

Well-Posedness of evolution variational inequalities with applications

Charles Castaing* Christiane Godet-Thobie† Manuel D.P. Monteiro Marques ‡ Lionel Thibault §

March 5, 2022

Abstract

We are concerned in the present work with the existence and uniqueness of absolutely continuous solutions to a class of evolution problems governed by time-dependent subdifferential operators of the form

$$f(t) + Bu(t) - A \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t))$$

with various applications.

1 Introduction

In this work we are concerned with the existence and uniqueness of absolutely continuous solution to an evolution inclusion in a separable Hilbert space H in the form

$$f(t) + Bu(t) - A \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)), t \in [0, T]. \quad (1.1)$$

Above $f : [0, T] \rightarrow H$ is a continuous mapping, $B : H \rightarrow H$ is an operator, $A : H \rightarrow H$ is a linear continuous coercive and symmetric operator, $\varphi : [0, T] \times H \rightarrow]-\infty, +\infty]$ is a normal lower semicontinuous convex integrand, and $\partial\varphi(t, \cdot)$ is the subdifferential of $\varphi(t, \cdot)$. Problem (1.1) is interpreted as an evolution variational

*IMAG, Univ Montpellier, CNRS, Montpellier II, 34095, Case courrier 051, Montpellier Cedex 5, France. E-Mail: charles.castaing@gmail.com

†Université de Bretagne Occidentale, Laboratoire de Mathématiques de Bretagne Atlantique, CNRS UMR 6205, 6, avenue Victor Le Gorgeu, CS 9387, e-mail: christiane.godet-thobie@univ-brest.fr

‡CMAF and Faculdade de Ciências de Lisboa, Av. Prof. Gama Pinto 2, P. 1600 Lisboa, Portugal. E-Mail: mmarques@lmc.fc.ul.pt

§Université de Montpellier, Institut Montpelliérian Alexander Grothendick Montpellier, France. Email:lionel.thibault@umontpellier.fr

inequality (EVI) with the velocity inside the subdifferential. Generally, the model for parabolic evolution inclusion is a differential inclusion of the form

$$B(t, u(t)) \in \frac{du}{dt}(t) + A(t)u(t) + \partial\varphi(t, u(t)), \quad t \in [0, T], \quad (1.2)$$

where $A(t)$ is a time dependent maximal monotone operator, $B(t, u)$ defined for $(t, u) \in [0, T] \times H$ is Lipschitz with respect to u . Then the existence and uniqueness of absolutely continuous solution to (1.2) is known in some particular cases in the literature, see e.g [4, 16] and Barbu and Rascanu in [6] dealing with existence of *generalized solutions* for parabolic variational inequalities with singular inputs and operators of the form

$$f(t) + \frac{dM}{dt}(t) \in \frac{du}{dt}(t) + Au(t) + \partial\varphi(u(t)),$$

where A is a linear coercive operator and φ is a lower semicontinuous convex function. There is an increasing activity around problem (1.2) since it contains several new applications such as sweeping process, relaxed problem and Skorohod problem etc. In this framework, problem (1.1) constitutes a new variational evolution inequality with the velocity inside the subdifferential **in contrast to** problem (1.2). Likewise problem (1.2), the study of (1.1) leads to several applications in a new setting such as the sweeping process, Skorohod problem, second order evolution and fractional differential equation [16]. Although (1.1) **deals with the** deterministic case, it is a **step towards the** Skorohod problem in the stochastic setting, see the recent articles by Castaing-Raynaud de Fitte [15, 17], Rascanu [31], and L.Maticiuc, A. Rascanu, L. Slominski and M.Topolewski [22] for references on this stochastic subject. Let us mention the current situation of problem (1.1) in the literature. In [16] **it was dealt with** the existence of absolutely continuous solutions to variational evolution inequalities in separable Hilbert space H of **the forms**

$$f(t) - Au(t) \in \partial\varphi(t, \frac{du}{dt}(t)) \quad (1.3)$$

$$f(t) - Au(t) \in N_{C(t)}(\frac{du}{dt}(t)), \quad (1.4)$$

where $f : [0, T] \rightarrow H$ is a continuous mapping, $A : H \rightarrow H$ is a linear continuous coercive symmetric operator, $\varphi : [0, T] \times H \rightarrow]-\infty, +\infty]$ is a **normal convex integrand**, and $N_{C(t)}(x)$ denotes the normal cone to a closed convex moving set $C(t) \subset H$. Some related variants of problem (1.4) dealing with two positive operators A and B are **given in** a series of papers by Adly et al [1, 2, 3]. We note that there is a new variant of problem (1.1) **in a recent work** by Bacho, Emmerich and Mielke [7] dealing with the following inclusion

$$B(t, u(t)) \in \partial\varphi(t, \frac{du}{dt}(t))$$

or more generally with two subdifferentials, namely

$$B(t, u(t)) \in \partial\varphi(t, \frac{du}{dt}(t)) + \partial\psi(t, \frac{du}{dt}(t)),$$

where B is a continuous mapping. In Mielke's paper, in order to solve the problem, it is proposed an algorithm **due to De Giorgi combined with regularization of subdifferentials**. Consult also a recent article by Migorski, Sofonea and Zeng [24] dealing with the inclusion of the form

$$-\frac{du}{dt}(t) \in N_{C(t)}(A\frac{du}{dt}(t) + Bu(t))$$

where $B : H \rightarrow H$ is Lipschitz continuous. Note that in [24] B is not assumed to be a positive operator in contrast to the results obtained by Adly et al [1, 2, 3].

Our aim in the present paper is to develop several variants of problem (1.1) along with diverse applications via some related variational limits.

2 Preliminaries

Throughout the paper, H is a *real separable Hilbert space*.

If I is an interval of \mathbb{R} the spaces $L^1_H(I)$ and $L^\infty_H(I)$ denote the usual spaces with respect the Lebesgue measure endowed with their canonical norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively. For any subset $Q \subset I$ the function $\mathbf{1}_Q$ is defined by $\mathbf{1}_Q(t) = 1$ if $t \in Q$ and $\mathbf{1}_Q(t) = 0$ otherwise.

Given a convex function $\varphi : H \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, its effective domain $\text{dom } \varphi$ is given by

$$\text{dom } \varphi := \{x \in H : \varphi(x) < +\infty\},$$

so the function φ is proper whenever $\text{dom } \varphi \neq \emptyset$ and φ does not take the value $-\infty$. At any $x \in H$ where φ is finite its subdifferential $\partial\varphi(x)$ is defined by

$$\partial\varphi(x) := \{\zeta \in H : \langle \zeta, y - x \rangle - \varphi(x) \leq \varphi(y), \forall y \in H\}.$$

If $f(x)$ is not finite $\partial\varphi(x) = \emptyset$. Considering the Legendre-Fenchel conjugate $\varphi^* : H \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ with

$$\varphi^*(y) := \sup\{\langle y, x \rangle - f(x) : x \in H\},$$

it is known that, when φ is a proper lower semicontinuous convex function, φ^* is also proper lower semicontinuous and

$$y \in \partial\varphi(x) \Leftrightarrow \langle y, x \rangle = \varphi(x) + \varphi^*(y).$$

Given a nonempty closed convex subset S of H , its indicator function $\delta(\cdot, S) : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by $\delta(x, S) = 0$ if $x \in S$ and $\delta(x, S) = +\infty$ if $x \notin S$.

$H \setminus S$. Clearly, $\delta(\cdot, S)$ is a proper lower semicontinuous convex function. Its subdifferential is called the *normal cone* of S , and obviously

$$N_S(x) = \{\zeta \in H : \langle \zeta, y - x \rangle \leq 0, \forall y \in S\} \quad \text{if } x \in S \quad (2.1)$$

and $N_S(x) = \emptyset$ if $x \in H \setminus S$. The Legendre-Fenchel conjugate $\delta^*(\cdot, S) : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is the *support function* of S , so

$$\delta^*(y, S) = \sup_{x \in S} \langle y, x \rangle \quad \text{for all } y \in H.$$

If S, S' are both nonempty closed bounded convex sets of H , the Hausdorff distance between S and S' can be defined by

$$\text{haus}(S, S') = \sup_{x \in H} |d_S(x) - d_{S'}(x)|,$$

and it is known that (see, e.g., *****)

$$\text{haus}(S, S') = \sup_{\|u\| \leq 1} |\delta^*(u, S) - \delta^*(u, S')|, \quad (2.2)$$

which entails

$$|\delta^*(y, S) - \delta^*(y, S')| \leq \|y\| \text{haus}(S, S') \quad \text{for all } y \in H. \quad (2.3)$$

We recall and summarize two useful results, see for example [5, Corollary 2.9, Corollary 2.10]. Remind that a linear operator $A : H \rightarrow H$ is *coercive* if there is a real $\omega > 0$ such that

$$\langle Ax, x \rangle \geq \omega \|x\|^2 \quad \text{for all } x \in H. \quad (2.4)$$

Proposition 2.1. *Let $A : H \rightarrow H$ be a linear continuous and coercive operator.*

(a) *If $\varphi : H \rightarrow [0, \infty]$ is a proper lower semicontinuous convex function, then for each $f \in H$ the problem $f \in Ay + \partial\varphi(y)$ admits a unique solution y .*

(b) *If K is a closed convex subset in H , then for each $f \in H$ the problem $f \in Ay + N_K(y)$ admits a unique solution y .*

3 Preparatory variational limit theorems

Proposition 3.1. *Let H be a separable Hilbert space and $I = [0, T]$. Let $C : I \rightrightarrows H$ be a closed convex valued scalarly measurable multimapping for which there is some real $r > 0$ such that $C(t) \subset r\mathbb{B}_H$ for all $t \in I$. Let A be a linear continuous coercive symmetric operator on H and let B be a linear continuous compact operator on H . Let $(\theta_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions from I into I such that for each $t \in I$ one has $\theta_n(t) \rightarrow t$ and $\text{haus}(C(\theta_n(t)), C(t)) \rightarrow 0$ as $n \rightarrow \infty$.*

Let $(f_n, f)_{n \in \mathbb{N}}$ be a bounded sequence in $L_H^\infty(I)$ with $\|f_n(t)\| \leq \beta, \|f(t)\| \leq \beta$ ($\beta > 0$) for all $n \in \mathbb{N}$ such that $f_n(t)$ converges to $f(t)$ for each $t \in [0, T]$.
Let $(v_n, v)_{n \in \mathbb{N}}$ be a bounded sequence in $L_H^\infty(I)$ with $\|v_n(t)\| \leq \gamma, \|v(t)\| \leq \gamma$ ($\gamma > 0$) for all $n \in \mathbb{N}$ such that $(v_n(t))_n$ converges weakly to $v(t)$ for each $t \in I$.
Let $(\zeta_n)_{n \in \mathbb{N}}$ be an integrable sequence in $L_H^1(I)$ such that $\zeta_n(t) \in C(\theta_n(t))$ for all $t \in I$ and such that $(\zeta_n)_{n \in \mathbb{N}} \sigma(L_H^1, L_H^\infty)$ converges in $L_H^1(I)$ to ζ . Assume that

$$f_n(t) + Bv_n(t) - A\zeta_n(t) \in N_{C(\theta_n(t))}(\zeta_n(t)) \quad \text{for all } n \in \mathbb{N}, \text{ a.e. } t \in I.$$

Then for a.e. $t \in I$ one has

$$\zeta(t) \in C(t) \quad \text{and} \quad f(t) + Bv(t) - A\zeta(t) \in N_{C(t)}(\zeta(t)).$$

Proof. We first verify that $u(t) \in C(t)$ a.e. $t \in I$. Indeed, for every measurable set $Z \subset I$ and for any $x \in H$, the function $\mathbf{1}_Z x \in L_H^\infty$. By the inequality

$$\langle x, \zeta_n(t) \rangle \leq \delta^*(x, C(\theta_n(t)))$$

integrating on Z gives

$$\int_I \langle \mathbf{1}_Z x, \zeta_n(t) \rangle dt = \int_Z \langle x, \dot{u}_n(t) \rangle dt \leq \int_Z \delta^*(x, C(\theta_n(t))) dt.$$

Passing to the upper limit in this inequality we obtain

$$\begin{aligned} \int_Z \langle \mathbf{1}_Z x, \dot{u}(t) \rangle dt &\leq \limsup_n \int_Z \delta^*(x, C(\theta_n(t))) dt \\ &\leq \int_Z \limsup_n \delta^*(x, C(\theta_n(t))) \leq \int_Z \delta^*(x, C(t)) dt. \end{aligned}$$

This being true for any Lebesgue measurable set $Z \subset I$ we deduce that for every $x \in H$

$$\langle x, \dot{u}(t) \rangle \leq \delta^*(x, C(t)) \quad \text{a.e. } t \in I.$$

By the separability of H and the weak compactness and convexity of $C(t)$ (see, e.g., Castaing-Valadier [19, Proposition III- 35]), we get the desired inclusion $\dot{u}(t) \in C(t)$ a.e. $t \in I$

For each $t \in I$ keeping in mind that $v_n(t) \rightarrow v(t)$ weakly in H and B is a linear continuous compact operator, we see that $Bv_n(t) \rightarrow Bv(t)$ strongly in H , so that $Bv_n(\cdot) \rightarrow Bv(\cdot)$ weakly in $L_H^1(I)$. Indeed, let any $h \in L_H^\infty(I)$. Then we have

$$\begin{aligned} \left| \int_0^T \langle h(t), Bv_n(t) \rangle dt - \int_0^T \langle h(t), Bv(t) \rangle dt \right| &\leq \int_0^T |\langle h(t), Bv_n(t) - Bv(t) \rangle| dt \\ &\leq |h|_\infty \int_0^T \|Bv_n(t) - Bv(t)\| dt. \end{aligned}$$

As $\int_0^T \|Bu_n(\delta_n(t)) - Bu(t)\|dt \rightarrow 0$ as $n \rightarrow \infty$ (by Lebesgue dominated convergence theorem), our assertion follows. Similarly as A is symmetric, we note that $A\dot{\zeta}_n \rightarrow A\dot{\zeta}$ weakly in $L^1_H(I)$. As a main consequence $f_n + Bv_n - A\zeta_n \rightarrow f + Bu - A\zeta$ weakly in $L^1_H(I)$. Then given any Lebesgue measurable subset $Z \subset I$ we may apply the lower semicontinuity of integral convex functional ([18], Theorem 8.1.16) to deduce that

$$\int_Z \delta^*(f(t) + Bv(t) - A\zeta(t), C(t))dt \leq \liminf_n \int_Z \delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t))dt. \quad (3.1)$$

This need a careful look. Indeed, we note that $(t, x) \mapsto \delta^*(x, C(t))$ is a normal lower semicontinuous convex integrand defined on $[0, T] \times H$ and $\delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t))$ is measurable and integrable since taking some real constant α with $\|f_n(t) + Bv_n(t) - A\zeta_n(t)\| \leq \alpha$ for all $n \in \mathbb{N}$ and $t \in I$ we have

$$|\delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t))| \leq \rho(t)\|f_n(t) + Bv_n(t) - A\zeta_n(t)\| \leq \alpha\rho.$$

Furthermore

$$\delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t)) \geq \langle f_n(t) + Bv_n(t) - A\zeta_n(t), \bar{u}(t) \rangle$$

where $\bar{u}(t)$ is a measurable selection of $C(\cdot)$. Write

$$\begin{aligned} & |\delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t)) - \delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(\theta_n(t)))| \\ & \leq \|f_n(t) + Bv_n(t) - A\zeta_n(t)\| \text{haus}(C(t), C(\theta_n(t))) \\ & \leq \alpha \text{haus}(C(t), C(\theta_n(t))), \end{aligned}$$

so that

$$\begin{aligned} & \liminf_n \int_Z \delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(\theta_n(t)))dt \\ & \geq \liminf_n \int_Z \delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(t))dt \\ & \geq \int_Z \delta^*(f(t) + Bv(t) - A\zeta(t), C(t))dt. \end{aligned} \quad (3.2)$$

Let us set $\psi_A(x) = \langle Ax, x \rangle$ if $x \in r\mathbb{B}_H$ and $\psi_A(x) = +\infty$ if $x \notin r\mathbb{B}_H$. Then it is clear ψ is a positive lower semicontinuous convex integrand. Apply again the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the positive normal convex integrand ψ_A we obtain

$$\liminf_n \int_Z \psi_A(\zeta_n(t))dt \geq \int_Z \psi_A(\zeta(t))dt,$$

that is,

$$\liminf_n \int_Z \langle A\zeta_n(t), \zeta_n(t) \rangle dt \geq \int_Z \langle A\zeta(t), \zeta(t) \rangle dt. \quad (3.3)$$

Since we already saw that $Bv_n(t) \rightarrow Bv(t)$ strongly in H for each $t \in I$, we also have

$$\lim_n \int_Z \langle Bv_n(t), \zeta_n(t) \rangle dt = \int_Z \langle Bv(t), \zeta(t) \rangle dt. \quad (3.4)$$

Similarly we have

$$\lim_n \int_Z \langle f_n(t), \zeta_n(t) \rangle dt = \int_Z \langle f(t), \zeta(t) \rangle dt \quad (3.5)$$

because f_n is uniformly bounded and pointwise strongly converge to f and $\zeta_n \rightarrow \zeta$ weakly in $L_H^1(I)$ by noting that a bounded sequence in $L_H^\infty(I)$ which pointwise converges to 0, converges to 0 uniformly on any uniformly integrable subset of $L_H^1([0, T])$, in other terms it converges to 0 with respect to the Mackey topology $\tau(L_H^\infty(I), L_H^1(i))$ (see[9]).¹ Now integrating on $Z \subset [0, T]$ the inequality (here measurability and integrability are guaranteed)

$$\begin{aligned} & \delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(\theta_n(t))) \\ & + \langle -f_n(t) - Bv_n(t), \zeta_n(t) \rangle + \langle A\zeta_n(t), \zeta_n(t) \rangle \leq 0 \end{aligned}$$

gives

$$\begin{aligned} & \int_Z \delta^*(f_n(t) + Bv_n(t) - A\zeta_n(t), C(\theta_n(t))) dt \\ & + \int_Z \langle Av_n(t), \zeta_n(t) \rangle dt + \int_Z \langle -f_n(t) - Bv_n(t), \zeta_n(t) \rangle dt \leq 0, \quad (3.6) \end{aligned}$$

so passing to the limit as $n \rightarrow \infty$ in this equality and using (3.1)—(3.6) yield

$$\int_Z [\delta^*(f(t) + Bv(t) - A\zeta(t), C(t)) + \langle A\zeta(t) - Bv(t) - f(t), \zeta(t) \rangle] dt \leq 0.$$

As $t \mapsto \delta^*(f(t) + Bv(t) - A\zeta(t), C(t)) + \langle A\zeta(t) - Bv(t) - f(t), \zeta(t) \rangle$ is integrable and as the latter inequality holds true for any Lebesgue measurable set $Z \subset I$, it follows that for a.e. $t \in I$

$$\delta^*(f(t) + Bv(t) - A\zeta(t), C(t)) + \langle A\zeta(t) - Bv(t) - f(t), \zeta(t) \rangle \leq 0.$$

This and the inclusion $\zeta(t) \in C(t)$ a.e. $t \in I$ allow us to conclude that

$$f(t) + Bv(t) - A\zeta(t) \in N_{C(t)}(\zeta(t)) \quad \text{a.e. } t \in I$$

according to the definition of the normal cone. □

¹If $H = \mathbb{R}^e$, one may invoke a classical fact that on bounded subsets of L_H^∞ the topology of convergence in measure coincides with the topology of uniform convergence on uniformly integrable sets, i.e. on relatively weakly compact subsets, alias the Mackey topology. This is a lemma due to Grothendieck [21] [Ch.5 §4 no 1 Prop. 1 and exercise]

An easy adaptation of the arguments in the above proposition furnishes the following variant.

Proposition 3.2. *Let H be a separable Hilbert space and $I = [0, T]$. Let $C : I \rightrightarrows H$ be a closed convex valued scalarly measurable multimapping for which there is some real $r > 0$ such that $C(t) \subset r\mathbb{B}_H$ for all $t \in I$. Let A be a linear continuous coercive symmetric operator on H and let $B : H \rightarrow H$ be a Lipschitz mapping. Let $(\theta_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions from I into I such that for each $t \in I$ one has $\theta_n(t) \rightarrow t$ and $\text{haus}(C(\theta_n(t)), C(t)) \rightarrow 0$ as $n \rightarrow \infty$.*

Let $(f_n, f)_{n \in \mathbb{N}}$ be a bounded sequence in $L_H^\infty(I)$ with $\|f_n(t)\| \leq \beta, \|f(t)\| \leq \beta$ ($\beta > 0$) for all $n \in \mathbb{N}$ such that $f_n(t)$ converges to $f(t)$ for each $t \in [0, T]$. Let $(v_n, v)_{n \in \mathbb{N}}$ be a bounded sequence in $L_H^\infty(I)$ with $\|v_n(t)\| \leq \gamma, \|v(t)\| \leq \gamma$ ($\gamma > 0$) for all $n \in \mathbb{N}$ such that $(v_n(t))_n$ converges to $v(t)$ for each $t \in I$. Let $(\zeta_n)_{n \in \mathbb{N}}$ be an integrable sequence in $L_H^1(I)$ such that $\zeta_n(t) \in C(\theta_n(t))$ for all $t \in I$ and such that $(\zeta_n)_{n \in \mathbb{N}} \sigma(L_H^1, L_H^\infty)$ converges in $L_H^1(I)$ to ζ . Assume that

$$f_n(t) + Bv_n(t) - A\zeta_n(t) \in N_{C(\theta_n(t))}(\zeta_n(t)) \quad \text{for all } n \in \mathbb{N}, t \in I.$$

Then for a.e. $t \in I$ one has

$$\zeta(t) \in C(t) \quad \text{and} \quad f(t) + Bv(t) - Azeta(t) \in N_{C(t)}(\zeta(t)).$$

Another variant of Proposition 3.1 is available by assuming that C is compact valued and the operator B is a Lipschitz mapping: $\|Bx - By\| \leq M\|x - y\|$. These results convex related to the evolution variational inequality of the form $f(t) + Bv(t) - Au(t) \in N_{C(t)}(u(t))$ a.e t stated in Theorem 4.1 and Theorem 4.3. The following result is a stability result related to the evolution variational inequality of the form $f(t) + \nabla g(u(t)) - Au(t) \in N_{C(t)}(\dot{u}(t))$ here ∇g is a specific gradient of a convex continuous Gateaux differentiable such that $g(v(t))$ is absolutely continuous for $v : [0, T] \rightarrow H$ absolutely continuous. There is no confusion with the linear compact operator B considered in Theorem 1.1 and Lipschitz operator B considered in Theorem 1.2.

Proposition 3.3. *$H = \mathbb{R}^e$. Let $C : [0, T] \rightarrow H$ be a convex compact valued measurable mapping such that $C(t) \subset r(t)\overline{B}_H$, for all $t \in [0, T]$ for some $r \in L_{\mathbb{R}^+}^1([0, T])$. Let A be linear continuous coercive symmetric operator on H and $B = \nabla g$ where ∇g is the gradient of a convex continuous Gateaux differentiable function $g : H \rightarrow \mathbb{R}^+$ such that $g(v(t))$ is absolutely continuous for $v : [0, T] \rightarrow H$ absolutely continuous.*

Let $(f_n, f)_{n \in \mathbb{N}}$ be a bounded sequence in $L_H^\infty([0, T])$ with $\|f_n(t)\| \leq \beta, \|f(t)\| \leq \beta$ ($\beta > 0$) for all $n \in \mathbb{N}$ such that $f_n(t)$ pointwise converges to $f(t)$ for each $t \in [0, T]$. Let $(u_n, u)_{n \in \mathbb{N}}$ be a sequence of absolutely continuous mappings

$$u_n(t) = u_0 + \int_0^t \dot{u}_n(s) ds, \dot{u}_n(t) \in C(t)$$

$$u(t) = u_0 + \int_0^t \dot{u}(s) ds, \dot{u}(t) \in C(t)$$

such that $u_n(t) \rightarrow u(t)$ uniformly in H and $\dot{u}_n \rightarrow \dot{u}$ weakly in $L_H^1[0, T]$. Assume that $f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t) \in N_{C(t)}(\dot{u}_n(t))$ for all $n \in \mathbb{N}$ and for all $t \in [0, T]$, then $f(t) + \nabla g(u(t)) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$ a.e.

Proof. Note that f_n are uniformly bounded and pointwise converges to $f(t)$ then we have

$$(2.2.1) \quad \lim_n \int_Z \langle f_n, \dot{u}_n \rangle dt = \int_Z \langle f, \dot{u} \rangle dt$$

for every measurable set $Z \subset [0, T]$. By integrating on Z (we are ensured that the functions given are measurable) the inequality

$$\delta^*(f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t), C_n(t)) + \langle A\dot{u}_n(t) - \nabla g(u_n(t)) - f_n, \dot{u}_n(t) \rangle \leq 0$$

we get

$$(2.2.2) \quad \int_Z \delta^*(f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t), C_n(t)) dt + \int_Z \langle A\dot{u}_n - f_n, \dot{u}_n(t) \rangle dt - \int_Z \langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle dt \leq 0.$$

Set $g_n(t) = f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t)$. We claim that $Au_n(t) \rightarrow Au(t)$ weakly in $L_H^1([0, T])$ and $\nabla g(u_n(t)) \rightarrow \nabla g(u(t))$ weakly in $L_H^1([0, T])$. So $g_n(t) = f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t) \rightarrow g(t) := f(t) + \nabla g(u(t)) - A\dot{u}(t)$ weakly in $L_H^1([0, T])$. Indeed, as $u_n(t) \rightarrow u(t)$ pointwise, $\langle x, Au_n(t) \rangle \rightarrow \langle x, Au(t) \rangle$. As consequence, for any $h \in L_H^\infty([0, T])$, we have $\langle h(t), Au_n(t) \rangle \rightarrow \langle h(t), Au(t) \rangle$ pointwise. But then the uniformly bounded sequence of bounded measurable functions ($\langle h, Au_n \rangle$) pointwise converge to the bounded measurable function $\langle h, Au \rangle$. As consequence, $\langle h, Au_n \rangle \rightarrow \langle h, Au \rangle$ weakly in $L_{\mathbb{R}}^1([0, T])$. This shows that $Au_n \rightarrow Au$ weakly in $L_H^1([0, T])$. Similarly we show that $\nabla g(u_n(t)) \rightarrow \nabla g(u(t))$ weakly in $L_H^1([0, T])$. Now we have to consider the term $\langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle$ by using the special property of ∇g . In fact u_n is absolutely continuous with derivative \dot{u}_n and $g(u_n)$ is absolutely continuous, so that by Moreau-Valadier [28],

$$\langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle = \frac{d}{dt} g(u_n(t))$$

From this fact, it is easy to deduce that

$$(2.2.3) \quad \begin{aligned} \liminf_n \int_0^T \langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle dt &= \liminf_n \int_0^T \frac{d}{dt} g(u_n(t)) dt \\ &= \liminf_n (g(u_n(T)) - g(u_n(0))) \geq g(u(T)) - g(u(0)) = \int_0^T \frac{d}{dt} g(u(t)) dt \end{aligned}$$

$$= \int_0^T \langle \nabla g(u(t)), \dot{u}(t) \rangle dt$$

Let us set $\varphi(t, x) = \langle Ax, x \rangle$ if $x \in K$ and $\varphi(t, x) = +\infty$ if $x \notin K$. Then it is clear φ is a positive lower semicontinuous convex normal integrand. By the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the positive normal convex integrand φ we obtain, for every Lebesgue measurable set $Z \subset [0, T]$

$$\liminf_n \int_Z \varphi(t, \dot{u}_n(t)) dt \geq \int_Z \varphi(t, \dot{u}(t)) dt$$

that is

$$(2.2.4) \quad \liminf_n \int_Z \langle A\dot{u}_n(t), \dot{u}_n(t) \rangle dt \geq \int_Z \langle A\dot{u}(t), \dot{u}(t) \rangle dt$$

To finish the proof we apply the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the normal convex integrand $(t, x) \mapsto \delta^*(x, C(t))$ by noting that

(j) $\delta^*(g_n(t), C(t))$ is minored by $\langle h(t), g_n(t) \rangle$; with $h \in S_C^\infty$.

(jj) the minored sequence $\langle g_n(t), h(t) \rangle$ is uniformly integrable.

Then we are ensured by the lower semicontinuity of the integral convex functional theorem ([18], Theorem 8.1.6)

$$(2.2.5) \quad \liminf_n \int_Z \delta^*(g_n(t), C(t)) dt \geq \int_Z \delta^*(g(t), C(t)) dt$$

By combining (2.2.3) – (2.2.5) we get

$$\int_0^T \delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) dt + \int_0^T \langle -f(t) + \nabla g(u(t)) + A\dot{u}(t), \dot{u}(t) \rangle dt \leq 0$$

But since $\dot{u}(t) \in C(t)$ a.e. we have

$$\delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) \geq \langle f(t) + \nabla g(u(t)) - A\dot{u}(t), \dot{u}(t) \rangle$$

a.e. that implies

$$\int_0^T [\delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) + \langle -f(t) - \nabla g(u(t)) + A\dot{u}(t), \dot{u}(t) \rangle] dt = 0$$

so we conclude that

$$\delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) = \langle f(t) + \nabla g(u(t)) - A\dot{u}(t), \dot{u}(t) \rangle$$

a.e. with $\dot{u}(t) \in C(t)$ a.e., just proving that $f(t) + \nabla g(u(t)) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$ a.e. \square

4 Well-posedness of inclusion (1.1)

Our main proofs in this section are build upon the variational inequalities in Proposition 2.1 and the variational limits in Section 3 as well as upon an explicit catching-up algorithm (alias Moreau's algorithm). We stress the fact that our algorithm and tools are self contained apart from the use of the mentioned variational inequalities.

Theorem 4.1. *Let $f : [0, T] \rightarrow H$ be a continuous mapping and let $v : [0, T] \rightarrow \mathbb{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : [0, T] \rightarrow H$ be a weakly compact convex valued multimapping such that*

$$\text{haus}(C(t), C(\tau)) \leq |v(t) - v(\tau)| \quad \text{for all } t, \tau \in [0, T].$$

Let $A : H \rightarrow H$ be a linear continuous coercive symmetric operator and let $B : H \rightarrow H$ be a linear continuous compact operator. Then, for any $u_0 \in H$, the evolution inclusion

$$\begin{cases} f(t) + Bu(t) - Au(t) \in N_{C(t)}(\frac{du}{dt}(t)) \\ u(0) = u_0 \end{cases}$$

admits a unique $W_H^{1,\infty}([0, T])$ solution $u : [0, T] \rightarrow H$. Further, one has $\|\dot{u}(t)\| \leq \rho$, where $\rho := \sup\{\|y\| : y \in C([0, T])\}$.

Proof. Put $I := [0, T]$ and denote $\beta := \max\{\|f(t)\| : t \in I\}$ (by continuity of f). Noticing that the multimapping $C(\cdot)$ is upper semicontinuous from I into H endowed with the weak topology, the set $C(I)$ is weakly compact, and hence $\rho := \sup\{\|y\| : y \in C(I)\}$ is finite and $L := \rho\mathbb{B}_H$ is weakly compact and convex.

Step I. Construction of a sequence $(u_n)_n$

We will use the Moreau's catching-up algorithm [26]. We consider for each $n \in \mathbb{N}$ the following partition of the interval $]0, T]$ given by

$$t_i^n = i \frac{T}{n} := i\eta_n \text{ for } 0 \leq i \leq n, \quad I_i^n :=]t_i^n, t_{i+1}^n] \text{ for } 0 \leq i \leq n-1.$$

Put $u_0^n = u_0$ and $f_i^n = f(t_i^n)$ for all $i = 1, \dots, n$. By Proposition 2.1(b), there is $z_1^n \in C(t_1^n) \subset L$ such that

$$f_1^n + Bu_0^n - Az_1^n \in N_{C(t_1^n)}(z_1^n).$$

Put $u_1^n = u_0^n + \eta_n z_1^n$. Suppose that $u_0^n, u_1^n, \dots, u_i^n, z_1^n, z_2^n, \dots, z_i^n$ are constructed. As above by Proposition 2.1(b) there exists $z_{i+1}^n \in C(t_{i+1}^n) \subset L$ such that

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in N_{C(t_{i+1}^n)}(z_{i+1}^n),$$

and we set $u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$. Then by induction there are finite sequences $(u_i^n)_{i=0}^n$ and $(z_i^n)_{i=1}^n$ such that

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in N_{C(t_{i+1}^n)}(z_{i+1}^n) \quad (4.1)$$

$$u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n.$$

From $(u_i^n)_{i=0}^n, (z_i^n)_{i=1}^n, (f_i^n)_{i=0}^n$, we construct two sequences of mappings $(u_n)_{n \in \mathbb{N}}$ and $(f_n)_{n \in \mathbb{N}}$ from I to H , by setting $f_n(0) = f_1^n, u_n(0) = u_0^n$ and for each $i = 0, \dots, n-1$ we set $f_n(t) = f_{i+1}^n$ and

$$f_n(t) = f_{i+1}^n \quad \text{and} \quad u_n(t) = u_i^n + \frac{t - t_i^n}{\eta_n} (u_{i+1}^n - u_i^n) \quad \text{for } t \in]t_i^n, t_{i+1}^n].$$

Keeping in mind that $C(t) \subset L = \rho \mathbb{B}_H$ we have $u_i \in C(t_{i+1}^n) \subset \rho \mathbb{B}_H$, so

$$\left\| \frac{1}{\eta_n} (u_{i+1}^n - u_i^n) \right\| = \|z_{i+1}^n\| \leq \rho.$$

From this it is clear that $u_n(\cdot)$ is Lipschitz continuous on I with ρ as a Lipschitz constant. This Lipschitz property of $u_n(\cdot)$ ensures that $\|u_n(t)\| \leq \|u_0\| + \rho T$ and $u_n(t) = u_0 + \int_{t_0}^t \dot{u}_n(s) ds$ for every $t \in I$. We also note that $\|f_n(t)\| \leq \beta$ for all $n \in \mathbb{N}$ and $t \in I$. Now, let us define the step functions $\theta_n, \delta_n : I \rightarrow I$ by $\theta_n(0) = \delta_n(0) = 0$ and

$$\theta_n(t) = t_{i+1}^n, \quad \delta_n(t) = t_i^n \quad \text{if } t \in]t_i^n, t_{i+1}^n],$$

so the inclusion (4.1) becomes

$$f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t) \in N_{C(\theta_n(t))}(\dot{u}_n(t)) \quad \text{a.e. } t \in I.$$

For each $t \in I$ we observe that there is some $i \in \{0, \dots, n-1\}$ such that $t \in [t_i^n, t_{i+1}^n[$, and then

$$|\theta_n(t) - t| \rightarrow 0 \quad \text{and} \quad |\delta_n(t) - t| \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

We also note that the latter inclusion above yields

$$\delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(\theta_n(t))) + \langle -f_n(t) - Bu_n(\delta_n(t)) + A\dot{u}_n(t), \dot{u}_n(t) \rangle \leq 0$$

with

$$\dot{u}_n(t) \in C(\theta_n(t)) \subset L \quad \text{a.e. } t \in I,$$

so that $\dot{u}_n \in S_L^1$ where $S_L^1 := \{\xi \in L_H^1(I) : \xi(t) \in L \text{ a.e. } t \in I\}$.

Step II. Convergence to a solution.

We note that S_L^1 is a weakly compact convex set of $L_H^1(I)$ (see, e.g., [18] and the references therein). Set

$$\mathcal{X} := \{\xi : I \rightarrow H : \xi(t) = u_0 + \int_0^t \dot{\xi}(s) ds, t \in [0, T]; \dot{\xi} \in S_L^1\}.$$

It is clear that \mathcal{X} is convex, equicontinuous and weakly compact [32] in $C_H([0, T])$ (see [32]). As $u_n \in \mathcal{X}$, one can extract from $(u_n)_n$ a (not relabelled) subsequence which pointwise weakly converges to $u : I \rightarrow H$ such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$ for all $t \in I$ and such that $(\dot{u}_n)_n$ $\sigma(L_H^1(I), L_H^\infty(I))$ -converges to $\dot{u} \in S_L^1$. Further, the inequality

$$\|u_n(\delta_n(t)) - u_n(t)\| \leq \rho|\delta_n(t) - t|$$

assures us that $(u_n(\delta_n(t)))_n$ converges weakly in H for each $t \in I$. This and the $\sigma(L_H^1(I), L_H^\infty(I))$ convergence of $(\dot{u}_n)_n$ to \dot{u} in $L_H^1(I)$ along with the inclusion (4.1) allow us (according to the pointwise convergence of $(f_n)_n$ to f and the estimates from the hypotheses) to obtain that for a.e. $t \in I$ the inclusions $\dot{u}(t) \in C(t)$ and

$$f(t) + Bu(t) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$$

hold true. This says that $u(\cdot)$ is solution of the inclusion of the theorem.

STEP III. Uniqueness.

The uniqueness of solutions follows easily from the coerciveness of the operator A . Indeed let u_1 and u_2 be two solutions. An easy computation gives

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle + \langle Bu_2(t) - Bu_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle \leq 0,$$

so that

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle \leq |B| \|u_2(t) - u_1(t)\| \|\dot{u}_2(t) - \dot{u}_1(t)\|.$$

By coerciveness of A we deduce that

$$\omega \|\dot{u}_2(t) - \dot{u}_1(t)\|^2 \leq |B| \|u_2(t) - u_1(t)\| \|\dot{u}_2(t) - \dot{u}_1(t)\|.$$

This entails that

$$\|\dot{u}_2(t) - \dot{u}_1(t)\| \leq \frac{|B|}{\omega} \|u_2(t) - u_1(t)\| \leq \frac{|B|}{\omega} \int_0^t \|\dot{u}_2(s) - \dot{u}_1(s)\| ds.$$

By Gronwall lemma $\dot{u}_1(t) = \dot{u}_2(t)$ a.e. $t \in I$, and so $u_1(t) = u_2(t)$ for every $t \in I$ since $u_1(t) = u_0 + \int_0^t \dot{u}_1(s)ds$ and $u_2(t) = u_0 + \int_0^t \dot{u}_2(s)ds$. \square

Remark 4.2. The tools developed above allow to obtain further variants. The fact that $C(t)$ is weakly compact is required, and mainly the coerciveness of A and the compactness assumption for the operator B . An inspection of the proof of Theorem 4.1, shows that the compactness assumption on B is required to prove the Fatou property,

$$\liminf_n \int_Z \langle B(u_n(\delta_n(t)), \dot{u}_n(t)) \rangle dt \geq \int_Z \langle Bu(t), \dot{u}(t) \rangle dt$$

So as a possible variant we may substitute the bounded operator B by the gradient ∇g of a positive convex continuous Gateaux differentiable function $g : H \rightarrow \mathbb{R}$

such that $g(v(t))$ is absolutely continuous for $v : [0, T] \rightarrow H$ absolutely continuous, so that by invoking the chain rule formula, see Moreau-Valadier, [28], we have the equality

$$\langle \nabla g(v(t)), \dot{v}(t) \rangle = \frac{d}{dt} g(v(t))$$

Hence by using this fact and the tool developed in Theorem 1.1, we obtain a variant of Theorem 1.1 by noting that

$$\begin{aligned} \liminf_n \int_0^T \langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle dt &= \liminf_n \int_0^T \frac{d}{dt} g(u_n(t)) dt \\ &\geq \int_0^T \frac{d}{dt} g(u(t)) dt = \int_0^T \langle \nabla g(u(t)), \dot{u}(t) \rangle dt \end{aligned}$$

It is obvious that a linear continuous operator and a gradient do not enjoy similar properties, showing the interest of the new variant we give further. This remark has some importance in further developments.

Now we present a variant dealing with the existence and uniqueness of absolutely continuous solution to the evolution inclusion of the form

$$f(t) + Bu(t) - A\dot{u}(t) \in N_{C(t)}(t, \frac{du}{dt}(t))$$

where f is a continuous mapping $f : I \rightarrow H$, A is a coercive symmetric operator, and $B : H \rightarrow H$ is a Lipschitz mapping.

Theorem 4.3. *Let $f : [0, T] \rightarrow H$ be a continuous mapping and let $v : [0, T] \rightarrow \mathbb{R}^+$ be a non-negative nondecreasing continuous function with $v(0) = 0$. Let $C : [0, T] \rightarrow H$ be a convex compact valued multimapping such that $\text{haus}(C(t), C(\tau)) \leq |v(t) - v(\tau)|$ for all $t, \tau \in [0, T]$. Let $A : H \rightarrow H$ be a linear continuous coercive symmetric operator and let $B : H \rightarrow H$ be a Lipschitz mapping, that is, for some real constant $M > 0$, $\|Bx - By\| \leq M\|x - y\|$ for all $x, y \in H$ for some positive constant M . Then, for any $u_0 \in H$, the evolution inclusion*

$$\begin{cases} f(t) + Bu(t) - A\dot{u}(t) \in N_{C(t)}(\frac{du}{dt}(t)) \\ u(0) = u_0 \end{cases}$$

admits a unique $W_H^{1,\infty}([0, T])$ solution $u : [0, T] \rightarrow H$. Further, $\|\dot{u}(t)\| \leq \rho$ a.e. $t \in [0, T]$, where $\rho := \max\{\|y\| : y \in C([0, T])\}$.

Proof. Put $I := [0, T]$ and denote $\beta := \max\{\|f(t)\| : t \in I\}$ (by continuity of f). Noticing that the multimapping $C(\cdot)$ is upper semicontinuous from I into H endowed with the norm topology, the set $C(I)$ is norm compact, and hence $\rho := \sup\{\|y\| : y \in C(I)\}$ is finite and $L := \overline{\text{co}}(C(I))$ is convex and norm compact.

Step I. The sequence $(u_n)_n$ is constructed as in Theorem 4.1.

Step II. With the strongly compact set $L = \overline{\text{co}}(C(I))$ at hands, we see that the set \mathcal{X} in the proof of Theorem 4.1 is strongly compact in $C_H(I)$. Since $u_n \in \mathcal{X}$ we can extract from $(u_n)_n$ a (not relabelled) sequence which pointwise converges to $u : I \rightarrow H$ such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$, for all $t \in I$ and $(\dot{u}_n)_n \sigma(L_H^1([0, T]), L_H^\infty([0, T]))$ -converges to $\dot{u} \in S_L^1$. The inequality

$$\|u_n(\delta_n(t)) - u_n(t)\| \leq \rho|\delta_n(t) - t|$$

ensures that the sequence $(u_n(\delta_n(t)))_n$ strongly converges to $u(t)$ for each $t \in I$. Consequently, we can follow **Step II** in the proof of Theorem 4.1 by applying Proposition 3.2 in place of Proposition 3.1, to arrive to the fact that $u(\cdot)$ is a solution of the inclusion in the present theorem.

Step III. The arguments for the uniqueness are the same as for Theorem 4.1. \square

We present another variant dealing with the existence and uniqueness of absolutely continuous solution to the evolution inclusion of the form

$$f(t) + Bu(t) - A\dot{u}(t) \in \partial\varphi(t, \frac{du}{dt}(t))$$

where f is a bounded continuous mapping $f : I \rightarrow H$, A is a coercive symmetric operator, and $B : H \rightarrow H$ be a linear continuous mapping $\partial\varphi$ is the subdifferential of a normal lower semicontinuous convex integrand φ .

Theorem 4.4. *Let H be a separable Hilbert space. Let K be a convex compact subset of H . Let $\varphi : [0, T] \times K \rightarrow]-\infty, +\infty]$ be a normal lower semicontinuous convex integrand such that*

(i) $\{\varphi(\cdot, u(\cdot)), u \in S_K^1\}$ *is uniformly integrable.*

(ii) $\varphi(t, x) \leq \varphi(\tau, x) + |v(t) - v(\tau)|$ *for all $t, \tau \in [0, T], x \in K$ where $v : [0, T] \rightarrow \mathbb{R}^+$ is a positive nondecreasing continuous function with $v(0) = 0$.*

Let $A : H \rightarrow H$ be a linear continuous coercive symmetric operator and $B : H \rightarrow H$ be a linear continuous mapping. Then, for any $u_0 \in H$, the evolution inclusion

$$f(t) + Bu(t) - A\dot{u}(t) \in \partial\varphi(t, \frac{du}{dt}(t))$$

$$u(0) = u_0$$

admits a unique $W_H^{1,\infty}([0, T])$ solution $u : [0, T] \rightarrow H$.

Proof. We will use again the Moreau's catching-up algorithm. We consider for each $n \in \mathbb{N}$ the following partition of the interval $I = [0, T]$.

$t_i^n = i\frac{T}{n} := i\eta_n$ for $0 \leq i \leq n$. $I_i^n :=]t_i^n, t_{i+1}^n]$ for $0 \leq i \leq n-1$.

Put $u_0^n = u_0$ and $f_i^n = f(t_i^n)$ for all $i = 1, \dots, n$. By Proposition 1.1 1), there is $z_1^n \in K$ such that

$$f_1^n + Bu_0^n - Az_1^n \in \partial\varphi(t_1^n, z_1^n).$$

Put $u_1^n = u_0^n + \eta_n z_1^n$. Suppose that $u_0^n, u_1^n, \dots, u_i^n, z_1^n, z_2^n, \dots, z_i^n$ are constructed. As above by Proposition 1.1 1) there exists $z_{i+1}^n \in K$ such that

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in \partial\varphi(t_{i+1}^n, z_{i+1}^n).$$

and we set $u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$. Then by induction there are finite sequences $(u_i^n)_{i=0}^n$ and $(z_i^n)_{i=1}^n$ such that

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in \partial\varphi(t_{i+1}^n, z_{i+1}^n)$$

$$u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$$

From $(u_i^n)_{i=0}^n, (z_i^n)_{i=1}^n, (f_i^n)_{i=0}^n$, we construct two sequences u_n from $[0, T]$ to H , f_n from $[0, T]$ to H , by setting $f_n(0) = f_1^n, u_n(0) = u_0^n$ and for each $i = 0, \dots, n-1$ we set $f_n(t) = f_{i+1}^n$ and

$$u_n(t) = u_i^n + \frac{t - t_i^n}{\eta_n} (u_{i+1}^n - u_i^n)$$

for $t \in]t_i^n, t_{i+1}^n]$. Clearly, the mapping $u_n(\cdot)$ is Lipschitz continuous on $[0, T]$, and ρ is a Lipschitz constant of $u_n(\cdot)$ on $[0, T]$ since for every $t \in]t_i^n, t_{i+1}^n]$

$$\dot{u}_n(t) = \frac{u_{i+1}^n - u_i^n}{\eta_n} = z_{i+1}^n \in K \subset \rho\overline{B}_H.$$

Furthermore, for every $t \in [0, T]$, one has $u_n(t) = u_0 + \int_0^t \dot{u}_n(s) ds$, hence $\|u_n(t)\| \leq \|u_0\| + \rho T$. We have

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in \partial\varphi(t_{i+1}^n, z_{i+1}^n).$$

Now, let us define the step functions $\theta_n, \delta_n : I \rightarrow I$ by

$$\theta_n(t) = t_{i+1}^n, \quad \delta_n(t) = t_i^n$$

if $t \in]t_i^n, t_{i+1}^n]$ and $\theta_n(0) = \delta_n(0) = 0$, and observe that for each $t \in I$, there is $i \in \{0, \dots, n-1\}$ such that $t \in]t_i^n, t_{i+1}^n]$, and then,

$$|\theta_n(t) - t| \rightarrow 0 \quad \text{and} \quad |\delta_n(t) - t| \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

So, the last inclusion becomes

$$f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t))$$

a.e. $t \in [0, T]$. We note that $\|u_n(t)\| \leq \|u_0\| + \rho T$, $\|f^n(\cdot)\| \leq \beta$ for all $t \in [0, T]$ and $u_n(t) = u_0 + \int_0^t \dot{u}_n(s) ds$ for all $t \in [0, T]$ with $\dot{u}_n \in K$ a.e.

Step 2 Convergence of the algorithm and final conclusion

Let $S_K^1 := \{h \in L_H^1([0, T]) : h(t) \in K \text{ a.e.}\}$ and let

$$\mathcal{X} := \{v : [0, T] \rightarrow H : v(t) = u_0 + \int_0^t \dot{v}(s) ds, t \in [0, T]; \dot{v} \in S_K^1\}.$$

Then it is clear that S_K^1 is convex and weakly compact in $L_H^1([0, T])$ (see e.g. [18] and the references therein) and that \mathcal{X} is convex, equicontinuous and compact in $\mathcal{C}_H([0, T])$. As $(u_n) \subset \mathcal{X}$, one can extract from (u_n) a subsequence not relabelled which pointwise converges to $u : [0, T] \rightarrow H$ such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$, for all $t \in [0, T]$ and $(\dot{u}_n) \sigma(L_H^1([0, T]), L_H^\infty([0, T]))$ -converges to $\dot{u} \in S_K^1$. As φ is normal lower semicontinuous convex, the conjugate function $\varphi^* : [0, T] \times H \rightarrow]-\infty, +\infty]$

$$(1.3.1) \quad \varphi^*(t, y) = \sup_{x \in K} [\langle x, y \rangle - \varphi(t, x)]$$

is normal, see e.g. Castaing-Valadier [19] and satisfies

$$\varphi^*(t, y) \leq \varphi^*(\tau, y) + |v(t) - v(\tau)|$$

for all $t, \tau \in [0, T], y \in H$ using assumption (ii) ([29], Proposition 27). By using the normality of φ , the mappings $t \mapsto \varphi(\theta_n(t), \dot{u}_n(t))$ and $t \mapsto \varphi(t, \dot{u}_n(t))$ are measurable and integrable. By construction we have

$$g_n(t) := f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t))$$

so that by the normality of φ^* , the mapping $t \mapsto \varphi^*(\theta_n(t), g_n(t))$ is measurable and integrable. Further by (1.3.1) and (iii) we have

$$(1.3.2) \quad -\varphi(t, \dot{u}_n(t)) + \langle \dot{u}_n(t), g_n(t) \rangle \leq \varphi^*(t, g_n(t)) \leq \varphi^*(\theta_n(t), g_n(t)) + |v(t) - v(\theta_n(t))|$$

so that $t \mapsto -\varphi(t, \dot{u}_n(t)) + \langle \dot{u}_n(t), g_n(t) \rangle$ is uniformly integrable thank to (iii). We note that $(h_n(t) := f_n(t) - Bu_n(\delta_n(t)))$ is uniformly bounded and pointwise converges to $h(t) = f(t) - Bu(t)$ in H . Hence $(h_n(\cdot) - h(\cdot))$ is uniformly bounded and pointwise converges to 0, so that it converges to 0 uniformly on any uniformly integrable subset of $L_H^1([0, T])$, in other terms it converges to 0 with respect to the Mackey topology $\tau(L_H^\infty([0, T]), L_H^1([0, T]))$ so that, for every Lebesgue measurable set $B \subset [0, T]$,

$$\lim_{n \rightarrow \infty} \int_B \langle h_n(t) - h(t), \dot{u}_n(t) \rangle dt = 0$$

because (\dot{u}_n) is uniformly integrable. Consequently

$$(1.3.3) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \int_B \langle h_n(t), \dot{u}_n(t) \rangle dt \\ &= \lim_{n \rightarrow \infty} \int_B \langle h_n(t) - h(t), \dot{u}_n(t) \rangle dt + \lim_{n \rightarrow \infty} \int_B \langle h(t), \dot{u}_n(t) \rangle dt \\ &= \lim_{n \rightarrow \infty} \int_B \langle h(t), \dot{u}_n(t) \rangle dt = \int_B \langle h(t), \dot{u}(t) \rangle dt. \end{aligned}$$

As A is symmetric, we show that $A\dot{u}_n \rightarrow A\dot{u}(\cdot)$ weakly in $L_H^1(I)$. As consequence $g_n := f_n + Bu_n(\delta_n(\cdot)) - A\dot{u}_n(\cdot) \rightarrow g := f + Bu - A\dot{u}$ weakly in $L_H^1(I)$. Further,

let us set $\psi_A(x) = \langle Ax, x \rangle$ if $x \in K$ and $\psi(x) = +\infty$ if $x \notin K$. Then it is clear ψ is a positive lower semicontinuous convex integrand. Apply again the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the positive normal lower semi continuous convex integrand ψ_A we obtain,

$$\liminf_n \int_Z \psi_A(u_n(t))dt \geq \int_Z \psi_A(u(t))dt$$

that is

$$(1.3.4) \quad \liminf_n \int_Z \langle A\dot{u}_n(t), \dot{u}_n(t) \rangle dt \geq \int_Z \langle A\dot{u}(t), \dot{u}(t) \rangle dt$$

Now, we deduce using (1.3.2) and the lower semicontinuity of integral convex functional ([18], Theorem 8.1.6) applied to φ^* ,

$$(1.3.5) \quad \int_Z \varphi^*(t, g(t))dt \leq \liminf_n \int_Z \varphi^*(t, g_n(t))dt \leq \liminf_n \int_Z \varphi^*(\theta_n(t), g_n(t))dt$$

As

$$\varphi(t, \dot{u}_n(t)) \leq \varphi(\theta_n(t), \dot{u}_n(t)) + |v(t) - v(\theta_n(t))|$$

we deduce that

$$\liminf_n \int_Z \varphi(t, \dot{u}_n(t))dt \leq \liminf_n \int_Z \varphi(\theta_n(t), \dot{u}_n(t))dt.$$

As (\dot{u}_n) weakly converges to $\dot{u} \in L_H^1([0, T])$, by the lower semi continuity theorem ([18], Theorem 8.1.6) applied to the lower semicontinuity of convex integral functional associated with φ , we deduce that

$$(1.3.6) \quad \int_Z \varphi(t, \dot{u}(t))dt \leq \liminf_n \int_Z \varphi(\theta_n(t), \dot{u}_n(t))dt$$

with $\dot{u}(t) \in K$ a.e. and $t \mapsto \varphi(t, \dot{u}(t))$ is integrable. Now integrating on any Lebesgue measurable set Z in $[0, T]$ the equality

$$\varphi(\theta_n(t), \dot{u}_n(t)) + \varphi^*(\theta_n(t), g_n(t)) = \langle \dot{u}_n(t), h_n(t) \rangle$$

gives

$$\int_Z \varphi(\theta_n(t), \dot{u}_n(t))dt + \int_Z \varphi^*(\theta_n(t), g_n(t))dt = \int_Z \langle \dot{u}_n(t), h_n(t) \rangle dt.$$

By passing to the limit when n goes to ∞ in this equality using (1.3.3) – (1.3.6) gives

$$\int_Z \varphi(t, \dot{u}(t))dt + \int_Z \varphi^*(t, g(t))dt \leq \int_B \langle \dot{u}(t), g(t) \rangle dt.$$

By $t \mapsto \varphi(t, \dot{u}(t)) + \varphi^*(t, g(t)) - \langle \dot{u}(t), g(t) \rangle$ is integrable, we deduce that

$$\varphi(t, \dot{u}(t)) + \varphi^*(t, g(t)) - \langle \dot{u}(t), g(t) \rangle \leq 0$$

a.e. with $\dot{u}(t) \in K$ a.e. So we conclude that $\varphi(t, \dot{u}(t)) + \varphi^*(t, g(t)) = \langle \dot{u}(t), g(t) \rangle$ a.e., equivalently $g(t) = f(t) + Bu(t) - A\dot{u}(t) \in \partial\varphi(t, \dot{u}(t))$ a.e. and equivalently $\dot{u}(t) \in \partial\varphi^*(t, f(t) + Bu(t) - A\dot{u}(t))$ a.e. \square

Remarks 1) The uniqueness of solutions follows easily from the coerciveness of the operator A . Indeed let u_1 and u_2 two solutions. then by an easy computation,

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle + \langle Bu_2(t) - Bu_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle \leq 0$$

so that

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle \leq |B| \|u_2(t) - u_1(t)\| \|\dot{u}_2(t) - \dot{u}_1(t)\|$$

By coerciveness, we deduce that

$$\omega \|\dot{u}_2(t) - \dot{u}_1(t)\|^2 \leq |B| \|u_2(t) - u_1(t)\| \|\dot{u}_2(t) - \dot{u}_1(t)\|$$

Whence

$$\|\dot{u}_2(t) - \dot{u}_1(t)\| \leq \frac{|B|}{\omega} \|u_2(t) - u_1(t)\| \leq \frac{|B|}{\omega} \int_0^t \|\dot{u}_2(s) - \dot{u}_1(s)\| ds$$

By Gronwall lemma $\dot{u}_1(t) = \dot{u}_2(t)$ a and so $u_1(t) = u_2(t)$ since $u_1(t) = u_0 + \int_0^t \dot{u}_1(s) ds, \forall t \in [0, T], u_2(t) = u_0 + \int_0^t \dot{u}_2(s) ds, \forall t \in [0, T]$.

2) Theorem 1.3 generalizes Theorem 6 in [16] dealing with finite dimensional space.

In view of further applications we provide some variational limits theorems below.

5 Some variational limit theorems

Proposition 5.1. *Let H be a separable Hilbert space. Let $C : [0, T] \rightarrow H$ be a convex weakly compact valued scalarly measurable mappings such that $C(t) \subset r(t)\overline{B}_H$, for all $t \in [0, T]$ for some $r \in L^1_{\mathbb{R}^+}([0, T])$. Let A be linear continuous coercive symmetric operator on H . Let B is a linear continuous compact operator on H .*

Let $(f_n, f)_{n \in \mathbb{N}}$ be a bounded sequence in $L^\infty_H([0, T])$ with $\|f_n(t)\| \leq \beta, \|f(t)\| \leq \beta$ ($\beta > 0$) for all $n \in \mathbb{N}$ such that $f_n(t)$ pointwise converges to $f(t)$ for each $t \in [0, T]$. Let $(v_n, v)_{n \in \mathbb{N}}$ be a bounded sequence in $L^\infty_H([0, T])$ with $\|v_n(t)\| \leq \gamma, \|v(t)\| \leq \gamma$ ($\gamma > 0$) for all $n \in \mathbb{N}$ such that $v_n(t)$ pointwise converges weakly to $v(t)$ for each $t \in [0, T]$. Let (u_n) be an integrable sequence in $L^1_H([0, T])$ such that $u_n(t) \in C(t)$ for all $t \in [0, T]$ such that (u_n) $\sigma(L^1_H, L^\infty_H)$ converges in $L^1_H([0, T])$ to u . Assume that $f_n(t) + Bv_n(t) - Au_n(t) \in N_{C(t)}(u_n(t))$ for all $n \in \mathbb{N}$ and for all $t \in [0, T]$, then $u(t) \in C(t)$ a.e. and $f(t) + Bv(t) - Au(t) \in N_{C(t)}(u(t))$ a.e.

Proof. We first check that $u(t) \in C(t)$ a.e. Indeed, we have $\langle x, u_n(t) \rangle \leq \delta^*(x, C(t))$, $\forall x \in H, \forall t \in [0, T]$. Then by integrating on any Lebesgue measurable set $Z \subset [0, T]$

$$\int_Z \langle x, u_n(t) \rangle dt \leq \int_Z \delta^*(x, C(t)) dt.$$

By using (iii) and by passing to the limit when n goes to ∞ gives

$$\int_Z \langle x, u(t) \rangle dt \leq \limsup_n \int_Z \delta^*(x, C(t)) dt.$$

Then we deduce that $\langle x, u(t) \rangle \leq \delta^*(x, C(t))$ a.e. so that by Castaing-Valadier ([19], Proposition III.35), we get $u(t) \in C(t)$ a.e. Note that this fact requires that $C(t)$ is convex weakly compact. Note that $f_n + Bv_n$ are uniformly bounded and pointwise converges to $f + Bv$ since B is compact operator. Since u_n is uniformly integrable and $\sigma(L_H^1, L_H^\infty)$ converges to u , we have by the Castaing trick given in the proof of Theorem 4.1

$$(2.1.1) \quad \lim_n \int_Z \langle f_n + Bv_n, u_n \rangle dt = \int_Z \langle f + Bv, u \rangle dt$$

for every measurable set $Z \subset [0, T]$. By integrating on Z (we are ensured that the functions given are measurable) the inequality

$$\delta^*(f_n(t) + Bv_n(t) - Au_n(t), C_n(t)) + \langle Av_n(t) - Bv_n(t) - f_n, u_n(t) \rangle \leq 0$$

we get

$$(2.1.2) \quad \int_Z \delta^*(f_n(t) + Bv_n(t) - Au_n(t), C_n(t)) dt + \int_Z \langle Au_n - f_n, u_n(t) \rangle dt - \int_Z \langle Bv_n(t), v_n(t) \rangle dt \leq 0.$$

Set $g_n = f_n + Bv_n - Au_n$. It is already seen that $Au_n \rightarrow Au$ weakly in $L_H^1([0, T])$ and $Bv_n \rightarrow Bv$ weakly in $L_H^1([0, T])$ so that $g_n \rightarrow g := f + Bv - Au$ weakly in $L_H^1([0, T])$. Let us set $\psi(t, x) = \langle Ax, x \rangle$ if $x \in K$ and $\psi(t, x) = +\infty$ if $x \notin K$. Then it is clear ψ is a positive lower semicontinuous convex normal integrand. By the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the positive normal convex integrand ψ we obtain, for every Lebesgue measurable set $Z \subset [0, T]$

$$\liminf_n \int_Z \psi(t, u_n(t)) dt \geq \int_Z \psi(t, u(t)) dt$$

that is

$$(2.1.3) \quad \liminf_n \int_Z \langle Au_n(t), u_n(t) \rangle dt \geq \int_Z \langle Au(t), u(t) \rangle dt$$

To finish the proof we apply the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the normal convex integrand

$(t, x) \mapsto \delta^*(x, C(t))$ by noting that

(j) $\delta^*(g_n(t), C(t))$ is minored by $\langle g_n(t), h(t) \rangle$ where $h \in S_C^1$.

(jj) the minored sequence $\langle g_n(t), h(t) \rangle$ is uniformly integrable.

Then by ([18], Theorem 8.1.6) we are ensured

$$(2.1.4) \quad \liminf_n \int_Z \delta^*(g_n(t), C(t)) dt \geq \int_Z \delta^*(g(t), C(t)) dt$$

By combining (2.1.2) – –(2.1.4) we get

$$\int_Z \delta^*(f(t) - Au(t) - Bv(t), C(t)) dt + \int_Z \langle -f(t) + Au(t) + Bv(t), u(t) \rangle dt \leq 0$$

for every Lebesgue measurable set Z . But since $u(t) \in C(t)$ a.e. we have

$$\delta^*(f(t) + Bv(t) - Au(t), C(t)) \geq \langle f(t) + Bv(t) - Au(t), u(t) \rangle$$

a.e. that is

$$\delta^*(f(t) - Bv(t) - Au(t), C(t)) + \langle -f(t) - Bv(t) + Au(t), u(t) \rangle \geq 0$$

a.e. so we conclude that

$$\delta^*(f(t) - Bv(t) - Au(t), C(t)) = \langle f(t) + Bv(t) - Au(t), u(t) \rangle = 0$$

a.e. with $u(t) \in C(t)$ a.e., just proving that $f(t) + Bv(t) - Au(t) \in N_{C(t)}(u(t))$ a.e. \square

A variant of Proposition 2.1 is available by assuming that C is convex compact valued and the operator B is a Lipschitz mapping: $\|Bx - By\| \leq M\|x - y\|$. These results related to the evolution variational inequality of the form $f(t) + Bv(t) - Au(t) \in N_{C(t)}(u(t))$ a.e stated in Theorem 1.1 and Theorem 1.2. The following result is a stability result related to the evolution variational inequality of the form $f(t) + \nabla g(u(t)) - Au(t) \in N_{C(t)}(\dot{u}(t))$ here ∇g is a specific gradient of a convex continuous Gateaux differentiable such that $g(v(t))$ is absolutely continuous for $v : [0, T] \rightarrow H$ absolutely continuous. There is no confusion with the linear compact operator B considered in Theorem 1.1 and Lipschitz operator B considered in Theorem 1.2.

Proposition 5.2. $H = \mathbb{R}^e$. Let $C : [0, T] \rightarrow H$ be a convex compact valued measurable mapping such that $C(t) \subset r(t)\bar{B}_H$, for all $t \in [0, T]$ for some $r \in L_{\mathbb{R}^+}^1([0, T])$. Let A be linear continuous coercive symmetric operator on H and $B = \nabla g$ where ∇g is the gradient of a convex continuous Gateaux differentiable function $g : H \rightarrow \mathbb{R}^+$ such that $g(v(t))$ is absolutely continuous for $v : [0, T] \rightarrow H$ absolutely continuous.

Let $(f_n, f)_{n \in \mathbb{N}}$ be a bounded sequence in $L_H^\infty([0, T])$ with $\|f_n(t)\| \leq \beta, \|f(t)\| \leq \beta$

($\beta > 0$) for all $n \in \mathbb{N}$ such that $f_n(t)$ pointwise converges to $f(t)$ for each $t \in [0, T]$.
Let $(u_n, \dot{u}_n)_{n \in \mathbb{N}}$ be a sequence of absolutely continuous mappings

$$u_n(t) = u_0 + \int_0^t \dot{u}_n(s) ds, \dot{u}_n(t) \in C(t)$$

$$u(t) = u_0 + \int_0^t \dot{u}(s) ds, \dot{u}(t) \in C(t)$$

such that $u_n(t) \rightarrow u(t)$ uniformly in H and $\dot{u}_n \rightarrow \dot{u}$ weakly in $L_H^1[0, T]$. Assume that $f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t) \in N_{C(t)}(\dot{u}_n(t))$ for all $n \in \mathbb{N}$ and for all $t \in [0, T]$, then $f(t) + \nabla g(u(t)) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$ a.e.

Proof. Note that f_n are uniformly bounded and pointwise converges to $f(t)$ then we have

$$(2.2.1) \quad \lim_n \int_Z \langle f_n, \dot{u}_n \rangle dt = \int_Z \langle f, \dot{u} \rangle dt$$

for every measurable set $Z \subset [0, T]$. By integrating on Z (we are ensured that the functions given are measurable) the inequality

$$\delta^*(f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t), C_n(t)) + \langle A\dot{u}_n(t) - \nabla g(u_n(t)) - f_n, \dot{u}_n(t) \rangle \leq 0$$

we get

$$(2.2.2) \quad \int_Z \delta^*(f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t), C_n(t)) dt + \int_Z \langle A\dot{u}_n - f_n, \dot{u}_n(t) \rangle dt - \int_Z \langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle dt \leq 0.$$

Set $g_n(t) = f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t)$. We claim that $Au_n(t) \rightarrow Au(t)$ weakly in $L_H^1([0, T])$ and $\nabla g(u_n(t)) \rightarrow \nabla g(u(t))$ weakly in $L_H^1([0, T])$. So $g_n(t) = f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t) \rightarrow g(t) := f(t) + \nabla g(u(t)) - A\dot{u}(t)$ weakly in $L_H^1([0, T])$. Indeed, as $u_n(t) \rightarrow u(t)$ pointwise, $\langle x, Au_n(t) \rangle \rightarrow \langle x, Au(t) \rangle$. As consequence, for any $h \in L_H^\infty([0, T])$, we have $\langle h(t), Au_n(t) \rangle \rightarrow \langle h(t), Au(t) \rangle$ pointwise. But then the uniformly bounded sequence of bounded measurable functions ($\langle h, Au_n \rangle$) pointwise converge to the bounded measurable function $\langle h, Au \rangle$. As consequence, $\langle h, Au_n \rangle \rightarrow \langle h, Au \rangle$ weakly in $L_{\mathbb{R}}^1([0, T])$. This shows that $Au_n \rightarrow Au$ weakly in $L_H^1([0, T])$. Similarly we show that $\nabla g(u_n(t)) \rightarrow \nabla g(u(t))$ weakly in $L_H^1([0, T])$. Now we have to consider the term $\langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle$ by using the special property of ∇g . In fact u_n is absolutely continuous with derivative \dot{u}_n and $g(u_n)$ is absolutely continuous, so that by Moreau-Valadier [28],

$$\langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle = \frac{d}{dt} g(u_n(t))$$

From this fact, it is easy to deduce that

$$\liminf_n \int_0^T \langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle dt = \liminf_n \int_0^T \frac{d}{dt} g(u_n(t)) dt$$

$$\begin{aligned}
(2.2.3) \quad &= \liminf_n (g(u_n(T)) - u_n(0)) \geq g(u(T)) - u(0) = \int_0^T \frac{d}{dt} g(u(t)) dt \\
&= \int_0^T \langle \nabla g(u(t)), \dot{u}(t) \rangle dt
\end{aligned}$$

Let us set $\varphi(t, x) = \langle Ax, x \rangle$ if $x \in K$ and $\varphi(t, x) = +\infty$ if $x \notin K$. Then it is clear φ is a positive lower semicontinuous convex normal integrand. By the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the positive normal convex integrand φ we obtain, for every Lebesgue measurable set $Z \subset [0, T]$

$$\liminf_n \int_Z \varphi(t, \dot{u}_n(t)) dt \geq \int_Z \varphi(t, \dot{u}(t)) dt$$

that is

$$(2.2.4) \quad \liminf_n \int_Z \langle A\dot{u}_n(t), \dot{u}_n(t) \rangle dt \geq \int_Z \langle A\dot{u}(t), \dot{u}(t) \rangle dt$$

To finish the proof we apply the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the normal convex integrand $(t, x) \mapsto \delta^*(x, C(t))$ by noting that

(j) $\delta^*(g_n(t), C(t))$ is minored by $\langle h(t), g_n(t) \rangle$; with $h \in S_C^\infty$.

(jj) the minored sequence $\langle g_n(t), h(t) \rangle$ is uniformly integrable.

Then we are ensured by the lower semicontinuity of the integral convex functional theorem ([18], Theorem 8.1.6)

$$(2.2.5) \quad \liminf_n \int_Z \delta^*(g_n(t), C(t)) dt \geq \int_Z \delta^*(g(t), C(t)) dt$$

By combining (2.2.3) – (2.2.5) we get

$$\int_0^T \delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) dt + \int_0^T \langle -f(t) + \nabla g(u(t)) + A\dot{u}(t), \dot{u}(t) \rangle dt \leq 0$$

But since $\dot{u}(t) \in C(t)$ a.e. we have

$$\delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) \geq \langle f(t) + \nabla g(u(t)) - A\dot{u}(t), \dot{u}(t) \rangle$$

a.e. that implies

$$\int_0^T [\delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) + \langle -f(t) - \nabla g(u(t)) + A\dot{u}(t), \dot{u}(t) \rangle] dt = 0$$

so we conclude that

$$\delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) = \langle f(t) + \nabla g(u(t)) - A\dot{u}(t), \dot{u}(t) \rangle$$

a.e. with $\dot{u}(t) \in C(t)$ a.e., just proving that $f(t) + \nabla g(u(t)) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$ a.e. \square

6 Applications

6.1 A Skorokhod problem

We present at first a new version of the Skorokhod problem in Castaing et al [15, 17] dealing with the sweeping process associated with an absolutely continuous (or continuous) closed convex moving set $C(t)$ in H . Here the novelty is the velocity is inside the subdifferential operator.

Theorem 6.1. *Let $I := [0, 1]$ and $H = \mathbb{R}^e$. Let $v : I \rightarrow \mathbb{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : I \rightrightarrows H$ be a convex compact valued mapping such that*

- (i) $C(t) \subset M\bar{B}_H$ for all $t \in I$ where M is a positive constant.
- (ii) $d_H(C(t), C(\tau)) \leq |v(t) - v(\tau)|$ for all $t, \tau \in I$.

Let $A : H \rightarrow H$ be a linear continuous coercive symmetric operator. Let $B : H \rightarrow H$ be a linear continuous operator. Let $z \in C^{1-var}([0, 1], \mathbb{R}^d)$ the space of continuous functions of bounded variation defined on $[0, 1]$ with values in \mathbb{R}^d . Let $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ the space of linear mappings f from \mathbb{R}^d to \mathbb{R}^e endowed with the operator norm

$$|f| := \sup_{x \in \mathbb{R}^d, \|x\|_{\mathbb{R}^d} = 1} |f(x)|_{\mathbb{R}^e}.$$

Let us consider a class of continuous integrand operator $b : [0, 1] \times \mathbb{R}^e \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ satisfying

- (a) $|b(t, x)| \leq M, \quad \forall (t, x) \in [0, 1] \times \mathbb{R}^e$.
- (b) $|b(t, x) - b(t, y)| \leq M\|x - y\|_{\mathbb{R}^e}, \quad \forall (t, x, y) \in [0, 1] \times \mathbb{R}^e \times \mathbb{R}^e$ with the perturbed Riemann-Stieljies integral $\int_0^t b(\tau, x(\tau)) dz_\tau$ defined on $x \in C([0, 1], \mathbb{R}^e)$.

Let $g : [0, 1] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$ be a continuous mapping satisfying:

- (i) $\|g(s, x)\| \leq M$ for all $(s, x) \in [0, 1] \times \mathbb{R}^e$.
- (ii) $\|g(s, x) - g(s, y)\| \leq M\|x - y\|$ for all $(s, x, y) \in [0, 1] \times \mathbb{R}^e \times \mathbb{R}^e$ with the perturbed Lebesgue integral $\int_0^t g(s, x(s)) ds$ defined on in $C([0, 1], \mathbb{R}^e)$.

Let $a \in C(0)$. Then there exist a BVC function $x : [0, 1] \rightarrow H$ and an absolutely continuous mapping function $u : [0, 1] \rightarrow H$ satisfying

$$\begin{cases} x(0) = u(0) = a \\ x(t) = h(t) + k(t) + Bu(t), \quad \forall t \in I \\ h(t) = \int_0^t b(\tau, x(\tau)) dz_\tau, \quad \forall t \in I \\ k(t) = \int_0^t g(s, x(s)) ds, \quad \forall t \in I \\ \int_0^t b(s, x(s)) dz_s + \int_0^t g(s, x(s)) ds + Bu(t) - Au(t) \in N_{C(t)}(\dot{u}(t)) \text{ a.e. } t \in I \end{cases}$$

Proof. Let $a \in C(0)$. Let us set for all $t \in I = [0, 1]$

$$x^0(t) = a, \quad h^1(t) = \int_0^t b(\tau, a) dz_\tau$$

then by Proposition 2.2 in Friz-Victoir [20], we have

$$\left| \int_0^t b(\tau, a) dz_\tau \right| \leq |b(\cdot, a)|_{\infty: [0,1]} |z|_{1-var: [0,t]}.$$

Moreover

$$\int_0^t b(\tau, a) dz_\tau - \int_0^s b(\tau, a) dz_\tau = \int_s^t b(\tau, a) dz_\tau$$

so that by condition (a)

$$\|h^1(t) - h^1(s)\| \leq M |z|_{1-var: [s,t]}$$

for all $0 \leq s \leq t \leq 1$ and in particular

$$\|h^1(t)\| \leq M |z|_{1-var: [0,t]} \leq M |z|_{1-var: [0,T]}$$

for all $t \in [0, 1]$. Let us set for all $t \in I = [0, 1]$

$$x^0(t) = a, \quad k^1(t) = \int_0^t g(s, x^0(s)) ds,$$

then k^1 is continuous with $\|k^1(t)\| \leq M$ for all $t \in I$. By an easy computation, using condition (i) and (ii) we have the estimate $\|k^1(t) - k^1(\tau)\| \leq M|t - \tau|$, for all $\tau, t \in I$. By Theorem 1.1 there is a unique absolutely continuous mapping $u^1 : I \rightarrow H$ solution of the problem

$$\begin{cases} u^1(0) = a \\ h^1(t) + k^1(t) + Bu^1(t) - Au^1(t) \in N_{C(t)}(\dot{u}^1(t)) \text{ a.e.} \end{cases}$$

with $u^1(t) = a + \int_0^t \dot{u}^1(s) ds$, $\forall t \in I$ and $\|\dot{u}^1(t)\| \leq M$, a.e. Set

$$x^1(t) = h^1(t) + k^1(t) + Bu^1(t) = \int_0^t b(\tau, x^0(\tau)) dz_\tau + \int_0^t g(t, s, x^0(s)) ds + Bu^1(t).$$

Then x^1 is BVC with $x^1(0) = a$. Now we construct x^n by induction as follows. Let for all $t \in I$

$$h^n(t) = \int_0^t b(\tau, x^{n-1}(\tau)) dz_\tau$$

$$k^n(t) = \int_0^t g(s, x^{n-1}(s)) ds.$$

Then k^n is equi-Lipschitz: $\|k^n(t) - k^n(\tau)\| \leq M|t - \tau|$, for all $\tau, t \in I$ with $\|k^n(t)\| \leq M$ for all $t \in I$. By Proposition 2.2 in Friz-Victoir [20] we have the estimate

$$\|h^n(t) - h^n(s)\| \leq M |z|_{1-var: [s,t]}$$

for all $0 \leq s \leq t \leq 1$ and in particular

$$\|h^n(t)\| \leq M|z|_{1-var:[0,t]} \leq M|z|_{1-var:[0,1]}$$

for all $0 \leq t \leq 1$. By Theorem 1.1 there is a unique absolutely continuous mapping $u^n : I \rightarrow H$ solution of the problem

$$\begin{cases} u^n(0) = a, \\ h^n(t) + k^n(t) + Bu^n(t) - Au^n(t) \in N_{C(t)}(\dot{u}^n(t)) \text{ a.e.} \end{cases}$$

with $u^n(t) = a + \int_0^t \dot{u}^n(s)ds \forall t \in I$ and $\|\dot{u}^n(t)\| \leq M$ a.e. Set for all $t \in I$

$$x^n(t) = h^n(t) + k^n(t) + Bu^n(t) = \int_0^t b(\tau, x^{n-1}(\tau))dz_\tau + \int_0^t g(s, x^{n-1}(s))ds + Bu^n(t)$$

so that x^n is BVC. As (u^n) is equi-absolutely continuous we may assume that (u^n) converges uniformly to an absolutely continuous mapping $u : I \rightarrow H$ with $u(t) = a + \int_0^t \dot{u}(s)ds, \forall t \in I$. We may also assume that \dot{u}^n weakly converges in L_H^1 to \dot{u} , and by Ascoli theorem we may assume that k^n converges uniformly to a continuous mapping $k : I \rightarrow H$. Now, recall that

$$\|h^n(t) - h^n(s)\| \leq M|z|_{1-var:[s,t]}$$

for all $0 \leq s \leq t \leq T$ by using Proposition 2.2 in Friz-Victoir [20], and our assumption (a) on the mapping b . So h^n is bounded and equicontinuous. By Ascoli theorem, we may assume that h^n converge uniformly to a continuous mapping h . Similarly k^n is bounded and equi-Lipschitz. By Ascoli theorem, we may assume that k^n converge uniformly to a continuous mapping k . Hence $x^n(t) = h^n(t) + k^n(t) + Bu^n(t)$ converge uniformly to $x(t) := h(t) + k(t) + Bu(t)$, and $b(\cdot, x^{n-1}(\cdot))$ converges uniformly to $b(\cdot, x(\cdot))$ using the Lipschitz condition (b). Then by Friz-Victoir [20] (Proposition 2.7) $\int_0^t b(\tau, x^{n-1}(\tau))dz_\tau$ converge uniformly to $\int_0^t b(\tau, x(\tau))dz_\tau$. By hypothesis (i), $g(s, x^{n-1}(s))$ pointwise converge to $g(s, x(s))$. Hence $\int_0^t g(s, x^{n-1}(s))ds \rightarrow \int_0^t g(s, x(s))ds$ for each $t \in I$ by Lebesgue theorem. So by identifying the limit

$$\begin{aligned} \lim_{n \rightarrow \infty} x^n(t) &= \lim_{n \rightarrow \infty} \int_0^t b(\tau, x^{n-1}(\tau))dz_\tau + \lim_{n \rightarrow \infty} \int_0^t g(s, x^{n-1}(s))ds + \lim_{n \rightarrow \infty} Bu^n(t) \\ &= \int_0^t b(\tau, x(\tau))dz_\tau + \int_0^t g(t, s, x(s))ds + Bu(t) = x(t). \end{aligned}$$

From the inclusion

$$h^n(t) + k^n(t) + Bu^n(t) - Au^n(t) \in N_{C(t)}(\dot{u}^n(t)) \text{ a.e.}$$

and the above convergence, repeating the argument involving variational techniques in Proposition 1.2 we get

$$h(t) + k(t) + Bu(t) - Au(t) \in N_{C(t)}(\dot{u}(t)) \text{ a.e.}$$

The proof is therefore complete. \square

Our tools allow to state several variants of Theorem 3.1 according to the nature of the perturbation and the operator. Actually Theorem 3.1 hold if $B : H \rightarrow H$ is a Lipschitz mapping: $\|Bx - By\| \leq M\|x - y\|, \forall x, y \in H$ for some positive constant M . Also Theorem 3.1 holds if we replace B by the gradient ∇g of a positive convex continuous Gateaux differentiable function $g : H \rightarrow \mathbb{R}$ such that $g(v(t))$ is absolutely continuous for $v : [0, T] \rightarrow H$ absolutely continuous,

6.2 Towards to some application in Optimal Control problem

In the previous results we have developed the Skorohod problem associated the sweeping process with Riemann-Stieltjes integral perturbation. This led to some problem in Optimal Control .

Proposition 6.1. *Let $I := [0, 1]$ and $H = \mathbb{R}^e$. Let $v : I \rightarrow \mathbb{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : I \rightrightarrows \mathbb{R}^e$ be a convex compact valued mapping such that*

(i) $C(t) \subset M\overline{B}_{\mathbb{R}^e}$ for all $t \in I$ where M is a positive constant.

(ii) $d_{\mathbb{R}^e}(C(t), C(\tau)) \leq |v(t) - v(\tau)|$ for all $t, \tau \in I$.

Let $A : \mathbb{R}^e \rightarrow \mathbb{R}^e$ be a linear continuous coercive symmetric operator. Let $B : \mathbb{R}^e \rightarrow \mathbb{R}^e$ be a linear continuous operator. Let $z \in C^{1-var}([0, 1], \mathbb{R}^d)$ the space of continuous functions of bounded variation defined on $[0, 1]$ with values in \mathbb{R}^d . Let $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ the space of linear mappings f from \mathbb{R}^d to \mathbb{R}^e endowed with the operator norm

$$|f| := \sup_{x \in \mathbb{R}^d, \|x\|_{\mathbb{R}^d} = 1} |f(x)|_{\mathbb{R}^e}.$$

Let us consider a class of continuous integrand operator $b : [0, 1] \times \mathbb{R}^e \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ satisfying

(a) $|b(t, x)| \leq M, \quad \forall (t, x) \in [0, 1] \times \mathbb{R}^e$.

(b) $|b(t, x) - b(t, y)| \leq M\|x - y\|_{\mathbb{R}^e}, \quad \forall (t, x, y) \in [0, 1] \times \mathbb{R}^e \times \mathbb{R}^e$ with the perturbed Riemann-Stieltjes integral $\int_0^t b(\tau, x(\tau)) dz_\tau$ defined on $x \in C([0, 1], \mathbb{R}^e)$.

Let $V : \mathbb{R}^d \rightarrow \mathbb{R}^e$ be a bounded continuous mapping. Let $L : [0, 1] \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e \rightarrow [0, \infty[$ be a lower semicontinuous integrand such that $L(t, x, y, \cdot)$ is convex on \mathbb{R}^e for every $(t, x, y) \in [0, 1] \times \mathbb{R}^e \times \mathbb{R}^e$. Then the problem of minimizing the cost function $\int_0^1 L(t, x(t), y(t), \dot{y}(t)) dt$ subject to

$$\begin{cases} dx_t = V(x_t) dz_t, & t \in [0, 1] \\ x_0 = \psi \in \mathbb{R}^e \\ y(0) = y_0 \in C(0) \\ \int_0^t b(\tau, x(\tau)) dz_\tau + By(t) - Ay(t) \in N_{C(t)}\left(\frac{dy}{dt}(t)\right), & \text{a.e. } t \in [0, T]. \end{cases}$$

has an optimal solution.

Proof. Let us consider a minimizing sequence (x_n, y_n) that is

$$\lim_{n \rightarrow \infty} \int_0^T L(t, x_n(t), y_n(t), \dot{y}_n(t)) dt = \inf_{(u, v) \in \mathcal{Y}} \int_0^T L(t, u(t), v(t), \dot{v}(t)) dt$$

where \mathcal{Y} is the solutions set (x, y) to the above dynamical system. First by [20], Theorem 3.4) we assert that the $C^{1-var}([0, T], \mathbb{R}^e)$ -solution set to

$$\begin{cases} dx_t = V(x_t)dz_t, & t \in [0, 1] \\ x_0 = \psi \in \mathbb{R}^e \end{cases}$$

is compact in $C([0, T], \mathbb{R}^e)$ and so is the $W^{1,\infty}([0, T], \mathbb{R}^e)$ -solution set to

$$\begin{cases} y(0) = y_0 \in C(0) \\ \int_0^t b(\tau, x(\tau))dz_\tau + By(t) - A\dot{y}(t) \in N_{C(t)}(\frac{dy}{dt}(t)), & a.e. t \in [0, T]. \end{cases}$$

this need a **careful**. We may ensure that

(i) $x_n \rightarrow x \in C^{1-var}([0, T], \mathbb{R}^e)$ with $x_t = \psi + \int_0^t V(x_s)dz_s$.

(ii) $y_n \rightarrow y \in W^{1,\infty}([0, T], \mathbb{R}^e)$ and $\dot{y}_n \rightarrow \dot{y}$ weakly in $L^1_{\mathbb{R}^e}([0, T])$.

Applying the lower semicontinuity of the integral functional ([18] , Theorem 8.16) gives

$$\liminf_n \int_0^T L(t, x_n(t), y_n(t), \dot{y}_n(t))dt \geq \int_0^T L(t, x(t), y(t), \dot{y}(t))dt$$

From the inclusion

$$\int_0^t b(\tau, x_n(\tau))dz_\tau + By_n(t) - A\dot{y}_n(t) \in N_{C(t)}(\frac{dy_n}{dt}(t))$$

and the fact that $\lim_n \int_0^t b(\tau, x_n(\tau))dz_\tau = \int_0^t b(\tau, x(\tau))dz_\tau$ uniformly (cf the proof of Theorem 3.1) we conclude by using Proposition 2.1

$$\int_0^t b(\tau, x(\tau))dz_\tau + By(t) - A\dot{y}(t) \in N_{C(t)}(\frac{dy}{dt}(t)) a.e.$$

□

Several variants of the preceding theorem are available using Theorem 1.2-1.3 and Proposition 2.2 -2.3

In the following we will examine a Bolza and Relaxation problem associated with a sweeping process associated with a compact Lipschitzean moving set $C(t)$: $d_{\mathbb{R}^e}(C(t), C(s)) \leq \alpha|t - s|$ and Young integral perturbation. First, we need some notation and background on Young integral and Young measures in this special context.

Young integral Let $z \in C^{1-var}([0, T], \mathbb{R}^d)$ the space bounded variation continuous defined on $[0, T]$ with values in \mathbb{R}^d . Let $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ the space of linear mappings f from \mathbb{R}^d to \mathbb{R}^e endowed with the operator norm

$$|f| := \sup_{x \in \mathbb{R}^d, \|x\|_{\mathbb{R}^d} = 1} |f(x)|_{\mathbb{R}^e}.$$

Let us consider a class of continuous integrand operator $b : [0, T] \times \mathbb{R}^e \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ satisfying

$$\mathcal{B}_1 : |b(t, x)| \leq M, \quad \forall x \in \mathbb{R}^e$$

$$\mathcal{B}_2 : |b(t, x) - b(\tau, y)| \leq \rho(t) - \rho(\tau) + M\|x - y\|_{\mathbb{R}^e}, \quad 0 \leq \tau \leq t \leq T, \quad \forall x, y \in \mathbb{R}^e$$

where $\rho : [0, T] \rightarrow \mathbb{R}^+$ is a positive nondecreasing continuous function and M is a positive constant. If $u_n : [0, T] \rightarrow \mathbb{R}^e$ is a uniformly bounded and uniformly bounded in variation then the sequence $y_n(t) = b(t, u_n(t))$ is continuous uniformly bounded and uniformly bounded in variation from $[0, T]$ to $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$, shortly $y_n \in C^{1-var}([0, T], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e))$. Indeed we have

$$|y_n(t) - y_n(\tau)| \leq \rho(t) - \rho(\tau) + M\|u_n(t) - u_n(\tau)\|_{\mathbb{R}^e}$$

for all $\tau \leq t \leq T$, so that $\sup_n |y_n|_{1-var;[s,t]} < \infty$ for all $n \in \mathbb{N}$ and $0 \leq s \leq t \leq T$. As consequence the Young integral $\int_0^t y_n(s) dz_s$ of y_n again z is well-defined and belong to $C^{1-var}([0, T], \mathbb{R}^e)$ according to Friz-Victoir [20], with the following estimates

$$\begin{aligned} \left\| \int_s^t y_n(\tau) dz_\tau \right\| &\leq \frac{1}{1 - 2^{1-\theta}} |z|_{1-var;[s,t]} |y_n|_{1-var;[s,t]} + |y_n(s)| \|z(t) - z(s)\|_{\mathbb{R}^d} \\ &\leq \frac{1}{1 - 2^{1-\theta}} |z|_{1-var;[s,t]} |y_n|_{1-var;[s,t]} + M \|z(t) - z(s)\|_{\mathbb{R}^d} \end{aligned}$$

for all $0 \leq s \leq t \leq T$ with $\theta = 2$ and

$$\left| \int_0^\cdot y_n(\tau) dz_\tau \right|_{1-var;[s,t]} \leq C(1, 1) |z|_{1-var;[s,t]} (|y_n|_{1-var;[s,t]} + |y_n|_{\infty;[s,t]})$$

for all $0 \leq s \leq t \leq T$. As consequence

$$\begin{aligned} \left\| \int_s^t y_n dz \right\| &\leq \frac{1}{1 - 2^{1-\theta}} |z|_{1-var;[s,t]} |y_n|_{1-var;[s,t]} + |y_n(s)| \|z(t) - z(s)\|_{\mathbb{R}^d} \\ &\leq \frac{1}{1 - 2^{1-\theta}} |z|_{1-var;[s,t]} |y_n|_{1-var;[s,t]} + M \|z(t) - z(s)\|_{\mathbb{R}^d} \\ &\leq \frac{1}{1 - 2^{1-\theta}} |z|_{1-var;[s,t]} \sup_n |y_n|_{1-var;[s,t]} + \sup_{s \in [0, T]} M \|z(t) - z(s)\|_{\mathbb{R}^d} \end{aligned}$$

for all $0 \leq s \leq t \leq T$ with $\theta = 2$ and

$$\left| \int_0^\cdot y_n dz \right|_{1-var;[s,t]} \leq C(1, 1) |z|_{1-var;[s,t]} \sup_n (|y_n|_{1-var;[s,t]} + |y_n|_{\infty;[s,t]})$$

for all $0 \leq s \leq t \leq T$. Shortly the sequence $g_n(\cdot) = \int_0^\cdot y_n dz$ of $C^{1-var}([0, T], \mathbb{R}^e)$ mappings is uniformly bounded, equicontinuous and uniformly bounded in variation. The above facts hold in the case when $u_n : [0, T] \rightarrow \mathbb{R}^e$ is uniformly bounded

and equi-Lipschitz and have some importance for our purpose.

Let E is a separable reflexive. Let us consider a convex weakly compact valued mapping $K : [0, 1] \Rightarrow \overline{B}_E$ with bounded right continuous retraction in the sense, there is a bounded and right continuous function $\rho : [0, 1] \rightarrow \mathbb{R}^+$ such that $d_H(K(t), K(\tau)) \leq \rho(\tau) - \rho(t), \forall t \leq \tau \in [0, 1]$ and such that its graph is Borel, that is, $\text{Graph}(C) \in \mathcal{B}([0, T]) \otimes \mathcal{B}(E)$. We consider the control sets given by

$$S_K^{BVRC} := \{u : [0, 1] \rightarrow E, u \text{ is BVRC}, u(t) \in K(t), \forall t \in [0, 1]\}$$

$$S_K^\infty := \{u \in L^\infty([0, 1], E, d\lambda), u(t) \in K(t), \forall t \in [0, T]\}$$

By J.J. Moreau ([27], Prop.5 d, p. 198) and Valadier [33] these sets are non empty and $cl S_C^{BVRC} = S_C^\infty$, here cl denotes the closure with respect to the $\sigma(L_E^\infty, L_{E^*}^1)$ -topology. Shortly S_K^{BVRC} is dense in S_K^∞ with respect to this topology. Then we have the following relaxation results in a control problem governed by an EVI sweeping process given above. We aim to present a Bolza problem and relaxation problem in a Optimal Control where the control belong to S_K^{BVRC} and S_K^∞ . Here $A : \mathbb{R}^e \rightarrow \mathbb{R}^e$ is a linear continuous coercive symmetric operator and let B is a linear continuous operator on \mathbb{R}^e and $b : [0, T] \times \mathbb{R}^e \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ satisfying $\mathcal{B}_1, \mathcal{B}_2$.

Theorem 6.2. *Let $E = \mathbb{R}^e$. Let us consider the problem*

$$\inf_{(\mathcal{P}_{S_K^{BVRC}})} \int_0^T \left\langle \int_0^t b(s, u(s)) dz_s, \zeta(t) \right\rangle dt$$

associated with the dynamical system

$$(\mathcal{P}_{S_K^{BVRC}}) \begin{cases} \int_0^t \zeta(s) ds + Bu(t) + A\dot{u}(t) \in N_{C(t)}(\dot{u}(t)), t \in I, \zeta \in S_K^{BVRC} \\ u(0) = a \in C(0) \end{cases}$$

and the problem

$$\inf_{(\mathcal{P}_{V_K^\infty})} \int_0^T \left\langle \int_0^t b(s, u(s)) dz_s, \zeta(t) \right\rangle dt$$

associated with the dynamic system

$$(\mathcal{P}_{V_K^\infty}) \begin{cases} \int_0^t \zeta(s) ds + Bu(t) + A\dot{u}(t) \in N_{C(t)}(\dot{u}(t)), t \in I, \zeta \in V_K^\infty \\ u(0) = a \in C(0) \end{cases}$$

Then we have $\inf(\mathcal{P}_{V_K^\infty}) = \inf(\mathcal{P}_{S_K^{BVRC}})$ As consequence

$$\inf_{(\mathcal{P}_{V_K^\infty})} \int_0^T \left\langle \int_0^t b(s, u(s)) dz_s, \zeta(t) \right\rangle dt$$

is a minimum.

Proof. The inequality $\inf(\mathcal{P}_{\mathcal{V}_K^\infty}) \leq \inf(\mathcal{P}_{\mathcal{S}_K^{BVRC}})$ is a simple consequence of the Valadier result above mentioned by observing that a BVRC function is Borel. Take and any $\zeta \in \mathcal{V}_K^\infty$. As \mathcal{S}_K^{BVRC} is dense in \mathcal{V}_K^∞ with respect to the $\sigma(L_E^\infty, L_E^1)$ topology, there exists a sequence (ζ_n) in \mathcal{S}_K^{BVRC} such that $(\zeta_n) \sigma(L_E^\infty, L_E^1)$ converge to ζ . For simplicity set $f_n(t) = \int_0^t \zeta_n(s) ds$, and $f(t) = \int_0^t \zeta(s) ds$ for all $t \in [0, T]$. Then it is clear that f_n and f continuous uniformly bounded with $f_n(t) \rightarrow f(t)$ for every $t \in [0, T]$. Let u_n be the unique Lipschitz solution to

$$\begin{cases} \int_0^t \zeta_n(s) ds + Bu_n(t) + A\dot{u}_n(t) \in N_{C(t)}(\dot{u}_n(t)), t \in I, \\ u_n(0) = a \in C(0) \end{cases}$$

and let v be the unique Lipschitz solution to

$$\begin{cases} \int_0^t \zeta(s) ds + Bv(t) + A\dot{v}(t) \in N_{C(t)}(\dot{v}(t)), t \in I, \\ v(0) = a \in C(0) \end{cases}$$

In view Theorem 1.1 and Proposition 2.1 the sequence (u_n) is equi-Lipchitz and converges uniformly to v . For simplicity set $g_n(t) = \int_0^t b(s, u_n(s)) dz_s, \forall t \in [0, T]$. Apply the estimates given the preliminary fact, $g_n(t) = \int_0^t b(s, u_n(s)) dz_s$ is uniformly bounded, equicontinuous and uniformly bounded in variation. By the Lipschitz condition $b(\cdot, u_n(\cdot))$ converge uniformly to $b(\cdot, v(\cdot))$. As $b(\cdot, u_n(\cdot))$ and $b(\cdot, v(\cdot))$ are bounded and uniformly bounded in variation, by Friz-Victoir ([20] Proposition 6.12) $g_n(t)$ converge uniformly to $g(t) = \int_0^t b(s, v(s)) dz_s$ so that

$$\lim_{n \rightarrow \infty} \int_0^T \langle \int_0^t b(s, u_n(s)) dz_s, \zeta_n(t) \rangle dt = \int_0^T \langle \int_0^t b(s, v(s)) dz_s, \zeta(t) \rangle dt$$

As

$$\int_0^T \langle \int_0^t b(s, u_n(s)) dz_s, \zeta_n(t) \rangle dt \geq \inf_{(\mathcal{P}_{\mathcal{S}_K^{BVRC}})} \int_0^T \langle \int_0^t b(s, u(s)) dz_s, \zeta(t) \rangle dt$$

for all $n \in \mathbb{N}$, it follows by taking the limit that

$$\int_0^T \langle \int_0^t b(s, v(s)) dz_s, \zeta(t) \rangle dt \geq \inf(\mathcal{P}_{\mathcal{S}_K^{BVRC}})$$

This holds for every $\zeta \in \mathcal{V}_K^\infty$ hence $\inf(\mathcal{P}_{\mathcal{V}_K^\infty}) \geq \inf(\mathcal{P}_{\mathcal{S}_K^{BVRC}})$ \square

For simplicity we consider a simple perturbation control integral such as $\int_0^t \zeta(s) ds$. Theorem 3.2 is valid iwth the perturbation control integral $\int_0^t \mathcal{D}\zeta(s) ds$ where $\mathcal{D} : \mathbb{R}^e \rightarrow \mathbb{R}^e$ is a linear continuous operator. Several variants of Theorem 3.2 are available using Theorem 1.2-1.3 and Proposition 2.2 -2.3.

Young measures For the sake of completeness, we summarize some useful facts concerning Young measures. Let (Ω, \mathcal{F}, P) be a complete probability space.

Let X be a Polish space and let $C^b(X, \mathbb{R})$ be the space of all bounded continuous functions defined on X . Let $\mathcal{M}_+^1(X)$ be the set of all Borel probability measures on X equipped with the narrow topology. A Young measure $\nu : \Omega \rightarrow \mathcal{M}_+^1(X)$ is, by definition, a *scalarly measurable* mapping from Ω into $\mathcal{M}_+^1(X)$, that is, for every $f \in C^b(X, \mathbb{R})$, the mapping $\omega \mapsto \langle f, \nu_\omega \rangle := \int_X f(x) d\nu_\omega(x)$ is \mathcal{F} -measurable. A sequence (ν^n) in the space of Young measures $\mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}_+^1(X))$ *stably converges* to a Young measure $\nu \in \mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}_+^1(X))$ if the following holds:

$$\lim_{n \rightarrow \infty} \int_A \left(\int_X f(x) d\nu_\omega^n(x) \right) dP(\omega) = \int_A \left(\int_X f(x) d\nu_\omega(x) \right) dP(\omega)$$

for every $A \in \mathcal{F}$ and for every $f \in C^b(X, \mathbb{R})$. We recall and summarize some results for Young measures.

In the remainder Z is a compact metric space, $\mathcal{M}_+^1(Z)$ is the space of all probability Radon measures on Z . We will endow $\mathcal{M}_+^1(Z)$ with the narrow topology so that $\mathcal{M}_+^1(Z)$ is a compact metrizable space. Let us denote by $\mathcal{Y}(I; \mathcal{M}_+^1(Z))$ the space of all Young measures defined on I endowed with the stable topology so that $\mathcal{Y}(I; \mathcal{M}_+^1(Z))$ is a compact metrizable space with respect to this topology. By the Portmanteau Theorem for Young measures [18, Theorem 2.1.3], a sequence (ν^n) in $\mathcal{Y}(I; \mathcal{M}_+^1(Z))$ stably converges to $\nu \in \mathcal{Y}(I; \mathcal{M}_+^1(Z))$ if

$$\lim_{n \rightarrow \infty} \int_0^T \left(\int_Z h_t(z) d\nu_t^n(z) \right) dt = \int_0^T \left(\int_Z h_t(z) d\nu_t(z) \right) dt$$

for all $h \in L^1(I, C(Z))$; here $C(Z)$ denotes the space of all continuous real valued functions defined on Z endowed with the norm of uniform convergence. Finally let Γ be a measurable mapping defined on I with nonempty compact values in Z and S_Γ the set of all Lebesgue measurable selections of Γ (alias *original controls*). Let $C : [0, T] \rightarrow \mathbb{R}^e$ compact valued Lipschitz mapping and let $f : I \times Z \rightarrow \mathbb{R}^e$ be a function satisfying

- (1) for every fixed $t \in I$, $f(t, \cdot)$ is continuous on Z ,
- (2) for every $z \in Z$, $f(\cdot, z)$ is Lebesgue-measurable on I ;
- (3) there is a constant $M > 0$ such that $\|f(t, z)\| \leq M$ for all (t, z) in $I \times Z$.

We aim to present some relaxation problems in the framework of Optimal Control Theory. We consider the evolution inclusion (\mathcal{PO}) associated with original controls

$$(\mathcal{PO}) \begin{cases} \int_0^t f(s, \zeta(s)) ds + Bu_\zeta(t) - A\dot{u}_\zeta(t) \in N_{C(t)}(\dot{u}_\zeta(t)), \text{ a.e. } t \in I, \\ u_\zeta(0) = u_0 \in C(0) \end{cases}$$

where ζ belongs to the set $\mathcal{Z} := S_\Gamma^1$ of all original controls, which means that ζ is a Lebesgue-measurable selection of Γ , and the evolution inclusion (\mathcal{PR}) associated with relaxed controls

$$(\mathcal{PR}) \begin{cases} \int_0^t [\int_Z f(s, z) \nu_s(dz)] ds + Bu_\nu(t) - A\dot{u}_\nu(t) \in N_{C(t)}(\dot{u}_\nu(t)), \text{ a.e. } t \in I \\ u_\nu(0) = u_0 \in C(0) \end{cases}$$

where ν belongs to the set $\mathcal{R} := S_\Sigma$ of all relaxed controls, which means that ν is a Lebesgue-measurable selection of the mapping Σ defined by

$$\Sigma(t) := \{\sigma \in \mathcal{M}_+^1(Z) : \sigma(\Gamma(t)) = 1\}$$

for all $t \in I$. Note that, for $\nu \in \mathcal{R}$, the mapping

$$h_\nu : (t, z) \mapsto \int_Z f(t, z) \nu_t(dz)$$

inherits the properties

- (1) for every fixed $t \in I$, $h_\nu(t, \cdot)$ is continuous on Z ;
 - (2) for every $z \in Z$, $h_\nu(\cdot, z)$ is Lebesgue-measurable on I ;
 - (3) there is a constant $M > 0$ such that $\|h_\nu(t, z)\| \leq M$ for all (t, z) in $I \times Z$;
- Consequently, for each $\zeta \in \mathcal{Z}$ (resp. $\nu \in \mathcal{R}$), the evolution inclusion (\mathcal{PO}) (resp. (\mathcal{PR})) has a unique Lipschitz continuous solution. Moreover, there is an a priori bound for the Lipschitz ratio of solutions which easily implies that the solution sets (\mathcal{SO}) and (\mathcal{SR}) (to (\mathcal{PO}) and (\mathcal{PR})) are equi-Lipschitz.

Now we will state some topological properties of the solution sets (\mathcal{SO}) and (\mathcal{SR}) , namely we obtain the typical relaxation result that the former is dense in the latter.

Theorem 6.3. *Assume that for every $t \in I$, let $C : [0, 1] \rightarrow \mathbb{R}^e$ compact valued Lipschitz mapping. Let $f : I \times Z \rightarrow \mathbb{R}^e$ be a mapping satisfying (1), (2), (3). Then the following hold:*

(a) *the solution set (\mathcal{SR}) to*

$$(\mathcal{PR}) \begin{cases} \int_0^t [\int_Z f(s, z) \nu_s(dz)] ds + Bu_\nu(t) - A\dot{u}_\nu(t) \in N_{C(t)}(\dot{u}_\nu(t)), \text{ a.e. } t \in I, \nu \in \mathcal{Z} \\ u_\nu(0) = u_0 \in C(0) \end{cases}$$

is nonempty and compact in $C(I, \mathbb{R}^e)$.

(b) *the solution set (\mathcal{SO}) to*

$$(\mathcal{PO}) \begin{cases} \int_0^t f(s, \zeta(s)) ds + Bu_\zeta(t) - A\dot{u}_\zeta(t) \in N_{C(t)}(\dot{u}_\zeta(t)), \text{ a.e. } t \in I, \zeta \in \mathcal{Z} \\ u_\zeta(0) = u_0 \in C(0) \end{cases}$$

is dense in (\mathcal{SR}) with respect to the topology of uniform convergence.

Proof. (a) By Theorem 1.1, the solution set (\mathcal{SR}) is bounded equi-Lipschitz. Then (\mathcal{SR}) is relatively compact in $C(I, H)$, by Ascoli's theorem. Therefore, if $(\nu^n) \subset \mathcal{R}$, there is a subsequence still denoted by (u_{ν^n}) which converges uniformly to a Lipschitz continuous function u^∞ with $\|\dot{u}^\infty(t)\| \leq K$ a.e. and such that also (\dot{u}_{ν^n}) $\sigma(L^1(I, \mathbb{R}^e; dt), L^\infty(I, \mathbb{R}^e; dt))$ -converges to \dot{u}^∞ . As \mathcal{R} is compact metrizable for the stable topology, we may assume that (ν^n) stably converges to $\nu^\infty \in \mathcal{R}$. Since the continuous functions

$$g_n(t) := \int_0^t [\int_Z f(s, z) \nu_s^n(dz)] ds$$

and

$$g(t) := \int_0^t \left[\int_Z f(s, z) \nu_s^\infty(dz) \right] ds$$

are uniformly bounded and with $g_n(t) \rightarrow g(t)$, from the inclusion

$$g_n(t) + Bu_\nu^n(t) - A\dot{u}_\nu(t) \in N_{C(t)}(\dot{u}_\nu^n(t))$$

and Proposition 2.1, we deduce that

$$g(t) + Bu_{\nu^\infty}(t) - A\dot{u}_{\nu^\infty}(t) \in N_{C(t)}(\dot{u}_{\nu^\infty}(t))$$

This proves the first part of the theorem.

(b) The second part follows by continuity and density since \mathcal{Z} is dense in \mathcal{R} with respect to the stable topology ([18], Lemma 7.1.1). \square

Theorem 6.4. *Let us consider the problem*

$$\inf_{(\mathcal{P}_\mathcal{R})} \int_0^T \left\langle \int_0^t b(s, u(s)) dz_s, \int_0^t \left[\int_Z f(s, z) \nu_s(dz) \right] ds \right\rangle dt$$

associated with the dynamic system

$$(\mathcal{P}_\mathcal{R}) \begin{cases} \int_0^t \left[\int_Z f(s, z) \nu_s(dz) \right] ds + Bu_\nu(t) - A\dot{u}_\nu(t) \in N_{C(t)}(\dot{u}_\nu(t)), \text{ a.e. } t \in I, \nu \in \mathcal{Z} \\ u_\nu(0) = u_0 \in C(0) \end{cases}$$

and the problem

$$\int_0^T \left\langle \int_0^t b(s, u(s)) dz_s, \int_0^t h(s, \zeta(s)) ds \right\rangle dt$$

associated with the dynamic system

$$(\mathcal{P}_\mathcal{O}) \begin{cases} \int_0^t f(s, \zeta(s)) ds + Bu_\zeta(t) - A\dot{u}_\zeta(t) \in N_{C(t)}(\dot{u}_\zeta(t)), \\ u(0) = a \in C(0) \end{cases}$$

Then we have $\inf(\mathcal{P}_\mathcal{R}) = \inf(\mathcal{P}_\mathcal{O})$ and

$$\inf_{(\mathcal{P}_\mathcal{R})} \int_0^T \left\langle \int_0^t b(s, u(s)) dz_s, \int_0^t \left[\int_Z h(s, z) \nu_s(dz) \right] ds \right\rangle dt$$

is a minimum.

Proof. The inequality $\inf(\mathcal{P}_\mathcal{R}) \geq \inf(\mathcal{P}_\mathcal{O})$ is clear. Let $\nu \in \mathcal{R}$, $\lim_n \int_0^t h(s, \zeta_n(s)) ds = \int_0^t \left[\int_Z h(s, z) \nu_s(dz) \right] ds$ for all $t \in [0, T]$. Let u_n be the unique Lipschitz solution to

$$\begin{cases} -\dot{u}_n(t) \in N_{C(t)}u_n(t) + \int_0^t h(s, \zeta_n(s)) ds, \text{ } t \in I \\ u_n(0) = a \in C(0) \end{cases}$$

and let v be the unique Lipschitz solution to

$$\begin{cases} -\dot{v}(t) \in N_{C(t)}v(t) + \int_0^t [\int_Z h(s, z)\nu_s(dz)]ds, t \in I \\ v(0) = a \in C(0). \end{cases}$$

In view of the first step of the proof of Theorem 4.3, the sequence (u_n) converges uniformly to v . For simplicity set $g_n(t) = \int_0^t b(s, u_n(s))dz_s, \forall t \in [0, T]$. Apply the estimates given the preliminary fact, $g_n(t) = \int_0^t b(s, u_n(s))dz_s$ is uniformly bounded, equicontinuous and uniformly bounded in variation. By the Lipschitz condition $b(\cdot, u_n(\cdot))$ converge uniformly to $b(\cdot, v(\cdot))$. As $b(\cdot, u_n(\cdot))$ and $b(\cdot, v(\cdot))$ are bounded and uniformly bounded in variation, by Friz-Victoir ([20] Proposition 6.12) $g_n(t)$ converge uniformly to $g(t) = \int_0^t b(s, v(s))dz_s$. For simplicity set $k_n(t) = \int_0^t h(s, \zeta_n(s))ds$ and $k(t) = \int_0^t [\int_Z h(s, z)\nu_s(dz)]ds$ so that $\langle g_n(t), k_n(t) \rangle = \langle g(t), k(t) \rangle$. As $g_n(t), g(t), k_n(t), k(t)$ are uniformly bounded, we deduce that so that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^T \langle \int_0^t b(s, u_n(s))dz_s, \int_0^t h(s, \zeta_n(s))ds \rangle dt &= \lim_{n \rightarrow \infty} \int_0^T \langle g_n(t), k_n(t) \rangle dt \\ &= \int_0^T \langle g(t), k(t) \rangle dt = \langle \int_0^t b(s, v(s))dz_s, \int_0^t [\int_Z h(s, z)\nu_s(dz)]ds \rangle dt \end{aligned}$$

As

$$\int_0^T \langle \int_0^t b(s, u_n(s))dz_s, \int_0^t h(s, \zeta_n(s))ds \rangle dt \geq \inf_{(\mathcal{P}_O)} \int_0^T \langle \int_0^t b(s, u(s))dz_s, \int_0^t h(s, \zeta(s))ds \rangle dt$$

for all $n \in \mathbb{N}$, it follows by taking the limit that

$$\int_0^T \langle \int_0^t b(s, v(s))dz_s, \int_0^t [\int_Z h(s, z)\nu_s(dz)]ds \rangle dt \geq \inf(\mathcal{P}_O)$$

This holds for every $\nu \in \mathcal{R}$ hence $\inf(\mathcal{P}_R) \geq \inf(\mathcal{P}_O)$. \square

6.3 Towards fractional inclusion coupled with EVI and sweeping process

Now $I = [0, 1]$ and we investigate a class of boundary value problem governed by a fractional differential inclusion (FDI) (3.1) in a separable Hilbert space H coupled with the evolution inclusion governed by the above EVI (3.3) and sweeping process (3.4).

$$D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = u(t), t \in I, \quad (6.1)$$

$$I_{0+}^\beta h(t) |_{t=0} := \lim_{t \rightarrow 0} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s) ds = 0, \quad h(1) = I_{0+}^\gamma h(1) = \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} h(s) ds, \quad (6.2)$$

$$f(t, h(t) + Bu(t) - A\dot{u}(t)) \in \partial\varphi(t, \dot{u}(t)) \text{ a.e.} \quad (6.3)$$

and

$$f(t, h(t) + Bu(t) - A\dot{u}(t)) \in N_{C(t)}(\dot{u}(t)) \text{ a.e.} \quad (6.4)$$

where $\alpha \in]1, 2]$, $\beta \in [0, 2 - \alpha]$, $\lambda \geq 0$, $\gamma > 0$ are given constants, D^α is the standard Riemann-Liouville fractional derivative, Γ is the gamma function.

For the convenience of the reader, we begin with a few reminders of the concepts that will be used in the rest of the paper.

Definition 6.1 (Fractional Bochner integral). *Let E be a separable Banach space. Let $f : I = [0, 1] \rightarrow E$. The fractional Bochner-integral of order $\alpha > 0$ of the function f is defined by*

$$I_{0+}^\alpha f(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds, \quad t > 0.$$

In the above definition, the sign “ \int ” denotes the classical Bochner integral.

Lemma 6.5 ([30]). *Let $f \in L^1([0, 1], E, dt)$. We have*

- (i) *If $\alpha \in]0, 1[$ then $I_{0+}^\alpha f$ exists almost everywhere on I and $I_{0+}^\alpha f \in L^1(I, E, dt)$.*
- (ii) *If $\alpha \in [1, \infty)$ then $I_{0+}^\alpha f \in C_E(I)$.*

Definition 6.2. *Let E be a separable Banach space. Let $f \in L^1(I, E, dt)$. We define the Riemann-Liouville fractional derivative of order $\alpha > 0$ of f by*

$$D^\alpha f(t) := D_{0+}^\alpha f(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} f(t) = \frac{d^n}{dt^n} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(s) ds,$$

where $n = [\alpha] + 1$.

We denote by $W_{B,E}^{\alpha,1}(I)$ the space of all continuous functions in $C_E(I)$ such that their Riemann-Liouville fractional derivative of order $\alpha - 1$ are continuous and their Riemann-Liouville fractional derivative of order α are Bochner integrable.

Green function and its properties

Let $\alpha \in]1, 2]$, $\beta \in [0, 2 - \alpha]$, $\lambda \geq 0$, $\gamma > 0$ and $G : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be a function defined by

$$G(t, s) = \varphi(s) I_{0+}^{\alpha-1}(\exp(-\lambda t)) + \begin{cases} \exp(\lambda s) I_{s+}^{\alpha-1}(\exp(-\lambda t)), & 0 \leq s \leq t \leq 1, \\ 0, & 0 \leq t \leq s \leq 1, \end{cases} \quad (6.5)$$

where

$$\varphi(s) = \frac{\exp(\lambda s)}{\mu_0} \left[\left(I_{s+}^{\alpha-1+\gamma}(\exp(-\lambda t)) \right) (1) - \left(I_{s+}^{\alpha-1}(\exp(-\lambda t)) \right) (1) \right] \quad (6.6)$$

with

$$\mu_0 = \left(I_{0+}^{\alpha-1}(\exp(-\lambda t)) \right) (1) - \left(I_{0+}^{\alpha-1+\gamma}(\exp(-\lambda t)) \right) (1). \quad (6.7)$$

We recall and summarize a useful result ([14]).

Lemma 6.6. *Let E be a separable Banach space. Let G be the function defined by (6.5)-(6.7).*

(i) $G(\cdot, \cdot)$ satisfies the following estimate

$$|G(t, s)| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{1 + \Gamma(\gamma + 1)}{|\mu_0| \Gamma(\alpha) \Gamma(\gamma + 1)} + 1 \right) = M_G.$$

(ii) If $u \in W_{B,E}^{\alpha,1}([0, 1])$ satisfying boundary conditions (4.2), then

$$u(t) = \int_0^1 G(t, s) (D^\alpha u(s) + \lambda D^{\alpha-1} u(s)) ds \quad \text{for every } t \in [0, 1].$$

(iii) Let $f \in L_E^1([0, 1])$ and let $u_f : [0, 1] \rightarrow E$ be the function defined by

$$u_f(t) := \int_0^1 G(t, s) f(s) ds \quad \text{for } t \in [0, 1].$$

Then

$$I_{0+}^\beta u_f(t)|_{t=0} = 0 \quad \text{and} \quad u_f(1) = (I_{0+}^\gamma u_f)(1).$$

Moreover $u_f \in W_{B,E}^{\alpha,1}([0, 1])$ and we have

$$(D^{\alpha-1} u_f)(t) = \int_0^t \exp(-\lambda(t-s)) f(s) ds + \exp(-\lambda t) \int_0^1 \varphi(s) f(s) ds \quad \text{for } t \in [0, 1], \quad (6.8)$$

$$(D^\alpha u_f)(t) + \lambda (D^{\alpha-1} u_f)(t) = f(t) \quad \text{for all } t \in [0, 1]. \quad (6.9)$$

From Lemma 6.6 we summarize a crucial fact.

Lemma 6.7. *Let E be a separable Banach space. Let $f \in L^1(I, E, dt)$. Then the boundary value problem*

$$\begin{cases} D^\alpha u(t) + \lambda D^{\alpha-1} u(t) = f(t), & t \in I \\ I_{0+}^\beta u(t)|_{t=0} = 0, & u(1) = I_{0+}^\gamma u(1) \end{cases}$$

has a unique $W_{B,E}^{\alpha,1}(I)$ -solution defined by

$$u(t) = \int_0^1 G(t, s) f(s) ds, \quad t \in I.$$

Theorem 6.8. *Let E be a separable Banach space. Let $X : I \rightarrow E$ be a convex compact valued measurable multifunction such that $X(t) \subset \gamma \overline{B}_E$ for all $t \in I$, where γ is a positive constant and S_X^1 be the set of all measurable selections of X . Then the $W_{B,E}^{\alpha,1}(I)$ -solutions set of problem*

$$\begin{cases} D^\alpha u(t) + \lambda D^{\alpha-1} u(t) = f(t), f \in S_X^1, \text{ a.e. } t \in I \\ I_{0+}^\beta u(t)|_{t=0} = 0, \quad u(1) = I_{0+}^\gamma u(1) \end{cases} \quad (6.10)$$

is compact in $C_E(I)$.

Proof. By virtue of Lemma 6.7 the $W_{B,E}^{\alpha,1}([0,1])$ -solutions set \mathcal{X} to the above inclusion is characterized by

$$\mathcal{X} = \{u_f : I \rightarrow E, u_f(t) = \int_0^1 G(t,s)f(s)ds, f \in S_X^1, t \in I\}$$

Claim: \mathcal{X} is bounded, convex, equicontinuous and **compact** in $C_E(I)$.

From definition of the Green function G , it is not difficult to show that $\{u_f : f \in S_X^1\}$ is bounded, equicontinuous in $C_E(I)$. Indeed let (u_{f_n}) be a sequence in \mathcal{X} . We note that, for each $n \in \mathbb{N}$, we have $u_{f_n} \in W_{B,E}^{\alpha,1}(I)$, and

$$u_{f_n}(t) = \int_0^1 G(t,s)f_n(s)ds, \quad t \in I,$$

with

- $I_{0+}^\beta u_{f_n}(t)|_{t=0} = 0, \quad u_{f_n}(1) = I_{0+}^\gamma u(1),$
- $(D^{\alpha-1}u_{f_n})(t) = \int_0^t \exp(-\lambda(t-s))f_n(s)ds + \exp(-\lambda t) \int_0^1 \varphi(s)f_n(s)ds, \quad t \in I,$
- $(D^\alpha u_{f_n})(t) + \lambda(D^{\alpha-1}u_{f_n})(t) = f_n(t), \quad t \in I.$

For $t_1, t_2 \in I, t_1 < t_2$, we have

$$\begin{aligned} u_{f_n}(t_2) - u_{f_n}(t_1) &= \int_0^1 G(t,s)(f_n(t_2,s) - f_n(t_1,s))ds \\ &= \int_0^1 \varphi(s)f_n(s)ds \left(\int_0^{t_2} \frac{e^{-\lambda\tau}}{\Gamma(\alpha-1)}(t_2-\tau)^{\alpha-2}d\tau - \int_0^{t_1} \frac{e^{-\lambda\tau}}{\Gamma(\alpha-1)}(t_1-\tau)^{\alpha-2}d\tau \right) \\ &+ \int_0^{t_2} e^{\lambda s} \left(\int_s^{t_2} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} e^{-\lambda\tau} d\tau \right) f(s)ds - \int_0^{t_1} e^{\lambda s} \left(\int_s^{t_1} \frac{e^{-\lambda\tau}}{\Gamma(\alpha-1)}(t_1-\tau)^{\alpha-2}d\tau \right) f(s)ds \\ &= \int_0^1 \phi(s)f(s)ds \left[\int_0^{t_1} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2} - (t_1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau + \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \right] \\ &+ \int_0^{t_1} e^{\lambda s} \left(\int_s^{t_1} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2} - (t_1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \right) f(s)ds \\ &+ \int_0^{t_1} e^{\lambda s} \left(\int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \right) f(s)ds + \int_{t_1}^{t_2} e^{\lambda s} \left(\int_s^{t_2} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} e^{-\lambda\tau} d\tau \right) f(s)ds. \end{aligned}$$

Then, we get

$$\begin{aligned} \|u_{f_n}(t_2) - u_{f_n}(t_1)\| &\leq \int_0^1 \left(|\varphi(s)| + e^{\lambda s} \right) |X(s)| ds \int_0^{t_1} e^{-\lambda\tau} \frac{(t_1 - \tau)^{\alpha-2} - (t_2 - \tau)^{\alpha-2}}{\Gamma(\alpha - 1)} d\tau \\ &\quad + \int_0^1 \left(|\varphi(s)| + e^{\lambda s} \right) |X(s)| ds \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2 - \tau)^{\alpha-2}}{\Gamma(\alpha - 1)} d\tau \\ &\quad + \int_{t_1}^{t_2} e^{\lambda s} |X(s)| ds \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2 - \tau)^{\alpha-2}}{\Gamma(\alpha - 1)} d\tau. \end{aligned}$$

It is easy to obtain, after an integration by part, that

$$\int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2 - \tau)^{\alpha-2}}{\Gamma(\alpha - 1)} d\tau = e^{-\lambda t_1} \frac{(t_2 - t_1)^{\alpha-2}}{\Gamma(\alpha)} + \lambda \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2 - \tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau \leq \frac{1 + \lambda}{\Gamma(\alpha)} (t_2 - t_1)^{\alpha-1}$$

and

$$\int_0^{t_1} e^{-\lambda\tau} \frac{(t_1 - \tau)^{\alpha-2} - (t_2 - \tau)^{\alpha-2}}{\Gamma(\alpha - 1)} d\tau \leq \int_0^{t_1} \frac{(t_1 - \tau)^{\alpha-2} - (t_2 - \tau)^{\alpha-2}}{\Gamma(\alpha - 1)} d\tau = \frac{(t_2 - t_1)^{\alpha-1} + t_1^{\alpha-1} - t_2^{\alpha-1}}{\Gamma(\alpha)}$$

Using the inequality that $|a^p - b^p| \leq |a - b|^p$ for all $a, b \geq 0$ and $0 < p \leq 1$, we yield

$$\int_0^{t_1} e^{-\lambda\tau} \frac{(t_2 - \tau)^{\alpha-2} - (t_1 - \tau)^{\alpha-2}}{\Gamma(\alpha - 1)} d\tau \leq \frac{2}{\Gamma(\alpha)} (t_2 - t_1)^{\alpha-1}$$

Then, since $\alpha \in]1, 2]$, we can increase $\|u_{f_n}(t_2) - u_{f_n}(t_1)\|$ by

$$\|u_{f_n}(t_2) - u_{f_n}(t_1)\| \leq K |t_2 - t_1|^{\alpha-1}$$

with $K = \int_0^1 [(3 + \lambda)|\phi(s)| + (4 + 2\lambda)e^{\lambda s}] |X(s)| ds$. This shows that $\{u_{f_n} : n \in \mathbb{N}\}$ is equicontinuous in $C_E(I)$. Moreover, for each $t \in I$ the set $\{u_{f_n}(t) : n \in \mathbb{N}\}$, is contained in the convex compact set $\int_0^1 G(t, s)X(s)ds$ [?, 19] so that \mathcal{X} is relatively compact in $C_E(I)$ as claimed. So, we can assume that

$$\lim_{n \rightarrow \infty} u_{f_n} = u_\infty \in C_E(I)$$

As S_X^1 is $\sigma(L_E^1, L_{E^*}^\infty)$ -compact e.g [19] we may assume that (f_n) $\sigma(L_E^1, L_{E^*}^\infty)$ -converges to $f_\infty \in S_X^1$. so that u_{f_n} weakly converges to u_{f_∞} in $C_E(I)$ where $u_{f_\infty}(t) = \int_0^1 G(t, s)f_\infty(s)ds$ and so for every $t \in I$,

$$u_\infty(t) = w\text{-}\lim_{n \rightarrow \infty} u_{f_n}(t) = w\text{-}\lim_{n \rightarrow \infty} \int_0^1 G(t, s)f_n(s)ds = \int_0^1 G(t, s)f_\infty(s)ds = u_{f_\infty}(t),$$

and

$$\begin{aligned} w\text{-}\lim_{n \rightarrow \infty} (D^{\alpha-1}u_{f_n})(t) &= w\text{-}\lim_{n \rightarrow \infty} \left[\int_0^t \exp(-\lambda(t-s))f_n(s)ds + \exp(-\lambda t) \int_0^1 \varphi(s)f_n(s)ds \right] \\ &= \int_0^t \exp(-\lambda(t-s))f_\infty(s)ds + \exp(-\lambda t) \int_0^1 \varphi(s)f_\infty(s)ds \\ &= (D^{\alpha-1}u_{f_\infty})(t), \quad t \in I. \end{aligned}$$

This means $u_\infty \in \mathcal{X}$, and the proof of the theorem is complete. \square

Theorem 6.9. *Let $I := [0, 1]$ and H be a separable Hilbert space. Let K be a convex compact subset of H . Let $\varphi : [0, T] \times K \rightarrow [-\infty, +\infty]$ be a normal lower semicontinuous convex integrand such that*

(i) $\{\varphi(\cdot, u(\cdot)), u \in S_K^1\}$ is uniformly integrable.

(ii) $\varphi(t, x) \leq \varphi(\tau, x) + |v(t) - v(\tau)|$ for all $t, \tau \in [0, T], x \in K$ where $v : [0, T] \rightarrow \mathbb{R}^+$ is a positive nondecreasing continuous function with $v(0) = 0$.

Let $A : H \rightarrow H$ be a linear continuous coercive symmetric operator and $B : H \rightarrow H$ be a linear continuous mapping.

Let $f : I \times H \rightarrow H$ be a bounded continuous mapping $\|f(t, x)\| \leq M$ for all $(t, x) \in [0, 1] \times H$. Then for any $u_0 \in H$, there exist a $W_{B,H}^{\alpha,1}([0, 1])$ mapping $x : [0, 1] \rightarrow H$ and an absolutely continuous mappings $u : [0, 1] \rightarrow H$ satisfying

$$\begin{cases} u(0) = u_0 \in H \\ D^\alpha x(t) + \lambda D^{\alpha-1} x(t) = u(t), \quad t \in [0, 1] \\ I_{0+}^\beta x(t)|_{t=0} = 0, \quad x(1) = I_{0+}^\gamma x(1) \\ f(t, x(t)) + Bu(t) - A \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)), \quad \text{a.e. } t \in [0, 1] \end{cases}$$

Proof. Let us consider the convex compact (cf Theorem 3.5) subset \mathcal{X} in the Banach space $\mathcal{C}_H([0, 1])$ defined by

$$\mathcal{X} := \{u_f : [0, 1] \rightarrow H : u_f(t) = \int_0^1 G(t, s)f(s)ds, \quad f \in S_{u_0+K}^1, \quad t \in [0, 1]\}$$

where $S_{u_0+K}^1$ denotes the set of all integrable selections of the convex compact valued constant multifunction $u_0 + K$. For each $h \in \mathcal{X}$, by Theorem 4.4 and the assumptions on f , there is a unique absolutely continuous solution v_h to the inclusion

$$\begin{cases} v_h(0) = u_0 \in H \\ f(t, h(t)) + Bv_h(t) - A \frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t)), \quad \text{a.e. } t \in [0, 1] \end{cases}$$

with $\frac{dv_h}{dt}(t) \in K$ a.e. so that $v_h(t) = u_0 + \int_0^t \frac{dv_h}{ds}(s)ds \in u_0 + \int_0^t K ds \subset u_0 + K, \forall t \in [0, 1]$.

Now for each $h \in \mathcal{X}$ let us consider the mapping defined by

$$\Phi(h)(t) := \int_0^t G(t, s)v_h(s)ds,$$

for $t \in [0, 1]$. Then it is clear that $\Phi(h) \in \mathcal{X}$. Now we check that Φ is continuous. It is sufficient to show that, if (h_n) uniformly converges to h in \mathcal{X} , then the absolutely continuous solution v_{h_n} associated with h_n

$$\begin{cases} v_{h_n}(0) = u_0 \in H \\ f(t, h_n(t)) + Bv_{h_n}(t) - A \frac{dv_{h_n}}{dt}(t) \in \partial\varphi(t, \frac{dv_{h_n}}{dt}(t)), \quad \text{a.e. } t \in [0, 1] \end{cases}$$

uniformly converges to the absolutely continuous solution v_h associated with h

$$\begin{cases} v_h(0) = u_0 \in H \\ f(t, h(t)) + Bv_h(t) - A\frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t)), \text{ a.e. } t \in [0, 1] \end{cases}$$

As (v_{h_n}) is equi-absolutely continuous with $v_{h_n}(t) \in u_0 + \int_0^t ds K ds \subset u_0 + K, \forall t \in [0, 1]$ we may assume that (v_{h_n}) uniformly converges to an absolutely continuous mapping z . Since $v_{h_n}(t) = u_0 + \int_{]0,t]} \frac{dv_{h_n}}{ds}(s) ds, t \in [0, 1]$ and $\frac{dv_{h_n}}{ds}(s) \in K, \text{ a.e. } s \in [0, 1]$, we may assume that $(\frac{dv_{h_n}}{dt})$ weakly converges in $L_H^1([0, 1])$ to $w \in L_H^1[0, 1]$ with $w(t) \in K, t \in [0, 1]$ so that

$$\lim_n v_{h_n}(t) = u_0 + \int_0^t w(s) ds := u(t), t \in [0, T].$$

By identifying the limits, we get

$$u(t) = z(t) = u_0 + \int_0^t w(s) ds$$

with $\dot{u} = w$. Therefore by applying the arguments in the above variational limit result we get

$$f(t, h(t)) + Bu(t) - A\frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)), \text{ a.e. } t \in [0, 1]$$

with $u(0) = u_0 \in H$, so that by uniqueness $u = v_h$. Since $h_n \rightarrow h$, we have

$$\begin{aligned} \Phi(h_n)(t) - \Phi(h)(t) &= \int_0^1 G(t, s)v_{h_n}(s) ds - \int_0^1 G(t, s)v_h(s) ds \\ &= \int_0^1 G(t, s)[v_{h_n}(s) - v_h(s)] ds \\ &\leq \int_0^1 M_G \|v_{h_n}(s) - v_h(s)\| ds \end{aligned}$$

As $\|v_{h_n}(\cdot) - v_h(\cdot)\| \rightarrow 0$ uniformly, by using Lemma 3.3(i) we conclude that

$$\sup_{t \in [0, 1]} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 M_G \|v_{h_n}(\cdot) - v_h(\cdot)\| ds \rightarrow 0$$

so that $\Phi(h_n) \rightarrow \Phi(h)$ in $\mathcal{C}_H([0, 1])$. Since $\Phi : \mathcal{X} \rightarrow \mathcal{X}$ is continuous Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that

$$h(t) = \Phi(h)(t) = \int_0^1 G(t, s)v_h(s) ds,$$

with

$$\begin{cases} v_h(0) = u_0 \in H \\ D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = v_h(t), \quad t \in [0, 1] \\ I_{0+}^\beta h(t)|_{t=0} = 0, \quad h(1) = I_{0+}^\gamma h(1) \\ f(t, h(t)) + Bv_h(t) - A \frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t)), \quad \text{a.e. } t \in [0, 1] \end{cases}$$

So by putting $x = h$ and $u = v_h$ we conclude that (x, u) solves the dynamic EVI

$$\begin{cases} u(0) = u_0 \in H \\ D^\alpha x(t) + \lambda D^{\alpha-1} x(t) = u(t), \quad t \in [0, 1] \\ I_{0+}^\beta x(t)|_{t=0} = 0, \quad x(1) = I_{0+}^\gamma x(1) \\ f(t, x(t)) + Bu(t) - A \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)), \quad \text{a.e. } t \in [0, 1] \end{cases}$$

The proof is complete. \square

Further variants of the above results are available. For instance, we are able to state the existence of solution to the dynamic system

$$\begin{cases} D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = u(t), \quad t \in [0, 1] \\ I_{0+}^\beta h(t)|_{t=0} = 0, \quad h(1) = I_{0+}^\gamma h(1) \\ u(0) = u_0 \\ f(t, h(t)) + Bu(t) - A \frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t)), \quad \text{a.e. } t \in [0, 1] \end{cases}$$

Our tools allow to treat other variants by considering other class of FDI given in [10, 11, 12, 13, 14]. We study below an example of a Caputo fractional differential inclusion governed by an EVI. For the sake of completeness, we recall some needed properties for the fractional calculus and provide a series of lemmas on the fractional integral. Throughout we assume $\alpha \in]1, 2]$.

Definition 6.3. *The Caputo fractional derivative of order $\gamma > 0$ of a function $h : I = [0, T] \rightarrow H$, ${}^c D^\gamma h : [0, T] \rightarrow H$, is defined by*

$${}^c D^\gamma h(t) = \frac{1}{\Gamma(n - \gamma)} \int_0^t \frac{h^{(n)}(s)}{(t - s)^{1-n+\gamma}} ds.$$

Here $n = [\gamma] + 1$ and $[\gamma]$ denotes the integer part of γ .

Denote by

$$W_H^{\alpha, \infty}(I) = \{u \in C_H^1(I) : {}^c D^{\alpha-1} u \in C_H(I); {}^c D^\alpha u \in L_H^\infty(I)\},$$

where ${}^c D^{\alpha-1} u$ and ${}^c D^\alpha u$ are the fractional Caputo derivatives of order $\alpha - 1$ and α of u , respectively.

We summarize some properties of a Green function given in Lemma 2.1 of [13].

Lemma 6.10. Let $I = [0, T]$ and let $G : I \times I \rightarrow \mathbb{R}$ be a function defined by

$$G(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{1+t}{T+2} \left[\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right], & \text{if } 0 \leq s < t, \\ -\frac{1+t}{T+2} \left[\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right] & \text{if } t \leq s < T. \end{cases}$$

Let $f \in L_H^\infty(I)$. Then the system defined by

$$\begin{cases} {}^c D^\alpha u(t) = f(t), & t \in [0, T] \\ u(0) - \frac{du}{dt}(0) = 0 \\ u(T) + \frac{du}{dt}(T) = 0 \end{cases}$$

has a unique $W_H^{\alpha, \infty}([0, T])$ -solution u given by $u(t) = \int_0^T G(t, s)f(s)ds$, $\forall t \in I$. with $|G(t, s)| \leq M_G := \frac{2T^{\alpha-1} + (\alpha-1)T^{\alpha-2}}{\Gamma(\alpha)}$.

We recall and summarize a crucial lemma (Lemma 3.5 of [13]) for our next theorem.

Lemma 6.11. Let $X : [0, T] \rightrightarrows H$ be a convex compact valued measurable mapping such that $|X(t)| \leq \gamma(t) < +\infty$, $\forall t \in I$ with $\gamma \in L^1(I)$. Then the $W_H^{\alpha, \infty}(I)$ -solutions set \mathcal{X} to

$$\begin{cases} {}^c D^\alpha u(t) \in X(t), & t \in I \\ u(0) - \frac{du}{dt}(0) = 0 \\ u(T) + \frac{du}{dt}(T) = 0, \end{cases}$$

is a convex compact in $\mathcal{C}_H(I)$.

Now comes an existence result with a Caputo fractional differential inclusion driven by maximal monotone operator.

Theorem 6.12. Let $I := [0, 1]$ and H be a separable Hilbert space. Let K be a convex compact subset of H . Let $\varphi : [0, T] \times K \rightarrow]-\infty, +\infty]$ be a normal lower semicontinuous convex integrand such that

(i) $\{\varphi(\cdot, u(\cdot)), u \in S_K^1\}$ is uniformly integrable.

(ii) $\varphi(t, x) \leq \varphi(\tau, x) + |v(t) - v(\tau)|$ for all $t, \tau \in [0, T]$, $x \in K$ where $v : [0, T] \rightarrow \mathbb{R}^+$ is a positive nondecreasing continuous function with $v(0) = 0$.

Let $A : H \rightarrow H$ be a linear continuous coercive symmetric operator and $B : H \rightarrow H$ be a linear continuous mapping.

Let $f : I \times H \rightarrow H$ be a mapping such that

(i) $f(\cdot, x, y)$ is $\mathcal{L}(I)$ measurable for all $(x, y) \in H \times H$;

(ii) $f(t, \cdot, \cdot)$ is continuous on $H \times H$ for all $t \in I$;

(iii) $\|f(t, x, y)\| \leq M$ for all $(t, x, y) \in I \times H \times H$.

Then given $a \in H$, there is a $W_H^{\alpha, \infty}(I)$ mapping $x : I \rightarrow H$ and an absolutely

continuous mapping $v : I \rightarrow H$ satisfying

$$\begin{cases} {}^c D^\alpha x(t) = v(t), & t \in I \\ x(0) - \frac{dx}{dt}(0) = 0 \\ x(T) + \frac{dx}{dt}(T) = 0 \\ v(0) = a \\ f(t, x(t)) + Bu(t) - A \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)), & \text{a.e. } t \in [0, 1] \end{cases}$$

Proof. For any continuous mapping $h : I \rightarrow E$, the mapping $f_h : I \times E \rightarrow E$ defined by $f_h(t) := f(t, h(t))$ for all $t \in I$ is $\mathcal{L}(I)$ measurable and satisfies $|f_h(t)| \leq M$ for all $t \in I$. Then by Theorem 4.4 and the assumptions on f , there is a unique absolutely continuous solution v_h to the inclusion

$$\begin{cases} v_h(0) = u_0 \in H \\ f(t, h(t)) + Bv_h(t) - A \frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t)), & \text{a.e. } t \in [0, 1] \end{cases}$$

with $\frac{dv_h}{dt}(t) \in K$ a.e. so that $v_h(t) = u_0 + \int_0^t \frac{dv_h}{ds}(s) ds \in u_0 + \int_0^t K ds \subset u_0 + K, \forall t \in [0, 1]$. with v_h uniformly bounded and equi-absolutely continuous: $\frac{dv_h}{dt} \in K\overline{B}_H$, where K is positive constant, so for some real constant $L > 0$ one has $\|v_h(t)\| \leq L$ for all $t \in I$. Now let us consider the set \mathcal{X} defined by

$$\mathcal{X} := \{\xi_f : I \rightarrow E : f \in S_{L\overline{B}_E}^1\},$$

each mapping ξ_f being given for every $t \in I$ by

$$\xi_f(t) = \int_0^1 G(t, s) f(s) ds, \quad f \in S_{L\overline{B}_E}^1$$

where G is the Green function given in Lemma 6.10. We note that \mathcal{X} is convex compact in $\mathcal{C}_E(I)$ by Lemma 6.11. Now for each $h \in \mathcal{X}$, by Theorem 4.4 again denote by u_h the unique absolutely continuous solution of the differential inclusion

$$f(t, h(t)) + Bu_h(t) - A \frac{du_h}{dt}(t) \in \partial\varphi(t, \frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I, u_h(0) = a \in \mathbb{R}^e.$$

For each $h \in \mathcal{X}$ let us set (again with the above Green function G)

$$\Phi(h)(t) = \int_0^T G(t, s) u_h(s) ds, \quad \text{for all } t \in I.$$

Then it is clear that $\Phi(h) \in \mathcal{X}$ because $\|u_h(t)\| \leq L$ for all $t \in I$. Hence $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E(I)$ because $\Phi(\mathcal{X}) \subset \mathcal{X}$. Now we check that Φ is continuous relative to \mathcal{X} . It is enough to show that, if $(h_n)_n$ converges uniformly to h in \mathcal{X} , then the sequence $(u_{h_n})_n$, where each u_{h_n} is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = a \in \mathbb{R}^e \\ f(t, h_n(t)) + Bu_{h_n}(t) - A \frac{du_{h_n}}{dt}(t) \in \partial\varphi(t, \frac{du_{h_n}}{dt}(t)), & \text{a.e. } t \in I, \end{cases}$$

uniformly converges to the unique absolutely continuous solution u_h of the differential inclusion

$$\begin{cases} u_h(0) = a \in \mathbb{R}^e \\ f(t, h(t)) + Bu_h(t) - A \frac{du_h}{dt}(t) \in \partial\varphi(t, \frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

We note that $(u_{h_n})_n$ is equicontinuous since for every $n \in \mathbb{N}$ one has $\|\dot{u}_{h_n}(t)\| \leq K$ for almost all $t \in I$. Further, $\{u_{h_n}(t) : n \in \mathbb{N}\}$ is included in the compact set $L\bar{B}_E$ for every $t \in I$. The Arzelà-Ascoli theorem tells us that $\{u_{h_n} : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}_E(I)$. So by extracting a subsequence, we may suppose that $(u_{h_n})_n$ converges uniformly on I to some mapping $\zeta : I \rightarrow E$ with

$$\zeta(t) = \zeta(0) + \int_0^t \dot{\zeta}(s) ds \quad \text{for all } t \in I,$$

along with $(\dot{u}_{h_n})_n$ converging weakly in $L^1_H(I)$ to $\dot{\zeta}$ with $\|\dot{\zeta}(t)\| \leq K$ for a.e. $t \in I$. Note that

$$f(t, u_{h_n}(t)) \rightarrow f(t, \zeta(t)) \quad \text{for all } t \in I.$$

For simplicity, denote

$$z_n(t) := f(t, u_{h_n}(t)), \quad \text{for all } t \in I,$$

$$z(t) = f(t, \zeta(t)), \quad \text{for all } t \in I.$$

First, we notice that these mappings are Lebesgue measurable. Second, by condition (iii) and the uniform boundedness of $(u_{h_n})_n$ and u_h , we also notice that $(z_n)_n$ and z are uniformly bounded, since

$$\|z_n(t)\| \leq M \quad \text{and} \quad \|z(t)\| \leq M, \quad \text{for all } n \in \mathbb{N} \text{ and } t \in I.$$

Consequently, $(z_n)_n$ is a sequence of measurable and uniformly bounded mappings which converges pointwise to the measurable mapping z . Therefore, the sequence $(\dot{u}_{h_n} + z_n)_n$ converges weakly to $\dot{\zeta} + z$ in $L^1_E(I)$. This combined with the variational limit theorem gives

$$f(t, h(t)) + B\zeta(t) - A \frac{d\zeta}{dt}(t) \in \partial\varphi(t, \frac{d\zeta}{dt}(t)), \quad \text{a.e. } t \in I$$

. So using the uniqueness of solution of the latter differential inclusion we obtain that $\zeta = u_h$. Now let us write by Lemma 6.10 and boundedness of the Green function G

$$\begin{aligned} \Phi(h_n)(t) - \Phi(h)(t) &= \int_0^1 G(t, s)u_{h_n}(s) ds - \int_0^1 G(t, s)u_h(s) ds \\ &= \int_0^1 G(t, s)[u_{h_n}(s) - u_h(s)] ds \\ &\leq \int_0^1 M_G \|u_{h_n}(s) - u_h(s)\| ds. \end{aligned}$$

Since $\|u_{h_n}(\cdot) - u_h(\cdot)\| \rightarrow 0$ uniformly on I as $n \rightarrow \infty$, we deduce that

$$\sup_{t \in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 M_G \|u_{h_n}(\cdot) - u_h(\cdot)\| ds \rightarrow 0,$$

which entails that $\Phi(h_n) \rightarrow \Phi(h)$ uniformly on I , as desired. Then $\Phi : \mathcal{X} \rightarrow \mathcal{X}$ is continuous hence by the Schauder theorem Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that for every $t \in I$

$$h(t) = \Phi(h)(t) = \int_0^1 G(t, s) u_h(s) ds,$$

with

$$\begin{cases} u_h(0) = a \\ f(t, h(t)) + B u_h(t) - A \frac{du_h}{dt}(t) \in \partial \varphi(t, \frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

Coming back to Lemma 6.10 and applying the above notations, this means that we have just shown that there exists a mapping $h \in W_{B,H}^{\alpha, \infty}(I)$ satisfying

$$\begin{cases} {}^c D^\alpha h(t) = u_h(t), \quad t \in I \\ h(0) - \frac{dh}{dt}(0) = 0 \\ h(T) + \frac{dh}{dt}(T) = 0 \\ u_h(0) = a \quad t \in I \\ f(t, h(t)) + B u_h(t) - A \frac{du_h}{dt}(t) \in \partial \varphi(t, \frac{du_h}{dt}(t)), \quad \text{a.e. } t \in I. \end{cases}$$

The proof of the theorem is then complete □

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