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# Development of the differential calculus schema for two-variable functions

María Trigueros<sup>1</sup>, Rafael Martínez-Planell<sup>2</sup>, and Vahid Borji<sup>3</sup>

<sup>1</sup>Benemérita Universidad Autónoma de Puebla, México, mtriguerosg@gmail.com;

<sup>2</sup>Universidad de Puerto Rico en Mayagüez, Puerto Rico; <sup>3</sup>Charles University in Prague, Department of Mathematics Education, Czech Republic

*We use Action-Process-Object-Schema theory (APOS) to study the development of the differential calculus Schema for two-variable functions. This allows us to obtain information about students' constructions and also gives us information about the notion of Schema. We performed semi-structured interviews with a group of eleven students that had completed an introductory multivariable calculus course. We use data from two students to exemplify the types of constructions observed. To analyze the data, we use the Schema development triad and the notions of correspondence, transformation, and equivalence relations between Schema components. Our study contributes to a better understanding of these notions, how students relate differential calculus concepts, and how to support their learning and Schema development.*

*Keywords: Teaching and learning of specific topics in university mathematics, teaching and learning of analysis and calculus, APOS theory, schema, two-variable functions.*

## INTRODUCTION

The notion of Schema is an important component of APOS Theory. However, there is not much research using Schema to analyze students' constructions of different topics. The first research to further the study of the notion of Schema was by Clark et al. (1997), who introduced the notion of the Schema triad from Piaget and Garcia's work (1982) into APOS theory to study the development of the chain rule Schema for one-variable functions. More recently, Trigueros (2019) studied the development of the linear algebra Schema, underscoring for the first time the types of relations between Schema components. She operationalized the use of Schemas into the study of its constituent components and types of relations between them. Further research needs to explore the notion of Schema and its application to the study of the teaching and learning of mathematics. One of the goals of this study is to contribute to the discussion of Schemas in APOS theory.

This study is part of a second research cycle based on previous work by Martínez-Planell et al. (2015, 2017) and Trigueros et al. (2018). These studies explore students' constructions of plane, partial derivatives, tangent planes, directional derivatives, and the total differential, applying, for the most part, the APO part of the theory. The second goal of this study is to contribute to the second cycle of research on students' understanding of the differential calculus of two-variable functions.

## THEORETICAL FRAMEWORK

The notion of Schema from APOS Theory is used as the theoretical framework in this study. Schema is a structure defined as the individual's collection of Actions, Processes, Objects, and other previously constructed Schema which are linked by different types of relations and that he/she brings to bear upon a problem situation related to a specific mathematical concept or topic (Arnon et al., 2014). Using the Schema point of view, the data analysis focuses on the possible changes in the components of the Schema and particularly on the relations that an individual shows to have constructed among the components of the Schema. A Schema can be considered coherent when the individual can discern between those problems that are within its scope and those that are not. A Schema may become an Object through thematization. This mechanism involves the possibility to apply new Actions or Processes to the Schema. An important characteristic of Schemas is that they are in continuous development. They are dynamic structures, but this development can be described through three recognizable stages, Intra-, Inter-, and Trans-, characterized by differences in the type of relations constructed among the Schema's components.

Any relation constructed between the components of a Schema may be classified as a correspondence, transformation, or equivalence relation. This classification of relations was introduced by Trigueros (2019) and is further detailed and explored here. *Correspondence relations* are those that result from the superficial comparison of structures in terms of similarities or differences. They may arise from the repeated observation of pairs of component structures that appear jointly in problem-solving situations, so the individual knows they are somehow or other related, but is not yet able to justify the relation. *Transformation relations* are developed when the individual discovers that some structures in the Schema can be grouped and related to each other in terms of other structures. These other structures play a role in the reasoning and justification of the connections in accordance with disciplinary practices. A relation between two components of such a grouping or that plays a role in explaining or justifying the interrelation between other component structures of a grouping, will be considered a transformation relation. Transformation relations can be distinguished when the relation is somehow justified. *Equivalence or conservation relations* involve the conservation of properties in which one structure is dependent upon the others. Equivalence relations can be distinguished when one structure is interchangeably used when solving a problem situation involving the other in accordance with disciplinary practices.

We will consider that a Schema is at the *Intra-stage* of development when its components are mainly isolated from one another and the type of relations that exist between components are for the most part correspondence relations. At an *Inter-stage* of Schema development, groupings of different components start to appear. At this stage of Schema development, transformation relations will be more prevalent so that the individual will be able to justify some of the relations by referring to others. At the

*Trans-stage* of Schema development, all the different components are related, conservation relations appear and there is evidence of coherence of the Schema.

Research questions for this study are: How can the study of the Differential Calculus Schema development inform us about students' learning? How does the study of students' construction of the Differential Calculus Schema for two-variable functions enable us to better understand the types of transformations between Schema components?

## **METHODOLOGY**

The *component structures* of the Differential Calculus Schema we consider are slope, partial derivative, directional derivative, tangent plane, total differential, function, and gradient. These are the same components that were studied in the first research cycle, now explicitly including the pre-requisite components of functions and slope, and adding gradient vector, for which questions were added to the new interview instrument. While clearly there are other components that may be called upon in a problem situation involving the differential multivariable calculus, our interview instrument will only allow us to examine the chosen components.

For *correspondence relations*, we look for relations that only point to commonalities between Schema components or appear to result from memorized procedures. For example: given a graphical representation of a tangent plane to the graph of a function at a point, a student might be able to correctly do computations to approximate the change in the values of the function for a small change in input values but might not recognize it is only an approximation and not the exact value of the change, or might not be able to explain why it is an approximation. Such a student would be grouping function, tangent plane, and perhaps even total differential components in a correspondence relation.

For *transformation relations*, we look for some reasoning explaining or justifying the relation in accordance with disciplinary practices. In the example above, a student that is able to recognize that his computations will only give an approximation, or who argues that close to the base point the tangent plane and surface are very close to each other, will be giving evidence consistent with a transformation relation grouping the components of function and tangent plane.

For *equivalence relations*, we look for evidence of one structure being used interchangeably for another in a way that agrees with disciplinary practices, consistently when solving different problem situations. For example, a student who is asked to approximate the value of a function near a base point, given graphical information of the tangent plane might immediately recur to the total differential at the base point, using the Schema components of tangent plane and total differential interchangeably or a student might be able to understand the role partial and directional derivatives play in terms of local change of the function. The student would be giving evidence of an equivalence relation among Schema components.

Eleven students spanning the range from above-average (3), average (5), to below-average (3), as chosen by the professor, were interviewed twice (Interviews I, II) two weeks after completing an introductory multivariable calculus course at a mid-tier Iranian university. The two interviews were held on different days and each one lasted approximately 1 hr. Interviews were video and audio recorded, transcribed, individually analyzed by the researchers, and discussed among them until consensus was reached. Interviews were analyzed in terms of the relations established between the seven chosen Differential Calculus Schema components.

There were six problems in Interview I and eight in Interview II. Some of these problems were multipart. Students were familiar with some problem types (see below: I-3b, I-4, I-6a, I-7c) and unfamiliar with others (II-1, II-6). In this report we will be referring to the following problems, which we label to include the interview and problem number:

I-3b. Suppose the graph of  $z = f(x, y)$  is as follows (Figure 1a). State the sign (positive, negative, zero) of  $D_{(-2,1)}f(4,0)$ . Justify your answer.

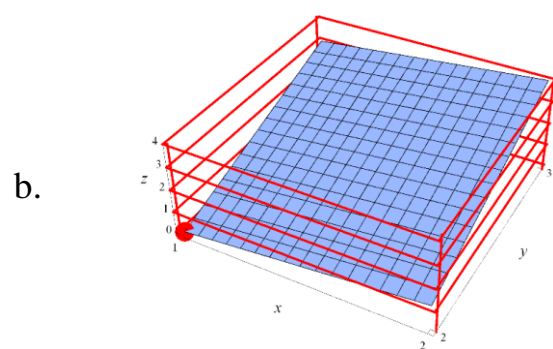
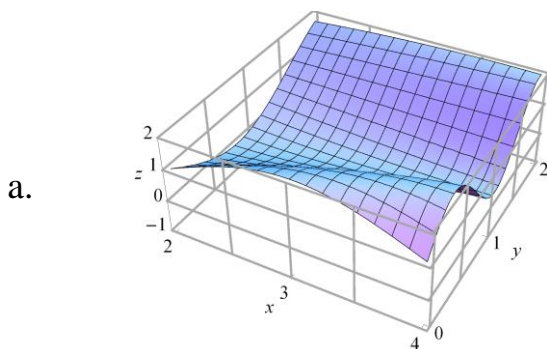
I-4. The following is a table of values of a differentiable function  $f$  of two variables (Figure 4). Approximate the value of  $\frac{\partial f}{\partial y}(1,2)$  the best you can.

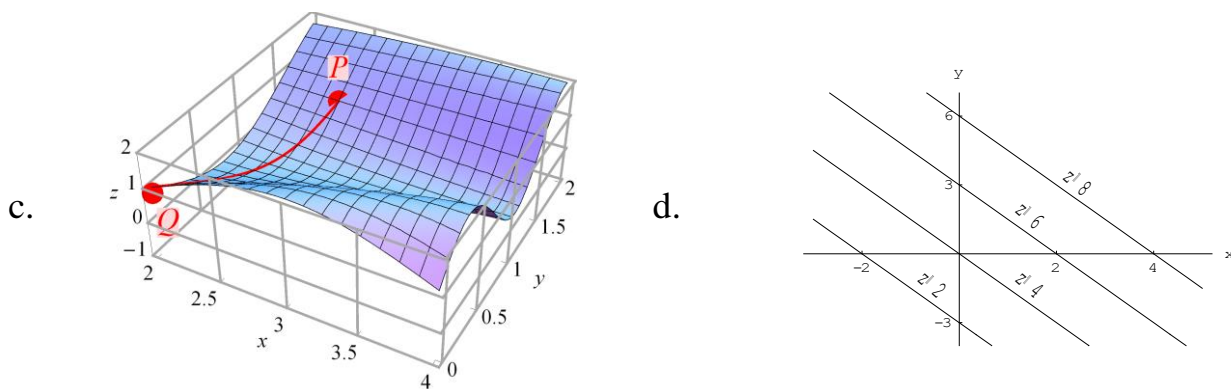
I-6a. The plane in the figure below (Figure 1b) is tangent to the graph of a differentiable function  $z = f(x, y)$  at a point  $(1,2,0)$ . What can you say about the change in the value of the function if  $x$  increases 0.02 units and  $y$  decreases 0.02 units?

II-1. The graph of  $z = f(x, y)$  is given in the figure below (Figure 1a). If  $H(t) = \frac{\partial f}{\partial x}(t, 0)$ , draw how the graph of  $H$  may look for  $2 \leq t \leq 4$ .

II-6 Suppose point P moves towards point Q at a constant speed along the curve that joints them (Figure 1c). How may the graph of  $D_{(1,-1)}f(P)$  as a function of time look?

II-7c. The following figure (Figure 1d) shows the contour diagram of the tangent plane of a function  $z = f(x, y)$  at the point  $(0,3)$ . Approximate the value of  $f(0.04, 2.97)$  the best you can.





**Figure 1: Figures for the interview problems**

## RESULTS

We show the results of two students, Hamid and Koorosh, relative to their Differential Calculus Schema development. With this, we show how we operationalized the study of Schemas by defining a collection of components and investigating the type of relations between components that have been established by students.

We found that Hamid interconnected all seven component structures with equivalence relations. So, we use him as an example of the Trans-DC (Differential Calculus) Schema development. We only show some excerpts from the interview to exemplify how he generally responded as it will not be possible to address all Schema components in this report.

In II-1, Hamid was given the graph of a function and was asked for the graph of  $H(t) = \frac{\partial f}{\partial x}(t,0)$  (Figure 2). He showed to have an equivalence relation grouping the components of partial derivative, slope, and function.

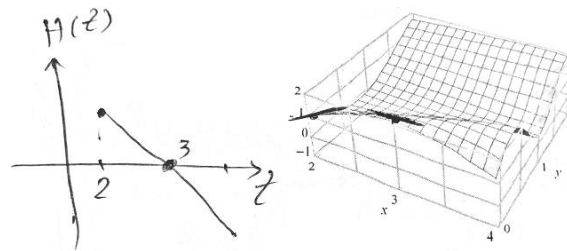
Hamid: I draw two axes, the vertical axis named  $H(t)$  and the horizontal one named  $t$ . I need to compare the slope of tangent lines to graph  $f$  at the points  $(t,0)$  in the  $x$  direction when  $t$  goes from 2 to 4.

Interviewer: You should draw H.

Hamid: Okay, the derivative with respect to  $x$  at the point  $(2,0)$  is the slope of this tangent line, the slope is positive so in the graph  $H(t)$  we have a point here above the  $t$  axis and umm for  $t=2$ . If I continue in such a way for  $t=3$  we have the point  $(3,0)$ , based on the figure the tangent line is a horizontal line like this so the slope is 0. I now plot point  $(3,0)$  in the graph. From  $t=2$  to  $t=3$  the slopes of the tangent lines are decreasing. From  $t=3$  to  $t=4$  the values of slopes of the tangent lines are negative and they going to more negative and more negative, so we have a decreasing curve like this from  $t=3$  to  $t=4$ .

In this problem, Hamid considered partial derivative and slope as interchangeable (“I need to compare the slope of tangent lines...”). This suggests that the components of

function, slope, and partial derivative are grouped in an equivalence relation where the properties of partial derivatives are conserved in the slopes of tangent lines.



**Figure 2: Hamid's work on problem II-1**

In I-6a, Hamid was given the graph of a tangent plane and was asked to discuss change in value of a function. He grouped the components of tangent plane, function, total differential, partial derivative, and slope. Hamid related the components of tangent plane (given) and function (requested) through the total differential while treating total differential and tangent plane as interchangeable.

Hamid: I know  $df = f_x dx + f_y dy$ . Here we have  $dx = 0.02$  and  $dy = -0.02$ . I have to find the values of  $f_x$  and  $f_y$  at the point  $(1, 2, 0)$ . Since it's a tangent line to the function  $f$  at the point  $(1, 2, 0)$  so  $f_x$  is  $m_x$  and  $f_y$  is  $m_y$ . Based on the figure  $m_x$  is 1 over umm 2 minus 1 which is 1 so it will be 1, so  $m_x$  is 1, and  $m_y$  is 3 units to the up over 3 minus 2 which is 1 umm it will be 3 over 1 which is 3, so  $m_y$  is 3. The change in the value of the function is 0.02 times 1 plus  $-0.02$  times 3 and umm the answer is  $-0.04$ .

Note that he justified the relation between tangent plane, partial derivative, and total differential when, while computing total differential, he says "since it's a tangent line [part of the tangent plane] to the function  $f$  at the point  $(1, 2, 0)$  so  $f_x$  is  $m_x$  and  $f_y$  is  $m_y$ ." As we previously saw, in another problem (II-1) he evidenced an equivalence relation between slope and partial derivative, and in the next problem (II-7c) will show such a relation between tangent plane and function.

In problem II-7c Hamid was given a contour diagram of the tangent plane of a function  $z = f(x, y)$  at the point  $(0, 3)$ . He was asked to approximate the value of  $f(0.04, 2.97)$  as best as he could. In this problem, Hamid treated function and tangent plane as interchangeable in recognizing that approximate change is conserved. He gave evidence consistent with an equivalence relation grouping function and tangent plane.

Hamid: The point  $(0.04, 2.97)$  is very close to the point  $(0, 3)$ , so to find the value of  $f$  at the point  $(0.04, 2.97)$  I can find the value of  $z$  at the point  $(0.04, 2.97)$  on the tangent plane.

Interviewer: Why?

Hamid: Because the graph of  $f$  and the graph of tangent plane are very similar to each other for the small neighbourhood of the point  $(0,3)$ . I mean the tangent plane is an approximation for the function  $f$ .

Interviewer: Okay, approximate the value of  $f(0.04,2.97)$ .

Hamid: At the point  $(0,3)$  the value of  $z$  or  $f(0,3)$  based on the contour diagram is 6. The equation of the tangent plane is  $z - 6 = 1(x - 0) + \frac{2}{3}(y - 3)$  so we have  $z = 6 + x + \frac{2}{3}(y - 3)$ . If I put  $x=0.04$  and  $y=2.97$  then I have  $z$  equal to  $z = 6 + 0.04 + \frac{2}{3}(2.97 - 3)$ , and this is an approximation for  $f(0.04,2.97)$ .

Hamid was not just doing a rote computation, he justified the relation when he said: “because the graph of  $f$  and the graph of tangent plane are very similar to each other for the small neighbourhood of the point  $(0,3)$  ...”

While we only showed a few examples, Hamid related all components of the Differential Calculus Schema with equivalence relations. The interview instrument we designed does not allow us to inquire into the coherence of the Schema. Students knew they were being asked about the Differential Calculus Schema. They did not have to decide if a problem situation fell in the scope of the Schema. Nevertheless, we tentatively classify him as in the Trans-DC stage of Schema development.

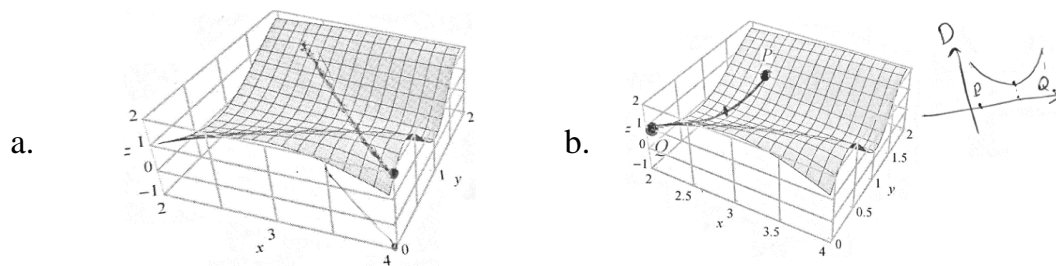
Now we consider Koorosh as an example of a student at the Inter-DC stage of Schema development. At this stage transformation relations are starting to form, grouping different components of the Schema. There will still be some components related by correspondence relations and there may be some unrelated components. Again, we are only able to show a few examples of his work.

In problem I-3b, Koorosh gave evidence of a transformation relation grouping components of function, directional derivative, and slope. When given the graph of a function (Figure 3a) and asked for the sign of a directional derivative:

Koorosh: We are at this point umm if we move in the direction umm -2 units in the  $x$  direction and 1 unit in the  $y$  direction it's like we are moving in this direction (see Figure 3a for his direction). Looking at the figure we see the tangent line in this direction has a positive slope because the values of  $z$  increase so the sign of  $D$  which is the directional derivative is positive.

Note that he justified his answer. So, we classified this relation between function, directional derivative, and slope as a transformation relation. We did not classify it as an equivalence relation since, in problem II-6, when asked for the graph of  $D_{(1,-1)}f(P)$  as a function of time as  $P$  moves towards point  $Q$  (Figure 3b), he did not evidence an equivalence between directional derivative and slope as may be seen in his answer; here, the directional derivative is initially negative but his graph of this derivative starts above the horizontal axis. There were however many instances where he showed the construction of correspondence relations as the next example shows.





**Figure 3: Koorosh's work on problems I-3b and II-6**

In problem I-4, Koorosh was given a table of values (see Figure 4) and was asked to approximate the value of  $\frac{\partial f}{\partial y}(1,2)$  the best he could.

Koorosh: I need to compute the change of  $z$  over the change of  $y$  by holding  $x$  equal to 1. When  $y$  changes from 1 to 2 then  $z$  changes from 6 to 10, therefore  $f_y$  is  $(10 - 6) / (2 - 1)$  and it's 4.

Interviewer: Is your answer the best approximation?

Koorosh: It's the derivative in the  $y$  direction and so my answer seems to be correct.

$x / y$	1	2	2.01
0	5	7	7.04
1	6	10	10.06
2	8	14	14.10

**Figure 4: Table of values for problem I-4**

Observe that Koorosh seems to be rigidly applying a procedure he could not justify. The difference between correspondence, transformation, and equivalence relation in this problem is best understood by comparing with the response of Hamid.

Hamid: It's the partial derivative with respect to  $y$ . I need to find it at the point  $(1,2)$ . I need to fix the  $x$  equal to 1 so we are in the third row of the table where  $x$  is 1. For this row when  $y$  is 2 umm  $f(1,2)$  is 10. When  $y$  changes from 2 to 2.01 then the value of the function changes from 10 to 10.06 so the derivative at the point  $(1,2)$  and with respect to  $y$  is approximately equal to  $(10.06 - 10) / (2.01 - 2)$  which is 6.

Here, Hamid aimed to relate the component of function (the given table) with that of partial derivative (the request of the problem). To do so, he related the components of partial derivative and slope as seen in the computation of slope as a quotient of "change". Here we consider average rate of change as another way to think about slope. The structures of slope and partial derivative are grouped and related to each other in terms of the structure of function (the table). Function plays a role in the reasoning of the connection when he recognizes the partial derivative "is approximately equal" to the slope he computed and uses the closest point to the base point in doing so. So, the relation between the slope, partial derivative, and function components in this problem

satisfies the definition of transformation relation. However, considering Hamid's overall performance, he consistently treated slope and partial derivative interchangeably in his other problem solving so we may classify this relation as an equivalence relation.

We may also compare Koorosh with Hamid in problem II-7c where he uses a contour diagram of the tangent plane to approximate a functional value:

Koorosh: The value of  $f$  at the point  $(0,3)$  is 6. When  $x$  increases 0.04 units then  $z$  changes as 0.04 times 1 which is 0.04 so  $f(x=0.04,3)$  will be 6.04. Now if  $y$  decreases as  $3-2.97$  which is 0.03 then  $z$  decreases umm 0.03 times  $2/3$ . So the value of  $f$  at  $(x=0.04, y=2.97)$  will be  $6.04-(0.03)2/3$ .

Observe that Koorosh could do computations relating tangent plane (the given contour diagram) with function (the requested value), but his argument lacks justification, suggesting a correspondence relation. Similarly, in problem I-6a, when given a graphical representation of the tangent plane and asked about the change in the value of the function if  $x$  increases 0.02 units and  $y$  decreases 0.02 units:

Koorosh: I first find  $f_y$  umm it's the change in  $z$  over the change in  $y$  which is umm using this line it will be  $(3-0)/(3-2)$  which is 3. Now I find  $f_x$  which is  $(4-3)/(2-1)$  umm which is 1.

Interviewer: You need to find the change in the function.

Koorosh: To compute it I use the differential formula which is  $dz = dx + 3dy$ . So it will be  $dz = 0.02 + 3(-0.02)$  umm the answer is  $-0.04$  and this means the function decreases 0.04 units.

Again, Koorosh grouped several Schema components: tangent plane (given graph), function (the requested approximation), total differential (what he chose to use to do the computations), partial derivative (to compute total differential), and slope (to compute partial derivative). He did this without any justification and without showing awareness in this or any other problem of the interview instruments of the relation between tangent plane and total differential, other than to do computations.

## DISCUSSION AND CONCLUSION

The cases of Hamid and Koorosh exemplify how Schemas may be used to better understand students' construction in the differential calculus of two-variable functions. The types of responses these students showed were very different from each other. We found that Hamid showed evidence of a Trans-DC Schema development. This is assuming a Schema coherence we can't justify or inquire into with our instruments. Koorosh showed evidence consistent with an Inter-DC Schema development. Although it has not been stressed in this report, a closer examination of the relations established or not, can inform instruction and result in improvement of didactical activities.

This study enabled us to consider the definitions of the types of relations between Schema components and interpret them in a new context. In doing so, we gained a deeper understanding of the relations involved in the development of the Differential Calculus Schema. In future studies, we need to be more insistent in teasing out explanations and justifications from students in order to better distinguish the difference between the types of relations. Our study contributes to furthering the understanding of Schemas, their stages of development, and the correspondence, transformation, and equivalence relations among Schema components together with understanding how students' constructions may evolve in the course. Considering the work of all eleven students, we find there are many possible ways in which the DC-Schema stage of development appears, varying from weak to strong transition levels from Intra-DC to Inter-DC, and from Inter-DC to Trans-DC stages of Schema development. We also found that students showed a deeper understanding of the different Schema components in this second research cycle than in the first research cycle (Martínez-Planell et al., 2015). A constraint we have in the present study is that the interview instrument was not designed to allow for an investigation into Schema coherence. More research is necessary to study Differential Calculus Schema coherence.

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