq \Gamma$  it follows that  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Gamma$ .

**THEOREM 5.10 (MLL<sup>ℳ</sup> COMPLETENESS).** *Let  $S$  be a cut-free net and  $\Gamma$  be a sequent. For any self dual and approximable interpretation basis  $\mathcal{B}$ , if  $S$  belongs to  $\llbracket \Gamma \rrbracket_{\mathcal{B}}$  then  $S$  represents a proof of  $\Gamma$  in  $\text{MLL}^{\mathfrak{X}}$ .*

**PROOF SKETCH.** Let  $\Gamma = A_1, \dots, A_n$  be a sequent. Consider a  $S$  a cut free net in  $\llbracket \Gamma \rrbracket_{\mathcal{B}}$ , since the basis is self-dual the syntactical part of  $S$  is borned by  $\Gamma$ , hence  $S \vdash \Delta$  such that  $\theta\Delta \leq \Gamma$  for some substitution  $\theta$ .  $S$  belonging to the interpretation of  $\Gamma$  equivalently means:

$$S \perp \llbracket A_1^\perp \rrbracket \parallel \dots \parallel \llbracket A_n^\perp \rrbracket.$$

By adequacy for any index  $i$ ,  $\llbracket A_i^\perp : \text{MLL}^{\mathfrak{X}} \rrbracket \subseteq \llbracket A_i^\perp \rrbracket$ . Furthermore  $\Delta \leq \Gamma$  hence  $\Delta = B_1, \dots, B_n$  such that  $B_i \leq A_i$  for each  $i$  and thus  $B_i^\perp \leq A_i^\perp$ . As a consequence  $\llbracket B_i^\perp : \text{MLL}^{\mathfrak{X}} \rrbracket \subseteq \llbracket A_i^\perp : \text{MLL}^{\mathfrak{X}} \rrbracket$ . Finally from theorem 5.7 we have  $\text{tests}(B_i) \subseteq \llbracket B_i^\perp : \text{MLL}^{\mathfrak{X}} \rrbracket$ . Hence we conclude

$$S \perp \text{tests}(B_1^\perp) \parallel \dots \parallel \text{tests}(B_n^\perp).$$

Since  $S \sim \Delta$  by theorem 5.8 we conclude  $S \vdash_{\text{MLL}^\times} \Delta$  and so using remark 25,  $S \vdash_{\text{MLL}^\times} \Gamma$ .  $\square$

*Notation 16.* Given an interpretation basis  $\mathcal{B}$  and a type  $\mathbf{A}$  we let  $\mathcal{B}\{X \mapsto \mathbf{A}\}$  denote the basis which maps  $X$  to  $\mathbf{A}$ ,  $X^\perp$  to  $\mathbf{A}^\perp$  and any propositional  $Y$  that isn't  $X$  or its dual to  $\llbracket Y \rrbracket_{\mathcal{B}}$ .

*Definition 5.11 (Intersection and union type).* Let  $\mathcal{B}$  be an interpretation basis, and  $\Omega$  a set of types with one output. Given  $\Gamma$  a sequent of MLL formulas and  $X$  a propositional variable the *intersection type* and *union type* on  $\Omega$  of  $\Gamma$  in  $X$  w.r.t. to  $\mathcal{B}$  are defined as follow:

$$\llbracket \bigcap_{X \in \Omega} \Gamma \rrbracket_{\mathcal{B}} \triangleq \bigcap_{R \in \Omega} \llbracket \Gamma \rrbracket_{\mathcal{B}\{X \mapsto R\}}, \quad \llbracket \bigcup_{X \in \Omega} \Gamma \rrbracket_{\mathcal{B}} \triangleq \left( \bigcup_{R \in \Omega} \llbracket \Gamma \rrbracket_{\mathcal{B}\{X \mapsto R\}} \right)^{\perp\perp}$$

*Remark 26.* We can naturally consider the intersection and unions on types that are not build by sequential operations, for instance on parallel composition of types or on functional composition of types. For example the following type;

$$\bigcap_{R \in \Omega} \llbracket A_1 \rrbracket_{\mathcal{B}\{X \mapsto R\}} \parallel \dots \parallel \llbracket A_n \rrbracket_{\mathcal{B}\{X \mapsto R\}}.$$

*Definition 5.12.* A net is proof like whenever its daimon links are only of the form  $\langle \triangleright_{\times} p_1, p_2 \rangle$ , e.g isomorphic to  $\times_2$ .

**THEOREM 5.13.** *Let  $S$  be a cut-free and proof-like net,  $\Gamma$  be a sequent and  $\Omega$  denote the set of all types with one output. For any self dual and approximable interpretation basis  $\mathcal{B}$ , if  $S$  belongs to  $\bigcap_{X \in \mathcal{V}} \llbracket \bigcap_{X \in \Omega} \Gamma \rrbracket_{\mathcal{B}}$  then  $S$  represents a proof of  $\Gamma$  in MLL.*

## 6 ADEQUACY AT (PRE-)SECOND ORDER

We now present how our approach can be extended to second order linear logic  $\text{MLL}_2$ , and provide an adequate model for  $\text{MLL}_2$ . The hypergraphs are now build with the multiplicative links of definition 2.10 to which we add two new kind of unary connective links, namely  $\langle p \triangleright_{\vee} q \rangle$  and  $\langle p \triangleright_{\exists} q \rangle$  which respectively corresponds to universal and existential quantification. In that context, the modules and nets are hypergraphs build from this wider set of links which follow the same specification as for multiplicative nets (see the definitions 2.10 and 2.11).

Furthermore we need to extend cut elimination in the presence of these new links, for now we give a dummy version of cut elimination similar to the elimination in the multiplicative case, namely the cuts  $(\forall/\exists)$  and  $(\exists/\times)$  and  $(\forall/\times)$  can be eliminated as follow, while the other kind of cut are treated as clashing cuts;

- $\langle p \triangleright_{\vee} p' \rangle + \langle p', q' \triangleright_{\text{cut}} \rangle + \langle q \triangleright_{\exists} q' \rangle \rightarrow \langle p, q \triangleright_{\text{cut}} \rangle$ .
- $\langle p \triangleright_{\vee} p' \rangle + \langle p', q \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\times} q \rangle \rightarrow \langle p, q \triangleright_{\text{cut}} \rangle$ .
- $\langle p \triangleright_{\exists} p' \rangle + \langle p', q \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\times} q \rangle \rightarrow \langle p, q \triangleright_{\text{cut}} \rangle$ .

*Definition 6.1 (realizers of  $\text{MLL}_2$ ).* Let  $\mathcal{B}$  be an interpretation basis. The set of *realizers* of a formula  $A$  of  $\text{MLL}_2$  is defined by adding the two following cases to definition 3.13 (the net  $S$  has one conclusion):

$$\llbracket \forall X A \rrbracket \triangleq \{S + \langle S(1) \triangleright_{\vee} q \rangle \mid S \in \llbracket \bigcap_{X \in \Omega} A \rrbracket\}^{\perp\perp}$$

$$\llbracket \exists X A \rrbracket \triangleq \{S + \langle S(1) \triangleright_{\exists} q \rangle \mid S \in \llbracket \bigcup_{X \in \Omega} A \rrbracket\}^{\perp\perp}$$

*Remark 27.* Proposition 4.2 ensures that  $\times_2$  belongs to  $\mathbf{A} > \mathbf{A}^\perp$  for any type  $\mathbf{A}$ . In particular for any interpretation basis  $\mathcal{B}$  it belongs

to the intersection  $\bigcap_{R \in \Omega} (\llbracket X \rrbracket_{\mathcal{B}\{X \mapsto R\}} > \llbracket X^\perp \rrbracket_{\mathcal{B}\{X \mapsto R\}})$  and is the only *cut free* net occurring in that intersection.

This implies that  $\langle \triangleright_{\times} p_1, p_2 \rangle + \langle p_1, p_2 \triangleright_{\times} p \rangle$  belongs to

$$\bigcap_{R \in \Omega} \left( \llbracket X \rrbracket_{\mathcal{B}\{X \mapsto R\}} \wp \llbracket X^\perp \rrbracket_{\mathcal{B}\{X \mapsto R\}} \right).$$

Hence, by definition this net extended with a universal quantifier link is an element of  $\llbracket \forall X, X \wp X^\perp \rrbracket_{\mathcal{B}}$ .

**THEOREM 6.2 (ADEQUACY FOR  $\text{MLL}_2$ ).** *Given  $S$  a proof in  $\text{MLL}_2$  of a sequent  $\Gamma$ . For any regular interpretation basis  $\mathcal{B}$ ,  $S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$ .*

## 7 COMPUTABILITY OF TYPES

This section shows that the adequate model we constructed enjoys a property of finiteness, namely each type is the orthogonal of a finite set. As mentionned in the introduction, this hints towards a correctness criterion for  $\text{MLL}_2$  which will be the object of the last section. Given a type  $\mathbf{A}$  and a type  $S$  how difficult is to assert that  $S$  belongs to  $\mathbf{A}$ . Orthogonality provides a natural tool to talk about the computability of such a problem. Given a sequent  $\Gamma = A_1, \dots, A_n$  we denote  $\bar{\Gamma}$  the sequent  $A_1^\perp, \dots, A_n^\perp$ .

*Definition 7.1.* A type  $\mathbf{A}$  is *computable* whenever there exists a finite set  $B$  such that  $\mathbf{A} = B^\perp$ . A type  $\mathbf{A}$  is *syntactically computable* if for each sequent  $\Gamma$  there exists a finite set  $A^\perp(\Gamma)$  of nets witnessing  $\bar{\Gamma}$  such that, for any net  $S$  witnessing  $\Gamma$ ;  $S \in \mathbf{A} \Leftrightarrow S \perp A^\perp(\Gamma)$ .

*Remark 28.* A computable type  $\mathbf{A} = B^\perp$  is a type such that the statement  $S \in \mathbf{A}$  can be decided interactively by testing the orthogonality with each element of the finite set  $B$ .

This notion is strongly related to the correctness criterion of proof nets [5]; given a sequent  $\Gamma$  and a proof system  $\wp$  the computability of the set  $\llbracket \Gamma : \wp \rrbracket = B^\perp$  provides a way to assert in *finite time*<sup>5</sup> if a net  $S$  is a proof net of  $\Gamma$  or not.<sup>6</sup>

**PROPOSITION 7.2.** *The functional composition  $\mathbf{A} > \mathbf{B}$  and the parallel composition  $\mathbf{A} \parallel \mathbf{B}$  of two syntactically computable types are syntactically computable.*

**PROPOSITION 7.3.** *The union  $(\bigcup_{i \in I} \mathbf{A}_i)^{\perp\perp}$  of a family of computable types is computable. The intersection  $\bigcap_{i \in I} \mathbf{A}_i$  of a finite family of computable types is computable.*

*Notation 17.* We write abusively  $\times$  the type  $\{\times_1\}^{\perp\perp}$  and define the following particular types:

$$\otimes_{\times} = \{\langle \triangleright_{\times} p_1, p_2 \rangle + \langle p_1, p_2 \triangleright_{\otimes} p \rangle\}^{\perp\perp}$$

$$\wp_{\times} = \{\langle \triangleright_{\times} p_1 \rangle + \langle \triangleright_{\times} p_2 \rangle + \langle p_1, p_2 \triangleright_{\wp} p \rangle\}^{\perp\perp}$$

**PROPOSITION 7.4.** *Let  $\omega$  be the set of types  $\{\otimes_{\times}, \wp_{\times}, \times, \times^\perp\}$  and  $\mathcal{B}$  be a basis which maps any  $X$  to  $\times$ . For any hyper-sequent  $\mathcal{H}$ :*

$$\llbracket \bigcap_{X \in \omega} \mathcal{H} \rrbracket_{\mathcal{B}} = \llbracket \bigcap_{X \in \Omega} \mathcal{H} \rrbracket_{\mathcal{B}}.$$

<sup>5</sup>However there can be a high complexity cost in order to decide  $B$ .

<sup>6</sup>In particular, in the case of  $\text{MLL}^\times$  using the theorem 5.8 we have shown that the nets which represent a cut-free proof of a formula  $F$  are the cut-free elements of a type  $\mathbf{A}$  which are orthogonal to a finite set tests  $(F)$ , the set of tests of  $F$ , see definition 5.6.

*Remark 29.* Since the set  $\omega$  is a finite set, the previous propositions implies that for an approximable and self dual basis, the intersection type  $\llbracket \bigcap_{X \in \Omega} \mathcal{H} \rrbracket_{\mathcal{B}}$  is computable.

*Remark 30.* The preservation of computability of the preconstructions  $\llbracket, \succ, \cap$  and  $\cup$  naturally extend to the preservation of computability for the construction on types, namely  $\exists, \otimes, \forall$  and  $\exists$ . In the case quantifiers this is because (1) a daimon link against a  $\exists$  link can match the reduction of a  $\forall$  link against  $\exists$  link, and vice versa, and (2) the dummy elimination rules introduced for the elimination of the quantifier links can always be performed first during cut elimination, e.g. they commute to the left during cut elimination.

## 8 SECOND ORDER NETS

Confirming the hint of the existence of correctness criterion for second order multiplicative linear logic nets provided by the finiteness of the union and intersection types, this section is dedicated to the definition of a novel correctness criterion for  $\text{MLL}_2^{\exists}$  nets, e.g. nets with exists-link and forall-links.

The criterion we present is based on the contractibility criterion for MLL (Danos [2]) as formulated in [1]. Our criterion is distinct from Girard's [6] for several reasons; (1) It is designed for untyped proof structures of  $\text{MLL}_2^{\exists}$ , (2) we are not bound to Girard's restriction of the criterion to proof structures having only closed formulas as conclusions. Furthermore, the contractibility criterion always reduces the size of the net and enjoys a quadratic complexity. As a consequence, increasing the number of dependencies of a universal quantifier do not increase the number of switchings (and so the time complexity) exponentially (which is the case in Girard's criterion).

In this section we introduce hypergraphs equipped with pointers, in order to handle second order quantification and to interpret proofs of  $\text{MLL}_2^{\exists}$ . The challenge in the definition is to add quantifiers while keeping an untyped setting. Pointers are used to reintroduce exactly the needed information to deal with quantifiers (which is usually implicitly given by types).

In previous sections, the vertices of an hypergraph belonged to a set positions  $\mathcal{P}$ . In order to handle pointers we enrich the set of positions as  $\mathcal{P} \cup \mathcal{P}^{\bullet}$  where the elements of  $\mathcal{P}^{\bullet}$  are called *pointer ports* or pointer positions. We will usually denote pointer ports as  $p, q, s, \dots$ . The elements of  $\mathcal{P} \cup \mathcal{P}^{\bullet}$  are called *positions*, while elements of  $\mathcal{P}$  will be called *regular positions*. A *pointer* or *pointer-link* is an link with one source which is a pointer port and one target which is a position, and the label  $\mathbf{p}$ . Moreover, we will distinguish between existential and universal pointers. We will denote a universal (resp. existential) pointer  $\langle s \triangleright_{\mathbf{p}} t \rangle$  by  $\langle s \xrightarrow{\forall} t \rangle$  (resp.  $\langle s \xrightarrow{\exists} t \rangle$ ). A *pointer structure* is an hypergraph made only of pointers. An hypergraph with pointers  $\mathbf{P}$  is an hypergraph  $\mathcal{H}$  summed with a pointer structure  $\mathbf{P}$ . A pointer *belongs* to an hypergraph  $\mathcal{H}$  when its source and target are positions of  $\mathcal{H}$ .

As for multiplicative nets, we define *modules* for second order multiplicative linear logic as sums of the following links:

$$\langle p, q \triangleright_{\text{cut}} \rangle \quad \langle \triangleright_{\exists} \bar{p} \rangle \quad \langle p_1, p_2 \triangleright_{\otimes} p \rangle \quad \langle p_1, p_2 \triangleright_{\exists} p \rangle \\ \langle p \triangleright_{\forall} q, s \rangle \quad \langle p \triangleright_{\exists} q, s \rangle \quad \langle s \xrightarrow{\forall} t \rangle \quad \langle s \xrightarrow{\exists} t \rangle.$$

Pointer positions are involved exclusively in existential, universal, and pointer links. We distinguish universal and existential

pointer links and the semantics of these pointers pointers will be distinct. A *pre-net* is a module  $\mathcal{H}$  with pointers  $\mathbf{P}$  such that:

- Existential pointers sources are pointer positions involved in existential links, and their target lies *above* their sources, i.e. there exists a directed path from the target to the source; moreover, given two existential links  $e_1$  and  $e_2$  with respective pointer positions  $s_1$  and  $s_2$  such that  $e_2$  is above  $e_1$  a pointer of source  $s_1$  cannot point on a position which is above the target of a pointer of source  $s_2$ ;
- Universal pointers sources are pointer positions involved in universal links, and their targets are either conclusions of  $\exists$  links or pointer positions of existential links;
- A given position is the target of at most one existential pointer link;
- $\mathcal{H}$  is target-disjoint, source-disjoint, and target-surjective.

*Remark 31.* In a second order net  $\mathcal{H} + \mathbf{P}$  only the underlying hypergraph  $\mathcal{H}$  need to be target and source disjoint, pointers have no such restriction in particular two pointers may have the same source such as  $\langle s \rightarrow t \rangle$  and  $\langle s \rightarrow t' \rangle$  while occurring in the same net. Similarly two distinct pointers may share their target, this may occur for instance when two distinct  $\forall$  links point to the same position, indicating it should contain a formula depending on the two corresponding quantified variables. Similarly, a  $\forall$  pointer link may share its target with an  $\exists$  pointer link, indicating that the existential witness in that position contains the universally quantified propositional variable.

### 8.1 Trees, pointers, and approximations

Pre-nets still lack an important property ensured by types: that all existential witnesses are the same. Without this property, we would allow an existential quantifier to point to two distinct witnesses with different principal connectives. We overcome the challenge of expressing this constraint without introducing types by defining a notion of *coherence* between positions in a pre-net. Intuitively coherent positions are positions which can host a unifying witness.

In order to formulate properly this constraint, we need to introduce the notion of *tree with pointers*. Along this section a tree is a tree with pointers.

*Definition 8.1.* Given a position  $p$ , *trees with pointers* of conclusion  $p$  or  $p$ -trees are inductively defined as follows:

- A unary link  $\langle \triangleright_{\exists} p \rangle$  is a  $p$ -tree;
- If  $\mathcal{S}$  is  $p$ -tree *disjoint* with  $\mathcal{S}'$  a  $p'$ -tree and  $q$  is a fresh position, then  $\mathcal{S} \cup \mathcal{S}' + \langle p, p' \triangleright_{\otimes} q \rangle$  and  $\mathcal{S} \cup \mathcal{S}' + \langle p, p' \triangleright_{\exists} q \rangle$  is a  $q$ -tree;
- If  $\mathcal{S}$  is a  $p$ -tree,  $q$  is a fresh regular position and  $s$  a fresh pointer position, then  $\mathcal{S} + \langle p \triangleright_{\forall} q, s \rangle$  and  $\mathcal{S} + \langle p \triangleright_{\exists} q, s \rangle$  is a  $q$ -tree.
- If  $\mathcal{S}$  is a  $p$ -tree,  $t$  is a regular position in  $\mathcal{S}$ , and  $s$  is a pointer position:
  - if  $s$  is conclusion of an  $\exists$  link, then  $\mathcal{S} + \langle s \xrightarrow{\exists} t \rangle$  is a  $p$ -tree;
  - if  $s$  is **not** conclusion of an  $\exists$  link (hence possibly a fresh pointer position) and  $p'$  is either the conclusion of a  $\exists$  link or the pointer position of an  $\exists$  link, then  $\mathcal{S} + \langle s \xrightarrow{\forall} t \rangle$  is a  $p$ -tree.

Given a pre-net  $\mathcal{H} + \mathbf{P}$  and a position  $p$ , we now define in a straightforward way *the tree (with pointers) above  $p$  in  $\mathcal{H} + \mathbf{P}$* . This will then be used to define the syntactic tree above the vertex, including the pointers of  $\forall$  links that may point to vertices in that syntactic tree.

*Definition 8.2.* Given a pre-net  $\mathcal{H} + \mathbf{P}$  and a position  $p$ , *the tree (with pointers) above  $p$  in  $\mathcal{H} + \mathbf{P}$* , written  $\mathcal{S}_{\mathcal{H}+\mathbf{P}}(p)$ , is defined as the largest  $p$ -tree  $\mathcal{S}$  that is contained in  $\mathcal{H} + \mathbf{P}$  when its daimon links are removed.

We now define a notion of *approximation* on trees. We first need to introduce useful notations that will simplify statements.

*Definition 8.3.* Given a pre-net  $\mathcal{H} + \mathbf{P}$  and a position  $p$ , we define  $P_{\forall}(\rightarrow p)$  (resp.  $P_{\exists}(\rightarrow p)$ ) as the set of all universal (resp. existential) pointer links in  $\mathbf{P}$  of target  $p$ . We similarly define  $P_{\forall}(p \rightarrow)$  (resp.  $P_{\exists}(p \rightarrow)$ ) as the set of all universal (resp. existential) pointer links in  $\mathbf{P}$  of source  $p$ .

*Notation 18.* Given  $p$  and  $q$  two positions we denote  $[p := q]$  the function which takes a position  $s$  and returns  $q$  if  $s = p$  and returns  $s$  otherwise. We adopt an infix notation  $s[p := q]$  to denote the image of  $s$  under  $[p := q]$ . We lift these functions to pointer links, given a link  $\langle s \rightarrow t \rangle$  defining  $\langle s \rightarrow t \rangle[p := q]$  as  $\langle s[p := q] \rightarrow t[p := q] \rangle$ . Furthermore we lift this notation to sets of pointers  $X$  so that  $X[p := q]$  denotes the images of the elements of  $X$  under  $[p := q]$ .

*Definition 8.4.* The approximation relation  $<$  on trees is defined as the transitive closure of the relation shown in Figure 5, where in the first two lines the positions  $p, q$  do not appear in left hand tree and in the last two lines the positions  $p, s$  do not appear in left hand tree. Note that when pointers are renamed, two pointers may become equal. In this case, only one link is kept.

A set of trees  $\mathcal{S}_1, \dots, \mathcal{S}_k$  is *coherent* when there exists a tree  $\mathcal{R}$ , a *witness*, such that  $\mathcal{S}_i < \mathcal{R}$  up to a renaming of positions.

We note that verifying that a set of trees  $\mathcal{S}_1, \dots, \mathcal{S}_n$  or respective conclusion  $p_1, \dots, p_n$  is coherent is not as costly as it could be thought at first sight. One can simply compare the tree from their conclusions while bookkeeping renamings of positions. Inductively, one performs the following steps:

- check if there exists a type of link such that all  $p_i$  are either targets of this type of link (type c) or target of a  $\forall$  link (type a) and deduce needed renamings (e.g. that  $p_1, \dots, p_n$  should be identified);
- check if the sets  $P_{\exists}(\rightarrow p_i)$  are all equal (up to renaming);
- check if the sets  $P_{\forall}(\rightarrow p_i)$  for  $p_i$  of type a are equal (up to renaming), and bookkeep one of them to check, when available, that it is equal, for each  $p_i$  of type c, to

$$\cup_{r' \text{ above } p_i} P_{\forall}(\rightarrow r') \setminus \cup_{r' \text{ above } p_i} P_{\forall}(r' \rightarrow).$$

The algorithm just described can furthermore be used to construct a witness, which in fact is minimal.

**LEMMA 8.5.** *Given a coherent set of trees  $\mathcal{S}_1, \dots, \mathcal{S}_k$ , there exists a minimal witness, noted  $\text{Wit}(\mathcal{S}_1, \dots, \mathcal{S}_k)$ .*

We are now ready to define nets as pre-nets for which  $\exists$  links possess a witness.

*Definition 8.6.* A pre-net  $\mathcal{H} + \mathbf{P}$  is a *net* if for all link  $\langle p \triangleright_{\exists} q, s \rangle$  in  $\mathcal{H}$ , the set of trees  $\mathcal{S} = \{\mathcal{S}_{\mathcal{H}+\mathbf{P}}(r) \mid r \in P_{\exists}(s \rightarrow)\}$  is coherent. The minimal witness  $\text{Wit}(\mathcal{S})$  is called the minimal witness of the link  $\langle p \triangleright_{\exists} q, s \rangle$ , and sometimes abusively written as  $\text{Wit}(\langle p \triangleright_{\exists} q, s \rangle)$ .

## 8.2 Contractibility criterion

The next step is to define the notion of *contraction* for second order nets which will be used to define the correctness criterion.

*Definition 8.7.* The contraction relation on second order nets is defined by the contextual closure of the following reduction steps (rules are shown in Figure 7 where we replaced sums by sets to ease readability, and illustrated in Figure 6):

- If  $P_{\forall}(\rightarrow p) \neq \emptyset$  and  $P_{\exists}(\rightarrow p) \neq \emptyset$ , we write  $\langle s \xrightarrow{\exists} p \rangle$  the unique element of  $P_{\exists}(\rightarrow p)$  and we can apply the pointers rule (Figure 7e);
- The tensor (Figure 7a) and parr rule (Figure 7b) can be applied without any side condition.
- If  $P_{\forall}(\rightarrow p)$  is empty, we can apply the existential rule (Figure 7c);
- If either  $P_{\forall}(s \rightarrow)$  is empty or contains only the pointer link  $\langle s \xrightarrow{\forall} p \rangle$ , we can apply the universal rule (Figure 7d).

Note that two pointers may become equal after applying a rule. In this case, only one link is kept.

Multiplicative rules for contractibility can be understood as Curien's contractibility rules for MLL [1], extended to handle pointers. This rewriting of pointers in the net is crucial: the existential pointers which were on premisses are destroyed (we lose the information of the existential witness), whereas the universal pointers are moved to the conclusion of the contracted link – keeping track of the dependence. The existential link contraction can always be performed, and removes any pointers targetting its pointer position: this is the only rule erasing universal pointers. The universal link contraction can only be performed when no pointers exists from this link to a position that is not above it. This intuitively corresponds to the usual condition on the context in the universal quantifier introduction rule. The last rule is the essential one: if a conclusion  $p$  of a daimon link is at the same time the target of an existential pointer and the target of (possibly multiple) universal pointers, we move the universal pointers to the pointer position of the existential link pointing to  $p$ . This records the dependence of the existential witness on the universal quantifiers.

*Definition 8.8.* A net  $\mathcal{H} + \mathbf{P}$  with  $n$  conclusions is contractible if there exists a sequence of contractibility reduction rules ending on  $\mathfrak{X}_n$ , i.e.  $\mathcal{S} \rightarrow_c^* \mathfrak{X}_n$ .

**Remark 32.** The contractibility rewriting preserves the number of conclusions of a net. During contraction steps, the number of links always decreases except in the case of Figure 7e in which the number of universal pointers pointing to a regular position decrease; this makes the rewriting strongly normalizing.

We are now ready to state the main theorem of this Section: the contractibility correctness criterion for second order nets.

**Remark 33.** Our way of dealing with existential quantifiers in nets forces witnesses to appear as *principal* formulas in the proof. Consequently, not all sequent calculus proofs are represented directly in

$$\begin{aligned}
1277 & \langle \triangleright_{\boxtimes} r \rangle + P_{\forall}(\rightarrow p)[p := r] + P_{\forall}(\rightarrow q)[q := r] < \langle \triangleright_{\boxtimes} p \rangle + P_{\forall}(\rightarrow p) + P_{\exists}(\rightarrow p) + \langle \triangleright_{\boxtimes} q \rangle + P_{\forall}(\rightarrow q) + P_{\exists}(\rightarrow q) + \langle p, q \triangleright_{\boxtimes} r \rangle \\
1278 & \langle \triangleright_{\boxtimes} r \rangle + P_{\forall}(\rightarrow p)[p := r] + P_{\forall}(\rightarrow q)[q := r] < \langle \triangleright_{\boxtimes} p \rangle + P_{\forall}(\rightarrow p) + P_{\exists}(\rightarrow p) + \langle \triangleright_{\boxtimes} q \rangle + P_{\forall}(\rightarrow q) + P_{\exists}(\rightarrow q) + \langle p, q \triangleright_{\otimes} r \rangle \\
1279 & \langle \triangleright_{\boxtimes} r \rangle + P_{\forall}(\rightarrow p)[p := r] < \langle \triangleright_{\boxtimes} p \rangle + P_{\forall}(\rightarrow p) + P_{\exists}(\rightarrow p) + \langle p \triangleright_{\supset} q, s \rangle + P_{\forall}(\rightarrow s) \\
1280 & \langle \triangleright_{\boxtimes} r \rangle + P_{\forall}(\rightarrow p)[p := r] < \langle \triangleright_{\boxtimes} p \rangle + P_{\forall}(\rightarrow p) + P_{\exists}(\rightarrow p) + \langle p \triangleright_{\vee} q, s \rangle + P_{\forall}(s \rightarrow) \\
1281 & \\
1282 & 
\end{aligned}$$

Figure 5: Approximation reduction steps

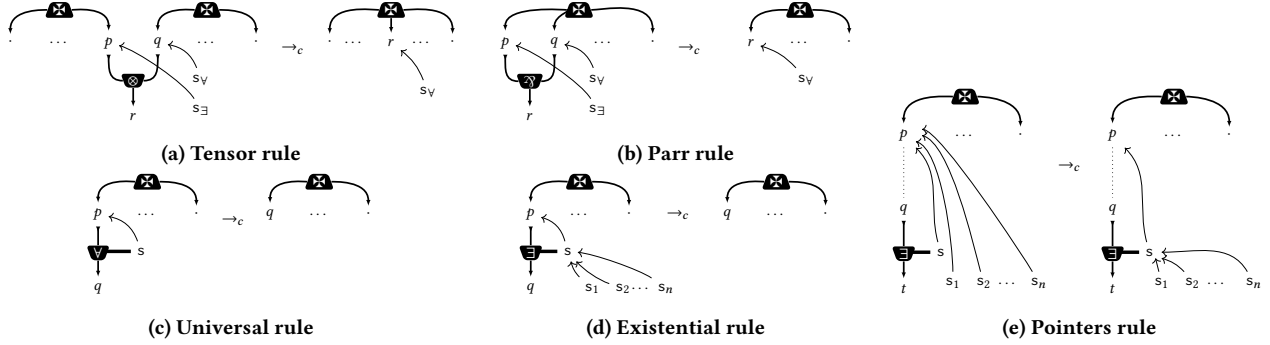


Figure 6: Contraction rules illustrated

$$\begin{aligned}
1296 & \left( \begin{array}{l} \langle \triangleright_{\boxtimes} p, \bar{p} \rangle \\ \langle \triangleright_{\boxtimes} q, \bar{q} \rangle \\ \langle p, q \triangleright_{\otimes} r \rangle \\ P_{\forall}(\rightarrow p) \\ P_{\exists}(\rightarrow p) \\ P_{\forall}(\rightarrow q) \\ P_{\exists}(\rightarrow q) \end{array} \right) \rightarrow_c \left( \begin{array}{l} \langle \triangleright_{\boxtimes} r, \bar{r}, \bar{q} \rangle \\ P_{\forall}(\rightarrow p)[p := r] \\ P_{\forall}(\rightarrow q)[q := r] \end{array} \right) \rightarrow_c \left( \begin{array}{l} \langle \triangleright_{\boxtimes} p, q, \bar{p} \rangle \\ \langle p, q \triangleright_{\supset} r \rangle \\ P_{\forall}(\rightarrow p) \\ P_{\exists}(\rightarrow p) \\ P_{\forall}(\rightarrow q) \\ P_{\exists}(\rightarrow q) \end{array} \right) \rightarrow_c \left( \begin{array}{l} \langle \triangleright_{\boxtimes} p, \bar{p} \rangle \\ \langle p \triangleright_{\supset} q, s \rangle \\ P_{\forall}(\rightarrow p) \\ P_{\exists}(\rightarrow p) \\ P_{\forall}(s \rightarrow) \end{array} \right) \rightarrow_c \langle \triangleright_{\boxtimes} q, \bar{p} \rangle \\
1297 & \left( \begin{array}{l} \langle \triangleright_{\boxtimes} p, \bar{p} \rangle \\ \langle p \triangleright_{\supset} q, s \rangle \\ P_{\forall}(\rightarrow p) \\ P_{\exists}(\rightarrow p) \\ P_{\forall}(s \rightarrow) \end{array} \right) \rightarrow_c \left( \begin{array}{l} \langle \triangleright_{\boxtimes} p, \bar{p} \rangle \\ P_{\forall}(\rightarrow p)[p := q] \setminus \{(s \rightarrow p)\} \end{array} \right) \rightarrow_c \left( \begin{array}{l} \langle \triangleright_{\boxtimes} p, \bar{p} \rangle \\ P_{\forall}(\rightarrow p) \\ P_{\exists}(\rightarrow p) \end{array} \right) \rightarrow_c \left( \begin{array}{l} \langle \triangleright_{\boxtimes} p, \bar{p} \rangle \\ P_{\forall}(\rightarrow p)[p := s] \\ P_{\exists}(\rightarrow p) \end{array} \right) \\
1298 & \text{(a) Tensor rule} \quad \text{(b) Parr rule} \quad \text{(c) Existential rule} \quad \text{(d) Universal rule} \quad \text{(e) Pointers rule} \\
1299 & \\
1300 & 
\end{aligned}$$

Figure 7: Contraction rules

our syntax. Proofs introducing existential quantifiers with a witness appearing only as a subformula (e.g. a subformula of an axiom's conclusion) need to be partially  $\eta$ -expanded for all witnesses to appear as principal formulas. With this mild requirement, we can show that second order nets represent second order sequent calculus.

**THEOREM 8.9 (CONTRACTIBILITY CRITERION).** *Given a net  $\mathcal{H} + \mathbf{P}$ , the following assertions are equivalent:*

- (1)  $\mathcal{H} + \mathbf{P}$  is contractible and its
- (2)  $\mathcal{H} + \mathbf{P}$  is sequentialisable: there exists a proof  $\pi$  in  $\text{MLL}_2^{\boxtimes}$  of a sequent  $\Gamma$  such that  $\mathcal{H} + \mathbf{P}$  represents  $\pi$ ,

**PROOF SKETCH.** **To show 1  $\Rightarrow$  2,** we start from a contractible net and consider the sequence of rules leading to the final  $\boxtimes$  rule. We then argue that each reduction rule can be followed in reverse: if the redex is sequentialisable, then the redex is. Formally, we show that a contractible net is the interpretation of *any sequent with the right structure*, where the structure describes constraints on the formulas: dependencies on some variables, syntactical structure.

The initial step is straightforward: a net  $\boxtimes_n$  is the interpretation of the proof of  $\vdash A_1, \dots, A_n$  using one daimon rule, for any set of formulas  $A_1, \dots, A_n$ .

Now, if a net  $R$  represent the proof of any sequent  $\vdash \Gamma$  with constraints, and  $R$  is obtained from  $R'$  through a tensor rule constructing a daimon of conclusions  $P, A_1, \dots, A_k, B_1, \dots, B_m$ , one can then impose the additional structural constraint  $P = A \otimes B$  imposing that  $P$  has a tensor as principal formula. The rule application furthermore allows to recover dependency constraints from the universal pointers: if  $P$  depended on the universal variables  $X_1, \dots, X_p$ , the rule can be used to redirect those dependencies to  $A$  and  $B$ . Lastly, the rule may have deleted an existential pointer: in

that case, we recover structural constraints as the pointed formula should be equal to the existential witness of the source.

A similar argument can be followed for the parr, universal, and existential rules. We get from the rule application a structural constraint on the formulas involved, as well as information from the pointers.

Lastly, the pointers rule allows to recover the universal dependencies of witnesses, leading to structural constraints; however the redex and the redex represent the same  $\text{MLL}_2$  proofs.

**To show 2  $\Rightarrow$  1,** we proceed by induction on the proof  $\pi$ , looking at its last rule. Whenever the last rule is daimon rule the implication directly follows. If the last rule is a  $\boxtimes$ - or  $\otimes$ -rule the induction hypothesis allows us to conclude as in [1].

In the following, we let given a proof  $\pi$  in  $\text{MLL}_2^{\boxtimes}$  we let  $\langle \pi \rangle$  denote a net which represents  $\pi$ . If the last rule is the introduction of  $\exists$ -connective then  $S$  is of the form  $\langle \langle \pi_0 \rangle \rangle + \langle p \triangleright_{\exists} q, s \rangle + P_{\exists}(s \rightarrow)$ , the new existential link points to the positions which corresponds to the abstracted formula by the quantifier. By induction  $\langle \langle \pi_0 \rangle \rangle \rightarrow_c \langle \triangleright_{\boxtimes} \bar{p}, p \rangle$  hence  $S \rightarrow \langle \triangleright_{\boxtimes} \bar{p}, p \rangle + \langle p \triangleright_{\exists} q, s \rangle + P_{\exists}(s \rightarrow)$  and the remaining pointers can only be in  $P_{\exists}(s \rightarrow)$ . Indeed we can perform a contraction step and reduce the net to  $\langle \triangleright_{\boxtimes} \bar{p}, q \rangle$  meaning that  $S$  is contractible.

If the last rule is a universal quantification,  $S = \langle \langle \pi_0 \rangle \rangle + \langle p \triangleright_{\forall} q, s \rangle + P_{\forall}(s \rightarrow)$  eventually the new link points to positions of  $\langle \langle \pi_0 \rangle \rangle$ , but since the quantified variable cannot appear in the context the pointers occur above a position which is pointed by an exists link. As a consequence performing the same contraction as for  $\langle \langle \pi_0 \rangle \rangle \rightarrow \langle \triangleright_{\boxtimes} \bar{p}, p \rangle$  together with some more rerouting contraction we obtain  $S \rightarrow \langle \triangleright_{\boxtimes} \bar{p}, p \rangle + \langle p \triangleright_{\forall} q \mid + \rangle P_{\forall}(s \rightarrow)$  and the remaining

1393 pointers may only point to  $p$ , hence the contraction rule be applied  
 1394 and  $S$  reduce to  $\langle \triangleright_{\mathfrak{X}} \bar{p}, q \rangle$  and so is contractible.  $\square$

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## A SECOND ORDER MULTIPLICATIVE LINEAR LOGIC

We recall the basic notions of formulas and proofs for second order multiplicative linear logic (denoted  $MLL_2$ ) – and thus also of its sub-fragment; multiplicative linear logic (denoted  $MLL$ ). We define the notion of hypersequent used for realizers. Furthermore we also give the denition of the proof system  $MLL^{\boxtimes}$  e.g. the multiplicative fragment with generalized axiom.

### A.1 Formulas

We assume that we are given a set of propositional variables  $\mathcal{V}$  together with an involution  $(\cdot)^\perp : \mathcal{V} \rightarrow \mathcal{V} : X \mapsto X^\perp$  on the propositional variables.

*Definition A.1 (formulas of  $MLL_2$ ).* The formulas of second order multiplicative linear logic are defined as follows:

$$A, B ::= X, X^\perp \mid A \otimes B \mid A \wp B \mid \forall X A \mid \exists X A$$

The involution  $(\cdot)^\perp$  is lifted to formulas using *de morgan laws*:

$$\begin{aligned} (A \otimes B)^\perp &\triangleq A^\perp \wp B^\perp & (\forall X A)^\perp &\triangleq \exists X A^\perp \\ (A \wp B)^\perp &\triangleq A^\perp \otimes B^\perp & (\exists X A)^\perp &\triangleq \forall X A^\perp \end{aligned}$$

*Remark 34.* The formulas of the multiplicative fragment of linear logic are the formulas of  $MLL_2$  which contain no quantifiers. In other words the formulas generated by the following induction;

$$A, B ::= X, X^\perp \mid A \otimes B \mid A \wp b$$

*Definition A.2 (Substitution).* The substitution of a propositional variable  $X$  by a formula  $F$  of  $MLL$  in a formula  $A$  of  $MLL_2$ , is denoted  $A[X \leftarrow F]$ , and is defined inductively:

$$\begin{aligned} X[X \leftarrow F] &\triangleq F \\ Y[X \leftarrow F] &\triangleq Y \\ (A \wp B)[X \leftarrow F] &\triangleq A[X \leftarrow F] \wp B[X \leftarrow F] \\ (A \otimes B)[X \leftarrow F] &\triangleq A[X \leftarrow F] \otimes B[X \leftarrow F] \\ (\forall X A)[X \leftarrow F] &\triangleq \forall X A \\ (\forall Y A)[X \leftarrow F] &\triangleq \forall Y A[X \leftarrow F] \\ (\exists X A)[X \leftarrow F] &\triangleq \exists X A \\ (\exists Y A)[X \leftarrow F] &\triangleq \exists Y A[X \leftarrow F] \end{aligned}$$

Where  $Y$  is a propositional variable that is distinct from  $X$ , i.e.  $X \neq Y$ .

**PROPOSITION A.3 (SUBSTITUTION AND NEGATION COMMUTE).** *Given  $A$  and  $F$  two formulas respectively from  $MLL_2$  and  $MLL$ , together with  $X$  a propositional variable.*

$$A[X \leftarrow F]^\perp = A^\perp[X \leftarrow F]$$

*Definition A.4.* The *universally (resp. existentially) bounded* variables of a  $MLL_2$  formula is given by induction;

$$\begin{aligned} U(X) &\triangleq \emptyset & E(X) &\triangleq \emptyset \\ U(A \otimes B) &\triangleq U(A) \cup U(B) & E(A \otimes B) &\triangleq E(A) \cup E(B) \\ U(A \wp B) &\triangleq U(A) \cup U(B) & E(A \wp B) &\triangleq E(A) \cup E(B) \\ U(\forall X A) &\triangleq U(A) \cup \{X\} & E(\forall X A) &\triangleq E(A) \\ U(\exists X A) &\triangleq U(A) & E(\exists X A) &\triangleq E(A) \cup \{X\} \end{aligned}$$

A propositional variable is *bounded* in a formula  $F$  whenever it's universally or existentially bounded in  $F$ .

*Definition A.5 ( $\alpha$ -equivalence).* Two formulas of  $\text{MLL}_2$  are  $\alpha$ -equivalent which we denote  $F \equiv_\alpha G$  if they have the same main connective and satisfy one of the inductive cases:

$$\begin{aligned} X \equiv_\alpha Y &\Leftrightarrow X = Y \text{ or } X = Y^\perp \\ A \wp B \equiv_\alpha A' \wp B' &\Leftrightarrow A \equiv_\alpha A' \text{ and } B \equiv_\alpha B' \\ A \otimes B \equiv_\alpha A' \otimes B' &\Leftrightarrow A \equiv_\alpha A' \text{ and } B \equiv_\alpha B' \\ \forall X A \equiv_\alpha \forall Y A' &\Leftrightarrow A \equiv_\alpha A' [Y \leftarrow X] \\ \exists X A \equiv_\alpha \exists Y A' &\Leftrightarrow A \equiv_\alpha A' [Y \leftarrow X] \end{aligned}$$

*Remark 35.* formulas of the second order multiplicative fragment of linear logic are considered modulo  $\alpha$ -equivalence, in order to avoid that a formula occurs both free and bounded in a same formula. Any formula  $F$  containing variables that are simultaneously bounded and free is  $\alpha$ -equivalent to a formula where variables cannot be both free and bounded. Intuitively this is because  $\alpha$ -equivalence allows to rename bounded variables occurring in  $F$  by any other propositional variables, thus we may choose a propositional variable that does not occur yet in  $F$ .

For instance in the formula  $F \triangleq X \wp \exists X.X^\perp$  the propositional variable  $X$  occurs both free and bounded but  $F$  is  $\alpha$ -equivalent to  $X \wp \exists Y.Y$ . Indeed  $\exists X.X^\perp$  and  $\exists Y.Y$  are  $\alpha$ -equivalent since  $X^\perp$  and  $Y[Y \leftarrow X] = X$  are by definition  $\alpha$ -equivalent.

*Remark 36.* By working modulo  $\alpha$ -equivalence we can assume that propositional variables cannot be simultaneously universally and existentially bounded. This is because we may rename any bounded propositional variable with a fresh propositional variable, in particular, this implies that distinct quantifiers can always bind distinct propositional variables.

## A.2 Sequents and Hypersequents

*Definition A.6 (hypersequent).* A hypersequent  $\mathcal{H}$  of  $\text{MLL}_2$  formulas (resp.  $\text{MLL}$ ) is an inductively defined structure as follows;

$$\mathcal{H}_1, \mathcal{H}_2 ::= A \mid \mathcal{H}_1, \mathcal{H}_2 \mid \mathcal{H}_1 \parallel \mathcal{H}_2$$

Where  $A$  is any formula of  $\text{MLL}_2$  (resp.  $\text{MLL}$ ).

*Remark 37.* Note that the hypersequents we consider in that work are not only of the form  $\Gamma_1 \parallel \dots \parallel \Gamma_n$  where each  $\Gamma_i$  is a sequent. For instance,  $A, (B \parallel C)$  is a possible hypersequent.

*Definition A.7 (Sequent).* A sequent is a multiset of formula i.e. a map  $\Gamma : \mathcal{F}_{\text{MLL}} \rightarrow \mathbb{N}$  of finite support. The *representant* of a sequent  $\Gamma$  is a sequence of  $\text{MLL}$  formulas  $\gamma : (A_1, \dots, A_n)$  such that

$$\Gamma = \bar{\gamma} : A \mapsto \text{card}\{1 \leq i \leq n \mid A_i = A\}.$$

Given  $\Gamma = (A_1, \dots, A_n)$  the representant of a sequent we denote by  $\Gamma(i)$  the formula  $A_i$ . The *size* of a sequent  $\Gamma$  denoted  $|\Gamma|$  is the number of formulas it contains, e.g. the length of its representants (as sequences of formulas).

*Remark 38.* We will use the notion of sequent loosely, implicitly, a sequent will be given by one of its representant e.g. a sequence of formulas.

*Definition A.8 (Flattening of an hypersequent).* The *flattening* of an hypersequent, denoted  $\downarrow \mathcal{H}$ , is defined inductively;

$$\begin{aligned} \downarrow A &\triangleq A. \\ \downarrow (\mathcal{H}_1, \mathcal{H}_2) &\triangleq \downarrow \mathcal{H}_1, \downarrow \mathcal{H}_2. \\ \downarrow (\mathcal{H}_1 \parallel \mathcal{H}_2) &\triangleq \downarrow \mathcal{H}_1, \downarrow \mathcal{H}_2. \end{aligned}$$

## A.3 Proofs

*Definition A.9.* A proof in  $\text{MLL}_2$  of a sequent  $\Gamma$  of formulas in  $\text{MLL}_2$  is a tree of root  $\Gamma$  constructed by the following rules;

$$\begin{array}{c} \frac{}{A, A^\perp} \text{ax} \quad \frac{\Gamma, A \quad \Delta, A^\perp}{\Gamma, \Delta} \text{cut} \quad \frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B} \otimes \quad \frac{\Gamma, A, B}{\Gamma, A \wp B} \wp \quad \frac{\Gamma, A, B, \Delta}{\Gamma, B, A, \Delta} \text{ex} \\ \frac{\Gamma, A}{\Gamma, \exists X A} \exists \quad \frac{\Gamma, A}{\Gamma, \forall X A} \forall \end{array}$$

The universal quantifier rule ( $\forall$ ) can only be applied whenever  $X$  does not occur free in  $\Gamma$ .

*Remark 39.* The proofs of  $\text{MLL}$  are the trees constructed only using the first rows.

*Definition A.10.* A proof in  $\text{MLL}^\bowtie$  of a sequent  $\Gamma$  of formulas in  $\text{MLL}^\bowtie$  is a tree of root  $\Gamma$  constructed by the following rules;

$$\frac{}{\Gamma} \bowtie \quad \frac{\Gamma, A \quad \Delta, A^\perp}{\Gamma, \Delta} \text{cut} \quad \frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B} \otimes \quad \frac{\Gamma, A, B}{\Gamma, A \wp B} \wp \quad \frac{\Gamma, A, B, \Delta}{\Gamma, B, A, \Delta} \text{ex}$$

The daimon rule ( $\bowtie$ ) can introduce any kind of sequent.

*Remark 40.* The proofs of  $\text{MLL}$  are the trees constructed using only the first row of the definition A.9. Equivalently they are proofs of  $\text{MLL}^\bowtie$  such that the daimon rule introduces only sequents of the form  $A, A^\perp$ .

*Remark 41.* The presence of the exchange rule is equivalent to assuming that each of the rule may be applied to any of the formulas in the sequents – not necessarily only to the last or first formulas.

## B COMPLEMENTS TO SECTION 2

### B.1 Complements to section 2.1

We recall the notion of homomorphism and isomorphism between hypergraphs.

*Definition B.1 (Homomorphism and isomorphisms of hypergraphs).* An *homomorphism* between two (labelled) hypergraphs  $\mathcal{H}_1 = (V_1, E_1, s_1, t_1)$  and  $\mathcal{H}_2 = (V_2, E_2, s_2, t_2)$  is a pair of bijective maps  $\langle f^V, f^E \rangle$ , such that

- (COMMUTATION) For any edge  $e_1$  of  $\mathcal{H}_1$ ,  $s_2(f^E(e_1)) = f^V(s_1(e_1))$ , i.e. the following diagram commutes:

$$\begin{array}{ccc} e_1 & \xrightarrow{s_1} & s_1(e_1) \\ f^E \downarrow & & \downarrow f^V \\ e_2 & \xrightarrow{s_2} & s_2(e_2) \end{array}$$

- (LABEL-PRESERVING) The function  $f^E$  preserves the labels of the links, i.e. for any link  $e$  in  $\mathcal{H}_1$ ;  $\ell(e) = \ell(f^E(e))$

The homomorphism is an isomorphism if both  $f^V$  and  $f^E$  are bijective.

### B.2 Complements to section 2.2

In that section we recall how multiplicative nets (e.g. multiplicative proof structures using the original terminology from [5]) may be defined inductively.

**PROPOSITION B.2.** *An ordered hypergraph is a multiplicative net if and only if it belongs to the following inductively defined set  $\text{Net}$ :*



- $(\langle \triangleright_{\times} p_1, \dots, p_n \rangle, p_1 < \dots < p_n)$  belongs to Net. □
- If  $(S, a)$  belongs to Net with  $a = (p_1, \dots, p_n)$  then for any  $i \in \{1, \dots, n-1\}$  and fresh position  $p$ ,  $(S + \langle p_i, p_{i+1} \triangleright_{\times} p \rangle, a')$  and  $(S + \langle p_i, p_{i+1} \triangleright_{\otimes} p \rangle, a')$  belong to Net, where  $a' = a[i \leftarrow p, i+1 \leftarrow \epsilon]$ .
- If  $(S, a)$  belongs to Net with  $a = (p_1, \dots, p_n)$  then for any  $i \in \{1, \dots, n-1\}$ ,  $(S + \langle p_i, p_{i+1} \triangleright_{\text{cut}} \rangle, a')$  belongs to Net, where  $a' = a[i+1 \leftarrow \epsilon, i \leftarrow \epsilon]$ .
- If  $(S, a)$  belongs to Net with  $a = (p_1, \dots, p_n)$ , then for any  $i \in \{1, \dots, n-1\}$  the net  $(S, a')$  belongs to Net, where  $a' = a[i \leftarrow p_{i+1}, i+1 \leftarrow p_i]$ .
- If  $(S_1, a)$  and  $(S_2, b)$  belong to Net and have disjoint positions i.e.  $V_1 \cap V_2 = \emptyset$ . then their sum  $(S_1 + S_2, a \cdot b)$  belongs to Net.

PROOF.  $1 \Rightarrow 2$ . Assume that  $(S, a)$  is a multiplicative net and we want to show it belongs to Net. To do so we will operate by induction on the number of links in  $S$ .

- If  $S$  has only one link, since  $S$  is surjective all its position must be the target of some link. Thus  $S$  can only be of the form  $\langle \triangleright_{\times} p_1, \dots, p_n \rangle$ , the order  $a$  may then be any total order on  $p_1, \dots, p_n$ .  
By definition  $(S, a_0)$  is in Net when  $a_0 = (p_1, \dots, p_n)$ . Then by doing exchanges we can rearrange the sequence  $a_0$  into any sequence on  $\{p_1, \dots, p_n\}$  in particular in  $a$ . Since Net is stable under this operation, this concludes that  $(S, a)$  is in Net.
- $S$  may be a sum of daimons, again the previous argument will work.
- $S$  may contain a terminal binary link such as  $l_1 ::= \langle p_1, p_2 \triangleright_{\times} p \rangle$  or  $l_2 ::= \langle p_1, p_2 \triangleright_{\otimes} p \rangle$  or  $l_3 ::= \langle p_1, p_2 \triangleright_{\text{cut}} \rangle$ . Thus  $S$  is of the form  $S' + l_i$ , and in that case  $S'$  has conclusions  $p_1, p_2, q_1, \dots, q_k$ . The order on the conclusion of  $S'$  may be any. We can call the induction hypothesis on  $S'$  and claim that  $(S', a')$  belong to Net. Then  $(S, a'[p_1 \leftarrow p, p_2 \leftarrow \epsilon])$  (the case for cut is similar) is by construction still in the class. Note that  $a'[p_1 \leftarrow p, p_2 \leftarrow \epsilon]$  is made of the conclusions of  $S$  thus after a series of exchange it can be reorganized as any sequence  $a$  on the conclusion of  $S$ , so we can conclude.

$2 \Rightarrow 1$ . We will do so by the induction given by the definition of Net.

- If  $S$  is daimon  $\langle \triangleright_{\times} p_1, \dots, p_n \rangle$  with the order  $(p_1, \dots, p_n)$  It is surjective, affine and linear so indeed it is a multiplicative net.
- If  $S_1$  and  $S_2$  are two affine linear are surjective nets, note that their disjoint sum (i.e. we assume  $V_1 \cap V_2 = \emptyset$ ) will still be surjective affine and linear.
- If we assume that  $(S, a)$  is a multiplicative net then performing an exchange will still make it a multiplicative net.
- If we add one terminal link to  $S$  note that surjectivity is preserved because the added position  $p$  belongs to the target of the new link. Furthermore linearity is preserved because the position of  $S$  don't become the target of any new link. Then affine property is preserved because the points that now belong to a new source where conclusions in  $S$ , thus it still true that any position belongs to at most the source of one link.

## C COMPLEMENTS TO SECTION 3

### C.1 Proofs of section 3.1

*Definition C.1 (Functional and injective relation).* A binary relation between two sets  $A$  and  $B$  is a subset of  $A \times B$ . A binary relation  $\mathcal{R} \subset A \times B$  is

- *functional* whenever for any  $(a, b) \in \mathcal{R}$  there is no other pair  $(a, b')$  with  $b \neq b'$  that belongs to  $\mathcal{R}$ .
- *injective* whenever for any  $(a, b) \in \mathcal{R}$  there is no other pair  $(a', b)$  with  $a \neq a'$  that belongs to  $\mathcal{R}$ .

A relation  $\mathcal{R}$  is defined on an element  $a \in A$  whenever there exists  $b \in B$  such that  $(a, b)$  belongs to  $\mathcal{R}$ .

*Remark 42.* Equivalently to the definition 3.1 an interface between two nets  $S$  and  $T$  of respective conclusions  $p_1, \dots, p_n$  and  $q_1, \dots, q_k$ . as partial injective maps  $\sigma : \{p_1, \dots, p_n\} \rightarrow \{q_1, \dots, q_k\}$ . However the notion of map is highly asymmetrical thus we cannot express easily the property of commutativity without mentioning the inverse of the map  $\sigma$ .

On the other hand, working with the previous definition, given  $S$  and  $T$  two nets and  $\sigma$  an interface between  $S$  and  $T$ , the interaction along  $\sigma$  is a commutative operation, e.g.  $S ::_{\sigma} T$  is equal to  $T ::_{\sigma} S$ .

**PROPOSITION 3.2 (INTERACTION AS AN ACTION).** *Given  $S, T$  and  $R$  three multiplicative nets, and  $\sigma$  an interface between  $S$  and  $T \parallel R$ .  $S ::_{\sigma} (T \parallel R)$  is equal to  $(S ::_{\sigma|T} T) ::_{\sigma|R} R$ .*

PROOF. of proposition 3.2. First simply unfold the definition;

$$S ::_{\sigma} (T + R) \triangleq S + T + R \sum_{(p,q) \in \sigma} \langle p, q \triangleright_{\text{cut}} \rangle.$$

Now note that the interface  $\sigma$  is partitioned in two sets, the positions that are related to conclusions of  $R$  and the positions that are related to conclusions of  $T$ . Thus the sum can be rewritten as:

$$\sum_{(p,q) \in \sigma} \langle p, q \triangleright_{\text{cut}} \rangle = \sum_{(p,q) \in \sigma|T} \langle p, q \triangleright_{\text{cut}} \rangle + \sum_{(p,q) \in \sigma|R} \langle p, q \triangleright_{\text{cut}} \rangle.$$

Hence since the sum is commutative and associative the interaction  $S ::_{\sigma} (T + R)$  is equal to the following.

$$S + \sum_{(p,q) \in \sigma|T} \langle p, q \triangleright_{\text{cut}} \rangle + T + \sum_{(p,q) \in \sigma|R} \langle p, q \triangleright_{\text{cut}} \rangle + R.$$

The first part of this expression corresponds to  $S ::_{\sigma|T} T$ . To conclude note that since  $\sigma|T$  relates conclusions of  $S$  to conclusion of  $T$  and since  $\text{out}(S ::_{\sigma|T} T)$  contains  $\text{out}(T)$ , it follows that  $\sigma|T$  is an interface between  $S ::_{\sigma|T} T$  and  $R$ . Thus  $S ::_{\sigma} (T + R)$  is equal to  $(S ::_{\sigma|T} T) ::_{\sigma|R} R$ . □

### C.2 Rewriting properties of nets

**PROPOSITION C.2 (NON-DETERMINISTIC NON HOMOGENEOUS CUT ELIMINATION COMMUTES TO THE RIGHT).** *Given  $S$  some net containing a non homogeneous cut link  $c$  of a daimon link against a  $\times$ -link. The diagram below commutes, for any cut kind of cut link  $c'$ ;*

$$\begin{array}{ccc} S & \xrightarrow{c} & S_1 \\ \downarrow c' & & \downarrow c' \\ S'_1 & \xrightarrow{c'} & S_2 \end{array}$$

Where the dotted arrows are the existence of a reduction.

PROOF. First note that the commutation of the cut elimination steps always hold whenever the two cuts involve a different part of the net. Furthermore if  $c'$  is a multiplicative cut the commutation holds. Thus  $c'$  is a glueing or non homogeneous cut involving the same daimon link as  $c$ . Let us treat each cases;

- In the first case, in all generality the reduction will be of the form:

$$\begin{aligned} & \langle \triangleright_{\otimes} p_1, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\otimes} r \rangle + \langle \triangleright_{\otimes} q_1, \dots, q_k \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2, q_1 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\otimes} p_1^1, p_2, A \rangle + \langle \triangleright_{\otimes} p_1^2, B \rangle + \langle \triangleright_{\otimes} q_1, \dots, q_k \rangle + \langle p_1^1, r_2 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle + \langle p_2, q_1 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\otimes} p_1^1, q_1, \dots, q_k, A \rangle + \langle \triangleright_{\otimes} p_1^2, B \rangle + \langle p_1^1, r_2 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle \end{aligned}$$

By consistently choosing the partition during the elimination of the cut  $c$  this can be matched by first starting with the elimination of the glueing cut  $c'$ .

$$\begin{aligned} & \langle \triangleright_{\otimes} p_1, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\otimes} r \rangle + \langle \triangleright_{\otimes} q_1, \dots, q_k \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2, q_1 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\otimes} p_1, q_1, \dots, q_k, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\otimes} r \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\otimes} p_1^1, q_1, \dots, q_k, A \rangle + \langle \triangleright_{\otimes} p_1^2, B \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle \end{aligned}$$

- In the first non homogeneous case the redex is of the following form

$$\langle \triangleright_{\otimes} p_1, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\otimes} r \rangle + \langle q_1, q_2 \triangleright_{\otimes} q \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2, q \triangleright_{\text{cut}} \rangle.$$

After one step of cut elimination this becomes in all generality, given that  $A = \{a_1, \dots, a_k\}$  and  $B = \{b_1, \dots, b_l\}$  partition  $p_2, \dots, p_n$ ,

$$\langle \triangleright_{\otimes} p_1^1, p_2, A' \rangle + \langle \triangleright_{\otimes} p_1^2, B \rangle + \langle q_1, q_2 \triangleright_{\otimes} q \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle + \langle p_2, q \triangleright_{\text{cut}} \rangle$$

Without loss of generality assume that  $p_2$  occurs in the class  $A = \{p_2\} \cup A'$ , then after one step of cut elimination this becomes;

$$\langle \triangleright_{\otimes} p_1^1, p_2^1, p_2^2, A' \rangle + \langle \triangleright_{\otimes} p_1^2, B \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle.$$

Indeed one can obtain the same redex by first eliminating  $c'$ , when eliminating the  $\otimes$ -link we need to make a consistent choice e.g. the partition of  $p_2^1, p_2^2, \dots, p_n$  made of the two classes  $B$  and  $A' \cup \{p_2^1, p_2^2\}$ :

$$\begin{aligned} & \langle \triangleright_{\otimes} p_1, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\otimes} r \rangle + \langle q_1, q_2 \triangleright_{\otimes} q \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2, q \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\otimes} p_1, p_2^1, p_2^2, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\otimes} r \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\otimes} p_1, p_2^1, p_2^2, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\otimes} r \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\otimes} p_1^1, p_2^1, p_2^2, A' \rangle + \langle \triangleright_{\otimes} p_1^2, B \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle \end{aligned}$$

- In the non homogeneous second case both cuts are made of  $\otimes$  links against a daimon thus in all generality a reduction is of the following form form;

$$\begin{aligned} & \langle \triangleright_{\otimes} p_1, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\otimes} r \rangle + \langle q_1, q_2 \triangleright_{\otimes} q \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2, q \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\otimes} p_1^1, p_2, A \rangle + \langle \triangleright_{\otimes} p_1^2, B \rangle + \langle q_1, q_2 \triangleright_{\otimes} q \rangle + \langle p_2, q \triangleright_{\text{cut}} \rangle + \langle p_1^1, r_2 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\otimes} p_1^1, p_2^1, A_1 \rangle + \langle \triangleright_{\otimes} p_2^2, A_2 \rangle + \langle \triangleright_{\otimes} p_1^2, B \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle \end{aligned}$$

Indeed starting by eliminating the cut  $c'$  we can obtain the same redex by making consistent choices in the partitions:

$$\begin{aligned} & \langle \triangleright_{\otimes} p_1, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\otimes} r \rangle + \langle q_1, q_2 \triangleright_{\otimes} q \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2, q \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\otimes} p_2^1, p_1, A_1 \cup B \rangle + \langle \triangleright_{\otimes} p_2^2, A_2 \rangle + \langle r_1, r_2 \triangleright_{\otimes} r \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\otimes} p_1^1, p_2^1, A_1 \rangle + \langle \triangleright_{\otimes} p_2^2, A_2 \rangle + \langle \triangleright_{\otimes} p_1^2, B \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle \end{aligned}$$

□

PROPOSITION C.3 (DETERMINISTIC NON HOMOGENEOUS CUT ELIMINATION COMMUTES TO THE LEFT). Given  $S$  some net containing a non homogeneous cut link  $c$  of a daimon link against a  $\otimes$ -link. The diagram below commutes, for any cut kind of cut link  $c'$ ;

$$\begin{array}{ccc} S & \xrightarrow{c'} & S_1 \\ \downarrow c & & \downarrow c \\ S'_1 & \xrightarrow{c'} & S_2 \end{array}$$

Where the dotted arrows are the existence of a reduction.

PROOF. We reason as in the proof for proposition C.2, if the two cuts involved different link of the net we conclude. Furthermore if  $c'$  is an homogeneous cut the proposition hold. Thus we assume that  $c$  and  $c'$  are both non-homogeneous and involve the same daimon link.

- If  $c'$  is also a tensor link its clear that the elimination of the two cuts commute.
- If  $c'$  is a  $\otimes$  link we can call the previous proposition C.2 on the cut  $c'$ , claiming that its elimination commutes on the right with any step of cut elimination. In particular any elimination of  $c'$  followed by any elimination of  $c$  can be matched by an elimination of  $c$  followed by an elimination of  $c'$ . Thus we conclude.

□

PROPOSITION C.4 (GENERAL RIGHT COMMUTATION). For any net  $S$  containing an irreversible cut  $c$ , and given  $\alpha^*$  a series of cut elimination (cuts occurring in  $S$ ) then the following diagram commutes;

$$\begin{array}{ccc} S & \xrightarrow{c} & S_1 \\ \downarrow \alpha^* & & \downarrow \alpha^* \\ S'_1 & \xrightarrow{c} & S_2 \end{array}$$

PROOF. This follows from a simple induction on  $\alpha^*$ . If  $\alpha^*$  is made of only one cut the proposition C.2 gets us the conclusion. Otherwise we decompose the sequence of reductions as  $\alpha^* = \beta \cdot \beta^*$  and by applying the proposition C.2 we obtain the following diagram.

$$\begin{array}{ccc} S & \xrightarrow{c} & S_1 \\ \downarrow \beta & & \downarrow \beta \\ S'_1 & \xrightarrow{c} & S_2 \\ & & \downarrow \beta^* \\ & & S_3 \end{array}$$

Calling the induction hypothesis on  $\beta^*$  we can complete the diagram as follows and conclude.

$$\begin{array}{ccc} S & \xrightarrow{c} & S_1 \\ \downarrow \beta & & \downarrow \beta \\ S'_1 & \xrightarrow{c} & S_2 \\ & & \downarrow \beta^* \\ S'_2 & \xrightarrow{c} & S_3 \end{array}$$

□

*Remark 43.* Moreover, following a similar method, we can easily establish that multiplicative glueing and clashing cuts commute to the left with any kind of cut.

This means that whenever two net  $S$  and  $T$  are orthogonal and a reversible cut  $c$  occurs in  $S :: T$  we can chose to start the reduction the reduction  $S :: T \rightarrow \langle \triangleright_{\boxtimes} \rangle$  by eliminating  $c$ . Thus the redex  $R$  of  $S :: T$  obtained after eliminating  $c$  still verifies that  $R \rightarrow \langle \triangleright_{\boxtimes} \rangle$ .

**PROPOSITION C.5 (GENERAL LEFT COMMUTATION).** *For any net  $S$  containing an reversible cut  $c$ , and given  $\alpha^*$  a series of cut elimination of cuts occuring in  $S$  then the following diagram commutes;*

$$\begin{array}{ccc} S & \xrightarrow{\alpha^*} & S_1 \\ \downarrow c & & \downarrow c \\ S'_1 & \xrightarrow{\alpha^*} & S_2 \end{array}$$

**PROOF.** As for the previous proposition it follows from a simple induction on  $\alpha^*$ . If  $\alpha^*$  is made of only one cut the proposition C.3 gets us to conclude. Otherwise we decompose the sequence of reductions as  $\alpha^* = \beta^* \cdot \beta$  and by applying the proposition C.3 together with the induction hypothesis we obtain the following diagram.

$$\begin{array}{ccccc} S & \xrightarrow{\beta^*} & S_1 & \xrightarrow{\beta} & S_2 \\ \downarrow c & & \downarrow c & & \downarrow c \\ S'_1 & \xrightarrow{\beta^*} & S_2 & \xrightarrow{\beta} & S_3 \end{array}$$

□

**Definition C.6 (Simulation).** A net  $S$  *simulates* a net  $T$ , if for any reduction  $T \rightarrow R$  there exists a reduction  $S \rightarrow^* R$ . We denote this by  $S > T$ .

Two nets  $S$  and  $T$  are *bisimilar* whenever  $S > T$  and  $T > S$ .

A *matching* of two nets  $S$  and  $T$  is a pair  $(p, q)$  of a conclusion of  $S$  and a conclusion of  $T$ . A net  $S$  *simulates* a net  $T$  in a matching  $(p, q)$  for a link  $l = \langle t_1, \dots, t_n \triangleright_{\ell} q_1, \dots, q_k \rangle$  whenever there exists a conclusion  $q_i$  such that  $S ::_{p, q_i} l$  simulates  $T ::_{p, q_i} l$ . We denote  $S >_{\ell}^{p, q} T$ .

**PROPOSITION C.7 (STANDARD SIMULATION).** *The following simulations hold;*

- $\langle \triangleright_{\boxtimes} \vec{p}, p_1, p_2 \rangle + \langle p_1, p_2 \triangleright_{\boxtimes} p \rangle \sim_{\otimes, \boxtimes}^{p, q} \langle \triangleright_{\boxtimes} \vec{p}, q \rangle$ .
- $\langle \triangleright_{\boxtimes} \vec{p}_1, p_1 \rangle + \langle \triangleright_{\boxtimes} \vec{p}_2, p_2 \rangle + \langle p_1, p_2 \triangleright_{\otimes} p \rangle \sim_{\boxtimes}^{p, q} \langle \triangleright_{\boxtimes} \vec{p}_1, \vec{p}_2, q \rangle$ .
- $\langle \triangleright_{\boxtimes} \vec{p}_1, p_1 \rangle + \langle \triangleright_{\boxtimes} \vec{p}_2, p_2 \rangle + \langle p_1, p_2 \triangleright_{\otimes} p \rangle \sim_{\boxtimes}^{p, q} \langle \triangleright_{\boxtimes} \vec{p}_1, \vec{p}_2, q \rangle$ .

### C.3 Proofs of section 3.3

**PROPOSITION 3.6 (SYMMETRY OF TYPES).** *For any type  $A$  with one conclusion; (1) there exists a net  $S$  with a terminal  $\boxtimes$ -link that belongs to  $A$  or  $A^\perp$ , and (2) there exists a net  $S$  with a terminal  $\otimes$ -link that belongs to  $A$  or  $A^\perp$ .*

**PROOF.** of proposition 3.6. First let us note that there cannot be only nets with daimon conclusion in  $A \cup A^\perp$ .

Assume that it is the case. Then  $S :: S'$  reduces to  $\langle \triangleright_{\boxtimes} \rangle$ . The redex corresponding to the glueing cut can be made into a multiplicative one.

Thus one of the two assertions must hold. To conclude we will show that they are equivalent.

1. Assume that some net  $S$  exists in with a terminal  $\otimes$  belongs to say  $A$  without loss of generality. If a net with a terminal  $\boxtimes$  belongs to  $A$  we can conclude. Assume otherwise e.g. only nets with terminal tensor link or daimon link belong to  $A$ .

Assume in order to reach a contradiction, that only net with terminal daimon link occur in  $A^\perp$ ,

Let  $R$  be a net of  $A^\perp$ , e.g. for any net  $S$  in  $A$  the interaction  $S :: R$  reduces to  $\langle \triangleright_{\boxtimes} \rangle$ . Consider  $T = R + \langle p_1, p_2 \triangleright_{\boxtimes} p \rangle$  where the position  $p$  the conclusion of  $R$ , has been duplicated, and connected to the same daimon link.

The succesfull  $S :: R \rightarrow \langle \triangleright_{\boxtimes} \rangle$  reduction needs at some point to eliminate the reversible cut  $c$ . But proposition C.3 ensure that the elimination of reversible cuts commutes on the left; thus the successful reduction may be rearranged as  $S :: R \rightarrow S' :: R' \rightarrow \langle \triangleright_{\boxtimes} \rangle$ . where  $S' :: R'$  is the redex obtained after eliminating the reversible cut  $c$  in  $S :: R$ .

Note that  $S :: T$  also reduces to  $S' :: R'$  after eliminating the reversible cut  $c$ , hence it reduces to  $\langle \triangleright_{\boxtimes} \rangle$ . This shows that  $S \perp T$  for any  $A$  meaning  $T$  is in  $A^\perp$ .

2. Assume that some net  $S$  exists in with a terminal  $\boxtimes$  belongs to, say  $A$  without loss of generality. Again we assume that in  $A^\perp$  only nets with a terminal daimon link exists. Let  $R$  be some net in  $A^\perp$  then consider  $T = R + \langle \triangleright_{\boxtimes} p_1 \rangle + \langle \triangleright_{\boxtimes} p_2, \vec{p} \rangle + \langle p_1, p_2 \triangleright_{\otimes} p \rangle$  where  $p$  the conclusion of  $R$  has been duplicated and connected to disctints daimon links.

For any net  $S$  belonging to  $A$  with a terminal  $\boxtimes$  link since the irreversible cuts commute to the right the interaction  $S :: R$  may reduce to  $\langle \triangleright_{\boxtimes} \rangle$  by eliminating the irreversible cut  $c$  as late as possible. In particular after eliminating the cuts with the sequence  $\alpha^*$  it becomes  $S' + R' + \text{red}$ , where  $S'$  and  $R'$  are cut free and the redex with the irreversible cut only remain. Hence  $S' + R' + \text{red} \rightarrow \langle \triangleright_{\boxtimes} \rangle$ .

Furthermore by eliminating the cuts as in the sequence  $\alpha^*$  the interaction  $S :: T$  reduces to  $S' :: R' :: \text{red}_1$ . The redexes  $\text{red}_1$  can match the reduction of the redex  $\text{red}$  hence if  $S' :: R' + \text{red}$  reduce to  $\langle \triangleright_{\boxtimes} \rangle$  then so does  $S :: R + \text{red}_1$ . As a consequence if  $S$  is orthogonal to  $R$  then  $S$  is also orthogonal to  $T$ . We conclude that  $T$  belongs to  $A^\perp$ . □

**PROPOSITION 3.8 (DUALITY).** *Given two types  $A$  and  $B$ :  $(A \parallel B)^\perp = A^\perp > B^\perp$  and  $(A > B)^\perp = A^\perp \parallel B^\perp$ .*

**PROOF.** of proposition 3.8 Consider a net  $S$  orthogonal to  $A \parallel B$  then for any  $a$  in  $A$  and  $b$  in  $B$  the interaction  $S :: (a \parallel b)$  reduces to  $\boxtimes_0$ .

Since  $S :: (a \parallel b)$  is equal to  $S :: a :: b$  and that the orthogonality hold for any pair  $(a, b)$ , in particular for any net  $b$  in  $B$  the interaction  $(S :: a) :: b$  reduce to  $\boxtimes_0$ . This means that  $S :: a$  is orthogonal to  $B$  hence for any  $a$  in  $A$  the interaction  $S :: a$  belongs to  $B^\perp$ . Since  $A = (A^\perp)^\perp$  by definition this means  $S$  belongs to  $A^\perp > B^\perp$ .

On the other consider a net  $S$  in  $A^\perp > B^\perp$  is such that for any  $a$  in  $A = (A^\perp)^\perp$  the interaction  $S :: a$  belongs to  $B^\perp$  e.g. is orthogonal to  $B$  and so for any  $b$  in  $B$  the interaction  $S :: a :: b$  reduce to  $\boxtimes_0$ .

Since  $S :: a :: b$  is equal to  $S :: (a \parallel b)$  this show that  $S$  is orthogonal to  $A \parallel B$ . Hence we showed  $(A \parallel B)^\perp$  is equal to  $A^\perp \succ B^\perp$ .

As a consequence  $(A \succ B)^\perp$  which is  $(A^{\perp\perp} \succ B^{\perp\perp})^\perp$  corresponds to  $(A^\perp \parallel B^\perp)^{\perp\perp}$  and since the parallel construction is a type this is equal to  $A^\perp \parallel B^\perp$ .  $\square$

**PROPOSITION 3.11 (DUALITY).** *Given  $A$  and  $B$  two types with one conclusion,  $(A \otimes B)^\perp = A^\perp \wp B^\perp$  and  $(A \wp B)^\perp = A^\perp \otimes B^\perp$ .*

**PROOF.** of proposition 3.11. As for the preconstructions one equality imply the other, say  $(A \otimes B)^\perp = A^\perp \wp B^\perp$  hold for any pair of types. Then  $A^\perp \otimes B^\perp$  which is  $(A^\perp \otimes B^\perp)^{\perp\perp}$  corresponds to  $(A^{\perp\perp} \wp B^{\perp\perp})^\perp$  and thus  $(A \wp B)^\perp$ .

The inclusion of  $A^\perp \wp B^\perp$  in  $(A \otimes B)^\perp$  is straightforward, since after eliminating the multiplicative cut we obtain an interaction  $S :: T$  where  $S$  belongs to  $A^\perp \succ B^\perp$  and  $T$  belongs to  $A \parallel B$  and that the construction are orthogonal – proposition 3.8. In other words we show

$$\{S + \langle S(1), S(2) \triangleright_\otimes p \rangle \mid S \in A \parallel B\} \subseteq \{T + \langle S(1), S(2) \triangleright_\wp p \rangle \mid S \in A^\perp \succ B^\perp\}^\perp.$$

and so since bi-orthogonality preserves inclusion;

$$A \otimes B \subseteq \{T + \langle S(1), S(2) \triangleright_\wp p \rangle \mid S \in A^\perp \succ B^\perp\}^\perp.$$

Since tri-orthogonal is has taking the orthogonal this shows;

$$A \otimes B \subseteq (A^\perp \wp B^\perp)^\perp.$$

Consider  $T$  a net orthogonal to the nets belonging to  $\{S + \langle S(1), S(2) \triangleright_\otimes p \rangle \mid S \in A \parallel B\}$ . If  $T = T_0 + l$  has a terminal  $\wp$ -link  $l$  then after one step of cut elimination we obtain  $T_0 :: S$  which must reduce to  $\langle \triangleright_{\wp} \bar{p} \rangle$ . As a consequence  $T_0$  is orthogonal to  $A \parallel B$  e.g. it belongs to  $A^\perp \succ B^\perp$  and so  $T$  belongs to  $A^\perp \wp B^\perp$ .

If  $T$  has for terminal link a daimon link; after some reduction of  $T$  we obtain a net  $T_1$  which is  $T$  with its output  $p$  duplicated to interact with the inputs of the tensor. After reducing the the reversible cut in the interaction  $T :: S$  we obtain a net  $T_1 :: S$  and orthogonality must hold still since the reversible cut commute to the left, hence  $T_1$  is orthogonal to  $A \parallel B$  and thus belongs to  $A^\perp \succ B^\perp$ .

This implies that  $T_1 + \langle p_1, p_2 \triangleright_\wp p \rangle$  belongs to  $A^\perp \wp B^\perp$ . Since daimon links simulate well-formed  $\wp$ -links this lead to  $T$  belonging to  $A^\perp \wp B^\perp$ .

Hence we have showed the other inclusion – obtained by passing the following inclusion to the duals;

$$\{S + \langle S(1), S(2) \triangleright_\otimes p \rangle \mid S \in A \parallel B\}^\perp \subseteq A^\perp \wp B^\perp.$$

Again this can be rewritten as;

$$(A \otimes B)^\perp \subseteq A^\perp \wp B^\perp.$$

Passing to the orthogonal this means;

$$(A^\perp \wp B^\perp)^\perp \subseteq A \otimes B$$

$\square$

## D COMPLEMENTS TO SECTION 4

### D.1 Proof Nets

We recall here the notion of proof net, e.g. nets that represent a proof. We give that notion for MLL<sub>2</sub> proofs which is easily adapted to the other fragments considered in our work, namely MLL and MLL<sup>ax</sup>.

**Definition D.1 (Representation of a proof).** An untyped multiplicative net  $S$  represents a proof  $\pi$  of MLL<sub>2</sub> denoted  $\pi \equiv_{\mathcal{R}} S$  or  $S \equiv_{\mathcal{R}} \pi$ , whenever the relation given by the following induction hold;

$$\frac{}{A, A^\perp} \text{ax} \equiv_{\mathcal{R}} \langle \triangleright_{\wp} p_1, p_2 \rangle$$

$p_1$  and  $p_2$  is a pair of

$$\frac{\frac{\frac{}{\vdash A, \Gamma} \pi_1 \quad \frac{}{\vdash A^\perp, \Delta} \pi_2}{\vdash \Gamma, \Delta} \text{cut}}{\vdash A, \Gamma} \equiv_{\mathcal{R}} S_1 + S_2 + \langle S_1(1), S_2(1) \triangleright_{\text{cut}} \rangle$$

Whenever  $\pi_1 \equiv_{\mathcal{R}} S_1$  a

$$\frac{\frac{\frac{}{\vdash A, \Gamma} \pi_1 \quad \frac{}{\vdash B, \Delta} \pi_2}{\vdash \Gamma, \Delta, A \otimes B} \otimes}{\vdash A, \Gamma} \equiv_{\mathcal{R}} S_1 + S_2 + \langle S_1(1), S_2(1) \triangleright_\otimes p \rangle$$

whenever  $\pi_1 \equiv_{\mathcal{R}} S_1$  position.

$$\frac{\frac{\frac{}{\vdash A, B, \Gamma} \pi}{\vdash A \wp B, \Gamma} \wp}{\vdash A, B, \Gamma} \equiv_{\mathcal{R}} S_0 + \langle S(1), S(2) \triangleright_\wp p \rangle$$

whenever  $\pi \equiv_{\mathcal{R}} S$  and

$$\frac{\frac{\frac{}{\vdash A, \Gamma} \pi}{\vdash \forall X A, \Gamma} \forall}{\vdash A, \Gamma} \equiv_{\mathcal{R}} S + \langle S(1) \triangleright_\forall p \rangle$$

whenever  $\pi \equiv_{\mathcal{R}} S$  and

$$\frac{\frac{\frac{}{\vdash A, \Gamma} \pi}{\vdash \exists X A, \Gamma} \exists}{\vdash A, \Gamma} \equiv_{\mathcal{R}} S + \langle S(1) \triangleright_\exists p \rangle$$

whenever  $\pi \equiv_{\mathcal{R}} S$  and

$$\frac{\frac{\frac{}{\vdash \Gamma, B, A, \Delta} \pi}{\vdash \Gamma, A, B, \Delta} \text{ex}}{\vdash \Gamma, B, A, \Delta} \equiv_{\mathcal{R}} (S, a)$$

where  $\pi \equiv_{\mathcal{R}} (S, a')$  a other hand,  $a'(\Gamma) =$

**Remark 44.** The previous definition naturally adapts to proofs of MLL and to proofs of MLL<sup>ax</sup> by considering the following base case in the induction;

$$\frac{}{A_1, \dots, A_n} \text{ax} \equiv_{\mathcal{R}} \langle \triangleright_{\wp} p_1, \dots, p_n \rangle \text{ whenever } p_1, \dots, p_n \text{ are distinct positions.}$$

### D.2 The Merge Operator

**Definition D.2 (Merge of two nets).** The merge of two daimon links  $d = \langle \triangleright_{\wp} \bar{p} \rangle$  and  $d' = \langle \triangleright_{\wp} \bar{q} \rangle$  is the daimon link  $d \bowtie d' = \langle \triangleright_{\wp} \bar{p}, \bar{q} \rangle$ .

The merge of two disjoint sums of daimon links  $\sum_{i \in I} d_i + \sum_{i \in I} d_j$  is a sum  $d_k \bowtie d_{k'} + \sum_{i \in I, i \neq k} d_i + \sum_{j \in J, j \neq k'} d_j$  for any  $k \in I$  and  $k' \in J$ .

Given two disjoint nets  $S$  and  $T$  with daimon links  $\mathcal{D}_S$  and  $\mathcal{D}_T$ . A merge of  $S$  and  $T$  is a net  $R$  which corresponds to  $\mathcal{B}_S + \mathcal{B}_T + \mathcal{D}_R$  where  $\mathcal{D}_R$  is  $\mathcal{D}_S + \mathcal{D}_T$  where one of the daimon link of  $S$  has been merged with a daimon link of  $T$ .

**Notation 19.** Given two sets of nets  $A$  and  $B$  we will let  $A \bowtie B$  denote the set of nets  $S$  which are merges of a net of  $A$  with a net of  $B$ .

Furthermore given two nets  $S$  and  $T$  their merge  $S \bowtie T$  is not unique. We will sometimes use the notation  $S \bowtie T$  abusively treating as a net but one should keep in mind that in practice we consider one of the possible merge of the two nets.

PROPOSITION D.3. *Given  $S$  and  $T$  two orthogonal nets then  $S \bowtie \mathfrak{X}_1 :: T$  reduces to  $\mathfrak{X}_1$ .*

PROOF. By induction on the pair  $(c, i)$  where  $c$  is the number of connectives in  $S + T$  which are above conclusions of  $S$  or of  $T$ ; and  $i$  is the number of initial positions in  $S + T$ .

- The cases  $(0, i)$  are straightforward. Only glueing cuts occur and these cuts commute to the left. Furthermore, eliminating a glueing cut decreases the size of the net.
- For the cases  $(c, i)$  make the number of connectives decrease by performing a step of cut elimination, in particular three cases occurs corresponding to the possible cut eliminated,  $(\mathfrak{Y}/\otimes)$ ,  $(\mathfrak{Y}/\mathfrak{X})$  or  $(\otimes/\mathfrak{X})$ . We will treat only the cases where the connective occurs in  $S$ ; it is easy to adapt the following proof for the case when the connectives occurs in  $T$ .

- Say that  $S :: T$  occurs a multiplicative cut thus the two nets may be written as  $S_0 + \alpha :: T_0 + \beta$  where  $\alpha$  is the  $\mathfrak{Y}$  link and  $\beta$  the  $\otimes$  link. Since multiplicative cut commutes to the left and since  $S :: T$  reduces to  $\langle \triangleright_{\mathfrak{X}} \rangle$  we can ensure that the multiplicative redex  $S_0 :: T_0$  reduces also to  $\langle \triangleright_{\mathfrak{X}} \rangle$  e.g.  $S_0$  and  $T_0$  are orthogonal. Furthermore note that  $S_0 + \alpha \bowtie \mathfrak{X}_1 :: T_0 + \beta$  corresponds to  $S_0 \bowtie \mathfrak{X}_1 + \alpha :: T_0 + \beta$  and so after a step of cut elimination reduces to  $S_0 \bowtie \mathfrak{X}_1 :: T_0$ . Since  $S_0 \perp T_0$  and that the number of connectives as decreased, we can apply the induction hypothesis and ensure that  $S_0 \bowtie \mathfrak{X}_1 :: T_0$  reduces to  $\mathfrak{X}_1$  hence it is also true for  $S \bowtie \mathfrak{X}_1 :: T$ .
- Say that a non-homogeneous cut  $(\mathfrak{X}/\otimes)$  occurs in the interaction of  $S :: T$  and assume the tensor link occurs in  $S$ . Then the interaction can be written as  $S_0 + \alpha :: T$  after eliminating that reversible cut we obtain the ent  $S_0 :: T'$  where  $T'$  is  $T$  with a duplicated output, and we have  $S_0 \perp T'$ . Furthermore  $S_0 + \alpha \bowtie \mathfrak{X}_1 :: T$  reduces to  $S_0 \bowtie \mathfrak{X}_1 :: T'$  applying the induction hypothesis this reduces to  $\mathfrak{X}_1$ .
- Say that only cuts  $(\mathfrak{X}/\mathfrak{Y})$  occurs in the interaction of  $S :: T$  (if not we can fall back onto one of the previous cases) and assume the  $\mathfrak{Y}$ -links occurs in  $S$ . Furthermore let us denote  $S_r$  the redex of  $S$  in which all reversible cuts have been eliminated, similarly  $T_r$  denotes such a redex for  $T$ . Then  $S \perp T$  if and only if  $S_r \perp T_r$ ,  $S_r$  may be written as  $S_r^* + \alpha$  where  $\alpha$  is one of its terminal  $\mathfrak{Y}$ -link. Then eliminating the cut involving  $\alpha$  the net  $S_r :: T_r$  reduces to  $S_r^* :: T_r'$  where  $T_r'$  is  $T_r$  in which one of its daimon as been splitted. Since in  $S_r :: T_r$  there are only irreversible cuts it follows that  $S_r \perp T_r$  implies  $S_r^* \perp T_r'$ . Then note that the reversible form of  $S \bowtie \mathfrak{X}_1$  corresponds to  $S_r \bowtie \mathfrak{X}_1$  thus,  $S \bowtie \mathfrak{X}_1 :: T$  reduces to  $S_r \bowtie \mathfrak{X}_1 :: T_r$ . Furthermore  $S_r \bowtie \mathfrak{X}_1 :: T_r$  reduces to  $S_r^* \bowtie \mathfrak{X}_1 :: T_r'$  and so by induction to  $\mathfrak{X}_1$ , and so we conclude.  $\square$

PROPOSITION D.4 (MERGES BELONGS TO THE COMPOSITIONAL CONSTRUCTION). *Given  $\mathbf{A}$  and  $\mathbf{B}$  two types;*

$$(\mathbf{A} \bowtie \mathbf{B})^{\perp\perp} \subseteq \mathbf{A} \cdot \mathbf{B}.$$

PROOF. Consider  $S$  a net in  $\mathbf{A} \bowtie \mathbf{B}$ .  $S$  may therefore be written as  $a \bowtie b$  for two elements  $a \in \mathbf{A}$  and  $b \in \mathbf{B}$ . The net  $a \bowtie \langle \triangleright_{\mathfrak{X}} p \rangle + \langle p, q \triangleright_{\text{cut}} \rangle \langle \triangleright_{\mathfrak{X}} q \rangle \bowtie b$  reduces to  $a \bowtie b$  by a glueing cut, since the glueing cut can always be performed first, an orthogonal net to  $a \bowtie \langle \triangleright_{\mathfrak{X}} p \rangle + \langle p, q \triangleright_{\text{cut}} \rangle \langle \triangleright_{\mathfrak{X}} q \rangle \bowtie b$  is a net orthogonal to  $a \bowtie b$ .

Now consider  $\bar{a}$  some net in  $\mathbf{A}^\perp$ . Since  $a :: \bar{a}$  reduces to  $\langle \triangleright_{\mathfrak{X}} p \rangle$ , by proposition D.3  $a \bowtie \langle \triangleright_{\mathfrak{X}} p \rangle :: \bar{a}$  reduces to  $\langle \triangleright_{\mathfrak{X}} p \rangle$ . Similarly  $b \bowtie \langle \triangleright_{\mathfrak{X}} q \rangle :: \bar{b}$  reduces to  $\langle \triangleright_{\mathfrak{X}} q \rangle$ . Since cut elimination is local;

$$\begin{aligned} a \bowtie \langle \triangleright_{\mathfrak{X}} p \rangle + \langle p, q \triangleright_{\text{cut}} \rangle \langle \triangleright_{\mathfrak{X}} q \rangle \bowtie b &:: (\bar{a} \parallel \bar{b}) \\ &= (a \bowtie \langle \triangleright_{\mathfrak{X}} p \rangle :: \bar{a}) + \langle p, q \triangleright_{\text{cut}} \rangle (\langle \triangleright_{\mathfrak{X}} q \rangle \bowtie b :: \bar{b}) \\ &\rightarrow \langle \triangleright_{\mathfrak{X}} p \rangle + \langle p, q \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\mathfrak{X}} q \rangle \\ &\rightarrow \langle \triangleright_{\mathfrak{X}} \rangle \end{aligned}$$

This shows that  $a \bowtie \langle \triangleright_{\mathfrak{X}} p \rangle + \langle p, q \triangleright_{\text{cut}} \rangle \langle \triangleright_{\mathfrak{X}} q \rangle \bowtie b :: (\bar{a} \parallel \bar{b})$  reduces to  $\mathfrak{X}_0$  and thus so does  $a \bowtie b :: (\bar{a} \parallel \bar{b})$  since the last glueing cut of the reduction may as been performed as first due to the commutation results. This shows that  $a \bowtie b$  is orthogonal to  $\mathbf{A}^\perp \parallel \mathbf{B}^\perp$  and so belongs to  $\mathbf{A} \cdot \mathbf{B}$ . Hence we have shown the inclusion  $\mathbf{A} \bowtie \mathbf{B} \subseteq \mathbf{A} \cdot \mathbf{B}$  and therefore  $(\mathbf{A} \bowtie \mathbf{B})^{\perp\perp} \subseteq \mathbf{A} \cdot \mathbf{B}$  since the righth member of the inclusion is closed under bi-orthogonal.  $\square$

PROPOSITION D.5. *For any integer  $n$  the daimon link  $\mathfrak{X}_n$  belongs to  $\{\mathfrak{X}_1\}^{\perp\perp} \succ \dots \succ \{\mathfrak{X}_1\}^{\perp\perp}$  (composition of  $n$  components).*

PROOF. By induction. For the base case note that  $\mathfrak{X}_1$  belongs to  $\{\mathfrak{X}_1\}^{\perp\perp}$  for any net. Now if we assume  $\mathfrak{X}_n$  belongs to the composition of size  $n$  that is  $\{\mathfrak{X}_1\}^{\perp\perp} \succ \dots \succ \{\mathfrak{X}_1\}^{\perp\perp}$  since  $\mathfrak{X}_1$  belongs to  $\{\mathfrak{X}_1\}^{\perp\perp}$  the previous proposition ensure that  $\mathfrak{X}_n \bowtie \mathfrak{X}_1$  belongs to the composition of size  $n + 1$  of  $\{\mathfrak{X}_1\}^{\perp\perp} \succ \dots \succ \{\mathfrak{X}_1\}^{\perp\perp}$ . But  $\mathfrak{X}_n \bowtie \mathfrak{X}_1$  is exactly  $\mathfrak{X}_{n+1}$  which allows us to conclude.  $\square$

### D.3 Proofs of section 4

Notation 20. We denote by  $\mathfrak{X}_2$  a daimon with two outputs e.g.  $\langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle$ . Given an integer  $n$  we denote  $I_{\mathfrak{X}}^n$  the sum  $\sum_{1 \leq i \leq n} \langle \triangleright_{\mathfrak{X}} p_i, q_i \rangle$  with the order  $(p_1 < \dots < p_n < q_1 < \dots < q_n)$  on the conclusions. In particular  $I_{\mathfrak{X}}^1$  corresponds to  $\mathfrak{X}_2$ .

LEMMA D.6. *For any orthogonal nets  $S_1$  and  $S_2$  with  $n$  output  $(I_{\mathfrak{X}}^n :: S_1) :: S_2 \rightarrow^* \langle \triangleright \rangle$ .*

PROOF. We do so by induction on the size of the nets. If the nets are made only of daimon this is trivial.  $\square$

PROPOSITION 4.2. *For any types with one output  $\mathbf{A} \subseteq \mathbf{B}^\perp$ ;  $\mathfrak{X}_2$  belongs to  $\mathbf{A}^\perp \succ \mathbf{B}^\perp$ .*

PROOF. This is a special case of the previous lemma when  $n = 2$  the interface  $I_{\mathfrak{X}}^2$  is  $\mathfrak{X}$ .

Given two types such that  $\mathbf{A} \subseteq \mathbf{B}^\perp$  note that any pair of net  $S \in \mathbf{A}$  and  $T \in \mathbf{B}$  will satisfies  $S \perp T$  and thus  $\mathfrak{X}_2$  will be orthogonal to  $\mathbf{A} \parallel \mathbf{B}$  and so belong to  $\mathbf{A}^\perp \succ \mathbf{B}^\perp$ .  $\square$

Remark 45. In particular for any type  $\mathbf{A}$ ,  $\mathfrak{X}_2$  belongs to  $\mathbf{A} \succ \mathbf{A}^\perp$ .

2321 COROLLARY D.7. *For any interpretation basis  $\mathcal{B}$ , if  $\mathcal{B}$  is standard*  
 2322 *in  $X$ ,  $\mathfrak{X}_2$  belongs to  $\llbracket X, X^\perp \rrbracket_{\mathcal{B}}$ .*

2323 PROOF. If the basis  $\mathcal{B}$  is standard in  $X$  we have the inclusion  
 2324  $\llbracket X \rrbracket_{\mathcal{B}}^\perp \subseteq \llbracket X^\perp \rrbracket_{\mathcal{B}}$ , which implies by the previous proposition, that  
 2325  $\mathfrak{X}_2$  belongs to  $\llbracket X \rrbracket_{\mathcal{B}}^{\perp\perp} \supseteq \llbracket X^\perp \rrbracket_{\mathcal{B}}$  and so since we deal with types to  
 2326  $\llbracket X \rrbracket_{\mathcal{B}} \supseteq \llbracket X^\perp \rrbracket_{\mathcal{B}}$ .  $\square$

2327 *Definition D.8 (Constraint).* A constraint is a set of sequents of  
 2328 multiplicative formulas. A sequent satisfies a constraint  $C$  if it  
 2329 belongs to  $C$ .  
 2330

2331 *Remark 46.* For instance the axiom rule of MLL only introduces  
 2332 sequents which belong to the constraint  $\{A, A^\perp \mid A \in \mathcal{F}_{\text{MLL}}\}$ . On  
 2333 the other hand the constraint for MLL $^{\mathfrak{X}}$  corresponds to the set of  
 2334 all sequents.  
 2335

2336 *Notation 21.* Given a constraint  $C$  we denote  $S \vdash_C \Gamma$  whenever  
 2337  $S$  represents a proof of  $\Gamma$  in MLL $^{\mathfrak{X}}$  such that the daimon rules  
 2338 introduces only sequents in  $C$ .  
 2339

2340 *Definition D.9 (Adequate interpretation basis).* An interpretation  
 2341 basis  $\mathcal{B}$  is adequate with a constraint  $C$  on MLL sequents whenever  
 2342 for any sequent  $\Gamma = A_1, \dots, A_n$  satisfying the constraint  $C$  the  
 2343 daimon with  $n$  outputs realizes  $\Gamma$ .  
 2344

2345 PROPOSITION D.10 (SOUNDNESS). *Given an adequate interpreta-*  
 2346 *tion basis  $\mathcal{B}$  with respect to a constraint  $C$ , for any multiplicative net*  
 2347  *$S$  and a sequent  $\Gamma$ ,*

$$2348 \quad S \vdash_C \Gamma \Rightarrow S \Vdash_{\mathcal{B}} \Gamma.$$

2349 PROOF. of Proposition D.10. We proceed by induction on the  
 2350 size of the proof that  $S$  represents.  
 2351

2352 BASE If the proof is of size one then  $S$  is made only of one  
 2353 daimon link. Furthermore by assumption  $S \vdash_C \Gamma$ , since  
 2354 the basis  $\mathcal{B}$  is sound with respect to the constraint this  
 2355 implies that  $S \Vdash_{\mathcal{B}} \Gamma$ .

2356 IND-1 Assume that the last rule in the represented proof is a  
 2357  $\mathfrak{X}$ -rule. Thus the sequent is of the form  $\Gamma, A \mathfrak{X} B$ , and by  
 2358 assumption  $S \vdash_C \Gamma, A \mathfrak{X} B$ .

2359 Assume without loss of generality that the active formula  
 2360 in the last rule is  $A \mathfrak{X} B$ . Say the conclusion of  $S$  are ordered  
 2361 as  $q_1, \dots, q_n, p$  then since  $p$  is given the type  $A \mathfrak{X} B$  and  
 2362 that this formula is active; it must be that  $S$  is of the form  
 2363  $S' + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle$  such that  $S' \vdash_C \Gamma, A, B$ .  
 2364 Calling the induction hypothesis we can deduce  $S' \Vdash_{\mathcal{B}}$   
 2365  $\Gamma, A, B$ . By definition, this is equivalent that for any  $\gamma$  in  
 2366  $\llbracket \Gamma \rrbracket_{\mathcal{B}}^\perp$ , we have  $S' :: \gamma \Vdash_{\mathcal{B}} A, B$ , indeed it follows that  
 2367  $S' :: \gamma + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle \Vdash_{\mathcal{B}} A \mathfrak{X} B$ .  
 2368 Furthermore interaction and sum commute hence,  $S' ::$   
 2369  $\gamma + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle = S' + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle :: \gamma$  and so  
 2370 corresponds to  $S :: \gamma$ . Since this hold for any  $\gamma \in \llbracket \Gamma \rrbracket_{\mathcal{B}}^\perp$   
 2371 this allows us to conclude that  $S \Vdash_{\mathcal{B}} \Gamma, A \mathfrak{X} B$ .

2372 IND-2 Assume that the represented proof by  $S$  as for last rule  
 2373 the introduction of a tensor. The proved sequent is of the  
 2374 form  $\Gamma, A \otimes B, \Delta$  where  $A \otimes B$  is the active formula in the  
 2375 last rule.

2376 Thus  $S$  is of the form  $S_1 \otimes S_2$  where  $S_1 \vdash_C \Gamma, A$  and  $S_2 \vdash_C$   
 2377  $\Delta, B$ .  
 2378

2379 Calling the induction hypothesis we obtain  $S_1 \Vdash_{\mathcal{B}} \Gamma, A$   
 2380 and  $S_2 \Vdash_{\mathcal{B}} \Delta, B$ . Thus for any  $\gamma \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$  and  $\delta \in \llbracket \Delta \rrbracket_{\mathcal{B}}$   $S_1 ::$   
 2381  $\gamma \Vdash_{\mathcal{B}} A$  and  $S_2 :: \delta \Vdash_{\mathcal{B}} B$ . In particular this means that the  
 2382 tensor union of the two nets  $S_1 :: \gamma + S_2 :: \delta + \langle p_1, p_2 \triangleright_{\otimes} p \rangle$   
 2383 is in  $\llbracket A \rrbracket_{\mathcal{B}} \otimes \llbracket B \rrbracket_{\mathcal{B}}$  and thus in  $\llbracket A \otimes B \rrbracket_{\mathcal{B}}$ .  
 2384 Note that  $S_1 :: \gamma + S_2 :: \delta + \langle p_1, p_2 \triangleright_{\otimes} p \rangle$  corresponds to  
 2385  $S_1 + S_2 + \langle p_1, p_2 \triangleright_{\otimes} p \rangle :: \gamma \parallel \delta$ . Thus  $S_1 + S_2 + \langle p_1, p_2 \triangleright_{\otimes}$   
 2386  $p \rangle :: \gamma \parallel \delta \Vdash_{\mathcal{B}} A \otimes B$ . Since this hold for any  $\gamma$  and  $\delta$  we  
 2387 conclude that  $S_1 + S_2 + \langle p_1, p_2 \triangleright_{\otimes} p \rangle \Vdash_{\mathcal{B}} \Gamma, \Delta, A \otimes B$  e.g.  
 2388  $S \Vdash_{\mathcal{B}} \Gamma, \Delta, A \otimes B$ .  $\square$   
 2389

2390 PROPOSITION D.11. *Given  $\mathcal{B}$  an approximable basis for any for-*  
 2391 *mula  $A$  of MLL the type  $\llbracket A \rrbracket_{\mathcal{B}}$  is approximable.*

2392 PROOF. By induction on  $A$ . In the case where  $A$  is atomic this  
 2393 follows from the definition.  
 2394

- 2395 • For a formula  $A \otimes B$ . Let us show  $\mathfrak{X}_1$  is orthogonal to  
 2396  $\llbracket A \otimes B \rrbracket_{\mathcal{B}}^\perp$ . The interpretation  $\llbracket A \otimes B \rrbracket_{\mathcal{B}}^\perp$  corresponds to  
 2397  $(\llbracket A \rrbracket_{\mathcal{B}} \otimes \llbracket B \rrbracket_{\mathcal{B}})^\perp$  and so  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \mathfrak{X} \llbracket B \rrbracket_{\mathcal{B}}^\perp$ . By induction  $\llbracket A \rrbracket_{\mathcal{B}}$   
 2398 and  $\llbracket B \rrbracket_{\mathcal{B}}$  are both approximable. In other words,  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \subseteq$   
 2399  $\{\mathfrak{X}_1\}^\perp$  and  $\llbracket B \rrbracket_{\mathcal{B}}^\perp \subseteq \{\mathfrak{X}_1\}^\perp$ .

2400 To be orthogonal to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \mathfrak{X} \llbracket B \rrbracket_{\mathcal{B}}^\perp$  is to be orthogonal to the  
 2401 nets  $S + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle$  where  $S$  belongs to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \supseteq \llbracket B \rrbracket_{\mathcal{B}}^\perp$ .  
 2402 After one step of cut elimination  $\mathfrak{X}_1 :: S + \langle p_1, p_2 \triangleright_{\mathfrak{X}}$   
 2403  $p \rangle$  reduces to  $\mathfrak{X}_2 :: S$  which is  $\mathfrak{X}_1 \parallel \mathfrak{X}_1 :: S$ . Since, by  
 2404 induction,  $\mathfrak{X}_1$  belongs to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp S :: \mathfrak{X}_1$  belongs to  $\llbracket B \rrbracket_{\mathcal{B}}^\perp$ .  
 2405 By induction,  $\llbracket B \rrbracket_{\mathcal{B}}^\perp$  is orthogonal to  $\mathfrak{X}_1$  hence  $S :: \mathfrak{X}_1 ::$   
 2406  $\mathfrak{X}_1$  reduces to  $\langle \triangleright_{\mathfrak{X}} \rangle$ .  
 2407

- 2408 • Now consider a formula of the form  $A \mathfrak{X} B$ . Again let us  
 2409 show that  $\mathfrak{X}_1$  is orthogonal to  $\llbracket A \mathfrak{X} B \rrbracket_{\mathcal{B}}^\perp$  e.g. to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \otimes \llbracket B \rrbracket_{\mathcal{B}}^\perp$ .  
 2410 To be orthogonal to that type is to be orthogonal to the nets  
 2411  $\bar{a} \parallel \bar{b} + \langle p_1, p_2 \triangleright_{\otimes} p \rangle$ . After one step of cut elimination  
 2412  $\mathfrak{X}_1 :: \bar{a} \parallel \bar{b} + \langle p_1, p_2 \triangleright_{\otimes} p \rangle$  reduces to  $\mathfrak{X}_2 :: \bar{a} \parallel \bar{b}$ . But  
 2413  $\mathfrak{X}_2$  is  $\mathfrak{X}_1 \bowtie \mathfrak{X}_1$  and by induction both  $\llbracket A \rrbracket_{\mathcal{B}}$  and  $\llbracket B \rrbracket_{\mathcal{B}}$   
 2414 contains  $\mathfrak{X}_1$  thus  $\llbracket A \rrbracket_{\mathcal{B}} \supseteq \llbracket B \rrbracket_{\mathcal{B}}$  contains  $\mathfrak{X}_2$ . Equivalently  
 2415 this means that  $\mathfrak{X}_2$  is orthogonal to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \parallel \llbracket B \rrbracket_{\mathcal{B}}^\perp$ , hence  
 2416  $\mathfrak{X}_2 \perp \bar{a} \parallel \bar{b}$ . This concludes to show that  $\mathfrak{X}_1$  belongs to  
 2417  $\llbracket A \mathfrak{X} B \rrbracket_{\mathcal{B}}^\perp$ .  $\square$   
 2418

2419 THEOREM 4.3. *Let  $S$  be a multiplicative net,  $\mathcal{B}$  an interpreta-*  
 2420 *tion basis, and  $\Gamma$  a sequent:*

- 2421 • if  $\mathcal{B}$  is regular, then  $S \vdash_{\text{MLL}} \Gamma \Rightarrow S \Vdash_{\mathcal{B}} \Gamma$ ,  
 2422
- 2423 • if  $\mathcal{B}$  is approximable, then  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Gamma \Rightarrow S \Vdash_{\mathcal{B}} \Gamma$ .  
 2424

2425 PROOF. of Theorem 4.3. We must treat two cases namely that of  
 2426 MLL and of MLL $^{\mathfrak{X}}$ . The proposition D.10 ensure that to conclude  
 2427 it is enough to show the base case of the induction.  
 2428

- 2429 • For MLL $^{\mathfrak{X}}$ , note that if  $\mathcal{B}$  is approximable then any inter-  
 2430 pretation  $\llbracket A \rrbracket_{\mathcal{B}}$  is approximable e.g. contains  $\mathfrak{X}_1$ . We can re-  
 2431 formulate this as  $\{\mathfrak{X}_1\}^{\perp\perp} \subseteq \llbracket A \rrbracket_{\mathcal{B}}$ . As a consequence since  
 2432 the construction on types preserve inclusion  $\{\mathfrak{X}_1\}^{\perp\perp} \supseteq \dots \supseteq$   
 2433  $\{\mathfrak{X}_1\}^{\perp\perp}$  is included in  $\llbracket A_1 \rrbracket_{\mathcal{B}} \supseteq \dots \supseteq \llbracket A_n \rrbracket_{\mathcal{B}}$  for any sequent  
 2434  $A_1, \dots, A_n$ . Using proposition D.5 this implies that  $\mathfrak{X}_n$  is  
 2435 contained in that type e.g. in  $\llbracket A_1, \dots, A_n \rrbracket_{\mathcal{B}}$ .  
 2436

- In the case of MLL by the corollary D.7  $\mathfrak{X}_2$  belongs to  $\llbracket X, X^\perp \rrbracket_{\mathcal{B}}$  for any basis.

□

## E COMPLEMENTS TO SECTION 5

### E.1 Proof of Theorem 5.8 – Reformulating Danos Regnier

*Definition E.1 (Switching graph).* Given a multiplicative which does not contain  $\mathfrak{Y}$ -links, we can associate a undirected graph denoted  $G(S)$  as follow;

- A vertex of  $G(S)$  is either a position of the net  $S$ , or a daimon or a cut link.
- There is an edge between two vertices whenever;
  - The two vertex  $u$  and  $v$  are positions of  $S$  and belong to the same tensor link.
  - $u$  is a cut link and  $v$  is an input of  $u$  (or vice versa).
  - $u$  is a daimon link and  $v$  is an output of  $u$  (or vice versa).

*Definition E.2.* Given  $E$  a set and  $X$  and  $Y$  two partitions of  $E$ . The graph induced by  $X$  and  $Y$  is the graph  $G(X, Y) = (V, E, b)$  such that;

- The vertex of  $G(X, Y)$  are the classes of  $X$  and  $Y$ .
- The edges of the graph are exactly the elements of  $E$ .
- The border of an edge  $e \in E$  is the pair  $\{c, c'\}$  of the class  $c$  in  $X$  containing  $e$  and the class  $c'$  in  $Y$  containing  $e$ .

Two partitions  $X$  and  $Y$  of a set  $E$  are *orthogonal* whenever their induced graph is acyclic and connected.

Then the classical results of Danos Regnier [3] can be stated as follow, either mentioning the acyclic and connected switching graphs, or the orthogonality of the induced partitions.

**THEOREM E.3.** *Given a multiplicative net  $S$  the propositions are equivalent;*

- $S$  is correct.
- For any switching  $\sigma S$  of  $S$  the graph  $G(\sigma S)$  is acyclic and connected.
- For any switching  $\sigma S$  the partitions  $sS^{\mathfrak{X}}$  and  $\uparrow \sigma S$  are orthogonal.

*Definition E.4 (image of a partition).* Given a partition  $P = \{C_1, \dots, C_n\}$  of a set  $X$  and a function  $f : X \rightarrow Y$  the image of  $P$  by  $f$  is the set  $\{f(C_1), \dots, f(C_n)\}$ , it is denoted  $f(P)$ .

**PROPOSITION E.5 (BIJECTIONS PRESERVE PARTITIONS).** *Given  $f : X \rightarrow Y$  a bijection between two sets. For any partition  $P$  of  $X$  the image  $f(P)$  is still a partition.*

*Definition E.6 (Image of a partition).* An image of a set  $\{p_1, \dots, p_n\}$  of positions is denoted  $X^{\mathfrak{X}}$  and is a daimon  $\langle \triangleright_{\mathfrak{X}} p_1, \dots, p_n \rangle$ . The image of a partition  $P = X_1, \dots, X_k$  of a set of positions  $X$ , is denoted  $P^{\mathfrak{X}}$ , and corresponds to the sum  $X_1^{\mathfrak{X}} + \dots + X_k^{\mathfrak{X}}$ .

*Remark 47.* The image of a set of position is *not unique* since the order of the output of the daimon link may be reordered. As a consequence the image of a partition is also not unique.

**PROPOSITION E.7.** *Given two partitions  $P$  and  $Q$  of a set of positions  $X$ . The two assertions are equivalent;*

- The partitions  $P$  and  $Q$  are orthogonal.
- Given a set  $Y$  disjoint from  $X$ , and  $\sigma : X \rightarrow Y$  a bijection, The interaction  $P^{\mathfrak{X}} ::_{\sigma} Q^{\mathfrak{X}}$  reduces to  $\langle \triangleright_{\mathfrak{X}} \rangle$ . In particular,  $P^{\mathfrak{X}}$  and  $Q^{\mathfrak{X}}$  are orthogonal.

**PROOF.** By induction on the size of the set  $X$ . □

**PROPOSITION E.8.** *Given two cut free nets  $S$  and  $T$  witnessing respectively  $A_1, \dots, A_n$  and  $A_1^\perp, \dots, A_n^\perp$ . The assertions are equivalent;*

- $S$  and  $T$  are orthogonal.
- $S^{\mathfrak{X}}$  and  $T^{\mathfrak{X}}$  are orthogonal.

**PROOF.** The multiplicative cuts get eliminated and can always be performed first due to the commutation results. Furthermore since the nets witness dual formulas no non homogeneous cut will occur. □

**COROLLARY E.9.** *Given  $S$  a net witnessing  $A$ . For a switching  $\sigma S$  of  $S$ , the assertions are equivalent;*

- The partitions  $s(S^{\mathfrak{X}})$  and  $\sigma S$  are orthogonal.
- $S$  is orthogonal to any net  $T$  witnessing  $A^\perp$  and such that  $T^{\mathfrak{X}} = (\sigma S)^{\mathfrak{X}}$ .

Using this corollary and the established result of Danos Regnier we obtain the following;

**PROPOSITION E.10.** *Given  $S$  a multiplicative net with one conclusion witnessing  $A$ . The assertions are equivalent;*

- $S$  is correct.
- $S$  is orthogonal to  $\text{tests}(A)$ .

We can indeed generalize this result the case where  $S$  has multiple conclusions, as usual we do this by transformin the net with multiple conclusions in a net with one conclusion by adding a bunch of  $\mathfrak{Y}$ -links.

*Definition E.11 (General connectives).* A generalized  $\mathfrak{Y}$ -link on the positions  $p_1, \dots, p_n$  is a module denoted  $\langle p_0, \dots, p_n \triangleright_{\mathfrak{Y}} p \rangle$  and defined by the following induction;

- $\langle p_0, p_1 \triangleright_{\mathfrak{Y}^1} p \rangle = \langle p_0, p_1 \triangleright_{\mathfrak{Y}} p^1 \rangle$ .
- For any  $n > 0$  we defined  $\langle p_1, \dots, p_{n+2} \triangleright_{\mathfrak{Y}^{n+1}} p \rangle = \langle p^{n-1}, p_n \triangleright_{\mathfrak{Y}^n} p \rangle + \langle p^n, p_{n+1} \triangleright_{\mathfrak{Y}} p^{n+1} \rangle$ .

Similarly we defined the generalized tensor links  $\langle p_0, \dots, p_n \triangleright_{\otimes} p \rangle$ .

**PROPOSITION E.12.** *Let  $S$  be a net with conclusions  $p_0, \dots, p_n, q_1, \dots, q_k$ . Let  $S_1, \dots, S_n, T_1, \dots, T_k$  be  $n$  nets with one conclusion the assertions are equivalent;*

- (1)  $S \perp S_1 \parallel \dots \parallel S_n \parallel T_1 \parallel \dots \parallel T_k$ .
- (2)  $S + \langle p_0, \dots, p_n \triangleright_{\mathfrak{Y}^n} p \rangle \perp S_1 + \dots + S_n + \langle S_1(1), \dots, S_n(1) \triangleright_{\otimes} q \rangle \parallel T_1 \parallel \dots \parallel T_k$

**PROOF.** By a simple induction of the size of the generalized  $\mathfrak{Y}$  connective. □

This easily leads to a proof of the Danos Regnier theorem as stated in section 5.

**THEOREM 5.8 (DANOS REGNIER TESTS, [3]).** *Given a cut-free net  $S \vdash A_1, \dots, A_n$ , the two following assertions are equivalent:*

- $S \vdash_{\text{MLL}^{\mathfrak{X}}} A_1, \dots, A_n$ ;
- $S$  is orthogonal to  $\text{tests}(A_1) \parallel \dots \parallel \text{tests}(A_n)$ .

## E.2 Correctness of Tests

**Definition E.13.** Given a net  $S$  and  $\sigma S$  one of its switching, the *node measure* of  $S$  in  $\sigma$  is the number of nodes of the graph  $G(\sigma S)$  respectively the *edge measure* of  $S$  in  $\sigma$  is the number of edges of the graph  $G(\sigma S)$ . We respectively denote them  $N(\sigma S)$  and  $E(\sigma S)$ .

**PROPOSITION E.14.** *Given a net  $S$  for any switching  $\sigma S$  and  $\sigma' S$ ;  $N(\sigma S) = N(\sigma' S)$  and  $E(\sigma S) = E(\sigma' S)$*

**Remark 48.** As a consequence we can define the node measure and edge measure of a net  $S$  as the node measure and edge measure of one of its switching. We denote it by  $N(S)$  and  $E(S)$ .

**PROPOSITION E.15.** *Given a graph  $G = (V, E)$  such that  $\text{card}(V) - \text{card}(E) = 1$  the assertions are equivalent;*

- $G$  is acyclic.
- $G$  is connected.

**PROPOSITION E.16.** *Whenever a graph  $G = (V, E)$  is acyclic and connected we have  $\text{card}(V) - \text{card}(E) = 1$ .*

**THEOREM E.17.** *Cut-elimination preserves correctness.*

**PROOF.** Note that the step of cut elimination preserve connectedness of the switching graphs, and furthermore that they preserve the measure  $N(S) - E(S)$  thus the proposition E.15 allows us to conclude.  $\square$

**PROPOSITION E.18.** *The switching rewriting  $\rightarrow_{\mathfrak{A}}$  preserves correctness.*

**PROOF.** Indeed since whenever  $S \rightarrow_{\mathfrak{A}} S'$  the switching of  $S'$  are include in the switching of  $S$ . Thus if all the switching of  $S$  are acyclic and connected it is also true in particular for those of  $S'$   $\square$

**PROPOSITION E.19.** *Eta-expansion preserves correctness.*

**PROOF.** Note that the obtained net preserve connection, furthermore the measure  $N(S) - E(S)$  is preserved.  $\square$

**PROPOSITION E.20.** *For any two correct nets  $S$  and  $T$  if  $T$  has one output, the interaction  $S :: T$  remains correct.*

**PROOF.** Indeed in the interaction  $S :: T$  the connectivity is preserved. Furthermore  $N(S) + N(T) + 1 - (E(S) + E(T) + 2)$  is equal to  $3 - 2$  and so is equal to 1.  $\square$

**Definition E.21 (Syntax tree).** A net  $\mathcal{N}$  is a *representation* of an MLL formula  $F$ , when it satisfies the inductive condition:

- Whenever  $F$  is an atomic formula,  $\mathcal{N}$  is an hypergraph consisting of only one position  $p$ .
- Given that  $\square = \otimes$  or  $\square = \wp$ ; whenever  $F = F_1 \square F_2$ , for any nets  $\mathcal{N}_1$  and  $\mathcal{N}_2$  the net  $\mathcal{N}_1 + \mathcal{N}_2 + \langle p_1, p_2 \triangleright_l p \rangle$  is a representation of  $F$  if  $\mathcal{N}_1$  represents  $F_1$  and has for conclusion  $p_1$ ,  $\mathcal{N}_2$  representing  $F_2$  and has for conclusion  $p_2$ , and the label  $l$  corresponds to  $\square$ , while  $p$  is a fresh position.

A *syntax tree* is the representation of a formula. Given a formula  $A$  we denote  $\text{syn}(A)$  the set of representations of  $A$ .

**PROPOSITION E.22.** *Given a net  $r$  with  $n$  conclusions that is the sum of the syntactical representation of formulas containing only tensors.  $r :: \sum_{1 \leq i \leq n} \mathfrak{A}_1$  reduces to  $\uparrow r^{\mathfrak{A}} + \sum_{1 \leq i \leq n} \langle p_i, q_i \triangleright_{\text{cut}} \rangle$ .*

**THEOREM 5.7 (CORRECTNESS OF TESTS).** *Any test of a formula  $A$  is a proof in MLL $^{\mathfrak{A}}$  of the negation  $A^\perp$ .*

**PROOF.** of Theorem 5.7. Consider a formula  $A$  of MLL. Consider the correct net  $\mathfrak{A}_2 = \langle \triangleright_{\mathfrak{A}} p_1, p_2 \rangle$  and perform a bunch of eta expansion to obtain a correct net  $S$  witness  $A^\perp$ ,  $A$ . In that case  $S$  is of the form  $r + r' + \sum_{1 \leq i \leq m} \langle \triangleright_{\mathfrak{A}} r_i, r'_i \rangle$  where  $r$  is a representation of  $A$  and  $r'$  is a representation of  $A^\perp$  and the  $r_i$ 's are the initial positions of  $r$  and similarly for the  $r'_i$ 's. Perform a bunch of switching rewriting witnessing  $A$  e.g. on the connectives above  $p_2$ . we obtain a net  $S'$  which remains correct with conclusion  $p_1, q_1, \dots, q_k$ . Now make  $S'$  interact with unary daimons on its conclusion  $q_1, \dots, q_k$ , e.g. consider

$$T = S' :: \{q_i \mapsto t_i, 1 \leq i \leq n\} \sum_{1 \leq i \leq k} \langle \triangleright_{\mathfrak{A}} t_i \rangle.$$

Again this transformation preserve correctness, thus  $T$  is correct. Now since cut elimination preserve correctness any redex of  $T$  will be correct. In particular these cuts reduces to the partitions of  $\uparrow(\sigma A)$ , e.g. we obtain,

$$r + \sum_{1 \leq i \leq m} \langle \triangleright_{\mathfrak{A}} r_i, r'_i \rangle + \sum_{1 \leq i \leq n} \langle r'_i, q_i \triangleright_{\text{cut}} \rangle (\uparrow r')^{\mathfrak{A}}.$$

After reducing the glueing we obtain a net of the form  $r + (\uparrow r')^{\mathfrak{A}}$  where  $r$  and  $r'$  are representations of dual formulas. This net is correct again since cut elimination preserve correctness. This net is a test of  $A^\perp$  hence we have showed that any test of  $A^\perp$  is correct and we can conclude.  $\square$

## E.3 Decomposition

**PROPOSITION E.23.** *Given two sets of nets  $A$  and  $B$ ;*

- $A \parallel B = A^{\perp\perp} \parallel B^{\perp\perp}$ .
- $A \succ B = A^{\perp\perp} \succ B^{\perp\perp}$ .
- $A \wp B = A^{\perp\perp} \wp B^{\perp\perp}$ .
- $A \otimes B = A^{\perp\perp} \otimes B^{\perp\perp}$ .

**PROOF.** We treat each cases independently.

- By duality,  $(A \parallel B)^\perp = A^\perp \succ B^\perp$  While  $(A^{\perp\perp} \parallel B^{\perp\perp})^\perp = A^{\perp\perp\perp} \succ B^{\perp\perp\perp}$  e.g. tri-orthogonality corresponds to orthogonality [8] this is  $A^\perp \succ B^\perp$  and so  $(A \parallel B)^\perp$ .

The two orthogonal of the two types are equal thus they are equal  $A \parallel B = A^{\perp\perp} \parallel B^{\perp\perp}$ .

- Say a net belongs to  $A \succ B$  equivalently it belongs to  $(A^\perp \parallel B^\perp)^\perp$  which since tri-orthogonal is orthogonality is  $(A^{\perp\perp\perp} \parallel B^{\perp\perp\perp})^\perp$  which is  $A^{\perp\perp} \succ B^{\perp\perp}$ .
- Note that  $A \wp B$  is equal to  $(A^\perp \otimes B^\perp)^\perp$  and thus to  $(A^{\perp\perp\perp} \otimes B^{\perp\perp\perp})^\perp$  which is exactly  $A^{\perp\perp} \wp B^{\perp\perp}$ .
- Similarly to the previous case.  $\square$

**PROPOSITION E.24 (DECOMPOSITION).** *Let  $\mathcal{B}$  be an interpretation basis,  $\mathcal{H}$  be an hypersequent,  $A, B$  two formulas and  $S$  be a multiplicative net.*

- If  $S = S_0 + l$  has a terminal  $\wp$  link  $l$  above its last conclusion and  $S \in \llbracket \mathcal{H}, A \wp B \rrbracket_{\mathcal{B}}$ ; then  $S_0$  belongs to  $\llbracket \mathcal{H}, A, B \rrbracket_{\mathcal{B}}$
- If  $S = S_0 + l$  has a terminal  $\wp$  link  $l$  above its last conclusion and  $S \in \llbracket \mathcal{H} \parallel A \wp B \rrbracket_{\mathcal{B}}$ ; then  $S_0$  belongs to  $\llbracket \mathcal{H} \parallel (A, B) \rrbracket_{\mathcal{B}}$



- If  $S = S_0 + l$  has a terminal  $\otimes$  link  $l$  above its last conclusion and  $S \in \llbracket \mathcal{H}, A \otimes B \rrbracket_{\mathcal{B}}$ ; then  $S_0$  belongs to  $\llbracket \mathcal{H}, (A \parallel B) \rrbracket_{\mathcal{B}}$
- If  $S = S_0 + l$  has a terminal  $\otimes$  link  $l$  above its last conclusion and  $S \in \llbracket \mathcal{H} \parallel A \otimes B \rrbracket_{\mathcal{B}}$ ; then  $S_0$  belongs to  $\llbracket \mathcal{H} \parallel A \parallel B \rrbracket_{\mathcal{B}}$

PROOF. • Consider  $S$  a net with a terminal  $\varnothing$ -link  $l$  decomposing  $S = S_0 + l$  such that  $l$  outputs the only conclusion of  $S$ . Say  $S$  belongs to  $\llbracket A \varnothing B \rrbracket_{\mathcal{B}}$  equivalently  $S$  is orthogonal to  $\llbracket A \rrbracket_{\mathcal{B}}^{\perp} \otimes \llbracket B \rrbracket_{\mathcal{B}}^{\perp}$ , since a multiplicative cut can always be performed first this implies that  $S_0$  is orthogonal to  $\llbracket A \rrbracket_{\mathcal{B}}^{\perp} \parallel \llbracket B \rrbracket_{\mathcal{B}}^{\perp}$  thus  $S_0$  belongs to  $\llbracket A \rrbracket_{\mathcal{B}}, \llbracket B \rrbracket_{\mathcal{B}}$ . This reasoning easily adapts to  $\llbracket \mathcal{H}, A, B \rrbracket_{\mathcal{B}}$ .

- The reasonment for a net in  $\llbracket A \otimes B \rrbracket_{\mathcal{B}}$  is similar and also easily adapts to the case  $\llbracket \mathcal{H}, A \otimes B \rrbracket_{\mathcal{B}}$ .
- Consider on the other hand a net  $S$  in  $\llbracket \mathcal{H} \parallel A \otimes B \rrbracket_{\mathcal{B}}$ , such that  $S = S_0 + l$  where  $l$  is a tensor link and outputs the last conclusion of  $S$ . Equivalently  $S$  is a net orthogonal to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^{\perp}, \llbracket A \otimes B \rrbracket_{\mathcal{B}}^{\perp}$  e.g.  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^{\perp}, \llbracket A \rrbracket_{\mathcal{B}}^{\perp} \otimes \llbracket B \rrbracket_{\mathcal{B}}^{\perp}$ . Since the multiplicative cut commute to the left and using proposition E.23 it follows that  $S_0$  is orthogonal to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^{\perp}, \llbracket A \rrbracket_{\mathcal{B}}^{\perp}, \llbracket B \rrbracket_{\mathcal{B}}^{\perp}$ . Equivalently  $S_0$  belongs to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}} \parallel \llbracket A \rrbracket_{\mathcal{B}} \parallel \llbracket B \rrbracket_{\mathcal{B}}$ .
- Similarly we treat the case  $\llbracket \mathcal{H} \parallel A \varnothing B \rrbracket_{\mathcal{B}}$ .

□

## E.4 Contracted, open and self dual Types

*Definition E.25 (Contracted type).* A type  $A$  is *contracted* whenever for any  $S$  in  $A$ , there exists a sequent  $\Gamma$  made only of atomic formulas such that  $S < \Gamma$ .

*Remark 49.* A type  $A$  is contracted equivalently whenever any net in  $A$  cannot have connectives has terminal link.

*Definition E.26 (type interface).* A *type interface* is a sequence  $(s_1, \dots, s_n)$  of symbols  $s_i$  which belong to  $\{\varnothing, \varnothing, \otimes\}$ . The *type interface* of a net  $S$  with conclusions  $(p_1, \dots, p_n)$  is the sequence  $(s_1, \dots, s_n)$  where each  $s_i$  is the label of the link which outputs  $p_i$ .

A type interface  $s$  is *coherent* with a type  $A$  whenever there exists a net  $S$  in  $A$  with interface  $s$ . A sequence  $(X_1, \dots, X_n)$  of subsets  $X_i$  of  $\{\varnothing, \varnothing, \otimes\}$  is *coherent* with a type  $A$  whenever any interface  $(s_1, \dots, s_n)$  with  $s_i \in X_i$  is coherent with  $A$ .

*Definition E.27 (Open type).* A type is *open* whenever it is coherent with  $(\{\varnothing, \otimes\}, \dots, \{\varnothing, \otimes\})$ .

PROPOSITION E.28. *Given two open types  $A$  and  $B$ ;*

- *Their functional composition  $A > B$  is still open.*
- *Their parallel composition  $A \parallel B$  is still open.*

PROOF. Indeed the parallel  $A \parallel B$  remains open since it contains in particular the nets of the form  $a \parallel b$  with  $a \in A$  and  $b \in B$ . Similarly this remains true for  $A > B$  since it contains the merge  $A \bowtie B$  (see the appendix D.2). □

PROPOSITION E.29. *Whenever a type  $A$  is open its dual is contracted.*

PROOF. Since a net  $S$  in  $A^{\perp}$  is orthogonal to any net in  $A$  in order to avoid clashing cuts the outputs of  $S$  cannot be the outputs of connectives. □

PROPOSITION E.30. *A self dual type has only nets with no connectives has conclusions.*

PROOF. A net  $S$  in a self dual type  $A$  must verify  $S \perp S$ . If  $S$  has a terminal connective then the interaction  $S :: S$  gives rise to a clashing cut and cut elimination fails. □

PROPOSITION E.31. *The orthogonal of a self dual type  $A$  with one output is an open type.*

PROOF. Since  $A$  is self dual it contains no net which have a terminal connective. By the proposition 3.6 this means that  $A^{\perp}$  must contain nets with  $\varnothing$  link and  $\otimes$  link as conclusions. Thus  $A^{\perp}$  is an open type. □

*Remark 50.* The previous proposition E.30 together with proposition 3.6 implies in particular that there cannot be any type  $A$  that is invariant under orthogonality, e.g. such that  $A = A^{\perp}$ . In particular this means that the inclusion  $A \subseteq A^{\perp}$  of a self dual type is always strict.

PROPOSITION E.32. *Given a self dual interpretation basis  $\mathcal{B}$  for any atomic hypersequent  $\mathcal{H}$  the type  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}$  is open.*

PROOF. This follows from proposition E.28 and proposition E.31 by induction on the size of the hypersequent. □

## E.5 Truncation Lemma

Recall that a *substitution* is a map  $\theta$  which maps propositional variables to formulas. Naturally substitutions can be lifted by induction to formulas and to sequents;  $\theta(A \square B) \triangleq \theta A \square \theta B$  and  $\theta(\Gamma, A) = \theta(\Gamma), \theta A$ . A sequent  $\Gamma$  is an *instance* of a sequent  $\Delta$  whenever there exists a substitution  $\theta$  such that  $\theta \Delta = \Gamma$ . In that case we denote  $\Delta \leq \Gamma$ .

*Definition E.33 (Approximation of a sequent).* A net  $S$  is an (*sub*)-*approximation* of a sequent  $\Gamma$ , denoted  $S < \Gamma$ , if it witnesses a sequent  $\Delta \leq \Gamma$ .

There is a simple inductive process to associate a sequent  $\downarrow \mathcal{H}$  to an hypersequent  $\mathcal{H}$ ;  $\downarrow A \triangleq A$  while  $\downarrow \mathcal{H}_1, \mathcal{H}_2 \triangleq \downarrow \mathcal{H}_1, \downarrow \mathcal{H}_2$  and  $\downarrow \mathcal{H}_1 \parallel \mathcal{H}_2 \triangleq \downarrow \mathcal{H}_1, \downarrow \mathcal{H}_2$ .

*Remark 51.* A sequent that is the truncation of a sequent of  $n$  formulas must also contain  $n$  formulas.

Furthermore, if two representation of sequents  $\Gamma = A_1, \dots, A_n$  and  $\Delta = B_1, \dots, B_n$  are such that  $\Delta \leq \Gamma$  this implies that for each index  $1 \leq i \leq n$  we have  $B_i \leq A_i$  specifically  $A_i[X_1 \mapsto F_1, \dots, X_k \mapsto F_k] = B_i$ .

LEMMA E.34 (TRUNCATION). *Let  $\mathcal{B}$  be a self dual basis. For any net  $S$ ;  $S \in \llbracket \mathcal{H} \rrbracket_{\mathcal{B}} \Rightarrow S < \downarrow \mathcal{H}$ .*

PROOF. of lemma E.34. By induction on the hypersequent using the measure  $(c, n)$  where  $c$  is the number of connectives in the hypersequent and  $n$  is the size of the hypersequent.

- If the hypersequent is made of one atomic formula  $X$  then  $S \in \llbracket X \rrbracket_{\mathcal{B}}$  implies that  $S$  belongs to a contracted type, e.g. the outputs of  $S$  are outputs of a daimon. This by definition means  $S < X$ .

- Say the hypersequent is made only of atomic formulas. The hypersequent may be of the form  $\mathcal{H}_1, \mathcal{H}_2$  then  $\llbracket \mathcal{H}_1, \mathcal{H}_2 \rrbracket_{\mathcal{B}} = (\llbracket \mathcal{H}_1 \rrbracket_{\mathcal{B}}^{\perp} \parallel \llbracket \mathcal{H}_2 \rrbracket_{\mathcal{B}}^{\perp})^{\perp}$  since,  $\llbracket \mathcal{H}_1 \rrbracket_{\mathcal{B}}^{\perp}$  and  $\llbracket \mathcal{H}_2 \rrbracket_{\mathcal{B}}^{\perp}$  are open types their parallel composition remains an open type.  $\llbracket \mathcal{H}_1, \mathcal{H}_2 \rrbracket_{\mathcal{B}}$  is the orthogonal of an open type hence it is a contracted type. Thus any sequent in that type as outputs which comes from a daimon link and thus is the approximation of any sequent of size  $n$  in particular  $S \prec \downarrow \mathcal{H}$ . We do a similar reasoning when  $\mathcal{H} = \mathcal{H}_1 \parallel \mathcal{H}_2$ .
- Case of non-atomic hypersequent with a virgula as main connective. Assume that  $S$  has a terminal connective link, say the hypersequent is of the form  $\mathcal{H}, A \wp B$  such that  $S = S_0 + l$  where  $l$  is a  $\wp$ -link and is the last conclusion of  $S$ . By proposition E.24 it follows that  $S_0$  belongs to  $\llbracket \mathcal{H}, A, B \rrbracket_{\mathcal{B}}$  the measure of that hypersequent as decreased and so we apply the induction hypothesis;  $S_0 \prec \downarrow (\mathcal{H}, A, B)$  indeed it follows that  $S \prec \downarrow (\mathcal{H}, A \wp B)$ . A similar argument works for an hypersequent of the form  $\mathcal{H}, A \otimes B$ .
- Case of non-atomic hypersequent with a parallel as main connective. On the other hand say  $S$  belongs to the interpretation  $\llbracket \mathcal{H} \parallel A \wp B \rrbracket_{\mathcal{B}}$  then it is orthogonal to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^{\perp} \succ \llbracket A \wp B \rrbracket_{\mathcal{B}}^{\perp}$  and thus  $S_0$  is orthogonal to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^{\perp} \succ \llbracket A, B \rrbracket_{\mathcal{B}}^{\perp}$ . Equivalently using proposition E.24  $S_0$  belongs to  $\llbracket \mathcal{H} \parallel A, B \rrbracket_{\mathcal{B}}$  the size of the hypersequent as decreased and so we can apply the induction hypothesis;  $S_0 \prec \downarrow (\mathcal{H}, A, B)$  and thus  $S \prec \downarrow (\mathcal{H}, A \wp B)$ . A similar argument works for an hypersequent of the form  $\mathcal{H} \parallel A \otimes B$ . Similarly we treat the case of  $\mathcal{H}, A \otimes B$ .

□

## E.6 Proofs of section 5.2

PROPOSITION E.35. *Let  $S$  a multiplicative net and  $\Delta$  and  $\Gamma$  two sequents such that  $\Delta \leq \Gamma$ ;*

$$S \vdash_{\text{MLL}^{\times}} \Delta \Rightarrow S \vdash_{\text{MLL}^{\times}} \Gamma.$$

PROOF. This is done by induction on the represented proof by  $S$ , specifically by looking at the last rule of the proof. The inductions steps are trivial. The base case hold since the daimon rule may introduce any sequents whenever it introduces  $\Delta$  it may as well introduce a sequent of the same size, say  $\Gamma$ . □

PROPOSITION E.36. *Given a net  $S$  and a sequent  $\Gamma$ . If  $S \prec \Gamma$  then there exists a truncation  $\Delta$  of  $S$  such that  $S \vdash \Delta$ .*

Notation 22. Given a proof system  $\mathcal{S}$  (which would be either MLL,  $\text{MLL}^{\times}$  or  $\text{MLL}_2$  here), for a sequent  $\Gamma$  in  $\text{MLL}_2$  we denote by  $\llbracket \Gamma : \mathcal{S} \rrbracket$  the set of nets which represent a proof in  $\mathcal{S}$  of the sequent  $\Gamma$ .

Remark 52. Proposition E.35 implies that whenever  $A \leq B$  the set  $\llbracket A : \text{MLL}^{\times} \rrbracket$  is contained in  $\llbracket B : \text{MLL}^{\times} \rrbracket$ .

THEOREM 5.10 (MLL<sup>×</sup> COMPLETENESS). *Let  $S$  be a cut-free net and  $\Gamma$  be a sequent. For any self dual and approximable interpretation basis  $\mathcal{B}$ , if  $S$  belongs to  $\llbracket \Gamma \rrbracket_{\mathcal{B}}$  then  $S$  represents a proof of  $\Gamma$  in  $\text{MLL}^{\times}$ .*

PROOF. of Theorem 5.10. Since the basis is self dual any net  $S$  in  $\llbracket \Gamma \rrbracket_{\mathcal{B}}$  is an approximation of  $\downarrow \Gamma$  by the lemma E.34, since  $\Gamma$  is a simple sequent, this means  $S$  is an approximation of  $\Gamma$ . This means

using proposition E.36 that  $S$  is the witness of a sequent  $\Delta$  such that  $\Delta \leq \Gamma$ . Denote  $\Gamma = A_1, \dots, A_n$  and  $\Delta = B_1, \dots, B_n$ .

Furhtermore  $S$  belongs to  $\llbracket \Gamma \rrbracket_{\mathcal{B}}$  which equivalently means that,

$$S \perp \llbracket A_1^{\perp} \rrbracket_{\mathcal{B}} \parallel \dots \parallel \llbracket A_n^{\perp} \rrbracket_{\mathcal{B}}.$$

Since adequacy is true for  $\text{MLL}^{\times}$  (theorem 4.3), for any index  $i$  the interpretation  $\llbracket A_i^{\perp} \rrbracket_{\mathcal{B}}$  contains the proofs of  $A_i^{\perp}$  therefore,

For any index  $i$  using the remark 52 since  $B_i \leq A_i$  it follows that  $B_i^{\perp} \leq A_i^{\perp}$  and thus  $\llbracket B_i^{\perp} : \text{MLL}^{\times} \rrbracket \subseteq \llbracket A_i^{\perp} : \text{MLL}^{\times} \rrbracket$ .

$$S \perp \llbracket B_1^{\perp} : \text{MLL}^{\times} \rrbracket \parallel \dots \parallel \llbracket B_n^{\perp} : \text{MLL}^{\times} \rrbracket.$$

In particular we have shown that the tests of a formula  $F$  are proofs of  $F^{\perp}$  thus tests( $B_i$ )  $\subseteq \llbracket B_i^{\perp} : \text{MLL}^{\times} \rrbracket$  for any index  $i$ . Hence,

$$S \perp \text{tests}(B_1) \parallel \dots \parallel \text{tests}(B_n).$$

Since  $S$  is a witness of  $\Delta$  we conclude using proposition ?? that  $S \vdash_{\text{MLL}^{\times}} \Delta$ . In particular using the proposition E.35 it follows that  $S \vdash_{\text{MLL}^{\times}} \Gamma$ . □

PROOF. of Theorem 5.13 Since the basis is self dual and approximable the theorem 5.10 ensure that  $S$  is a proof in  $\text{MLL}^{\times}$  of  $\Gamma$ . To conclude it remains to show that the binary daimons realise only sequents of the form  $X, X^{\perp}$ .

Indeed consider a binary daimon  $\wp_2$  from the proposition 4.2 it belongs to  $A \wp A^{\perp}$  for any type  $A$ . Thus it belongs to  $\llbracket \bigcap_{X \in \Omega} X, X^{\perp} \rrbracket_{\mathcal{B}}$ . But indeed  $S$  does not belongs to  $\llbracket \bigcap_{Y \in \Omega} \bigcap_{X \in \Omega} X, Y \rrbracket_{\mathcal{B}}$  since then  $S$  belongs to  $A \succ A$  for any type. Chosing  $A$  with the right measure  $\wp_2$  cannot belong to  $A \succ A$  and thus  $S$  does not belong to  $\llbracket \bigcap_{Y \in \Omega} \bigcap_{X \in \Omega} X, Y \rrbracket_{\mathcal{B}}$ . This show that the binary daimons of  $S$  realise only sequents of the form  $X, X^{\perp}$  hence  $\wp_2$  is a proof of MLL, this indeed is preserved by induction. □

## F COMPLEMENTS TO SECTION 6

PROPOSITION F.1. *Let  $\mathcal{H}$  be an hyper-sequent and  $X$  be a propositional variable. If  $X$  does not occur free in  $\mathcal{H}$  then for any type with one conclusion  $R$ ;  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}} = \llbracket \mathcal{H} \rrbracket_{\mathcal{B}\{X \mapsto R\}}$ .*

PROOF. of proposition F.1. By induction on the couple  $(c, n)$  where  $c$  is the number of connectives in  $\mathcal{H}$  and  $n$  is the size of the hypersequent. Consider  $R$  some type with one output.

- Base case  $(0, 1)$  e.g. hypersequent made of one formula. Note that  $\llbracket Y \rrbracket_{\mathcal{B}\{X \mapsto R\}}$  is equal to  $\llbracket Y \rrbracket_{\mathcal{B}}$  by definition.
- We show that whenever the property hold for  $(c, n)$  and  $(c', n')$  it holds for  $(c + c', n + n')$ . To do so we show the parallel operator and virgula's of hypersequent preserve the property.

Say  $\mathcal{H} = \mathcal{H}_1 \parallel \mathcal{H}_2$  then  $\llbracket \mathcal{H}_1 \parallel \mathcal{H}_2 \rrbracket_{\mathcal{B}\{X \mapsto R\}}$  corresponds to  $\llbracket \mathcal{H}_1 \rrbracket_{\mathcal{B}\{X \mapsto R\}} \parallel \llbracket \mathcal{H}_2 \rrbracket_{\mathcal{B}\{X \mapsto R\}}$  by induction hypothesis on  $\mathcal{H}_1$  and  $\mathcal{H}_2$  this is equal to  $\llbracket \mathcal{H}_1 \rrbracket_{\mathcal{B}} \parallel \llbracket \mathcal{H}_2 \rrbracket_{\mathcal{B}}$  which is  $\llbracket \mathcal{H}_1 \parallel \mathcal{H}_2 \rrbracket_{\mathcal{B}}$ . Similarly we treat the case  $\mathcal{H} = \mathcal{H}_1, \mathcal{H}_2$

- Show that connectives can be decomposed, increasing the size of the hypersequent but decreasing  $c$ . We show that if the property hold for the cases  $(c, 1)$  and  $(c', 1)$  it holds for the cases  $(c + c' + 1, 1)$ .

Say the hypersequent is  $\llbracket A \wp B \rrbracket_{\mathcal{B}\{X \mapsto R\}}$ ; if a net  $S = S_0 + l$  where  $l$  denotes the  $\wp$ -link which outputs the only conclusion belongs to  $\llbracket A \wp B \rrbracket_{\mathcal{B}\{X \mapsto R\}}$  then  $S_0$  belongs

$\llbracket A, B \rrbracket_{\mathcal{B}\{X \mapsto R\}}$  applying induction hypothesis this is equal to  $\llbracket A, B \rrbracket_{\mathcal{B}}$  and thus  $S$  belongs to  $\llbracket A \wp B \rrbracket_{\mathcal{B}}$ .

If  $S$  has for its last conclusion  $p$  the output of a daimon link then we can add a  $\wp$ -link to  $S$  and duplicate the conclusion  $p$  obtaining  $S + l$  and ensure that  $S + l$  belongs to  $\llbracket A \wp B \rrbracket_{\mathcal{B}}$ . Then since a daimon link can simulate a well-formed  $\wp$ -link below a daimon this implies that  $S$  belongs to  $\llbracket A \wp B \rrbracket_{\mathcal{B}}$ . For the case of  $\llbracket A \otimes B \rrbracket_{\mathcal{B}\{X \mapsto R\}}$  the argument is similar.

- The base together with the third case allow us to show the property hold for any hypersequent of measure  $(c, 1)$ . This together with the first and second case allows us to show it hold for any hypersequent of measure  $(c, n)$ .

□

**THEOREM 6.2 (ADEQUACY FOR MLL<sub>2</sub>).** *Given  $S$  a proof in MLL<sub>2</sub> of a sequent  $\Gamma$ . For any regular interpretation basis  $\mathcal{B}$ ,  $S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$ .*

**PROOF.** of Theorem 6.2. Assume  $S$  represents a proof  $\pi$  of  $\Gamma$ . We proceed by induction on  $\pi$  looking at the last rule of  $\pi$ .

- If  $\pi$  is an axiom rule then  $\pi$  is a proof of a sequent  $A, A^\perp$  for some formula  $A$ . On the other hand  $S$  is of the form  $\langle \triangleright_{\mathfrak{X}} p, q \rangle$  e.g. corresponds to  $\mathfrak{X}_2$ . The proposition 4.2 ensures that  $\mathfrak{X}_2$  belongs to  $\mathbf{A} \succ \mathbf{A}^\perp$  for any type  $\mathbf{A}$  hence it belongs to  $\llbracket X, X^\perp \rrbracket_{\mathcal{B}}$  for any basis  $\mathcal{B}$ .

- If the last rule of  $\pi$  is the introduction of a  $\wp$  connective or a tensor connective we reason as in the proof of MLL adequacy.

- If the last rule of  $\pi$  is an existential quantifier, then  $S$  can be decomposed as  $S_0 + l$  where  $l$  is the existential link corresponding to the last rule of  $\pi$ . Then  $S$  is a proof of  $\Gamma, \exists X A$  and  $S_0$  is a proof of  $\Gamma, A$ .

$S_0$  is a representation of a smaller proof than  $S$ , thus we can apply the induction hypothesis on  $S_0$  and claim that  $S_0$  belongs to  $\llbracket \Gamma, A \rrbracket_{\mathcal{B}}$  for any basis  $\mathcal{B}$ . This means that for any  $\gamma$  in  $\llbracket \Gamma \rrbracket_{\mathcal{B}}^\perp$  the interaction  $S_0 :: \gamma$  belongs to  $\llbracket A \rrbracket_{\mathcal{B}}$ . Therefore  $S_0 :: \gamma$  belongs to  $\bigcup_{R \in \Omega} \llbracket A \rrbracket_{\mathcal{B}\{X \mapsto R\}}$  and so to

$\left( \bigcup_{R \in \Omega} \llbracket A \rrbracket_{\mathcal{B}\{X \mapsto R\}} \right)^{\perp\perp}$ . As a consequence  $S_0 :: \gamma + l$  e.g.  $S_0 + l :: \gamma$  which is  $S :: \gamma$ , belongs to  $\llbracket \exists X A \rrbracket_{\mathcal{B}}$ .

Since  $\gamma$  range in  $\llbracket \Gamma \rrbracket_{\mathcal{B}}^\perp$  we can conclude that  $S$  belongs to  $\llbracket \Gamma, \exists X A \rrbracket_{\mathcal{B}}$ . Indeed this hold for any interpretation basis  $\mathcal{B}$ .

- If the last rule of  $\pi$  is the introduction of a universal quantifier. Then  $S$  can be decomposed as  $S_0 + l$  where  $l$  is the link corresponding to the universal quantification.  $S_0$  is a proof of  $\Gamma, A$  and furthermore since we can apply a universal quantification,  $X$  does not occur free in  $\Gamma$ .

Applying the induction hypothesis  $S_0$  belongs to  $\llbracket \Gamma, A \rrbracket_{\mathcal{B}}$  for any basis  $\mathcal{B}$ . In particular for any type  $R$ , the net  $S_0$  belongs to  $\llbracket \Gamma, A \rrbracket_{\mathcal{B}\{X \mapsto R\}}$ . This means that  $S_0$  belongs to the intersection  $\bigcap_{R \in \Omega} \llbracket \Gamma, A \rrbracket_{\mathcal{B}\{X \mapsto R\}}$  which is  $\bigcap_{R \in \Omega} \llbracket \Gamma \rrbracket_{\mathcal{B}\{X \mapsto R\}} \succ \llbracket A \rrbracket_{\mathcal{B}\{X \mapsto R\}}$ .

As a consequence, for any type  $R$ , for any net  $\gamma$  in  $\llbracket \Gamma \rrbracket_{\mathcal{B}\{X \mapsto R\}}^\perp$  the interaction  $S_0 :: \gamma$  belongs to  $\llbracket A \rrbracket_{\mathcal{B}\{X \mapsto R\}}$ . Since the universal quantification can be applied  $X$  does not occur free in the sequent  $\Gamma$  and thus calling proposition F.1  $\llbracket \Gamma \rrbracket_{\mathcal{B}\{X \mapsto R\}}^\perp$  is equal to  $\llbracket \Gamma \rrbracket_{\mathcal{B}}^\perp$ . Thus we have showed that  $S_0$  belongs to  $\llbracket \Gamma \rrbracket_{\mathcal{B}} \succ \llbracket A \rrbracket_{\mathcal{B}\{X \mapsto R\}}$  for any type  $R$ . Hence  $S_0$  belongs to

$\llbracket \Gamma \rrbracket_{\mathcal{B}} \succ \bigcap_{R \in \Omega} \llbracket A \rrbracket_{\mathcal{B}\{X \mapsto R\}}$  which is  $\llbracket \Gamma, \bigcap_{X \in \Omega} A \rrbracket_{\mathcal{B}}$ . We can therefore conclude that  $S$  belongs to  $\llbracket \Gamma, \forall X A \rrbracket_{\mathcal{B}}$ .

□

## G COMPLEMENTS TO SECTION 7

### G.1 Measures of Types

**Definition G.1.** The size of an interface  $\sigma \subseteq \text{out}(S) \times \text{out}(T)$  is its cardinal.

**Definition G.2.** A *measure* is a map  $\mu : \text{Mod} \rightarrow \mathbb{R}$  which is invariant under cut elimination. The *weight* of a measure is the value of  $\mu(\mathfrak{X}_0)$ , we denote it  $w(\mu)$ . The *measure of an interface*  $\sigma$  is the measure of the module  $\sum_{(p,q) \in \sigma} \langle p, q \triangleright_{\text{cut}} \rangle$ , it is denoted  $\mu(\sigma)$ . A measure is;

- *linear* whenever  $\mu(S \parallel T) = \mu(S) + \mu(T)$ , for any modules  $S$  and  $T$ .
- *stable* whenever two interface  $\sigma$  and  $\sigma'$  of the same size have the same measure  $\mu(\sigma) = \mu(\sigma')$ .

**PROPOSITION G.3.** *Given a linear and stable measure  $\mu$ , for any two nets  $S$  and  $T$  with  $n$  outputs;*

$$S \perp T \Rightarrow \mu(S) + \mu(T) + \mu(\text{id}_n) = w(\mu).$$

**PROPOSITION G.4.** *Given a linear and stable measure  $\mu$ , and a type  $\mathbf{A}$  such that  $\mathbf{A}$  and  $\mathbf{A}^\perp$  are both non empty;*

- *All the nets in  $\mathbf{A}$  have the same measure.*
- *All the nets in  $\mathbf{A}^\perp$  have the same measure.*

**PROOF.** For any pair of nets  $S$  and  $T$  which belongs respectively to  $\mathbf{A}$  and  $\mathbf{A}^\perp$  the equality hold  $\mu(S) + \mu(T) + \mu(\text{id}_n) = w(\mu)$ . Thus  $\mu(S) = w(\mu) - (\mu(T) + \mu(\text{id}_n))$  and this must hold for any net in  $\mathbf{A}$ , thus the measure is invariant on  $\mathbf{A}$ .

Similarly any net in  $\mathbf{A}^\perp$  must verify  $\mu(T) = w(\mu) - (\mu(S) + \mu(\text{id}_n))$ . This means that  $\mu$  is also invariant on  $\mathbf{A}^\perp$ . □

**Definition G.5 (compatible measures).** Two nets  $S$  and  $T$  with  $n$  outputs have *compatible measures* whenever;

$$\mu(S) + \mu(T) + \mu(\text{id}_n) = w(\mu)$$

The analogue notion is also defined for types.

**Definition G.6 (balanced type).** A type  $\mathbf{A}$  is *balanced* when  $\mathbf{A}$  and  $\mathbf{A}^\perp$  are both non empty. The measure  $\mu(\mathbf{A})$  of a balanced type is the (unique) measure of the nets it contains. A balanced type is *symmetrical* whenever  $\mu(\mathbf{A}) = \mu(\mathbf{A}^\perp)$ .

**Remark 53.** A symmetrical type  $\mathbf{A}$  with  $n$  outputs must have the measure  $1/2 \times (w(\mu) - \mu(\text{id}_n))$

**Definition G.7 (Separator).** A set of types  $\{\mathbf{B}, \mathbf{B}^\perp\}$  is a *separator* for a measure  $\mu$ , whenever  $\mathbf{B}$  is an asymmetrical balanced type.

**PROPOSITION G.8.** *Let  $\omega$  be a separator, and  $\Gamma = \Delta, X^k, X^{\perp k'}$  a sequent where  $X$  does not occur free in  $\Delta$ .*

*Given a daimon link  $\mathfrak{X}_n = \langle \triangleright_{\mathfrak{X}} p_1, \dots, p_n \rangle$  if  $\mathfrak{X}_n$  belongs to  $\llbracket \bigcap_{X \in \omega} \Gamma \rrbracket_{\mathcal{B}}$  then  $\Gamma$  contains the same number of atomic variable  $X$  and  $X^\perp$ .*

PROOF. Say  $\omega = \{\mathbf{B}, \mathbf{B}^\perp\}$ . We denote  $k$  (resp.  $k'$ ) the number of occurrence of  $X$  (resp.  $X^\perp$ ) in  $\Gamma$ . By assumption  $\mathfrak{X}_n$  belongs to  $\llbracket \Gamma \rrbracket_{\mathcal{B}\{X \mapsto \mathbf{B}\}}$  and so it is orthogonal to  $\llbracket A_1 \rrbracket_{\mathcal{B}\{X \mapsto \mathbf{B}\}} \parallel \dots \parallel \llbracket A_n \rrbracket_{\mathcal{B}\{X \mapsto \mathbf{B}\}}$ . The measure of that set is the sum  $\sum_{1 \leq i \leq n} \mu(\llbracket A_i \rrbracket_{\mathcal{B}\{X \mapsto \mathbf{B}\}})$  indeed this writing  $\Gamma$  as  $B_1, \dots, B_m, X^k, X^{\perp k'}$  where each  $B_i$  is a formula that does not contain free occurrences of  $X$  or  $X^\perp$ , the previous sum is equal to,

$$\sum_{1 \leq i \leq m} \mu(\llbracket B_i \rrbracket_{\mathcal{B}\{X \mapsto \mathbf{B}\}}) + k \times \mu(\llbracket X \rrbracket_{\mathcal{B}\{X \mapsto \mathbf{B}\}}) + k' \times \mu(\llbracket X^\perp \rrbracket_{\mathcal{B}\{X \mapsto \mathbf{B}\}})$$

Which indeed is

$$\sum_{1 \leq i \leq m} \mu(\llbracket B_i \rrbracket_{\mathcal{B}\{X \mapsto \mathbf{B}\}}) + k \times \mu(\mathbf{B}) + k' \times \mu(\mathbf{B}^\perp)$$

Since  $X$  does not occur free in  $B_1, \dots, B_m$  calling proposition F.1, in particular this is equal to

$$\sum_{1 \leq i \leq m} \mu(\llbracket B_i \rrbracket_{\mathcal{B}}) + k \times \mu(\mathbf{B}) + k' \times \mu(\mathbf{B}^\perp)$$

Furthmore  $\mathfrak{X}_n$  also belongs to  $\llbracket \Gamma \rrbracket_{\mathcal{B}\{X \mapsto \mathbf{B}^\perp\}}$ , applying the same argument this means that its measure is also equal to;

$$\sum_{1 \leq i \leq m} \mu(\llbracket B_i \rrbracket_{\mathcal{B}}) + k \times \mu(\mathbf{B}^\perp) + k' \times \mu(\mathbf{B}).$$

Since the two sums must be equal it follows that

$$k \times \mu(\mathbf{B}^\perp) + k' \times \mu(\mathbf{B}) = k \times \mu(\mathbf{B}) + k' \times \mu(\mathbf{B}^\perp)$$

and thus

$$k' \times \mu(\mathbf{B}) - k' \times \mu(\mathbf{B}^\perp) = k \times \mu(\mathbf{B}) - k \times \mu(\mathbf{B}^\perp)$$

Indeed we can rewrite this

$$k \times (\mu(\mathbf{B}) - \mu(\mathbf{B}^\perp)) = k' \times (\mu(\mathbf{B}) - \mu(\mathbf{B}^\perp))$$

Since the type  $\mathbf{B}$  is asymmetrical the difference of the measure is not null it follows that we can divide both sides by the differences of the measure and conclude  $k = k'$ . In other words in  $\Gamma$  the propositional variable  $X$  occurs as many times as  $X^\perp$  occurs.  $\square$

*Remark 54.* The previous proposition in particular applies to any sequent constituted only of propositional variable. In fact this generalizes to any atomic hypersequent of the form  $\Gamma_1 \parallel \dots \parallel \Gamma_n$ . This is because  $\llbracket \bigcap_{X \in \omega} \mathcal{H}_1 \parallel \mathcal{H}_2 \rrbracket_{\mathcal{B}}$  is equal to  $\llbracket \bigcap_{X \in \omega} \mathcal{H}_1 \rrbracket_{\mathcal{B}} \parallel \llbracket \bigcap_{X \in \omega} \mathcal{H}_2 \rrbracket_{\mathcal{B}}$  so a simple induction can be performed.

*Definition G.9.* The switching measure of a net  $S$  corresponds to the integer  $N(S) - E(S)$  where  $N(S)$  is the number of nodes of the switching graphs of  $S$  and  $E(S)$  is the number of edges of the switching graphs of  $S$ .

PROPOSITION G.10. *The switching measure is a linear and stable measure of weight 1.*

## G.2 Properties of Intersection and Union

PROPOSITION G.11. *Given  $(A_i)_{i \in I}$  a family of types;*

- *The intersection  $\bigcap_{i \in I} A_i$  is still a type.*
- *$(\bigcup_{i \in I} A_i)^\perp$  is equal to  $\bigcap_{i \in I} A_i^\perp$ .*
- *$(\bigcap_{i \in I} A_i)^\perp$  is equal to  $(\bigcup_{i \in I} A_i^\perp)^\perp$ .*

PROOF. We treat each point independently;  $\square$

- For any index  $i$  the intersection  $\bigcap_{i \in I} A_i$  is included in the type  $A_i$ . Since orthogonality invert the inclusion of sets this implies  $A_i^\perp$  is contained in  $(\bigcap_{i \in I} A_i)^\perp$ .

Therefore a net  $S$  which is orthogonal to  $(\bigcap_{i \in I} A_i)^\perp$ , is also orthogonal to  $A_i^\perp$  for any index  $i \in I$ . This means that  $S$  belongs to  $A_i^{\perp\perp}$  which since we consider a family of types is exactly  $A_i$ .

Thus we showed that  $(\bigcap_{i \in I} A_i)^{\perp\perp}$  is included in  $\bigcap_{i \in I} A_i$ . Since the other inclusion always hold we conclude.

- Let  $S$  be a net in  $\bigcap_{i \in I} A_i^\perp$ . For any index  $i$  and any net  $a$  in  $A_i$  the net  $S$  is orthogonal to  $a$ . This shows that  $S$  belongs to  $(\bigcup_{i \in I} A_i)^\perp$  and thus the first inclusion hold. On the other hand let  $S$  be a net orthogonal to  $\bigcup_{i \in I} A_i$  then in particular for any type  $A_i$  the net  $S$  is orthogonal to  $A_i$  and so belong to  $A_i^\perp$ . This means that  $S$  belongs to  $\bigcap_{i \in I} A_i^\perp$ .
- This is a direct consequence of the previous point.  $(\bigcap A_i)^\perp$  corresponds to  $(\bigcap A_i^{\perp\perp})^\perp$  and thus to  $(\bigcup A_i^\perp)^{\perp\perp}$ .  $\square$

PROPOSITION G.12. *Given  $(A_i)_{i \in I}$  a family of sets of nets with the same number of outputs;*

- *$(\bigcap_{i \in I} A_i)^\perp$  is equal to  $(\bigcap_{i \in I} A_i^{\perp\perp})^\perp$ .*
- *$(\bigcup_{i \in I} A_i)^\perp$  is equal to  $(\bigcup_{i \in I} A_i^{\perp\perp})^\perp$ .*

PROOF. We treat both cases independently;

- The inclusions  $A_i \subseteq A_i^{\perp\perp}$  always hold thus the intersections are included  $\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} A_i^{\perp\perp}$ . Since orthogonality inverts inclusion it follows that  $(\bigcap_{i \in I} A_i^{\perp\perp})^\perp \subseteq (\bigcap_{i \in I} A_i)^\perp$ . On the other hand, consider  $S$  a net orthogonal to  $(\bigcap_{i \in I} A_i)^\perp$  e.g. belonging to  $\bigcup_{i \in I} A_i^\perp$ . Hence, for any index  $i$ ,  $S$  is orthogonal to  $A_i$ , in particular then  $S$  is orthogonal to any net which belong to  $\bigcap_{i \in I} A_i$ .

Thus any net orthogonal to  $(\bigcap_{i \in I} A_i)^\perp$  is orthogonal to  $S$ . Hence we have showed the inclusion  $(\bigcup_{i \in I} A_i^\perp)^\perp \subseteq (\bigcap_{i \in I} A_i)^\perp$ . Since orthogonality inverts inclusion this yields the inclusion  $(\bigcap_{i \in I} A_i)^\perp \subseteq (\bigcup_{i \in I} A_i^\perp)^\perp$ .

Hence using the previous point we obtained,

$$\left( \bigcap_{i \in I} A_i \right)^\perp = \left( \bigcup_{i \in I} A_i^\perp \right)^\perp = \left( \bigcap_{i \in I} A_i^{\perp\perp} \right)^\perp$$

- One of the inclusion is trivial namely  $(\bigcup_{i \in I} A_i)^\perp$  contains  $(\bigcup_{i \in I} A_i^{\perp\perp})^\perp$ .

On the other hand say a net  $S$  belongs to  $(\bigcup_{i \in I} A_i)^\perp$ , then for any index  $i$  the net  $S$  is orthogonal to  $A_i$ , and so belongs to  $A_i^\perp$ . Since this is true for any index  $i$  this means that  $S$  belongs to  $\bigcap_{i \in I} A_i^\perp$  which corresponds to  $(\bigcap_{i \in I} A_i^{\perp\perp})^\perp$  and thus to  $(\bigcup_{i \in I} A_i^{\perp\perp})^\perp$ . Thus we have shown:

$$\left( \bigcup_{i \in I} A_i \right)^\perp = \left( \bigcup_{i \in I} A_i^{\perp\perp} \right)^\perp.$$

$\square$

*Remark 55.* A consequence of proposition G.11 is that the intersections of types are still types. Thus this set–theoretic operation is a valid construction on types. Later on, intersections (and unions) will appear in order to define the realizers of quantified formulas.

*Remark 56.* The (finite) union of types on the contrary is not guaranteed to still be a type. For instance in the case of two types  $\mathbf{A}$  and  $\mathbf{B}$  that don't cover the whole set of nets, and such that their orthogonals don't intersect.

The inclusion of  $\mathbf{A} \cup \mathbf{B}$  in  $(\mathbf{A} \cup \mathbf{B})^{\perp\perp}$  is always guaranteed. On the other hand, using proposition G.11 the opposite inclusion would mean in particular that  $(\mathbf{A}^{\perp} \cap \mathbf{B}^{\perp})^{\perp}$  is included in  $\mathbf{A} \cup \mathbf{B}$ . Using our assumption that would mean that  $\emptyset^{\perp}$  is included in the union of  $\mathbf{A}$  and  $\mathbf{B}$ . Since  $\emptyset^{\perp}$  is the whole set of nets this would mean that  $\mathbf{A} \cup \mathbf{B}$  covers the whole set of nets, this goes against our assumption.

As a consequence we cannot have a 'De Morgan' law for set–constructions e.g. the equality of  $(\mathbf{A} \cap \mathbf{B})^{\perp}$  and  $\mathbf{A}^{\perp} \cup \mathbf{B}^{\perp}$ . Since on one hand  $(\mathbf{A} \cap \mathbf{B})^{\perp}$  is guaranteed to be a type while  $\mathbf{A}^{\perp} \cup \mathbf{B}^{\perp}$  can fail to be a type.

*Remark 57.* Finally we point out that the proposition G.11 affirms that for two types  $(\mathbf{A}^{\perp} \cup \mathbf{B}^{\perp})^{\perp}$  corresponds to  $\mathbf{A}^{\perp\perp} \cap \mathbf{B}^{\perp\perp}$  and thus to  $\mathbf{A} \cap \mathbf{B}$ . Applying orthogonality this yields that  $(\mathbf{A} \cap \mathbf{B})^{\perp}$  corresponds to  $(\mathbf{A}^{\perp} \cup \mathbf{B}^{\perp})^{\perp\perp}$ .

### G.3 Proofs of Section 7

*Definition G.13 (computable type).* A type is *computable* whenever there exists a finite set  $B$  such that  $\mathbf{A} = B^{\perp}$ .

**PROPOSITION 7.2.** *The functional composition  $\mathbf{A} \succ \mathbf{B}$  and the parallel composition  $\mathbf{A} \parallel \mathbf{B}$  of two syntactically computable types are syntactically computable.*

**OF THE FIRST STATEMENT.** Consider two types  $\mathbf{A}$  and  $\mathbf{B}$  which are finitely testable, this means there exists two finite sets of nets  $X_A$  and  $X_B$  such that  $\mathbf{A} = X_A^{\perp}$  and  $\mathbf{B} = X_B^{\perp}$ . The following set of nets is still finite and of cardinality  $\text{card}(X_A) \times \text{card}(X_B)$ ;

$$X = \{a \parallel b \mid a \in X_A, b \in X_B\}$$

Furthermore  $X^{\perp}$  corresponds to the orthogonal  $(X_A \parallel X_B)^{\perp}$ . By duality this type is equal to  $X_A^{\perp} \succ X_B^{\perp}$  identifying  $\mathbf{A}$  and  $\mathbf{B}$  this corresponds to  $\mathbf{A} \succ \mathbf{B}$ .  $\square$

*Notation 23.* Given a sequent  $\Gamma = A_1, \dots, A_n$  we denote  $\bar{\Gamma}$  the sequent  $\bar{A}_1, \dots, \bar{A}_n$ .

**PROPOSITION 7.2.** *The functional composition  $\mathbf{A} \succ \mathbf{B}$  and the parallel composition  $\mathbf{A} \parallel \mathbf{B}$  of two syntactically computable types are syntactically computable.*

**OF THE SECOND STATEMENT.** Consider  $S$  with conclusion  $p_1, \dots, p_n, q_1, \dots, q_m$  some net in  $\mathbf{A} \parallel \mathbf{B}$  witnessing  $\Gamma, \Delta$  such that the  $n$  first conclusions correspond to  $\Gamma$  and the remaining conclusions correspond to  $\Delta$ .

Consider the set  $\mathbf{A}(\Gamma) \bowtie \mathbf{B}(\Delta)$ . for any net  $T$  in that merge the interaction  $S :: T$  is made only of glueing and multiplicative cuts. If  $S$  is orthogonal to  $\mathbf{A}(\Gamma) \bowtie \mathbf{B}(\Delta)$ , the conclusions  $p_i$  and  $q_i$  of  $S$  are not related by a daimon link. Otherwise a cycle can be created while interacting with a well chosen merge. Hence  $S$  can be written as  $a(p_1, \dots, p_n) \parallel b(q_1, \dots, q_m)$ . Given some merge  $a' \bowtie b'$  then  $a' \bowtie b' :: (a \parallel b)$  can be rewritten as  $a' :: a \bowtie b' :: b$ .

Since the cuts involved in that interaction are all reversible we can ensure that  $a$  is orthogonal to  $\mathbf{A}(\Gamma)$  and  $b$  is orthogonal to  $\mathbf{B}(\Delta)$ .  $\square$

**PROPOSITION G.14.** *Given  $(A_i)_{i \in I}$  a family of sets of nets with the same number of outputs;*

- $(\bigcap_{i \in I} A_i)^{\perp}$  is equal to  $(\bigcap_{i \in I} A_i^{\perp\perp})^{\perp}$ .
- $(\bigcup_{i \in I} A_i)^{\perp}$  is equal to  $(\bigcup_{i \in I} A_i^{\perp\perp})^{\perp}$ .

**PROOF.** We treat both cases independently;

- The inclusions  $A_i \subseteq A_i^{\perp\perp}$  always hold thus the intersections are included  $\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} A_i^{\perp\perp}$ . Since orthogonality inverts inclusion it follows that  $(\bigcap_{i \in I} A_i^{\perp\perp})^{\perp} \subseteq (\bigcap_{i \in I} A_i)^{\perp}$ . On the other hand, consider  $S$  a net orthogonal to  $\bigcup_{i \in I} A_i^{\perp}$  hence for any index  $i$ ,  $S$  is orthogonal to  $A_i$ , in particular then  $S$  is orthogonal to any net which belong to  $\bigcap_{i \in I} A_i$ . Thus any net orthogonal to  $(\bigcap_{i \in I} A_i)^{\perp}$  is orthogonal to  $S$ . Hence we have showed the inclusion  $(\bigcup_{i \in I} A_i^{\perp})^{\perp} \subseteq (\bigcap_{i \in I} A_i)^{\perp\perp}$ . Since orthogonality inverts inclusion this yields the inclusion  $(\bigcap_{i \in I} A_i)^{\perp} \subseteq (\bigcup_{i \in I} A_i^{\perp})^{\perp\perp}$ . Hence using the previous point we obtained,

$$\left(\bigcap_{i \in I} A_i\right)^{\perp} = \left(\bigcup_{i \in I} A_i^{\perp}\right)^{\perp\perp} = \left(\bigcap_{i \in I} A_i^{\perp\perp}\right)^{\perp}$$

- One of the inclusion is trivial namely  $(\bigcup_{i \in I} A_i)^{\perp}$  contains  $(\bigcup_{i \in I} A_i^{\perp\perp})^{\perp}$ . On the other hand say a net  $S$  belongs to  $(\bigcup_{i \in I} A_i)^{\perp}$ , then for any index  $i$  the net  $S$  is orthogonal to  $A_i$ , and so belongs to  $A_i^{\perp}$ . Since this is true for any index  $i$  this means that  $S$  belongs to  $\bigcap_{i \in I} A_i^{\perp}$  which corresponds to  $(\bigcap_{i \in I} A_i^{\perp})^{\perp\perp}$  and thus to  $(\bigcup_{i \in I} A_i^{\perp\perp})^{\perp}$ . Thus we have shown:

$$\left(\bigcup_{i \in I} A_i\right)^{\perp} = \left(\bigcup_{i \in I} A_i^{\perp\perp}\right)^{\perp}$$

**PROPOSITION 7.3.** *The union  $(\bigcup_{i \in I} \mathbf{A}_i)^{\perp\perp}$  of a family of computable types is computable. The intersection  $\bigcap_{i \in I} \mathbf{A}_i$  of a finite family of computable types is computable.*

**PROOF.** of proposition 7.3. We treat both cases independently.

- By equational reasoning with the previous propositions;

$$\begin{aligned}
\bigcap_{i \in I} \mathbf{A}_i &= \left( \bigcap_{i \in I} \mathbf{A}_i \right)^{\perp\perp} \\
&= \left( \bigcup_{i \in I} \mathbf{A}_i^\perp \right)^{\perp\perp\perp} \\
&= \left( \bigcup_{i \in I} \mathbf{A}_i^\perp \right)^\perp \\
&= \left( \bigcup_{i \in I} t_i^{\perp\perp} \right)^\perp \\
&= \left( \bigcup_{i \in I} t_i \right)^\perp.
\end{aligned}$$

The cardinal of the union  $\bigcup_{i \in I} t_i$  is bounded by  $\sum_{i \in I} \text{card}(t_i)$ , hence since each  $t_i$  is finite and since the set of indexes  $I$  is finite that sum is finite and thus the cardinal of  $\bigcup_{i \in I} t_i$  is also finite. The intersection  $\bigcap_{i \in I} \mathbf{A}_i$  is the orthogonal of a finite set thus it is a computable type.

- Using the previous proposition we can obtain the following series of equalities;

$$\begin{aligned}
\left( \bigcup_{i \in I} \mathbf{A}_i \right)^{\perp\perp} &= \left( \bigcap_{i \in I} \mathbf{A}_i^\perp \right)^\perp \\
&= \left( \bigcap_{i \in I} t_i^{\perp\perp} \right)^\perp \\
&= \left( \bigcap_{i \in I} t_i \right)^\perp.
\end{aligned}$$

The cardinal of the intersection  $\bigcap_{i \in I} t_i$  is bounded by  $\min_{i \in I} \text{card}(t_i)$  since for any  $i$  the cardinal of  $t_i$  is finite then the minimum of the cardinals must also be finite. As a consequence  $\bigcap_{i \in I} t_i$  has a finite cardinality. The union  $(\bigcup_{i \in I} \mathbf{A}_i)^{\perp\perp}$  is the orthogonal of a finite set, hence it is a computable type.  $\square$

**Notation 24.** Let  $\mathbf{A}$  be a type we denote  $cf(\mathbf{A})$  the set of cut free nets in  $\mathbf{A}$ .

**Definition G.15.** A type  $\mathbf{A}$  is *effective* whenever  $\mathbf{A}^\perp$  is equal to  $cf(\mathbf{A})^\perp$ .

An interpretation basis is *effective* whenever it maps atomic formulas to effective types.

**Remark 58.** Equivalently a type  $\mathbf{A}$  is effective whenever  $\mathbf{A}^{\perp\perp} = cf(\mathbf{A})^{\perp\perp}$ .

**PROPOSITION G.16.** For any two effective types  $\mathbf{A}$  and  $\mathbf{B}$  the types  $\mathbf{A} \wp \mathbf{B}$ ,  $\mathbf{A} \otimes \mathbf{B}$ ,  $\mathbf{A} \succ \mathbf{B}$  and  $\mathbf{A} \parallel \mathbf{B}$  are still effective.

**PROOF.** This is a direct consequence from proposition E.23, noting that an effective type  $\mathbf{A}$  satisfies  $\mathbf{A} = cf(\mathbf{A})^{\perp\perp}$ .  $\square$

**PROPOSITION G.17.** Given an effective interpretation basis  $\mathcal{B}$  and  $\Gamma$  some sequent the interpretation  $\llbracket \Gamma \rrbracket_{\mathcal{B}}$  is effective.

**PROOF.** This is easily obtained by induction on the size of the sequent, using the proposition G.16.  $\square$

**Remark 59.** A consequence of the previous theorem is that if a the set of cut free nets of two types  $\llbracket \Gamma \rrbracket_{\mathcal{B}}$  and  $\llbracket \Gamma \rrbracket_{\mathcal{B}}$  are equal then these types are equal.

**PROPOSITION G.18.** Given  $(\mathbf{A}_i)_{i \in I}$  a family of effective types, then  $\bigcup_{i \in I} \mathbf{A}_i$  and  $\bigcap_{i \in I} \mathbf{A}_i$  are effective types.

**PROOF.** This is a direct consequence of remark 58 and proposition G.14.  $\square$

**Notation 25.** Given a formula  $A$  we define by induction the notation  $A^n$  whenever  $n$  is an integer;

- $A^0$  denotes the empty sequent.
- $A^{n+1}$  denotes  $A^n, A$ .

**Notation 26.** Let  $X$  be a propositional variable, we denote by  $C_X$  the set of sequents of MLL of the form  $\Gamma, X^n, X^{\perp n}$  for some integer  $n$  and such that  $\Gamma$  does not contain an occurrence of  $X$ .

**PROPOSITION G.19.** For any propositional variable  $X$  any approximable basis  $\mathcal{B}$ , and any net  $S$

$$S \vdash_{C_X} \Gamma \Rightarrow S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}.$$

**PROOF.** As for soundness in the case of MLL and MLL $^{\boxtimes}$  we only need to prove the base case. Thus we need to prove that for any sequent  $\Gamma, X^n, X^{\perp n}$  a daimon link  $\boxtimes_{k+2n}$  belongs to  $\llbracket \Gamma, X^n, X^{\perp n} \rrbracket_{\mathcal{B}}$ .  $\square$

**Remark 60.** In particular this ensure that any net representing a proof of MLL $^{\boxtimes}$  that introduce only sequents of  $C_X$  belongs to  $\llbracket \bigcap_{X \in \Omega} \Gamma \rrbracket_{\mathcal{B}}$ , whenever  $\mathcal{B}$  is an approximable basis.

**Notation 27.** Recall that a multiplicative net  $S$  is an ordered hypergraphs hence it is equipped with an order on its conclusion. Whenever the conclusions of  $S$  are  $\{p_1, \dots, p_n\}$  and  $(p_{\sigma(n)}, \dots, p_{\sigma(1)})$  is an ordering of that set we denote  $S[(p_{\sigma(n)}, \dots, p_{\sigma(1)})]$  to explicitly refer to the order of the conclusion of  $S$  as  $(p_{\sigma(n)}, \dots, p_{\sigma(1)})$ .

**Remark 61.** Given an hypersequent  $\mathcal{H}$  the size of the hypersequent  $\mathcal{H}$  is equal to the number of conclusions the net in  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}$  contains – this can be showed by a simple inclusion on  $\mathcal{H}$ .

**PROPOSITION G.20.** Given  $S[\overline{p}_1, \overline{p}_2, \overline{p}_3]$  a net made of daimon links only.  $\mathcal{H}_1, \mathcal{H}_2$  and  $\mathcal{H}_3$  three hypersequents such that for each index  $1 \leq i \leq n$  the size of  $\mathcal{H}_i$  is the size of the sequence  $\overline{p}_i$ .

If  $S[\overline{p}_1, \overline{p}_2, \overline{p}_3]$  belongs to  $\llbracket (\mathcal{H}_1, (\mathcal{H}_2 \parallel \mathcal{H}_3)) \rrbracket_{\mathcal{B}}$  then  $S[\overline{p}_1, \overline{p}_2, \overline{p}_3]$  belongs to  $\llbracket (\mathcal{H}_1, \mathcal{H}_2) \parallel \mathcal{H}_3 \rrbracket_{\mathcal{B}}$  or  $S[\overline{p}_1, \overline{p}_3, \overline{p}_2] \in \llbracket (\mathcal{H}_1, \mathcal{H}_3) \parallel \mathcal{H}_2 \rrbracket_{\mathcal{B}}$ .

**Remark 62.** The previous proposition ensure that a net made of daimon links only belongs to some hypersequent  $\mathcal{H}$  only if it belongs to an hypersequent of the form  $\Gamma_1 \parallel \dots \parallel \Gamma_n$  where each  $\Gamma_i$  are sequents. This can be obtained by performing an induction on the number of non external parallel links in  $\mathcal{H}$ . This together with the remark 54 provides a proof of the following proposition.

**PROPOSITION G.21.** Let  $\omega$  be a separator (see appendix G.1). Given  $\Gamma$  some sequent and  $\mathcal{B}$  an approximable, self dual, and effective basis, denote  $\mathcal{B}_X = \{\llbracket X \rrbracket_{\mathcal{B}}, \llbracket X^\perp \rrbracket_{\mathcal{B}}\}$ .

For any cut free net  $S$ ;

$$S \in \llbracket \bigcap_{X \in \omega \cup \mathcal{B}_X} \Gamma \rrbracket_{\mathcal{B}} \Rightarrow S \vdash_{C_X} \Gamma.$$

**PROOF.** The net  $S$  belongs to  $\llbracket \bigcap_{X \in \omega \cup \mathcal{B}_X} \Gamma \rrbracket_{\mathcal{B}}$  thus in particular  $S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$ .  $\mathcal{B}$  is an approximable and self dual basis therefore by theorem 5.10,  $S$  represents a proof of  $\Gamma$  in  $\text{MLL}^{\mathfrak{X}}$ .

To conclude we must ensure the daimons of  $S$  only introduces sequents  $\Delta$  which belongs to  $C_X$ , Any of the daimons of  $S$  must belongs to a type  $\llbracket \bigcap_{X \in \omega \cup \mathcal{B}_X} \Delta \rrbracket_{\mathcal{B}}$  and so in particular  $\llbracket \bigcap_{X \in \omega} \Delta \rrbracket_{\mathcal{B}}$  where  $\Delta$  is a sequent. We can apply therefore proposition G.8.  $\square$

*Remark 63.* Proposition G.19 ensure that any cut-free proof in  $\text{MLL}^{\mathfrak{X}}$  introducing only sequents following the constraint  $C_X$  is included in  $\llbracket \bigcap_{X \in \Omega} \Gamma \rrbracket_{\mathcal{B}}$ , when  $\Omega$  denote the set of all types.

Furthermore  $\llbracket \bigcap_{X \in \Omega} \Gamma \rrbracket_{\mathcal{B}}$  is necessarily included in  $\llbracket \bigcap_{X \in \omega} \Gamma \rrbracket_{\mathcal{B}}$  since  $\omega \subset \Omega$ . On the other hand the completeness result show that any cut free net in  $\llbracket \bigcap_{X \in \omega} \Gamma \rrbracket_{\mathcal{B}}$  is in fact a proof in  $\text{MLL}^{\mathfrak{X}}$  introducing only sequents following the constraint  $C_X$ .

This shows that the intersections  $\llbracket \bigcap_{X \in \Omega} \Gamma \rrbracket_{\mathcal{B}}$  and  $\llbracket \bigcap_{X \in \omega} \Gamma \rrbracket_{\mathcal{B}}$  contain the same cut free nets. As a consequence since these types are effective, they are equal.

**PROPOSITION 7.4.** Let  $\omega$  be the set of types  $\{\otimes_{\mathfrak{X}}, \wp_{\mathfrak{X}}, \mathfrak{X}, \mathfrak{X}^{\perp}\}$  and  $\mathcal{B}$  be a basis which maps any  $X$  to  $\mathfrak{X}$ . For any hyper-sequent  $\mathcal{H}$ :

$$\llbracket \bigcap_{X \in \omega} \mathcal{H} \rrbracket_{\mathcal{B}} = \llbracket \bigcap_{X \in \Omega} \mathcal{H} \rrbracket_{\mathcal{B}}.$$

**PROOF.** Using the previous remark.  $\square$

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