Well-Posedness of evolution variational inequalities with applications

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Abstract

We are concerned in the present work with the existence and uniqueness of absolutely continuous solutions to a class of evolution problems governed by time-dependent subdifferential operators of the form

$$f(t) + Bu(t) - A\frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t))$$

with various applications.

1 Introduction

In this work we are concerned with the existence and uniqueness of absolutely continuous solution to an evolution inclusion in a separable Hilbert space H in the form

$$f(t) + Bu(t) - A\frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)), t \in [0, T]. \tag{1.1}$$

Above $f : [0, T] \to H$ is a continuous mapping, $B : H \to H$ is an operator, $A : H \to H$ is a linear continuous coercive and symmetric operator, $\varphi : [0, T] \times H \to]-\infty, +\infty]$ is a normal lower semicontinuous convex integrand, and $\partial\varphi(t, .)$ is the subdifferential of $\varphi(t, .)$. Problem (1.1) is interpreted as an evolution variational

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inequality (EVI) with the velocity inside the subdifferential. Generally, the model for parabolic evolution inclusion is a differential inclusion of the form

$$B(t,u(t)) \in \frac{du}{dt}(t) + A(t)u(t) + \partial\varphi(t,u(t)), \quad t \in [0,T], \tag{1.2}$$

where $A(t)$ is a time dependent maximal monotone operator, $B(t,u)$ defined for $(t,u) \in [0,T]\times H$ is Lipschitz with respect to u. Then the existence and uniqueness of absolutely continuous solution to (1.2) is known in some particular cases in the literature, see e.g [4, 19] and Barbu and Rascanu in [7] dealing with existence of *generalized solutions* for parabolic variational inequalities with singular inputs and operators of the form

$$f(t) + \frac{dM}{dt}(t) \in \frac{du}{dt}(t) + Au(t) + \partial\varphi(u(t)),$$

where A is a linear coercive operator and φ is a lower semicontinuous convex function. There is an increasing activity around problem (1.2) since it contains several new applications such as sweeping process, relaxed problem and Skorohod problem etc. In this framework, problem (1.1) constitutes a new variational evolution inequality with the velocity inside the subdifferential in constrast to problem (1.2). Likewise problem (1.2), the study of (1.1) leads to several applications in a new setting such as the sweeping process, Skorohod problem, second order evolution and fractional differential equation [19]. Although (1.1) deals with the deterministic case, it is a step towards the Skorohod problem in the stochastic setting, see the recent articles by Castaing-Raynaud de Fitte [18, 20], Rascanu [38], and L.Maticiuc, A. Rascanu, L. Slominski and M.Topolewski [27] for references on this stochastic subject. Let us mention the current situation of problem (1.1) in the literature. In [19] it was dealt with the existence of absolutely continuous solutions to variational evolution inequalities in separable Hilbert space H of the forms

$$f(t) - Au(t) \in \partial\varphi(t, \frac{du}{dt}(t)) \tag{1.3}$$

$$f(t) - Au(t) \in N_{C(t)}(\frac{du}{dt}(t)), \tag{1.4}$$

where $f : [0,T] \to H$ is a continuous mapping, $A : H \to H$ is a linear continuous coercive symmetric operator, $\varphi : [0,T] \times H \to]-\infty, +\infty]$ is a normal convex integrand, and $N_{C(t)}(x)$ denotes the normal cone to a closed convex moving set $C(t) \subset H$. Some related variants of problem (1.4) dealing with two positive operators A and B are given in a series of papers by Adly et al [1, 2, 3]. We note that there is a new variant of problem (1.1) in a recent work by Bacho, Emmrich and Mielke [8] dealing with the following inclusion

$$B(t,u(t)) \in \partial\varphi(t, \frac{du}{dt}(t))$$

or more generally with two subdifferentials, namely

$$B(t,u(t)) \in \partial\varphi(t,\frac{du}{dt}(t)) + \partial\psi(t,\frac{du}{dt}(t)),$$

where B is a continuous mapping. In Mielke's paper, in order to solve the problem, it is proposed an algorithm due to De Giorgi combined with regularization of subdifferentials. Consult also a recent article by Migorski, Sofonea and Zeng [29] dealing with the inclusion of the form

$$-\frac{du}{dt}(t) \in N_{C(t)}(A\frac{du}{dt}(t) + Bu(t))$$

where $B : H \to H$ is Lipschitz continuous. Note that in [29] B is not assumed to be a positive operator in contrast to the results obtained by Adly et al [1, 2, 3].

Our aim in the present paper is to develop several variants of problem (1.1) along with diverse applications via some related variational limits.

2 Preliminaries

Throughout the paper, H is a *real separable Hilbert space* and E is a separable Banach space with $\mathbb{B}_E$ as its closed unit ball centered at zero.

If I is an interval of $\mathbb{R}$ the spaces $L^1(I,E,dt)$ ($L^1_E(I)$, for short) and $L^\infty(I,E,dt)$ ($L^\infty_E(I)$, for short) denote the usual spaces with respect to the Lebesgue measure endowed with their canonical norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively. Similarly, $\mathcal{C}(I,E)$ or $\mathcal{C}_E(I)$ will be the space of continuous mappings from I into E. For any subset $Q \subset I$ the function $\mathbf{1}_Q$ is defined by $\mathbf{1}_Q(t) = 1$ if $t \in Q$ and $\mathbf{1}_Q(t) = 0$ otherwise.

Given a convex function $\varphi : H \to \mathbb{R} \cup \{-\infty, +\infty\}$, its effective domain $\operatorname{dom}\varphi$ is the set

$$\operatorname{dom}\varphi := \{x \in H : \varphi(x) < +\infty\},$$

so the function φ is proper whenever $\operatorname{dom}\varphi \neq \emptyset$ and φ does not take the value $-\infty$. At any $x \in H$ where φ is finite its subdifferential $\partial\varphi(x)$ is defined by

$$\partial\varphi(x) := \{\zeta \in H : \langle \zeta, y - x\rangle - \varphi(x) \leq \varphi(y),\ \forall y \in H\}.$$

If $f(x)$ is not finite $\partial\varphi(x) = \emptyset$. Considering the Legendre-Fenchel conjugate $\varphi^* : H \to \mathbb{R} \cup \{-\infty, +\infty\}$ with

$$\varphi^*(y) := \sup\{\langle y, x\rangle - f(x) : x \in H\},$$

it is known that, when φ is a proper lower semicontinuous convex function, φ^* is also proper lower semicontinuous and

$$y \in \partial\varphi(x) \ \Leftrightarrow\ \langle y, x\rangle = \varphi(x) + \varphi^*(y).$$

Given a nonempty closed convex subset S of H, its indicator function $\delta(\cdot, S) : H \to \mathbb{R} \cup \{+\infty\}$ is defined by $\delta(x, S) = 0$ if $x \in S$ and $\delta(x, S) = +\infty$ if $x \in H \setminus S$. Clearly, $\delta(\cdot, S)$ is a proper lower semicontinuous convex function. Its subdifferential is called the *normal cone* of S, and obviously

$$N_S(x) = \{\zeta \in H : \langle \zeta, y - x\rangle \leq 0,\ \forall y \in S\} \quad \text{if } x \in S \tag{2.1}$$

and $N_S(x) = \emptyset$ if $x \in H \setminus S$. The Legendre-Fenchel conjugate $\delta^*(\cdot, S) : H \to \mathbb{R} \cup \{+\infty\}$ coincides with the *support function* of S, that is,

$$\delta^*(y, S) = \sup_{x \in S} \langle y, x\rangle \quad \text{for all } y \in H.$$

If S, S' are both nonempty closed bounded convex sets of H, the Hausdorff distance between S and S' can be defined by

$$\operatorname{haus}(S, S') = \sup_{x \in H} |d_S(x) - d_{S'}(x)|,$$

and it is known that (see, e.g., [24, 41])

$$\operatorname{haus}(S, S') = \sup_{\|u\| \leq 1} |\delta^*(u, S) - \delta^*(u, S')|, \tag{2.2}$$

which entails

$$|\delta^*(y, S) - \delta^*(y, S')| \leq \|y\| \operatorname{haus}(S, S') \quad \text{for all } y \in H. \tag{2.3}$$

We recall and summarize two useful results, see for example [6, Corollary 2.9, Corollary 2.10]. Remind that a linear operator $A : H \to H$ is *coercive* if there is a real $\omega > 0$ such that

$$\langle Ax, x\rangle \geq \omega \|x\|^2 \quad \text{for all } x \in H. \tag{2.4}$$

Proposition 2.1. *Let $A : H \to H$ be a linear continuous and coercive operator.*
(a) If $\varphi : H \to [0, \infty]$ is a proper lower semicontinuous convex function, then for each $f \in H$ the problem $f \in Ay + \partial\varphi(y)$ admits a unique solution y.
(b) If K is a closed convex subset in H, then for each $f \in H$ the problem $f \in Ay + N_K(y)$ admits a unique solution y.

3 Preparatory variational limit theorems

Proposition 3.1. *Let H be a separable Hilbert space and $I = [0, T]$. Let $C : I \rightrightarrows H$ be a closed convex valued scalarly measurable multimapping for which there is some real $r > 0$ such that $C(t) \subset r\mathbb{B}_H$ for all $t \in I$. Let A be a continuous symmetric coercive linear operator on H and let B be a continuous compact linear*

operator on H. Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions from I into I such that for each $t \in I$ one has $\theta_n(t) \to t$ and haus $\big(C(\theta_n(t)), C(t)\big) \to 0$ *as* $n \to \infty$.

Let $(f_n, f)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_H(I)$ with $\|f_n(t)\| \le \beta, \|f(t)\| \le \beta$ $(\beta > 0)$ for all $n \in \mathbb{N}$ such that $(f_n(t))_n$ converges to $f(t)$ for each $t \in I$.
Let $(v_n, v)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_H(I)$ with $\|v_n(t)\| \le \gamma, \|v(t)\| \le \gamma$ $(\gamma > 0)$ for all $n \in \mathbb{N}$ such that $(v_n(t))_n$ converges weakly to $v(t)$ for each $t \in I$.
Let $(\zeta_n)_{n\in\mathbb{N}}$ be an integrable sequence in $L^1_H(I)$ such that $\zeta_n(t) \in C(\theta_n(t))$ for all $t \in I$ and such that $(\zeta_n)_{n\in\mathbb{N}}$ $\sigma(L^1_H, L^\infty_H)$ converges in $L^1_H(I)$ to ζ. Assume that

$$f_n(t) + Bv_n(t) - A\,\zeta_n(t) \in N_{C(\theta_n(t))}(\zeta_n(t)) \quad \text{for all } n \in \mathbb{N},\ \text{a.e. } t \in I.$$

Then for a.e. $t \in I$ one has

$$\zeta(t) \in C(t) \quad \text{and} \quad f(t) + Bv(t) - A\,\zeta(t) \in N_{C(t)}(\zeta(t)).$$

Proof. We first verify that $\zeta(t) \in C(t)$ a.e.$t \in I$. Indeed, for every measurable set $Z \subset I$ and for any $x \in H$, the function $\mathbf{1}_Z x \in L^\infty_H(I)$. By the inequality

$$\langle x, \zeta_n(t)\rangle \le \delta^*(x, C(\theta_n(t)))$$

integrating on Z gives

$$\int_I \langle \mathbf{1}_Z x, \zeta_n(t), \rangle dt = \int_Z \langle x, \zeta_n(t)\rangle dt \le \int_Z \delta^*(x, C(\theta_n(t)))dt.$$

Passing to the upper limit in this inequality we obtain

$$\begin{aligned}\int_Z \langle \mathbf{1}_Z x, \zeta(t)\rangle dt &\le \limsup_n \int_Z \delta^*(x, C(\theta_n(t)))dt \\ &\le \int_Z \limsup_n \delta^*(x, C(\theta_n(t))) \le \int_Z \delta^*(x, C(t))dt.\end{aligned}$$

This being true for any Lebesgue measurable set $Z \subset I$ we deduce that for every $x \in H$

$$\langle x, \dot{u}(t)\rangle \le \delta^*(x, C(t)) \ \text{ a.e. } t \in I.$$

By the separability of H and the weak compactness and convexity of $C(t)$ (see, e.g., Castaing-Valadier [24, Proposition III- 35]), we get the desired inclusion $\zeta(t) \in C(t)$ a.e. $t \in I$

For each $t \in I$ keeping in mind that $v_n(t) \to v(t)$ weakly in H and B is a continuous compact linear operator, we see that $Bv_n(t) \to Bv(t)$ strongly in H, so that $Bv_n(.) \to Bv(.)$ weakly in $L^1_H(I)$. Indeed, let any $h \in L^\infty_H(I)$. Then we have

$$\begin{aligned}\left|\int_0^T \langle h(t), Bv_n(t)\rangle dt - \int_0^T \langle h(t), Bv(t)\rangle dt\right| &\le \int_0^T |\langle h(t), Bv_n(t) - Bv(t)\rangle| dt \\ &\le |h|_\infty \int_0^T \|Bv_n(t) - Bv(t)\| dt.\end{aligned}$$

As $\int_0^T \|Bv_n(\delta_n(t))-Bv(t)\|dt \to 0$ as $n \to \infty$ (by Lebesgue dominated convergence theorem), our assertion follows. Similarly as A is symmetric, we note that $A\zeta_n \to A\zeta$ weakly in $L^1_H(I)$. As a main consequence $f_n+Bv_n-A\zeta_n \to f+Bu-A\zeta$ weakly in $L^1_H(I)$. Then given any Lebesgue measurable subset $Z \subset I$ we may apply the lower semicontinuity of convex integral functional ([22], Theorem 8.1.16) to deduce that

$$\int_Z \delta^*(f(t)+Bv(t)-A\zeta(t), C(t))dt \le \liminf_n \int_Z \delta^*(f_n(t)+Bv_n(t)-A\zeta_n(t), C(t))dt. \tag{3.1}$$

This need a careful look. Indeed, we note that $(t,x) \mapsto \delta^*(x, C(t))$ is a normal lower semicontinuous convex integrand defined on $[0,T] \times H$ and $\delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t))$ is measurable and bounded by an integrable function independent of $n \in \mathbb{N}$ since taking some real constant α with $\|f_n(t) + Bv_n(t) - A\zeta_n(t)\| \le \alpha$ for all $n \in \mathbb{N}$ and $t \in I$ we have

$$|\delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t))| \le r\|f_n(t) + Bv_n(t) - A\zeta_n(t)\| \le \alpha r.$$

Then writing

$$\begin{aligned}
&\big|\delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t)) - \delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(\theta_n(t)))\big| \\
&\le \|f_n(t) + Bv_n(t) - A\zeta_n(t))\| \operatorname{haus}\big(C(t), C(\theta_n(t))\big) \\
&\le \alpha \operatorname{haus}\big(C(t), C(\theta_n(t))\big),
\end{aligned}$$

we can see that

$$\begin{aligned}
&\liminf_n \int_Z \delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(\theta_n(t)))dt \\
&\ge \liminf_n \int_Z \delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t))dt \\
&\ge \int_Z \delta^*(f(t) + Bv(t) - A\zeta(t), C(t))dt.
\end{aligned} \tag{3.2}$$

Let us set $\psi_A(x) = \langle Ax, x\rangle$ if $x \in r\mathbb{B}_H$ and $\psi_A(x) = +\infty$ if $x \notin r\mathbb{B}_H$. Then it is clear that ψ_A is a positive lower semicontinuous convex integrand. Apply again the lower semicontinuity of the convex integral functional ([22], Theorem 8.1.6) associated with the positive normal convex integrand ψ_A we obtain

$$\liminf_n \int_Z \psi_A(\zeta_n(t))dt \ge \int_Z \psi_A(\zeta(t))dt,$$

that is,

$$\liminf_n \int_Z \langle A\zeta_n(t), \zeta_n(t))dt \ge \int_Z \langle A\zeta(t), \zeta(t)dt. \tag{3.3}$$

Since we already saw that $Bv_n(t) \to Bv(t)$ strongly in H for each $t \in I$, we also have by the Lebesgue dominated convergence theorem

$$\lim_n \int_Z \langle Bv_n(t), \zeta_n(t)\rangle dt = \int_Z \langle Bv(t), \zeta(t)\rangle dt. \tag{3.4}$$

Further, by the Lebesgue dominated convergence theorem again the inequality

$$|\langle f_n(t), \zeta_n(t)\rangle - \langle f(t), \zeta_n(t)\rangle| \leq r\|f_n(t) - f(t)\|$$

gives $\int_Z \big(\langle f_n(t), \zeta_n(t)\rangle - \langle f(t), \zeta_n(t)\rangle\big)dt \to 0$. Also as $n \to \infty$

$$\int_Z \langle f(t), \zeta_n(t)\rangle dt \to \int_Z \langle f(t), \zeta(t)\rangle dt$$

since $\zeta_n \to \zeta$ weakly in $L^1_H(I)$. It ensues that

$$\lim_n \int_Z \langle f_n(t), \zeta_n(t)\rangle dt = \int_Z \langle f(t), \zeta(t)\rangle dt. \tag{3.5}$$

Now putting $q_n(t) := f_n(t) + Bv_n(t) - A\zeta_n(t)$ and integrating on $Z \subset [0,T]$ the inequality (here measurability and integrability are guaranted)

$$\delta^*(q_n(t), C(\theta_n(t))) + \langle -f_n(t) - Bv_n(t), \zeta_n(t)\rangle + \langle A\zeta_n(t), \zeta_n(t)\rangle \leq 0$$

gives

$$\int_Z \delta^*(q_n(t), C(\theta_n(t)))dt + \int_Z \langle Av_n(t), \zeta_n(t)\rangle dt + \int_Z \langle -f_n(t) - Bv_n(t), \zeta_n(t)\rangle dt \leq 0,$$

so passing to the limit as $n \to \infty$ in this equality and using (3.1)—(3.5) yield

$$\int_Z \big[\delta^*(f(t) + Bv(t) - A\zeta(t), C(t)) + \langle A\zeta(t) - Bv(t) - f(t), \zeta(t)\rangle\big]dt \leq 0.$$

As $t \mapsto \delta^*(f(t) + Bv(t) - A\zeta(t), C(t)) + \langle A\zeta(t) - Bv(t) - f(t), \zeta(t)\rangle$ is integrable and as the latter inequality holds true for any Lebesgue measurable set $Z \subset I$, it follows that for a.e. $t \in I$

$$\delta^*(f(t) + Bv(t) - A\zeta(t), C(t)) + \langle A\zeta(t) - Bv(t) - f(t), \zeta(t)\rangle \leq 0.$$

This and the inclusion $\zeta(t) \in C(t)$ a.e. $t \in I$ allow us to conclude that

$$f(t) + Bv(t) - A\zeta(t)) \in N_{C(t)}(\zeta(t)) \text{ a.e. } t \in I$$

according to the description (2.1) of the normal cone. □

An easy adaptation of the arguments in the above proposition furnishes the following variant.

Proposition 3.2. *Let H be a separable Hilbert space and $I = [0,T]$. Let $C : I \rightrightarrows H$ be a closed convex valued scalarly measurable multimapping for which there is some real $r > 0$ such that $C(t) \subset r\mathbb{B}_H$ for all $t \in I$. Let A be a continuous symmetric linear operator coercive on H and let $B : H \to H$ be a Lipschitz*

mapping. Let $(\theta_n)_{n\in\mathbb{N}}$ *be a sequence of measurable functions from* I *into* I *such that for each* $t \in I$ *one has* $\theta_n(t) \to t$ *and* $\operatorname{haus}\big(C(\theta_n(t)), C(t)\big) \to 0$ *as* $n \to \infty$.

Let $(f_n, f)_{n\in\mathbb{N}}$ *be a bounded sequence in* $L^\infty_H(I)$ *with* $\|f_n(t)\| \le \beta, \|f(t)\| \le \beta$ $(\beta > 0)$ *for all* $n \in \mathbb{N}$ *such that* $(f_n(t))_n$ *converges to* $f(t)$ *for each* $t \in I$.
Let $(v_n, v)_{n\in\mathbb{N}}$ *be a bounded sequence in* $L^\infty_H(I)$ *with* $\|v_n(t)\| \le \gamma$, $\|v(t)\| \le \gamma$ $(\gamma > 0)$ *for all* $n \in \mathbb{N}$ *such that* $(v_n(t))_n$ *converges to* $v(t)$ *for each* $t \in I$. *Let* $(\zeta_n)_{n\in\mathbb{N}}$ *be an integrable sequence in* $L^1_H(I)$ *such that* $\zeta_n(t) \in C(\theta_n(t))$ *for all* $t \in I$ *and such that* $(\zeta_n)_{n\in\mathbb{N}}$ $\sigma(L^1_H, L^\infty_H)$ *converges in* $L^1_H(I)$ *to* ζ. *Assume that*

$$f_n(t) + Bv_n(t) - A\,\zeta_n(t) \in N_{C(\theta_n(t))}(\zeta_n(t)) \quad \text{for all } n \in \mathbb{N}, t \in I.$$

Then for a.e. $t \in I$ *one has*

$$\zeta(t) \in C(t) \quad \text{and} \quad f(t) + Bv(t) - A\,\zeta(t) \in N_{C(t)}(\zeta(t)).$$

Another variant of Proposition 3.1 is available with the gradient ∇g of a convex function g in place of the mapping B.

Proposition 3.3. *Let* H *be a separable Hilbert space and* $I = [0, T]$. *Let* $C : I \rightrightarrows H$ *be a closed convex valued scalarly measurable multimapping for which there is some real* $r > 0$ *such that* $C(t) \subset r\mathbb{B}_H$ *for all* $t \in I$. *Let* A *be a continuous symmetric linear coercive operator on* H *and let* $g : H \to \mathbb{R}$ *be a convex function Lipschitz on bounded sets and continuously differentiable on* H. *Let* $(\theta_n)_{n\in\mathbb{N}}$ *be a sequence of measurable functions from* I *into* I *such that for each* $t \in I$ *one has* $\theta_n(t) \to t$ *and* $\operatorname{haus}\big(C(\theta_n(t)), C(t)\big) \to 0$ *as* $n \to \infty$.

Let $(f_n, f)_{n\in\mathbb{N}}$ *be a bounded sequence in* $L^\infty_H(I)$ *with* $\|f_n(t)\| \le \beta, \|f(t)\| \le \beta$ $(\beta > 0)$ *for all* $n \in \mathbb{N}$ *such that* $(f_n(t))_n$ *converges to* $f(t)$ *for each* $t \in I$.
Let $(u_n, u)_{n\in\mathbb{N}}$ *be a sequence of absolutely continuous mappings*

$$u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds, \;\; \dot{u}_n(t) \in C(t)$$

$$u(t) = u_0 + \int_0^t \dot{u}(s)ds, \;\; \dot{u}(t) \in C(t)$$

such that $(u_n)_n$ *converges to* u *uniformly on* I *and* $(\dot{u}_n)_n$ *conveges to* $\dot{u}$ *weakly in* $L^1_H(I)$. *Assume that for every* $n \in \mathbb{N}$

$$f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t) \in N_{C(\theta_n(t))}(\dot{u}_n(t)) \text{ a.e. } t \in I.$$

Then for a.e. $t \in I$ *one has*

$$\dot{u}(t) \in C(t) \quad \text{and} \quad f(t) + \nabla g(u(t)) - A\,\dot{u}(t) \in N_{C(t)}(\dot{u}(t)).$$

Proof. First, we justify that $\dot{u}(t) \in C(t)$ a.e. $t \in I$. We proceed as in the proof of Proposition 3.1. Take any measurable Lebesgue set $Z \subset I$ and any $x \in H$. The function $\mathbf{1}_Z x \in L^\infty_H(I)$. Writing

$$\langle x, \dot{u}_n(t)\rangle \le \delta^*(x, C(\theta_n(t))),$$

we see that

$$\int_I \langle \mathbf{1}_Z x, \dot{u}_n(t), \rangle dt = \int_Z \langle x, \dot{u}_n(t) \rangle dt \leq \int_Z \delta^*(x, C(\theta_n(t))) dt.$$

Passing to the upper limit we obtain

$$\int_Z \langle \mathbf{1}_Z x, \dot{u}(t) \rangle dt \leq \limsup_n \int_Z \delta^*(x, C(\theta_n(t))) dt$$
$$\leq \int_Z \limsup_n \delta^*(x, C(\theta_n(t))) \leq \int_Z \delta^*(x, C(t)) dt.$$

This being true for any Lebesgue measurable set $Z \subset I$ we deduce that for every $x \in H$

$$\langle x, \dot{u}(t) \rangle \leq \delta^*(x, C(t)) \ {}_t ext a.e.\ t \in I.$$

By the separability of H and the weak compactness and convexity of $C(t)$, we get the desired inclusion $\dot{u}(t) \in C(t)$ a.e. $t \in I$.

Now let $\gamma > 0$ be a Lipschitz constant of g on $(\|u_0\| + r'T)\mathbb{B}_H$ for some $r' > r$. Take any Lebesgue measurable set $Z \subset I$. Since $(f_n)_n$ is uniformly bounded and pointwise converges to f, we have

$$\lim_n \int_Z \langle f_n(t), \dot{u}_n(t) \rangle dt = \int_Z \langle f(t), \dot{u}(t) \rangle dt. \tag{3.6}$$

Also by integrating on Z (we are ensured that the functions given are measurable) the inequality

$$\delta^*\big(f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t), C(\theta_n(t))\big) + \langle A\dot{u}_n(t) - \nabla g(u_n(t)) - f_n(t), \dot{u}_n(t) \rangle \leq 0$$

it ensues that with $q_n(t) := f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t)$

$$\int_Z \delta^*(q_n(t), C(\theta_n(t)))dt + \int_Z \langle A\dot{u}_n(t) - f_n(t), \dot{u}_n(t) \rangle dt - \int_Z \langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle dt \leq 0. \tag{3.7}$$

We claim that $Au_n(\cdot) \to Au(\cdot)$ weakly in $L^1_H(I)$ and $\nabla g(u_n(\cdot)) \to \nabla g(u(\cdot))$ weakly in $L^1_H(I)$, so

$$q_n(\cdot) = f_n(\cdot) + \nabla g(u_n(\cdot)) - A\dot{u}_n(\cdot) \to q(\cdot) := f(\cdot) + \nabla g(u(\cdot)) - A\dot{u}(\cdot)$$

weakly in $L^1_H(I)$. Indeed, for any $h \in L^\infty_H(I)$ the weak convergence in $L^1_H(I)$ of $(\dot{u}_n)_n$ to $\dot{u}$ says that $\int_0^T \langle Ah(t), \dot{u}_n(t) \rangle dt \to \int_0^T \langle Ah(t), \dot{u}(t) \rangle dt$, which means

$$\int_0^T \langle h(t), A\dot{u}_n(t) \rangle dt \to \int_0^T \langle h(t), A\dot{u}(t) \rangle dt.$$

This property for every $h \in L^\infty_H(I)$ translates the weak convergence in $L^1_H(I)$ of $(A\dot{u}_n)_n$ to $A\dot{u}$. Concerning $(\nabla g(u_n(\cdot)))_n$ it converges strongly in $L^1_H(I)$ to $\nabla g(u(\cdot))$

(by Lebesgue convergence theorem) since $\nabla g(u_n(t)) \to \nabla g(u(t))$ for each $t \in I$ and $\|\nabla g(u_n(t))\| \le \gamma$ (by the γ-Lipschitz property of g on $(\|u_0\| + r'T)\mathbb{B}_H$). The desired weak convergence in $L^1_H(I)$ of $(q_n)_n$ to q is justified. Let us consider now the term $\langle \nabla g(u_n(t), \dot{u}_n(t)\rangle$. The Lipschitz property of g on $(\|u_0\| + rT)\mathbb{B}_H$ assures us that $g \circ u_n$ and $g \circ u$ are absolutely continuous on I and at any $t \in I$ where u and all u_n are derivable and are derivable (by classical chain rule)

$$\langle \nabla g(u(t)), \dot{u}(t)\rangle = \frac{d}{dt} g(u(t)) \quad \text{and} \quad \langle \nabla g(u_n(t)), \dot{u}_n(t)\rangle = \frac{d}{dt} g(u_n(t)).$$

From this we deduce that

$$\begin{aligned}
&\int_0^T \langle \nabla g(u(t)), \dot{u}(t)\rangle dt \\
&= \int_0^T \frac{d}{dt} g(u(t)) dt = g(u(T)) - g(u(0)) = \lim_n \big(g(u_n(T)) - g(u_n(0))\big) \\
&= \lim_n \int_0^T \frac{d}{dt} g(u_n(t)) dt = \lim_n \int_0^T \langle \nabla g(u_n(t)), \dot{u}_n(t)\rangle dt. \qquad (3.8)
\end{aligned}$$

Regarding $\int_0^T \langle A\dot{u}_n(t), \dot{u}_n(t)\rangle dt$ let us set $\psi_A(t,x) = \langle Ax, x\rangle$ for all $x \in H$. It is clear that ψ_A is a positive lower semicontinuous convex normal integrand. By the lower semicontinuity of convex integral functional (see [22, Theorem 8.1.6]) we obtain

$$\liminf_n \int_0^T \psi_A(t, \dot{u}_n(t)) dt \ge \int_0^T \psi_A(t, \dot{u}(t)) dt,$$

that is,

$$\liminf_n \int_0^T \langle A\dot{u}_n(t), \dot{u}_n(t)\rangle dt \ge \int_0^T \langle A\dot{u}(t), \dot{u}(t)\rangle dt. \qquad (3.9)$$

The last step is concerned with $\int_0^T \delta^*(q_n(t), C(t)) dt$. Note that $\|q_n(t)\| \le \beta + \gamma + r\|A\|$, hence $\delta^*(q_n(t), C(t)) \ge -r(\beta + \gamma + r\|A\|)$. Using this and the lower semicontinuous convex normal integrand $(t,x) \mapsto \delta^*(x, C(t))$ we obtain by [22, Theorem 8.1.6] again that

$$\liminf_n \int_0^T \delta^*(q_n(t), C(t)) dt \ge \int_0^T \delta^*(g(t), C(t)) dt.$$

Since

$$\big|\delta^*\big(q_n(t), C(\theta_n(t)\big) - \delta^*\big(q_n(t), C(t)\big)\big| \le (\beta+\gamma+r\|A\|)\,\mathrm{haus}\big(C(\theta_n(t)), C(t)\big) =: \varepsilon_n(t)$$

with $\int_0^T \varepsilon_n(t) \to 0$ as $n \to \infty$, we are ensured that

$$\liminf_n \int_0^T \delta^*\big(q_n(t), C(t)\big) dt \ge \int_0^T \delta^*\big(g(t), C(t)\big) dt. \qquad (3.10)$$

Putting together (3.37)—(3.39) yields

$$\int_0^T \delta^*(f(t)+\nabla g(u(t))-A\dot{u}(t),C(t))dt+\int_0^T \langle -f(t)+\nabla g(u(t))+A\dot{u}(t),\dot{u}(t)\rangle dt \leq 0. \tag{3.11}$$

On the other hand, the inclusion $\dot{u}(t) \in C(t)$ says that for a.e. $t \in I$

$$\delta^*(f(t)+\nabla g(u(t))-A\dot{u}(t),C(t)) \geq \langle f(t)+\nabla g(u(t))-A\dot{u}(t),\dot{u}(t)\rangle.$$

Taking this into account in (3.40) we deduce for a.e. $t \in I$

$$\delta^*\big(f(t)+\nabla g(u(t))-A\dot{u}(t),C(t)\big)-\langle f(t+\nabla g(u(t))-A\dot{u}(t),\dot{u}(t) \leq 0$$

The latter inequality and the inclusion $\dot{u}(t) \in C(t)$ guarantees that for a.e. $t \in I$

$$f(t)+\nabla g(u(t))-A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$$

according to the description (2.1) of the normal cone. □

Remark 3.1. *Concerning the chain rule for $g \circ u$ and $g \circ u_n$ above, one can more in Moreau-Valadier [35] under weaker assumptions for the convex function g.*

When $\theta_n(t) = t$ (i.e., $C(\theta_n(t)) = C(t)$) in Proposition 3.1 the uniform boundedness condition $C(t) \subset r\mathbb{B}_H$ can be relaxed as shown in the next proposition. Let us prove first a lemma.

Lemma 3.2. *Let H be a separable Hilbert space and $I = [0,T]$. Let $C : I \rightrightarrows H$ be a scalarly measurable multimapping with closed convex weakly locally compact values which contain no line. Let $(\zeta_n)_{n\in\mathbb{N}}$ be sequence in $L^1_H(I)$ such that $\zeta_n(t) \in C(t)$ for all $t \in I$ and $n \in \mathbb{N}$ and such that $(\zeta_n)_{n\in\mathbb{N}}$ $\sigma(L^1_H, L^\infty_H)$ converges in $L^1_H(I)$ to ζ. Then $\zeta(t) \in C(t)$ for a.e. $t \in I$.*

Proof. Let $(e_p)_{p\in\mathbb{N}}$ be a dense sequence in H. Take any measurable set $Z \subset I$ and any $p \in \mathbb{N}$, and note that the mapping $\mathbf{1}_Z e_p \in L^\infty_H(I)$. Considering the inequality (due to the inclusion $\zeta_n(t) \in C(t)$)

$$\langle e_p, \zeta_n(t)\rangle \leq \delta^*(e_p, C(t))$$

and integrating on Z ensure that

$$\int_I \langle \mathbf{1}_Z e_p, \zeta_n(t),\rangle dt = \int_Z \langle e_p, \zeta_n(t)\rangle dt \leq \int_Z \delta^*(e_p, C(t))dt.$$

Passing to the limit in the latter inequality assures us that

$$\int_Z \langle \mathbf{1}_Z e_p, \zeta(t)\rangle dt \leq \int_Z \delta^*(e_p, C(t))dt.$$

This being true for any Lebesgue measurable set $Z \subset I$, it follows that for every $p \in \mathbb{N}$

$$\langle e_p, \zeta(t)\rangle \leq \delta^*(e_p, C(t)) \text{ a.e. } t \in I.$$

As H is separable and $C(t)$ is closed convex weakly locally compact and contains no line, by (Castaing-Valadier [24, Proposition III- 35]), we get the desired inclusion $\zeta(t) \in C(t)$ a.e. $t \in I$. □

Proposition 3.4. *Let H be a separable Hilbert space and $I = [0, T]$. Let $C : I \rightrightarrows H$ be a scalarly measurable multimapping with closed convex weakly locally compact values which contain no line and for which there exist $r \in L^\infty_{\mathbb{R}^+}(I)$ such that $C(t) \cap r(t)\mathbb{B}_H \neq \emptyset$ for all $t \in I$. Let $A : H \to H$ be a continuous symmetric linear coercive operator and let $B : H \to H$ be a continuous compact linear operator. Let $(f_n, f)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_H(I)$ with $\|f_n(t)\| \leq \beta, \|f(t)\| \leq \beta$ $(\beta > 0)$ for all $n \in \mathbb{N}$ such that $(f_n(t))_n$ converges to $f(t)$ for each $t \in I$.*
Let $(v_n, v)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_H(I)$ with $\|v_n(t)\| \leq \gamma, \|v(t)\| \leq \gamma$ $(\gamma > 0)$ for all $n \in \mathbb{N}$ such that $(v_n(t))_n$ converges weakly to $v(t)$ for each $t \in I$. Let $(\zeta_n)_{n\in\mathbb{N}}$ be an equi-integrable sequence in $L^1_H(I)$ such that $\zeta_n(t) \in C(t)$ for all $t \in I$ and $n \in \mathbb{N}$ and such that $(\zeta_n)_{n\in\mathbb{N}}$ $\sigma(L^1_H, L^\infty_H)$ converges in $L^1_H(I)$ to ζ. Assume that

$$f_n(t) + Bv_n(t) - A\,\zeta_n(t) \in N_{C(t)}(\zeta_n(t)) \quad \text{for all } n \in \mathbb{N},\ \text{a.e. } t \in I.$$

Then for a.e. $t \in I$ one has

$$\zeta(t) \in C(t) \quad \text{and} \quad f(t) + Bv(t) - A\,\zeta(t) \in N_{C(t)}(\zeta(t)).$$

Proof. First, Lemma 3.2 justifies that $\zeta(t) \in C(t)$ for a.e. $t \in I$.

For each $t \in I$ keeping in mind that $v_n(t) \to v(t)$ weakly in H and B is a continuous compact linear operator, we see that $Bv_n(t) \to Bv(t)$ strongly in H. The Lebesgue dominated convergence theorem says that $\int_0^T \|Bv_n(t)) - Bv(t)\|\, dt \to 0$, i.e., $Bv_n(\cdot) \to Bv(\cdot)$ strongly in $L^1_H(I)$. We can also see that $A\zeta_n \to A\zeta$ weakly in $L^1_H(I)$ since for any $h \in L^\infty_H(I)$

$$\int_0^T \langle h(t), A\zeta_n(t)\rangle\, dt = \int_0^T \langle Ah(t), \zeta_n(t)\rangle\, dt \to \int_0^T \langle Ah(t), \zeta(t)\rangle\, dt = \int_0^T \langle h(t), A\zeta(t)\rangle\, dt.$$

As a main consequence $f_n + Bv_n - A\zeta_n \to f + Bu - A\zeta$ weakly in $L^1_H(I)$.

Given any Lebesgue measurable subset $Z \subset I$ we may apply the lower semicontinuity of convex integral functional in [22, Theorem 8.1.16] to derive that

$$\int_Z \delta^*(f(t)+Bv(t)-A\zeta(t), C(t))dt \leq \liminf_n \int_Z \delta^*(f_n(t)+Bv_n(t)-A\zeta_n(t), C(t))dt. \tag{3.12}$$

This needs a careful look. Choose a measurable selection s of $t \mapsto C(t) \cap r(t)\mathbb{B}_H$, so $s \in L^\infty_H(I)$. We note that $(t, x) \mapsto \delta^*(x, C(t))$ is a normal lower semicontinuous convex integrand defined on $I \times H$ and $\delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t))$ is

measurable and minored by $\rho_n(t) := \langle s(t), f_n(t) + Bv_n(t) - A\zeta_n(t)\rangle$. Further, it is easy to check that $(\rho_n(\cdot))$ is equi-integrable in $L^1_{\mathbb{R}}(I)$. Then by [22, Theorem 8.1.6] we deduce that

$$\begin{aligned}&\liminf_n \int_Z \delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t))dt\\ &\geq \int_Z \delta^*(f(t) + Bv(t) - A\zeta(t), C(t))dt,\end{aligned}$$

which confirms (3.32).

Let us set $\psi_A(x) = \langle Ax, x\rangle$ if $x \in C(t)$ and $\psi_A(x) = +\infty$ if $x \notin C(t)$. Then it is clear that ψ_A is a positive lower semicontinuous convex integrand. Apply again the lower semicontinuity of the convex integral functional ([22], Theorem 8.1.6) associated with the positive normal convex integrand ψ_A we obtain

$$\liminf_n \int_Z \psi_A(\zeta_n(t))\, dt \geq \int_Z \psi_A(\zeta(t))\, dt,$$

that is,

$$\liminf_n \int_Z \langle A\zeta_n(t), \zeta_n(t))dt \geq \int_Z \langle A\zeta(t), \zeta(t)dt. \tag{3.13}$$

Further, we have

$$\lim_n \int_Z \langle Bv_n(t), \zeta_n(t)\rangle\, dt = \int_Z \langle Bv(t), \zeta(t)\rangle\, dt, \tag{3.14}$$

$$\lim_n \int_Z \langle f_n(t), \zeta_n(t)\rangle dt = \int_Z \langle f(t), \zeta(t)\rangle dt. \tag{3.15}$$

The two latter equality features require a careful justification. Indeed, since $(Bv_n)_n$ is uniformly bounded and pointwise strongly converges to Bv and $(f_n)_n$ is uniformly bounded and pointwise strongly converges to f and (ζ_n) is equi-integrable with $\zeta_n \to \zeta$ weakly in $L^1_H(I)$, we see that both sequences $(Bv_n - Bv)_n$ and $(f_n - f)_n$ converge to 0 uniformly on any uniformly integrable subset of $L^1_H([0,T])$, in other terms they converge to 0 with respect to the Mackey topology $\tau(L^\infty_H(I), L^1_H(I))$. Consequently, both (3.34) and (3.35) hold true, as claimed above. [1]

On the other hand, the inclusion $f_n(t) = Bv_n(t) - A\zeta_n(t) \in N_{C(t)}(\zeta_n(t))$ for a.e. $t \in I$ ensures that

$$\delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t)) - \langle f_n(t) + Bv_n(t) - A\zeta_n(t), \zeta_n(t)\rangle \leq 0.$$

[1] If $H = \mathbb{R}^e$, here one may invoke a classical fact that on bounded subsets of L^∞_H the topology of convergence in measure coincides with the topology of uniform convergence on uniformly integrable sets, i.e. on relatively weakly compact subsets, alias the Mackey topology. This is a lemma due to Grothendieck [26] [Ch.5 §4 no 1 Prop. 1 and exercice].

Integrating this inequality on I gives

$$\int_0^T \delta^*(f_n(t)+Bv_n(t)-A\zeta_n(t),C(t))dt+\int_0^T\langle A\zeta_n(t),\zeta_n(t)\rangle dt \\ +\int_0^T\langle -f_n(t)-Bv_n(t),\zeta_n(t)\rangle dt\le 0.$$

Passing to the limit inferior as $n\to\infty$ and using (3.32) we obtain

$$\int_0^T\big[\delta^*(f(t)+Bv(t)-A\zeta(t),C(t))+\langle A\zeta(t)-Bv(t)-f(t),\zeta(t)\rangle\big]dt\le 0.$$

This and the inclusion $\zeta(t)\in C(t)$ a.e. $t\in I$ allow us to conclude that

$$f(t)+Bv(t)-A\zeta(t))\in N_{C(t)}(\zeta(t)) \text{ a.e. } t\in I$$

according to the description (2.1) of the normal cone.

□

Similarly to Proposition 3.4, when $C(\theta_n(t)=C(t)$, the next result relaxes the condition $C(t)\subset r\mathbb{B}_H$ in Proposition 3.10.

Proposition 3.5. *Let H be a separable Hilbert space and $I=[0,T]$. Let $C: I\rightrightarrows H$ be a scalarly measurable multimapping with closed convex weakly locally compact values containing no line, for which there exist $r\in L^\infty_{\mathbb{R}^+}(I)$ such that $C(t)\cap r(t)\mathbb{B}_H\neq\emptyset$ for all $t\in I$. Let $A:H\to H$ be a continuous symmetric linear coerciveoperator and let $g:H\to\mathbb{R}$ be a convex function Lipschitz on bounded sets and continuously differentiable on H.*

Let $(f_n,f)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_H(I)$ with $\|f_n(t)\|\le\beta,\|f(t)\|\le\beta$ $(\beta>0)$ for all $n\in\mathbb{N}$ such that $(f_n(t))_n$ converges to $f(t)$ for each $t\in I$.
Let $(u_n,u)_{n\in\mathbb{N}}$ be a sequence of absolutely continuous mappings

$$u_n(t)=u_0+\int_0^t\dot u_n(s)ds,\ \ \dot u_n(t)\in C(t)$$

$$u(t)=u_0+\int_0^t\dot u(s)ds,\ \ \dot u(t)\in C(t)$$

such that $(u_n)_n$ converges to u uniformly on I and $(\dot u_n)_n$ converges to $\dot u$ weakly in $L^1_H(I)$. Assume that for every $n\in\mathbb{N}$

$$f_n(t)+\nabla g(u_n(t))-A\dot u_n(t)\in N_{C(t)}(\dot u_n(t)) \text{ a.e. } t\in I.$$

Then for a.e. $t\in I$ one has

$$\dot u(t)\in C(t)\quad\text{and}\quad f(t)+\nabla g(u(t))-A\,\dot u(t)\in N_{C(t)}(\dot u(t)).$$

Proof. The inclusion $\dot{u}(t) \in C(t)$ for a.e. $t \in I$ is justified by Lemma 3.2.

Take a real $r' > r$ and let $\gamma > 0$ be a Lipschitz constant of g on $(\|u_0\| + r'T)\mathbb{B}_H$. Take any Lebesgue measurable set $Z \subset I$. Let us consider the term $\langle \nabla g(u_n(t)), \dot{u}_n(t)\rangle$. The Lipschitz property of g on $(\|u_0\|+r'T)\mathbb{B}_H$ assures us that $g \circ u_n$ and $g \circ u$ are absolutely continuous on I and at any $t \in I$ where u and all u_n are derivable

$$\langle \nabla g(u(t)), \dot{u}(t)\rangle = \frac{d}{dt} g(u(t)) \quad \text{and} \quad \langle \nabla g(u_n(t)), \dot{u}_n(t)\rangle = \frac{d}{dt} g(u_n(t)).$$

From this we deduce that

$$\begin{aligned}
&\int_0^T \langle \nabla g(u(t)), \dot{u}(t)\rangle dt \\
&= \int_0^T \frac{d}{dt} g(u(t)) dt = g(u(T)) - g(u(0)) = \lim_n \big(g(u_n(T)) - g(u_n(0))\big) \\
&= \lim_n \int_0^T \frac{d}{dt} g(u_n(t)) dt = \lim_n \int_0^T \langle \nabla g(u_n(t)), \dot{u}_n(t)\rangle dt. \qquad (3.16)
\end{aligned}$$

Since $(f_n)_n$ is uniformly bounded and pointwise converges to f and $(\dot{u}_n)_n$ weakly converges in $L^1_H([0,T])$ to $\dot{u}$, we have

$$\lim_n \int_Z \langle f_n(t), \dot{u}_n(t)\rangle dt = \int_Z \langle f(t), \dot{u}(t)\rangle dt. \qquad (3.17)$$

This fact is mainly justified in the proof of Proposition 3.4 and is not a consequence of Lebesgue theorem. Also the assumption of normal cone inclusion can be reformulated for a.e. $t \in I$ as

$$\delta^*\big(f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t), C(t)\big)\big) + \langle A\dot{u}_n(t) - \nabla g(u_n(t)) - f_n(t), \dot{u}_n(t)\rangle \leq 0$$

with $q_n(t) := f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t)$, which gives

$$\int_Z \delta^*(q_n(t), C(t)) dt + \int_Z \langle A\dot{u}_n(t) - f_n(t), \dot{u}_n(t)\rangle dt - \int_Z \langle \nabla g(u_n(t)), \dot{u}_n(t)\rangle dt \leq 0, \qquad (3.18)$$

(noting that the above functions are measurable). We claim that $A\dot{u}_n(\cdot) \to A\dot{u}(\cdot)$ weakly in $L^1_H(I)$ and $\nabla g(u_n(\cdot)) \to \nabla g(u(\cdot))$ weakly in $L^1_H(I)$, so

$$q_n(\cdot) = f_n(\cdot) + \nabla g(u_n(\cdot)) - A\dot{u}_n(\cdot) \to q(\cdot) := f(\cdot) + \nabla g(u(\cdot)) - A\dot{u}(\cdot)$$

weakly in $L^1_H(I)$. Indeed, for any $h \in L^\infty_H(I)$ the weak convergence in $L^1_H(I)$ of $(\dot{u}_n)_n$ to $\dot{u}$ says that $\int_0^T \langle Ah(t), \dot{u}_n(t)\rangle dt \to \int_0^T \langle Ah(t), \dot{u}(t)\rangle dt$, which means

$$\int_0^T \langle h(t), A\dot{u}_n(t)\rangle dt \to \int_0^T \langle h(t), A\dot{u}(t)\rangle dt.$$

This property for every $h \in L^\infty_H(I)$ translates the weak convergence in $L^1_H(I)$ of $(A\dot{u}_n)_n$ to $A\dot{u}$. Concerning $(\nabla g(u_n(\cdot)))_n$ it converges strongly in $L^1_H(I)$ to $\nabla g(u(\cdot))$ (by Lebesgue convergence theorem) since $\nabla g(u_n(t)) \to \nabla g(u(t))$ for each $t \in I$ and $\|\nabla g(u_n(t))\| \leq \gamma$ (by the γ-Lipschitz property of g on $(\|u_0\| + r'T)\mathbb{B}_H$). Altogether, the weak convergence in $L^1_H(I)$ of $(q_n)_n$ to q is justified.

Regarding $\int_0^T \langle A\dot{u}_n(t), \dot{u}_n(t)\rangle dt$ let us set $\psi_A(t,x) = \langle Ax, x\rangle$ for all $x \in C(t)$ and $\psi_A(t,x) = +\infty$ for $x \notin C(t)$. It is clear that ψ_A is a positive lower semicontinuous convex normal integrand. By the lower semicontinuity of convex integral functional (see [22, Theorem 8.1.6]) we obtain

$$\liminf_n \int_0^T \psi_A(t, \dot{u}_n(t))dt \geq \int_0^T \psi_A(t, \dot{u}(t))dt,$$

that is,

$$\liminf_n \int_0^T \langle A\dot{u}_n(t), \dot{u}_n(t)\rangle dt \geq \int_0^T \langle A\dot{u}(t), \dot{u}(t)\rangle dt. \tag{3.19}$$

The last step deals with $\int_0^T \delta^*(q_n(t), C(t))dt$. Fix a measurable selection $s(\cdot)$ of $t \mapsto C(t) \cap r(t)\mathbb{B}_H$, set $\rho_n(t) := \langle s(t), q_n(t)\rangle$, and observe that $\delta^*(q_n(t), C(t)) \geq \rho_n(t)$. It is not difficult to check that $(\rho_n)_n$ is equi-integrable in $L^1_{\mathbb{R}}(I)$. From this and the lower semicontinuous convex normal integrand $(t,x) \mapsto \delta^*(x, C(t))$ it follows by [22, Theorem 8.1.6] again that

$$\liminf_n \int_0^T \delta^*(q_n(t), C(t))dt \geq \int_0^T \delta^*(q(t), C(t))dt.$$

The latter inequality and (3.16)—(3.19) yield

$$\int_0^T \delta^*(f(t)+\nabla g(u(t))-A\dot{u}(t), C(t))dt + \int_0^T \langle -f(t)+\nabla g(u(t))+A\dot{u}(t), \dot{u}(t)\rangle dt \leq 0. \tag{3.20}$$

On the other hand, the inclusion $\dot{u}(t) \in C(t)$ entails that for a.e. $t \in I$

$$\delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) \geq \langle f(t) + \nabla g(u(t)) - A\dot{u}(t), \dot{u}(t)\rangle.$$

Taking this into account in (3.40) we deduce for a.e. $t \in I$

$$\delta^*\big(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)\big) - \langle f(t + \nabla g(u(t)) - A\dot{u}(t), \dot{u}(t) \leq 0.$$

The latter inequality and the inclusion $\dot{u}(t) \in C(t)$ guarantees that for a.e. $t \in I$

$$f(t) + \nabla g(u(t)) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$$

according to the description (2.1) of the normal cone. □

We pass now to the situation of a subdifferential $\partial\varphi(t,\cdot)$ in place of the normal cone $N_{C(t)}(\cdot)$.

Proposition 3.6. *Let H be a separable Hilbert space and $I = [0,T]$ Let $\varphi : [0,T] \times H \to]-\infty, +\infty]$ be a normal lower semicontinuous convex integrand for which there exists a convex weakly compact set Γ such that:*
(i) for all $t \in I$, $dom\varphi_t := \Gamma$;
(ii $\{\varphi(.,u(.)), u \in S^1_\Gamma\}$ is uniformly integrable;
(iii) $\varphi(t,x) \leq \varphi(\tau,x) + |v(t) - v(\tau)|$ for all $t, \tau \in [0,T], x \in \Gamma$ where $v : [0,T] \to \mathbb{R}^+$ is a positive nondecreasing continuous function with $v(0) = 0$.
Let $A : H \to H$ be a continuous symmetric linear coerciveoperator and let $g : H \to \mathbb{R}$ be a convex function Lipschitz on bounded sets and continuously differentiable on H. Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions from I into I such that for each $t \in I$ one has $\theta_n(t) \to t$.
Let $(f_n, f)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_H(I)$ with $\|f_n(t)\| \leq \beta, \|f(t)\| \leq \beta$ $(\beta > 0)$ for all $n \in \mathbb{N}$ such that $(f_n(t))_n$ converges to $f(t)$ for each $t \in I$.
Let $(v_n, v)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_H(I)$ with $\|v_n(t)\| \leq \gamma, \|v(t)\| \leq \gamma$ $(\gamma > 0)$ for all $n \in \mathbb{N}$ such that $(v_n(t))_n$ converges strongly to $v(t)$ for each $t \in I$.
Let $(u_n, u)_{n\in\mathbb{N}}$ be a sequence of absolutely continuous mappings

$$u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds, \;\; \dot{u}_n(t) \in \Gamma$$

$$u(t) = u_0 + \int_0^t \dot{u}(s)ds,$$

such that $(u_n)_n$ converges to u uniformly on I and $(\dot{u}_n)_n$ converges to $\dot{u}$ weakly in $L^1_H(I)$. Assume that for every $n \in \mathbb{N}$

$$f_n(t) + \nabla g(v_n(t)) - A\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t)) \text{ a.e. } t \in I.$$

Then for a.e. $t \in I$ one has

$$\dot{u}(t) \in \Gamma \quad \text{and} \quad f(t) + \nabla g(v(t)) - A\,\dot{u}(t) \in \partial\varphi(t, \dot{u}(t)).$$

Proof. First, Lemma 3.2 tells us that $\dot{u}(t) \in \Gamma$ for a.e. $t \in I$.

Let $\kappa > 0$ be a Lipschitz constant of g on $\gamma\mathbb{B}_H$, so that ∇g is bounded on $\gamma\mathbb{B}_H$ and $\nabla g(v_n) \to \nabla g(v)$ strongly . Take any Lebesgue measurable set $Z \subset I$. Since $(f_n)_n$ is uniformly bounded and pointwise converges to f and $(\dot{u}_n)_n$ weakly converges in $L^1_H([0,T])$ to $\dot{u}$, we have

$$\lim_n \int_Z \langle f_n(t), \dot{u}_n(t)\rangle\, dt = \int_Z \langle f(t), \dot{u}(t)\rangle\, dt. \tag{3.21}$$

This fact has been already justified in the proof of Proposition 3.4. As φ is normal lower semicontinuous convex integrand, the conjugate function $\varphi^* : I \times H \to]-\infty, +\infty]$

$$\varphi^*(t,y) = \sup_{x\in H}[\langle x, y\rangle - \varphi(t,x)] \tag{3.22}$$

is normal, see e.g Castaing-Valadier [24], and satisfies

$$\varphi^*(t,y) \leq \varphi^*(\tau,y) + |v(t) - v(\tau)|$$

for all $t, \tau \in I$, $y \in H$ using assumption (ii) ([36], Proposition 27). By using the normality of φ, the functions $t \mapsto \varphi(\theta_n(t), \dot{u}_n(t))$ and $t \mapsto \varphi(t, \dot{u}_n(t))$ are measurable and integrable.By assumption we have

$$q_n(t) := f_n(t) + \nabla g(v_n(t)) - A\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t))$$

so that by the normality of φ^*, the function $t \mapsto \varphi^*(\theta_n(t), q_n(t))$ is measurable and integrable, the measurability and integrability of $\nabla g(v_n(t))$ is already ensured as above. We also note that $\langle q_n, \dot{u}_n \rangle$ is measurable and integrable and the sequence $(\langle q_n, \dot{u}_n \rangle)$ is uniformly integrable.

Further, by (3.42) and condition (ii) we have

$$-\varphi(t, \dot{u}_n(t)) + \langle \dot{u}_n(t), q_n(t) \rangle \leq \varphi^*(t, q_n(t)) \leq \varphi^*(\theta_n(t), q_n(t)) + |v(t) - v(\theta_n(t))|, \tag{3.23}$$

so that $t \mapsto -\varphi(t, \dot{u}_n(t)) + \langle \dot{u}_n(t), q_n(t) \rangle$ is uniformly integrable thanks to (ii). We note that, for $h_n(t) := f_n(t) + \nabla g(v_n(t))$, $(h_n)_n$ is uniformly bounded and pointwise converges to h given by $h(t) = f(t) + \nabla g(v(t)$ in H. Hence $h_n \to h$ strongly in $L^1_H(I)$. As consequence, for every measurable set Z in I, we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_Z \langle h_n(t), \dot{u}_n(t) \rangle \, dt \\
&= \lim_{n\to\infty} \int_Z \langle h_n(t) - h(t), \dot{u}_n(t) \rangle \, dt + \lim_{n\to\infty} \int_Z \langle h(t), \dot{u}_n(t) \rangle \, dt \\
&= \lim_{n\to\infty} \int_Z \langle h(t), \dot{u}_n(t) \rangle dt = \int_Z \langle h(t), \dot{u}(t) \rangle \, dt.
\end{aligned} \tag{3.24}$$

This fact has been justified in the proof of Proposition 3.4. As A is symmetric, we also show that $A\dot{u}_n(\cdot) \to A\dot{u}(\cdot)$ weakly in $L^1_H(I)$. As consequence $q_n = f_n + \nabla g(v_n(.)) - A\dot{u}_n(.) \to q := f + \nabla g(v(.)) - A\dot{u}$ weakly in $L^1_H(I)$. Further, let us set $\psi_A(x) = \langle Ax, x \rangle$ if $x \in \Gamma$ and $\psi(x) = +\infty$ if $x \notin \Gamma$. Then it is clear ψ is a positive lower semicontinuous convex integrand. Apply again the lower semicontinuity of the integral convex functional ([22], Theorem 8.1.6) associated with the positive normal lower semi continuous convex integrand ψ_A we obtain

$$\liminf_n \int_Z \psi_A(u_n(t)) dt \geq \int_Z \psi_A(u(t)) \, d, t$$

that is,

$$\liminf_n \int_Z \langle A\dot{u}_n(t), \dot{u}_n(t)) \, dt \geq \int_Z \langle A\dot{u}(t), \dot{u}(t) \, dt. \tag{3.25}$$

Now, we deduce using (3.43) and the lower semicontinuity of integral convex functional (see [22, Theorem 8.1.6]) applied to φ^*,

$$\int_Z \varphi^*(t,q(t))\,dt \le \liminf_n \int_Z \varphi^*(t,q_n(t))\,dt \le \liminf_n \int_Z \varphi^*(\theta_n(t),q_n(t))dt \quad (3.26)$$

This fact is justified because $\varphi^*(t,q_n(t)) \ge -\varphi(t,\dot{u}_n(t)) + \langle \dot{u}_n(t), q_n(t)\rangle$, and the sequence $(-\varphi(t,\dot{u}_n(t)) + \langle \dot{u}_n(t), q_n(t)\rangle)$ is uniformly integrable. By

$$\varphi(t,\dot{u}_n(t)) \le \varphi(\theta_n(t),\dot{u}_n(t)) + |v(t) - v(\theta_n(t))|$$

we also have that

$$\liminf_n \int_Z \varphi(t,\dot{u}_n(t))\,dt \le \liminf_n \int_Z \varphi(\theta_n(t),\dot{u}_n(t))\,dt.$$

As $(\dot{u}_n)_n$ weakly converges to $\dot{u} \in L^1_H(I)$, by the lower semi continuity theorem ([22], Theorem 8.1.6) applied to the lower semicontinuous convex integral functional associated with φ, we derive that

$$\int_Z \varphi(t,\dot{u}(t))\,dt \le \liminf_n \int_Z \varphi(\theta_n(t),\dot{u}_n(t))\,dt \quad (3.27)$$

with $\dot{u}(t) \in \Gamma$ a.e. and $t \mapsto \varphi(t,\dot{u}(t))$ is integrable. Now integrating on the Lebesgue measurable subset Z of I the equality

$$\varphi(\theta_n(t),\dot{u}_n(t)) + \varphi^*(\theta_n(t),q_n(t)) = \langle \dot{u}_n(t), q_n(t)\rangle$$

gives

$$\int_Z \varphi(\theta_n(t),\dot{u}_n(t))dt + \int_Z \varphi^*(\theta_n(t),q_n(t))dt + \int_Z \langle A\dot{u}_n(t),\dot{u}_n(t)dt. = \int_Z \langle \dot{u}_n(t), h_n(t)\rangle dt.$$

Passing to the limit as $n \to \infty$ in this equality and using (3.44)–(3.47) give

$$\int_Z \varphi(t,\dot{u}(t))\,dt + \int_Z \varphi^*(t,q(t))\,dt \le \int_Z \langle \dot{u}(t), q(t)\rangle\,dt.$$

By the measurability of the non negative function $t \mapsto \varphi(t,\dot{u}(t)) + \varphi^*(t,q(t)) - \langle \dot{u}(t), q(t)\rangle$, we deduce that for almost every $t \in I$

$$\varphi(t,\dot{u}(t)) + \varphi^*(t,q(t)) - \langle \dot{u}(t), q(t)\rangle \le 0$$

along with $\dot{u}(t) \in \Gamma$. So, it follows for almost every $t \in I$ that $\varphi(t,\dot{u}(t)) + \varphi^*(t,q(t)) = \langle \dot{u}(t), q(t)\rangle$, or equivalently

$$q(t) = f(t) + \nabla g(u(t)) - A\dot{u}(t) \in \partial\varphi(t,\dot{u}(t)).$$

The proof is finished. □

Remark Proposition 3.6 hold true if we replace the gradient ∇g by a linear continuous operator $B : H \to H$.

We finish this section with some new variants which have also some importance in further developments.

Proposition 3.7. *Let H be a separable Hilbert space and $I = [0,T]$. Let $C : I \rightrightarrows H$ be a convex weakly compact valued multimapping for which there is some nondecreasing continuous function $r : I \to \mathbb{R}^+$ such that $haus(C(t), C(\tau)) \leq |r(t) - r(\tau)|$ for all $\tau, t \in I$ Let A be a linear continuous coercive symmetric operator on H and let $B = \nabla g$ where ∇g is the gradient of a convex continuous Gateaux differentiable function $g : H \to \mathbb{R}^+$ such that $g(v(t))$ is absolutely continuous for $v : [0,T] \to H$ absolutely continuous. Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions from I into I such that for each $t \in I$ one has $\theta_n(t) \to t$.*
Let $(u_n, u)_{n\in\mathbb{N}}$ be a bounded sequence of absolutely continuous mapping on H such that $(u_n(t))_n$ converges pointwise strongly to u and such that $(\dot{u}_n)_{n\in\mathbb{N}}$ $\sigma(L^1_H, L^\infty_H)$ converges in $L^1_H(I)$ to u. Assume that

$$-\dot{u}_n(t) \in N_{C(\theta_n(t))}(A\dot{u}_n(t) + \nabla g(u_n(t))) \text{for all } n \in \mathbb{N},\ \text{a.e. } t \in I.$$

Then for a.e. $t \in I$ one has $-\dot{u}(t) \in N_{C(t)}(A\dot{u}(t) + \nabla g(u(t)))$.

Proof. As $-\dot{u}_n(t) \in N_{C(\theta_n(t))}(A\dot{u}_n(t) + \nabla g(u_n(t)))$ then

$$\delta^*(-\dot{u}_n(t), C(\theta_n(t))) + \langle \dot{u}_n(t), A\dot{u}_n(t) + \nabla g(u_n(t))\rangle \leq 0$$

By integrating on $Z \in \mathcal{L}(I)$ (we are ensured that the functions given are measurable) this inequality we get

$$\int_Z \delta^*(-\dot{u}_n(t), C(\theta_n(t)))dt + \int_Z \langle \dot{u}_n(t), A\dot{u}_n(t) + \nabla g(u_n(t))\rangle dt \leq 0. \tag{3.7.1}$$

Now we have to consider first the term $\langle \nabla g(u_n(t), \dot{u}_n(t)\rangle$ by using the special property of ∇g. In fact u_n is absolutely continuous with derivative $\dot{u}_n$ and $g(u_n)$ is absolutely continuous , so that by Moreau-Valadier [35] ,

$$\langle \dot{u}_n(t), \nabla g(u_n(t))\rangle = \frac{d}{dt} g(u_n(t))$$

From this fact, it is easy to deduce that

$$\liminf_n \int_0^T \langle \nabla g(u_n(t)), \dot{u}_n(t)\rangle dt = \liminf_n \int_0^T \frac{d}{dt} g(u_n(t))\rangle dt$$

$$= \liminf_n (g(u_n(T) - u_n(0)) \geq g(u(T) - u(0)) = \int_0^T \frac{d}{dt} g(u(t))\rangle dt \tag{3.7.2}$$

$$= \int_0^T \langle \nabla g(u(t)), \dot{u}(t) \rangle dt$$

Further let $\Psi(x) = \langle x, Ax \rangle$ for all $x \in H$. Then Ψ is positive lower semicontinuous convex on H. As $\dot{u}_n$ weakly converges converges in $L^1_H(I)$ to $\dot{u}$, again by ([22], Theorem 8.1.6) we get we have

$$\liminf_n \int_Z \Psi(\dot{u}_n(t))dt \geq \int_Z \Psi(\dot{u}(t))dt \tag{3.7.3}$$

As it is readly seen that $A\dot{u}_n$ weakly converge to $A\dot{u}$ and $\nabla g(u_n(t))) \to \nabla g(u(t)))$, then let $e \in H$, we have

$$\langle e, A\dot{u}_n(t) + \nabla g(u_n(t)))\rangle \leq \delta^*(e, C(\theta_n(t)))$$

Hence for $Z \in \mathcal{L}(I)$

$$\int_Z \langle e, A\dot{u}_n(t) + \nabla g(u_n(t)))\rangle dt \leq \int_Z \delta^*(e, C(\theta_n(t)))dt$$

Passing to the limit in this inequality we get

$$\int_Z \langle e, A\dot{u}(t) + \nabla g(u(t)))\rangle dt \leq \int_Z \delta^*(e, C(t))dt$$

By [24] we conclude that $A\dot{u}(t) + \nabla g(ut))) \in C(t)$ By (ii) using the Hormander formula we have the estimation

$$\int_Z \delta^*(-\dot{u}_n(t), C(t))dt \leq \int_Z \delta^*(-\dot{u}_n(t), C(\theta_n(t)dt + \int_Z ||\dot{u}_n(t)|||r(\theta_n(t) - r(t)|dt$$

with $\int_Z ||\dot{u}_n(t)|||r(\theta_n(t) - r(t)|dt \to 0$ using the fact that $\rho_n(t) := |r(\theta_n(t) - r(t)| \to 0$ and $||\dot{u}_n(t)||$ is uniformly integrable. Here the measurability of the functions given these integrals is granted; in fact it is easily seen that C is scalarly continuous. So that

$$\liminf_n \int_Z \delta^*(-\dot{u}_n(t), C(\theta_n(t))dt \geq \liminf_n \int_Z \delta^*(-\dot{u}_n(t), C(t))dt$$

To finish the proof we apply the lower semicontinuity of the integral convex functional ([22], Theorem 8.1.6) associated with the normal convex integrand $(t, x) \mapsto \delta^*(x, C(t))$ by noting that
(j) $\delta^*(-\dot{u}_n(t), C(t))$ is minored by $\langle -\dot{u}_n(t), h(t)\rangle$ with $h(t) = A\dot{u}(t) + \nabla g(u(t))$,
(jj) the minored sequence $(\langle -\dot{u}_n(t), h(t)\rangle)$ is uniformly integrable.
Then we are ensured by
(3.7.4)

$$\liminf_n \int_Z \delta^*(-\dot{u}_n(t), C(\theta_n(t))dt \geq \liminf_n \int_Z \delta^*(-\dot{u}_n(t), C(t))dt \geq \int_Z \delta^*(-\dot{u}(t), C(t))dt$$

By combining $(3.7.1) - -(3.7.4)$ we get

$$\int_0^T \delta^*(-\dot{u}(t), C(t))dt + \langle \dot{u}(t), A\dot{u}(t) + \nabla g(u(t))\rangle dt \leq 0. \tag{3.7.5}$$

as

$$\delta^*(-\dot{u}(t), C(t)) \geq \langle -\dot{u}(t), Au(t) + \nabla g(u(t))\rangle$$

a.e. that implies

$$\int_0^T [\delta^*(-\dot{u}(t), C(t)) + \langle \dot{u}(t), Au(t) + \nabla g(u(t))\rangle]dt = 0$$

so we conclude that

$$\delta^*(-\dot{u}(t), C(t)) + \langle \dot{u}(t), Au(t) + \nabla g(u(t))\rangle = 0$$

a.e. with $A\dot{u}(t) + \nabla g(u(t)) \in C(t)$, just proving that $-\dot{u}(t) \in N_{C(t)}(A\dot{u}(t) + \nabla g(u(t)))$ a.e. □

Application We give an example of the existence of absolutelty continuous solution to the evolution inclusion of the form $-\dot{u}(t) \in N_{C(t)}(A\dot{u}(t) + \nabla g(u(t)))$ where C is a convex compact valued continuous mapping and A is a linear continuous coercive: $\langle Ax, x\rangle \geq \gamma||x||^2$ symmetric operator in $H = \mathbb{R}^e$.

Further variational limit involving time dependent maximal monotone operator

We introduce in the following the definition and some properties of maximal monotone operators needed in the proofs of our results, Let $A : D(A) \subset H \rightrightarrows H$ be a set-valued operator. The domain, the range and the graph of A are the following sets

$$D(A) = \{x \in H : \ Ax \neq \emptyset\},$$

$$R(A) = \{y \in H : \ \exists\, x \in D(A), \ y \in Ax\} = \cup\{Ax : \ x \in D(A)\},$$

$$gph(A) = \{(x, y) \in H \times H : \ x \in D(A), \ y \in Ax\}.$$

We say that $A : D(A) \subset H \rightrightarrows H$ is monotone, if $\langle y_1 - y_2, x_1 - x_2\rangle \geq 0$ whenever $(x_i, y_i) \in gph(A)$, $i = 1, 2$. It is maximal monotone, if its graph could not be contained strictly in the graph of any other monotone operator, in this case, for all $\lambda > 0$, $R(I_H + \lambda A) = H$, where I_H stands for the identity mapping of H.
If A is a maximal monotone operator then, for every $x \in D(A)$, Ax is non-empty, closed and convex. So that, the projection of the origin into Ax, $A^0(x)$, exists and is unique.
For $\lambda > 0$, we define the resolvent and the Yosida approximation of A respectively by, $J_\lambda^A = (I_H + \lambda A)^{-1}$ and $A_\lambda = \frac{1}{\lambda}\left(I_H - J_\lambda^A\right)$. These operators are both single-valued and defined on the whole space H, and we have

$$J_\lambda^A x \in D(A) \ \text{ and } \ A_\lambda(x) \in A(J_\lambda^A x), \text{ for every } \ x \in H, \tag{3.28}$$

$$||A_\lambda(x)|| \leq ||A^0 x|| \ \ \forall x \in D(A). \tag{3.29}$$

Let $A : D(A) \subset H \rightrightarrows H$ and $B : D(B) \subset H \rightrightarrows H$ be two maximal monotone operators, then we denote by $dis(A, B)$ the pseudo-distance between A and B defined by

$$dis(A, B) = \sup \left\{ \frac{\langle y - y', x' - x \rangle}{1 + ||y|| + ||y'||} : \ (x, y) \in gph(A), \ (x', y') \in gph(B) \right\}. \tag{3.30}$$

Clearly, $dis(A, B) \in [0, +\infty], dis(A, B) = dis(B, A)$ and $dis(A, B) = 0$ iff $A = B$.

Lemma 3.3. *Let A_n $(n \in \mathbb{N})$, A be maximal monotone operators of H such that $dis(A_n, A) \to 0$. Suppose also that $x_n \in D(A_n)$ with $x_n \to x$ and $y_n \in A_n(x_n)$ with $y_n \to y$ weakly for some $x, y \in H$. Then $x \in D(A)$ and $y \in Ax$.*

Let $I := [0, T.]$ Let $t \to A_t : D(\mathcal{A}_t) \to 2^H$ be a time dependent maximal monotone operator. We say that $A_t : D(A_t) \to 2^H$ is *continuous in variation* if there exist nondecreasing continuous function $\rho : I \to \mathbb{R}^+$ with $\rho(0) = 0$ such that $dist(A_t, A_\tau) \leq |\rho(t) - \rho(\tau)|$ for all $t, \tau \in I$.

Proposition 3.8. *Let H be a separable Hilbert space and $I = [0, T]$. Let $C : I \rightrightarrows H$ be a convex weakly compact valued multimapping for which there is some nondecreasing continuous function $r : I \to \mathbb{R}^+$ such that $haus(C(t), C(\tau)) \leq |r(t) - r(\tau)|$ for all $\tau, t \in I$. Let $t \to \mathcal{A}_t : D(\mathcal{A}_t) \to 2^H$ be a continuous in variation time dependent maximal monotone operator: $dist(\mathcal{A}_t, \mathcal{A}_\tau) \leq |\rho(t) - \rho(\tau)|$ for all $t, \tau \in I$. Let $B = \nabla g$ where ∇g is the gradient of a convex continuous Gateaux differentiable function $g : H \to \mathbb{R}^+$ such that $g(v(t))$ is absolutely continuous for $v : [0, T] \to H$ absolutely continuous. Let $(\theta_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions from I into I such that for each $t \in I$ one has $\theta_n(t) \to t$*
Let $(u_n, u)_{n \in \mathbb{N}}$ be a bounded sequence of absolutely continuous mapping on H such that $(u_n(t))_n$ converges pointwise strongly to u and such that $(\dot{u}_n)_{n \in \mathbb{N}}$ $\sigma(L^1_H, L^\infty_H)$ converges in $L^1_H(I)$ to $\dot{u}$. Let (v_n) be a sequence of bounded measurable mapping pointwise converging to a bounded measurable mapping v. Assume that

$$-\dot{u}_n(t) \in N_{C(\theta_n(t))}(v_n(t) + \nabla g(u_n(t))) \text{ for all } n \in \mathbb{N}, \text{ a.e. } t \in I.$$

$$v_n(t) \in \mathcal{A}_{\theta_n(t)} u_n(t)$$

Then for a.e. $t \in I$ one has a.e. $t \in [0, T]$

$$-\dot{u}(t) \in N_{C(t)}(v(t) + \nabla g(u(t)))$$

$$v(t) \in \mathcal{A}_t u(t)$$

Proof. As $-\dot{u}_n(t) \in N_{C(\theta_n(t))}(v_n(t) + \nabla g(u_n(t)))$ then

$$\delta^*(-\dot{u}_n(t), C(\theta_n(t))) + \langle \dot{u}_n(t), v_n(t) + \nabla g(u_n(t)) \rangle \leq 0$$

By integrating on $Z \in \mathcal{L}(I)$ (we are ensured that the functions given are measurable) this inequality we get

$$\int_Z \delta^*(-\dot{u}_n(t), C(\theta_n(t)))dt + \int_Z \langle \dot{u}_n(t), v_n(t) + \nabla g(u_n(t))\rangle dt \leq 0. \tag{3.7.1}$$

Now we have to consider first the term $\langle \nabla g(u_n(t), \dot{u}_n(t)\rangle$ by using the special property of ∇g. In fact u_n is absolutely continuous with derivative $\dot{u}_n$ and $g(u_n)$ is absolutely continuous , so that by Moreau-Valadier [35] ,

$$\langle \dot{u}_n(t), \nabla g(u_n(t))\rangle = \frac{d}{dt} g(u_n(t))$$

From this fact, it is easy to deduce that

$$\liminf_n \int_0^T \langle \nabla g(u_n(t)), \dot{u}_n(t)\rangle dt = \liminf_n \int_0^T \frac{d}{dt} g(u_n(t))\rangle dt$$

$$= \liminf_n (g(u_n(T) - u_n(0)) \geq g(u(T) - u(0)) = \int_0^T \frac{d}{dt} g(u(t))\rangle dt \tag{3.7.2}$$

$$= \int_0^T \langle \nabla g(u(t)), \dot{u}(t)\rangle dt$$

Further we have

$$\lim_n \int_Z \langle \dot{u}_n(t), v_n(t)\rangle dt = \int_Z \langle \dot{u}(t), v(t)\rangle dt. \tag{3.7.3}$$

This need a careful look. Indeed, since v_n is uniformly bounded and pointwise strongly converge to v and $\dot{u}_n$ is uniformly integrable and $\dot{u}_n \to \dot{u}$ weakly in $L^1_H(I)$ we ensure that $v_n - v \to 0$ uniformly on any uniformly integrable subset of $L^1_H([0,T])$, in other terms it converges to 0 with respect to the Mackey topology $\tau(L^\infty_H(I), L^1_H(I))$, as consequence (3.7.3) hold. If $H = \mathbb{R}^e$, here one may invoke a classical fact that on bounded subsets of L^∞_H the topology of convergence in measure coincides with the topology of uniform convergence on uniformly integrable sets, i.e. on relatively weakly compact subsets, alias the Mackey topology. This is a lemma due to Grothendieck [26] [Ch.5 §4 no 1 Prop. 1 and exercice]. By (ii) using the Hormander formula we have the estimation

$$\int_Z \delta^*(-\dot{u}_n(t), C(t))dt \leq \int_Z \delta^*(-\dot{u}_n(t), C(\theta_n(t)dt + \int_Z ||\dot{u}_n(t)|||r(\theta_n(t) - r(t)|dt$$

with $\int_Z ||\dot{u}_n(t)|||r(\theta_n(t) - r(t)|dt \to 0$ using the fact that $\rho_n(t) := |r(\theta_n(t) - r(t)| \to 0$ and $||\dot{u}_n(t)||$ is uniformly integrable. Here the measurability of the

functions given these integrals is granted; in fact it is easily seen that C is scalarly continuous. So that

$$\liminf_n \int_Z \delta^*(-\dot{u}_n(t), C(\theta_n(t))dt \geq \liminf_n \int_Z \delta^*(-\dot{u}_n(t), C(t))dt$$

To finish the proof we apply the lower semicontinuity of the integral convex functional ([22], Theorem 8.1.6) associated with the normal convex integrand $(t,x) \mapsto \delta^*(x, C(t))$ by noting that
(j) $\delta^*(-\dot{u}_n(t), C(t))$ is minored by $\langle h(t), -\dot{u}_n(t)\rangle$; with $h(t) = v(t) + \nabla g(u(t)) \in C(t)$ for all $t \in I$.
(jj) the minored sequence $(\langle h(t), -\dot{u}_n(t)\rangle)$ is uniformly integrable with $\langle h(t), -\dot{u}_n(t)\rangle \leq \delta^*(-\dot{u}_n(t), C(t))$.
In fact the last inclusion is easily ensured by noting that $v_n(t) + \nabla g(u_n(t)) \in C(\theta_n(t))$ and $v_n(t) + \nabla g(u_n(t)) \to v(t) + \nabla g(u(t))$ and the multimapping C is scalarly continuous. Then we are ensured
(3.7.4)

$$\liminf_n \int_Z \delta^*(-\dot{u}_n(t), C(\theta_n(t))dt \geq \liminf_n \int_Z \delta^*(-\dot{u}_n(t), C(t))dt \geq \int_Z \delta^*(-\dot{u}(t), C(t))dt$$

By combining (3.7.1) $-$ $-$(3.7.4) we get

$$\int_0^T \delta^*(-\dot{u}(t), C(t))dt + \langle \dot{u}(t), v(t) + \nabla g(u(t))\rangle dt \leq 0. \tag{3.7.5}$$

as we have

$$\delta^*(-\dot{u}(t), C(t)) \geq \langle -\dot{u}(t), v(t) + \nabla g(u(t))\rangle$$

a.e. that implies

$$\int_0^T [\delta^*(-\dot{u}(t), C(t)) + \langle \dot{u}(t), v(t) + \nabla g(u(t))\rangle]dt = 0$$

so we conclude that

$$\delta^*(-\dot{u}(t), C(t)) + \langle \dot{u}(t), v(t) + \nabla g(u(t))\rangle = 0$$

a.e. with $v(t) + \nabla g(u(t)) \in C(t)$, just proving that $-\dot{u}(t) \in N_{C(t)}(v(t) + \nabla g(u(t)))$ a.e. It remain to check that $v(t) \in \mathcal{A}_t u(t)$. Indeed, as $\theta_n(t) - t \to 0$ and $v_n(t) \in \mathcal{A}_{\theta_n(t)} u_n(t)$, by Lemma 3.5, we get $u(t) \in D(\mathcal{A}_t)$ and $v(t) \in \mathcal{A}_t u(t)$. □

As further application we consider a class of time measurable maximal monotone perturbation $t \to A_t : H \to H$ satisfying the conditions.

$(\mathcal{A}_1)$ $t \to J_\lambda(t,x)$ is $\mathcal{L}(I)$-measurable for every $\lambda > 0$ and for every $x \in H$
$(\mathcal{A}_2)$ $|A^0(t,x)| \leq c(1 + ||x||)$ for all $(t,x) \in I \times H$.
$(\mathcal{A}_3)$ For $\lambda \in]0,1], \lambda \to 0^+$, for fixed $r > 0$, for fixed $t \in I$, $\sup_{|z| \leq r} |A_{\lambda_n}(t,z) -$

$A^0(t,z)| \to 0$.
$(\mathcal{A}_4)$ For fixed $t \in I$, $A^0(t,.)$ is continuous on H.
It is important to note a useful consequence of these properties. Let (u_n) a uniformly bounded sequence of Lebesgue measurable mappings pointwise converging to measurable mapping u. Let us consider the measurable mapping $v_n(t) = A_{\lambda_n}(t, u_n(t))$. Then $(v_n(t))$ pointwise converge to a bounded measurable mapping $v : I \to H$. Indeed, let $v_n(t) = A_{\lambda_n}(t, u_n(t))$, $v_m(t) = A_{\lambda_m}(t, u_m(t))$, then let us write for every $t \in I$

$$|v_m(t) - v_n(t)| \leq |A_{\lambda_m}(t, u_m(t)) - A^0(t, u_m(t))| + |A^0(t, u_m(t)) - A^0(t, u_n(t))|$$
$$+|A^0(t, u_n(t)) - A_{\lambda_n}(t, u_n(t))| \to 0$$

when $m, n \to \infty$. Then $(v_n(t))$ is a Cauchy sequence which converges uniformly to $v(t)$. Here the measurability of these mappings is ensured thank to $(\mathcal{A}_1)$. Note that $(\mathcal{A}_1)$. is not necessary when dealing with fixed maximal monotone $A : H \to H$. If $t \to A_t : H \to H$ is a measurable single valued maximal monotone operator satisfying $(\mathcal{A}_1)$ $(\mathcal{A}_2)$, then $A_{\lambda_n}(t, u_n(t))) = A(t, J_\lambda(t, u_n(t))) \to A(t, u(t)) := v(t)$ As an example, consider $A = \nabla g$ the gradient of a C^1 convex Lipschitz. At this point, in further application,we only need $(\mathcal{A}_1)$ $(\mathcal{A}_2)$ and the condition $(\mathcal{A}_\mathcal{C})$: For any uniformly bounded sequence of Lebesgue measurable mappings (u_n) pointwise converging to measurable mapping u, for $\lambda \in]0, 1], \lambda_n \to 0^+$, then $A_{\lambda_n}(t, u_n(t))$ pointwise converges to a bounded measurable mapping $v(t)$.

Theorem 3.4. *Assume that $H = \mathbb{R}^e, I = [0, T]$. Let $C : I \times H \rightrightarrows H$ be a closed convex valued multimapping for which there is some nondecreasing continuous function $r : I \to \mathbb{R}^+$ such that $haus(C(t,x), C(\tau,y)) \leq |r(t) - r(\tau)| + a||x - y||$ for all $\tau, t \in I$ and for all $x, y \in H$ with $r(0) = 0$. Let B be a linear operator continuous coercive: $\langle Bx, x\rangle \geq \gamma||x||^2 \geq 0$ for all $x \in H$, symmetric operator on H.*
Let $A : H \to 2^H$ be a maximal monotone operator satisfying
$(\mathcal{A})_1$ $|A^0(x)| \leq c(1 + ||x||)$ for all $x \in H$.
$(\mathcal{A})_2$:If $\lambda_n \to 0^+$, if (u_n, u) is a bounded sequence of measurable mappings on H converging pointwise to u, then $t \to A_{\lambda_n} u_n(t)$ pointwise converge to $v \in L^\infty_H(I)$.
$(\mathcal{H})_3$: $A_{\lambda_n} x \in C(0, x)$ for all $n \in \mathbb{N}$ and for all $x \in H$.
Then, for any $u_0 \in H$, there exist $v \in L^\infty_H(I)$ and an absolutely continuous mapping $u : I \to H$ satisfying

$$\begin{cases} -\frac{du}{dt}(t) \in N_{C(t,u(t))}(Bu(t) + v(t)) a.e. \\ u(0) = u_0 \in H \\ v(t) \in Au(t) a.e. \end{cases}$$

Further, one has $\|\dot{u}(t)\| \leq \rho := \frac{r(T)}{\gamma}$.

Proof. Put $I := [0,T]$.We note that by Hormander formula (see e.g. Castaing-Valadier [24])
(1.1.1)
$|\delta^*(e,C(t,x))-\delta^*(e,C(\tau,y))| \leq ||e||d_H(C(t,x),C(\tau,y)) \leq ||e||(|r(t)-r(\tau)|+a||x-y||)$

so that C is scalarly continuous.

Step I. Construction of a sequence $(u_n)_n$
We will use the Moreau 's catching-up algorithm [33]. We consider for each $n \in \mathbb{N}$ the following partition of the interval $]0,T]$ given by

$$t_i^n = i\frac{T}{n} := i\eta_n \text{ for } 0 \leq i \leq n, \quad I_i^n :=]t_i^n, t_{i+1}^n] \text{ for } 0 \leq i \leq n-1.$$

Put $u_0^n = u_0 \in H$. By $(\mathcal{H}_2)$. There is $z_1^n \in H$ such that

$$-z_1^n \in N_{C(t_1^n,u_0^n)}(Bz_1^n + A_{\lambda_n}u_0^n).$$

with $A_{\lambda_n}u_0^n \in C(0,u_0^n)$. Put $u_1^n = u_0^n + \eta_n z_1^n$. We have

$$\gamma||z_1^n||^2 \leq \langle Bz_1^n, z_1^n\rangle = \langle [Bz_1^n + A_{\lambda_n}u_0^n - v] + [v - A_{\lambda_n}u_0^n], z_1^n\rangle$$

$$\leq \langle v - A_{\lambda_n}u_0^n, z_1^n\rangle \leq ||v - A_{\lambda_n}u_0^n||||z_1^n|| \leq d_H(C(0,u_0^n),C(t_1^n,u_0^n)||z_1^n||$$

for all $v \in C(t_1^n,u_0^n)$. So

$$||z_1^n|| \leq \frac{r(T)}{\gamma}$$

Suppose that $u_0^n, u_1^n, .., u_i^n, z_1^n, z_2^n, ..z_i^n$ are constructed. As above by condition $(\mathcal{H}2)$

$$-z_{i+1}^n \in N_{C(t_{i+1}^n,u_i^n)}(Bz_{i+1}^n + A_{\lambda_n}u_i^n),$$

with $A_{\lambda_n}u_i^n \in C(0,u_i^n)$, and we set $u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$. Then by induction there are finite sequences $(u_i^n)_{i=0}^n$ and $(z_i^n)_{i=1}^n$ such that

$$-z_{i+1}^n \in N_{C(t_{i+1}^n,u_i^n)}(Bz_{i+1}^n + A_{\lambda_n}u_i^n) \tag{3.31}$$

$$u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n.$$

From $(u_i^n)_{i=0}^n$, $(z_i^n)_{i=1}^n$, we construct a sequence of mappings $(u_n)_{n\in\mathbb{N}}$ from I to H, $u_n(0) = u_0^n$ and for each $i = 0, .., ..n-1$

$$u_n(t) = u_i^n + \frac{t-t_i^n}{\eta_n}(u_{i+1}^n - u_i^n) \quad \text{for } t \in]t_i^n, t_{i+1}^n].$$

Furthermore we have the estimate

$$\gamma||z_{i+1}^n||^2 \leq \langle Bz_{i+1}^n, z_{i+1}^n\rangle = \langle [Bz_{i+1}^n + A_{\lambda_n}u_i^n - v] + [v - A_{\lambda_n}u_i^n], z_{i+1}^n\rangle$$

$$\leq \langle v - A_{\lambda_n} u_i^n, z_{i+1}^n \rangle \leq ||v - A_{\lambda_n} u_i^n||, ||z_{i+1}^n|| \leq d_H(C(0, u_i^n), C(t_{i+1}^n, u_i^n))||z_{i+1}^n||$$

for all $v \in C(t_{i+1}^n, u_i^n)$. Whence

$$||z_{i+1}^n|| \leq \frac{1}{\gamma} d_H(C(0, u_i^n), C(t_{i+1}^n, u_i^n)) \leq \frac{r(t_{i+1}^n) - r(0)}{\gamma} \leq \frac{r(T)}{\gamma}$$

Keeping in mind that

$$\left\| \frac{1}{\eta_n}(u_{i+1}^n - u_i^n) \right\| = \|z_{i+1}^n\| \leq \rho := \frac{r(T)}{\gamma}$$

From this we construct sequence $u_n(\cdot)$ is Lipschitz continuous on I with ρ as a Lipschitz constant. This Lipschitz property of $u_n(\cdot)$ ensures that $\|u_n(t)\| \leq \|u_0\| + \rho T$ and $u_n(t) = u_0 + \int_{t_0}^t \dot{u}_n(s)ds$ for every $t \in I$ with for every $t \in]t_i^n, t_{i+1}^n]$

$$\dot{u}_n(t) = \frac{u_{i+1}^n - u_i^n}{\eta_n} = z_{i+1}^n \in \rho \overline{B}_H.$$

Furthermore, for every $t \in [0, T]$, one has $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$, hence $||u_n(t)|| \leq ||u_0|| + \rho T$.

Now, let us define the step functions $\theta_n, \delta_n : I \longrightarrow I$ by $\theta_n(0) = \delta_n(0) = 0$ and

$$\theta_n(t) = t_{i+1}^n, \quad \delta_n(t) = t_i^n \quad \text{if } t \in]t_i^n, t_{i+1}^n],$$

so the inclusion (4.1) becomes

$$-\dot{u}_n(t) \in N_{C(\theta_n(t), u_n(\delta_n(t)))}(B\dot{u}_n(t) + A_{\lambda_n} u_n(\delta_n(t))) \text{ a.e. } t \in I$$

For each $t \in I$ we observe that there is some $i \in \{0, ..., n-1\}$ such that $t \in [t_i^n, t_{i+1}^n[$, and then

$$|\theta_n(t) - t| \to 0 \text{ and } |\delta_n(t) - t| \to 0 \text{ as } n \to +\infty.$$

We also note that the latter inclusion above yields

$$\delta^*(-\dot{u}_n(t), C(\theta_n(t), u_n(\delta_n(t)))) + \langle \dot{u}_n(t), B\dot{u}_n(t) + A_{\lambda_n} u_n(\delta_n(t)) \rangle \leq 0$$

with $\dot{u}_n(t) \in \rho \overline{B}_H$ a.e. $t \in I$, so that $\dot{u}_n \in S^1_{\rho \overline{B}_H}$ where $S^1_{\rho \overline{B}_H} = \{\xi \in L^1_H(I) : \xi(t) \in \rho \overline{B}_H \text{ a.e. } t \in I\}$. a.e. $t \in [0, T]$. We note that $||u_n(t)|| \leq ||u_0|| + \rho T$, for all $t \in [0, T]$ and $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$ for all $t \in [0, T]$ so that

$$|A_{\lambda_n} u_n(\delta_n(t))| \leq |A^0(u_n(\delta_n(t))| \leq c(1 + ||u_n(\delta_n(t))|| \leq c(1 + ||u_0|| + \rho T)$$

Step II. Convergence to a solution.
We note that S^1_L is a weakly compact convex set of $L^1_H(I)$. Set

$$\mathcal{X} := \{\xi : I \to H : \xi(t) = u_0 + \int_0^t \dot{\xi}(s)ds, \, t \in [0, T]; \, \dot{v} \in S^1_L\}.$$

It is clear that $\mathcal{X}$ is convex, equicontinuous and compact. As $u_n \in \mathcal{X}$, one can extract from $(u_n)_n$ a (not relabeled) subsequence which converges uniformly to $u : I \to H$ such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$ for all $t \in I$ and such that $(\dot{u}_n)_n$ $\sigma(L^1_H(I), L^\infty_H(I))$-converges to $\dot{u} \in S^1_L$. Further, the inequality

$$\|u_n(\delta_n(t)) - u_n(t)\| \leq \rho|\delta_n(t) - t|$$

assures us that $\big(u_n(\delta_n(t))\big)_n$ converges to $u(t)$ to H for each $t \in I$. This and the $\sigma(L^1_H(I), L^\infty_H(I))$ convergence of $(\dot{u}_n)_n$ to $\dot{u}$ in $L^1_H(I)$ along with the inclusion (4.1) allow us to obtain that for a.e. $t \in I$ the required inclusion. This need a **careful look**. Indeed, for every Lebesgue measurable set $Z \subset I$ and for any $e \in H$, the function $\mathbf{1}_Z e \in L^\infty_H(I)$. Considering the inequality

$$\langle e, B\dot{u}_n(t) + A_{\lambda_n} u_n(\delta_n(t))\rangle \leq \delta^*(e, C(\theta_n(t), u_n(\delta_n(t))))$$

and integrating on Z gives

$$\int_I \langle \mathbf{1}_Z e, B\dot{u}_n(t) + A_{\lambda_n} u_n(\delta_n(t))\rangle dt \leq \int_Z \delta^*(x, C(\theta_n(t), u_n(\delta_n(t))))dt.$$

Passing to the limit in this inequality and using the scalar continuity of $C(\cdot)$ and the fact that $B\dot{u}_n \to B\dot{u}$ weakly in $L^1_H(I)$, $A_{\lambda_n}(u_n(\delta_n(.)))$ pointwise converge to $v \in L^\infty_H(I)$ we obtain

$$\int_I \langle \mathbf{1}_Z e, B\dot{u}(t) + v(t)\rangle dt \leq \limsup_n \int_Z \delta^*(e, C(\theta_n(t), u_n(\delta_n(t)))dt$$

$$\leq \int_Z \limsup_n \delta^*(e, C(\theta_n(t), u_n(\delta_n(t)))) \leq \int_Z \delta^*(e, C(t, u(t)))dt,$$

which implies

$$\langle e, B\dot{u}(t) + v(t)\rangle \leq \delta^*(e, C(t, u(t))) \text{ a.e. } t \in I.$$

By the separability of H and the compactness of $C(t, u(t))$ (see, e.g., Castaing-Valadier [24, Proposition III- 35]), we get the desired inclusion $B\dot{u}(t)+v(t) \in C(t)$ a.e. As consequence, we have

$$\lim_n \int_0^T \langle \dot{u}_n(t), A_{\lambda_n} u_n(\delta_n(t))\rangle dt = \int_0^T \langle \dot{u}(t), v(t)\rangle dt.$$

because $A_{\lambda_n} u_n(\delta_n(.))$ $\tau(L^\infty_H(I), L^1_H(I))$ (Mackey) converges to v in L^∞_H using Grothendieck Lemma, recalling that $|A_{\lambda_n} u_n(\delta_n(.))| \leq c(1 + ||u_0|| + \rho T)$. As above we may apply the lower semicontinuity of integral convex functional ([22], Theorem 8.1.16) to deduce that

$$\int_Z \delta^*(-\dot{u}(t), C(t, u(t)))dt \leq \liminf_n \int_Z \delta^*(-\dot{u}_n(t), C(t, u(t)))dt$$

$$\leq \liminf_n \int_Z \delta^*(-\dot{u}_n(t), C(\theta_n(t), u_n(\delta_n(t))))dt \tag{1.1.2}$$

by noting that $B\dot{u}(t) + v(t) \in C(t)$ with $B\dot{u} + v \in L^\infty_H(I)$.

Let us set $\psi_B(x) = \langle Bx, x\rangle$ if $x \in \rho\overline{B}_H$ and $\psi(x) = +\infty$ if $x \notin \rho\overline{B}_H$. Then it is clear ψ is a positive lower semicontinuous convex integrand. Apply again the lower semicontinuity of the integral convex functional ([22], Theorem 8.1.6) associated with the positive normal convex integrand ψ_B we obtain,

$$\liminf_n \int_Z \psi_B(\dot{u}_n(t))dt \geq \int_Z \psi_B(\dot{u}(t))dt$$

that is

$$\liminf_n \int_Z \langle B\dot{u}_n(t), \dot{u}_n(t))dt \geq \int_Z \langle B\dot{u}(t), \dot{u}(t)dt. \tag{1.1.3}$$

By integrating on measurable subset $Z \subset [0, T]$ the inequality (here measurability and integrability are guaranted)

$$\delta^*(-\dot{u}_n(t), C(\theta_n(t), u_n(\delta_n(t)))) + \langle \dot{u}_n(t), B\dot{u}_n(t) + A_{\lambda_n} u_n(\delta_n(t))\rangle dt \leq 0$$

gives

(1.1.4)

$$\int_Z \delta^*(-\dot{u}_n(t), C(\theta_n(t), u_n(\delta_n(t))))dt + \int_Z \langle B\dot{u}_n(t), \dot{u}_n(t)\rangle dt + \int_Z \langle \dot{u}_n(t), A_{\lambda_n} u_n(\delta_n(t))\rangle dt \leq 0$$

by passing to the limit when n goes to ∞ in this equality using (1.1.1)—(1.1.7) gives

$$\int_0^T [\delta^*(-\dot{u}(t), C(t, u(t))) + \langle \dot{u}(t), B\dot{u}(t) + v(t)\rangle]dt \leq 0. \tag{1.1.5}$$

As $t \mapsto \delta^*(-\dot{u}(t), C(t, u(t))) + \langle \dot{u}(t), B\dot{u}(t) + v(t)\rangle$ is integrable, by (1.1.5) and $B\dot{u}(t) + v(t) \in C(t, u(t))$, we have

$$\delta^*(-\dot{u}(t), C(t, u(t))) \geq \langle -\dot{u}(t), B\dot{u}(t) + v(t)\rangle$$

that implies

$$\int_0^T [\delta^*(-\dot{u}(t), C(t, u(t))) + \langle \dot{u}(t), B\dot{u}(t) + v(t)\rangle]dt = 0. \tag{1.1.6}$$

By (1.1.6) we get finally

$$\delta^*(-\dot{u}(t), C(t, u(t))) + \langle \dot{u}(t), B\dot{u}(t) + v(t)\rangle = 0$$

a.e. with $B\dot{u}(t)+v(t) \in C(t,u(t))$ a.e. so we conclude that $-\dot{u}(t)) \in N_{C(t,u(t))}(B\dot{u}(t)+v(t))$ a.e. It remain to check that $v(t) \in Au(t)$ a.e. Indeed $J_{\lambda_n}u_n(\delta_n(t)) \to u(t)$ by writing

$$||J_{\lambda_n}u_n(\delta_n(t)) - u(t)|| \leq ||J_{\lambda_n}u_n(\delta_n(t)) - J_{\lambda_n}u(t)|| \ + ||J_{\lambda_n}u(t) - u(t)||$$

$$\leq ||u_n(\delta_n(t)) - u(t)|| + ||J_{\lambda_n}u(t) - u(t)|| \to 0$$

From $v_n(t) = A_{\lambda_n}u_n(\delta_n(t)) \in AJ_{\lambda_n}u_n(\delta_n(t))$ we have that

$$(v_n(t), J_{\lambda_n}u_n(\delta_n(t))) \in graphA$$

As $graphA$ is closed that implies $v(t) \in Au(t)$ a.e.

□

Proposition 3.9. *Let H be a separable Hilbert space and $I = [0,T]$. Let $C : I \rightrightarrows H$ be a scalarly measurable multimapping with closed convex weakly locally compact values which contain no line and for which there exist $r \in L^{\infty}_{\mathbb{R}^+}(I)$ such that $C(t) \cap r(t)\overline{B}_H \neq \emptyset$ for all $t \in I$. Let $B : H \to H$ be a continuous symmetric linear coercive operator. Let $A : H \to 2^H$ be a maximal monotone operator satisfying*
$(\mathcal{H})_1$ $|A^0(x)| \leq c(1 + ||x||)$ for all $x \in H$. Let $(f_n, f)_{n\in\mathbb{N}}$ be a bounded sequence in $L^{\infty}_H(I)$ with $\|f_n(t)\| \leq \beta, \|f(t)\| \leq \beta$ $(\beta > 0)$ for all $n \in \mathbb{N}$ such that $(f_n(t))_n$ converges to $f(t)$ for each $t \in I$.
Let $(v_n, v)_{n\in\mathbb{N}}$ be a bounded sequence in $L^{\infty}_H(I)$ with $\|v_n(t)\| \leq \gamma, \|v(t)\| \leq \gamma$ $(\gamma > 0)$ for all $n \in \mathbb{N}$ such that $(v_n(t))_n$ converges to $v(t)$ for each $t \in I$. Let $(\zeta_n)_{n\in\mathbb{N}}$ be an equi-integrable sequence in $L^1_H(I)$ such that $\zeta_n(t) \in C(t)$ for all $t \in I$ and $n \in \mathbb{N}$ and such that $(\zeta_n)_{n\in\mathbb{N}}$ $\sigma(L^1_H, L^{\infty}_H)$ converges in $L^1_H(I)$ to ζ. Assume that

$$(i) f_n(t) + A_{\lambda_n}v_n(t) - B\,\zeta_n(t) \in N_{C(t)}(\zeta_n(t)) \quad \text{for all } n \in \mathbb{N}, \text{ a.e. } t \in I.$$

and (ii) $\lambda_n \to 0$, $\lim_n A_{\lambda_n}v_n(t) \to w(t)$ pointwise, where w is a measurable mapping. Then for a.e. $t \in I$ one has

$$\zeta(t) \in C(t) \quad \text{and} \quad f(t) + w(t) - B\,\zeta(t) \in N_{C(t)}(\zeta(t)).$$

$$w(t) \in Av(t)$$

Proof. We first verify that $\zeta(t) \in C(t)$ a.e. $t \in I$. Let $(e_p)_{p\in\mathbb{N}}$ be a dense sequence in H. Take any measurable set $Z \subset I$ and any $p \in \mathbb{N}$, and note that the mapping $\mathbf{1}_Z e_p \in L^{\infty}_H(I)$. Considering the inequality By the inequality (due to the inclusion $\zeta_n(t) \in C(t)$)

$$\langle e_p, \zeta_n(t)\rangle \leq \delta^*(e_p, C(t))$$

and integrating on Z ensure that

$$\int_I \langle \mathbf{1}_Z e_p, \zeta_n(t), \rangle dt = \int_Z \langle e_p, \zeta_n(t)\rangle dt \leq \int_Z \delta^*(e_p, C(t))dt.$$

Passing to the limit in the latter inequality assures us that

$$\int_Z \langle \mathbf{1}_Z e_p, \zeta(t)\rangle dt \leq \int_Z \delta^*(e_p, C(t))dt.$$

This being true for any Lebesgue measurable set $Z \subset I$, it follows that for every $p \in \mathbb{N}$

$$\langle e_p, \zeta(t)\rangle \leq \delta^*(e_p, C(t)) \text{ a.e. } t \in I.$$

As H is separable and $C(t)$ is closed convex weakly locally compact and contains no line, by (Castaing-Valadier [24, Proposition III- 35]), we get the desired inclusion $\zeta(t) \in C(t)$ a.e. $t \in I$.

For each $t \in I$ keeping in mind that $v_n(t) \to v(t)$ strongly in H and A is a maximal montone operator satisfying $(\mathcal{H})_1$ we see that $|A_{\lambda_n} v_n(t)| \leq c(1+\gamma)$ strongly in H so that by our assumption $w(t) = \lim_n A_{\lambda_n} v_n(t) \in c(1+\gamma)\overline{B}_H$. Then the uniformly bounded sequence $w_n := A_{\lambda_n} v_n(.) \to w$ for the Mackey topology $\tau(L^\infty_H(I), L^1_H(I)$ .As consequence

$$\lim_n \int_0^T \langle A_{\lambda_n} v_n(t), \zeta_n(t)dt = \lim_n \int_0^T \langle w(t), \zeta(t)dt$$

We can also see that $B\zeta_n \to B\zeta$ weakly in $L^1_H(I)$ since for any $h \in L^\infty_H(I)$

$$\int_0^T \langle h(t), B\zeta_n(t)\rangle\, dt = \int_0^T \langle Bh(t), \zeta_n(t)\rangle\, dt \to \int_0^T \langle Bh(t), \zeta(t)\rangle\, dt = \int_0^T \langle h(t), B\zeta(t)\rangle\, dt.$$

As a main consequence $f_n + w_n - B\zeta_n \to f + w - B\zeta$ weakly in $L^1_H(I)$.

Given any Lebesgue measurable subset $Z \subset I$ we may apply the lower semicontinuity of convex integral functional in [22, Theorem 8.1.16] to derive that

$$\int_Z \delta^*(f(t) + w(t) - B\zeta(t), C(t))dt \leq \liminf_n \int_Z \delta^*(f_n(t) + w_n(t) - B\zeta_n(t), C(t))dt. \tag{3.32}$$

This needs a careful look. Choose a bounded measurable selection s of $t \mapsto C(t) \cap r(t)\overline{B}_H$, so $s \in L^\infty_H(I)$. We note that $(t,x) \mapsto \delta^*(x, C(t))$ is a normal lower semicontinuous convex integrand defined on $I \times H$ and $\delta^*(f_n(t) + w_n(t) - B\zeta_n(t), C(t))$ is measurable and minored by $\rho_n(t) := \langle s(t), f_n(t) + w_n(t)) - B\zeta_n(t)\rangle$. Further, it is easy to check that $(\rho_n(\cdot))$ is equi-integrable in $L^1_{\mathbb{R}}(I)$. Then by [22, Theorem 8.1.6] we deduce that

$$\begin{aligned}&\liminf_n \int_Z \delta^*(f_n(t) + w_n(t) - B\zeta_n(t), C(t))dt\\ &\geq \int_Z \delta^*(f(t) + w(t) - B\zeta(t), C(t))dt,\end{aligned}$$

which confirms (3.32).

Let us set $\psi_B(x) = \langle Bx, x\rangle$ if $x \in C(t)$ and $\psi_B(x) = +\infty$ if $x \notin C(t)$. Then it is clear that ψ_B is a positive lower semicontinuous convex integrand. Apply again the lower semicontinuity of the convex integral functional ([22], Theorem 8.1.6) associated with the positive normal convex integrand ψ_B we obtain

$$\liminf_n \int_Z \psi_B(\zeta_n(t))\, dt \geq \int_Z \psi_B(\zeta(t))\, dt,$$

that is,

$$\liminf_n \int_Z \langle B\zeta_n(t), \zeta_n(t))dt \geq \int_Z \langle B\zeta(t), \zeta(t)dt. \tag{3.33}$$

Further, we have

$$\lim_n \int_Z \langle w_n(t)), \zeta_n(t)\rangle\, dt = \int_Z \langle w(t), \zeta(t)\rangle\, dt, \tag{3.34}$$

$$\lim_n \int_Z \langle f_n(t), \zeta_n(t)\rangle dt = \int_Z \langle f(t), \zeta(t)\rangle dt. \tag{3.35}$$

The two latter equality features require a careful justification. using the fact that a uniformly bounded sequence of measurable mapping (g_n) that pointwise converge to a bounded measurable mapping g,then (g_n) converge to g with respect to the Mackey topology $\tau(L^\infty_H(I), L^1_H(I)$. Consequently, both (3.34) and (3.35) hold true, as claimed above, by noting that ζ_n is equi-integrable [2]

On the other hand, the inclusion $w_n(t) - B\zeta_n(t) \in N_{C(t)}(\zeta_n(t))$ for a.e. $t \in I$ ensures that

$$\delta^*(f_n(t) + w_n(t) - B\zeta_n(t), C(t)) - \langle f_n(t) + w_n(t) - B\zeta_n(t), \zeta_n(t)\rangle \leq 0.$$

Integrating this inequality on I gives

$$\int_0^T \delta^*(f_n(t) + w_n(t) - B\zeta_n(t), C(t))dt + \int_0^T \langle B\zeta_n(t), \zeta_n(t)\rangle dt + \int_0^T \langle -f_n(t) - w_n(t), \zeta_n(t)\rangle dt \leq 0.$$

Passing to the limit inferior as $n \to \infty$ and using (3.32) we obtain

$$\int_0^T \big[\delta^*(f(t) + w(t) - A\zeta(t), C(t)) + \langle B\zeta(t) - w(t) - f(t), \zeta(t)\rangle\big] dt \leq 0.$$

This and the inclusion $\zeta(t) \in C(t)$ a.e. $t \in I$ allow us to conclude that

$$f(t) + w(t) - B\zeta(t)) \in N_{C(t)}(\zeta(t)) \text{ a.e. } t \in I$$

[2] If $H = \mathbb{R}^e$, here one may invoke a classical fact that on bounded subsets of L^∞_H the topology of convergence in measure coincides with the topology of uniform convergence on uniformly integrable sets, i.e. on relatively weakly compact subsets, alias the Mackey topology. This is a lemma due to Grothendieck [26] [Ch.5 §4 no 1 Prop. 1 and exercice].

according to the description (2.1) of the normal cone. It remain to check that $w(t) \in Av(t)$ a.e. Indeed $J_{\lambda_n} v_n(t) \to v(t)$ by writing

$$||J_{\lambda_n} v_n(t) - v(t)|| \leq ||J_{\lambda_n} v_n(t)) - J_{\lambda_n} v(t)|| \ + ||J_{\lambda_n} v(t) - v(t)||$$

$$\leq ||v_n(t)) - v(t)|| + ||J_{\lambda_n} v(t) - v(t)|| \to 0$$

From $w_n(t) = A_{\lambda_n} v_n(t) \in A J_{\lambda_n} v_n(t)$ we have that

$$(w_n(t), J_{\lambda_n} v_n(t)) \in graph A$$

As $graph A$ is weakly strongly sequentially closed that implies $w(t) \in Av(t)$ a.e. □

Proposition 3.10. *Let H be a separable Hilbert space and $I = [0,T]$. Let $C : I \rightrightarrows H$ be a closed convex valued scalarly measurable multimapping for which there is some real $r > 0$ such that $C(t) \subset r\overline{B}_H$ for all $t \in I$. Let B be a continuous symmetric linear coercive operator on H and let $A : H \to 2^H$ be a maximal monotone operator satisfying*
$(\mathcal{H})_1$ $|A^0(x)| \leq c(1 + ||x||)$ for all $x \in H$. Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions from I into I such that for each $t \in I$ one has $\theta_n(t) \to t$ and haus $\big(C(\theta_n(t)), C(t)\big) \to 0$ *as $n \to \infty$.*

Let $(f_n, f)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_H(I)$ with $\|f_n(t)\| \leq \beta, \|f(t)\| \leq \beta$ $(\beta > 0)$ for all $n \in \mathbb{N}$ such that $(f_n(t))_n$ converges to $f(t)$ for each $t \in I$.
Let $(u_n, u)_{n\in\mathbb{N}}$ be a sequence of absolutely continuous mappings

$$u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds, \ \ \dot{u}_n(t) \in C(t)$$

$$u(t) = u_0 + \int_0^t \dot{u}(s)ds,$$

such that $(u_n)_n$ converges to u uniformly on I and $(\dot{u}_n)_n$ converges to $\dot{u}$ weakly in $L^1_H(I)$. Assume that for every $n \in \mathbb{N}$

$$f_n(t) + A_{\lambda_n} u_n(t) - B\dot{u}_n(t) \in N_{C(\theta_n(t))}(\dot{u}_n(t)) \text{ a.e. } t \in I.$$

and $\lambda_n \to 0$, $\lim_n A_{\lambda_n} u_n(t) \to w(t)$ pointwise, where w is a measurable mapping. Then for a.e. $t \in I$ one has

$$\dot{u}(t) \in C(t) \quad \text{and} \quad f(t) + w(t) - B\,\dot{u}(t) \in N_{C(t)}(\dot{u}(t)).$$

$$w(t) \in Au(t)$$

Proof. First, we justify that $\dot{u}(t) \in C(t)$ a.e. $t \in I$. Take any measurable Lebesgue set $Z \subset I$ and any $x \in H$. The function $\mathbf{1}_Z x \in L^\infty_H(I)$. Writing

$$\langle x, \dot{u}_n(t)\rangle \leq \delta^*(x, C(\theta_n(t))),$$

we see that

$$\int_I \langle \mathbf{1}_Z x, \dot{u}_n(t), \rangle dt = \int_Z \langle x, \dot{u}_n(t) \rangle dt \leq \int_Z \delta^*(x, C(\theta_n(t))) dt.$$

Passing to the upper limit we obtain

$$\int_Z \langle \mathbf{1}_Z x, \dot{u}(t) \rangle dt \leq \limsup_n \int_Z \delta^*(x, C(\theta_n(t))) dt$$
$$\leq \int_Z \limsup_n \delta^*(x, C(\theta_n(t))) \leq \int_Z \delta^*(x, C(t)) dt.$$

This being true for any Lebesgue measurable set $Z \subset I$ we deduce that for every $x \in H$

$$\langle x, \dot{u}(t) \rangle \leq \delta^*(x, C(t)) \ {}_t ext a.e.\ t \in I.$$

By the separability of H and the weak compactness and convexity of $C(t)$, we get the desired inclusion $\dot{u}(t) \in C(t)$ a.e. $t \in I$.

Since $(f_n)_n$ is uniformly bounded and pointwise converges to f, we have

$$\lim_n \int_Z \langle f_n(t), \dot{u}_n(t) \rangle dt = \int_Z \langle f(t), \dot{u}(t) \rangle dt. \tag{3.36}$$

this fact is explained. Also by integrating on Z (we are ensured that the functions given are measurable) the inequality it ensues that with $q_n(t) := f_n(t) + A_{\lambda_n} u_n(t) - B\dot{u}_n(t)$

$$\int_Z \delta^*(q_n(t), C(\theta_n(t))) dt + \int_Z \langle B\dot{u}_n(t) - f_n(t), \dot{u}_n(t) \rangle dt - \int_Z \langle A_{\lambda_n} u_n(t), \dot{u}_n(t) \rangle dt \leq 0. \tag{3.37}$$

We claim that $Bu_n(\cdot) \to Bu(\cdot)$ weakly in $L^1_H(I)$ and as above $w_n(t) : A_{\lambda_n} u_n(t) \to w(t)$ for with respect to the Mackey topology $\tau(L^\infty_H(I), L^1_H(I)$in $L^1_H(I)$, so

$$q_n(\cdot) = f_n(\cdot) + A_{\lambda_n}(u_n(\cdot)) - B\dot{u}_n(\cdot) \to q(\cdot) := f(\cdot) + w(.) - B\dot{u}(\cdot)$$

weakly in $L^1_H(I)$. Indeed, for any $h \in L^\infty_H(I)$ the weak convergence in $L^1_H(I)$ of $(\dot{u}_n)_n$ to $\dot{u}$ says that $\int_0^T \langle Bh(t), \dot{u}_n(t) \rangle dt \to \int_0^T \langle Bh(t), \dot{u}(t) \rangle dt$, which means

$$\int_0^T \langle h(t), B\dot{u}_n(t) \rangle dt \to \int_0^T \langle h(t), B\dot{u}(t) \rangle dt.$$

This property for every $h \in L^\infty_H(I)$ translates the weak convergence in $L^1_H(I)$ of $(B\dot{u}_n)_n$ to $B\dot{u}$. Concerning $(A_{\lambda_n}(u_n(\cdot)))_n$ it converges $\tau(L^\infty_H(I), L^1_H(I)$ to w as have already seen.So weak convergence in $L^1_H(I)$ of $(q_n)_n$ to q is justified.

Regarding $\int_0^T \langle B\dot{u}_n(t), \dot{u}_n(t) \rangle dt$ let us set $\psi_B(t, x) = \langle Bx, x \rangle$ if $x \in C(t)$ and $\psi_B(t, x) = 0$ if $x \notin C(t)$. It is clear that ψ_B is a positive lower semicontinuous convex normal integrand. By the lower semicontinuity of convex integral functional (see [22, Theorem 8.1.6]) we obtain

$$\liminf_n \int_0^T \psi_B(t, \dot{u}_n(t)) dt \geq \int_0^T \psi_B(t, \dot{u}(t)) dt,$$

that is,

$$\liminf_n \int_0^T \langle B\dot{u}_n(t), \dot{u}_n(t)\rangle dt \geq \int_0^T \langle B\dot{u}(t), \dot{u}(t)\rangle dt. \tag{3.38}$$

The last step is concerned with $\int_0^T \delta^*(q_n(t), C(t))dt$. Note that $\|q_n(t)\| \leq \beta+\gamma+r\|B\|$, hence $\delta^*(q_n(t), C(t)) \geq -r(\beta+\gamma+r\|B\|)$. Using this and the lower semicontinuous convex normal integrand $(t,x) \mapsto \delta^*(x, C(t))$ we obtain by [22, Theorem 8.1.6] again that

$$\liminf_n \int_0^T \delta^*(q_n(t), C(t))dt \geq \int_0^T \delta^*(g(t), C(t))dt.$$

Since

$$\big|\delta^*\big(q_n(t), C(\theta_n(t)\big) - \delta^*\big(q_n(t), C(t)\big)\big| \leq (\beta+\gamma+r\|B\|)\,\mathrm{haus}\big(C(\theta_n(t)), C(t)\big) =: \varepsilon_n(t)$$

with $\int_0^T \varepsilon_n(t) \to 0$ as $n \to \infty$, we are ensured that

$$\liminf_n \int_0^T \delta^*\big(q_n(t), C(t)\big)dt \geq \int_0^T \delta^*\big(g(t), C(t)\big)dt. \tag{3.39}$$

Putting together (3.37)—(3.39) yields

$$\int_0^T \delta^*(f(t)+w(t)-B\dot{u}(t), C(t))dt + \int_0^T \langle -f(t)+w(t)+B\dot{u}(t), \dot{u}(t)\rangle dt \leq 0. \tag{3.40}$$

On the other hand, the inclusion $\dot{u}(t) \in C(t)$ says that for a.e. $t \in I$

$$\delta^*(f(t)+w(t)-B\dot{u}(t), C(t)) \geq \langle f(t)+w(t)-B\dot{u}(t), \dot{u}(t)\rangle.$$

Taking this into account in (3.40) we deduce for a.e. $t \in I$

$$\delta^*\big(f(t)+w(t)-B\dot{u}(t), C(t)\big) - \langle f(t+w(t)-B\dot{u}(t), \dot{u}(t) \leq 0$$

The latter inequality and the inclusion $\dot{u}(t) \in C(t)$ guarantees that for a.e. $t \in I$

$$f(t)+w(t)-B\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$$

according to the description (2.1) of the normal cone. It remain to check that $w(t) \in Au(t)$ a.e. Indeed $J_{\lambda_n}u_n(t) \to u(t)$ by writing

$$||J_{\lambda_n}u_n(t) - u(t)|| \leq ||J_{\lambda_n}u_n(t)) - J_{\lambda_n}u(t)|| \ + ||J_{\lambda_n}u(t) - u(t)||$$

$$\leq ||u_n(t)) - u(t)|| + ||J_{\lambda_n}u(t) - u(t)|| \to 0$$

From $w_n(t) = A_{\lambda_n}u_n(t) \in AJ_{\lambda_n}u_n(t)$ we have that

$$(w_n(t), J_{\lambda_n}u_n(t)) \in graphA$$

As $graphA$ is weakly strongly sequentially closed that implies $w(t) \in Au(t)$ a.e. □

Theorem 3.5. *Let $f : [0,T] \to H = \mathbb{R}^e$ be a continuous mapping and let $v : [0,T] \to \mathbb{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : [0,T] \to H$ be a compact convex valued multimapping such that*

$$\text{haus}\,(C(t), C(\tau)) \leq |v(t) - v(\tau)| \quad \textit{for all}\, t, \tau \in [0,T].$$

Let $B : H \to H$ be a linear continuous coercive symmetric operator and let $A : H \to 2^H$ be a maximal monotone operator satisfying
$(\mathcal{A})_1$ $|A^0(x)| \leq c(1 + ||x||)$ for all $x \in H$.
$(\mathcal{A})_2$:If $\lambda_n \to 0^+$, if (u_n, u) is a bounded sequence of measurable mappings on H converging pointwise to u, then $t \to A_{\lambda_n} u_n(t)$ pointwise converge to $v \in L^\infty_H(I)$.
Then, for any $u_0 \in H$, there exists an $W^{1,\infty}_H([0,T])$ solution $u : [0,T] \to H$ and a bounded measurable mapping $w : I \to H$ such that

$$\begin{cases} f(t) + w(t) - B\,\frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t)) \\ u(0) = u_0 \\ w(t) \in Au(t) \end{cases}$$

Further, one has $\|\dot{u}(t)\| \leq \rho$, where $\rho := \sup\{\|y\| : y \in C([0,T])\}$.

Proof. Put $I := [0,T]$ and denote $\beta := \max\{\|f(t)\| : t \in I\}$ (by continuity of f). Noticing that the multimapping $C(\cdot)$ is upper semicontinuous from I into H the set $C(I)$ is compact, and hence $\rho := \sup\{\|y\| : y \in C(I)\}$ is finite and $L := \rho\overline{B}_H$ is compact and convex.

Step I. Construction of a sequence $(u_n)_n$
We will use the Moreau 's catching-up algorithm [33]. We consider for each $n \in \mathbb{N}$ A_{λ_n} the Yosida approxiamtion of A and the following partition of the interval $]0,T]$ given by

$$t^n_i = i\frac{T}{n} := i\eta_n \text{ for } 0 \leq i \leq n, \quad I^n_i :=]t^n_i, t^n_{i+1}] \text{ for } 0 \leq i \leq n-1.$$

Put $u^n_0 = u_0$ and $f^n_i = f(t^n_i)$ for all $i = 1, .., n$. By Proposition 2.1(b), there is $z^n_1 \in C(t^n_1) \subset L$ such that

$$f^n_1 + A_{\lambda_n} u^n_0 - Bz^n_1 \in N_{C(t^n_1)}(z^n_1).$$

Put $u^n_1 = u^n_0 + \eta_n z^n_1$. Suppose that $u^n_0, u^n_1, .., u^n_i, z^n_1, z^n_2, ..z^n_i$ are constructed. As above by Proposition 2.1(b) there exists $z^n_{i+1} \in C(t^n_{i+1}) \subset L$ such that

$$f^n_{i+1} + A_{\lambda_n} u^n_i - Bz^n_{i+1} \in N_{C(t^n_{i+1})}(z^n_{i+1}),$$

and we set $u^n_{i+1} = u^n_i + \eta_n z^n_{i+1}$. Then by induction there are finite sequences $(u^n_i)^n_{i=0}$ and $(z^n_i)^n_{i=1}$ such that

$$f^n_{i+1} + A_{\lambda_n} u^n_i - Bz^n_{i+1} \in N_{C(t^n_{i+1})}(z^n_{i+1}) \tag{3.41}$$

$$u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n.$$

From $(u_i^n)_{i=0}^n$, $(z_i^n)_{i=1}^n$ $(f_i^n)_{i=0}^n$, we construct two sequences of mappings $(u_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ from I to H, by setting $f_n(0) = f_1^n$, $u_n(0) = u_0^n$ and for each $i = 0, .., ..n-1$ we set $f_n(t) = f_{i+1}^n$ and

$$f_n(t) = f_{i+1}^n \quad \text{and} \quad u_n(t) = u_i^n + \frac{t - t_i^n}{\eta_n}(u_{i+1}^n - u_i^n) \quad \text{for } t \in]t_i^n, t_{i+1}^n].$$

Keeping in mind that $C(t) \subset L = \rho\overline{B}_H$ we have $u_i \in C(t_{i+1}^n) \subset \rho\overline{B}_H$, so

$$\left\| \frac{1}{\eta_n}(u_{i+1}^n - u_i^n) \right\| = \|z_{i+1}^n\| \leq \rho.$$

From this it is clear that $u_n(\cdot)$ is Lipschitz continuous on I with ρ as a Lipschitz constant. This Lipschitz property of $u_n(\cdot)$ ensures that $\|u_n(t)\| \leq \|u_0\| + \rho T$ and $u_n(t) = u_0 + \int_{t_0}^t \dot{u}_n(s)ds$ for every $t \in I$. We also note that $\|f_n(t)\| \leq \beta$ for all $n \in \mathbb{N}$ and $t \in I$. Now, let us define the step functions $\theta_n, \delta_n : I \longrightarrow I$ by $\theta_n(0) = \delta_n(0) = 0$ and

$$\theta_n(t) = t_{i+1}^n, \quad \delta_n(t) = t_i^n \quad \text{if } t \in]t_i^n, t_{i+1}^n],$$

so the inclusion (4.1) becomes

$$f_n(t) + A_{\lambda_n} u_n(\delta_n(t)) - B\dot{u}_n(t) \in N_{C(\theta_n(t))}(\dot{u}_n(t)) \text{ a.e. } t \in I \tag{0.5.1}$$

For each $t \in I$ we observe that there is some $i \in \{0, ..., n-1\}$ such that $t \in [t_i^n, t_{i+1}^n[$, and then

$$|\theta_n(t) - t| \to 0 \text{ and } |\delta_n(t) - t| \to 0 \text{ as } n \to +\infty.$$

We also note that the latter inclusion above yields

$$\delta^*(f_n(t)+A_{\lambda_n}u_n(\delta_n(t))-B\dot{u}_n(t), C(\theta_n(t)))+\langle -f_n(t)-A_{\lambda_n}u_n(\delta_n(t))-Bdotu_n(t), \dot{u}_n(t)\rangle \leq 0$$

with

$$\dot{u}_n(t) \in C(\theta_n(t)) \subset L \text{ a.e. } t \in I,$$

so that $\dot{u}_n \in S_L^1$ where $S_L^1 := \{\xi \in L_H^1(I) : \xi(t) \in L \text{ a.e. } t \in I\}$.

Step II. Convergence to a solution.
We note that S_L^1 is a weakly compact convex set of $L_H^1(I)$ (see, e.g., [22] and the references therein). Set

$$\mathcal{X} := \{\xi : I \to H : \xi(t) = u_0 + \int_0^t \dot{\xi}(s)ds,\ t \in [0,T];\ \dot{v} \in S_L^1\}.$$

It is clear that $\mathcal{X}$ is convex, equicontinuous and compact in $C_H([0,T])$ As $u_n \in \mathcal{X}$, one can extract from $(u_n)_n$ a (not relabeled) subsequence which pointwise

converges to $u : I \to H$ (i.e., $u_n(t) \to u(t)$ in H for each $t \in I$) such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$ for all $t \in I$ and such that $(\dot{u}_n)_n$ $\sigma(L^1_H(I), L^\infty_H(I))$-converges to $\dot{u} \in S^1_L$. Further, the inequality

$$\|u_n(\delta_n(t)) - u_n(t)\| \leq \rho|\delta_n(t) - t|$$

assures us that $\big(u_n(\delta_n(t))\big)_n$ converges in H for each $t \in I$. This and the $\sigma(L^1_H(I), L^\infty_H(I))$ convergence of $(\dot{u}_n)_n$ to $\dot{u}$ in $L^1_H(I)$ along with the inclusion (4.1) allow us (according to the pointwise convergence of $(f_n)_n$ to f and the estimates from the hypotheses) to obtain that for a.e. $t \in I$ the inclusions $\dot{u}(t) \in C(t)$

$$f(t) + w(t) - B\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$$

and

$$w(t) \in Au(t)$$

Our first task is to prove the inclusion $\dot{u}(t) \in C(t)$ a.e. $t \in I$. Indeed, for every Lebesgue measurable set $Z \subset I$ and for any $x \in H$, the function $\mathbf{1}_Z x \in L^\infty_H$. Considering the inequality

$$\langle x, \dot{u}_n(t)\rangle \leq \delta^*(x, C(\theta_n(t)))$$

and integrating on Z gives

$$\int_I \langle \mathbf{1}_Z x, \dot{u}_n(t), \rangle dt = \int_Z \langle x, \dot{u}_n(t)\rangle dt \leq \int_Z \delta^*(x, C(\theta_n(t)))dt.$$

Passing to the limit in this inequality and using the scalar upper semicontinuity of $C(\cdot)$ we obtain

$$\int_I \langle \mathbf{1}_Z x, \dot{u}(t)\rangle dt \leq \limsup_n \int_Z \delta^*(x, C(\theta_n(t)))dt$$

$$\leq \int_Z \limsup_n \delta^*(x, C(\theta_n(t))) \leq \int_Z \delta^*(x, C(t))dt,$$

which is equivalent to

$$\langle x, \dot{u}(t)\rangle \leq \delta^*(x, C(t)) \text{ a.e. } t \in I.$$

By the separability of H and the weak compactness of $C(t)$ (se, e.g., Castaing-Valadier [24, Proposition III- 35]), we get the desired inclusion $\dot{u}(t) \in C(t)$ a.e. Let $h \in L^\infty_H(I)$. We first note that $A_{\lambda_n} u_n(\delta_n(t)) \to w(t)$ by $\mathcal{A}_2$ so that $(w_n(t) : A_{\lambda_n} u_n(\delta_n(t)))$ is a uniformly bounded measurable sequence pointwise converging to w. As consequence, we have

$$\lim_n \int_0^T \langle \dot{u}_n(t), w_n(t)\rangle dt = \int_0^T \langle \dot{u}(t), w(t)\rangle dt.$$

Similarly we note that $B\dot{u}_n \to B\dot{u}$ weakly in $L^1_H(I)$. As a main consequence $f_n + A_{\lambda_n}u_n(\delta_n(t)) - B\dot{u}_n \to f + w - B\dot{u}$ weakly in $L^1_H(I)$. Then we may apply the lower semicontinuity of integral convex functional ([22], Theorem 8.1.16) to deduce that

(1.1.2)

$$\int_Z \delta^*(f(t)+w(t)-B\dot{u}(t), C(t))dt \leq \liminf_n \int_Z \delta^*(f_n(t)+A_{\lambda_n}u_n(\delta_n(t))-B\dot{u}_n(t), C(t))dt$$

for every Lebesque measurable set $Z \subset [0,T]$. This need a careful look. Indeed, we note that $(t,x) \mapsto \delta^*(x, C(t))$ is a normal lower semicontinuous convex integrand defined on $[0,T] \times H$ and $\delta^*(f_n(t) + A_{\lambda_n}u_n(\delta_n(t)) - B\dot{u}_n(t), C(t))$ is measurable and integrable:

$$|\delta^*(f_n(t)+A_{\lambda_n}u_n(\delta_n(t))-B\dot{u}_n(t), C(t))| \leq ||f_n(t)+A_{\lambda_n}u_n(\delta_n(t))-B\dot{u}_n(t)|||L| \leq Constant$$

furthermore

$$\delta^*(f_n(t) + A_{\lambda_n}u_n(\delta_n(t)) - B\dot{u}_n(t), C(t)) \geq \langle f_n(t) + A_{\lambda_n}u_n(\delta_n(t)) - B\dot{u}_n(t), \overline{u}(t)\rangle$$

where $\overline{u}(t)$ is a measurable selection of C (note that C is scalarly upper semicontinuous multimapping from $[0,T]$ to L). Further we have by (1.1.1)

$$|\delta^*(f_n(t)+A_{\lambda_n}u_n(\delta_n(t))-B\dot{u}_n(t), C(t))-\delta^*(f_n(t)+A_{\lambda_n}u_n(\delta_n(t))-B\dot{u}_n(t), C(\theta_n(t)))|$$

$$\leq ||f_n(t) + A_{\lambda_n}u_n(\delta_n(t)) - B\dot{u}_n(t))||d_H(C(t), C(\theta_n(t)))$$

$$\leq ||f_n(t) + A_{\lambda_n}u_n(\delta_n(t)) - B\dot{u}_n(t)|||v(t) - v(\theta_n(t))| \leq Constant|v(t) - v(\theta_n(t))|$$

so that

$$\liminf_n \int_Z \delta^*(f_n(t) + A_{\lambda_n}u_n(\delta_n(t)) - B\dot{u}_n(t), C(\theta_n(t)))dt$$

$$\geq \liminf_n \int_Z \delta^*(f_n(t) + A_{\lambda_n}u_n(\delta_n(t)) - B\dot{u}_n(t), C(t))dt$$

(1.1.3)
$$\geq \int_Z \delta^*(f(t) + w(t) - B\dot{u}(t), C(t))dt.$$

Let us set $\psi_B(x) = \langle Bx, x\rangle$ if $x \in L$ and $\psi(x) = +\infty$ if $x \notin L$. Then it is clear ψ_B is a positive lower semicontinuous convex integrand. Apply again the lower semicontinuity of the integral convex functional ([22], Theorem 8.1.6) associated with the positive normal convex integrand ψ_B we obtain,

$$\liminf_n \int_Z \psi_B(\dot{u}_n(t))dt \geq \int_Z \psi_B(\dot{u}(t))dt$$

that is

(1.1.4)
$$\liminf_n \int_Z \langle B\dot{u}_n(t), \dot{u}_n(t))dt \geq \int_Z \langle B\dot{u}(t), \dot{u}(t)dt.$$

Whence we have

$$\lim_n \int_Z \langle A_{\lambda_n} u_n(\delta_n(t)), \dot{u}_n(t)\rangle dt = \int_Z \langle w(t)), \dot{u}(t)\rangle dt. \tag{1.1.5}$$

Similarly we have

$$\lim_n \int_Z \langle f_n(t), \dot{u}_n(t)\rangle dt = \int_Z \langle f(t), \dot{u}(t)\rangle dt \tag{1.1.6}$$

because f_n is uniformly bounded and pointwise strongly converge to f and $\dot{u}_n \to \dot{u}$ weakly in $L^1_H([0,T])$ by noting that a bounded sequence is $L^\infty_H(I)$ which pointwise converges to 0, converges to 0 uniformly on any uniformly integrable subset of $L^1_H([0,T])$, in other terms it converges to 0 with respect to the Mackey topology $\tau(L^\infty_H([0,T]), L^1_H([0,T]))$ (see[10]).[3] Now integrating on $Z \subset [0,T]$ the inequality (here measurability and integrability are guaranted)

$$\delta^*(f_n(t) + A_{\lambda_n} u_n(\delta_n(t)) - B\dot{u}_n(t), C(\theta_n(t)))$$

$$+\langle -f_n(t) - A_{\lambda_n} u_n(\delta_n(t)), \dot{u}_n(t)\rangle + \langle B\dot{u}_n(t), \dot{u}_n(t) \leq 0$$

gives

$$\int_Z \delta^*(f_n(t) + A_{\lambda_n} u_n(\delta_n(t)) - B\dot{u}_n(t), C(\theta_n(t)))) dt$$

$$+ \int_Z \langle B\dot{u}_n(t), \dot{u}_n(t)\rangle dt + \int_Z \langle -f_n(t) - A_{\lambda_n} u_n(\delta_n(t)), \dot{u}_n(t)\rangle dt \leq 0 \tag{1.1.7}$$

by passing to the limit when n goes to ∞ in this equality using (1.1.1)—(1.1.7) gives

$$\int_Z [\delta^*(f(t) + w(t) - B\dot{u}(t), C(t)) + \langle B\dot{u}(t) - w(t) - f(t), \dot{u}(t)\rangle] dt \leq 0. \tag{1.1.8}$$

As $t \mapsto \delta^*(f(t) + w(t) - B\dot{u}(t), C(t)) + \langle B\dot{u}(t) - w(t) - f(t), \dot{u}(t)\rangle$ is integrable, by (1.1.8) it follow

$$\delta^*(f(t) + w(t) - B\dot{u}(t), C(t)) + \langle B\dot{u}(t) - w(t) - f(t), \dot{u}(t)\rangle \leq 0, \quad a.e. \tag{1.1.9}$$

As $\dot{u}(t) \in C(t)$, we have

$$\delta^*(f(t) + w(t) - B\dot{u}(t), C(t)) \geq \langle f(t) + w(t) - B\dot{u}(t), \dot{u}(t)\rangle$$

[3] If $H = \mathbb{R}^e$, one may invoke a classical fact that on bounded subsets of L^∞_H the topology of convergence in measure coincides with the topology of uniform convergence on uniformly integrable sets, i.e. on relatively weakly compact subsets, alias the Mackey topology. This is a lemma due to Grothendieck [26] [Ch.5 §4 no 1 Prop. 1 and exercice]

that is

$$\delta^*(f(t)+w(t)-B\dot{u}(t),C(t))+\langle -f(t)-w(t)+B\dot{u}(t),\dot{u}(t)\geq 0. \tag{1.1.10}$$

By (1.1.9) and (1.1.10) we get finally

$$\delta^*(f(t)+w(t)-B\dot{u}(t),C(t))=\langle f(t)+w(t)-B\dot{u}(t),\dot{u}(t)\rangle$$

a.e. with $\dot{u}(t)\in C(t)$ a.e. so we conclude that $f(t)+w(t)-B\dot{u}(t))\in N_{C(t)}(\dot{u}(t))$ a.e. It remain to check $w(t)\in Av(t)$ a.e. Indeed $J_{\lambda_n}u_n(\delta_n(t))-\to u(t)$ by writing

$$||J_{\lambda_n}u_n(\delta_n(t))-u(t)||\leq ||J_{\lambda_n}u_n(\delta_n(t))-J_{\lambda_n}u(t)|| \ +||J_{\lambda_n}u(t)-u(t)||$$

$$\leq ||u_n(\delta_n(t))-u(t)||+||J_{\lambda_n}u(t)-u(t)||\to 0$$

From $w_n(t)=A_{\lambda_n}u_n(\delta_n(t))\in AJ_{\lambda_n}u_n(\delta_n(t))$ we have that

$$(w_n(t),J_{\lambda_n}u_n(\delta_n(t)))\in graphA$$

As $graphA$ is weakly strongly sequentially closed that implies $w(t)\in Au(t)$ a.e. □

Remark We cannot expect to have uniqueness of solution. In case when H is separable Hilbert space the result hold true if we replace the operator A is replaced by a linear compact operator, uniqueness of solutions hold true using the coerciveness of the operator B, then result is read as : there exist an absolutely continuous mapping $u:I\to H$ such tthat for a.e.

$$\begin{cases} f(t)+Au(t)-B\frac{du}{dt}(t)\in N_{C(t)}(\frac{du}{dt}(t)) \\ u(0)=u_0 \end{cases}$$

If $A:I\to H$ is a continuous mapping with the growth condition $||Ax||\leq c(1+||x||)$, the result is read as : there exist an absolutely continuous mapping $u:I\to H$ such that for a.e.

$$\begin{cases} f(t)+Au(t)-B\frac{du}{dt}(t)\in N_{C(t)}(\frac{du}{dt}(t)) \\ u(0)=u_0 \end{cases}$$

Proposition 3.11. *Let H be a separable Hilbert space and $I=[0,T]$ Let $\varphi:[0,T]\times H\to]-\infty,+\infty]$ be a normal lower semicontinuous convex integrand for which there exists a convex weakly compact set Γ such that:*
(i) for all $t\in I$, $dom\varphi_t:=\Gamma$;
(ii $\{\varphi(.,u(.)),u\in S^\infty_\Gamma\}$ is uniformly integrable;
(iii) $\varphi(t,x)\leq\varphi(\tau,x)+|v(t)-v(\tau)|$ for all $t,\tau\in[0,T],x\in\Gamma$ where $v:[0,T]\to\mathbb{R}^+$ is a positive nondecreasing continuous function with $v(0)=0$.
Let $B:H\to H$ be a continuous symmetric linear coerciveoperator, let $A_t:H\to$

2^H be a time measurable maximal monotone operator satisfying
$(\mathcal{A})_1$ $|A^0(t,x)| \leq c(1+||x||)$ for all $(t,x) \in I \times H$.
Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions from I into I such that for each $t \in I$ one has $\theta_n(t) \to t$.
Let $(f_n, f)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_H(I)$ with $\|f_n(t)\| \leq \beta, \|f(t)\| \leq \beta$ $(\beta > 0)$ for all $n \in \mathbb{N}$ such that $(f_n(t))_n$ converges to $f(t)$ for each $t \in I$.

Let $(u_n, u)_{n\in\mathbb{N}}$ be a sequence of absolutely continuous mappings

$$u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds, \;\; \dot{u}_n(t) \in \Gamma$$

$$u(t) = u_0 + \int_0^t \dot{u}(s)ds,$$

such that $(u_n)_n$ converges to u uniformly on I and $(\dot{u}_n)_n$ converges to $\dot{u}$ weakly in $L^1_H(I)$. Assume that for every $n \in \mathbb{N}$

$$(^*) f_n(t) + A_{\lambda_n}(t, u_n(t)) - B\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t)) \text{ a.e. } t \in I.$$

$(^{**})$ $\lambda_n \in]0,1] \to 0$ and $A_{\lambda_n}(t, u_n(t)) \to v(t)$ pointwise strongly
Then for a.e. $t \in I$ one has

$$\dot{u}(t) \in \Gamma, \quad f(t) + v(t) - B\,\dot{u}(t) \in \partial\varphi(t, \dot{u}(t)) \text{ and } v(t) \in A(t, u(t))$$

Proof. First, Lemma 3.2 tells us that $\dot{u}(t) \in \Gamma$ for a.e. $t \in I$.

First, we justify that $\dot{u}(t) \in \Gamma$ a.e. $t \in I$. Let (e_p) be a dense sequence in H. For every measurable set $Z \subset I$ and for any $e_p \in H$, the function $\mathbf{1}_Z e_p \in L^\infty_H(I)$. By the inequality

$$\langle e_p, \dot{u}_n(t)\rangle \leq \delta^*(e_p, \Gamma)$$

integrating on Z gives

$$\int_I \langle \mathbf{1}_Z e_p, \dot{u}_n(t)\rangle dt = \int_Z \langle e_p, \dot{u}_n(t)\rangle dt \leq \int_Z \delta^*(e_p, \Gamma)dt.$$

Passing to the upper limit in this inequality we obtain

$$\int_Z \langle \mathbf{1}_Z e_p, \dot{u}(t)\rangle dt \leq \int_Z \delta^*(e_p, \Gamma)dt.$$

This being true for any Lebesgue measurable set $Z \subset I$ we deduce that for every $e_p \in H$

$$\langle e_p, \dot{u}(t)\rangle \leq \delta^*(e_p, \Gamma) \text{ a.e. } t \in I.$$

As H is separable and Γ is closed convex weakly locally weakly compact which contain no line, by (Castaing-Valadier [24, Proposition III- 35]), we get the desired inclusion $\dot{u}(t) \in \Gamma$ a.e. $t \in I$. Take any Lebesgue measurable set $Z \subset I$.

Since $(f_n)_n$ is uniformly bounded and pointwise converges to f and $(\dot{u}_n)_n$ weakly converges in $L^1_H([0,T])$ to $\dot{u}$, we have

$$\lim_n \int_Z \langle f_n(t), \dot{u}_n(t)\rangle \, dt = \int_Z \langle f(t), \dot{u}(t)\rangle \, dt. \tag{3.42}$$

This fact has been already justified in the proof of Proposition 3.4. As φ is normal lower semicontinuous convex integrand, the conjugate function $\varphi^* : I \times H \to]-\infty, +\infty]$

$$\varphi^*(t,y) = \sup_{x\in H}[\langle x, y\rangle - \varphi(t,x)] \tag{3.43}$$

is normal, see e.g Castaing-Valadier [24], and satisfies

$$\varphi^*(t,y) \le \varphi^*(\tau, y) + |v(t) - v(\tau)|$$

for all $t, \tau \in I$, $y \in H$ using assumption (ii) ([36], Proposition 27). By using the normality of φ, the functions $t \mapsto \varphi(\theta_n(t), \dot{u}_n(t))$ and $t \mapsto \varphi(t, \dot{u}_n(t))$ are measurable and integrable.By assumption we have

$$q_n(t) := f_n(t) + A_{\lambda_n}(t, u_n(t)) - B\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t))$$

so that by the normality of φ^*, the function $t \mapsto \varphi^*(\theta_n(t), q_n(t))$ is measurable and integrable, the measurability and integrability of $A_{\lambda_n}(t, u_n(t))$ is already ensured as above. We also note that $\langle q_n, \dot{u}_n\rangle$ is measurable and integrable and the sequence $(\langle q_n, \dot{u}_n\rangle)$ is uniformly integrable.

Further, by (3.42) and condition (ii) we have

$$-\varphi(t, \dot{u}_n(t)) + \langle \dot{u}_n(t), q_n(t)\rangle \le \varphi^*(t, q_n(t)) \le \varphi^*(\theta_n(t), q_n(t)) + |v(t) - v(\theta_n(t))|, \tag{3.44}$$

so that $t \mapsto -\varphi(t, \dot{u}_n(t)) + \langle \dot{u}_n(t), q_n(t)\rangle$ is uniformly integrable thanks to (ii). We note that, for $h_n(t) := f_n(t) + A_{\lambda_n}(t, u_n(t))$, $(h_n)_n$ is uniformly bounded and pointwise converges to h given by $h(t) = f(t) + v(t)$ in H. As consequence, for every measurable set Z in I, we have

$$\lim_{n\to\infty} \int_Z \langle h_n(t), \dot{u}_n(t)\rangle \, dt = \int_Z \langle h(t), \dot{u}(t)\rangle \, dt. \tag{3.45}$$

This fact has been justified in the proof of Proposition 3.4. As B is symmetric, we also show that $B\dot{u}_n(\cdot) \to A\dot{u}(\cdot)$ weakly in $L^1_H(I)$. As consequence $q_n = f_n + A_{\lambda_n}(t, u_n(t)) - B\dot{u}_n(.) \to q := f + v(t) - B\dot{u}$ weakly in $L^1_H(I)$. Further, let us set $\psi_B(x) = \langle Bx, x\rangle$ if $x \in \Gamma$ and $\psi(x) = +\infty$ if $x \notin \Gamma$. Then it is clear ψ is a positive lower semicontinuous convex integrand. Apply again the lower semicontinuity of the integral convex functional ([22] Theorem 8.1.6) associated with the positive normal lower semi continuous convex integrand ψ_A we obtain

$$\liminf_n \int_Z \psi_B(u_n(t))dt \ge \int_Z \psi_B(u(t))\, d, t$$

that is,

$$\liminf_n \int_Z \langle B\dot{u}_n(t), \dot{u}_n(t)) \, dt \geq \int_Z \langle B\dot{u}(t), \dot{u}(t) \, dt. \tag{3.46}$$

Now, we deduce using (3.43) and the lower semicontinuity of integral convex functional (see [22, Theorem 8.1.6]) applied to φ^*,

$$\int_Z \varphi^*(t, q(t)) \, dt \leq \liminf_n \int_Z \varphi^*(t, q_n(t)) \, dt \leq \liminf_n \int_Z \varphi^*(\theta_n(t), q_n(t)) dt \tag{3.47}$$

This fact is justified because $\varphi^*(t, q_n(t)) \geq -\varphi(t, \dot{u}_n(t)) + \langle \dot{u}_n(t), q_n(t) \rangle$, and the sequence $(-\varphi(t, \dot{u}_n(t)) + \langle \dot{u}_n(t), q_n(t) \rangle)$ is uniformly integrable. By

$$\varphi(t, \dot{u}_n(t)) \leq \varphi(\theta_n(t), \dot{u}_n(t)) + |v(t) - v(\theta_n(t))|$$

we also have that

$$\liminf_n \int_Z \varphi(t, \dot{u}_n(t)) \, dt \leq \liminf_n \int_Z \varphi(\theta_n(t), \dot{u}_n(t)) \, dt.$$

As $(\dot{u}_n)_n$ weakly converges to $\dot{u} \in L^1_H(I)$, by the lower semi continuity theorem ([22], Theorem 8.1.6) applied to the lower semicontinuous convex integral functional associated with φ, we derive that

$$\int_Z \varphi(t, \dot{u}(t)) \, dt \leq \liminf_n \int_Z \varphi(\theta_n(t), \dot{u}_n(t)) \, dt \tag{3.48}$$

with $\dot{u}(t) \in \Gamma$ a.e. and $t \mapsto \varphi(t, \dot{u}(t))$ is integrable. Now integrating on the Lebesgue measurable subset Z of I the equality

$$\varphi(\theta_n(t), \dot{u}_n(t)) + \varphi^*(\theta_n(t), q_n(t)) = \langle \dot{u}_n(t), q_n(t) \rangle$$

gives

$$\int_Z \varphi(\theta_n(t), \dot{u}_n(t)) dt + \int_Z \varphi^*(\theta_n(t), q_n(t)) dt + \int_Z \langle A\dot{u}_n(t), \dot{u}_n(t) dt. = \int_Z \langle \dot{u}_n(t), h_n(t) \rangle dt.$$

Passing to the limit as $n \to \infty$ in this equality and using (3.44)–(3.47) give

$$\int_Z \varphi(t, \dot{u}(t)) \, dt + \int_Z \varphi^*(t, q(t)) \, dt \leq \int_Z \langle \dot{u}(t), q(t) \rangle \, dt.$$

By the measurability of the non negative function $t \mapsto \varphi(t, \dot{u}(t)) + \varphi^*(t, q(t)) - \langle \dot{u}(t), q(t) \rangle$, we deduce that for almost every $t \in I$

$$\varphi(t, \dot{u}(t)) + \varphi^*(t, q(t)) - \langle \dot{u}(t), q(t) \rangle \leq 0$$

along with $\dot{u}(t) \in \Gamma$. So, it follows for almost every $t \in I$ that $\varphi(t, \dot{u}(t)) + \varphi^*(t, q(t)) = \langle \dot{u}(t), q(t) \rangle$, or equivalently

$$q(t) = f(t) + v(t) - A\dot{u}(t) \in \partial\varphi(t, \dot{u}(t)).$$

It remain to check $v(t) \in A(t,u(t))$ a.e. Indeed $J_{\lambda_n}(t,u_n(t)) - \to u(t)$ by writing

$$||J_{\lambda_n}(t,u_n(t)) - u(t)|| \leq ||J_{\lambda_n}(t,u(t)) - J_{\lambda_n}(t,u(t))|| \ + ||J_{\lambda_n}(t,u(t)) - u(t)||$$

$$\leq ||u_n(t) - u(t)|| + ||J_{\lambda_n}(t,u(t)) - u(t)|| \to 0$$

As $||u_n(t) - J_{\lambda_n}(t,u_n(t))|| = \lambda_n ||A_{\lambda_n}(t,u_n(t))|| \leq \lambda_n |A^0(t,u_n(t)) \leq c(1+||u_n(t)||)$ with $\lambda_n < 1$, $J_{\lambda_n}(t,u_n(t))$ is uniformly bounded and pointwise converge to $u(t)$, so that $t \to J_{\lambda_n}(t,u_n(t))$ converge to u in $L^2_H(I)$. From $v_n(t) = A_{\lambda_n}(t,u_n(t)) \in A(t, J_{\lambda_n}(t,u_n(t))$ we show that $(v_n(t), J_{\lambda_n}(t,u_n(t)) \in graph A_t$, so that $(v_n, w_n) \in graph \mathcal{A}^*$ with $w_n(t) = J_{\lambda_n}(t,u_n(t))$. As $graph \mathcal{A}^*$ is sequentially strong weakly closed by Lemma 0.1, with $v_n \to v$ strongly hence weakly in $L^2_H(I)$ and $w_n \to u$ strongly in $L^2_H(I)$ applying Lemma 0.1 gives $(v,u) \in graph A^*$, that implies $v(t) \in A(t,u(t))$ a.e. The proof is finished. □

Theorem 3.6. *Let H be a separable Hilbert space.Let K be a convex compact subset of H. Let $\varphi : [0,T] \times K \to]-\infty, +\infty]$ be a normal lower semicontinuous convex integrand such that*
(i $\{\varphi(.,u(.)), u \in S^1_K\}$ is uniformly integrable.
(ii) $\varphi(t,x) \leq \varphi(\tau,x) + |v(t) - v(\tau)|$ for all $t, \tau \in [0,T], x \in K$ where $v : [0,T] \to \mathbb{R}^+$ is a positive nondecreasing continuous function with $v(0) = 0$.
Let $B : H \to H$ be a linear continuous coercive symmetric operator and let $A : H \to 2^H$ be a maximal monotone operator satisfying the properties
$(\mathcal{A})_1$ $|A^0 x)| \leq c(1 + ||x||)$ for all $(t,x) \in I \times H$.
$(\mathcal{A})_2$:If $\lambda_n \to 0^+$, if (u_n, u) is a bounded sequence of measurable mappings on H converging pointwise to u, then $A_{\lambda_n} u_n(.)$ pointwise converge to $v \in L^\infty_H(I)$
Let $f : I \to H$ be a continuous mapping.

Then, for any $u_0 \in H$, there is an absolutely continuous mapping $u : i \to H$ and a bounded measurable mapping $v : I \to H$ such that for a.e.,

$$f(t) + v(t) - B\dot{u}(t) \in \partial\varphi(t, \frac{du}{dt}(t))$$

$$u(0) = u_0$$

$$v(t) \in Au(t)$$

Further the solution set $\{(u,v)\}$ is compact in $C_H(I) \times$ w-$L^2_H(I)$.

Proof. We will use again the Moreau 's catching-up algorithm. We consider for each $n \in \mathbb{N}$ the following partition of the interval $I = [0,T]$.
$t^n_i = i\frac{T}{n} := i\eta_n$ for $0 \leq i \leq n$. $I^n_i :=]t^n_i, t^n_{i+1}]$ for $0 \leq i \leq n-1$.
Put $u^n_0 = u_0$ and $f^n_i = f(t^n_i)$ for all $i = 1, .., n$. By Proposition 1.1 1) , there is $z^n_1 \in K$ such that

$$f^n_1 + A_{\lambda_n} u^n_0 - Bz^n_1 \in \partial\varphi(t^n_1, z^n_1).$$

Put $u_1^n = u_0^n + \eta_n z_1^n$. Suppose that $u_0^n, u_1^n, .., u_i^n, z_1^n, z_2^n, ..z_i^n$ are constructed. As above by Proposition 1.1 1) there exists $z_{i+1}^n \in K$ such that

$$f_{i+1}^n + A_{\lambda_n} u_i^n - B z_{i+1}^n \in \partial\varphi(t_{i+1}^n, z_{i+1}^n).$$

and we set $u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$. Then by induction there are finite sequences $(u_i^n)_{i=0}^n$ and $(z_i^n)_{i=1}^n$ such that

$$f_{i+1}^n + A_{\lambda_n} u_i^n - B z_{i+1}^n \in \partial\varphi(t_{i+1}^n, z_{i+1}^n)$$

$$u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$$

From $(u_i^n)_{i=0}^n$, $(z_i^n)_{i=1}^n$ $(f_i^n)_{i=0}^n$, we construct two sequences u_n from $[0,T]$ to H, f_n from $[0,T]$ to H, by setting $f_n(0) = f_1^n$, $u_n(0) = u_0^n$ and for each $i = 0, .., ..n-1$ we set $f_n(t) = f_{i+1}^n$ and

$$u_n(t) = u_i^n + \frac{t - t_i^n}{\eta_n}(u_{i+1}^n - u_i^n)$$

for $t \in]t_i^n, t_{i+1}^n]$. Clearly, the mapping $u_n(.)$ is Lipschitz continuous on $[0,T]$, and ρ is a Lipschitz constant of $u_n(.)$ on $[0,T]$ since for every $t \in]t_i^n, t_{i+1}^n]$

$$\dot{u}_n(t) = \frac{u_{i+1}^n - u_i^n}{\eta_n} = z_{i+1}^n \in K \subset \rho\overline{B}_H.$$

Furthermore, for every $t \in [0,T]$, one has $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$, hence $||u_n(t)|| \leq ||u_0|| + \rho T$. We have

$$f_{i+1}^n + A_{\lambda_n} u_i^n - B z_{i+1}^n \in \partial\varphi(t_{i+1}^n, z_{i+1}^n).$$

Now, let us define the step functions $\theta_n, \delta_n : I \longrightarrow I$ by

$$\theta_n(t) = t_{i+1}^n, \quad \delta_n(t) = t_i^n$$

if $t \in]t_i^n, t_{i+1}^n]$ and $\theta_n(0) = \delta_n(0) = 0$, and observe that for each $t \in I$, there is $i \in \{0, ..., n-1\}$ such that $t \in [t_i^n, t_{i+1}^n[$, and then,

$$|\theta_n(t) - t| \to 0 \ \text{ and } \ |\delta_n(t) - t| \to 0 \ \ as\ n \to +\infty,$$

So, the last inclusion becomes

$$f_n(t) + A_{\lambda_n} u_n(\delta_n(t)) - B\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t))$$

a.e. $t \in [0,T]$. We note that $||u_n(t)|| \leq ||u_0|| + \rho T$, $||f^n(.)|| \leq \beta$ for all $t \in [0,T]$ and $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$ for all $t \in [0,T]$ with $\dot{u}_n \in K$ a.e.

Step 2 Convergence of the algorithm and final conclusion
Let $S_K^1 := \{h \in L_H^1([0,T]) : h(t) \in K \ a.e.\}$ and let

$$\mathcal{X} := \{v : [0,T] \to H : v(t) = u_0 + \int_0^t \dot{v}(s)ds,\ t \in [0,T];\ \dot{v} \in S_K^1\}.$$

Then it is clear that S_K^1 is convex and weakly compact in $L_H^1([0,T])$ (see e.g. [22] and the references therein) and that $\mathcal{X}$ is convex, equicontinuous and compact in $\mathcal{C}_H([0,T])$. As $(u_n) \subset \mathcal{X}$, one can extract from (u_n) a subsequence not relabelled which pointwise converges to $u : [0,T] \to H$ such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$, for all $t \in [0,T]$ and $(\dot{u}_n)$ $\sigma(L_H^1([0,T]), L_H^\infty([0,T]))$-converges to $\dot{u} \in S_K^1$. By using the normality of φ, the mappings $t \mapsto \varphi(\theta_n(t), \dot{u}_n(t))$ and $t \mapsto \varphi(t, \dot{u}_n(t))$ are measurable and integrable. By construction we have

$$g_n(t) := f_n(t) + A_{\lambda_n} u_n(\delta_n(t)) - B\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t)).$$

by $(\mathcal{A})_2$:let $\lambda_n \to 0^+$, then $A_{\lambda_n} u_n(\delta_n(t))$ pointwise converges to a bounded measurable mapping $v : I \to H$. For simplicity set

$$g(t) := f(t) + v(t) - B\dot{u}(t).$$

As $f_n \to f$ pointwise strongly, $u_n(\delta_n(.)) \to u(.))$ pointwise strongly, and $\dot{u}_n \to \dot{u}$ weakly in $L_E^1([0,T])$, $A_{\lambda_n} u_n(\delta_n(t))$ pointwise converges to v, a direct application of Proposition 3.11 gives

$$g(t) := f(t) + v(t) - B\dot{u}(t) \in \partial\varphi(t, \dot{u}(t))$$

a.e It remain to check $v(t) \in Au(t)$ a.e. Indeed $J_{\lambda_n}(t, u_n(t)) - \to u(t)$ by writing

$$||J_{\lambda_n} u_n(\delta_n(t)) - u(t)|| \leq ||J_{\lambda_n} u_n(\delta_n(t)) - J_{\lambda_n} u(t))|| \ + ||J_{\lambda_n} u(t)) - u(t)||$$

$$\leq ||u_n(\delta_n(t)) - u(t)|| + ||J_{\lambda_n}(t, u(t)) - u(t)|| \to 0$$

From $v_n(t) = A_{\lambda_n} u_n(\delta_n(t)) \in AJ_{\lambda_n} u_n(\delta_n(t))$ we show that $(v_n, w_n) \in graph\mathcal{A}$ with $w_n(t) = J_{\lambda_n} u_n(\delta_n(t)))$. As $graph\mathcal{A}$ is sequentially strong weakly closed, with $v_n \to v$ strongly pointwise in H and $w_n \to u$ strongly pointwise in H this gives $(v, u) \in graphA$, that implies $v(t) \in Au(t)$ a.e. and finish the proof. The compactness of solution set follows from the arguments given in the variational limit theorem, cf Proposition 0.7. □

Theorem 3.7. *Let H be a separable Hilbert space. Let K be a convex compact subset of H. Let $\varphi : [0,T] \times K \to]-\infty, +\infty]$ be a normal lower semicontinuous convex integrand such that*
(i $\{\varphi(., u(.)), u \in S_K^1\}$ is uniformly integrable.
(ii) $\varphi(t,x) \leq \varphi(\tau, x) + |v(t) - v(\tau)|$ for all $t, \tau \in [0,T], x \in K$ where $v : [0,T] \to \mathbb{R}^+$ is a positive nondecreasing continuous function with $v(0) = 0$.
Let $B : H \to H$ be a linear continuous coercive symmetric operator. Let $f : I \to H$ be a continuous mapping. Let $g : H \to \mathbb{R}$ be a convex function Lipschitz on bounded sets and continuously differentiable on H.
Then, for any $u_0 \in H$, there is an absolutely continuous mapping $u : I \to H$ such that for a.e.,

$$f(t) + \nabla g(u(t)) - B\dot{u}(t) \in \partial\varphi(t, \frac{du}{dt}(t))$$

$$u(0) = u_0$$

Further the solution set is compact in $C_H(I)$.

Proof. We will use again the Moreau 's catching-up algorithm. We consider for each $n \in \mathbb{N}$ the following partition of the interval $I = [0, T]$.
$t_i^n = i\frac{T}{n} := i\eta_n$ for $0 \leq i \leq n$. $I_i^n :=]t_i^n, t_{i+1}^n]$ for $0 \leq i \leq n-1$.
Put $u_0^n = u_0$ and $f_i^n = f(t_i^n)$ for all $i = 1, .., n$. By Proposition 1.1 1) , there is $z_1^n \in K$ such that

$$f_1^n + \nabla g(u_0^n) - Bz_1^n \in \partial\varphi(t_1^n, z_1^n).$$

Put $u_1^n = u_0^n + \eta_n z_1^n$. Suppose that $u_0^n, u_1^n, .., u_i^n, z_1^n, z_2^n, ..z_i^n$ are constructed. As above by Proposition 1.1 1) there exists $z_{i+1}^n \in K$ such that

$$f_{i+1}^n + \nabla g(u_i^n) - Bz_{i+1}^n \in \partial\varphi(t_{i+1}^n, z_{i+1}^n).$$

and we set $u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$. Then by induction there are finite sequences $(u_i^n)_{i=0}^n$ and $(z_i^n)_{i=1}^n$ such that

$$f_{i+1}^n + \nabla g(u_i^n) - Bz_{i+1}^n \in \partial\varphi(t_{i+1}^n, z_{i+1}^n)$$

$$u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$$

From $(u_i^n)_{i=0}^n$, $(z_i^n)_{i=1}^n$ $(f_i^n)_{i=0}^n$, we construct two sequences u_n from $[0, T]$ to H, f_n from $[0, T]$ to H, by setting $f_n(0) = f_1^n$, $u_n(0) = u_0^n$ and for each $i = 0, .., ..n-1$ we set $f_n(t) = f_{i+1}^n$ and

$$u_n(t) = u_i^n + \frac{t - t_i^n}{\eta_n}(u_{i+1}^n - u_i^n)$$

for $t \in]t_i^n, t_{i+1}^n]$. Clearly, the mapping $u_n(.)$ is Lipschitz continuous on $[0, T]$, and ρ is a Lipschitz constant of $u_n(.)$ on $[0, T]$ since for every $t \in]t_i^n, t_{i+1}^n]$

$$\dot{u}_n(t) = \frac{u_{i+1}^n - u_i^n}{\eta_n} = z_{i+1}^n \in K \subset \rho\overline{B}_H.$$

Furthermore, for every $t \in [0, T]$, one has $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$, hence $||u_n(t)|| \leq ||u_0|| + \rho T$. We have

$$f_{i+1}^n + \nabla g(u_i^n) - Bz_{i+1}^n \in \partial\varphi(t_{i+1}^n, z_{i+1}^n).$$

Now, let us define the step functions $\theta_n, \delta_n : I \longrightarrow I$ by

$$\theta_n(t) = t_{i+1}^n, \quad \delta_n(t) = t_i^n$$

if $t \in]t_i^n, t_{i+1}^n]$ and $\theta_n(0) = \delta_n(0) = 0$, and observe that for each $t \in I$, there is $i \in \{0, ..., n-1\}$ such that $t \in [t_i^n, t_{i+1}^n[$, and then,

$$|\theta_n(t) - t| \to 0 \ \text{ and } \ |\delta_n(t) - t| \to 0 \ \ as\ n \to +\infty,$$

So, the last inclusion becomes

$$f_n(t) + \nabla g(u_n(\delta_n(t))) - B\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t))$$

a.e. $t \in [0,T]$. We note that $||u_n(t)|| \leq ||u_0|| + \rho T$, $||f^n(.)|| \leq \beta$ for all $t \in [0,T]$ and $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$ for all $t \in [0,T]$ with $\dot{u}_n \in K$ a.e.

Step 2 Convergence of the algorithm and final conclusion
Let $S_K^1 := \{h \in L_H^1([0,T]) : h(t) \in K \ a.e.\}$ and let

$$\mathcal{X} := \{v : [0,T] \to H : v(t) = u_0 + \int_0^t \dot{v}(s)ds,\ t \in [0,T];\ \dot{v} \in S_K^1\}.$$

Then it is clear that S_K^1 is convex and weakly compact in $L_H^1([0,T])$ (see e.g. [22] and the references therein) and that $\mathcal{X}$ is convex, equicontinuous and compact in $\mathcal{C}_H([0,T])$. As $(u_n) \subset \mathcal{X}$, one can extract from (u_n) a subsequence not relabelled which pointwise converges to $u : [0,T] \to H$ such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$, for all $t \in [0,T]$ and $(\dot{u}_n)$ $\sigma(L_H^1([0,T]), L_H^\infty([0,T]))$-converges to $\dot{u} \in S_K^1$. By using the normality of φ, the mappings $t \mapsto \varphi(\theta_n(t), \dot{u}_n(t))$ and $t \mapsto \varphi(t, \dot{u}_n(t))$ are measurable and integrable. By construction we have

$$g_n(t) := f_n(t) + \nabla g(u_n(\delta_n(t))) - B\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t)).$$

For simplicity set

$$g(t) := f(t) + \nabla g(u(t)) - B\dot{u}(t).$$

As $f_n \to f$ pointwise strongly, $u_n(\delta_n(.)) \to u(.))$ pointwise strongly, and $\dot{u}_n \to \dot{u}$ weakly in $L_E^1([0,T])$, $\nabla g(u_n(\delta_n(t)))$ pointwise converges to $\nabla g(u(t))$, a direct application of Proposition 3.6 gives

$$g(t) := f(t) + \nabla g(u(t)) - B\dot{u}(t) \in \partial\varphi(t, \dot{u}(t))$$

a.e and finish the proof. □

Lemma 3.8. *Let a time measurable maximal monotone operator $t \to A_t : H \to H$ satisfying the conditions.*
($\mathcal{A}_1$) $t \to J_\lambda(t,x)$ is $\mathcal{L}(I)$-measurable for every $\lambda > 0$ and for every $x \in H$
($\mathcal{A}_2$) $|A^0(t,x)| \leq c(1 + ||x||)$ for all $(t,x) \in I \times H$.
If (u_n) is a uniformly bounded sequence of Lebesgue measurable mappings pointwise converging to measurable mapping u, for $\lambda \in]0,1], \lambda_n \to 0^+$, $A_{\lambda_n}(t, u_n(t))$ weakly converge in $L_H^2(I)$ to a bounded measurable mapping v, then $v(t) \in A(t, u(t))$ a.e.

Proof. Let us set $v_n(t) := A_{\lambda_n}(t, u_n(t))$. From Lemma we are ensured that (v_n) is bounded measurable and (v_n) wealy converge to v in $L_H^2(I)$. We have $v_n(t) := A_{\lambda_n}(t, u_n(t)) \in A(t, J_\lambda(t, u_n(t)))$ with $w_n(t) := J_\lambda(t, u_n(t))) \to u(t))$ pointwise. Indeed $J_{\lambda_n}(t, u_n(t)) - \to u(t)$ by writing

$$||J_{\lambda_n}(t, u_n(t)) - u(t)|| \leq ||J_{\lambda_n}(t, u(t)) - J_{\lambda_n}(t, u(t))|| \ + ||J_{\lambda_n}(t, u(t)) - u(t)||$$

$$\leq ||u_n(t) - u(t)|| + ||J_{\lambda_n}(t, u(t)) - u(t)|| \to 0$$

As $||u_n(t) - J_{\lambda_n}(t, u_n(t))|| = \lambda_n ||A_{\lambda_n}(t, u_n(t))|| \leq \lambda_n |A^0(t, u_n(t)) \leq c(1 + ||u_n(t)||)$ with $\lambda_n < 1$, $J_{\lambda_n}(t, u_n(t))$ is uniformly bounded and pointwise converge to $u(t)$, so that $t \to J_{\lambda_n}(t, u_n(t))$ converge to u in $L^2_H(I)$. Then from $v_n(t) \in A(t, w_n(t))$ with $v_n \to v$ weakly in $L^2_H(I)$ and $w_n \to u$ strongly in $L^2_H(I)$ and $(v_n, w_n) \in \mathcal{A}^*$. by Lemma 0.1, $(v, u) \in \mathcal{A}^*$ i.e $v(t) \in A(t, u(t))$ a.e. □

4 Well-posedness of inclusion (1.1)

Our main proofs in this section are build upon the variational inequalities in Proposition 2.1 and the variational limits in Section 3 as well as upon an explicit catching-up algorithm (alias Moreau's algorithm). We stress the fact that our algorithm and tools are self contained apart from the use of the mentioned variational inequalities.

Theorem 4.1. *Let $f : [0, T] \to H$ be a continuous mapping and let $v : [0, T] \to \mathbb{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : [0, T] \to H$ be a weakly compact convex valued multimapping such that*

$$\text{haus}\,(C(t), C(\tau)) \leq |v(t) - v(\tau)| \quad \textit{for all}\, t, \tau \in [0, T].$$

Let $A : H \to H$ be a linear continuous coercive symmetric operator and let $B : H \to H$ be a linear continuous compact operator. Then, for any $u_0 \in H$, the evolution inclusion

$$\begin{cases} f(t) + Bu(t) - A\,\frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t)) \\ u(0) = u_0 \end{cases}$$

admits a unique $W^{1,\infty}_H([0, T])$ solution $u : [0, T] \to H$. Further, one has $\|\dot{u}(t)\| \leq \rho$, where $\rho := \sup\{\|y\| : y \in C([0, T])\}$.

Proof. Put $I := [0, T]$ and denote $\beta := \max\{\|f(t)\| : t \in I\}$ (by continuity of f). Noticing that the multimapping $C(\cdot)$ is upper semicontinuous from I into H endowed with the weak topology, the set $C(I)$ is weakly compact, and hence $\rho := \sup\{\|y\| : y \in C(I)\}$ is finite and $L := \rho \mathbb{B}_H$ is weakly compact and convex.

Step I. Construction of a sequence $(u_n)_n$
We will use the Moreau 's catching-up algorithm [33]. We consider for each $n \in \mathbb{N}$ the following partition of the interval $]0, T]$ given by

$$t^n_i = i\frac{T}{n} := i\eta_n \text{ for } 0 \leq i \leq n, \quad I^n_i :=]t^n_i, t^n_{i+1}] \text{ for } 0 \leq i \leq n-1.$$

Put $u_0^n = u_0$ and $f_i^n = f(t_i^n)$ for all $i = 1, .., n$. By Proposition 2.1(b), there is $z_1^n \in C(t_1^n) \subset L$ such that

$$f_1^n + Bu_0^n - Az_1^n \in N_{C(t_1^n)}(z_1^n).$$

Put $u_1^n = u_0^n + \eta_n z_1^n$. Suppose that $u_0^n, u_1^n, .., u_i^n, z_1^n, z_2^n, ..z_i^n$ are constructed. As above by Proposition 2.1(b) there exists $z_{i+1}^n \in C(t_{i+1}^n) \subset L$ such that

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in N_{C(t_{i+1}^n)}(z_{i+1}^n),$$

and we set $u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$. Then by induction there are finite sequences $(u_i^n)_{i=0}^n$ and $(z_i^n)_{i=1}^n$ such that

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in N_{C(t_{i+1}^n)}(z_{i+1}^n) \tag{4.1}$$

$$u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n.$$

From $(u_i^n)_{i=0}^n$, $(z_i^n)_{i=1}^n$ $(f_i^n)_{i=0}^n$, we construct two sequences of mappings $(u_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ from I to H, by setting $f_n(0) = f_1^n$, $u_n(0) = u_0^n$ and for each $i = 0, .., ..n-1$ we set $f_n(t) = f_{i+1}^n$ and

$$f_n(t) = f_{i+1}^n \quad \text{and} \quad u_n(t) = u_i^n + \frac{t - t_i^n}{\eta_n}(u_{i+1}^n - u_i^n) \quad \text{for } t \in]t_i^n, t_{i+1}^n].$$

Keeping in mind that $C(t) \subset L = \rho\mathbb{B}_H$ we have $u_i \in C(t_{i+1}^n) \subset \rho\mathbb{B}_H$, so

$$\left\| \frac{1}{\eta_n}(u_{i+1}^n - u_i^n) \right\| = \|z_{i+1}^n\| \leq \rho.$$

From this it is clear that $u_n(\cdot)$ is Lipschitz continuous on I with ρ as a Lipschitz constant. This Lipschitz property of $u_n(\cdot)$ ensures that $\|u_n(t)\| \leq \|u_0\| + \rho T$ and $u_n(t) = u_0 + \int_{t_0}^t \dot{u}_n(s)ds$ for every $t \in I$. We also note that $\|f_n(t)\| \leq \beta$ for all $n \in \mathbb{N}$ and $t \in I$. Now, let us define the step functions $\theta_n, \delta_n : I \longrightarrow I$ by $\theta_n(0) = \delta_n(0) = 0$ and

$$\theta_n(t) = t_{i+1}^n, \quad \delta_n(t) = t_i^n \quad \text{if } t \in]t_i^n, t_{i+1}^n],$$

so the inclusion (4.1) becomes

$$f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t) \in N_{C(\theta_n(t))}(\dot{u}_n(t)) \text{ a.e. } t \in I.$$

For each $t \in I$ we observe that there is some $i \in \{0, ..., n-1\}$ such that $t \in [t_i^n, t_{i+1}^n[$, and then

$$|\theta_n(t) - t| \to 0 \text{ and } |\delta_n(t) - t| \to 0 \text{ as } n \to +\infty.$$

We also note that the latter inclusion above yields

$$\delta^*(f_n(t)+Bu_n(\delta_n(t))-A\dot{u}_n(t), C(\theta_n(t)))+\langle -f_n(t)-Bu_n(\delta_n(t))+A\dot{u}_n(t), \dot{u}_n(t)\rangle \leq 0$$

with

$$\dot{u}_n(t) \in C(\theta_n(t)) \subset L \text{ a.e. } t \in I,$$

so that $\dot{u}_n \in S_L^1$ where $S_L^1 := \{\xi \in L_H^1(I) : \xi(t) \in L \text{ a.e. } t \in I\}$.

Step II. Convergence to a solution.
We note that S_L^1 is a weakly compact convex set of $L_H^1(I)$ (see, e.g., [22] and the references therein). Set

$$\mathcal{X} := \{\xi : I \to H : \xi(t) = u_0 + \int_0^t \dot{\xi}(s)ds,\ t \in [0,T];\ \dot{v} \in S_L^1\}.$$

It is clear that $\mathcal{X}$ is convex, equicontinuous and weakly compact [40] in $C_H([0,T])$ (see [40]). As $u_n \in \mathcal{X}$, one can extract from $(u_n)_n$ a (not relabeled) subsequence which pointwise weakly converges to $u : I \to H$ (i.e., $u_n(t) \to u(t)$ weakly in H for each $t \in I$) such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$ for all $t \in I$ and such that $(\dot{u}_n)_n$ $\sigma(L_H^1(I), L_H^\infty(I))$-converges to $\dot{u} \in S_L^1$. Further, the inequality

$$\|u_n(\delta_n(t)) - u_n(t)\| \leq \rho|\delta_n(t) - t|$$

assures us that $\big(u_n(\delta_n(t))\big)_n$ converges weakly in H for each $t \in I$. This and the $\sigma(L_H^1(I), L_H^\infty(I))$ convergence of $(\dot{u}_n)_n$ to $\dot{u}$ in $L_H^1(I)$ along with the inclusion (4.1) allow us (according to the pointwise convergence of $(f_n)_n$ to f and the estimates from the hypotheses) to apply Proposition 3.1, with $v_n = u_n \circ \delta_n$ and $\zeta_n = \dot{u}_n$, to obtain that for a.e. $t \in I$ the inclusions $\dot{u}(t) \in C(t)$ and

$$f(t) + Bu(t) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$$

hold true. This says that $u(\cdot)$ is solution of the inclusion of the theorem.

STEP III. Uniqueness.
The uniqueness of solutions follows easily from the coerciveness of the operator A. Indeed let u_1 and u_2 be two solutions. An easy computation gives

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t)\rangle + \langle Bu_2(t) - Bu_1(t), \dot{u}_2(t) - \dot{u}_1(t)\rangle \leq 0,$$

so that

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t)\rangle \leq |B|\,\|u_2(t) - u_1(t)\|\,\|\dot{u}_2(t) - \dot{u}_1(t)\|.$$

By coerciveness of A we deduce that

$$\omega\|\dot{u}_2(t) - \dot{u}_1(t)\|^2 \leq |B|\,\|u_2(t) - u_1(t)\|\,\|\dot{u}_2(t) - \dot{u}_1(t)\|.$$

This entails that

$$\|\dot{u}_2(t) - \dot{u}_1(t)\| \leq \frac{|B|}{\omega}\|u_2(t) - u_1(t)\| \leq \frac{|B|}{\omega}\int_0^t \|\dot{u}_2(s) - \dot{u}_1(s)\|ds.$$

By Gronwall lemma $\dot{u}_1(t) = \dot{u}_2(t)$ a.e. $t \in I$, and so $u_1(t) = u_2(t)$ for every $t \in I$ since $u_1(t) = u_0 + \int_0^t \dot{u}_1(s)ds$ and $u_2(t) = u_0 + \int_0^t \dot{u}_2(s)ds$. □

Remark 4.2. The tools developed above allow to obtain further variants. The fact that $C(t)$ is weakly compact is required, and mainly the coerciveness of A and the compactness assumption for the operator B. An inspection of the proof of Theorem 4.1, shows that the compactness assumption on B is required to prove the Fatou property,

$$\liminf_n \int_Z \langle B(u_n(\delta_n(t)), \dot{u}_n(t)\rangle dt \geq \int_Z \langle Bu(t), \dot{u}(t)\rangle dt$$

So as a possible variant we may substitute the bounded operator B by the gradient ∇g of a positive convex continuous Gateaux differentiable function $g : H \to \mathbb{R}$ such that $g(v(t))$ is absolutely continuous for $v : [0,T] \to H$ absolutely continuous, so that by invoking the chain rule formula, see Moreau-Valadier, [35], we have the equality

$$\langle \nabla g(v(t)), \dot{v}(t)\rangle = \frac{d}{dt} g(v(t))$$

Hence by using this fact and the tool developed in Theorem 1.1, we obtain a variant of Theorem 1.1 by noting that

$$\liminf_n \int_0^T \langle \nabla g(u_n(t)), \dot{u}_n(t) dt = \liminf_n \int_0^T \frac{d}{dt} g(u_n(t)) dt$$

$$\geq \int_0^T \frac{d}{dt} g(u(t)) dt = \int_0^T \langle \nabla g(u(t)), \dot{u}(t) dt$$

It is obvious that a linear continuous operator and a gradient do not enjoy similar properties, showing the interest of the new variant we give further. This remark has some importance in further developments.

Now we present a variant dealing with the existence and uniqueness of absolutely continuous solution to the evolution inclusion of the form

$$f(t) + Bu(t) - A\dot{u}(t) \in N_{C(t)}(t, \frac{du}{dt}(t))$$

where f is a continuous mapping $f : I \to H$, A is a coercive symmetric operator, and $B : H \to H$ is a Lipschitz mapping.

Theorem 4.3. *Let $f : [0,T] \to H$ be a continuous mapping and let $v : [0,T] \to \mathbb{R}^+$ be a non-negative nondecreasing continuous function with $v(0) = 0$. Let $C : [0,T] \to H$ be a strongly compact convex valued multimapping such that* $\mathrm{haus}(C(t), C(\tau)) \leq |v(t) - v(\tau)|$ *for all $t, \tau \in [0,T]$. Let $A : H \to H$ be a linear continuous coercive symmetric operator and let $B : H \to H$ be a Lipschitz mapping, that is, for some real constant $M > 0$, $\|Bx - By\| \leq M\|x - y\|$ for all $x, y \in H$ for some positive constant M. Then, for any $u_0 \in H$, the evolution inclusion*

$$\begin{cases} f(t) + Bu(t) - A\frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t)) \\ u(0) = u_0 \end{cases}$$

admits a unique $W^{1,\infty}_H([0,T])$ *solution* $u:[0,T]\to H$. *Further,* $\|\dot u(t)\|\le\rho$ *a.e.* $t\in[0,T]$, *where* $\rho:=\max\{\|y\|:y\in C([0,T])\}$.

Proof. Put $I:=[0,T]$ and denote $\beta:=\max\{\|f(t)\|:t\in I\}$ (by continuity of f). Noticing that the multimapping $C(\cdot)$ is upper semicontinuous from I into H endowed with the norm topology, the set $C(I)$ is norm compact, and hence $\rho:=\sup\{\|y\|:y\in C(I)\}$ is finite and $L:=\overline{\rm co}\,\big(C(I)\big)$ is convex and norm compact.

Step I. The sequence $(u_n)_n$ is constructed as in Theorem 4.1.

Step II. With the strongly compact set $L=\overline{\rm co}\,\big(C(I)\big)$ at hands, we see that the set $\mathcal{X}$ in the proof of Theorem 4.1 is strongly compact in $C_H(I)$. Since $u_n\in\mathcal{X}$ we can extract from $(u_n)_n$ a (not relabeled) sequence which pointwise converges to $u:I\to H$ (i.e., $u_n(t)\to u(t)$ strongly in H for each $t\in I$) such that $u(t)=u_0+\int_0^t\dot u(s)ds$, for all $t\in I$ and $(\dot u_n)_n$ $\sigma(L^1_H([0,T]),L^\infty_H([0,T]))$-converges to $\dot u\in S^1_L$. The inequality

$$\|u_n(\delta_n(t))-u_n(t)\|\le\rho|\delta_n(t)-t|$$

ensures that the sequence $\big(u_n(\delta_n(t))\big)_n$ strongly converges to $u(t)$ for each $t\in I$. Consequently, we can follow **Step II** in the proof of Theorem 4.1 by applying Proposition 3.2 in place of Proposition 3.1, to arrive to the fact that $u(\cdot)$ is a solution of the inclusion in the present theorem.

Step III. The arguments for the uniqueness are the same as for Theorem 4.1. □

Similarly, in the proof of Theorem 4.1 employing Proposition 3.10 instead of Proposition 3.1 we easily obtain the following case with the gradient ∇g of a convex function g in place of B.

Theorem 4.4. *Let* $f:[0,T]\to H$ *be a continuous mapping and let* $v:[0,T]\to\mathbb{R}^+$ *be a positive nondecreasing continuous function with* $v(0)=0$. *Let* $C:[0,T]\to H$ *be a strongly compact convex valued multimapping such that*

$$\mathrm{haus}\,(C(t),C(\tau))\le|v(t)-v(\tau)|\quad\textit{for all}\,t,\tau\in[0,T].$$

Let $A:H\to H$ *be a linear continuous coercive symmetric operator and let* g *be a convex function on* H *Lipschitz on bounded sets and continuously differentiable on* H. *Then, for any* $u_0\in H$, *the evolution inclusion*

$$\begin{cases} f(t)+\nabla g(u(t))-A\,\frac{du}{dt}(t)\in N_{C(t)}(\frac{du}{dt}(t))\\ u(0)=u_0\end{cases}$$

admits at least a $W^{1,\infty}_H([0,T])$ *solution* $u:[0,T]\to H$. *Further, one has* $\|\dot u(t)\|\le\rho$, *where* $\rho:=\sup\{\|y\|:y\in C([0,T])\}$.

We present another variant dealing with the existence and uniqueness of absolutely continuous solution to the evolution inclusion of the form

$$f(t) + Bu(t) - A\dot{u}(t) \in \partial\varphi(t, \frac{du}{dt}(t))$$

where f is a bounded continuous mapping $f : I \to H$, A is a coercive symmetric operator, and $B : H \to H$ be a linear continuous mapping $\partial\varphi$ is the subdifferential of a normal lower semicontinuous convex integrand φ.

Theorem 4.5. *Let H be a separable Hilbert space.Let K be a convex compact subset of H. Let $\varphi : [0,T] \times K \to]-\infty, +\infty]$ be a normal lower semicontinuous convex integrand such that*
(i $\{\varphi(.,u(.)), u \in S^1_K\}$ is uniformly integrable.
(ii) $\varphi(t,x) \leq \varphi(\tau,x) + |v(t) - v(\tau)|$ for all $t, \tau \in [0,T], x \in K$ where $v : [0,T] \to \mathbb{R}^+$ is a positive nondecreasing continuous function with $v(0) = 0$.
Let $A : H \to H$ be a linear continuous coercive symmetric operator and $B : H \to H$ be a linear continuous mapping.

Then, for any $u_0 \in H$, the evolution inclusion

$$f(t) + Bu(t) - A\dot{u}(t) \in \partial\varphi(t, \frac{du}{dt}(t))$$

$$u(0) = u_0$$

admits a unique $W^{1,\infty}_H([0,T])$ solution $u : [0,T] \to H$.

Proof. We will use again the Moreau 's catching-up algorithm. We consider for each $n \in \mathbb{N}$ the following partition of the interval $I = [0,T]$.
$t^n_i = i\frac{T}{n} := i\eta_n$ for $0 \leq i \leq n$. $I^n_i :=]t^n_i, t^n_{i+1}]$ for $0 \leq i \leq n-1$.
Put $u^n_0 = u_0$ and $f^n_i = f(t^n_i)$ for all $i = 1, .., n$. By Proposition 1.1 1) , there is $z^n_1 \in K$ such that

$$f^n_1 + Bu^n_0 - Az^n_1 \in \partial\varphi(t^n_1, z^n_1).$$

Put $u^n_1 = u^n_0 + \eta_n z^n_1$. Suppose that $u^n_0, u^n_1, .., u^n_i, z^n_1, z^n_2, ..z^n_i$ are constructed. As above by Proposition 1.1 1) there exists $z^n_{i+1} \in K$ such that

$$f^n_{i+1} + Bu^n_i - Az^n_{i+1} \in \partial\varphi(t^n_{i+1}, z^n_{i+1}).$$

and we set $u^n_{i+1} = u^n_i + \eta_n z^n_{i+1}$. Then by induction there are finite sequences $(u^n_i)^n_{i=0}$ and $(z^n_i)^n_{i=1}$ such that

$$f^n_{i+1} + Bu^n_i - Az^n_{i+1} \in \partial\varphi(t^n_{i+1}, z^n_{i+1})$$

$$u^n_{i+1} = u^n_i + \eta_n z^n_{i+1}$$

From $(u_i^n)_{i=0}^n$, $(z_i^n)_{i=1}^n$ $(f_i^n)_{i=0}^n$, we construct two sequences u_n from $[0,T]$ to H, f_n from $[0,T]$ to H, by setting $f_n(0) = f_1^n$, $u_n(0) = u_0^n$ and for each $i = 0,..,..n-1$ we set $f_n(t) = f_{i+1}^n$ and

$$u_n(t) = u_i^n + \frac{t - t_i^n}{\eta_n}(u_{i+1}^n - u_i^n)$$

for $t \in]t_i^n, t_{i+1}^n]$. Clearly, the mapping $u_n(.)$ is Lipschitz continuous on $[0,T]$, and ρ is a Lipschitz constant of $u_n(.)$ on $[0,T]$ since for every $t \in]t_i^n, t_{i+1}^n]$

$$\dot{u}_n(t) = \frac{u_{i+1}^n - u_i^n}{\eta_n} = z_{i+1}^n \in K \subset \rho \overline{B}_H.$$

Furthermore, for every $t \in [0,T]$, one has $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$, hence $||u_n(t)|| \leq ||u_0|| + \rho T$. We have

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in \partial\varphi(t_{i+1}^n, z_{i+1}^n).$$

Now, let us define the step functions $\theta_n, \delta_n : I \longrightarrow I$ by

$$\theta_n(t) = t_{i+1}^n, \;\; \delta_n(t) = t_i^n$$

if $t \in]t_i^n, t_{i+1}^n]$ and $\theta_n(0) = \delta_n(0) = 0$, and observe that for each $t \in I$, there is $i \in \{0,...,n-1\}$ such that $t \in [t_i^n, t_{i+1}^n[$, and then,

$$|\theta_n(t) - t| \to 0 \;\text{ and }\; |\delta_n(t) - t| \to 0 \;\; as \; n \to +\infty,$$

So, the last inclusion becomes

$$f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t))$$

a.e. $t \in [0,T]$. We note that $||u_n(t)|| \leq ||u_0|| + \rho T$, $||f^n(.)|| \leq \beta$ for all $t \in [0,T]$ and $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$ for all $t \in [0,T]$ with $\dot{u}_n \in K$ a.e.

Step 2 Convergence of the algorithm and final conclusion
Let $S_K^1 := \{h \in L_H^1([0,T]) : h(t) \in K \; a.e.\}$ and let

$$\mathcal{X} := \{v : [0,T] \to H : v(t) = u_0 + \int_0^t \dot{v}(s)ds, \; t \in [0,T]; \; \dot{v} \in S_K^1\}.$$

Then it is clear that S_K^1 is convex and weakly compact in $L_H^1([0,T])$ (see e.g. [22] and the references therein) and that $\mathcal{X}$ is convex, equicontinuous and compact in $\mathcal{C}_H([0,T])$. As $(u_n) \subset \mathcal{X}$, one can extract from (u_n) a subsequence not relabelled which pointwise converges to $u : [0,T] \to H$ such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$, for all $t \in [0,T]$ and $(\dot{u}_n)$ $\sigma(L_H^1([0,T]), L_H^\infty([0,T]))$-converges to $\dot{u} \in S_K^1$. As φ is normal lower semicontinuous convex, the conjugate function $\varphi^* : [0,T] \times H \to]-\infty, +\infty]$

$$\varphi^*(t,y) = \sup_{x \in K}[\langle x, y\rangle - \varphi(t,x)]$$

is normal, see e.g Castaing-Valadier [24] and satisfies

$$\varphi^*(t,y) \leq \varphi^*(\tau,y) + |v(t) - v(\tau)|$$

for all $t, \tau \in [0,T], y \in H$ using assumption (ii) ([36], Proposition 27).

By using the normality of φ, the mappings $t \mapsto \varphi(\theta_n(t), \dot{u}_n(t))$ and $t \mapsto \varphi(t, \dot{u}_n(t))$ are measurable and integrable. By construction we have

$$g_n(t) := f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t)).$$

For simplicity set

$$g(t) := f(t) + Bu(t)) - A\dot{u}(t).$$

As $f_n \to f$ pointwise strongly, $u_n(\delta_n(.)) \to u(.))$ pointwise strongly, and $\dot{u}_n \to \dot{u}$ weakly in $L^1_E([0,T])$, a direct application of Proposition 3.11 and its remark gives

$$g(t) := f(t) + Bu(t)) - A\dot{u}(t) \in \partial\varphi(t, \dot{u}(t))$$

a.e and finish the proof. □

Remarks 1) The uniqueness of solutions follows easily from the coerciveness of the operator A. Indeed let u_1 and u_2 two solutions. then by an easy computation,

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t)\rangle + \langle Bu_2(t) - Bu_1(t), \dot{u}_2(t) - \dot{u}_1(t)\rangle \leq 0$$

so that

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t)\rangle \leq |B|||u_2(t) - u_1(t)||||\dot{u}_2(t) - \dot{u}_1(t)||$$

By coerciveness, we deduce that

$$\omega||\dot{u}_2(t) - \dot{u}_1(t)||^2 \leq |B|||u_2(t) - u_1(t)||||\dot{u}_2(t) - \dot{u}_1(t)||$$

Whence

$$||\dot{u}_2(t) - \dot{u}_1(t)|| \leq \frac{|B|}{\omega}||u_2(t) - u_1(t)|| \leq \frac{|B|}{\omega}\int_0^t ||\dot{u}_2(s-)\dot{u}_1(s)||ds$$

By Gronwall lemma $\dot{u}_1(t) = \dot{u}_2(t)$ a and so $u_1(t) = \dot{u}_2(t)$ since $u_1(t) = u_0 + \int_0^t \dot{u}_1(s)ds, \forall t \in [0,T]$, $u_2(t) = u_0 + \int_0^t \dot{u}_2(s)ds, \forall t \in [0,T]$.

2) Theorem 4.5 holds if we replace the operator B by the gradient ∇g of a smooth function g.

3) Theorem 4.5 generalizes Theorem 6 in [19] dealing with finite dimensional space.

5 Applications

5.1 A Skorokhod problem

We present at first a new version of the Skorokhod problem in Castaing et al [18, 20] dealing with the sweeping process associated with an absolutely continuous (or continuous) closed convex moving set $C(t)$ in H. Here the novelty is the velocity inside the nrmal cone operator. We will denote, as usual, by $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ the space of linear mappings Λ from $\mathbb{R}^d$ to $\mathbb{R}^e$ endowed with the operator norm

$$|\Lambda| := \sup_{x\in\mathbb{R}^d, \|x\|_{\mathbb{R}^d}=1} \|\Lambda(x)\|_{\mathbb{R}^e}.$$

Given a mapping $Q : I \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ on a compact interval I, it will be convenient to write

$$|Q(\cdot)|_{\infty:I} := \sup_{t\in I} |Q(t)|.$$

Theorem 5.1. *Let $I := [0, 1]$ and $H = \mathbb{R}^e$. Let $v : I \to \mathbb{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : I \rightrightarrows \mathbb{R}^e$ be a compact convex valued multimapping such that*
(i) there is a real constant $M > 0$ such that $C(t) \subset M\mathbb{B}_H$ for all $t \in I$;
(ii) $\text{haus}\,(C(t), C(\tau)) \le |v(t) - v(\tau)|$ for all $t, \tau \in I$.
Let $A : \mathbb{R}^e \to \mathbb{R}^e$ be a coercive symmetric linear operator and let $B : \mathbb{R}^e \to \mathbb{R}^e$ be a linear operator. Let $z \in C^{1-var}([0, 1], \mathbb{R}^d)$, the space of continuous mappings of bounded variation defined on $[0, 1]$ with values in $\mathbb{R}^d$. Let $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying for some real $M_b > 0$
(a) $|b(t, x)| \le M_b$ for all $(t, x) \in I \times \mathbb{R}^e$;
(b) $|b(t, x) - b(t, y)| \le M_b\|x - y\|_{\mathbb{R}^e}$ for all $(t, x, y) \in I \times \mathbb{R}^e \times \mathbb{R}^e$ with the perturbed Riemann-Stieljies integral $\int_0^t b(\tau, x(\tau))dz_\tau$ defined for $x \in C(I, \mathbb{R}^e)$.
Let $g : I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ be a continuous mapping satisfying for real $M_g > 0$:
(i) $\|g(t, s, x)\| \le M_g$ for all $(t, s, x) \in I \times I \times \mathbb{R}^e$;
(ii) $\|g(t, s, x) - g(t, s, y)\| \le M_g||x - y||$ for all $t, s \in I$, $x, y \in \mathbb{R}^e$ with the perturbed Lebesgue integral $\int_0^t g(t, s, x(s))ds$ defined on in $C(I, \mathbb{R}^e)$.
Let $a \in C(0)$. Then there exist a BVC mapping $x : [0, 1] \to \mathbb{R}^e$ and an absolutely continuous mapping $u : [0, 1] \to \mathbb{R}^e$ satisfying

$$\begin{cases} x(0) = u(0) = a \\ x(t) = h(t) + k(t) + Bu(t),\ \forall t \in I \\ h(t) = \int_0^t b(\tau, x(\tau))dz_\tau,\ \forall t \in I \\ k(t) = \int_0^t g(t, s, x(s))ds,\ \forall t \in I \\ \int_0^t b(s, x(s))dz_s + \int_0^t g(t, s, x(s))ds + Bu(t) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t)) \text{ a.e. } t \in I \end{cases}$$

Proof. Let $M := \max\{M_b, M_g\}$. Set for all $t \in I = [0,1]$

$$x^0(t) = a,\ h^1(t) = \int_0^t b(\tau, a)dz_\tau,$$

so by Proposition 2.2 in Friz-Victoir [25], we have

$$\left| \int_0^t b(\tau, a)dz_\tau \right| \leq |b(., a)|_{\infty:[0,1]} |z|_{1-var:[0,t]}.$$

Writing

$$\int_0^t b(\tau, a)dz_\tau - \int_0^s b(\tau, a)dz_\tau = \int_s^t b(\tau, a)dz_\tau,$$

we see by condition (a) that

$$\|h^1(t) - h^1(s)\| \leq M|z|_{1-var:[s,t]}$$

for all $0 \leq s \leq t \leq 1$, and in particular

$$\|h^1(t)\| \leq M|z|_{1-var:[0,t]} \leq M|z|_{1-var:[0,1]}$$

for all $t \in [0,1]$. Let us set for all $t \in I = [0,1]$

$$k^1(t) = \int_0^t g(t, s, x^0(s))ds \quad \text{for all } t \in I = [0,1]$$

and note that k^1 is continuous with $\|k^1(t)\| \leq M$ for all $t \in I$. By an easy computation, using conditions (i) and (ii) we have the estimate $\|k^1(t) - k^1(\tau)\| \leq M|t - \tau|$, for all $\tau, t \in I$. By Theorem 4.1 there is a unique absolutely continuous mapping $u^1 : I \to H$ solution of the problem

$$\begin{cases} u^1(0) = a \\ h^1(t) + k^1(t) + Bu^1(t) - A\dot{u}^1(t) \in N_{C(t)}(\dot{u}^1(t)) \text{ a.e. } t \in I \end{cases}$$

with $u^1(t) = a + \int_0^t \dot{u}^1(s)ds$ for all $t \in I$ and $\|\dot{u}^1(t)\| \leq M$ a.e. $t \in I$. Set

$$x^1(t) = h^1(t) + k^1(t) + Bu^1(t) = \int_0^t b(\tau, x^0(\tau)dz_\tau + \int_0^t g(t, s, x^0(s))ds + Bu^1(t).$$

Then x^1 is BVC with $x^1(0) = a$. Now we construct x^n by induction as follows. Let for all $t \in I$

$$h^n(t) = \int_0^t b(\tau, x^{n-1}(\tau))dz_\tau$$

$$k^n(t) = \int_0^t g\big(t, s, x^{n-1}(s)\big)ds.$$

Then $||k^n(t) - k^n(\tau)|| \leq M|t-\tau|$, for all $\tau, t \in I$ with $\|k^n(t)\| \leq M$ for all $t \in I$. By Proposition 2.2 in Friz-Victoir [25] we have the estimate

$$\|h^n(t) - h^n(s)\| \leq M|z|_{1-var:[s,t]} \tag{5.1}$$

for all $0 \leq s \leq t \leq 1$, and in particular

$$\|h^n(t)\| \leq M|z|_{1-var:[0,t]} \leq M|z|_{1-var:[0,1]} \tag{5.2}$$

for all $0 \leq t \leq 1$. By Theorem 1.1 there is a unique absolutely continuous mapping $u^n : I \to H$ solution of the problem

$$\begin{cases} u^n(0) = a, \\ h^n(t) + k^n(t) + Bu^n(t) - A\dot{u}^n(t) \in N_{C(t)}(\dot{u}^n(t)) \text{ a.e. } \in t \in I \end{cases}$$

with $u^n(t) = a + \int_0^t \dot{u}^n(s)ds$ for all $t \in I$ and $\|\dot{u}^n(t)\| \leq M$ a.e. $t \in I$. Set for all $t \in I$

$$x^n(t) = \ h^n(t)+k^n(t)+Bu^n(t) = \int_0^t b(\tau, x^{n-1}(\tau))dz_\tau + \int_0^t g\big(t, s, x^{n-1}(s)\big)ds + Bu^n(t)$$

so that x^n is BVC.

As $(u^n)_n$ is equi-Lipschitz continuous (with M as Lipschitz constant) we may suppose that $(u^n)_n$ converges uniformly to a Lipschitz continuous mapping $u : I \to H$ with $u(t) = a + \int_0^t \dot{u}(s)ds$ for all $t \in I$ and with $\|\dot{u}(t)\| \leq M$ for a.e. $t \in I$. We may also suppose that $(\dot{u}^n)_n$ weakly converges in $L^1_H(I)$ to $\dot{u}$, and by Arzela-Ascoli theorem we may suppose that $(k^n)_n$ converges uniformly to a continuous mapping $k : I \to H$. Thanks to (5.1) $(h^n)_n$ is bounded and equicontinuous. By Arzela-Ascoli theorem again, we may suppose that $(h^n)_n$ converge uniformly to a continuous mapping h. Similarly $(k^n)_n$ is bounded and equi-Lipschitz. By Arzeala-Ascoli theorem, we may suppose that $(k^n)_n$ converges uniformly to a continuous mapping k. Then putting $x^n(t) = h^n(t)+k^n(t)+Bu^n(t)$ and $x(t) := h(t)+k(t)+Bu(t)$, we see that $(x^n)_n$ converges uniformly to x, and $\big(b(., x^{n-1}(.))\big)_n$ converges uniformly to $b(., x(.))$ according to the Lipschitz condition (b). Therefore, by Friz-Victoir [25, Proposition 2.7] $\int_0^t b(\tau, x^{n-1}(\tau))dz_\tau$ converge uniformly in $t \in I$ to $\int_0^t b(\tau, x(\tau))dz_\tau$ as $n \to \infty$. By hypothesis (ii), $g(t, s, x^{n-1}(s))$ converges to $g(t, s, x(s))$ for every $(t, s) \in I \times I$, hence $\int_0^t g(t, s, x^{n-1}(s))ds \to \int_0^t g(t, s, x(s))ds$ for each $t \in I$ by Lebesgue dominated convergence theorem. So we can write

$$\begin{aligned} x(t) &= \lim_{n\to\infty} x^n(t) \\ &= \lim_{n\to\infty} \int_0^t b(\tau, x^{n-1}(\tau))dz_\tau + \lim_{n\to\infty} \int_0^t g(t, s, x^{n-1}(s))ds + \lim_{n\to\infty} Bu^n(t) \\ &= \int_0^t b(\tau, x(\tau))dz_\tau + \int_0^t g(t, s, x(s))ds + Bu(t). \end{aligned}$$

From the inclusion

$$h^n(t) + k^n(t) + Bu^n(t) - A\dot{u}^n(t)) \in N_{C(t)}(\dot{u}^n(t)) \text{ a.e. } t \in I$$

and the above convergence, applying Proposition 3.1 we obtain

$$h(t) + k(t) + Bu(t) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t)) \text{ a.e. } t \in I.$$

The proof is therefore complete. □

Our tools allow to state several variants of Theorem 5.1 according to the nature of the perturbation and the operator. Actually Theorem 5.1 holds true if $B : H \to H$ is a Lipschitz mapping, that is, there is some real constant $M > 0$ such that $\|Bx - By\| \leq M \|x - y\|$ for all $x, y \in H$. Theorem 5.1 is still valid if we replace B by the gradient ∇g of a positive convex function $g : H \to \mathbb{R}$ Lipschitz on bounded sets and continuously differentiable.

5.2 Towards an application in Optimal Control problem

In the previous results we have developed the Skorokhod problem associated with the sweeping process with Riemann-Stieltjes integral perturbation. This leads to study the following optimal control problem.

Proposition 5.1. *Let $I := [0,1]$ and $H = \mathbb{R}^e$. Let $v : I \to \mathbb{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : I \rightrightarrows \mathbb{R}^e$ be a compact convex valued multimapping such that*
(i) $C(t) \subset M\mathbb{B}_{\mathbb{R}^e}$ for all $t \in I$ where M is a positive constant;
(ii) $\text{haus}_{\mathbb{R}^e}(C(t), C(\tau)) \leq |v(t) - v(\tau)|$ for all $t, \tau \in I$.
Let $A : \mathbb{R}^e \to \mathbb{R}^e$ be a coercive symmetric linear operator and let $B : \mathbb{R}^e \to \mathbb{R}^e$ be a clinear operator. Let $z \in C^{1-var}([0,1], \mathbb{R}^d)$, the space of continuous functions of bounded variation defined on $[0,1]$ with values in $\mathbb{R}^d$. Let $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be the space of linear mappings Λ from $\mathbb{R}^d$ to $\mathbb{R}^e$ endowed with the operator norm

$$|\Lambda| := \sup_{x \in \mathbb{R}^d, \|x\|_{\mathbb{R}^d} = 1} \|\Lambda(x)\|_{\mathbb{R}^e}.$$

Let us consider a continuous integrand operator $b : [0,1] \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ satisfying
(a) $|b(t,x)| \leq M$ for all $(t,x) \in I \times \mathbb{R}^e$;
(b) $|b(t,x) - b(t,y)| \leq M\|x - y\|_{\mathbb{R}^e}$ for all $(t,x,y) \in I \times \mathbb{R}^e \times \mathbb{R}^e$.
Let $V : \mathbb{R}^d \to \mathbb{R}^e$ be a bounded continuous mapping. Let $L : [0,1] \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e \to [0, \infty[$ be a lower semicontinuous integrand such that $L(t,x,y,.)$ is convex on $\mathbb{R}^e$ for every $(t,x,y) \in [0,1] \times \mathbb{R}^e \times \mathbb{R}^e$. Then the problem of minimizing the cost function $\int_0^1 L(t, x(t), y(t), \dot{y}(t))dt$ subject to

$$\begin{cases} dx_t = V(x_t)dz_t, \ t \in [0,1] \\ x_0 = \psi \in \mathbb{R}^e \\ y(0) = y_0 \in C(0) \\ \int_0^t b(\tau, x(\tau))dz_\tau + By(t) - A\dot{y}(t) \in N_{C(t)}(\frac{dy}{dt}(t)), \ \ a.e. \ t \in [0,T] \end{cases}$$

has an optimal solution.

Proof. Let us consider a minimizing sequence $(x_n, y_n)_n$ in $\mathcal{Y}$, that is,

$$\lim_{n\to\infty} \int_0^T L(t, x_n(t), y_n(t), \dot{y}_n(t))dt = \inf_{(u,v)\in\mathcal{Y}} \int_0^T L(t, u(t), v(t), \dot{v}(t))dt,$$

where $\mathcal{Y}$ is the set of solutions (x, y) to the above dynamical system. First by [25, Theorem 3.4] we assert that the $C^{1-var}(I, \mathbb{R}^e)$-solution set to

$$\begin{cases} dx_t = V(x_t)dz_t, \ t \in I \\ x_0 = \psi \in \mathbb{R}^e \end{cases}$$

is compact in $C(I, \mathbb{R}^e)$ and so is the $W^{1,\infty}(I, \mathbb{R}^e)$-solution set to

$$\begin{cases} y(0) = y_0 \in C(0) \\ \int_0^t b(\tau, x(\tau))dz_\tau + By(t) - A\left(\frac{dy}{dt}(t)\right) \in N_{C(t)}(\frac{dy}{dt}(t)), \ \text{ a.e. } t \in I. \end{cases}$$

Then $(x_n)_n$ converges uniformly to some $x \in C^{1-var}(I, \mathbb{R}^e)$ with $x_t = \psi + \int_0^t V(x_s)dz_s$, and $(y_n)_n$ converges to $y \in W^{1,\infty}(I, \mathbb{R}^e)$ and $(\dot{y}_n)_n$ converges to $\dot{y}$ weakly in $L^1_{\mathbb{R}^e}(I)$.
Applying the lower semicontinuity of the integral functional ([22, Theorem 8.16]) gives

$$\liminf_n \int_0^1 L(t, x_n(t), y_n(t), \dot{y}_n(t))dt \geq \int_0^1 L(t, x(t), y(t), \dot{y}(t))dt.$$

From the inclusion

$$\int_0^t b(\tau, x_n(\tau))dz_\tau + By_n(t) - A\dot{y}_n(t) \in N_{C(t)}(\frac{dy_n}{dt}(t))$$

and the fact that $\lim_n \int_0^t b(\tau, x_n(\tau))dz_\tau = \int_0^t b(\tau, x(\tau))dz_\tau$ uniformly with respect to $t \in I$ (cf the proof of Theorem 5.1) we obtain by using Proposition 3.2 that

$$\int_0^t b(\tau, x(\tau))dz_\tau + By(t) - A\dot{y}(t) \in N_{C(t)}(\frac{dy}{dt}(t)) \ \text{ a.e. } t \in I.$$

We then conclude that (x, y) is an optimal solution. □

Several variants of the preceding theorem are available using Theorems 4.3 and 4.4 along with Propositions 3.2 and 3.10.

In the following we will examine a Bolza problem and its relaxation associated with a Young integral perturbation of a sweeping process with a Lipschitzian moving compact set $C(t)$, say $\text{haus}_{\mathbb{R}^e}(C(t), C(s)) \leq \alpha|t - s|$. First, we need some notation and background on Young integral and Young measures in this special

context.

Young integral. Let $z \in C^{1-var}([0,T],\mathbb{R}^d)$, that is, z is a bounded variation continuous mapping defined on $[0,T]$ with values in $\mathbb{R}^d$. We remind that $\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ denotes the space of linear mappings Λ from $\mathbb{R}^d$ to $\mathbb{R}^e$ endowed with the operator norm

$$|\Lambda| := \sup_{x\in\mathbb{R}^d, \|x\|_{\mathbb{R}^d}=1} |\Lambda(x)|_{\mathbb{R}^e}.$$

Let us consider a continuous integrand operator $b : [0,T]\times\mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ satisfying

$$(\mathcal{B}_1) : |b(t,x)| \le M, \quad \forall x \in \mathbb{R}^e$$

$$(\mathcal{B}_2) : |b(t,x)-b(\tau,y)| \le \rho(t)-\rho(\tau)+M\|x-y\|_{\mathbb{R}^e}, \ \ 0\le\tau\le t\le T, \ \ \forall x,y\in\mathbb{R}^e$$

where $\rho : [0,T]\to\mathbb{R}^+$ is a positive nondecreasing continuous function and M is a positive constant. If a sequence $(u_n)_n$ of continuous mappings from $[0,T]$ into $\mathbb{R}^e$ is uniformly bounded and uniformly bounded in variation, then the sequence $(y_n)_n$, with $y_n(t) = b(t,u_n(t))$, is formed with mappings which are continuous, uniformly bounded and uniformly bounded in variation from $[0,T]$ to $\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$, shortly $y_n \in C^{1-var}([0,T],\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e))$. Indeed we have

$$|y_n(t)-y_n(\tau)| \le \rho(t)-\rho(\tau)+M\|u_n(t)-u_n(\tau)\|_{\mathbb{R}^e}$$

for all $\tau\le t\le T$, so that $\sup_n |y_n|_{1-var;[s,t]} < \infty$ for all $0\le s\le t\le T$. As consequence the Young integral $\int_0^t y_n(s)dz_s$ of y_n against z is well-defined and belongs to $C^{1-var}([0,T],\mathbb{R}^e)$ according to Friz-Victoir [25], with the following estimates

$$\begin{aligned}\left\|\int_s^t y_n(\tau)dz_\tau\right\| &\le \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}|y_n|_{1-var;[s,t]} + |y_n(s)|\,\|z(t)-z(s)\|_{\mathbb{R}^d}\\ &\le \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}|y_n|_{1-var;[s,t]} + M\|z(t)-z(s)\|_{\mathbb{R}^d}\end{aligned}$$

for all $0\le s\le t\le T$ with $\theta = 2$ and

$$\left|\int_0^\cdot y_n(\tau)dz_\tau\right|_{1-var;[s,t]} \le C(1,1)|z|_{1-var;[s,t]}\left(|y_n|_{1-var;[s,t]} + |y_n|_{\infty;[s,t]}\right)$$

for all $0\le s\le t\le T$. As consequence

$$\begin{aligned}\left\|\int_s^t y_n dz\right\| &\le \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}|y_n|_{1-var;[s,t]} + |y_n(s)|\ \|z(t)-z(s)\|_{\mathbb{R}^d}\\ &\le \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}|y_n|_{1-var;[s,t]} + M||z(t)-z(s)||_{\mathbb{R}^d}\\ &\le \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}\sup_n|y_n|_{1-var;[s,t]} + \sup_{s\in[0,T]} M||z(t)-z(s)||_{\mathbb{R}^d}\end{aligned}$$

for all $0 \leq s \leq t \leq T$ with $\theta = 2$ and

$$\left| \int_0^{\cdot} y_n dz \right|_{1-var;[s,t]} \leq C(1,1)|z|_{1-var;[s,t]} \sup_n \left(|y_n|_{1-var;[s,t]} + |y_n|_{\infty;[s,t]} \right)$$

for all $0 \leq s \leq t \leq T$. Shortly the sequence $g_n(.) = \int_0^{\cdot} y_n dz$ of $C^{1-var}([0,T], \mathbb{R}^e)$ mappings is uniformly bounded, equicontinuous and uniformly bounded in variation.

Now let E be a separable reflexive Banach space. Let us consider a weakly compact convex valued multimapping $K : [0,1] \rightrightarrows \mathbb{B}_E$ with bounded right continuous retraction in the sense, there is a bounded and right continuous function $\rho : [0,1] \to \mathbb{R}^+$ such that

$$\text{haus}(K(t), K(\tau)) \leq \rho(\tau) - \rho(t) \quad \text{for all } t \leq \tau \in [0,1]$$

and such that its graph is Borelian, that is, $\text{gph}\,(C) \in \mathcal{B}([0,T]) \otimes \mathcal{B}(E)$. We consider the control sets given by

$$S_K^{BVRC} := \{u : [0,1] \to E, u \text{ is BVRC}, \ u(t) \in K(t), \ \forall t \in [0,1]\}$$

$$S_K^{\infty} := \{u \in L^{\infty}([0,1], E, dt), \ u(t) \in K(t), \ \ \forall t \in [0,T].\}$$

By J.J. Moreau ([34], Prop.5 d, p. 198) and Valadier [42] these sets are non empty and $\text{cl}\, S_K^{BVRC} = S_K^{\infty}$, here cl denotes the closure with respect to the $\sigma(L_E^{\infty}, L_{E^*}^1)$-topology. Shortly, S_K^{BVRC} is dense in S_K^{∞} with respect to this topology.

Our next theorem, say theorem 5.2, will present relaxation results for a Bolza optimal control problem governed by EVI sweeping processes of the types developed in the previous sections. The control will belong either in S_K^{BVRC} or S_K^{∞}. In the theorem $A : \mathbb{R}^e \to \mathbb{R}^e$ is a coercive symmetric linear operator while B is a linear operator on $\mathbb{R}^e$ and $b : [0,T] \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ is a mapping satisfying $(\mathcal{B}_1)$ and $(\mathcal{B}_2)$.

Theorem 5.2. *With the Hilbert space $\mathbb{R}^e$ consider the problem*

$$\inf_{(\mathcal{P}_{\mathcal{S}_K^{BVRC}})} \int_0^T \langle \int_0^t b(s, u(s)) dz_s, \zeta(t) \rangle dt$$

associated, for $I := [0,T]$, with the dynamical system

$$(\mathcal{P}_{\mathcal{S}_K^{BVRC}}) \begin{cases} \int_0^t \zeta(s)ds + Bu(t) + A\dot{u}(t) \in N_{C(t)}(\dot{u}(t)), t \in I, \zeta \in \mathcal{S}_K^{BVRC} \\ u(0) = a \in C(0) \end{cases}$$

as well as the problem

$$\inf_{(\mathcal{P}_{\mathcal{S}_K^\infty})} \int_0^T \langle \int_0^t b(s,u(s))dz_s, \zeta(t)\rangle dt$$

associated with the dynamic system

$$(\mathcal{P}_{\mathcal{S}_K^\infty}) \begin{cases} \int_0^t \zeta(s)ds + Bu(t) + A\dot{u}(t) \in N_{C(t)}(\dot{u}(t)), t \in I, \zeta \in \mathcal{S}_K^\infty \\ u(0) = a \in C(0). \end{cases}$$

Then one has $\inf(\mathcal{P}_{\mathcal{S}_K^\infty}) = \inf(\mathcal{P}_{\mathcal{S}_K^{BVRC}}$, *and as a consequence*

$$\inf_{(\mathcal{P}_{\mathcal{S}_K^\infty})} \int_0^T \langle \int_0^t b(s,u(s))dz_s, \zeta(t)\rangle dt$$

has a minimizer.

Proof. The inequality $\inf(\mathcal{P}_{\mathcal{S}_K^\infty}) \leq \inf(\mathcal{P}_{\mathcal{S}_K^{BVRC}})$ is a simple consequence of the Valadier result mentioned above since any BVRC mappin,g is Borelian. Take and any $\zeta \in \mathcal{V}_K^\infty$. As $\mathcal{S}_K^{BVRC}$ is sequentially dense in $\mathcal{V}_K^\infty$ with respect to the $\sigma(L_E^\infty, L_E^1)$ topology, there exists a sequence (ζ_n) in $\mathcal{S}_K^{BVRC}$ such that (ζ_n) $\sigma(L_E^\infty, L_E^1)$ converge to ζ. For simplicity set $f_n(t) = \int_0^t \zeta_n(s)ds$, and $f(t) = \int_0^t \zeta(s)ds$ for all $t \in [0,T]$. Then it is clear that f_n and f continuous uniformly bounded with $f_n(t) \to f(t)$ for every $t \in [0,T]$. Let u_n be the unique Lipschitz solution to

$$\begin{cases} \int_0^t \zeta_n(s)ds + Bu_n(t) + A\dot{u}_n(t) \in N_{C(t)}(\dot{u}_n(t)), t \in I, \\ u_n(0) = a \in C(0) \end{cases}$$

and let v be the unique Lipschitz solution to

$$\begin{cases} \int_0^t \zeta(s)ds + Bv(t) + A\dot{v}(t) \in N_{C(t)}(\dot{v}(t)), t \in I, \\ u(0) = a \in C(0) \end{cases}$$

In view Theorem 4.1 and Proposition 3.1 the sequence $(u_n)_n$ is equi-Lipchitz and converges uniformly to v. For simplicity set $g_n(t) = \int_0^t b(s,u_n(s))dz_s$ for all $t \in [0,T]$. Applying the foregoing estimates concerning the Young integral, the sequence $(g_n)_n$ is uniformly bounded, equicontinuous and uniformly bounded in variation. Indeed, by condition $(\mathcal{B}_2)$, the sequence $(b(\cdot,u_n(\cdot)))_n$ converge uniformly to $b(\cdot,v(\cdot))$. As $b(.,u_n(.))$ and $b(.,v(.))$ are bounded and uniformly bounded in variation, by Friz-Victoir ([25, Proposition 6.12]) $(g_n)_n$ converges uniformly to g defined by $g(t) = \int_0^t b(s,v(s))dz_s$, so that

$$\lim_{n\to\infty} \int_0^T \langle \int_0^t b(s,u_n(s))dz_s, \zeta_n(t)\rangle dt = \int_0^T \langle \int_0^t b(s,v(s)dz_s, \zeta(t)\rangle dt.$$

Since

$$\int_0^T \langle \int_0^t b(s,u_n(s))dz_s, \zeta_n(t)\rangle dt \geq \inf_{(\mathcal{P}_{\mathcal{S}_K^{BVRC}})} \int_0^T \langle \int_0^t b(s,u(s))dz_s, \zeta(t)\rangle dt$$

for all $n \in \mathbb{N}$, it follows by taking the limit that

$$\int_0^T \langle \int_0^t b(s,v(s))dz_s, \zeta(t)\rangle dt \geq \inf(\mathcal{P}_{\mathcal{S}_K^{BVRC}}).$$

This holds for every $\zeta \in \mathcal{V}_K^\infty$, hence $\inf(\mathcal{P}_{\mathcal{V}_K^\infty}) \geq \inf(\mathcal{P}_{\mathcal{S}_K^{BVRC}})$. □

For simplicity we considered in Theorem 5.2 a simple perturbation control integral in the form $\int_0^t \zeta(s)ds$. This theorem is still valid with the perturbation control integral $\int_0^t \mathcal{D}\zeta(s)ds$ where $\mathcal{D} : \mathbb{R}^e \to \mathbb{R}^e$ is a linear operator. Several other variants of Theorem 5.2 are available using Theorems 4.1 and 4.3 along with Propositions 3.2 and 3.10.

Young measures. For the sake of completeness of the next development of Theorem 5.3, we summarize some useful facts concerning Young measures. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let X be a Polish space and let $C^b(X, \mathbb{R})$ be the space of all bounded continuous functions defined on X. Let $\mathcal{M}_+^1(X)$ be the set of all Borel probability measures on X equipped with the narrow topology. A Young measure $\nu : \Omega \to \mathcal{M}_+^1(X)$ is, by definition, a *scalarly measurable* mapping from Ω into $\mathcal{M}_+^1(X)$, that is, for every $f \in C^b(X, \mathbb{R})$, the mapping $\omega \mapsto \langle f, \nu_\omega\rangle := \int_X f(x)\, d\nu_\omega(x)$ is $\mathcal{F}$-measurable. A sequence (ν^n) in the space of Young measures $\mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}_+^1(X))$ *stably converges* to a Young measure $\nu \in \mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}_+^1(X))$ if the following holds:

$$\lim_{n\to\infty} \int_A \left(\int_X f(x)\, d\nu_\omega^n(x) \right) dP(\omega) = \int_A \left(\int_X f(x)\, d\nu_\omega(x) \right) dP(\omega)$$

for every $A \in \mathcal{F}$ and for every $f \in C^b(X, \mathbb{R})$.

In the remainder Z is a compact metric space, $\mathcal{M}_+^1(Z)$ is the space of all Radon probability measures on Z. We will endow $\mathcal{M}_+^1(Z)$ with the narrow topology so that $\mathcal{M}_+^1(Z)$ is a compact metrizable space. For $I := [0,1]$ let us denote by $\mathcal{Y}(I; \mathcal{M}_+^1(Z))$ the space of all Young measures defined on I endowed with the stable topology so that $\mathcal{Y}(I; \mathcal{M}_+^1(Z))$ is a compact metrizable space with respect to this topology. By the Portmanteau Theorem for Young measures [22, Theorem 2.1.3], a sequence (ν^n) in $\mathcal{Y}(I; \mathcal{M}_+^1(Z))$ stably converges to $\nu \in \mathcal{Y}(I; \mathcal{M}_+^1(Z))$ if

$$\lim_{n\to\infty} \int_0^T \left(\int_Z h_t(z) d\nu_t^n(z) \right) dt = \int_0^T \left(\int_Z h_t(z) d\nu_t(z) \right) dt$$

for all $h \in L^1(I, \mathcal{C}_\mathbb{R}(Z))$; here $\mathcal{C}_\mathbb{R}(Z)$ denotes the space of all continuous real valued functions defined on Z endowed with the norm of uniform convergence. Finally

let Γ be a measurable multimapping defined on I with nonempty compact values in Z and let S_Γ be the set of all Lebesgue measurable selections of Γ (alias *original controls*).

Let $C : [0,T] \rightrightarrows \mathbb{R}^e$ be a compact valued Lipschitzian multimapping and let $f : I \times Z \to \mathbb{R}^e$ be a mapping satisfying
(1) for every fixed $t \in I, f(t,\cdot)$ is continuous on Z,
(2) for every $z \in Z, f(\cdot,z)$ is Lebesgue-measurable on I;
(3) there is a constant $M > 0$ such that $\|f(t,z)\| \leq M$ for all (t,z) in $I \times Z$.

We aim to present some relaxation problems in the framework of Optimal Control Theory. We consider the evolution inclusion $(\mathcal{PO})$ associated with original controls

$$(\mathcal{PO}) \begin{cases} \int_0^t f(s,\zeta(s))ds + Bu_\zeta(t) - A\dot{u}_\zeta(t) \in N_{C(t)}(\dot{u}_\zeta(t))), \text{ a.e. } t \in I, \\ u_\zeta(0) = u_0 \in C(0), \end{cases}$$

where ζ belongs to the set $\mathcal{Z} := S^1_\Gamma$ of all original controls, which means that ζ is a Lebesgue-measurable selection of Γ, and the evolution inclusion $(\mathcal{PR})$ associated with relaxed controls

$$(\mathcal{PR}) \begin{cases} \int_0^t [\int_Z f(s,z)\nu_s(dz)]ds + Bu_\nu(t) - A\dot{u}_\nu(t) \in N_{C(t)}(\dot{u}_\nu(t))), \text{ a.e. } t \in I \\ u_\nu(0) = u_0 \in C(0), \end{cases}$$

where ν belongs to the set $\mathcal{R} := S_\Sigma$ of all relaxed controls, which means that ν is a Lebesgue-measurable selection of the multimapping Σ defined by

$$\Sigma(t) := \left\{\sigma \in \mathcal{M}^1_+(Z) : \sigma(\Gamma(t)) = 1\right\}$$

for all $t \in I$. Note that, for $\nu \in \mathcal{R}$, the mapping

$$h_\nu : (t,z) \mapsto \int_Z f(t,z)\nu_t(dz)$$

inherits the properties
(1) for every fixed $t \in I, h_\nu(t,\cdot)$ is continuous on Z;
(2) for every $z \in Z, h_\nu(\cdot,z)$ is Lebesgue-measurable on I;
(3) there is a constant $M > 0$ such that $\|h_\nu(t,z)\| \leq M$ for all (t,z) in $I \times Z$.
Consequently, for each $\zeta \in \mathcal{Z}$ (resp. $\nu \in \mathcal{R}$), the evolution inclusion $(\mathcal{PO})$ (resp. $(\mathcal{PR})$) has a unique Lipschitz continuous solution. Moreover, there is an a priori bound for the Lipschitz ratio of solutions which easily implies that the solution sets $(\mathcal{SO})$ and $(\mathcal{SR})$ (to $(\mathcal{PO})$ and $(\mathcal{PR})$) are equi-Lipschitz.

We can now prove the following theorem establishing some topological properties of the solution sets $(\mathcal{S}_\mathcal{O})$ and $(\mathcal{S}_\mathcal{R})$, namely we obtain the typical relaxation result that the former is dense in the latter.

Theorem 5.3. *Let $I := [0,1]$ and let $C : I \rightrightarrows \mathbb{R}^e$ be a compact convex valued Lipschitz multimapping. Let $f : I \times Z \to \mathbb{R}^e$ be a mapping satisfying (1), (2), (3).*

Then the following hold:
(a) the solution set $(\mathcal{S}_{\mathcal{R}})$ *to*

$$(\mathcal{PR})\begin{cases} \int_0^t [\int_Z f(s,z)\nu_s(dz)]ds + Bu_\nu(t) - A\dot{u}_\nu(t) \in N_{C(t)}(\dot{u}_\nu(t))), \ a.e.\, t \in I, \ \nu \in \mathcal{R} \\ u_\nu(0) = u_0 \in C(0) \end{cases}$$

is nonempty and compact in $\mathrm{C}(I, \mathbb{R}^e)$.
(b) the solution set $(\mathcal{S}_{\mathcal{O}})$ *to*

$$(\mathcal{PO})\begin{cases} \int_0^t f(s,\zeta(s))ds + Bu_\zeta(t) - A\dot{u}_\zeta(t) \in N_{C(t)}(\dot{u}_\zeta(t))), \ a.e.\, t \in I, \ \zeta \in \mathcal{Z} \\ u_\zeta(0) = u_0 \in C(0) \end{cases}$$

is dense in $(\mathcal{S}_{\mathcal{R}})$ *with respect to the topology of uniform convergence.*

Proof. (a) By Theorem 4.1, the solution set $(\mathcal{S}_{\mathcal{R}})$ is bounded and equi-Lipschitz. Then $(\mathcal{S}_{\mathcal{R}})$ is relatively compact in $\mathcal{C}(I, \mathbb{R}^e)$, by Arzela-Ascoli theorem. Therefore, for $(\nu^n) \subset \mathcal{R}$, there is a subsequence still denoted by (u_{ν^n}) which converges uniformly to a Lipschitz continuous mapping u^∞ with $\|\dot{u}^\infty(t)\| \leq K$ a.e. $t \in I$ and such that also $(\dot{u}_{\nu^n})$ $\sigma(L^1(I, \mathbb{R}^e; dt), L^\infty(I, \mathbb{R}^e; dt))$-converges to $\dot{u}^\infty$. As $\mathcal{R}$ is compact and metrizable for the stable topology, we may suppose that (ν^n) stably converges to $\nu^\infty \in \mathcal{R}$. Since the continuous functions g_n and g, given for all $t \in I$ by

$$g_n(t) := \int_0^t [\int_Z f(s,z)\nu_s^n(dz)]ds$$

and

$$g(t) := \int_0^t [\int_Z f(s,z)\nu_s^\infty(dz)]ds,$$

are uniformly bounded, and since $g_n(t) \to g(t)$ for every $t \in I$, from the inclusion

$$g_n(t) + Bu_\nu^n(t) - A\dot{u}_\nu(t) \in N_{C(t)}(\dot{u}_\nu^n(t)))$$

and Proposition 3.2, we deduce that

$$g(t) + Bu_{\nu^\infty}(t) - A\dot{u}_{\nu^\infty}(t) \in N_{C(t)}(\dot{u}_{\nu^\infty}(t))).$$

This proves the first part of the theorem.
(b) The second part follows by continuity and density, since $\mathcal{Z}$ is dense in $\mathcal{R}$ with respect to the stable topology ([22, Lemma 7.1.1]). □

With notation and assumptions in Theorem 5.3

Theorem 5.4. *With notation and assumptions in Theorem 5.3 let us consider the problem*

$$\inf_{(\mathcal{P}_{\mathcal{R}})} \int_0^T \Big\langle \int_0^t b(s,u(s))dz_s, \int_0^t [\int_Z f(s,z)\nu_s(dz)]ds \Big\rangle dt$$

associated with the dynamic system

$$(\mathcal{P}_{\mathcal{R}})\begin{cases} \int_0^t [\int_Z f(s,z)\nu_s(dz)]ds + Bu_\nu(t) - A\dot{u}_\nu(t) \in N_{C(t)}(\dot{u}_\nu(t))), \ a.e.\, t \in I, \ \nu \in \mathcal{R} \\ u_\nu(0) = u_0 \in C(0) \end{cases}$$

and the problem

$$\int_0^T \Big\langle \int_0^t b(s,u(s))dz_s, \int_0^t h(s,\zeta(s))ds \Big\rangle dt$$

associated with the dynamic system

$$(\mathcal{P}_{\mathcal{O}})\begin{cases} \int_0^t f(s,\zeta(s))ds + Bu_\zeta(t) - A\dot{u}_\zeta(t) \in N_{C(t)}(\dot{u}_\zeta(t))), \ a.e.\ t \in I, \ \zeta \in \mathcal{Z} \\ u(0) = a \in C(0) \end{cases}$$

Then one has $\inf(\mathcal{P}_{\mathcal{R}}) = \inf(\mathcal{P}_{\mathcal{O}})$ *and*

$$\inf_{(\mathcal{P}_{\mathcal{R}})} \int_0^T \Big\langle \int_0^t b(s,u(s))dz_s, \int_0^t [\int_Z h(s,z)\nu_s(dz)]ds \Big\rangle dt$$

has a minimizer.

Proof. The inequality $\inf(\mathcal{P}_{\mathcal{R}}) \geq \inf(\mathcal{P}_{\mathcal{O}})$ is clear. Fix any $\nu \in \mathcal{R}$. Let $(\zeta_n)_n$ in $\mathcal{Z}$ with

$$\lim_n \int_0^t f(s,\zeta_n(s))ds = \int_0^t [\int_Z f(s,z)\nu_s(dz)]ds \text{ for all } t \in I.$$

Let u_n be the unique Lipschitz solution to

$$\begin{cases} -\dot{u}_n(t) \in N_{C(t)}u_n(t) + \int_0^t f(s,\zeta_n(s))ds, \text{ a.e. } t \in I \\ u_n(0) = a \in C(0) \end{cases}$$

and let v be the unique Lipschitz solution to

$$\begin{cases} -\dot{v}(t) \in N_{C(t)}v(t) + \int_0^t [\int_Z f(s,z)\nu_s(dz)]ds, \text{ a.e. } t \in I \\ v(0) = a \in C(0). \end{cases}$$

In view of the first step of the proof of Theorem 5.3 the sequence $(u_n)_n$ converges uniformly to v. For simplicity set $g_n(t) = \int_0^t b(s,u_n(s))dz_s$ for all $t \in I$. Apply the foregoing estimates related to the Young integral, the sequence $(g_n)_n$ is uniformly bounded, equicontinuous and uniformly bounded in variation. By condition $(\mathcal{B}_2)$ the sequence $(b(\cdot,u_n(\cdot)))_n$ converges uniformly to $b(\cdot,v(\cdot))$. As $b(\cdot,u_n(\cdot))$ and $b(\cdot,v(\cdot))$ are bounded and uniformly bounded in variation, by Friz-Victoir ([25, Proposition 6.12]) $(g_n)_n$ converges uniformly to g defined by $g(t) = \int_0^t b(s,v(s))dz_s$. For simplicity set $k_n(t) = \int_0^t f(s,\zeta_n(s))ds$ and $k(t) =$

$\int_0^t[\int_Z f(s,z)\nu_s(dz)]ds$, so that $\lim_n\langle g_n(t),k_n(t)\rangle = \langle g(t),k(t)\rangle$. Since $(g_n)_n$, g, $(k_n)_n$, k are uniformly bounded, we deduce that

$$\lim_{n\to\infty}\int_0^T\langle\int_0^t b(s,u_n(s))dz_s,\int_0^t h(s,\zeta_n(s))ds\rangle dt=\lim_{n\to\infty}\int_0^T\langle g_n(t),k_n(t)\rangle dt$$

$$=\int_0^T\langle g(t),k(t)\rangle dt=\langle\int_0^t b(s,v(s)dz_s,\int_0^t[\int_Z h(s,z)\nu_s(dz)]ds\rangle dt.$$

As

$$\int_0^T\langle\int_0^t b(s,u_n(s)dz_s,\int_0^t h(s,\zeta_n(s))ds\rangle dt\geq\inf(\mathcal{P}_{\mathcal{O}})$$

for all $n\in\mathbb{N}$, it follows by taking the limit that

$$\int_0^T\langle\int_0^t b(s,v(s))dz_s,\int_0^t[\int_Z h(s,z)\nu_s(dz)]ds\rangle dt\geq\inf(\mathcal{P}_{\mathcal{O}})$$

Since this holds for every $\nu\in\mathcal{R}$, we conclude that $\inf(\mathcal{P}_{\mathcal{R}})\geq\inf(\mathcal{P}_{\mathcal{O}})$. □

5.3 Towards fractional inclusion coupled with EVI and sweeping process

Now given $I=[0,1]$ we investigate a class of boundary value problems governed by a fractional differential inclusion (FDI) (5.3) in the separable Hilbert space H coupled with the evolution inclusion governed by the (EVI) (5.4) and sweeping process (5.6) below.

$$D^\alpha h(t)+\lambda D^{\alpha-1}h(t)=u(t),t\in I,$$

$$I_{0^+}^\beta h(t)\,|_{t=0}:=\lim_{t\to0}\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}h(s)ds=0,\ h(1)=I_{0^+}^\gamma h(1)=\int_0^1\frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)}h(s)ds, \tag{5.3}$$

$$f(t,h(t))+Bu(t)-A\dot{u}(t)\in\partial\varphi(t,\dot{u}(t))\text{ a.e. }t\in I$$

and

$$f(t,h(t))+Bu(t)-A\dot{u}(t)\in N_{C(t)}(\dot{u}(t))\text{ a.e. }t\in I,$$

where $\alpha\in]1,2]$, $\beta\in[0,2-\alpha]$, $\lambda\geq0,\gamma>0$ are given constants, D^α is the standard Riemann-Liouville fractional derivative , Γ is the Classical gamma function.

For the convenience of the reader, we begin with a few reminders of the concepts which will be used in this subsection.

Definition 5.1 (Fractional Bochner integral)**.** *Let E be a separable Banach space and $f:I=[0,1]\to E$. The fractional Bochner-integral of order $\alpha>0$ of the function f is defined by*

$$I_{0^+}^\alpha f(t):=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds,\ t>0.$$

In the above definition, the sign "$\int$" denotes the classical Bochner integral.

Lemma 5.5 ([37]). *Let $f \in L^1(I, E, dt)$ with $I = [0, 1]$.*

(a) If $\alpha \in]0, 1[$ then $I^{\alpha}_{0+} f$ exists almost everywhere on I and $I^{\alpha}_{0+} f \in L^1(I, E, dt)$.

(b) If $\alpha \in [1, \infty)$ then $I^{\alpha}_{0+} f \in \mathcal{C}_E(I)$.

Definition 5.2. *Let E be a separable Banach space and let $f \in L^1_E(I, E, dt)$ with $I := [0, 1]$. One defines the Riemann-Liouville fractional derivative of order $\alpha > 0$ of f by*

$$D^{\alpha} f(t) := D^{\alpha}_{0+} f(t) = \frac{d^n}{dt^n} I^{n-\alpha}_{0+} f(t) = \frac{d^n}{dt^n} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α.

We denote by $W^{\alpha,1}_{B,E}(I)$ the space of all continuous functions in $\mathcal{C}_E(I)$ such that their Riemann-Liouville fractional derivatives of order $\alpha - 1$ are continuous and their Riemann-Liouville fractional derivatives of order α are Bochner integrable.

Green function and its properties.

In all the rest of this subsection $\alpha \in]1, 2]$, $\beta \in [0, 2 - \alpha]$, $\lambda \geq 0, \gamma > 0$. Let $G : [0, 1] \times [0, 1] \to \mathbb{R}$ be the Green function defined by

$$G(t, s) = \varphi(s) I^{\alpha-1}_{0+}(\exp(-\lambda t)) + \begin{cases} \exp(\lambda s) I^{\alpha-1}_{s+}(\exp(-\lambda t)), & 0 \leq s \leq t \leq 1, \\ 0, & 0 \leq t \leq s \leq 1, \end{cases} \tag{5.4}$$

where

$$\varphi(s) = \frac{\exp(\lambda s)}{\mu_0} \left[\left(I^{\alpha-1+\gamma}_{s+}(\exp(-\lambda t)) \right)(1) - \left(I^{\alpha-1}_{s+}(\exp(-\lambda t)) \right)(1) \right] \tag{5.5}$$

with

$$\mu_0 = \left(I^{\alpha-1}_{0+}(\exp(-\lambda t)) \right)(1) - \left(I^{\alpha-1+\gamma}_{0+}(\exp(-\lambda t)) \right)(1). \tag{5.6}$$

We recall and summarize a useful result ([16]).

Lemma 5.6. *Let E be a separable Banach space and let G be the function defined by (5.4)-(5.6). For $\alpha \in]1, 2]$ the following hold:*

(a) $G(\cdot, \cdot)$ satisfies the estimate

$$|G(t, s)| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{1 + \Gamma(\gamma + 1)}{|\mu_0| \Gamma(\alpha) \Gamma(\gamma + 1)} + 1 \right) = M_G.$$

(b) If $u \in W^{\alpha,1}_{B,E}([0, 1])$ satisfying boundary conditions (5.3), then

$$u(t) = \int_0^1 G(t, s) \left(D^{\alpha} u(s) + \lambda D^{\alpha-1} u(s) \right) ds \quad \text{for every } t \in [0, 1].$$

(c) Let $f \in L^1_E([0,1])$ and let $u_f : [0,1] \to E$ be the function defined by

$$u_f(t) := \int_0^1 G(t,s) f(s) ds \quad for \quad t \in [0,1].$$

Then

$$I^{\beta}_{0^+} u_f(t)\,|_{t=0} = 0 \quad and \quad u_f(1) = \left(I^{\gamma}_{0^+} u_f\right)(1).$$

Moreover $u_f \in W^{\alpha,1}_{B,E}([0,1])$ and one has for every $t \in [0,1]$

$$\left(D^{\alpha-1} u_f\right)(t) = \int_0^t \exp(-\lambda(t-s)) f(s) ds + \exp(-\lambda t) \int_0^1 \varphi(s) f(s) ds,$$

$$\left(D^{\alpha} u_f\right)(t) + \lambda \left(D^{\alpha-1} u_f\right)(t) = f(t).$$

From Lemma 5.6 we derive a crucial feature.

Lemma 5.7. *Let E be a separable Banach space and let $f \in L^1(I, E, dt)$ with $I := [0,1]$. Then the boundary value problem*

$$\begin{cases} D^{\alpha} u(t) + \lambda D^{\alpha-1} u(t) = f(t), \quad t \in I \\ I^{\beta}_{0^+} u(t)\,|_{t=0} = 0, \quad u(1) = I^{\gamma}_{0^+} u(1) \end{cases}$$

has a unique $W^{\alpha,1}_{B,E}(I)$-solution defined by

$$u(t) = \int_0^1 G(t,s) f(s) ds, \; textforall\, t \in I.$$

Theorem 5.8. *Let E be a separable Banach space, $\alpha \in]1,2)$ and $I := [0,1]$. Let $X : I \rightrightarrows E$ be a compact convex valued measurable multimapping such that $X(t) \subset \gamma \mathbb{B}_E$ for all $t \in I$, where γ is a positive constant. Let S^1_X be the set of all measurable selections of X. Then the $W^{\alpha,1}_{B,E}(I)$-solutions set of problem*

$$\begin{cases} D^{\alpha} u(t) + \lambda D^{\alpha-1} u(t) = f(t), f \in S^1_X, \; a.e.\; t \in I \\ I^{\beta}_{0^+} u(t)\,|_{t=0} = 0, \quad u(1) = I^{\gamma}_{0^+} u(1) \end{cases}$$

is compact in $\mathcal{C}_E(I)$.

Proof. By virtue of Lemma 5.7 the $W^{\alpha,1}_{B,E}(I)$-solutions set $\mathcal{X}$ to the above inclusion is characterized by

$$\mathcal{X} = \{u_f : I \to E, \; u_f(t) = \int_0^1 G(t,s) f(s) ds, \; f \in S^1_X, \; t \in I\}. \tag{5.7}$$

Claim: $\mathcal{X}$ is bounded, convex, equicontinuous and **compact** in $\mathcal{C}_E(I)$.
From the definition of the Green function G, it is not difficult to show that

$\{u_f : f \in S^1_X\}$ is bounded, equicontinuous in $\mathcal{C}_E(I)$. To show first the relative compactness take any sequence $(u_{f_n})_n$ in $\mathcal{X}$. We note that, for each $n \in \mathbb{N}$, we have $u_{f_n} \in W^{\alpha,1}_{B,E}(I)$, and

$$u_{f_n}(t) = \int_0^1 G(t,s) f_n(s) ds, \quad t \in I,$$

with by Lemma 5.6

- $I^{\beta}_{0^+} u_{f_n}(t)|_{t=0} = 0, \quad u_{f_n}(1) = I^{\gamma}_{0^+} u(1),$
- $\left(D^{\alpha-1} u_{f_n}\right)(t) = \int_0^t \exp(-\lambda(t-s)) f_n(s) ds + \exp(-\lambda t) \int_0^1 \varphi(s) f_n(s) ds,\ t \in I,$
- $\left(D^{\alpha} u_{f_n}\right)(t) + \lambda \left(D^{\alpha-1} u_{f_n}\right)(t) = f_n(t),\ t \in I.$

Consider any $t_1, t_2 \in I$ with $t_1 < t_2$. Let us write

$$\begin{aligned}
&u_{f_n}(t_2) - u_{f_n}(t_1) = \int_0^1 G(t,s)(f_n(s) - f_n(s)) ds \\
&= \int_0^1 \varphi(s) f_n(s) ds \left(\int_0^{t_2} \frac{e^{-\lambda\tau}}{\Gamma(\alpha-1)} (t_2-\tau)^{\alpha-2} d\tau - \int_0^{t_1} \frac{e^{-\lambda\tau}}{\Gamma(\alpha-1)} (t_1-\tau)^{\alpha-2} d\tau \right) \\
&+ \int_0^{t_2} e^{\lambda s} \left(\int_s^{t_2} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} e^{-\lambda\tau} d\tau \right) f_n(s) ds \\
&- \int_0^{t_1} e^{\lambda s} \left(\int_s^{t_1} \frac{e^{-\lambda\tau}}{\Gamma(\alpha-1)} (t_1-\tau)^{\alpha-2} d\tau \right) f_n(s) ds,
\end{aligned}$$

which gives with $\Phi(t,\tau) := (t-\tau)^{\alpha-2}/\Gamma(\alpha-1)$

$$\begin{aligned}
&u_{f_n}(t_2) - u_{f_n}(t_1) \\
&= \int_0^1 \varphi(s) f_n(s) ds \left[\int_0^{t_1} e^{-\lambda\tau} \big(\Phi(t_2,\tau) - \Phi(t_1,\tau)\big) d\tau + \int_{t_1}^{t_2} e^{-\lambda\tau} \Phi(t_2,\tau) d\tau \right] \\
&+ \int_0^{t_1} e^{\lambda s} \left(\int_s^{t_1} e^{-\lambda\tau} \big(\Phi(t_2,\tau) - \Phi(t_1,\tau)\big) d\tau \right) f_n(s) ds \\
&+ \int_0^{t_1} e^{\lambda s} \left(\int_{t_1}^{t_2} e^{-\lambda\tau} \Phi(t_2,\tau) d\tau \right) f_n(s) ds + \int_{t_1}^{t_2} e^{\lambda s} \left(\int_s^{t_2} e^{-\lambda\tau} \Phi(t_2,\tau) d\tau \right) f_n(s) ds.
\end{aligned}$$

Then, putting $|X(s)| := \sup\{\|y\| : y \in X(s)\}$ it follows that

$$\begin{aligned}
&\|u_{f_n}(t_2) - u_{f_n}(t_1)\| \\
&\le \int_0^1 \left(|\varphi(s)| + e^{\lambda s} \right) |X(s)| ds \int_0^{t_1} e^{-\lambda\tau} \frac{(t_1-\tau)^{\alpha-2} - (t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \\
&+ \int_0^1 \left(|\varphi(s)| + e^{\lambda s} \right) |X(s)| ds \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \\
&+ \int_{t_1}^{t_2} e^{\lambda s} |X(s)| ds \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau.
\end{aligned}$$

It is easy to see, after an integration by parts, that

$$\int_{t_1}^{t_2} e^{-\lambda\tau}\frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau = e^{-\lambda t_1}\frac{(t_2-t_1)^{\alpha-2}}{\Gamma(\alpha)} + \lambda\int_{t_1}^{t_2} e^{-\lambda\tau}\frac{(t_2-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau$$
$$\leq \frac{1+\lambda}{\Gamma(\alpha)}(t_2-t_1)^{\alpha-1}$$

and

$$\int_0^{t_1} e^{-\lambda\tau}\frac{(t_1-\tau)^{\alpha-2}-(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau \leq \int_0^{t_1}\frac{(t_1-\tau)^{\alpha-2}-(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau$$
$$= \frac{(t_2-t_1)^{\alpha-1}+t_1^{\alpha-1}-t_2^{\alpha-1}}{\Gamma(\alpha)}.$$

Using for $p\in]0,1]$ the inequality $|a^p-b^p|\leq|a-b|^p$ for all $a,b\geq 0$, we deduce that

$$\int_0^{t_1} e^{-\lambda\tau}\frac{(t_2-\tau)^{\alpha-2}-(t_1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau \leq \frac{2}{\Gamma(\alpha)}(t_2-t_1)^{\alpha-1}.$$

Then, since $\alpha\in]1,2]$, we can estimate $\|u_{f_n}(t_2)-u_{f_n}(t_1)\|$ by

$$\|u_{f_n}(t_2)-u_{f_n}(t_1)\|\leq K|t_2-t_1|^{\alpha-1}$$

with $K=\int_0^1\left[(3+\lambda)|\phi(s)|+(4+2\lambda)e^{\lambda s}\right]|X(s)|ds$. This shows that $\{u_{f_n}:n\in\mathbb{N}\}$ is equicontinuous in $\mathcal{C}_E(I)$. Moreover, for each $t\in I$ the set $\{u_{f_n}(t):n\in\mathbb{N}\}$ is contained in the compact convex set $\int_0^1 G(t,s)X(s)ds$ [?, 24], so that $\mathcal{X}$ is relatively compact in $\mathcal{C}_E(I)$ as claimed. It remains to justify that $\mathcal{X}$ is closed in $\mathcal{C}_E(I)$. Let $(u_{f_n})_n$ in $\mathcal{X}$ converging to u_∞ in $\mathcal{C}_E(I)$. As S_X^1 is $\sigma(L_E^1,L_{E^*}^\infty)$-compact (see, e.g, [24]) we may suppose that $(f_n)_n$ $\sigma(L_E^1,L_{E^*}^\infty)$-converges to $f_\infty\in S_X^1$. Then $(u_{f_n})_n$ pointwise weakly converges to u_{f_∞}, with $u_\infty\in\mathcal{C}_E(I)$ given by $u_{f_\infty}(t)=\int_0^1 G(t,s)f_\infty(s)ds$. Therefore, for each $t\in I$

$$u_\infty(t)=w\text{-}\lim_{n\to\infty}u_{f_n}(t)=w\text{-}\lim_{n\to\infty}\int_0^1 G(t,s)f_n(s)ds$$
$$=\int_0^1 G(t,s)f_\infty(s)ds=u_{f_\infty}(t),$$

so $u_\infty=u_{f_\infty}$, and the desired closedness of $\mathcal{X}$ in $\mathcal{C}_E(I)$ is confirmed. The proof of the theorem is complete. □

We can now state and prove the theorem concerned with a fractional inclusion coupled with an EVI.

Theorem 5.9. *Let $I:=[0,1]$ and H be a separable Hilbert space. Let K be a compact convex equilibrated subset of H. Let $\varphi:I\times H\to]-\infty,+\infty]$ be a normal lower semicontinuous convex integrand such that* $\operatorname{dom}\varphi(t,\cdot)=K$ *for all $t\in I$ and*

(i $\{\varphi(.,u(.)), u \in S^1_K\}$ *is uniformly integrable;*
(ii) $\varphi(t,x) \le \varphi(\tau,x) + |v(t)-v(\tau)|$ *for all* $t,\tau \in I$, $x \in K$, *where* $v : I \to \mathbb{R}^+$ *is a positive nondecreasing continuous function with* $v(0)=0$.
Let $A : H \to H$ *be a continuous coercive symmetric linear operator and* $B : H \to H$ *be a continuous linear operator. Let* $f : I \times H \to H$ *be a bounded continuous mapping, say* $\|f(t,x)\| \le M$ *for all* $(t,x) \in I \times H$.

Then for any $u_0 \in H$, *there exist a* $W^{\alpha,1}_{B,H}([0,1])$ *mapping* $x : I \to H$ *and an absolutely continuous mapping* $u : I \to H$ *satisfying*

$$\begin{cases} u(0) = u_0 \in H \\ D^\alpha x(t) + \lambda D^{\alpha-1} x(t) = u(t), \ t \in [0,1] \\ I^\beta_{0^+} x(t)\,|_{t=0} = 0, \quad x(1) = I^\gamma_{0^+} x(1) \\ f(t,x(t)) + Bu(t) - A\frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)), \text{ a.e. } t \in [0,1]. \end{cases}$$

Proof. Let us consider the compact convex (cf Theorem 5.8 and (5.7)) subset $\mathcal{X}$ in the Banach space $\mathcal{C}_H(I)$ defined by

$$\mathcal{X} := \{u_f : I \to H : u_f(t) = \int_0^1 G(t,s) f(s) ds, \ f \in S^1_{u_0+K}, \ t \in I\},$$

where $S^1_{u_0+K}$ denotes the set of all integrable selections of the compact convex valued constant multimapping $u_0 + K$. For each $h \in \mathcal{X}$, by Theorem 4.5 and the assumptions on f, there is a unique absolutely continuous solution v_h to the inclusion

$$\begin{cases} v_h(0) = u_0 \in H \\ f(t,h(t)) + Bv_h(t) - A\frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t)), \text{ a.e. } t \in I \end{cases}$$

with $\frac{dv_h}{dt}(t) \in K$ a.e. $t \in I$, so that $v_h(t) = u_0 + \int_0^t \frac{dv_h}{ds}(s) ds \in u_0 + \int_0^t K ds \subset u_0 + K$ for all $t \in I$.

For each $h \in \mathcal{X}$ consider the mapping $\Phi(h)$ defined on I by

$$\Phi(h)(t) := \int_0^t G(t,s) v_h(s) ds \quad \text{for all } t \in I.$$

It is clear that $\Phi(h) \in \mathcal{X}$. Let us check that Φ is continuous on $\mathcal{X}$. It is sufficient to show that, if $(h_n)_n$ uniformly converges to h in $\mathcal{X}$, then for v_{h_n} denoting the absolutely continuous solution of the inclusion

$$\begin{cases} v_{h_n}(0) = u_0 \in H \\ f(t,h_n(t)) + Bv_{h_n}(t) - A\frac{dv_{h_n}}{dt}(t) \in \partial\varphi(t, \frac{dv_{h_n}}{dt}(t)), \text{ a.e. } t \in I, \end{cases}$$

the sequence $(v_{h_n})_n$ uniformly converges to the absolutely continuous solution v_h of the inclusion

$$\begin{cases} v_h(0) = u_0 \in H \\ f(t,h(t)) + Bv_h(t) - A\frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t)), \text{ a.e. } t \in I. \end{cases}$$

As $(v_{h_n})_n$ is equi-absolutely continuous with $v_{h_n}(t) \in u_0 + \int_0^t K ds \subset u_0 + K$ for all $t \in I$ we may suppose that $(v_{h_n})_n$ uniformly converges to an absolutely continuous mapping u on I. Since $v_{h_n}(t) = u_0 + \int_{]0,t]} \frac{dv_{h_n}}{ds}(s)ds$, for all $t \in I$ and $\frac{dv_{h_n}}{ds}(s) \in K$ a.e. $s \in I$, we may also suppose that $(\frac{dv_{h_n}}{dt})_n$ weakly converges in $L^1_H(I)$ to $w \in L^1_H(I)$ with $w(t) \in K$ for all $t \in I$, so that

$$\lim_n v_{h_n}(t) = u_0 + \int_0^t w(s)ds \text{ for all } t \in I.$$

Identifying the limits yields for every $t \in I$

$$u(t) = u_0 + \int_0^t w(s)ds,$$

so $\dot{u} = w$. Therefore, by applying the arguments in the above variational limit result Cf. Proposition 3.6we get

$$f(t, h(t)) + Bu(t) - A\frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)), \text{ a.e. } t \in I$$

with $u(0) = u_0 \in H$, so that by uniqueness $u = v_h$.

On the other hand, writing

$$\begin{aligned}\Phi(h_n)(t) - \Phi(h)(t) &= \int_0^1 G(t,s)v_{h_n}(s)ds - \int_0^1 G(t,s)v_h(s)ds \\ &= \int_0^1 G(t,s)[v_{h_n}(s) - v_h(s)]ds\end{aligned}$$

we see by Lemma 5.6(a) that

$$\sup_{t\in[0,1]} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 M_G \|v_{h_n}(\cdot) - v_h(\cdot)\|ds.$$

This inequality and the uniform convergence of $(v_{h_n})_n$ to v_h on I entail that $\Phi(h_n) \to \Phi(h)$ in $\mathcal{C}_H(I)$, so $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous. This continuity of $\Phi : \mathcal{X} \to \mathcal{X}$ on the comapct convex set $\mathcal{X}$ of $\mathcal{C}_H(I)$ tells us that Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that

$$h(t) = \Phi(h)(t) = \int_0^1 G(t,s)v_h(s)ds \text{ for all } t \in I$$

with

$$\begin{cases} v_h(0) = u_0 \in H \\ D^\alpha h(t) + \lambda D^{\alpha-1}h(t) = v_h(t), \ t \in [0,1] \\ I^\beta_{0+}h(t)\,|_{t=0} = 0, \quad h(1) = I^\gamma_{0+}h(1) \\ f(t, h(t)) + Bv_h(t) - A\frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t)), \text{ a.e. } t \in I. \end{cases}$$

So by putting $x = h$ and $u = v_h$ we conclude that (x, u) solves the dynamic EVI

$$\begin{cases} u(0) = u_0 \in H \\ D^\alpha x(t) + \lambda D^{\alpha-1} x(t) = u(t),\ t \in [0,1] \\ I^\beta_{0^+} x(t)\,|_{t=0} = 0, \quad x(1) = I^\gamma_{0^+} x(1) \\ f(t, x(t)) + Bu(t) - A\frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)),\ \text{a.e. } t \in [0,1] \end{cases}$$

The proof is complete. □

Further variants of the above results are available. For instance, we are able to state the existence of solution to the dynamic system

$$\begin{cases} D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = u(t),\ t \in [0,1] \\ I^\beta_{0^+} h(t)\,|_{t=0} = 0, \quad h(1) = I^\gamma_{0^+} h(1) \\ u(0) = u_0 \\ f(t, h(t)) + Bu(t) - A\frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t)),\ \text{a.e. } t \in [0,1] \end{cases}$$

Our tools also allow to treat other variants by considering other class of FDI given in [11, 13, 14, 15, 16].

We study below an example of a Caputo fractional differential inclusion governed by an EVI. For the sake of completeness, we recall some needed properties for the fractional calculus and provide a series of lemmas on the fractional integral. Throughout we assume $\alpha \in]1, 2]$.

Definition 5.3. *The Caputo fractional derivative of order $\gamma > 0$ of a function $h : I = [0, T] \to H$, ${}^cD^\gamma h : [0, T] \to H$, is defined by*

$${}^cD^\gamma h(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{1-n+\gamma}} ds.$$

Here $n = [\gamma] + 1$ and $[\gamma]$ denotes the integer part of γ.

Denote by

$$W^{\alpha,\infty}_H(I) = \{u \in C^1_H(I) : {}^cD^{\alpha-1}u \in \mathcal{C}_H(I);\ {}^cD^\alpha u \in L^\infty_H(I)\},$$

where ${}^cD^{\alpha-1}u$ and ${}^cD^\alpha u$ are the fractional Caputo derivatives of order $\alpha - 1$ and α of u, respectively.

We summarize some properties of a Green function given in Lemma 2.1 of [15].

Lemma 5.10. *Let $I = [0, T]$ and let $G : I \times I \to \mathbb{R}$ be a function defined by*

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{1+t}{T+2}\Big[\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big], & \text{if } 0 \le s < t, \\ -\frac{1+t}{T+2}\Big[\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big] & \text{if } t \le s < T. \end{cases}$$

Let $f \in L^\infty_H(I)$. Then the system defined by

$$\begin{cases} {}^cD^\alpha u(t) = f(t),\ t \in [0,T] \\ u(0) - \frac{du}{dt}(0) = 0 \\ u(T) + \frac{du}{dt}(T) = 0 \end{cases}$$

has a unique $W^{\alpha,\infty}_H([0,T])$-solution u given by $u(t) = \int_0^T G(t,s)f(s)ds$, $\forall t \in I$. with $|G(t,s)| \leq M_G := \frac{2T^{\alpha-1}+(\alpha-1)T^{\alpha-2}}{\Gamma(\alpha)}$.

We recall and summarize a crucial lemma (Lemma 3.5 of [15]) for our next theorem.

Lemma 5.11. *Let $X : [0,T] \rightrightarrows H$ be a convex compact valued measurable mapping such that $|X(t)| \leq \gamma(t) < +\infty$, $\forall t \in I$ with $\gamma \in L^1(I)$. Then the $W^{\alpha,\infty}_H(I)$-solutions set $\mathcal{X}$ to*

$$\begin{cases} {}^cD^\alpha u(t) \in X(t),\ t \in I \\ u(0) - \frac{du}{dt}(0) = 0 \\ u(T) + \frac{du}{dt}(T) = 0, \end{cases}$$

is convex compact in $\mathcal{C}_H(I)$.

Now comes an existence result with a Caputo fractional differential inclusion.

Theorem 5.12. *Let $I := [0,1]$ and H be a separable Hilbert space. Let K be a convex compact equilibrated subset of H. Let $\varphi : [0,1] \times K \to]-\infty, +\infty]$ be a normal lower semicontinuous convex integrand such that*
(i $\{\varphi(.,u(.)), u \in S^1_K\}$ is uniformly integrable.
(ii) $\varphi(t,x) \leq \varphi(\tau,x) + |v(t) - v(\tau)|$ for all $t,\tau \in [0,1], x \in K$ where $v : [0,1] \to \mathbb{R}^+$ is a positive nondecreasing continuous function with $v(0) = 0$.
Let $A : H \to H$ be a linear continuous coercive symmetric operator and $B : H \to H$ be a linear continuous mapping.
Let $f : I \times H \to H$ be a bounded mapping : $||f(t,x)|| \leq M$ for all $(t,x) \in [0,1] \times H$ such that
(i) $f(\cdot,x)$ is $\mathcal{L}(I)$ measurable for all $x \in H$,
(ii) $f(t,\cdot)$ is continuous on H for all $t \in I$.
Then given $a \in H$, there is a $W^{\alpha,\infty}_H(I)$ mapping $x : I \to H$ and an absolutely continuous mapping $u : I \to H$ satisfying

$$\begin{cases} {}^cD^\alpha x(t) = v(t),\ t \in I \\ x(0) - \frac{dx}{dt}(0) = 0 \\ x(1) + \frac{dx}{dt}(1) = 0 \\ v(0) = a \\ f(t,x(t)) + Bu(t) - A\frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)),\ \text{a.e. } t \in [0,1] \end{cases}$$

Proof. For any continuous mapping $h : I \to H$, the mapping $f_h : I \times H \to H$ defined by $f_h(t) := f(t,h(t))$ for all $t \in I$ is $\mathcal{L}(I)$ measurable and satisfies

$|f_h(t)| \leq M \quad$ for all $t \in I$. Then by Theorem 4.4 and the assumptions on f, there is a unique absolutely continuous solution v_h to the inclusion

$$\begin{cases} v_h(0) = a \in H \\ \int_0^t f(s, h(s))ds + Bv_h(t) - A\frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t)), \text{ a.e. } t \in [0,1] \end{cases}$$

with $\frac{dv_h}{dt}(t) \in K$ a.e. so that $v_h(t) = a + \int_0^t \frac{dv_h}{ds}(s)ds \in a + \int_0^t Kds \subset a + K, \forall t \in [0,1]$ with v_h uniformly bounded and equi-absolutely continuous: $\frac{dv_h}{dt} \in K$. Now let us consider the set $\mathcal{X}$ defined by

$$\mathcal{X} := \{\xi_f : I \to H : \ f \in S^1_{a+K}\},$$

each mapping ξ_f being given for every $t \in I$ by

$$\xi_f(t) = \int_0^1 G(t,s)f(s)\,ds, f \in S^1_{a+K}$$

where G is the Green function given in Lemma 5.10. We note that $\mathcal{X}$ is convex compact in $\mathcal{C}_E(I)$ by Lemma 5.11. Now for each $h \in \mathcal{X}$, by Theorem 4.4 again denote by u_h the unique absolutely continuous solution of the differential inclusion

$$\int_0^t f(s, h(s))ds + Bu_h(t) - A\frac{du_h}{dt}(t) \in \partial\varphi(t, \frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I$$

$$u_h(0) = a \in H.$$

For each $h \in \mathcal{X}$ let us set (again with the above Green function G)

$$\Phi(h)(t) = \int_0^1 G(t,s)u_h(s)\,ds, \quad \text{for all } t \in I.$$

Then it is clear that $\Phi(h) \in \mathcal{X}$ because $u_h(t) \in a + K$ for all $t \in I$. Hence $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E(I)$ because $\Phi(\mathcal{X}) \subset \mathcal{X}$. Now we check that Φ is continuous relative to $\mathcal{X}$. It is enough to show that, if $(h_n)_n$ converges uniformly to h in $\mathcal{X}$, then the sequence $(u_{h_n})_n$, where each u_{h_n} is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = a \in H \\ \int_0^t f(s, h_n(s))ds + Bu_{h_n}(t) - A\frac{du_{h_n}}{dt}(t) \in \partial\varphi(t, \frac{du_{h_n}}{dt}(t)), \quad \text{a.e. } t \in I, \end{cases}$$

uniformly converges to the unique absolutely continuous solution u_h of the differential inclusion

$$\begin{cases} u_h(0) = a \in H \\ \int_0^t f(s, h(s))ds + Bu_h(t) - A\frac{du_h}{dt}(t) \in \partial\varphi(t, \frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

We note that $(u_{h_n})_n$ is equicontinuous since for every $n \in \mathbb{N}$ one has $\dot{u}_{h_n}(t) \in K$ for almost all $t \in I$. Further, $\{u_{h_n}(t) : n \in \mathbb{N}\}$ is included in the compact set

$a + K$ for every $t \in I$. The Arzelà-Ascoli theorem tells us that $\{u_{h_n} : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}_H(I)$. So by extracting a subsequence, we may suppose that $(u_{h_n})_n$ converges uniformly on I to some mapping $\zeta : I \to H$ with

$$\zeta(t) = a + \int_0^t \dot{\zeta}(s)\, ds \quad \text{for all } t \in I,$$

along with $(\dot{u}_{h_n})_n$ converging weakly in $L^1_H(I)$ to $\dot{\zeta}$ with $\dot{\zeta}(t) \in K$ for a.e. $t \in I$. Note that

$$\int_0^t f(s, h_n(s))ds \to \int_0^t f(s, h(s)ds \quad \text{for all } t \in I.$$

This combined with the variational limit theorem (Cf. Proposition 5.3) gives

$$\int_0^t f(s, h(s))ds + B\zeta(t) - A\frac{d\zeta}{dt}(t)) \in \partial\varphi(t, \frac{d\zeta}{dt}(t)), \quad \text{a.e. } t \in I$$

So using the uniqueness of solution of the latter differential inclusion we obtain that $\zeta = u_h$. Now let us write by Lemma 5.10 and boundedness of the Green function G

$$\begin{aligned}
\Phi(h_n)(t) - \Phi(h)(t) &= \int_0^1 G(t,s)u_{h_n}(s)\, ds - \int_0^1 G(t,s)u_h(s)\, ds \\
&= \int_0^1 G(t,s)[u_{h_n}(s) - u_h(s)]\, ds \\
&\leq \int_0^1 M_G \|u_{h_n}(s) - u_h(s)\|\, ds.
\end{aligned}$$

Since $\|u_{h_n}(\cdot) - u_h(\cdot)\| \to 0$ uniformly on I as $n \to \infty$, we deduce that

$$\sup_{t\in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 M_G \|u_{h_n}(\cdot) - u_h(\cdot)\|\, ds \to 0,$$

which entails that $\Phi(h_n) \to \Phi(h)$ uniformly on I, as desired. Then $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous, hence Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that for every $t \in I$

$$h(t) = \Phi(h)(t) = \int_0^1 G(t,s)u_h(s)\, ds,$$

with

$$\begin{cases} u_h(0) = a \\ \int_0^t f(s, h(s))ds + Bu_h(t) - A\frac{du_h}{dt}(t) \in \partial\varphi(t, \frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

Coming back to Lemma 5.10 and applying the above notations, this means that we have just shown that there exists a mapping $h \in W^{\alpha,\infty}_{B,H}(I)$ satisfying

$$\begin{cases} {}^cD^\alpha h(t) = u_h(t),\ t \in I \\ h(0) - \frac{dh}{dt}(0) = 0 \\ h(1) + \frac{dh}{dt}(1) = 0 \\ u_h(0) = a,\ t \in I \\ \int_0^t f(s, h(s))ds + Bu_h(t) - A\frac{du_h}{dt}(t) \in \partial\varphi(t, \frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

The proof of the theorem is then complete □

We finish this section with two variants.

Theorem 5.13. *Let $I = [0,1]$ and let $v : [0,1] \to \mathbb{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : [0,1] \to H$ be a convex compact valued multimapping such that*

$$\text{haus}\,(C(t), C(\tau)) \leq |v(t) - v(\tau)| \quad \textit{for all } t, \tau \in [0,1].$$

Let $A : H \to H$ be a linear continuous coercive symmetric operator and let $B : H \to H$ be a linear continuous compact operator.
Let $f : I \times H \to H$ be a bounded mapping : $||f(t,x)|| \leq M$ for all $(t,x) \in [0,1] \times H$ such that
(i) $f(\cdot, x)$ is $\mathcal{L}(I)$ measurable for all $x \in H$,
(ii) $f(t, \cdot)$ is continuous on H for all $t \in I$.
Then given $a \in H$, there is a $W^{\alpha,\infty}_H(I)$ mapping $x : I \to H$ and an absolutely continuous mapping $u : I \to H$ satisfying

$$\begin{cases} {}^cD^\alpha x(t) = u(t),\ t \in I \\ x(0) - \frac{dx}{dt}(0) = 0 \\ x(1) + \frac{dx}{dt}(1) = 0 \\ v(0) = a \\ \int_0^t f(s, x(s))ds + Bu(t) - A\frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t)),\ \text{a.e. } t \in [0,1] \end{cases}$$

Proof. For any continuous mapping $h : I \to H$, the mapping $f_h : I \times H \to H$ defined by $f_h(t) := f(t, h(t))$ for all $t \in I$ is $\mathcal{L}(I)$ measurable and satisfies $|f_h(t)| \leq M$ for all $t \in I$. Then by Theorem 4.1 and the assumptions on f, there is a unique absolutely continuous solution v_h to the inclusion

$$\begin{cases} v_h(0) = a \in H \\ \int_0^t f(s, h(s))ds + Bv_h(t) - A\frac{dv_h}{dt}(t) \in N_{C(t)}(\frac{dv_h}{dt}(t)),\ \text{a.e. } t \in [0,1] \end{cases}$$

with $\frac{dv_h}{dt}(t) \in K$, where $K := \bigcup_{t\in[0,1]} C(t)$ a.e. so that $v_h(t) = a + \int_0^t \frac{dv_h}{ds}(s)ds \in a + \int_0^t \overline{co}K ds \subset a + \overline{co}K, \forall t \in [0,1]$ with v_h uniformly bounded and equi-absolutely continuous. Now let us consider the set $\mathcal{X}$ defined by

$$\mathcal{X} := \{\xi_f : I \to H :\ f \in S^1_{a+\overline{co}K}\},$$

each mapping ξ_f being given for every $t \in I$ by

$$\xi_f(t) = \int_0^1 G(t,s)f(s)\,ds, f \in S^1_{a+\overline{co}K}$$

where G is the Green function given in Lemma 5.10. We note that $\mathcal{X}$ is convex compact in $\mathcal{C}_E(I)$ by Lemma 5.11. Now for each $h \in \mathcal{X}$, by Theorem 4.1 again denote by u_h the unique absolutely continuous solution of the differential inclusion

$$\int_0^t f(s,h(s))ds + Bu_h(t) - A\frac{du_h}{dt}(t) \in N_{C(t)}(\frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I$$

$$u_h(0) = a \in H.$$

For each $h \in \mathcal{X}$ let us set (again with the above Green function G)

$$\Phi(h)(t) = \int_0^1 G(t,s)u_h(s)\,ds, \quad \text{for all } t \in I.$$

Then it is clear that $\Phi(h) \in \mathcal{X}$ because $u_h(t) \in a + \overline{co}K$ for all $t \in I$. Hence $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E(I)$ because $\Phi(\mathcal{X}) \subset \mathcal{X}$. Now we check that Φ is continuous relative to $\mathcal{X}$. It is enough to show that, if $(h_n)_n$ converges uniformly to h in $\mathcal{X}$, then the sequence $(u_{h_n})_n$, where each u_{h_n} is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = a \in H \\ \int_0^t f(s,h_n(s))ds + Bu_{h_n}(t) - A\frac{du_{h_n}}{dt}(t) \in N_{C(t)}(\frac{du_{h_n}}{dt}(t)), \quad \text{a.e. } t \in I, \end{cases}$$

uniformly converges to the unique absolutely continuous solution u_h of the differential inclusion

$$\begin{cases} u_h(0) = a \in H \\ \int_0^t f(s,h(s))ds + Bu_h(t) - A\frac{du_h}{dt}(t) \in N_{C(t)}(\frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

We note that $(u_{h_n})_n$ is equicontinuous since for every $n \in \mathbb{N}$ one has $\dot{u}_{h_n}(t) \in \overline{co}K$ for almost all $t \in I$. Further, $\{u_{h_n}(t) : n \in \mathbb{N}\}$ is included in the convex compact set $a+\overline{co}K$ for every $t \in I$. The Arzelà-Ascoli theorem tells us that $\{u_{h_n} : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}_H(I)$. So by extracting a subsequence, we may suppose that $(u_{h_n})_n$ converges uniformly on I to some mapping $\zeta : I \to H$ with

$$\zeta(t) = a + \int_0^t \dot{\zeta}(s)\,ds \quad \text{for all } t \in I,$$

along with $(\dot{u}_{h_n})_n$ converging weakly in $L^1_H(I)$ to $\dot{\zeta}$ with $\dot{\zeta}(t) \in \overline{co}K$ for a.e. $t \in I$. Note that

$$\int_0^t f(s,h_n(s))ds \to \int_0^t f(s,h(s)ds \quad \text{for all } t \in I.$$

This combined with the variational limit theorem (Cf. Proposition 3.1) gives

$$\int_0^t f(s,h(s))ds + B\zeta(t) - A\frac{d\zeta}{dt}(t)) \in N_{C(t)}(\frac{d\zeta}{dt}(t)), \quad \text{a.e. } t \in I$$

So using the uniqueness of solution of the latter differential inclusion we obtain that $\zeta = u_h$. Now let us write by Lemma 5.10 and boundedness of the Green function G

$$\begin{aligned}\Phi(h_n)(t) - \Phi(h)(t) &= \int_0^1 G(t,s)u_{h_n}(s)\,ds - \int_0^1 G(t,s)u_h(s)\,ds \\ &= \int_0^1 G(t,s)[u_{h_n}(s) - u_h(s)]\,ds \\ &\leq \int_0^1 M_G\|u_{h_n}(s) - u_h(s)\|\,ds.\end{aligned}$$

Since $\|u_{h_n}(\cdot) - u_h(\cdot)\| \to 0$ uniformly on I as $n \to \infty$, we deduce that

$$\sup_{t\in I}\|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 M_G\|u_{h_n}(\cdot) - u_h(\cdot)\|\,ds \to 0,$$

which entails that $\Phi(h_n) \to \Phi(h)$ uniformly on I, as desired. Then $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous, hence Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that for every $t \in I$

$$h(t) = \Phi(h)(t) = \int_0^1 G(t,s)u_h(s)\,ds,$$

with

$$\begin{cases} u_h(0) = a \\ \int_0^t f(s,h(s))ds + Bu_h(t) - A\frac{du_h}{dt}(t) \in N_{C(t)}(\frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

Coming back to Lemma 5.10 and applying the above notations, this means that we have just shown that there exists a mapping $h \in W^{\alpha,\infty}_{B,H}(I)$ satisfying

$$\begin{cases} {}^cD^\alpha h(t) = u_h(t),\ t \in I \\ h(0) - \frac{dh}{dt}(0) = 0 \\ h(1) + \frac{dh}{dt}(1) = 0 \\ u_h(0) = a,\ t \in I \\ \int_0^t f(s,h(s))ds + Bu_h(t) - A\frac{du_h}{dt}(t) \in N_{C(t)}(\frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

The proof of the theorem is then complete □

Theorem 5.14. *Let $I = [0,1]$ and let $v : [0,1] \to \mathbb{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : [0,1] \to H$ be a convex compact valued multimapping such that*

$$\text{haus}\,(C(t), C(\tau)) \leq |v(t) - v(\tau)| \quad \textit{for all } t, \tau \in [0,1].$$

Let $A : H \to H$ be a linear continuous coercive symmetric operator and let g be a convex function on H Lipschitz on bounded sets and continuously differentiable on H whose gradient is locally Lipschitz. Let $f : I \times H \to H$ be a bounded mapping : $||f(t,x)|| \leq M$ for all $(t,x) \in [0,1] \times H$ such that
(i) $f(\cdot, x)$ is $\mathcal{L}(I)$ measurable for all $x \in H$,
(ii) $f(t,\cdot)$ is continuous on H for all $t \in I$.
Then given $a \in H$, there is a $W_H^{\alpha,\infty}(I)$ mapping $x : I \to H$ and an absolutely continuous mapping $u : I \to H$ satisfying

$$\begin{cases} {}^cD^\alpha x(t) = u(t), \ t \in I \\ x(0) - \frac{dx}{dt}(0) = 0 \\ x(1) + \frac{dx}{dt}(1) = 0 \\ v(0) = a \\ \int_0^t f(s, x(s))ds + \nabla g(u(t)) - A\frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t)), \text{ a.e. } t \in [0,1] \end{cases}$$

Proof. The proof is omitted by repeating the arguments given in the proof of Theorem 5.13. Here we apply Theorem 4.4 with the variational limit given in Proposition 3.3. □

An easy inspection of the tool developed above lead to a second order evolution like mechanical problem with dry friction.

Theorem 5.15. *Let $I = [0,1]$ and let $v : [0,1] \to \mathbb{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : [0,1] \to H$ be a convex compact valued multimapping such that*

$$\text{haus}\,(C(t), C(\tau)) \leq |v(t) - v(\tau)| \quad \text{for all } t, \tau \in [0,1].$$

Let $A : H \to H$ be a linear continuous coercive symmetric operator and let g be a convex function on H Lipschitz on bounded sets and continuously differentiable on H whose gradient is locally Lipschitz.
Let $f : I \times H \to H$ be a bounded mapping : $||f(t,x)|| \leq M$ for all $(t,x) \in [0,1] \times H$ such that
(i) $f(\cdot, x)$ is $\mathcal{L}(I)$ measurable for all $x \in H$,
(ii) $f(t,\cdot)$ is continuous on H for all $t \in I$.
Then given $a, b \in H$, there is an absolutely continuous mapping $x : I \to H$ and an absolutely continuous mapping $u : I \to H$ satisfying

$$\begin{cases} x(t) = a + \int_0^t u(s)ds, \ t \in I \\ u(0) = b \\ \int_0^t f(s, x(s))ds + \nabla g(u(t)) - A\frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t)), \text{ a.e. } t \in [0,1] \end{cases}$$

Proof. For any continuous mapping $h : I \to H$, the mapping $f_h : I \times H \to H$ defined by $f_h(t) := f(t, h(t))$ for all $t \in I$ is $\mathcal{L}(I)$ measurable and satisfies

$|f_h(t)| \leq M \quad$ for all $t \in I$. Then by Theorem 4.4 and the assumptions on f, there is a unique absolutely continuous solution v_h to the inclusion

$$\begin{cases} v_h(0) = b \in H \\ \int_0^t f(s, h(s))ds + \nabla g(v_h(t)) - A\frac{dv_h}{dt}(t) \in N_{C(t)}(\frac{dv_h}{dt}(t)), \text{ a.e. } t \in [0,1] \end{cases}$$

with $\frac{dv_h}{dt}(t) \in K$, where $K := \bigcup_{t\in[0,1]} C(t)$ a.e. so that $v_h(t) = b + \int_0^t \frac{dv_h}{ds}(s)ds \in b + \int_0^t \overline{co}(K \cup 0)ds \subset b + \overline{co}(K \cup 0), \forall t \in [0,1]$ with v_h uniformly bounded and equi-absolutely continuous. Now let us consider the set $\mathcal{X}$ defined by

$$\mathcal{X} := \{\xi_f : I \to H : \ f \in S^1_{b+\overline{co}(K\cup 0)}\},$$

each mapping ξ_f being given for every $t \in I$ by

$$\xi_f(t) = a + \int_0^t f(s)\, ds, t \in I, f \in S^1_{b+\overline{co}(K\cup 0)}$$

We note that $\mathcal{X}$ is convex compact in $\mathcal{C}_H(I)$. Now for each $h \in \mathcal{X}$, by Theorem 4.4 again denote by u_h the unique absolutely continuous solution of the differential inclusion

$$\int_0^t f(s, h(s))ds + \nabla g(u_h(t)) - A\frac{du_h}{dt}(t) \in N_{C(t)}(\frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I$$

$$u_h(0) = b \in H.$$

For each $h \in \mathcal{X}$ let us set

$$\Phi(h)(t) = a + \int_0^t u_h(s)\, ds, \quad \text{for all } t \in I.$$

Then it is clear that $\Phi(h) \in \mathcal{X}$ because $u_h(t) \in b + \overline{co}K$ for all $t \in I$. Hence $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E(I)$ because $\Phi(\mathcal{X}) \subset \mathcal{X}$. Now we check that Φ is continuous relative to $\mathcal{X}$. It is enough to show that, if $(h_n)_n$ converges uniformly to h in $\mathcal{X}$, then the sequence $(u_{h_n})_n$, where each u_{h_n} is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = b \in H \\ \int_0^t f(s, h_n(s))ds + \nabla g u_{h_n}(t) - A\frac{du_{h_n}}{dt}(t) \in N_{C(t)}(\frac{du_{h_n}}{dt}(t)), \quad \text{a.e. } t \in I, \end{cases}$$

uniformly converges to the unique absolutely continuous solution u_h of the differential inclusion

$$\begin{cases} u_h(0) = b \in H \\ \int_0^t f(s, h(s))ds + \nabla g(u_h(t)) - A\frac{du_h}{dt}(t) \in N_{C(t)}(\frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

We note that $(u_{h_n})_n$ is equicontinuous since for every $n \in \mathbb{N}$ one has $\dot{u}_{h_n}(t) \in \overline{co}K$ for almost all $t \in I$. Further, $\{u_{h_n}(t) : n \in \mathbb{N}\}$ is included in the convex compact

set $a+\overline{co}K$ for every $t \in I$. The Arzelà-Ascoli theorem tells us that $\{u_{h_n} : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}_H(I)$. So by extracting a subsequence, we may suppose that $(u_{h_n})_n$ converges uniformly on I to some mapping $\zeta : I \to H$ with

$$\zeta(t) = b + \int_0^t \dot{\zeta}(s)\, ds \quad \text{for all } t \in I,$$

along with $(\dot{u}_{h_n})_n$ converging weakly in $L^1_H(I)$ to $\dot{\zeta}$ with $\dot{\zeta}(t) \in \overline{co}K$ for a.e. $t \in I$. Note that

$$\int_0^t f(s, h_n(s))ds \to \int_0^t f(s, h(s)ds \quad \text{for all } t \in I.$$

This combined with the variational limit theorem (Cf. Proposition 3.3) gives

$$\int_0^t f(s, h(s))ds + \nabla g(\zeta(t)) - A\frac{d\zeta}{dt}(t)) \in N_{C(t)}(\frac{d\zeta}{dt}(t)), \quad \text{a.e. } t \in I$$

So using the uniqueness of solution of the latter differential inclusion we obtain that $\zeta = u_h$. Now let us write

$$\begin{aligned}\Phi(h_n)(t) - \Phi(h)(t) &= \int_0^t u_{h_n}(s)\, ds - \int_0^t u_h(s)\, ds \\ &= \int_0^t ||u_{h_n}(s) - u_h(s)||\, ds\end{aligned}$$

Since $\|u_{h_n}(\cdot) - u_h(\cdot)\| \to 0$ uniformly on I as $n \to \infty$, we deduce that

$$\sup_{t\in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 \|u_{h_n}(\cdot) - u_h(\cdot)\|\, ds \to 0,$$

which entails that $\Phi(h_n) \to \Phi(h)$ uniformly on I, as desired. Then $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous, hence Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that for every $t \in I$

$$h(t) = \Phi(h)(t) = a + \int_0^t u_h(s)\, ds, t \in I$$

with

$$\begin{cases} u_h(0) = b \in H \\ \int_0^t f(s, h(s))ds + \nabla g(u_h(t)) - A\frac{du_h}{dt}(t) \in N_{C(t)}(\frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

So we have just shown that there is an absolutely continuous mapping $x : I \to H$ with $x(0) = a$, $\dot{x}(0) = b$ such that

$$\int_0^t f(s, x(s))ds + \nabla g(\dot{x}(t)) - A\ddot{x}(t) \in N_{C(t)}(\ddot{x}(t)), \quad \text{a.e. } t \in I$$

□

Theorem 5.16. *Let $I := [0,1]$ and H be a separable Hilbert space. Let K be a convex compact equilibrated subset of H. Let $\varphi : [0,1] \times K \to]-\infty, +\infty]$ be a normal lower semicontinuous convex integrand such that*
(i $\{\varphi(.,u(.)), u \in S^1_K\}$ is uniformly integrable.
(ii) $\varphi(t,x) \leq \varphi(\tau,x) + |v(t) - v(\tau)|$ for all $t,\tau \in [0,1], x \in K$ where $v : [0,1] \to \mathbb{R}^+$ is a positive nondecreasing continuous function with $v(0) = 0$.
Let $A : H \to H$ be a linear continuous coercive symmetric operator and let g be a convex function on H Lipschitz on bounded sets and continuously differentiable on H whose gradient is locally Lipschitz.
Let $f : I \times H \to H$ be a bounded mapping : $||f(t,x)|| \leq M$ for all $(t,x) \in [0,1] \times H$ such that
(i) $f(\cdot,x)$ is $\mathcal{L}(I)$ measurable for all $x \in H$,
(ii) $f(t,\cdot)$ is continuous on H for all $t \in I$.
Then given $a, b \in H$, there is an absolutely continuous mapping $x : I \to H$ and an absolutely continuous mapping $u : I \to H$ satisfying

$$\begin{cases} x(t) = a + \int_0^t u(s)ds, \ t \in I \\ u(0) = b \\ \int_0^t f(s,x(s))ds + \nabla g(u(t)) - A\frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)), \text{ a.e. } t \in [0,1] \end{cases}$$

Proof. For any continuous mapping $h : I \to H$, the mapping $f_h : I \times H \to H$ defined by $f_h(t) := f(t,h(t))$ for all $t \in I$ is $\mathcal{L}(I)$ measurable and satisfies $|f_h(t)| \leq M$ for all $t \in I$. Then by Theorem 4.5 and the assumptions on f, there is a unique absolutely continuous solution v_h to the inclusion

$$\begin{cases} v_h(0) = b \in H \\ \int_0^t f(s,h(s))ds + \nabla g(v_h(t)) - A\frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t)), \text{ a.e. } t \in [0,1] \end{cases}$$

with $\frac{dv_h}{dt}(t) \in K$ a.e. so that $v_h(t) = b + \int_0^t \frac{dv_h}{ds}(s)ds \in b + \int_0^t \overline{K}ds \subset b + K \forall t \in [0,1]$ with v_h uniformly bounded and equi-absolutely continuous. Now let us consider the set $\mathcal{X}$ defined by

$$\mathcal{X} := \{\xi_f : I \to H : \ f \in S^1_{b+K}\},$$

each mapping ξ_f being given for every $t \in I$ by

$$\xi_f(t) = a + \int_0^t f(s)\,ds, t \in I, f \in S^1_{b+K}$$

We note that $\mathcal{X}$ is convex compact in $\mathcal{C}_H(I)$. Now for each $h \in \mathcal{X}$, by Theorem 4.5 again denote by u_h the unique absolutely continuous solution of the differential inclusion

$$\int_0^t f(s,h(s))ds + \nabla g(u_h(t)) - A\frac{du_h}{dt}(t) \in \partial\varphi(t, \frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I$$

$$u_h(0) = b \in H.$$

For each $h \in \mathcal{X}$ let us set

$$\Phi(h)(t) = a + \int_0^t u_h(s)\, ds, \quad \text{for all } t \in I.$$

Then it is clear that $\Phi(h) \in \mathcal{X}$ because $u_h(t) \in b + K$ for all $t \in I$. Hence $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E(I)$ because $\Phi(\mathcal{X}) \subset \mathcal{X}$. Now we check that Φ is continuous relative to $\mathcal{X}$. It is enough to show that, if $(h_n)_n$ converges uniformly to h in $\mathcal{X}$, then the sequence $(u_{h_n})_n$, where each u_{h_n} is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = b \in H \\ \int_0^t f(s, h_n(s))ds + \nabla g u_{h_n}(t) - A\frac{du_{h_n}}{dt}(t) \in \partial\varphi(t, \frac{du_{h_n}}{dt}(t)), \quad \text{a.e. } t \in I, \end{cases}$$

uniformly converges to the unique absolutely continuous solution u_h of the differential inclusion

$$\begin{cases} u_h(0) = b \in H \\ \int_0^t f(s, h(s))ds + \nabla g(u_h(t)) - A\frac{du_h}{dt}(t) \in \partial\varphi(t, \frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

We note that $(u_{h_n})_n$ is equicontinuous since for every $n \in \mathbb{N}$ one has $\dot{u}_{h_n}(t) \in K$ for almost all $t \in I$. Further, $\{u_{h_n}(t) : n \in \mathbb{N}\}$ is included in the convex compact set $a + \overline{co}K$ for every $t \in I$. The Arzelà-Ascoli theorem tells us that $\{u_{h_n} : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}_H(I)$. So by extracting a subsequence, we may suppose that $(u_{h_n})_n$ converges uniformly on I to some mapping $\zeta : I \to H$ with

$$\zeta(t) = b + \int_0^t \dot{\zeta}(s)\, ds \quad \text{for all } t \in I,$$

along with $(\dot{u}_{h_n})_n$ converging weakly in $L^1_H(I)$ to $\dot{\zeta}$ with $\dot{\zeta}(t) \in K$ for a.e. $t \in I$. Note that

$$\int_0^t f(s, h_n(s))ds \to \int_0^t f(s, h(s)ds \quad \text{for all } t \in I.$$

This combined with the variational limit theorem (Cf. Proposition 3.6) gives

$$\int_0^t f(s, h(s))ds + \nabla g(\zeta(t)) - A\frac{d\zeta}{dt}(t)) \in N_{C(t)}(\frac{d\zeta}{dt}(t)), \quad \text{a.e. } t \in I$$

So using the uniqueness of solution of the latter differential inclusion we obtain that $\zeta = u_h$. Now let us write

$$\begin{aligned} \Phi(h_n)(t) - \Phi(h)(t) &= \int_0^t u_{h_n}(s)\, ds - \int_0^t u_h(s)\, ds \\ &= \int_0^t ||u_{h_n}(s) - u_h(s)||\, ds \end{aligned}$$

Since $\|u_{h_n}(\cdot) - u_h(\cdot)\| \to 0$ uniformly on I as $n \to \infty$, we deduce that

$$\sup_{t\in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 \|u_{h_n}(\cdot) - u_h(\cdot)\|\, ds \to 0,$$

which entails that $\Phi(h_n) \to \Phi(h)$ uniformly on I, as desired. Then $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous, hence Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that for every $t \in I$

$$h(t) = \Phi(h)(t) = a + \int_0^t u_h(s)\, ds, t \in I$$

with

$$\begin{cases} u_h(0) = b \in H \\ \int_0^t f(s, h(s))ds + \nabla g(u_h(t)) - A\frac{du_h}{dt}(t) \in \partial\varphi(t, \frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

So we have just shown that there is an absolutely continuous mapping $x : I \to H$ with $x(0) = a$, $\dot{x}(0) = b$ such that

$$\int_0^t f(s, x(s))ds + \nabla g(\dot{x}(t)) - A\ddot{x}(t) \in \partial\varphi(t, \ddot{x}(t)), \quad \text{a.e. } t \in I$$

□

Remark. Theorem 5.16 hold if we replace the gradient ∇g by a linear continuous operator $B : H \to H$.

Comments on second order variational problems We have stated the existence of solution to a class of second order evolution inclusion of the form

$$0 \in A\ddot{x}(t) - B\dot{x}(t) - \int_0^t f(s, x(s))ds + \partial\varphi(t, \ddot{x}(t))$$

$$0 \in A\ddot{x}(t) - \nabla g(\dot{x}(t)) - \int_0^t f(s, x(s))ds + \partial\varphi(t, \ddot{x}(t))$$

with given operator A and B and given gradient ∇g, perturbation f and time dependent subdifferential operator $\partial\varphi_t$ and also in the context of sweeping process

$$0 \in A\ddot{x}(t) - B\dot{x}(t)) - \int_0^t f(s, x(s))ds + N_{C(t)}(\ddot{x}(t))$$

$$0 \in A\ddot{x}(t) - \nabla g(\dot{x}(t)) - \int_0^t f(s, x(s))ds + N_{C(t)}(\ddot{x}(t))$$

Existence and uniqueness of solution for a general second order evolution inclusion in a separable Hilbert space of the form

$$0 \in \ddot{u}(t) + A(t)\dot{u}(t) + f(t, u(t)),\ t \in [0, T]$$

where $A(t)$ is a time dependent Lipschitz variation maximal monotone operator and the perturbation $f(t,.)$ is boundedly Lipschitz is stated in [4] . In particular, existence and uniqueness of solution to

$$0 = \ddot{u}(t) + A(t)\dot{u}(t) + \nabla\varphi(u(t)),\ t \in [0,T]$$

where $A(t)$ is a time dependent Lipschitz variation single valued maximal monotone operator and $\nabla\varphi$ is the gradient of a smooth Lipschitz function φ is stated, by constrast with classical cases dealing with some special fixed operators; cf. Attouch et al [5], Paoli [31] and Schatzman [32]. Problems of second order have some importance in Mechanics [30], which may require a more general evolution inclusion of the form

$$0 \in \ddot{u}(t) + A(t)\dot{u}(t) + \partial\varphi(u(t)),\ t \in [0,T]$$

here $\partial\varphi(u(t))$ denotes the subdifferential of a proper lower semicontinuous convex function φ at the point $u(t)$. Existence for this problem is stated in [17] via a variational approach. In this spirit, existence of solution for a second order problem dealing with time and state dependent maximal monotone $A_{t,x}$ and multivalued perturbation $F(t,x,y)$ of the form

$$-\ddot{u}(t) \in A_{t,u(t)}\dot{u}(t) + F(t,u(t),\dot{u}(t))$$

is given in [15]. Second order evolution inclusion under consideration require concise and original proofs. We will give below a main variational limit result which help to give a meaning of the *variational limit solution* to the second order evolution inclusion of the form

$$f(t) \in \ddot{u}(t) + A\dot{u}(t) + \partial\varphi(u(t))$$

where A is a linear continuous coercive symmetric operator and $\partial\varphi$ is the subdifferential of a lower convex lower semi continuous function and $f \in L^2_H([0,T])$. In this context our results contains novelties since the second velocity is in inside the subdifferential (resp. the normal cone). These variants are not comparable.

We recall below some notations and summarize some results which describe the limiting behavior of a bounded sequence in $L^1_H([0,T])$. See ([22], Proposition 6.5.17).

Proposition 5.2. *Let H be a separable Hilbert space. Let (ζ_n) be a bounded sequence in $L^1_H([0,T])$. Then the following hold:*
1) (ζ_n) (up to an extracted subsequence) stably converges to a Young measure ν that is, there exist a subsequence (ζ'_n) of (ζ_n) and a Young measure ν belonging to the space of Young measure $\mathcal{Y}([0,T];H_\sigma)$ with $t \mapsto bar(\nu_t) \in L^1_H([0,T])$ (here $bar(\nu_t)$ denotes the barycenter of ν_t) such that

$$\lim_{n\to\infty} \int_0^T h(t,\zeta'_n(t)))\,dt) = \int_0^T [\int_H h(t,x)\,\nu_t(dx)]\,dt$$

for all bounded Carathéodory integrands $h : [0,T] \times H_\sigma \to \mathbb{R}$,
2) (ζ_n) *(up to an extracted subsequence) weakly biting converges to an integrable function* $f \in L^1_H([0,T])$*, which means that, there is a subsequence* (ζ'_m) *of* (ζ_n) *and an increasing sequence of Lebesgue-measurable sets* (A_p) *with* $\lim_p \lambda(A_p) = 1$ *and* $f \in L^1_H([0,T])$ *such that, for each* p,

$$\lim_{m\to\infty} \int_{A_p} \langle h(t), \zeta'_m(t)\rangle \, dt = \int_{A_p} \langle h(t), f(t)\rangle \, dt$$

for all $h \in L^\infty_H([0,T])$,
3) (ζ_n) *(up to an extracted subsequence) Komlós converges to an integrable function* $g \in L^1_H([0,T])$*, which means that, there is a subsequence* $(\zeta_{\beta(m)})$ *and an integrable function* $g \in L^1_H([0,T])$*, such that*

$$\lim_{n\to\infty} \frac{1}{n} \Sigma_{j=1}^n \zeta_{\gamma(j)}(t) = g(t), \; a.e. \in [0,T],$$

for every subsequence $(f_{\gamma(n)})$ *of* $(f_{\beta(n)})$.
4) There is a filter $\mathcal{U}$ *finer than the Fréchet filter such that* $\mathcal{U} - \lim_n \zeta_n = l \in (L^\infty_H)'_{weak}$ *where* $(L^\infty_H)'_{weak}$ *is the second dual of* $L^1_H([0,T])$.
Let $w_{l_a} \in L^1_H([0,T])$ *be the density of the absolutely continuous part* l_a *of* l *in the decomposition* $l = l_a + l_s$ *in absolutely continuous part* l_a *and singular part* l_s.
If we have considered the same extracted subsequence in 1), 2), 3), 4), then one has

$$f(t) = g(t) = bar(\nu_t) = w_{l_a}(t) \; a.e. \; t \in [0,T]$$

By $W^{2,1}_{\mathbb{R}^d}([0,T])$ (resp. $W^{2,2}_{\mathbb{R}^d}([0,T])$ we denote the set of all continuous functions in $\mathcal{C}_{\mathbb{R}^d}([0,T])$ such that their first derivatives are continuous and their second derivatives belong to $L^1_{\mathbb{R}^d}([0,T])$ (resp. $L^2_{\mathbb{R}^d}([0,T])$) and by $W^{1,1}_{BV}([0,T])$ we denote the set of all continuous functions in $\mathcal{C}_{\mathbb{R}^d}([0,T])$ such that their first derivatives are of bounded variation (BV for short).

Let us recall a useful Gronwall type lemma [23].

Lemma 5.17. *(A Gronwall-like inequality.) Let* $p, q, r : [0,T] \to [0,\infty[$ *be three nonnegative Lebesgue integrable functions such that for almost all* $t \in [0,T]$

$$r(t) \leq p(t) + q(t) \int_0^t r(s) \, ds.$$

Then

$$r(t) \leq p(t) + q(t) \int_0^t [p(s) \exp(\int_s^t q(\tau) \, d\tau)] \, ds$$

for all $t \in [0,T]$.

Here is a main variational limit result which help to give a meaning of the variational limit solution to the second order evolution inclusion

$$f(t) \in \ddot{u}(t) + A\dot{u}(t) + \partial\varphi(u(t))$$

where A is a linear continuous coercive symmetric operator and $\partial\varphi$ is the subdifferential of a lower convex lower semi continuous function φ and $f \in L^2_H([0,T])$.

Proposition 5.3. *Assume that $\beta \in L^2_{\mathbb{R}^+}([0,T])$ and A is a linear continuous symmetric and coercive operator: $\langle Ax, x\rangle \geq M||x||^2$ for all $x \in \mathbb{R}^d$ where M is a positive constant. Let $n \in \mathbb{N}$ and $\varphi_n : \mathbb{R}^d \to \mathbb{R}^+$ be a C^1, convex, Lipschitz function and let φ_∞ be a nonnegative l.s.c proper function defined on $\mathbb{R}^d$ such that $\varphi_n(x) \leq \varphi_\infty(x)$ for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}^d$. Let $f^n \in L^2_{\mathbb{R}^d}([0,T])$ such that $||f_n(t)|| \leq \beta(t)$, $\forall n \in \mathbb{N}$, $\forall t \in [0,T]$. For each $n \in \mathbb{N}$, let u^n be a $W^{2,1}_{\mathbb{R}^d}([0,T])$-solution to the problem*

$$\begin{cases} f^n(t) \in \ddot{u}^n(t) + A\dot{u}^n(t) + \partial\varphi_n(u^n(t)), t \in [0,T] \\ u^n(0) = u^n_0;\ \dot{u}^n(0) = \dot{u}^n_0. \end{cases}$$

Assume that
(i) f^n $\sigma(L^2_{\mathbb{R}^d}, L^2_{\mathbb{R}^d})$-converges to $f^\infty \in L^2_{\mathbb{R}^d}([0,T])$,
(ii) φ_n epi-converges to φ_∞,
(iii) $\lim_n u^n(0) = u^\infty_0 \in dom\,\varphi_\infty$, $\lim_n \varphi_n(u^n(0)) = \varphi_\infty(u^\infty_0)$, and $\lim_n \dot{u}^n(0) = \dot{u}^\infty_0$,
(iv) There exist $r_0 > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\sup_{n\in\mathbb{N}} \sup_{v \in \overline{B}_{L^\infty_{\mathbb{R}^d}([0,T])}} \int_0^T \varphi_\infty(x_0 + r_0 v(t)) < +\infty$$

here $\overline{B}_{L^\infty_{\mathbb{R}^d}([0,T])}$ is the closed unit ball in $L^\infty_{\mathbb{R}^d}([0,T])$.
(a) Then up to extracted subsequences, (u^n) converges uniformly to an $W^{1,1}_{BV}([0,T])$-function u^∞ and $(\dot{u}^n)$ pointwisely converges to a BV function v^∞ with $v^\infty = \dot{u}^\infty$, and $(\ddot{u}^n)$ biting converges to a function $\zeta^\infty \in L^1_{\mathbb{R}^d}([0,T])$ so that the limit function $u^\infty, \dot{u}^\infty$ and the biting limit ζ^∞ satisfy the variational inclusion

$$f^\infty \in \zeta^\infty + A\dot{u}^\infty + \partial I_{\varphi_\infty}(u^\infty)$$

here $\partial I_{\varphi_\infty}$ denotes the subdifferential of the convex lower semicontinuous integral functional I_{φ_∞} defined on $L^\infty_{\mathbb{R}^d}([0,T])$

$$I_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt,\ \forall u \in L^\infty_{\mathbb{R}^d}([0,T]).$$

Furthermore $\lim_n \varphi_n(u^n(t)) = \varphi_\infty(u^\infty(t)) < \infty$ a.e. and $\lim_n \int_0^T \varphi_n(u^n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt$. Subsequently, the energy estimate holds true almost everywhere $t \in [0,T]$,

$$\varphi_\infty(u^\infty(t)) + \frac{1}{2}||\dot{u}^\infty(t)||^2 = \varphi_\infty(u^\infty_0) + \frac{1}{2}||\dot{u}^\infty_0||^2$$
$$- \int_0^t \langle A\dot{u}^\infty(s), \dot{u}^\infty(s)\rangle ds + \int_0^t \langle \dot{u}^\infty(s), f^\infty(s)\rangle ds$$

Further $(\ddot{u}^n)$ *weakly converges to the vector measure* $m \in \mathcal{M}^b_{\mathbb{R}^d}([0,T])$ *so that the limit functions* $u^\infty(.)$ *and the limit measure* m *satisfy the following variational inequality:*

$$\begin{aligned}\int_0^T \varphi_\infty(v(t))\,dt &\geq \int_0^1 \varphi_\infty(u^\infty(t))\,dt + \int_0^1 \langle -A\dot{u}^\infty(t) + f^\infty(t), v(t) - u^\infty(t)\rangle\,dt \\ &+ \langle -m, v - u^\infty\rangle_{(\mathcal{M}^b_{\mathbb{R}^d}([0,T]),\mathcal{C}_{\mathbb{R}^d}([0,T]))}.\end{aligned}$$

In other words, the vector measure $-m + [-A\dot{u}^\infty + f^\infty]\,dt$ *belongs to the subdifferential* $\partial J_{\varphi_\infty}(u^\infty)$ *of the convex functional integral* J_{φ_∞} *defined on* $\mathcal{C}_{\mathbb{R}^d}([0,T])$ *by* $J_{\varphi_\infty}(v) = \int_0^1 \varphi_\infty(t, v(t))\ dt$, $\forall v \in \mathcal{C}_{\mathbb{R}^d}([0,T])$.
(b) There are a filter $\mathcal{U}$ *finer than the Fréchet filter,* $l \in L^\infty_{\mathbb{R}^d}([0,T])'$ *such that*

$$\mathcal{U} - \lim_n [f^n - \ddot{u}^n - A\dot{u}^n] = l \in L^\infty_{\mathbb{R}^d}([0,T])'_{weak}$$

where $L^\infty_{\mathbb{R}^d}([0,T])'_{weak}$ *is the second dual of* $L^1_{\mathbb{R}^d}([0,T])$ *endowed with the topology* $\sigma(L^\infty_{\mathbb{R}^d}([0,T])', L^\infty_{\mathbb{R}^d}([0,T]))$ *and* $\mathbf{n} \in \mathcal{C}_{\mathbb{R}^d}([0,T])'_{weak}$ *such that*

$$\lim_n [f^n - \ddot{u}^n - A\dot{u}^n] = \mathbf{n} \in \mathcal{C}_{\mathbb{R}^d}([0,T])'_{weak}$$

here $\mathcal{C}_{\mathbb{R}^d}([0,T])'_{weak}$ *denotes the space* $\mathcal{C}_{\mathbb{R}^d}([0,T])'$ *endowed with the weak topology* $\sigma(\mathcal{C}_{\mathbb{R}^d}([0,T])', \mathcal{C}_{\mathbb{R}^d}([0,T]))$. *Let* l_a *be the density of the absolutely continuous part* l_a *of* l *in the decomposition* $l = l_a + l_s$ *in absolutely continuous part* l_a *and singular part* l_s. *Then*

$$l_a(h) = \int_0^T \langle h(t), f^\infty(t) - \zeta^\infty(t) - A\dot{u}^\infty(t)\rangle dt$$

for all $h \in L^\infty_{\mathbb{R}^d}([0,T])$ *so that*

$$I^*_{\varphi^*_\infty}(l) = I_{\varphi^*_\infty}(f^\infty - \zeta^\infty - A\dot{u}^\infty) + \delta^*(l_s, dom I_{\varphi_\infty})$$

here φ^*_∞ *is the conjugate of* φ_∞, $I_{\varphi^*_\infty}$ *the integral functional defined on* $L^1_{\mathbb{R}^d}([0,T])$ *associated with* φ^*_∞, $I^*_{\varphi_\infty}$ *the conjugate of the integral functional* I_{φ_∞}, $dom I_{\varphi_\infty} := \{u \in L^\infty_{\mathbb{R}^d}([0,T]) : I_{\varphi_\infty}(u) < \infty\}$ *and*

$$\langle \mathbf{n}, h\rangle = \int_0^T \langle f^\infty(t) - \zeta^\infty(t) - A\dot{u}^\infty(t), h(t)\rangle dt + \langle \mathbf{n}_s, h\rangle, \quad \forall h \in \mathcal{C}_{\mathbb{R}^d}([0,T]).$$

with $\langle \mathbf{n}_s, h\rangle = l_s(h)$, $\forall h \in \mathcal{C}_{\mathbb{R}^d}([0,T])$. *Further* $\mathbf{n}$ *belongs to the subdifferential* $\partial J_{\varphi_\infty}(u^\infty)$ *of the convex lower semicontinuous integral functional* J_{φ_∞} *defined on* $\mathcal{C}_{\mathbb{R}^d}([0,T])$

$$J_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt,\ \forall u \in \mathcal{C}_{\mathbb{R}^d}([0,T]).$$

(c) Consequently the density $f^\infty - \zeta^\infty - A\dot{u}^\infty$ of the absolutely continuous part n_a

$$\mathbf{n}_a(h) := \int_0^T \langle f^\infty(t) - \zeta^\infty(t) - A\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in \mathcal{C}_{\mathbb{R}^d}([0,T])$$

satisfies the inclusion

$$f^\infty(t) - \zeta^\infty(t) - A\dot{u}^\infty(t) \in \partial\varphi_\infty(u^\infty(t)), \quad \text{a.e..}$$

and for any nonnegative measure θ on $[0,T]$ with respect to which $\mathbf{n}_s$ is absolutely continuous

$$\int_0^T h_{\varphi_\infty^*}(\frac{d\mathbf{n}_s}{d\theta}(t))d\theta(t) = \int_0^T \langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\rangle d\theta(t)$$

here $h_{\varphi_\infty^*}$ denotes the recession function of φ_∞^*.

Proof. Step 1 $||\dot{u}^n(.)||$ and $\varphi_n(u_n(.))$ are uniformly bounded.
Multiplying scalarly the inclusion

$$f^n(t) - \ddot{u}^n(t) - A\dot{u}^n(t) \in \partial\varphi_n(u^n(t))$$

by $\dot{u}^n(t)$ and applying the chain rule theorem ([35], Theorem 2) yields

$$\langle \dot{u}^n(t), f^n(t)\rangle - \langle \dot{u}^n(t), \ddot{u}^n(t)\rangle - \langle \dot{u}^n(t), A\dot{u}_n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t))]$$

that is

$$-\langle A\dot{u}^n(t), \dot{u}^n(t)\rangle + \langle \dot{u}^n(t), f^n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t)) + \frac{1}{2}||\dot{u}^n(t)||^2]. \tag{3.3.1}$$

By integrating on $[0,t]$ this equality we get

$$\begin{aligned}
\varphi_n(u^n(t)) + \frac{1}{2}||\dot{u}^n(t)||^2 &= \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 \\
&\quad - \int_0^t \langle A\dot{u}^n(s), \dot{u}^n(s)\rangle ds + \int_0^t \langle \dot{u}^n(s), f^n(s)\rangle ds \\
&\leq \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 \\
+M\int_0^t ||\dot{u}^n(s)||^2 ds + ||f^n||_{L^2_{\mathbb{R}^d}([0,T])}(\int_0^t ||\dot{u}^n(s)||^2 ds)^{\frac{1}{2}} &\\
&\leq \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 \\
+M\int_0^t ||\dot{u}^n(s)||^2 ds + \frac{1}{2}||f^n||_{L^2_{\mathbb{R}^d}([0,T])}(1 + \int_0^t ||\dot{u}^n(s)||^2 ds) &\\
&\leq \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 \\
+M\int_0^t ||\dot{u}^n(s)||^2 ds + \frac{1}{2}||\beta||_{L^2_{\mathbb{R}}([0,T])}(1 + \int_0^t ||\dot{u}^n(s)||^2 ds). &
\end{aligned}$$

Then from (*iii*), the preceding estimate and the Gronwall like inequality (Lemma 3.1), it is immediate that

$$\sup_{n\geq 1}\sup_{t\in[0,T]}||\dot{u}^n(t)|| < +\infty \quad \text{and} \quad \sup_{n\geq 1}\sup_{t\in[0,T]}\varphi_n(u^n(t)) < +\infty. \tag{3.3.2}$$

Step 2 Estimation of $||\ddot{u}^n(.)||$. As

$$z^n(t) := f^n(t) - \ddot{u}^n(t) - A\dot{u}^n(t) \in \partial\varphi_n(u^n(t))$$

by the subdifferential inequality for convex lower semi continuous functions we have

$$\varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbb{R}^d$. Now let $v \in \overline{B}_{L^\infty_{\mathbb{R}^d}([0,T])}$, the closed unit ball of $L^\infty_{\mathbb{R}^d}[0,T]$). By taking $x = w(t) := x_0 + r_0 v(t)$ in the preceding inequality we get

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle w(t) - u^n(t), z^n(t)\rangle.$$

Integrating the preceding inequality gives

$$\begin{aligned}
&\int_0^T \langle x_0 + r_0 v(t) - u^n(t), z^n(t)\rangle dt \\
&= \int_0^T \langle x_0 - u^n(t), z^n(t)\rangle dt + r_0 \int_0^T \langle v(t), z^n(t)\rangle dt \\
&\leq \int_0^T \varphi_n(x_0 + r_0 v(t))dt - \int_0^T \varphi_n(u^n(t))dt.
\end{aligned}$$

Whence follows

$$\begin{aligned}
r_0 \int_0^T \langle v(t), z^n(t)\rangle dt &\leq \int_0^T \varphi_n(x_0 + r_0 v(t))dt \\
&- \int_0^T \varphi_n(u^n(t))dt - \int_0^T \langle x_0 - u^n(t), z^n(t)\rangle dt.
\end{aligned} \tag{3.3.3}$$

We compute the last integral in the preceding inequality. For simplicity, let us set $v^n(t) = u^n(t) - x_0$ for all $t \in [0,T]$. By integration by parts and taking account into (3.3.2) we have

$$\begin{aligned}
&-\int_0^T \langle x_0 - u^n(t), z^n(t)\rangle dt = -\int_0^T \langle v^n(t), \ddot{v}^n(t) + A\dot{v}^n(t)\rangle - f^n(t)\rangle dt \qquad (3.3.4)\\
&= -[\langle v^n(t), \dot{v}^n(t) + Av^n(t)]_0^T + \int_0^T \langle \dot{v}^n(t), \dot{v}^n(t) + Av^n(t)\rangle dt + \int_0^T \langle v^n(t), f^n(t)\rangle dt \\
&\leq -\langle v^n(T), \dot{v}^n(T)\rangle + \langle v^n(0), \dot{v}^n(0)\rangle - \langle Av^n(T), v^n(T)\rangle \\
&+\langle Av^n(0), v^n(0)\rangle + \int_0^T ||\dot{v}^n(t)||^2 dt + \int_0^T \langle \dot{v}^n(t), Av^n(t)\rangle dt + \int_0^T \langle v^n(t), f^n(t)\rangle dt.
\end{aligned}$$

By (3.3.2) $-$ $-$(3.3.4), we get

$$(3.3.5) \quad r_0 \int_0^T \langle v(t), z^n(t)\rangle dt \leq \int_0^T \varphi_\infty(x_0 + r_0 v(t))dt + L$$

for all $v \in \overline{B}_{L^\infty_{\mathbb{R}^d}([0,T])}$, here L is a generic positive constant independent of $n \in \mathbb{N}$. By (iv) and (3.3.5) we conclude that $(z^n = f^n - \ddot{u}^n - A\dot{u}^n)$ is bounded in $L^1_{\mathbb{R}^d}([0,T])$, then so is $(\ddot{u}^n)$. It turns out that the sequence $(\dot{u}^n)$ is uniformly bounded by using (3.3.2) and is bounded in variation. By Helly theorem, we may assume that $(\dot{u}^n)$ pointwisely converges to a BV function $v^\infty : [0,T] \to \mathbb{R}^d$ and the sequence (u^n) converges uniformly to an absolutely continuous function u^∞ with $\dot{u}^\infty = v^\infty$ a.e. At this point, it is clear that $(\dot{u}_n)$ converges in $L^1_{\mathbb{R}^d}([0,T])$ to v^∞, using (3.4.2) and the dominated convergence theorem. Hence $(A\dot{u}^n(.))$ converges in $L^1_{\mathbb{R}^d}([0,T])$ to $Av^\infty(.)$.

Step 3. Young measure limit and biting limit of $\ddot{u}_n$. As $(\ddot{u}_n)$ is bounded in $L^1_{\mathbb{R}^d}([0,T])$, we may assume that $(\ddot{u}^n)$ stably converges to a Young measure $\nu \in \mathcal{Y}([0,T];\mathbb{R}^d)$ with bar $(\nu) : t \mapsto$ bar $(\nu_t) \in L^1_{\mathbb{R}^d}([0,T])$ (here bar (ν_t) denotes the barycenter of ν_t). Further by Proposition 3.1, we may assume that $(\ddot{u}^n)$ biting converges to a function $\zeta^\infty : t \mapsto$ bar (ν_t) that is, there exists a decreasing sequence of Lebesgue-measurable sets (B_p) with $\lim_p \lambda(B_p) = 0$ such that the restriction of $(\ddot{u}_n)$ on each B_p^c converges weakly in $L^1_{\mathbb{R}^d}([0,T])$ to ζ^∞. Noting that $(A\dot{u}^n)$ converges in $L^1_{\mathbb{R}^d}([0,T])$ to Av^∞. It follows that the restriction of $(z^n = f^n - \ddot{u}^n - A\dot{u}^n)$ to each B_p^c weakly converges in $L^1_{\mathbb{R}^d}([0,T])$ to $z^\infty := f^\infty - \zeta^\infty - Av^\infty$, because (f^n) weakly converges in $L^1_{\mathbb{R}^d}([0,T])$ to f^∞, $(A\dot{u}^n)$ converges in $L^1_{\mathbb{R}^d}([0,T])$ to Av^∞ and $(\ddot{u}^n)$ biting converges to $\zeta^\infty \in L^1_{\mathbb{R}^d}([0,T])$. It follows that

$$(3.4.6) \quad \lim_n \int_B \langle -\ddot{u}^n - W^n(t), w(t) - u^n(t)\rangle = \int_B \langle -\text{bar } (\nu_t) - W(t), w(t) - u(t)\rangle dt$$

for every $B \in B_p^c \cap \mathcal{L}([0,T])$, and for every $w \in L^\infty_{\mathbb{R}^d}([0,T])$, where $W^n(t) = M\dot{u}^n(t) - f^n(t)$ and $W(t) = M\dot{u}^\infty - f^\infty$. Indeed, we note that $(w(t) - u^n(t))$ is a bounded sequence in $L^\infty_{\mathbb{R}^d}([0,1])$ which pointwisely converges to $w(t) - u^\infty(t)$, it converges uniformly on every uniformly integrable subset of $L^1_{\mathbb{R}^d}([0,T])$ by virtue of a Grothendieck Lemma [26], recalling here that the restriction of $-\ddot{u}^n - W^n$ on each B_p^c is uniformly integrable. Now, since φ_n lower epiconverges to φ_∞, for every Lebesgue-measurable set A in $[0,T]$, by virtue of Corollary 4.7 in [21], we have

$$(3.3.7) \qquad +\infty > \liminf_n \int_A \varphi_n(u^n(t))dt \geq \int_A \varphi_\infty(u^\infty(t))dt$$

Combining (3.3.2)-(3.3.5)-(3.3.6)-(3.3.7) and using the subdifferential inequality

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle -\ddot{u}^n(t) - W^n(t), w(t) - u^n(t)\rangle$$

gives

$$\int_B \varphi_\infty(w(t))\,dt \geq \int_B \varphi_\infty(u^\infty(t))\,dt + \int_B \langle -\text{bar }(\nu_t) - W(t), w(t) - u^\infty(t)\rangle\,dt.$$

This shows that $t \mapsto -\text{bar }(\nu_t) - W(t)$ is a subgradient at the point u^∞ of the convex integral functional I_{φ_∞} restricted to $L^\infty_{\mathbb{R}^d}(B_p^c)$, consequently,

$$-\text{bar }(\nu_t) - W(t) \in \partial\varphi_\infty(u^\infty(t)), \text{ a.e. on } B_p^c.$$

As this inclusion is true on each B_p^c and $B_p^c \uparrow [0,T]$, we conclude that

$$-\text{bar }(\nu_t) - W(t) \in \partial\varphi_\infty(u^\infty(t)), \text{ a.e. on } [0,T].$$

Step 4. Measure limit in $\mathcal{M}^b_{\mathbb{R}^d}([0,T])$ of $\ddot{u}^n$. As $(\ddot{u}_n)$ is bounded in $L^1_{\mathbb{R}^d}([0,T])$, we may assume that $(\ddot{u}^n)$ weakly converges to the vector measure $m \in \mathcal{M}^b_{\mathbb{R}^d}([0,T])$ so that the limit functions $u^\infty(.)$ and the limit measure m satisfy the following variational inequality:

$$\begin{aligned}\int_0^T \varphi_\infty(v(t))\,dt &\geq \int_0^1 \varphi_\infty(u^\infty(t))\,dt + \int_0^1 \langle -A\dot{u}^\infty(t) + f^\infty(t), v(t) - u^\infty(t)\rangle\,dt \\ &+ \langle -m, v - u^\infty\rangle_{(\mathcal{M}^b_E([0,T]),\mathcal{C}_{\mathbb{R}^d}([0,T]))}.\end{aligned}$$

In other words, the vector measure $-m + [-A\dot{u}^\infty + f^\infty]\,dt = -m - W.dt$ belongs to the subdifferential $\partial J_{\varphi_\infty}(u^\infty)$ of the convex functional integral J_{f_∞} defined on $\mathcal{C}_{\mathbb{R}^d}([0,T])$ by $J_{\varphi_\infty}(v) = \int_0^1 \varphi_\infty(v(t))\ dt$, $\forall v \in \mathcal{C}_{\mathbb{R}^d}([0,T])$. Indeed, let $w \in \mathcal{C}_{\mathbb{R}^d}([0,T])$. Integrating the subdifferential inequality

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle -\ddot{u}^n(t) - W^n(t), w(t) - u^n(t)\rangle$$

and noting that $\varphi_\infty(w(t)) \geq \varphi_n(w(t))$ gives immediately

$$\begin{aligned}\int_0^T \varphi_\infty(w(t))dt &\geq \int_0^T \varphi_n(w(t))dt \\ \geq \int_0^T \varphi_n(u^n(t))dt &+ \langle -\ddot{u}^n(t) - W^n(t), w(t) - u^n(t)\rangle dt\end{aligned}$$

We note that

$$\lim_n \int_0^T \langle -W^n(t), w(t) - u^n(t)\rangle dt = \int_0^T \langle -W(t), w(t) - u^\infty(t)\rangle dt$$

because $(W^n := A\dot{u}^n - f^n)$ is uniformly integrable, and weakly converges to $W := A\dot{u}^\infty - f^\infty$ and the bounded sequence in $w(t) - u^n(t)$ pointwise converges

to $w-u^\infty$ so that it converges uniformly on uniformly integrable subsets by virtue of Grothendieck lemma. Whence follows

$$\int_0^T \varphi_\infty(w(t))dt \geq \int_0^T \varphi_\infty(u^\infty(t))dt + \int_0^T \langle -W(t), w(t)-u^\infty(t)\rangle dt \\ \langle -m, w-u^\infty\rangle_{(\mathcal{M}^b_{\mathbb{R}^d}([0,T]),\mathcal{C}_{\mathbb{R}^d}([0,T]))}$$

which shows that that the vector measure $-m - W.dt$ is a subgradient at the point u^∞ of the of the convex integral functional J_{φ_∞} defined on $\mathcal{C}_{\mathbb{R}^d}([0,T]))$ by $J_{\varphi_\infty}(v) := \int_0^T \varphi_\infty(v(t))dt, \forall v \in \mathcal{C}_{\mathbb{R}^d}([0,T])$.

Step 5. **Claim** $\lim_n \varphi_n(u^n(t)) = \varphi_\infty(u^\infty(t)) < \infty$ a.e, and $\lim_n \int_0^T \varphi_n(u^n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt < \infty$, and subsequently, the energy estimate holds for a.e $t \in [0,T]$:

$$\varphi_\infty(u^\infty(t)) + \frac{1}{2}||\dot{u}^\infty(t)||^2 = \varphi_\infty(u^\infty(0)) + \frac{1}{2}||\dot{u}^\infty(0)||^2 \\ + \int_0^t \langle A\dot{u}^\infty(s), \dot{u}^\infty(s)\rangle ds - \int_0^t \langle \dot{u}^\infty(s), f^\infty(s)\rangle ds$$

With the above results and notations, applying the subdifferential inequality

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle -\ddot{u}^n(t) - W^n(t), w(t) - u^n(t)\rangle$$

with $w = u^\infty$, integrating on $[0,T]$, and passing to the limit when n goes to ∞, gives the inequality

$$\int_B \varphi_\infty(u^\infty(t))dt \geq \liminf_n \int_B \varphi_n(u^n(t))dt \\ \geq \int_B \varphi_\infty(u^\infty(t))dt \geq \limsup_n \int_B \varphi_n(u^n(t))dt$$

on $B \in B_p^c \cap \mathcal{L}([0,T])$ so that

$$\lim_n \int_B \varphi_n(u^n(t))dt = \int_B \varphi_\infty(u^\infty(t))dt \tag{3.3.8}$$

on $B \in B_p^c \cap \mathcal{L}([0,T])$. Now, from the chain rule theorem given in Step 1, recall that

$$\langle \dot{u}^n(t), f^n(t)\rangle - \langle \dot{u}^n(t), \ddot{u}^n(t) - A\dot{u}_n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t))]$$

that is

$$\langle \dot{u}^n(t), z^n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t))].$$

By the estimate (3.4.2) and the boundedness in $L^1_{\mathbb{R}^d}([0,T])$ of (z^n), it is immediate that $(\frac{d}{dt}[\varphi_n(u_n(t))])$ is bounded in $L^1_{\mathbb{R}}([0,T])$ so that $(\varphi_n(u_n(.))$ is bounded

in variation. By Helly theorem, we may assume that $(\varphi_n(u_n(.))$ pointwisely converges to a BV function ψ. By (3.4.2), $(\varphi_n(u_n(.))$ converges in $L^1_{\mathbb{R}}([0,T])$ to ψ. In particular, for every $k \in L^\infty_{\mathbb{R}^+}([0,T])$ we have

$$\lim_{n\to\infty} \int_0^T k(t)\varphi_n(u_n(t))dt = \int_0^T k(t)\psi(t)dt. \tag{3.3.9}$$

Combining (3.3.8)–(3.3.9) yields

$$\int_B \psi(t)\,dt = \lim_{n\to\infty} \int_B \varphi_n(u^n(t))\,dt = \int_B \varphi_\infty(u^\infty(t))\,dt$$

for all $\in B_p^c \cap \mathcal{L}([0,T])$. As this inclusion is true on each B_p^c and $B_p^c \uparrow [0,T]$, we conclude that

$$\psi(t) = \lim_n \varphi_n(u_n(t)) = \varphi_\infty(u^\infty(t)) \text{ a.e.}$$

Hence we get $\lim_n \varphi_n(u_n(t)) = \varphi_\infty(u^\infty(t))$ a.e. Subsequently, using (iii) the passage to the limit when n goes to ∞ in the equation

$$\varphi_n(u^n(t)) + \frac{1}{2}||\dot{u}^n(t)||^2 = \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2$$

$$- \int_0^t \langle A\dot{u}^n(s), \dot{u}^n(s)\rangle ds + \int_0^t \langle \dot{u}^n(s), f^n(s)\rangle ds$$

yields for a.e $t \in [0,T]$

$$\varphi_\infty(u^\infty(t)) + \frac{1}{2}||\dot{u}^\infty(t)||^2 = \varphi_\infty(u_0^\infty) + \frac{1}{2}||\dot{u}_0^\infty||^2$$

$$- \int_0^t \langle A\dot{u}^\infty(s), \dot{u}^\infty(s)\rangle ds + \int_0^t \langle \dot{u}^\infty(s), f^\infty(s)\rangle ds$$

By noting that (f^n) is uniformly integrable and $\dot{u}^n$ is uniformly bounded and pointwise converges to $\dot{u}^\infty$, by virtue of Grothendieck lemma [26], it converges uniformly on uniformly integrable (= relatively weakly compact) subsets of $L^1_{\mathbb{R}^d}([0,T])$, so that

$$\lim_n \int_0^t \langle \dot{u}^n(s), f^n(s)\rangle ds = \int_0^t \langle \dot{u}^\infty(s), f^\infty(s)\rangle ds.$$

Step 6. Localization of further limits and final step.
As $(z^n = f^n - \ddot{u}^n - A\dot{u}^n)$ is bounded in $L^1_{\mathbb{R}^d}([0,T])$ in view of Step 3, it is relatively compact in the second dual $L^\infty_{\mathbb{R}^d}([0,T])'$ of $L^1_{\mathbb{R}^d}([0,T])$ endowed with the weak topology $\sigma(L^\infty_{\mathbb{R}^d}([0,T])', L^\infty_{\mathbb{R}^d}([0,T]))$. Furthermore, (z^n) can be viewed as a bounded sequence in $\mathcal{C}_{\mathbb{R}^d}([0,T])'$. Hence there are a filter $\mathcal{U}$ finer than the Fréchet filter, $l \in L^\infty_{\mathbb{R}^d}([0,T])'$ and $\mathbf{n} \in \mathcal{C}_{\mathbb{R}^d}([0,T])'$ such that

$$\mathcal{U} - \lim_n z^n = l \in L^\infty_{\mathbb{R}^d}([0,T])'_{weak} \tag{3.3.10}$$

and

$$\lim_n z^n = \mathbf{n} \in \mathcal{C}_{\mathbb{R}^d}([0,T])'_{weak} \tag{3.3.11}$$

where $L^\infty_{\mathbb{R}^d}([0,T])'_{weak}$ is the second dual of $L^1_{\mathbb{R}^d}([0,T])$ endowed with the topology $\sigma(L^\infty_{\mathbb{R}^d}([0,T])', L^\infty_{\mathbb{R}^d}([0,T]))$ and $\mathcal{C}_{\mathbb{R}^d}([0,T])'_{weak}$ denotes the space $\mathcal{C}_{\mathbb{R}^d}([0,T])'$ endowed with the weak topology $\sigma(\mathcal{C}_{\mathbb{R}^d}([0,T])', \mathcal{C}_{\mathbb{R}^d}([0,T]))$, because $\mathcal{C}_{\mathbb{R}^d}([0,T])$ is a separable Banach space for the norm sup, so that we may assume by extracting subsequences that (z^n) weakly converges to $\mathbf{n} \in \mathcal{C}_{\mathbb{R}^d}([0,T])'$. Using Step 4, we note that $\mathbf{n} = -m - W.dt = -m - (A\dot{u}^\infty - f^\infty).dt$. Let l_a be the density of the absolutely continuous part l_a of l in the decomposition $l = l_a + l_s$ in absolutely continuous part l_a and singular part l_s, in the sense there is an decreasing sequence (A_n) of Lebesgue measurable sets in $[0,T]$ with $A_n \downarrow \emptyset$ such that $l_s(h) = l_s(1_{A_n}h)$ for all $h \in L^\infty_{\mathbb{R}^d}([0,T])$ and for all $n \geq 1$. As $(z^n = f^n - \ddot{u}^n - A\dot{u}^n)$ biting converges to $z^\infty = f^\infty - \zeta^\infty(t) - A\dot{u}^\infty$ in Step 4, it is already seen (cf. Proposition 3.1) that

$$l_a(h) = \int_0^T \langle h(t), f^\infty - \zeta^\infty(t) - M\dot{u}^\infty(t)\rangle dt$$

for all $h \in L^\infty_{\mathbb{R}^d}([0,T])$, shortly $z^\infty = f^\infty - \zeta^\infty(t) - A\dot{u}^\infty$ coincides a.e. with the density of the absolutely continuous part l_a. By ([24], [39]) we have

$$I^*_{\varphi_\infty}(l) = I_{\varphi^*_\infty}(f^\infty - \zeta^\infty - A\dot{u}^\infty) + \delta^*(l_s, domI_{\varphi_\infty})$$

here φ^*_∞ is the conjugate of φ_∞, $I_{\varphi^*_\infty}$ is the integral functional defined on $L^1_{\mathbb{R}^d}([0,T])$ associated with φ^*_∞, $I^*_{\varphi_\infty}$ is the conjugate of the integral functional I_{φ_∞} and

$$domI_{\varphi_\infty} := \{u \in L^\infty_{\mathbb{R}^d}([0,T]) : I_{\varphi_\infty}(u) < \infty\}.$$

Using the inclusion

$$z^\infty = f^\infty - \zeta^\infty - A\dot{u}^\infty \in \partial I_{\varphi_\infty}(u^\infty).$$

that is

$$I_{\varphi^*_\infty}(f^\infty - \zeta^\infty - A\dot{u}^\infty) = \langle f^\infty - \zeta^\infty - A\dot{u}^\infty, u^\infty\rangle - I_{\varphi_\infty}(u^\infty)$$

we see that

$$I^*_{\varphi_\infty}(l) = \langle f^\infty - \zeta^\infty - A\dot{u}^\infty, u^\infty\rangle - I_{\varphi_\infty}(u^\infty) + \delta^*(l_s, domI_{\varphi_\infty}).$$

Coming back to the inclusion $z^n(t) \in \partial\varphi_n(u^n(t))$, we have

$$\varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbb{R}^d$. By substituting x by $h(t)$ in this inequality, here $h \in L^\infty_{\mathbb{R}^d}([0,T])$, and by integrating

$$\int_0^T \varphi_n(h(t))\,dt \geq \int_0^T \varphi_n(u^n(t))\,dt + \int_0^T \langle h(t) - u^n(t), z^n(t)\rangle\,dt.$$

Arguing as in Step 4 by passing to the limit in the preceding inequality, involving the epiliminf property for integral functionals $\int_0^T \varphi_n(h(t))dt$ defined on $L^\infty_{\mathbb{R}^d}([0,T])$, it is easy to see that

$$\int_0^T \varphi_\infty(h(t))\,dt \geq \int_0^T \varphi_\infty(u^\infty(t))\,dt + \langle h - u^\infty, n\rangle.$$

Since this holds, in particular, when $h \in \mathcal{C}_{\mathbb{R}^d}([0,T])$, we conclude that n belongs to the subdifferential $\partial J_{\varphi_\infty}(u^\infty)$ of the convex lower semicontinuous integral functional J_{φ_∞} defined on $\mathcal{C}_{\mathbb{R}^d}([0,T])$

$$J_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt,\ \forall u \in \mathcal{C}_{\mathbb{R}^d}([0,T]).$$

Now let $B : \mathcal{C}_{\mathbb{R}^d}([0,T]) \to L^\infty_{\mathbb{R}^d}([0,T])$ be the continuous injection and let $B^* : L^\infty_{\mathbb{R}^d}([0,T])' \to \mathcal{C}_{\mathbb{R}^d}([0,T])'$ be the adjoint of B given by

$$\langle B^*l, h\rangle = \langle l, Bh\rangle = \langle l, h\rangle, \quad \forall l \in L^\infty_{\mathbb{R}^d}([0,T])', \quad \forall h \in \mathcal{C}_{\mathbb{R}^d}([0,T]).$$

Then we have $B^*l = B^*l_a + B^*l_s$, $l \in L^\infty_{\mathbb{R}^d}([0,T])'$ being the limit of $(z_n = f^n - \ddot{u}^n - A\dot{u}^n)$ under the filter $\mathcal{U}$ given in section 4 and $l = l_a + l_s$ being the decomposition of l in absolutely continuous part l_a and singular part l_s. It follows that

$$\langle B^*l, h\rangle = \langle B^*l_a, h\rangle + \langle B^*l_s, h\rangle = \langle l_a, h\rangle + \langle l_s, h\rangle$$

for all $h \in \mathcal{C}_{\mathbb{R}^d}([0,T])$. But it is already seen that

$$\langle l_a, h\rangle = \langle f^\infty - \zeta^\infty - A\dot{u}^\infty, h\rangle$$
$$= \int_0^T \langle f^\infty(t) - \zeta^\infty(t) - A\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in L^\infty_{\mathbb{R}^d}([0,T])$$

so that the measure B^*l_a is absolutely continuous

$$\langle B^*l_a, h\rangle = \int_0^T \langle f^\infty - \zeta^\infty(t) - A\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in \mathcal{C}_{\mathbb{R}^d}([0,T])$$

and its density $f^\infty - \zeta^\infty - A\dot{u}^\infty$ satisfies the inclusion

$$f^\infty(t) - \zeta^\infty(t) - A\dot{u}^\infty(t) \in \partial\varphi_\infty(u^\infty(t)), \quad \text{a.e..}$$

and the singular part B^*l_s satisfies the equation

$$\langle B^*l_s, h\rangle = \langle l_s, h\rangle, \quad \forall h \in \mathcal{C}_{\mathbb{R}^d}([0,T]).$$

As $B^*l = \mathbf{n}$, using (3.3.10)–(3.3.11), it turns out that $\mathbf{n}$ is the sum of the absolutely continuous measure $\mathbf{n}_a$ with

$$\langle \mathbf{n}_a, h\rangle = \int_0^T \langle f^\infty - \zeta^\infty(t) - A\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in \mathcal{C}_{\mathbb{R}^d}([0,T])$$

and the singular part $\mathbf{n}_s$ given by

$$\langle \mathbf{n}_s, h\rangle = \langle l_s, h\rangle, \quad \forall h \in \mathcal{C}_{\mathbb{R}^d}([0,T]).$$

which satisfies the property: for any nonnegative measure θ on $[0,T]$ with respect to which $\mathbf{n}_s$ is absolutely continuous

$$\int_0^T h_{\varphi^*_\infty}(\frac{d\mathbf{n}_s}{d\theta}(t))d\theta(t) = \int_0^T \langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\rangle d\theta(t)$$

here $h_{\varphi^*_\infty}$ denotes the recession function of φ^*_∞. Indeed, as $\mathbf{n}$ belongs to $\partial J_{\varphi_\infty}(u^\infty)$ by applying Theorem 5 in [39] we have

$$J^*_{\varphi_\infty}(n) = I_{\varphi^*_\infty}(\frac{d\mathbf{n}_a}{dt}) + \int_0^T h_{\varphi^*_\infty}(\frac{d\mathbf{n}_s}{d\theta}(t))d\theta(t) \tag{3.3.12}$$

with

$$I_{\varphi^*_\infty}(v) := \int_0^T \varphi^*_\infty(v(t))dt, \forall v \in L^1_{\mathbb{R}^d}([0,T]).$$

Recall that

$$\frac{d\mathbf{n}_a}{dt} = f^\infty - \zeta^\infty - A\dot{u}^\infty \in \partial I_{\varphi_\infty}(u^\infty)$$

that is

$$I_{\varphi^*_\infty}(\frac{d\mathbf{n}_a}{dt}) = \langle f^\infty - \zeta^\infty - A\dot{u}^\infty, u^\infty\rangle_{\langle L^1_{\mathbb{R}^d}([0,T]), L^\infty_{\mathbb{R}^d}([0,T])\rangle} - I_{\varphi_\infty}(u^\infty) \tag{3.3.13}$$

From (3.3.13) we deduce

$$\begin{aligned}
J^*_{\varphi_\infty}(n) &= \langle u^\infty, \mathbf{n}\rangle_{\langle \mathcal{C}_{\mathbb{R}^d}([0,T]), \mathcal{C}_{\mathbb{R}^d}([0,T])'\rangle} - J_{\varphi_\infty}(u^\infty) \\
&= \langle u^\infty, \mathbf{n}\rangle_{\langle \mathcal{C}_{\mathbb{R}^d}([0,T]), \mathcal{C}_{\mathbb{R}^d}([0,T])'\rangle} - I_{\varphi_\infty}(u^\infty) \\
&= \int_0^T \langle u^\infty(t), f^\infty(t) - \zeta^\infty(t) - A\dot{u}^\infty(t)\rangle dt \\
&\quad + \int_0^T \langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\rangle d\theta(t) - I_{\varphi_\infty}(u^\infty) \\
&= I_{\varphi^*_\infty}(\frac{d\mathbf{n}_a}{dt}) + \int_0^T \langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\rangle d\theta(t)).
\end{aligned}$$

Coming back to (3.3.12) we get the equality

$$\int_0^T h_{\varphi^*_\infty}(\frac{d\mathbf{n}_s}{d\theta}(t))d\theta(t) = \int_0^T \langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\rangle d\theta(t)).$$

□

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