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Well-Posedness of evolution variational inequalities with applications

Charles Castaing ^{*} Christiane Godet-Thobie [†] Manuel D.P. Monteiro Marques [‡]

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Abstract

We are concerned in the present work with the existence and uniqueness of absolutely continuous solutions to a class of evolution problems governed by time-dependent subdifferential operators of the form

$$f(t) + Bu(t) - A \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t))$$

with various applications.

1 Introduction

In this work we are concerned with the existence and uniqueness of absolutely continuous solution to an evolution inclusion in a separable Hilbert space H

$$(1.1) \quad f(t) + Bu(t) - A \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)), t \in [0, T]$$

where $f : [0, T] \rightarrow H$ is a continuous mapping, $B : H \rightarrow H$ an operator, $A : H \rightarrow H$ a linear continuous coercive and symmetric operator, $\varphi : [0, T] \times H \rightarrow [0, +\infty]$ is a normal lower semicontinuous convex integrand, $\partial\varphi(t, \cdot)$ is the subdifferential of $\varphi(t, \cdot)$. Problem (1.1) is interpreted as an evolution variational inequality (EVI) with the velocity inside the subdifferential. Generally, the model for parabolic evolution inclusion is a differential inclusion of the form

$$(1.2) \quad B(t, u(t)) \in \frac{du}{dt}(t) + A(t)u(t) + \partial\varphi(t, u(t)), \quad t \in [0, T]$$

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where $A(t)$ is a time dependent maximal monotone operator, $B(t, u)$ defined for $(t, u) \in [0, T] \times H$ is Lipschitz with respect to u . Then the existence and uniqueness of absolutely continuous solution to (1.2) is known in some particular cases in the literature, see e.g [4, 16] and Barbu and Rascanu in [6] dealing with existence of *generalized solutions* for parabolic variational inequalities with singular inputs and operators of the form

$$f(t) + \frac{dM}{dt}(t) \in \frac{du}{dt}(t) + Au(t) + \partial\varphi(u(t))$$

where A is a linear coercive operator and $\partial\varphi$ is the subdifferential associated with a convex lower semicontinuous function φ . There is an increasing activity around problem (1.2) since it contains several new applications such as sweeping process, relaxed problem and Skorohod problem etc. In this framework, problem (1.1) constitutes a new variational evolution inequality with the velocity inside the subdifferential in contrast with problem (1.2). Likewise problem (1.2), the study of (1.1) leads to several applications in a new setting such as the sweeping process, Skorohod problem, second order evolution and fractional differential equation [16]. Although (1.1) deal with deterministic case, it is a step forward to Skorohod problem in the stochastic setting, see the recent articles by Castaing-Raynaud de Fitte [15, 17], Rascanu [30], and L.Maticiuc, A. Rascanu, L. Slominski and M.Topolewski [22] for references on this stochastic subject. Let us mention the current situation of problem (1.1) in the literature. In [16] we deal with the existence of absolutely continuous solutions to variational evolution inequalities in separable Hilbert space H of the form

$$(1.3) \quad f(t) - Au(t) \in \partial\varphi(t, \frac{du}{dt}(t))$$

$$(1.4) \quad f(t) - Au(t) \in N_{C(t)}(\frac{du}{dt}(t))$$

where $f : [0, T] \rightarrow H$ is a continuous mapping, $A : H \rightarrow H$ is a linear continuous coercive symmetric operator, $\partial\varphi(t, x)$ is the subdifferential of a normal convex lower semicontinuous integrand $\varphi : [0, T] \times H \rightarrow [0, +\infty]$, $N_{C(t)}(x)$ denotes the normal cone to a closed convex moving set $C(t) \subset H$. Some related variants of problem (1.4) dealing with two positive operators A and B are given a series of papers by Adly et al [1, 2, 3]. We note that there is a new variant of problem (1.1) in some recent work by Bacho, Emmrich and Mielke [7] dealing with the following inclusion

$$(1.6) \quad B(t, u(t)) \in \partial\varphi(t, \frac{du}{dt}(t))$$

or more generally with two subdifferentials, namely

$$(1.7) \quad B(t, u(t)) \in \partial\varphi(t, \dot{u}(t)) + \partial\psi(t, \frac{du}{dt}(t))$$

Here B is a continuous mapping. In Mielke's paper, in order to solve the problem, it is proposed an algorithm due De Giorgi combined with regularised subdifferentials. Consult also a recent article by Migorski, Sofonea and Zeng [24] dealing with the inclusion of the form

$$-\dot{u}(t) \in N_{C(t)}(A\frac{du}{dt}(t) + Bu(t))$$

where $B : H \rightarrow H$ is Lipschitz continuous, note that here B is not assumed to be a positive operator by contrast with the results obtained by Adly et al [1, 2, 3]. After this prologue, we develop several variants of problem (1.1) and its applications via some related variational limits.

For the convenience we recall and summarize two useful results. See ([5], Corollary 2.9, Corollary 2.10).

Proposition 1.1. *1) If $A : H \rightarrow H$ is linear continuous and coercive: $\langle Ax, x \rangle \geq \omega \|x\|^2$ for all $x \in H$ for some $\omega > 0$ and if $\varphi : H \rightarrow [0, \infty]$ is lower semicontinuous convex proper, then, for $f \in H$, the problem $f \in Ay + \partial\varphi(y)$ admits a unique solution y .*

2) If $A : H \rightarrow H$ is linear continuous and coercive: $\langle Ax, x \rangle \geq \omega \|x\|^2$ for all $x \in H$ for some $\omega > 0$ and if K is a closed convex subset in H , then, for $f \in H$, the problem $f \in Ay + N_K(y)$ admits a unique solution y .

Our main proofs are build upon the above variational inequalities (Cf Proposition 1.1) and an explicit catching-up algorithm (alias Moreau's algorithm). We stress the fact that our algorithm and tools are self contained a part from the use of the mentioned variational inequalities. Further our work contain new results with various applications such as Skorohod problem and Fractional inclusion coupled with EVI and sweeping process.

Theorem 1.1. *Let $f : [0, T] \rightarrow H$ be a continuous mapping such that $\|f(t)\| \leq \beta$ for all $t \in [0, T]$, let $v : [0, T] \rightarrow \mathbf{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : [0, T] \rightarrow H$ be a convex weakly compact valued mapping such that $d_H(C(t), C(\tau)) \leq |v(t) - v(\tau)|$ for all $t, \tau \in [0, T]$. Let $A : H \rightarrow H$ be a linear continuous coercive symmetric operator and let $B : H \rightarrow H$ be a linear continuous compact operator. Then, for any $u_0 \in H$, the evolution inclusion*

$$f(t) + Bu(t) - A\dot{u}(t) \in N_{C(t)}(\frac{du}{dt}(t))$$

$$u(0) = u_0$$

admits a unique $W_H^{1,\infty}([0, T])$ solution $u : [0, T] \rightarrow H$.

Proof. We note that by Hormander formula (see e.g. Castaing-Valadier [19])

$$(1.1.1) \quad |\delta^*(x, C(t)) - \delta^*(x, C(\tau))| \leq \|x\| d_H(C(t), C(\tau)) \leq \|x\| \|v(t) - v(\tau)\|.$$

As a consequence C is scalarly continuous, a fortiori scalarly upper semicontinuous, so that C is upper semi continuous mapping from $[0, T]$ into H with respect to the weak topology, hence the image $C([0, T])$ of $[0, T]$ by C is weakly compact, in other words, $C(t) \subset L$ where L is a convex weakly compact subset of H , for all $t \in [0, T]$. We will use the Moreau 's catching-up algorithm [26] as follows. We consider for each $n \in \mathbf{N}$ the following partition of the interval $I = [0, T]$.

$$t_i^n = i \frac{T}{n} := i \eta_n \text{ for } 0 \leq i \leq n. \quad I_i^n :=]t_i^n, t_{i+1}^n] \text{ for } 0 \leq i \leq n-1.$$

Put $u_0^n = u_0$ and $f_i^n = f(t_i^n)$ for all $i = 1, \dots, n$. By Proposition 1.1 2) , there is $z_1^n \in C(t_1^n) \subset L$ such that

$$f_1^n + Bu_0^n - Az_1^n \in N_{C(t_1^n)}(z_1^n).$$

Put $u_1^n = u_0^n + \eta_n z_1^n$. Suppose that $u_0^n, u_1^n, \dots, u_i^n, z_1^n, z_2^n, \dots, z_i^n$ are constructed. As above by Proposition 1.1 2) there exists $z_{i+1}^n \in C(t_{i+1}^n) \subset L$ such that

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in N_{C(t_{i+1}^n)}(z_{i+1}^n).$$

and we set $u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$. Then by induction there are finite sequences $(u_i^n)_{i=0}^n$ and $(z_i^n)_{i=1}^n$ such that

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in N_{C(t_{i+1}^n)}(z_{i+1}^n)$$

$$u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$$

From $(u_i^n)_{i=0}^n, (z_i^n)_{i=1}^n, (f_i^n)_{i=0}^n$, we construct two sequences u_n from $[0, T]$ to H , f_n from $[0, T]$ to H , by setting $f_n(0) = f_1^n, u_n(0) = u_0^n$ and for each $i = 0, \dots, n-1$ we set $f_n(t) = f_{i+1}^n$ and

$$u_n(t) = u_i^n + \frac{t - t_i^n}{\eta_n} (u_{i+1}^n - u_i^n)$$

for $t \in]t_i^n, t_{i+1}^n]$. Clearly, the mapping $u_n(\cdot)$ is Lipschitz continuous on $[0, T]$, and ρ is a Lipschitz constant of $u_n(\cdot)$ on $[0, T]$ since for every $t \in]t_i^n, t_{i+1}^n]$

$$\dot{u}_n(t) = \frac{u_{i+1}^n - u_i^n}{\eta_n} = z_{i+1}^n \in C(t_{i+1}^n) \subset L \subset \rho \overline{B}_H.$$

Furthermore, for every $t \in [0, T]$, one has $u_n(t) = u_0 + \int_0^t \dot{u}_n(s) ds$, hence $\|u_n(t)\| \leq \|u_0\| + \rho T$. We have

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in N_{C(t_{i+1}^n)}(z_{i+1}^n).$$

Now, let us define the step functions $\theta_n, \delta_n : I \rightarrow I$ by

$$\theta_n(t) = t_{i+1}^n, \quad \delta_n(t) = t_i^n$$

if $t \in]t_i^n, t_{i+1}^n]$ and $\theta_n(0) = \delta_n(0) = 0$, and observe that for each $t \in I$, there is $i \in \{0, \dots, n-1\}$ such that $t \in [t_i^n, t_{i+1}^n[$, and then,

$$|\theta_n(t) - t| \rightarrow 0 \text{ and } |\delta_n(t) - t| \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

So, the last inclusion becomes

$$f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t) \in N_{C(\theta_n(t))}(\dot{u}_n(t))$$

a.e. $t \in [0, T]$. We note that $\|u_n(t)\| \leq \|u_0\| + \rho T$, $\|f^n(\cdot)\| \leq \beta$ for all $t \in [0, T]$ and $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$ for all $t \in [0, T]$ with $\dot{u}_n \in L$ a.e. Hence

$$\delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(\theta_n(t))) + \langle -f_n(t) - Bu_n(\delta_n(t)) + A\dot{u}_n(t), \dot{u}_n(t) \rangle \leq 0$$

with

$$\dot{u}_n(t) \in C(\theta_n(t)) \subset L$$

so that $\dot{u}_n \in S_L^1$ where $S_L^1 := \{v \in L_H^1([0, T]) : v(t) \in L \text{ a.e.}\}$ and $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$ for all $t \in [0, T]$ with $\dot{u}_n \in L$ a.e. We note that S_L^1 is a convex weakly compact set of $L_H^1([0, T])$ see (e.g [18] and the references therein). Let

$$\mathcal{X} := \{v : [0, T] \rightarrow H : v(t) = u_0 + \int_0^t \dot{v}(s)ds, t \in [0, T]; \dot{v} \in S_L^1\}$$

Then it is clear that \mathcal{X} is convex, equicontinuous and weakly compact [31] in $C_H([0, T])$. As $(u_n) \subset \mathcal{X}$. One can extract from (u_n) a subsequence not relabelled which pointwise weakly converges to $u : I \rightarrow H$ such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$, for all $t \in I$ and $(\dot{u}_n) \sigma(L_H^1([0, T]), L_H^\infty([0, T]))$ -converges to $\dot{u} \in S_L^1$. Our first task is to prove the inclusion $\dot{u}(t) \in C(t)$ a.e. Indeed, for every measurable set $Z \subset [0, T]$ and for any $x \in H$, the function $1_Z x \in L_H^\infty$. We have

$$\langle x, \dot{u}_n(t) \rangle \leq \delta^*(x, C(\theta_n(t))).$$

Integrating on Z gives

$$\int_Z \langle 1_Z x, \dot{u}_n(t) \rangle dt = \int_Z \langle x, \dot{u}_n(t) \rangle dt \leq \int_Z \delta^*(x, C(\theta_n(t))) dt.$$

Passing to the limit in this inequality

$$\begin{aligned} \int_Z \langle 1_Z x, \dot{u}(t) \rangle dt &\leq \limsup_n \int_Z \delta^*(x, C(\theta_n(t))) dt \\ &\leq \int_Z \limsup_n \delta^*(x, C(\theta_n(t))) \leq \int_Z \delta^*(x, C(t)) dt \end{aligned}$$

using the fact that C is scalarly upper semicontinuous and (7.1). So we deduce that

$$\langle x, \dot{u}(t) \rangle \leq \delta^*(x, C(t))$$

a.e. By the separability of H and the weak compactness of $C(t)$ by virtue of Castaing-Valadier ([19], Proposition III- 35) , we get $\dot{u}(t) \in C(t)$ a.e. Let $h \in L_H^\infty(I)$. We first note that

$$\begin{aligned} \left| \int_0^T \langle h(t), Bu_n(\delta_n(t)) \rangle dt - \int_0^T \langle h(t), Bu_n(t) \rangle dt \right| &\leq \int_0^T |\langle h(t), Bu_n(\delta_n(t)) - Bu_n(t) \rangle| dt \\ &\leq |h|_\infty |B| \rho \int_0^T |\delta_n(t) - t| dt. \end{aligned}$$

As consequence, we have

$$\liminf_n \int_0^T \langle \dot{u}_n(t), Bu_n(\delta_n(t)) \rangle dt \leq \liminf_n \int_0^T \langle \dot{u}_n(t), Bu_n(t) \rangle dt.$$

Now as $u_n(\delta_n(t)) \rightarrow u(t)$ pointwise weakly in H and B is a linear continuous compact operator, $Bu_n(\delta_n(t)) \rightarrow Bu(t)$ pointwise in H , so that $Bu_n(\delta_n(\cdot)) \rightarrow Bu(\cdot)$ weakly in $L_H^1(I)$. Indeed, let $h \in L_H^\infty(I)$. Then we have

$$\begin{aligned} \left| \int_0^T \langle h(t), Bu_n(\delta_n(t)) \rangle dt - \int_0^T \langle h(t), Bu(t) \rangle dt \right| &\leq \int_0^T |\langle h(t), Bu_n(\delta_n(t)) - Bu(t) \rangle| dt \\ &\leq |h|_\infty \int_0^T \|Bu_n(\delta_n(t)) - Bu(t)\| dt. \end{aligned}$$

As $\int_0^T \|Bu_n(\delta_n(t)) - Bu(t)\| dt \rightarrow 0$ when $n \rightarrow \infty$, our assertion follows. Similarly as A is symmetric, we note that $A\dot{u}_n \rightarrow A\dot{u}$ weakly in $L_H^1(I)$. As a main consequence $f_n + Bu_n(\delta_n(\cdot)) - A\dot{u}_n \rightarrow f + Bu - A\dot{u}$ weakly in $L_H^1(I)$. Then we may apply the lower semicontinuity of integral convex functional ([18], Theorem 8.1.16) to deduce that

(1.1.2)

$$\int_Z \delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) dt \leq \liminf_n \int_Z \delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(t)) dt$$

for every Lebesgue measurable set $Z \subset [0, T]$. This need a careful look. Indeed, we note that $(t, x) \mapsto \delta^*(x, C(t))$ is a normal lower semicontinuous convex integrand defined on $[0, T] \times H$ and $\delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(t))$ is measurable and integrable:

$$|\delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(t))| \leq \|f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t)\| |L| \leq \text{Constant}$$

furthermore

$$\delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(t)) \geq \langle f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), \bar{u}(t) \rangle$$

where $\bar{u}(t)$ is a measurable selection of C (note that C is scalarly upper semicontinuous multimapping from $[0, T]$ to L). Further we have by (1.1.1)

$$|\delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(t)) - \delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(\theta_n(t)))|$$

$$\begin{aligned} &\leq \|f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t)\| d_H(C(t), C(\theta_n(t))) \\ &\leq \|f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t)\| |v(t) - v(\theta_n(t))| \leq \text{Constant} |v(t) - v(\theta_n(t))| \end{aligned}$$

so that

$$\begin{aligned} &\liminf_n \int_Z \delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(\theta_n(t))) dt \\ &\geq \liminf_n \int_Z \delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(t)) dt \\ (1.1.3) \quad &\geq \int_Z \delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) dt. \end{aligned}$$

Let us set $\psi_A(x) = \langle Ax, x \rangle$ if $x \in L$ and $\psi(x) = +\infty$ if $x \notin L$. Then it is clear ψ is a positive lower semicontinuous convex integrand. Apply again the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the positive normal convex integrand ψ_A we obtain,

$$\liminf_n \int_Z \psi_A(\dot{u}_n(t)) dt \geq \int_Z \psi_A(\dot{u}(t)) dt$$

that is

$$(1.1.4) \quad \liminf_n \int_Z \langle A\dot{u}_n(t), \dot{u}_n(t) \rangle dt \geq \int_Z \langle A\dot{u}(t), \dot{u}(t) \rangle dt.$$

Now we use the fact that B is compact operator to assert that $Bu_n(\delta_n(t)) \rightarrow Bu(t)$ pointwise strongly in H . Whence it follows that

$$(1.1.5) \quad \lim_n \int_Z \langle Bu_n(\delta_n(t)), \dot{u}_n(t) \rangle dt = \int_Z \langle Bu(t), \dot{u}(t) \rangle dt.$$

Similarly we have

$$(1.1.6) \quad \lim_n \int_Z \langle f_n(t), \dot{u}_n(t) \rangle dt = \int_Z \langle f(t), \dot{u}(t) \rangle dt$$

because f_n is uniformly bounded and pointwise strongly converge to f and $\dot{u}_n \rightarrow \dot{u}$ weakly in $L_H^1([0, T])$ by noting that a bounded sequence is $L_H^\infty(I)$ which pointwise converges to 0, converges to 0 uniformly on any uniformly integrable subset of $L_H^1([0, T])$, in other terms it converges to 0 with respect to the Mackey topology $\tau(L_H^\infty([0, T]), L_H^1([0, T]))$ (see[9]).¹ Now integrating on $Z \subset [0, T]$ the inequality (here measurability and integrability are guaranteed)

$$\delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(\theta_n(t)))$$

¹If $H = \mathbf{R}^e$, one may invoke a classical fact that on bounded subsets of L_H^∞ the topology of convergence in measure coincides with the topology of uniform convergence on uniformly integrable sets, i.e. on relatively weakly compact subsets, alias the Mackey topology. This is a lemma due to Grothendieck [21] [Ch.5 §4 no 1 Prop. 1 and exercice]

$$+\langle -f_n(t) - Bu_n(\delta_n(t)), \dot{u}_n(t) \rangle + \langle A\dot{u}_n(t), \dot{u}_n(t) \rangle \leq 0$$

gives

$$(1.1.7) \quad \int_Z \delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(\theta_n(t))) dt \\ + \int_Z \langle A\dot{u}_n(t), \dot{u}_n(t) \rangle dt + \int_Z \langle -f_n(t) - Bu_n(\delta_n(t)), \dot{u}_n(t) \rangle dt \leq 0$$

by passing to the limit when n goes to ∞ in this equality using (1.1.1)—(1.1.7) gives

$$(1.1.8) \quad \int_Z [\delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) + \langle A\dot{u}(t) - Bu(t) - f(t), \dot{u}(t) \rangle] dt \leq 0.$$

As $t \mapsto \delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) + \langle A\dot{u}(t) - Bu(t) - f(t), \dot{u}(t) \rangle$ is integrable, by (1.1.8) it follow

$$(1.1.9) \quad \delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) + \langle A\dot{u}(t) - Bu(t) - f(t), \dot{u}(t) \rangle \leq 0, \quad a.e.$$

As $\dot{u}(t) \in C(t)$, we have

$$\delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) \geq \langle f(t) + Bu(t) - A\dot{u}(t), \dot{u}(t) \rangle$$

that is

$$(1.1.10) \quad \delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) + \langle -f(t) - Bu(t) + A\dot{u}(t), \dot{u}(t) \rangle \geq 0.$$

By (1.1.9) and (1.1.10) we get finally

$$\delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) = \langle f(t) + Bu(t) - A\dot{u}(t), \dot{u}(t) \rangle$$

a.e. with $\dot{u}(t) \in C(t)$ a.e. so we conclude that $f(t) + Bu(t) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$ a.e. \square

Remarks 1) The uniqueness of solutions follows easily from the coerciveness of the operator A . Indeed let u_1 and u_2 two solutions. then by an easy computation,

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle + \langle Bu_2(t) - Bu_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle \leq 0$$

so that

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle \leq |B| \|u_2(t) - u_1(t)\| \|\dot{u}_2(t) - \dot{u}_1(t)\|$$

By coerciveness, we deduce that

$$\omega \|\dot{u}_2(t) - \dot{u}_1(t)\|^2 \leq |B| \|u_2(t) - u_1(t)\| \|\dot{u}_2(t) - \dot{u}_1(t)\|$$

Whence

$$\|\dot{u}_2(t) - \dot{u}_1(t)\| \leq \frac{|B|}{\omega} \|u_2(t) - u_1(t)\| \leq \frac{|B|}{\omega} \int_0^t \|\dot{u}_2(s-) \dot{u}_1(s)\| ds$$

By Gronwall lemma $\dot{u}_1(t) = \dot{u}_2(t)$ and so $u_1(t) = u_2(t)$ since $u_1(t) = u_0 + \int_0^t \dot{u}_1(s) ds, \forall t \in [0, T], u_2(t) = u_0 + \int_0^t \dot{u}_2(s) ds, \forall t \in [0, T]$.

2) The tools developed above allows to obtain further variants. The fact that $C(t)$ is weakly compact is required, and mainly the coerciveness of A and the compactness assumption for the operator B . An inspection of the proof of Theorem 1.1, shows that the compactness assumption on B is required to prove the Fatou property,

$$\liminf_n \int_Z \langle B(u_n(\delta_n(t)), \dot{u}_n(t)) \rangle dt \geq \int_Z \langle Bu(t), \dot{u}(t) \rangle dt$$

So as a possible variant we may substitute the bounded operator B by the gradient ∇g of a positive convex continuous Gateaux differentiable function $g : H \rightarrow \mathbf{R}$ such that $g(v(t))$ is absolutely continuous for $v : [0, T] \rightarrow H$ absolutely continuous, so that by invoking the chain rule formula, see Moreau-Valadier, [27], we have the equality

$$\langle \nabla g(v(t)), \dot{v}(t) \rangle = \frac{d}{dt} g(v(t))$$

Hence by using this fact and the tool developed in Theorem 1.1, we obtain a variant of Theorem 1.1 by noting that

$$\begin{aligned} \liminf_n \int_0^T \langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle dt &= \liminf_n \int_0^T \frac{d}{dt} g(u_n(t)) dt \\ &\geq \int_0^T \frac{d}{dt} g(u(t)) dt = \int_0^T \langle \nabla g(u(t)), \dot{u}(t) \rangle dt \end{aligned}$$

It is obvious that a linear continuous operator and a gradient do not enjoy similar properties, showing the interest of the new variant we give further. These remarks have some importance in further developments.

Now we present a second variant dealing with the existence and uniqueness of absolutely continuous solution to the evolution inclusion of the form

$$f(t) + Bu(t) - A\dot{u}(t) \in N_{C(t)}(t, \frac{du}{dt}(t))$$

where f is a bounded continuous mapping $f : I \rightarrow H$, A is a coercive symmetric operator, and $B : H \rightarrow H$ is a Lipschitz mapping: $\|Bx - By\| \leq M\|x - y\|, \forall x, y \in H$ for some positive constant M .

Theorem 1.2. *Let $f : [0, T] \rightarrow H$ be a continuous mapping such that $\|f(t)\| \leq \beta$ for all $t \in [0, T]$, let $v : [0, T] \rightarrow \mathbf{R}^+$ be a positive nondecreasing continuous*

function with $v(0) = 0$. Let $C : [0, T] \rightarrow H$ be a convex compact valued mapping such that $d_H(C(t), C(\tau)) \leq |v(t) - v(\tau)|$ for all $t, \tau \in [0, T]$. Let $A : H \rightarrow H$ be a linear continuous coercive symmetric operator and let $B : H \rightarrow H$ be a Lipschitz mapping: $\|Bx - By\| \leq M\|x - y\|, \forall x, y \in H$ for some positive constant M . Then, for any $u_0 \in H$, the evolution inclusion

$$\begin{aligned} f(t) + Bu(t) - A\dot{u}(t) &\in N_{C(t)}\left(\frac{du}{dt}(t)\right) \\ u(0) &= u_0 \end{aligned}$$

admits a unique $W_H^{1,\infty}([0, T])$ solution $u : [0, T] \rightarrow H$.

Proof. We note that by Hormander formula (see e.g. Castaing-Valadier [19])

$$(1.2.1) \quad |\delta^*(x, C(t)) - \delta^*(x, C(\tau))| \leq \|x\|d_H(C(t), C(\tau)) \leq \|x\|\|v(t) - v(\tau)\|.$$

and the image $C([0, T])$ of $[0, T]$ by C is compact, in other words, $C(t) \subset L$ where L is a convex compact subset of H , for all $t \in [0, T]$. We will use the Moreau's catching-up algorithm [26]. We consider for each $n \in \mathbf{N}$ the following partition of the interval $I = [0, T]$.

$$t_i^n = i\frac{T}{n} := i\eta_n \text{ for } 0 \leq i \leq n. \quad I_i^n :=]t_i^n, t_{i+1}^n] \text{ for } 0 \leq i \leq n-1.$$

Put $u_0^n = u_0$ and $f_i^n = f(t_i^n)$ for all $i = 1, \dots, n$. By Proposition 6.1 2) there is $z_1^n \in C(t_1^n) \subset L$ such that

$$f_1^n + Bu_0^n - Az_1^n \in N_{C(t_1^n)}(z_1^n).$$

Put $u_1^n = u_0^n + \eta_n z_1^n$. Suppose that $u_0^n, u_1^n, \dots, u_i^n, z_1^n, z_2^n, \dots, z_i^n$ are constructed. As above by Proposition 6.1 2) there exists $z_{i+1}^n \in C(t_{i+1}^n) \subset L$ such that

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in N_{C(t_{i+1}^n)}(z_{i+1}^n).$$

and we set $u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$. Then by induction there are finite sequences $(u_i^n)_{i=0}^n$ and $(z_i^n)_{i=1}^n$ such that

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in N_{C(t_{i+1}^n)}(z_{i+1}^n)$$

$$u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$$

From $(u_i^n)_{i=0}^n, (z_i^n)_{i=1}^n, (f_i^n)_{i=0}^n$, we construct two sequences u_n from $[0, T]$ to H , f_n from $[0, T]$ to H , by setting $f_n(0) = f_1^n, u_n(0) = u_0^n$ and for each $i = 0, \dots, n-1$ we set $f_n(t) = f_{i+1}^n$ and

$$u_n(t) = u_i^n + \frac{t - t_i^n}{\eta_n}(u_{i+1}^n - u_i^n)$$

for $t \in]t_i^n, t_{i+1}^n]$. Clearly, the mapping $u_n(\cdot)$ is Lipschitz continuous on $[0, T]$, and ρ is a Lipschitz constant of $u_n(\cdot)$ on $[0, T]$ since for every $t \in]t_i^n, t_{i+1}^n]$

$$\dot{u}_n(t) = \frac{u_{i+1}^n - u_i^n}{\eta_n} = z_{i+1}^n \in C(t_{i+1}^n) \subset L \subset \rho\overline{B}_H.$$

Furthermore, for every $t \in [0, T]$, one has $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$ hence $\|u_n(t)\| \leq \|u_0\| + \rho T$. We have

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in N_{C(t_{i+1}^n)}(z_{i+1}^n).$$

Now, let us define the step functions $\theta_n, \delta_n : I \rightarrow I$ by

$$\theta_n(t) = t_{i+1}^n, \quad \delta_n(t) = t_i^n$$

if $t \in]t_i^n, t_{i+1}^n]$ and $\theta_n(0) = \delta_n(0) = 0$, and observe that for each $t \in I$, there is $i \in \{0, \dots, n-1\}$ such that $t \in]t_i^n, t_{i+1}^n[$, and then,

$$|\theta_n(t) - t| \rightarrow 0 \quad \text{and} \quad |\delta_n(t) - t| \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

So, the last inclusion becomes

$$f_n(t) + B(u_n(\delta_n(t))) - A\dot{u}_n(t) \in N_{C(\theta_n(t))}(\dot{u}_n(t))$$

a.e. $t \in [0, T]$. We note that $\|u_n(t)\| \leq \|u_0\| + \rho T$, $\|f^n(\cdot)\| \leq \beta$ for all $t \in [0, T]$ and $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$ for all $t \in [0, T]$ with $\dot{u}_n \in L$ a.e. Hence

$$\delta^*(f_n(t) + B(u_n(\delta_n(t))) - A\dot{u}_n(t), C(\theta_n(t))) + \langle -f_n(t) - B(u_n(\delta_n(t))) + A\dot{u}_n(t), \dot{u}_n(t) \rangle \leq 0$$

with

$$\dot{u}_n(t) \in C(\theta_n(t)) \subset L$$

so that $\dot{u}_n \in S_L^1$ where $S_L^1 := \{v \in L_H^1([0, T]) : v(t) \in L \text{ a.e.}\}$ and $u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds$ for all $t \in [0, T]$ with $\dot{u}_n \in L$ a.e. We note that S_L^1 is a convex weakly compact set of $L_H^1([0, T])$. Let

$$\mathcal{X} := \{v : [0, T] \rightarrow H : v(t) = u_0 + \int_0^t \dot{v}(s)ds, t \in [0, T]; \dot{v} \in S_L^1\}$$

Then it is clear that \mathcal{X} is convex, equicontinuous and compact in $C_H([0, T])$. As $(u_n) \subset \mathcal{X}$. One can extract from (u_n) a subsequence not relabelled which pointwise converges to $u : I \rightarrow H$ such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$, for all $t \in I$ and $(\dot{u}_n) \sigma(L_H^1([0, T]), L_H^\infty([0, T]))$ -converges to $\dot{u} \in S_L^1$. Our first task is to prove the inclusion $\dot{u}(t) \in C(t)$ a.e. Indeed, for every measurable set $Z \subset [0, T]$ and for any $x \in H$, the function $1_Z x \in L_H^\infty$. We have

$$\langle x, \dot{u}_n(t) \rangle \leq \delta^*(x, C(\theta_n(t))).$$

Integrating on Z gives

$$\int_Z \langle 1_Z x, \dot{u}_n(t), \rangle dt = \int_Z \langle x, \dot{u}_n(t) \rangle dt \leq \int_Z \delta^*(x, C(\theta_n(t))) dt.$$

Passing to the limit in this inequality

$$\begin{aligned} \int_Z \langle 1_Z x, \dot{u}(t) \rangle dt &\leq \limsup_n \int_Z \delta^*(x, C(\theta_n(t))) dt \\ &\leq \int_Z \limsup_n \delta^*(x, C(\theta_n(t))) \leq \int_Z \delta^*(x, C(t)) dt \end{aligned}$$

using the fact that C is scalarly upper semicontinuous. So we deduce that

$$\langle x, \dot{u}(t) \rangle \leq \delta^*(x, C(t))$$

a.e. By the separability of H and the compactness of $C(t)$ by virtue of Castaing-Valadier ([19], Proposition III- 35) , we get $\dot{u}(t) \in C(t)$ a.e. Let $h \in L_H^\infty(I)$. We first note that

$$\begin{aligned} \left| \int_0^T \langle h(t), B(u_n(\delta_n(t))) \rangle dt - \int_0^T \langle h(t), B(u_n(t)) \rangle dt \right| &\leq \int_0^T |\langle h(t), B(u_n(\delta_n(t))) - B(u_n(t)) \rangle| dt \\ &\leq |h|_\infty M \rho \int_0^T |\delta_n(t) - t| dt \end{aligned}$$

using the Lipschitz condition of B . As consequence, we have

$$\liminf_n \int_0^T \langle \dot{u}_n(t), B u_n(\delta_n(t)) \rangle dt \geq \liminf_n \int_0^T \langle \dot{u}_n(t), B u_n(t) \rangle dt$$

Now as $u_n(\delta_n(t)) \rightarrow u(t)$ pointwise in H and B is Lipschitz continuous, $B u_n(\delta_n(t)) \rightarrow B u(t)$ pointwise in H so that $B u_n(\delta_n(\cdot)) \rightarrow B u(\cdot)$ weakly in $L_H^1(I)$. Indeed, let $h \in L_H^\infty(I)$. Then we have

$$\begin{aligned} \left| \int_0^T \langle h(t), B u_n(\delta_n(t)) \rangle dt - \int_0^T \langle h(t), B u(t) \rangle dt \right| &\leq \int_0^T |\langle h(t), B u_n(\delta_n(t)) - B u(t) \rangle| dt \\ &\leq |h|_\infty \int_0^T M \|u_n(\delta_n(t)) - u(t)\| dt \end{aligned}$$

using the Lipschitz condition of B . As $\int_0^T M \|u_n(\delta_n(t)) - u(t)\| dt \rightarrow 0$ when $n \rightarrow \infty$, our assertion follows. Further as A is symmetric, we note that $A \dot{u}_n \rightarrow A \dot{u}(\cdot)$ weakly in $L_H^1(I)$. As a main consequence $f_n + B u_n(\delta_n(\cdot)) - A \dot{u}_n \rightarrow f + B u - A \dot{u}$ weakly in $L_H^1(I)$. Then we may apply the lower semicontinuity of integral convex functional ([18], Theorem 8.1.16) to deduce that

$$(1.2.2) \quad \int_Z \delta^*(f(t) + B u(t) - A \dot{u}(t), C(t)) dt \leq \liminf_n \int_Z \delta^*(f_n(t) + B u_n(\delta_n(t)) - A \dot{u}_n(t), C(t)) dt$$

for every Lebesgue measurable set $Z \subset [0, T]$. This need a careful look. Indeed, we note that $(t, x) \mapsto \delta^*(x, C(t))$ is a normal convex lower semicontinuous integrand

defined on $[0, T] \times H$ and $\delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(t))$ is measurable and integrable:

$$|\delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(t))| \leq \|f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t)\| \|L\| \leq \text{Constant}$$

furthermore

$$\delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(t)) \geq \langle f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), \bar{u}(t) \rangle$$

where $\bar{u}(t)$ is a measurable selection of C (note that C is scalarly upper semicontinuous multimapping from $[0, T]$ to L). Further we have by (1.2.1)

$$\begin{aligned} & |\delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(t)) - \delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(\theta_n(t)))| \\ & \leq \|f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t)\| d_H(C(t), C(\theta_n(t))) \\ & \leq \|f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t)\| |v(t) - v(\theta_n(t))| \leq \text{Constant} |v(t) - v(\theta_n(t))| \end{aligned}$$

so that

$$\begin{aligned} & \liminf_n \int_Z \delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(\theta_n(t))) dt \\ & \geq \liminf_n \int_Z \delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(t)) dt \\ (1.2.3) \quad & \geq \int_Z \delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) dt \end{aligned}$$

Let us set $\psi_A(x) = \langle Ax, x \rangle$ if $x \in L$ and $\psi(x) = +\infty$ if $x \notin L$. Then it is clear ψ is a positive lower semicontinuous convex integrand. Apply again the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the positive normal lower semicontinuous convex integrand ψ_A we obtain,

$$\liminf_n \int_Z \psi_A(\dot{u}_n(t)) dt \geq \int_Z \psi_A(\dot{u}(t)) dt$$

that is

$$(1.2.4) \quad \liminf_n \int_Z \langle A\dot{u}_n(t), \dot{u}_n(t) \rangle dt \geq \int_Z \langle A\dot{u}(t), \dot{u}(t) \rangle dt$$

Recall that $Bu_n(\delta_n(t)) \rightarrow Bu(t)$ pointwise in H . Whence it follows that

$$(1.2.5) \quad \lim_n \int_Z \langle Bu_n(\delta_n(t)), \dot{u}_n(t) \rangle dt = \int_Z \langle Bu(t), \dot{u}(t) \rangle dt$$

Similarly we have

$$(1.2.6) \quad \lim_n \int_Z \langle f_n(t), \dot{u}_n(t) \rangle dt = \int_Z \langle f(t), \dot{u}(t) \rangle dt$$

because f_n is uniformly bounded and pointwise converge to f and $\dot{u}_n \rightarrow \dot{u}$ weakly in $L_H^1([0, T])$, using the Castaing's trick [9] given in the proof of the preceding theorem. Now integrating on $Z \subset [0, T]$ the inequality here measurability and integrability are guaranteed)

$$\begin{aligned} & \delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(\theta_n(t))) \\ & + \langle -f_n(t) - Bu_n(\delta_n(t)), \dot{u}_n(t) \rangle + \langle A\dot{u}_n(t), \dot{u}_n(t) \rangle \leq 0 \end{aligned}$$

gives

$$(1.2.7) \quad \int_Z \delta^*(f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t), C(\theta_n(t))) dt + \int_Z \langle A\dot{u}_n(t), \dot{u}_n(t) \rangle dt + \int_Z \langle -f_n(t) - Bu_n(\delta_n(t)), \dot{u}_n(t) \rangle dt \leq 0$$

by passing to the limit when n goes to ∞ in this equality using (1.3.4)–(1.3.6) gives

$$(1.2.8) \quad \int_Z [\delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) + \langle A\dot{u}(t) - Bu(t) - f(t), \dot{u}(t) \rangle] dt \leq 0$$

As $t \mapsto \delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) + \langle A\dot{u}(t) - Bu(t) - f(t), \dot{u}(t) \rangle$ is integrable, by (1.2.7) it follow

$$\delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) + \langle A\dot{u}(t) - Bu(t) - f(t), \dot{u}(t) \rangle \leq 0, \quad a.e.$$

As $\dot{u}(t) \in C(t)$, we have

$$\delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) \geq \langle f(t) + Bu(t) - A\dot{u}(t), \dot{u}(t) \rangle$$

that is

$$(1.2.9) \quad \delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) + \langle -f(t) - Bu(t) + A\dot{u}(t), \dot{u}(t) \rangle \geq 0$$

By (1.2.8) and (1.2.9) we get finally

$$\delta^*(f(t) + Bu(t) - A\dot{u}(t), C(t)) = \langle f(t) + Bu(t) - A\dot{u}(t), \dot{u}(t) \rangle$$

a.e. with $\dot{u}(t) \in C(t)$ a.e. so we conclude that $f(t) + Bu(t) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$ a.e. The uniqueness of solutions follows easily from the coerciveness of the operator A . Indeed let u_1 and u_2 two solutions. then by an easy computation,

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle + \langle Bu_2(t) - Bu_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle \leq 0$$

so that

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle \leq M \|u_2(t) - u_1(t)\| \|\dot{u}_2(t) - \dot{u}_1(t)\|$$

By coerciveness, we deduce that

$$\omega \|\dot{u}_2(t) - \dot{u}_1(t)\|^2 \leq M \|u_2(t) - u_1(t)\| \|\dot{u}_2(t) - \dot{u}_1(t)\|$$

Whence

$$\|\dot{u}_2(t) - \dot{u}_1(t)\| \leq \frac{M}{\omega} \|u_2(t) - u_1(t)\| \leq \frac{M}{\omega} \int_0^t \|\dot{u}_2(s-) \dot{u}_1(s)\| ds$$

By Gronwall lemma $\dot{u}_1(t) = \dot{u}_2(t)$ and so $u_1(t) = \dot{u}_2(t)$ since $u_1(t) = u_0 + \int_0^t \dot{u}_1(s) ds, \forall t \in [0, T], u_2(t) = u_0 + \int_0^t \dot{u}_2(s) ds, \forall t \in [0, T].$ \square

We present a third variant dealing with the existence and uniqueness of absolutely continuous solution to the evolution inclusion of the form

$$f(t) + Bu(t) - A\dot{u}(t) \in \partial\varphi(t, \frac{du}{dt}(t))$$

where f is a bounded continuous mapping $f : I \rightarrow H$, A is a coercive symmetric operator, and $B : H \rightarrow H$ be a linear continuous mapping $\partial\varphi$ is the subdifferential of a normal lower semicontinuous convex integrand φ .

Theorem 1.3. *Let H be a separable Hilbert space. Let K be a convex compact subset of H . Let $\varphi : [0, T] \times K \rightarrow]-\infty, +\infty]$ be a normal lower semicontinuous convex integrand such that*

(i) $\{\varphi(\cdot, u(\cdot)), u \in S_K^1\}$ is uniformly integrable.

(ii) $\varphi(t, x) \leq \varphi(\tau, x) + |v(t) - v(\tau)|$ for all $t, \tau \in [0, T], x \in K$ where $v : [0, T] \rightarrow \mathbf{R}^+$ is a positive nondecreasing continuous function with $v(0) = 0$.

Let $A : H \rightarrow H$ be a linear continuous coercive symmetric operator and $B : H \rightarrow H$ be a linear continuous mapping. Then, for any $u_0 \in H$, the evolution inclusion

$$f(t) + Bu(t) - A\dot{u}(t) \in \partial\varphi(t, \frac{du}{dt}(t))$$

$$u(0) = u_0$$

admits a unique $W_H^{1,\infty}([0, T])$ solution $u : [0, T] \rightarrow H$.

Proof. We will use again the Moreau's catching-up algorithm. We consider for each $n \in \mathbf{N}$ the following partition of the interval $I = [0, T]$.

$t_i^n = i \frac{T}{n} := i\eta_n$ for $0 \leq i \leq n$. $I_i^n :=]t_i^n, t_{i+1}^n]$ for $0 \leq i \leq n-1$.

Put $u_0^n = u_0$ and $f_i^n = f(t_i^n)$ for all $i = 1, \dots, n$. By Proposition 1.1 1), there is $z_1^n \in K$ such that

$$f_1^n + Bu_0^n - Az_1^n \in \partial\varphi(t_1^n, z_1^n).$$

Put $u_1^n = u_0^n + \eta_n z_1^n$. Suppose that $u_0^n, u_1^n, \dots, u_i^n, z_1^n, z_2^n, \dots, z_i^n$ are constructed. As above by Proposition 1.1 1) there exists $z_{i+1}^n \in K$ such that

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in \partial\varphi(t_{i+1}^n, z_{i+1}^n).$$

and we set $u_{i+1}^n = u_i^n + \eta_n z_{i+1}^n$. Then by induction there are finite sequences $(u_i^n)_{i=0}^n$ and $(z_i^n)_{i=1}^n$ such that

$$\begin{aligned} f_{i+1}^n + Bu_i^n - Az_{i+1}^n &\in \partial\varphi(t_{i+1}^n, z_{i+1}^n) \\ u_{i+1}^n &= u_i^n + \eta_n z_{i+1}^n \end{aligned}$$

From $(u_i^n)_{i=0}^n, (z_i^n)_{i=1}^n, (f_i^n)_{i=0}^n$, we construct two sequences u_n from $[0, T]$ to H , f_n from $[0, T]$ to H , by setting $f_n(0) = f_1^n$, $u_n(0) = u_0^n$ and for each $i = 0, \dots, n-1$ we set $f_n(t) = f_{i+1}^n$ and

$$u_n(t) = u_i^n + \frac{t - t_i^n}{\eta_n} (u_{i+1}^n - u_i^n)$$

for $t \in]t_i^n, t_{i+1}^n]$. Clearly, the mapping $u_n(\cdot)$ is Lipschitz continuous on $[0, T]$, and ρ is a Lipschitz constant of $u_n(\cdot)$ on $[0, T]$ since for every $t \in]t_i^n, t_{i+1}^n]$

$$\dot{u}_n(t) = \frac{u_{i+1}^n - u_i^n}{\eta_n} = z_{i+1}^n \in K \subset \rho \bar{B}_H.$$

Furthermore, for every $t \in [0, T]$, one has $u_n(t) = u_0 + \int_0^t \dot{u}_n(s) ds$, hence $\|u_n(t)\| \leq \|u_0\| + \rho T$. We have

$$f_{i+1}^n + Bu_i^n - Az_{i+1}^n \in \partial\varphi(t_{i+1}^n, z_{i+1}^n).$$

Now, let us define the step functions $\theta_n, \delta_n : I \rightarrow I$ by

$$\theta_n(t) = t_{i+1}^n, \quad \delta_n(t) = t_i^n$$

if $t \in]t_i^n, t_{i+1}^n]$ and $\theta_n(0) = \delta_n(0) = 0$, and observe that for each $t \in I$, there is $i \in \{0, \dots, n-1\}$ such that $t \in]t_i^n, t_{i+1}^n]$, and then,

$$|\theta_n(t) - t| \rightarrow 0 \quad \text{and} \quad |\delta_n(t) - t| \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

So, the last inclusion becomes

$$f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t))$$

a.e. $t \in [0, T]$. We note that $\|u_n(t)\| \leq \|u_0\| + \rho T$, $\|f^n(\cdot)\| \leq \beta$ for all $t \in [0, T]$ and $u_n(t) = u_0 + \int_0^t \dot{u}_n(s) ds$ for all $t \in [0, T]$ with $\dot{u}_n \in K$ a.e.

Step 2 Convergence of the algorithm and final conclusion

Let $S_K^1 := \{h \in L_H^1([0, T]) : h(t) \in K \text{ a.e.}\}$ and let

$$\mathcal{X} := \{v : [0, T] \rightarrow H : v(t) = u_0 + \int_0^t \dot{v}(s) ds, t \in [0, T]; \dot{v} \in S_K^1\}.$$

Then it is clear that S_K^1 is convex and weakly compact in $L_H^1([0, T])$ (see e.g. [18] and the references therein) and that \mathcal{X} is convex, equicontinuous and compact in

$\mathcal{C}_H([0, T])$. As $(u_n) \subset \mathcal{X}$, one can extract from (u_n) a subsequence not relabelled which pointwise converges to $u : [0, T] \rightarrow H$ such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$, for all $t \in [0, T]$ and $(\dot{u}_n) \sigma(L_H^1([0, T]), L_H^\infty([0, T]))$ -converges to $\dot{u} \in S_K^1$. As φ is normal lower semicontinuous convex, the conjugate function $\varphi^* : [0, T] \times H \rightarrow H$

$$(1.3.1) \quad \varphi^*(t, y) = \sup_{x \in K} [\langle x, y \rangle - \varphi(t, x)]$$

is normal, see e.g Castaing-Valadier [19] and satisfies

$$\varphi^*(t, y) \leq \varphi^*(\tau, y) + |v(t) - v(\tau)|$$

for all $t, \tau \in [0, T], y \in H$ using assumption (ii) ([28], Proposition 27). By using the normality of φ , the mappings $t \mapsto \varphi(\theta_n(t), \dot{u}_n(t))$ and $t \mapsto \varphi(t, \dot{u}_n(t))$ are measurable and integrable. By construction we have

$$g_n(t) := f_n(t) + Bu_n(\delta_n(t)) - A\dot{u}_n(t) \in \partial\varphi(\theta_n(t), \dot{u}_n(t))$$

so that by the normality of φ^* , the mapping $t \mapsto \varphi^*(\theta_n(t), g_n(t))$ is measurable and integrable. Further by (1.3.1) and (iii) we have

$$(1.3.2) \quad -\varphi(t, \dot{u}_n(t)) + \langle \dot{u}_n(t), g_n(t) \rangle \leq \varphi^*(t, g_n(t)) \leq \varphi^*(\theta_n(t), g_n(t)) + |v(t) - v(\theta_n(t))|$$

so that $t \mapsto -\varphi(t, \dot{u}_n(t)) + \langle \dot{u}_n(t), g_n(t) \rangle$ is uniformly integrable thank to (iii). We note that $(h_n(t) := f_n(t) - Bu_n(\delta_n(t)))$ is uniformly bounded and pointwise converges to $h(t) = f(t) - Bu(t)$ in H . Hence $(h_n(\cdot) - h(\cdot))$ is uniformly bounded and pointwise converges to 0, so that it converges to 0 uniformly on any uniformly integrable subset of $L_H^1([0, T])$, in other terms it converges to 0 with respect to the Mackey topology $\tau(L_H^\infty([0, T]), L_H^1([0, T]))$ so that, for every Lebesgue measurable set $B \subset [0, T]$,

$$\lim_{n \rightarrow \infty} \int_B \langle h_n(t) - h(t), \dot{u}_n(t) \rangle dt = 0$$

because (\dot{u}_n) is uniformly integrable. Consequently

$$(1.3.3) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \int_B \langle h_n(t), \dot{u}_n(t) \rangle dt \\ &= \lim_{n \rightarrow \infty} \int_B \langle h_n(t) - h(t), \dot{u}_n(t) \rangle dt + \lim_{n \rightarrow \infty} \int_B \langle h(t), \dot{u}_n(t) \rangle dt \\ &= \lim_{n \rightarrow \infty} \int_B \langle h(t), \dot{u}_n(t) \rangle dt = \int_B \langle h(t), \dot{u}(t) \rangle dt. \end{aligned}$$

As A is symmetric, we show that $A\dot{u}_n \rightarrow A\dot{u}(\cdot)$ weakly in $L_H^1(I)$. As consequence $g_n := f_n + Bu_n(\delta_n(\cdot)) - A\dot{u}_n(\cdot) \rightarrow g := f + Bu - A\dot{u}$ weakly in $L_H^1(I)$. Further, let us set $\psi_A(x) = \langle Ax, x \rangle$ if $x \in K$ and $\psi(x) = +\infty$ if $x \notin K$. Then it is clear

ψ is a positive lower semicontinuous convex integrand. Apply again the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the positive normal lower semi continuous convex integrand ψ_A we obtain,

$$\liminf_n \int_Z \psi_A(u_n(t))dt \geq \int_Z \psi_A(u(t))dt$$

that is

$$(1.3.4) \quad \liminf_n \int_Z \langle A\dot{u}_n(t), \dot{u}_n(t) \rangle dt \geq \int_Z \langle A\dot{u}(t), \dot{u}(t) \rangle dt$$

Now, we have

$$\begin{aligned} \varphi^*(t, g_n(t)) &\leq \varphi^*(\theta_n(t), g_n(t)) + |v(t) - v(\theta_n(t))| \\ &= \langle \dot{u}_n(t), g_n(t) \rangle - \varphi(\theta_n(t), \dot{u}_n(t)) + |v(t) - v(\theta_n(t))| \\ &\leq \langle \dot{u}_n(t), g_n(t) \rangle + \varphi(t, \dot{u}_n(t)) + 2|v(t) - v(\theta_n(t))|. \end{aligned}$$

Whence

$$\begin{aligned} \int_0^T \varphi^*(t, g_n(t))dt &\leq \sup_n \int_0^T [|\langle \dot{u}_n(t), g_n(t) \rangle| + \varphi(t, \dot{u}_n(t)) + 2|v(t) - v(\theta_n(t))|]dt \\ &< Constant < \infty \end{aligned}$$

because

$$t \mapsto [|\langle \dot{u}_n(t), g_n(t) \rangle| + \varphi(t, \dot{u}_n(t)) + 2|v(t) - v(\theta_n(t))|]$$

is bounded in $L^1_{\mathbf{R}}([0, T])$ so that by noting that (g_n) weakly converges to g in $L^1_H([0, T])$ and applying the lower semicontinuity of integral convex functional ([18], Theorem 8.1.6) to φ^* , we deduce using (1.3.2) that

$$\int_Z \varphi^*(t, g(t))dt \leq \liminf_n \int_Z \varphi^*(t, g_n(t))dt < Constant < \infty$$

as consequence

$$(1.3.5) \quad \int_Z \varphi^*(t, g(t))dt \leq \liminf_n \int_Z \varphi^*(t, g_n(t))dt \leq \liminf_n \int_Z \varphi^*(\theta_n(t), g_n(t))dt$$

with $t \mapsto \varphi^*(t, g(t))$ integrable. From

$$\varphi(t, \dot{u}_n(t)) \leq \varphi(\theta_n(t), \dot{u}_n(t)) + |v(t) - v(\theta_n(t))|$$

we deduce that

$$\liminf_n \int_Z \varphi(t, \dot{u}_n(t))dt \leq \liminf_n \int_Z \varphi(\theta_n(t), \dot{u}_n(t))dt.$$

As (\dot{u}_n) weakly converges to $\dot{u} \in L^1_H([0, T])$, by the lower semi continuity theorem ([18], Theorem 8.1.6) applied to the lower semicontinuity of convex integral functional associated with φ , we deduce that

$$(1.3.6) \quad \int_Z \varphi(t, \dot{u}(t)) dt \leq \liminf_n \int_Z \varphi(\theta_n(t), \dot{u}_n(t)) dt$$

with $\dot{u}(t) \in K$ a.e. and $t \mapsto \varphi(t, \dot{u}(t))$ is integrable. Now integrating on any Lebesgue measurable set Z in $[0, T]$ the equality

$$\varphi(\theta_n(t), \dot{u}_n(t)) + \varphi^*(\theta_n(t), g_n(t)) = \langle \dot{u}_n(t), g_n(t) \rangle$$

gives

$$\int_Z \varphi(\theta_n(t), \dot{u}_n(t)) dt + \int_Z \varphi^*(\theta_n(t), g_n(t)) dt = \int_Z \langle \dot{u}_n(t), h_n(t) \rangle dt.$$

By passing to the limit when n goes to ∞ in this equality using (1.3.3) – (1.3.6) gives

$$\int_Z \varphi(t, \dot{u}(t)) dt + \int_Z \varphi^*(t, g(t)) dt \leq \int_B \langle \dot{u}(t), g(t) \rangle dt.$$

By $t \mapsto \varphi(t, \dot{u}(t)) + \varphi^*(t, g(t)) - \langle \dot{u}(t), g(t) \rangle$ is integrable, we deduce that

$$\varphi(t, \dot{u}(t)) + \varphi^*(t, g(t)) - \langle \dot{u}(t), g(t) \rangle \leq 0$$

a.e. with $\dot{u}(t) \in K$ a.e. So we conclude that $\varphi(t, \dot{u}(t)) + \varphi^*(t, g(t)) = \langle \dot{u}(t), g(t) \rangle$ a.e., equivalently $g(t) = f(t) + Bu(t) - A\dot{u}(t) \in \partial\varphi(t, \dot{u}(t))$ a.e. and equivalently $\dot{u}(t) \in \partial\varphi^*(t, f(t) + Bu(t) - A\dot{u}(t))$ a.e. \square

Remarks 1) The uniqueness of solutions follows easily from the coerciveness of the operator A . Indeed let u_1 and u_2 two solutions. then by an easy computation,

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle + \langle Bu_2(t) - Bu_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle \leq 0$$

so that

$$\langle A\dot{u}_2(t) - A\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t) \rangle \leq |B| \|u_2(t) - u_1(t)\| \|\dot{u}_2(t) - \dot{u}_1(t)\|$$

By coerciveness, we deduce that

$$\omega \|\dot{u}_2(t) - \dot{u}_1(t)\|^2 \leq |B| \|u_2(t) - u_1(t)\| \|\dot{u}_2(t) - \dot{u}_1(t)\|$$

Whence

$$\|\dot{u}_2(t) - \dot{u}_1(t)\| \leq \frac{|B|}{\omega} \|u_2(t) - u_1(t)\| \leq \frac{|B|}{\omega} \int_0^t \|\dot{u}_2(s) - \dot{u}_1(s)\| ds$$

By Gronwall lemma $\dot{u}_1(t) = \dot{u}_2(t)$ a and so $u_1(t) = u_2(t)$ since $u_1(t) = u_0 + \int_0^t \dot{u}_1(s) ds, \forall t \in [0, T], u_2(t) = u_0 + \int_0^t \dot{u}_2(s) ds, \forall t \in [0, T]$.

2) Theorem 1.3 generalizes Theorem 6 in [16] dealing with finite dimensional space.

In view of further applications we provide some variational limits theorems below.

2 Some variational limit theorems

Proposition 2.1. *Let H be a separable Hilbert space. Let $C : [0, T] \rightarrow H$ be a convex weakly compact valued scalarly measurable mappings such that $C(t) \subset r(t)\overline{B}_H$, for all $t \in [0, T]$ for some $r \in L^1_{\mathbf{R}^+}([0, T])$. Let A be linear continuous coercive symmetric operator on H . Let B is a linear continuous compact operator on H .*

Let $(f_n, f)_{n \in \mathbf{N}}$ be a bounded sequence in $L^\infty_H([0, T])$ with $\|f_n(t)\| \leq \beta, \|f(t)\| \leq \beta$ ($\beta > 0$) for all $n \in \mathbf{N}$ such that $f_n(t)$ pointwise converges to $f(t)$ for each $t \in [0, T]$. Let $(v_n, v)_{n \in \mathbf{N}}$ be a bounded sequence in $L^\infty_H([0, T])$ with $\|v_n(t)\| \leq \gamma, \|v(t)\| \leq \gamma$ ($\gamma > 0$) for all $n \in \mathbf{N}$ such that $v_n(t)$ pointwise converges weakly to $v(t)$ for each $t \in [0, T]$. Let (u_n) be an integrable sequence in $L^1_H([0, T])$ such that $u_n(t) \in C(t)$ for all $t \in [0, T]$ such that (u_n) $\sigma(L^1_H, L^\infty_H)$ converges in $L^1_H([0, T])$ to u . Assume that $f_n(t) + Bv_n(t) - Au_n(t) \in N_{C(t)}(u_n(t))$ for all $n \in \mathbf{N}$ and for all $t \in [0, T]$, then $u(t) \in C(t)$ a.e. and $f(t) + Bv(t) - Au(t) \in N_{C(t)}(u(t))$ a.e.

Proof. We first check that $u(t) \in C(t)$ a.e. Indeed, we have $\langle x, u_n(t) \rangle \leq \delta^*(x, C(t))$, $\forall x \in H, \forall t \in [0, T]$. Then by integrating on any Lebesgue measurable set $Z \subset [0, T]$

$$\int_Z \langle x, u_n(t) \rangle dt \leq \int_Z \delta^*(x, C(t)) dt.$$

By using (iii) and by passing to the limit when n goes to ∞ gives

$$\int_Z \langle x, u(t) \rangle dt \leq \limsup_n \int_Z \delta^*(x, C(t)) dt.$$

Then we deduce that $\langle x, u(t) \rangle \leq \delta^*(x, C(t))$ a.e. so that by Castaing-Valadier ([19], Proposition III.35), we get $u(t) \in C(t)$ a.e. Note that this fact requires that $C(t)$ is convex weakly compact. Note that $f_n + Bv_n$ are uniformly bounded and pointwise converges to $f + Bv$ since B is compact operator. Since u_n is uniformly integrable and $\sigma(L^1_H, L^\infty_H)$ converges to u , we have by the Castaing trick given in the proof of Theorem 1.1

$$(2.1.1) \quad \lim_n \int_Z \langle f_n + Bv_n, u_n \rangle dt = \int_Z \langle f + Bv, u \rangle dt$$

for every measurable set $Z \subset [0, T]$. By integrating on Z (we are ensured that the functions given are measurable) the inequality

$$\delta^*(f_n(t) + Bv_n(t) - Au_n(t), C_n(t)) + \langle Av_n(t) - Bv_n(t) - f_n, u_n(t) \rangle \leq 0$$

we get

$$(2.1.2) \quad \int_Z \delta^*(f_n(t) + Bv_n(t) - Au_n(t), C_n(t)) dt + \int_Z \langle Au_n - f_n, u_n(t) \rangle dt - \int_Z \langle Bv_n(t), v_n(t) \rangle dt \leq 0.$$

Set $g_n = f_n + Bv_n - Au_n$. It is already seen that $Au_n \rightarrow Au$ weakly in $L^1_H([0, T])$ and $Bv_n \rightarrow Bv$ weakly in $L^1_H([0, T])$ so that $g_n \rightarrow g := f + Bv - Au$ weakly in $L^1_H([0, T])$. Let us set $\psi(t, x) = \langle Ax, x \rangle$ if $x \in K$ and $\psi(t, x) = +\infty$ if $x \notin K$. Then it is clear ψ is a positive lower semicontinuous convex normal integrand. By the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the positive normal convex integrand ψ we obtain, for every Lebesgue measurable set $Z \subset [0, T]$

$$\liminf_n \int_Z \psi(t, u_n(t)) dt \geq \int_Z \psi(t, u(t)) dt$$

that is

$$(2.1.3) \quad \liminf_n \int_Z \langle Au_n(t), u_n(t) \rangle dt \geq \int_Z \langle Au(t), u(t) \rangle dt$$

To finish the proof we apply the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the normal convex integrand $(t, x) \mapsto \delta^*(x, C(t))$ by noting that

- (j) $\delta^*(g_n(t), C(t))$ is minored by $\langle g_n(t), h(t) \rangle$ where $h \in S^1_C$.
 - (jj) the minored sequence $\langle g_n(t), h(t) \rangle$ is uniformly integrable.
- Then by ([18], Theorem 8.1.6) we are ensured

$$(2.1.4) \quad \liminf_n \int_Z \delta^*(g_n(t), C(t)) dt \geq \int_Z \delta^*(g(t), C(t)) dt$$

By combining (2.1.2) – (2.1.4) we get

$$\int_Z \delta^*(f(t) - Au(t) - Bv(t), C(t)) dt + \int_Z \langle -f(t) + Au(t) + Bv(t), u(t) \rangle dt \leq 0$$

for every Lebesgue measurable set Z . But since $u(t) \in C(t)$ a.e. we have

$$\delta^*(f(t) + Bv(t) - Au(t), C(t)) \geq \langle f(t) + Bv(t) - Au(t), u(t) \rangle$$

a.e. that is

$$\delta^*(f(t) - Bv(t) - Au(t), C(t)) + \langle -f(t) - Bv(t) + Au(t), u(t) \rangle \geq 0$$

a.e. so we conclude that

$$\delta^*(f(t) - Bv(t) - Au(t), C(t)) = \langle f(t) + Bv(t) - Au(t), u(t) \rangle = 0$$

a.e. with $u(t) \in C(t)$ a.e., just proving that $f(t) + Bv(t) - Au(t) \in N_{C(t)}(u(t))$ a.e. \square

A variant of Proposition 2.1 is available by assuming that C is convex compact valued and the operator B is a Lipschitz mapping: $\|Bx - By\| \leq M\|x - y\|$. These results related to the evolution variational inequality of the form $f(t) + Bv(t) - Au(t) \in N_{C(t)}(u(t))$ a.e stated in Theorem 1.1 and Theorem 1.2. The following result is a stability result related to the evolution variational inequality of the form $f(t) + \nabla g(u(t)) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$ here ∇g is a specific gradient of a convex continuous Gateaux differentiable such that $g(v(t))$ is absolutely continuous for $v : [0, T] \rightarrow H$ absolutely continuous. There is no confusion with the linear compact operator B considered in Theorem 1.1 and Lipschitz operator B considered in Theorem 1.2.

Proposition 2.2. $H = \mathbf{R}^e$. Let $C : [0, T] \rightarrow H$ be a convex compact valued measurable mapping such that $C(t) \subset r(t)\bar{B}_H$, for all $t \in [0, T]$ for some $r \in L^1_{\mathbf{R}^+}([0, T])$. Let A be linear continuous coercive symmetric operator on H and $B = \nabla g$ where ∇g is the gradient of a convex continuous Gateaux differentiable function $g : H \rightarrow \mathbf{R}^+$ such that $g(v(t))$ is absolutely continuous for $v : [0, T] \rightarrow H$ absolutely continuous.

Let $(f_n, f)_{n \in \mathbf{N}}$ be a bounded sequence in $L^\infty_H([0, T])$ with $\|f_n(t)\| \leq \beta, \|f(t)\| \leq \beta$ ($\beta > 0$) for all $n \in \mathbf{N}$ such that $f_n(t)$ pointwise converges to $f(t)$ for each $t \in [0, T]$. Let $(u_n, u)_{n \in \mathbf{N}}$ be a sequence of absolutely continuous mappings

$$u_n(t) = u_0 + \int_0^t \dot{u}_n(s) ds, \dot{u}_n(t) \in C(t)$$

$$u(t) = u_0 + \int_0^t \dot{u}(s) ds, \dot{u}(t) \in C(t)$$

such that $u_n(t) \rightarrow u(t)$ uniformly in H and $\dot{u}_n \rightarrow \dot{u}$ weakly in $L^1_H[0, T]$. Assume that $f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t) \in N_{C(t)}(\dot{u}_n(t))$ for all $n \in \mathbf{N}$ and for all $t \in [0, T]$, then $f(t) + \nabla g(u(t)) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$ a.e.

Proof. Note that f_n are uniformly bounded and pointwise converges to $f(t)$ then we have

$$(2.2.1) \quad \lim_n \int_Z \langle f_n, \dot{u}_n \rangle dt = \int_Z \langle f, \dot{u} \rangle dt$$

for every measurable set $Z \subset [0, T]$. By integrating on Z (we are ensured that the functions given are measurable) the inequality

$$\delta^*(f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t), C_n(t)) + \langle A\dot{u}_n(t) - \nabla g(u_n(t)) - f_n, \dot{u}_n(t) \rangle \leq 0$$

we get

$$(2.2.2) \quad \int_Z \delta^*(f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t), C_n(t)) dt + \int_Z \langle A\dot{u}_n - f_n, \dot{u}_n(t) \rangle dt - \int_Z \langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle dt \leq 0.$$

Set $g_n(t) = f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t)$. We claim that $Au_n(t) \rightarrow Au(t)$ weakly in $L_H^1([0, T])$ and $\nabla g(u_n(t)) \rightarrow \nabla g(u(t))$ weakly in $L_H^1([0, T])$. So $g_n(t) = f_n(t) + \nabla g(u_n(t)) - A\dot{u}_n(t) \rightarrow g(t) := f(t) + \nabla g(u(t)) - A\dot{u}(t)$ weakly in $L_H^1([0, T])$. Indeed, as $u_n(t) \rightarrow u(t)$ pointwise, $\langle x, Au_n(t) \rangle \rightarrow \langle x, Au(t) \rangle$. As consequence, for any $h \in L_H^\infty([0, T])$, we have $\langle h(t), Au_n(t) \rangle \rightarrow \langle h(t), Au(t) \rangle$ pointwise. But then the uniformly bounded sequence of bounded measurable functions $(\langle h, Au_n \rangle)$ pointwise converge to the bounded measurable function $\langle h, Au \rangle$. As consequence, $\langle h, Au_n \rangle \rightarrow \langle h, Au \rangle$ weakly in $L_{\mathbf{R}}^1([0, T])$. This shows that $Au_n \rightarrow Au$ weakly in $L_H^1([0, T])$. Similarly we show that $\nabla g(u_n(t)) \rightarrow \nabla g(u(t))$ weakly in $L_H^1([0, T])$. Now we have to consider the term $\langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle$ by using the special property of ∇g . In fact u_n is absolutely continuous with derivative \dot{u}_n and $g(u_n)$ is absolutely continuous, so that by Moreau-Valadier [27],

$$\langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle = \frac{d}{dt} g(u_n(t))$$

From this fact, it is easy to deduce that

$$\begin{aligned} \liminf_n \int_0^T \langle \nabla g(u_n(t)), \dot{u}_n(t) \rangle dt &= \liminf_n \int_0^T \frac{d}{dt} g(u_n(t)) dt \\ (2.2.3) \quad &= \liminf_n (g(u_n(T)) - g(u_n(0))) \geq g(u(T)) - g(u(0)) = \int_0^T \frac{d}{dt} g(u(t)) dt \\ &= \int_0^T \langle \nabla g(u(t)), \dot{u}(t) \rangle dt \end{aligned}$$

Let us set $\varphi(t, x) = \langle Ax, x \rangle$ if $x \in K$ and $\varphi(t, x) = +\infty$ if $x \notin K$. Then it is clear φ is a positive lower semicontinuous convex normal integrand. By the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the positive normal convex integrand φ we obtain, for every Lebesgue measurable set $Z \subset [0, T]$

$$\liminf_n \int_Z \varphi(t, \dot{u}_n(t)) dt \geq \int_Z \varphi(t, \dot{u}(t)) dt$$

that is

$$(2.2.4) \quad \liminf_n \int_Z \langle A\dot{u}_n(t), \dot{u}_n(t) \rangle dt \geq \int_Z \langle A\dot{u}(t), \dot{u}(t) \rangle dt$$

To finish the proof we apply the lower semicontinuity of the integral convex functional ([18], Theorem 8.1.6) associated with the normal convex integrand $(t, x) \mapsto \delta^*(x, C(t))$ by noting that

(j) $\delta^*(g_n(t), C(t))$ is minored by $\langle h(t), g_n(t) \rangle$; with $h \in S_C^\infty$.

(jj) the minored sequence $\langle g_n(t), h(t) \rangle$ is uniformly integrable.

Then we are ensured by the lower semicontinuity of the integral convex functional theorem ([18], Theorem 8.1.6)

$$(2.2.5) \quad \liminf_n \int_Z \delta^*(g_n(t), C(t)) dt \geq \int_Z \delta^*(g(t), C(t)) dt$$

By combining (2.2.3) – (2.2.5) we get

$$\int_0^T \delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) dt + \int_0^T \langle -f(t) + \nabla g(u(t)) + A\dot{u}(t), \dot{u}(t) \rangle dt \leq 0$$

But since $\dot{u}(t) \in C(t)$ a.e. we have

$$\delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) \geq \langle f(t) + \nabla g(u(t)) - A\dot{u}(t), \dot{u}(t) \rangle$$

a.e. that implies

$$\int_0^T [\delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) + \langle -f(t) - \nabla g(u(t)) + A\dot{u}(t), \dot{u}(t) \rangle] dt = 0$$

so we conclude that

$$\delta^*(f(t) + \nabla g(u(t)) - A\dot{u}(t), C(t)) = \langle f(t) + \nabla g(u(t)) - A\dot{u}(t), \dot{u}(t) \rangle$$

a.e. with $\dot{u}(t) \in C(t)$ a.e., just proving that $f(t) + \nabla g(u(t)) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t))$ a.e. \square

3 Applications

3.1 A Skorokhod problem

We present at first a new version of the Skorokhod problem in Castaing et al [15, 17] dealing with the sweeping process associated with an absolutely continuous (or continuous) closed convex moving set $C(t)$ in H . Here the novelty is the velocity is inside the subdifferential operator.

Theorem 3.1. *Let $I := [0, 1]$ and $H = \mathbf{R}^e$. Let $v : I \rightarrow \mathbf{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : I \rightrightarrows H$ be a convex compact valued mapping such that*

(i) $C(t) \subset M\bar{B}_H$ for all $t \in I$ where M is a positive constant.

(ii) $d_H(C(t), C(\tau)) \leq |v(t) - v(\tau)|$ for all $t, \tau \in I$.

Let $A : H \rightarrow H$ be a linear continuous coercive symmetric operator. Let $B : H \rightarrow H$ be a linear continuous operator. Let $z \in C^{1-var}([0, 1], \mathbf{R}^d)$ the space of continuous functions of bounded variation defined on $[0, 1]$ with values in \mathbf{R}^d . Let $\mathcal{L}(\mathbf{R}^d, \mathbf{R}^e)$ the space of linear mappings f from \mathbf{R}^d to \mathbf{R}^e endowed with the operator norm

$$|f| := \sup_{x \in \mathbf{R}^d, \|x\|_{\mathbf{R}^d} = 1} |f(x)|_{\mathbf{R}^e}.$$

Let us consider a class of continuous integrand operator $b : [0, 1] \times \mathbf{R}^e \rightarrow \mathcal{L}(\mathbf{R}^d, \mathbf{R}^e)$ satisfying

(a) $|b(t, x)| \leq M, \quad \forall (t, x) \in [0, 1] \times \mathbf{R}^e.$

(b) $|b(t, x) - b(t, y)| \leq M \|x - y\|_{\mathbf{R}^e}, \quad \forall (t, x, y) \in [0, 1] \times \mathbf{R}^e \times \mathbf{R}^e$ with the perturbed Riemann-Stieljies integral $\int_0^t b(\tau, x(\tau)) dz_\tau$ defined on $x \in C([0, 1], \mathbf{R}^e)$.

Let $g : I \times I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$ be a continuous mapping satisfying:

(i) $\|g(t, s, x)\| \leq M$ for all $t, s, x \in \mathbf{R}^e$.

(ii) $\|g(t, s, x) - g(\tau, s, x)\| \leq M |t - \tau|$ for all $(t, \tau, s, x) \in I \times I \times I \times \mathbf{R}^e$ with the perturbed Lebesgue integral $\int_0^t g(t, s, x(s)) ds$ defined on in $C([0, 1], \mathbf{R}^e)$.

Let $a \in C(0)$. Then there exist a BVC function $x : [0, 1] \rightarrow H$ and an absolutely continuous mapping function $u : [0, 1] \rightarrow H$ satisfying

$$\begin{cases} x(0) = u(0) = a \\ x(t) = h(t) + k(t) + Bu(t), \quad \forall t \in I \\ h(t) = \int_0^t b(\tau, x(\tau)) dz_\tau, \quad \forall t \in I \\ k(t) = \int_0^t g(t, s, x(s)) ds, \quad \forall t \in I \\ \int_0^t b(s, x(s)) dz_s + \int_0^t g(t, s, x(s)) ds + Bu(t) - A\dot{u}(t) \in N_{C(t)}(\dot{u}(t)) \text{ a.e. } t \in I \end{cases}$$

Proof. Let $a \in C(0)$. Let us set for all $t \in I = [0, 1]$

$$x^0(t) = a, \quad h^1(t) = \int_0^t b(\tau, a) dz_\tau$$

then by Proposition 2.2 in Friz-Victoir [20], we have

$$\left| \int_0^t b(\tau, a) dz_\tau \right| \leq |b(\cdot, a)|_{\infty: [0, 1]} |z|_{1-var: [0, t] \cdot}$$

Moreover

$$\int_0^t b(\tau, a) dz_\tau - \int_0^s b(\tau, a) dz_\tau = \int_s^t b(\tau, a) dz_\tau$$

so that by condition (a)

$$\|h^1(t) - h^1(s)\| \leq M |z|_{1-var: [s, t]}$$

for all $0 \leq s \leq t \leq 1$ and in particular

$$\|h^1(t)\| \leq M |z|_{1-var: [0, t]} \leq M |z|_{1-var: [0, T]}$$

for all $t \in [0, 1]$. Let us set for all $t \in I = [0, 1]$

$$x^0(t) = a, \quad k^1(t) = \int_0^t g(t, s, x^0(s)) ds,$$

then k^1 is continuous with $\|k^1(t)\| \leq M$ for all $t \in I$. By an easy computation, using condition (i) and (ii) we have the estimate $\|k^1(t) - k^1(\tau)\| \leq 3M |t - \tau|$,

for all $\tau, t \in I$. By Theorem 1.1 there is a unique absolutely continuous mapping $u^1 : I \rightarrow H$ solution of the problem

$$\begin{cases} u^1(0) = a \\ h^1(t) + k^1(t) + Bu^1(t) - Au^1(t) \in N_{C(t)}(\dot{u}^1(t)) \text{ a.e.} \end{cases}$$

with $u^1(t) = a + \int_0^t \dot{u}^1(s)ds$, $\forall t \in I$ and $\|\dot{u}^1(t)\| \leq M$, a.e. Set

$$x^1(t) = h^1(t) + k^1(t) + Bu^1(t) = \int_0^t b(\tau, x^0(\tau))dz_\tau + \int_0^t g(t, s, x^0(s))ds + Bu^1(t).$$

Then x^1 is BVC with $x^1(0) = a$. Now we construct x^n by induction as follows. Let for all $t \in I$

$$\begin{aligned} h^n(t) &= \int_0^t b(\tau, x^{n-1}(\tau))dz_\tau \\ k^n(t) &= \int_0^t g(t, s, x^{n-1}(s))ds. \end{aligned}$$

Then k^n is equi-Lipschitz: $\|k^n(t) - k^n(\tau)\| \leq 3M|t - \tau|$, for all $\tau, t \in I$ with $\|k^n(t)\| \leq M$ for all $t \in I$. By Proposition 2.2 in Friz-Victoir [20] we have the estimate

$$\|h^n(t) - h^n(s)\| \leq M|z|_{1-var:[s,t]}$$

for all $0 \leq s \leq t \leq 1$ and in particular

$$\|h^n(t)\| \leq M|z|_{1-var:[0,t]} \leq M|z|_{1-var:[0,1]}$$

for all $0 \leq t \leq 1$. By Theorem 1.1 there is a unique absolutely continuous mapping $u^n : I \rightarrow H$ solution of the problem

$$\begin{cases} u^n(0) = a, \\ h^n(t) + k^n(t) + Bu^n(t) - Au^n(t) \in N_{C(t)}(\dot{u}^n(t)) \text{ a.e.} \end{cases}$$

with $u^n(t) = a + \int_{[0,t]} \dot{u}^n(s)ds$ $\forall t \in I$ and $\|\dot{u}^n(t)\| \leq M$ a.e., Set for all $t \in I$

$$x^n(t) = h^n(t) + k^n(t) + Bu^n(t) = \int_0^t b(\tau, x^{n-1}(\tau))dz_\tau + \int_0^t g(t, s, x^{n-1}(s))ds + Bu^n(t)$$

so that x^n is BVC. As (u^n) is equi-absolutely continuous we may assume that (u^n) converges uniformly to an absolutely continuous mapping $u : I \rightarrow H$ with $u(t) = a + \int_0^t \dot{u}(s)ds, \forall t \in I$. We may also assume that \dot{u}^n weakly converges in L^1_H to \dot{u} , and by Ascoli theorem we may assume that k^n converges uniformly to a continuous mapping $k : I \rightarrow H$. Now, recall that

$$\|h^n(t) - h^n(s)\| \leq M|z|_{1-var:[s,t]}$$

for all $0 \leq s \leq t \leq T$ by using Proposition 2.2 in Friz-Victoir [20], and our assumption (a) on the mapping b . So h^n is bounded and equicontinuous. By Ascoli theorem, we may assume that h^n converge uniformly to a continuous mapping h . Similarly k^n is bounded and equi-Lipschitz. By Ascoli theorem, we may assume that k^n converge uniformly to a continuous mapping k . Hence $x^n(t) = h^n(t) + k^n(t) + Bu^n(t)$ converge uniformly to $x(t) := h(t) + k(t) + Bu(t)$, and $b(\cdot, x^{n-1}(\cdot))$ converges uniformly to $b(\cdot, x(\cdot))$ using the Lipschitz condition (b). Then by Friz-Victoir [20] (Proposition 2.7) $\int_0^t b(\tau, x^{n-1}(\tau)) dz_\tau$ converge uniformly to $\int_0^t b(\tau, x(\tau)) dz_\tau$. By hypothesis (i), $g(t, s, x^{n-1}(s))$ pointwise converge to $g(t, s, x(s))$. Hence $\int_0^t g(t, s, x^{n-1}(s)) ds \rightarrow \int_0^t g(t, s, x(s)) ds$ for each $t \in I$ by Lebesgue theorem. So by identifying the limit

$$\begin{aligned} \lim_{n \rightarrow \infty} x^n(t) &= \lim_{n \rightarrow \infty} \int_0^t b(\tau, x^{n-1}(\tau)) dz_\tau + \lim_{n \rightarrow \infty} \int_0^t g(t, s, x^{n-1}(s)) ds + \lim_{n \rightarrow \infty} Bu^n(t) \\ &= \int_0^t b(\tau, x(\tau)) dz_\tau + \int_0^t g(t, s, x(s)) ds + Bu(t) = x(t). \end{aligned}$$

From the inclusion

$$h^n(t) + k^n(t) + Bu^n(t) - Ai^n(t) \in N_{C(t)}(\dot{u}^n(t)) \text{ a.e.}$$

and the above convergence, repeating the argument involving variational techniques in Proposition 1.2 we get

$$h(t) + k(t) + Bu(t) - Ai(t) \in N_{C(t)}(\dot{u}(t)) \text{ a.e.}$$

The proof is therefore complete. In Theorem 2.3 we have provided existence results of solution for a class of integral equation of Hammerstein type coupled with a sweeping process in the BV setting. Actually our tools allow to state several variants of Theorem 2.3 according to the nature of the perturbation and the operator. Actually Theorem 2.3 hold if $B : H \rightarrow H$ is a Lipschitz mapping: $\|Bx - By\| \leq M\|x - y\|, \forall x, y \in H$ for some positive constant M . Theorem 2.3 holds if we replace B by the gradient ∇g of a positive convex continuous Gateaux differentiable function $g : H \rightarrow \mathbf{R}$ such that $g(v(t))$ is absolutely continuous for $v : [0, T] \rightarrow H$ absolutely continuous,

□

3.2 Towards fractional inclusion coupled with EVI and sweeping process

Now $I = [0, 1]$ and we investigate a class of boundary value problem governed by a fractional differential inclusion (FDI) (3.1) in a separable Hilbert space H coupled with the evolution inclusion governed by the above EVI (3.3) and sweeping process (3.4).

$$D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = u(t), t \in I, \quad (3.1)$$

$$I_{0+}^{\beta} h(t) |_{t=0} := \lim_{t \rightarrow 0} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s) ds = 0, \quad h(1) = I_{0+}^{\gamma} h(1) = \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} h(s) ds, \quad (3.2)$$

$$f(t, h(t) + Bu(t) - Au(t)) \in \partial\varphi(t, \dot{u}(t)) \text{ a.e.} \quad (3.3)$$

and

$$f(t, h(t) + Bu(t) - Au(t)) \in N_{C(t)}(\dot{u}(t)) \text{ a.e.} \quad (3.4)$$

where $\alpha \in]1, 2]$, $\beta \in [0, 2 - \alpha]$, $\lambda \geq 0$, $\gamma > 0$ are given constants, D^{α} is the standard Riemann-Liouville fractional derivative, Γ is the gamma function.

For the convenience of the reader, we begin with a few reminders of the concepts that will be used in the rest of the paper.

Definition 3.1 (Fractional Bochner integral). *Let E be a separable Banach space. Let $f : I = [0, 1] \rightarrow E$. The fractional Bochner-integral of order $\alpha > 0$ of the function f is defined by*

$$I_{a+}^{\alpha} f(t) := \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds, \quad t > a.$$

In the above definition, the sign “ \int ” denotes the classical Bochner integral.

Lemma 3.2 ([29]). *Let $f \in L^1([0, 1], E, dt)$. We have*

- (i) *If $\alpha \in]0, 1[$ then $I^{\alpha} f$ exists almost everywhere on I and $I^{\alpha} f \in L^1(I, E, dt)$.*
- (ii) *If $\alpha \in [1, \infty)$ then $I^{\alpha} f \in C_E(I)$.*

Definition 3.2. *Let E be a separable Banach space. Let $f \in L^1(I, E, dt)$. We define the Riemann-Liouville fractional derivative of order $\alpha > 0$ of f by*

$$D^{\alpha} f(t) := D_{0+}^{\alpha} f(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} f(t) = \frac{d^n}{dt^n} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(s) ds,$$

where $n = [\alpha] + 1$.

We denote by $W_{B,E}^{\alpha,1}(I)$ the space of all continuous functions in $C_E(I)$ such that their Riemann-Liouville fractional derivative of order $\alpha - 1$ are continuous and their Riemann-Liouville fractional derivative of order α are Bochner integrable.

Green function and its properties

Let $\alpha \in]1, 2]$, $\beta \in [0, 2 - \alpha]$, $\lambda \geq 0$, $\gamma > 0$ and $G : [0, 1] \times [0, 1] \rightarrow \mathbf{R}$ be a function defined by

$$G(t, s) = \varphi(s) I_{0+}^{\alpha-1}(\exp(-\lambda t)) + \begin{cases} \exp(\lambda s) I_{s+}^{\alpha-1}(\exp(-\lambda t)), & 0 \leq s \leq t \leq 1, \\ 0, & 0 \leq t \leq s \leq 1, \end{cases} \quad (3.5)$$

where

$$\varphi(s) = \frac{\exp(\lambda s)}{\mu_0} \left[\left(I_{s^+}^{\alpha-1+\gamma}(\exp(-\lambda t)) \right) (1) - \left(I_{s^+}^{\alpha-1}(\exp(-\lambda t)) \right) (1) \right] \quad (3.6)$$

with

$$\mu_0 = \left(I_{0^+}^{\alpha-1}(\exp(-\lambda t)) \right) (1) - \left(I_{0^+}^{\alpha-1+\gamma}(\exp(-\lambda t)) \right) (1). \quad (3.7)$$

We recall and summarize a useful result ([14]).

Lemma 3.3. *Let E be a separable Banach space. Let G be the function defined by (3.5)-(3.7).*

(i) $G(\cdot, \cdot)$ satisfies the following estimate

$$|G(t, s)| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{1 + \Gamma(\gamma + 1)}{|\mu_0| \Gamma(\alpha) \Gamma(\gamma + 1)} + 1 \right) = M_G.$$

(ii) If $u \in W_{B,E}^{\alpha,1}([0, 1])$ satisfying boundary conditions (4.2), then

$$u(t) = \int_0^1 G(t, s) (D^\alpha u(s) + \lambda D^{\alpha-1} u(s)) ds \quad \text{for every } t \in [0, 1].$$

(iii) Let $f \in L_E^1([0, 1])$ and let $u_f : [0, 1] \rightarrow E$ be the function defined by

$$u_f(t) := \int_0^1 G(t, s) f(s) ds \quad \text{for } t \in [0, 1].$$

Then

$$I_{0^+}^\beta u_f(t) |_{t=0} = 0 \quad \text{and} \quad u_f(1) = (I_{0^+}^\gamma u_f)(1).$$

Moreover $u_f \in W_{B,E}^{\alpha,1}([0, 1])$ and we have

$$(D^{\alpha-1} u_f)(t) = \int_0^t \exp(-\lambda(t-s)) f(s) ds + \exp(-\lambda t) \int_0^1 \varphi(s) f(s) ds \quad \text{for } t \in [0, 1], \quad (3.8)$$

$$(D^\alpha u_f)(t) + \lambda (D^{\alpha-1} u_f)(t) = f(t) \quad \text{for all } t \in [0, 1]. \quad (3.9)$$

From Lemma 3.3 we summarize a crucial fact.

Lemma 3.4. *Let E be a separable Banach space. Let $f \in L^1(I, E, dt)$. Then the boundary value problem*

$$\begin{cases} D^\alpha u(t) + \lambda D^{\alpha-1} u(t) = f(t), & t \in I \\ I_{0^+}^\beta u(t) |_{t=0} = 0, & u(1) = I_{0^+}^\gamma u(1) \end{cases}$$

has a unique $W_{B,E}^{\alpha,1}(I)$ -solution defined by

$$u(t) = \int_0^1 G(t,s)f(s)ds, \quad t \in I.$$

Theorem 3.5. *Let E be a separable Banach space. Let $X : I \rightarrow E$ be a convex compact valued measurable multifunction such that $X(t) \subset \gamma \overline{B}_E$ for all $t \in I$, where γ is a positive constant and S_X^1 be the set of all measurable selections of X . Then the $W_{B,E}^{\alpha,1}(I)$ -solutions set of problem*

$$\begin{cases} D^\alpha u(t) + \lambda D^{\alpha-1} u(t) = f(t), f \in S_X^1, \text{ a.e. } t \in I \\ I_{0+}^\beta u(t)|_{t=0} = 0, \quad u(1) = I_{0+}^\gamma u(1) \end{cases} \quad (3.10)$$

is compact in $C_E(I)$.

Proof. By virtue of Lemma 3.4 the $W_{B,E}^{\alpha,1}([0,1])$ -solutions set \mathcal{X} to the above inclusion is characterized by

$$\mathcal{X} = \{u_f : I \rightarrow E, u_f(t) = \int_0^1 G(t,s)f(s)ds, f \in S_X^1, t \in I\}$$

Claim: \mathcal{X} is bounded, convex, equicontinuous and **compact** in $C_E(I)$.

From definition of the Green function G , it is not difficult to show that $\{u_f : f \in S_X^1\}$ is bounded, equicontinuous in $C_E(I)$. Indeed let (u_{f_n}) be a sequence in \mathcal{X} . We note that, for each $n \in \mathbb{N}$, we have $u_{f_n} \in W_{B,E}^{\alpha,1}(I)$, and

$$u_{f_n}(t) = \int_0^1 G(t,s)f_n(s)ds, \quad t \in I,$$

with

- $I_{0+}^\beta u_{f_n}(t)|_{t=0} = 0, \quad u_{f_n}(1) = I_{0+}^\gamma u_{f_n}(1),$
- $(D^{\alpha-1}u_{f_n})(t) = \int_0^t \exp(-\lambda(t-s))f_n(s)ds + \exp(-\lambda t) \int_0^1 \varphi(s)f_n(s)ds, \quad t \in I,$
- $(D^\alpha u_{f_n})(t) + \lambda (D^{\alpha-1}u_{f_n})(t) = f_n(t), \quad t \in I.$

For $t_1, t_2 \in I$, $t_1 < t_2$, we have

$$\begin{aligned}
u_{f_n}(t_2) - u_{f_n}(t_1) &= \int_0^1 G(t, s)(f_n(t_2, s) - f_n(t_1, s))ds \\
&= \int_0^1 \varphi(s)f_n(s)ds \left(\int_0^{t_2} \frac{e^{-\lambda\tau}}{\Gamma(\alpha-1)}(t_2-\tau)^{\alpha-2}d\tau - \int_0^{t_1} \frac{e^{-\lambda\tau}}{\Gamma(\alpha-1)}(t_1-\tau)^{\alpha-2}d\tau \right) \\
&+ \int_0^{t_2} e^{\lambda s} \left(\int_s^{t_2} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}e^{-\lambda\tau}d\tau \right) f(s)ds - \int_0^{t_1} e^{\lambda s} \left(\int_s^{t_1} \frac{e^{-\lambda\tau}}{\Gamma(\alpha-1)}(t_1-\tau)^{\alpha-2}d\tau \right) f(s)ds \\
&= \int_0^1 \phi(s)f(s)ds \left[\int_0^{t_1} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2} - (t_1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau + \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau \right] \\
&+ \int_0^{t_1} e^{\lambda s} \left(\int_s^{t_1} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2} - (t_1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau \right) f(s)ds \\
&+ \int_0^{t_1} e^{\lambda s} \left(\int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau \right) f(s)ds + \int_{t_1}^{t_2} e^{\lambda s} \left(\int_s^{t_2} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}e^{-\lambda\tau}d\tau \right) f(s)ds.
\end{aligned}$$

Then, we get

$$\begin{aligned}
\|u_{f_n}(t_2) - u_{f_n}(t_1)\| &\leq \int_0^1 (|\varphi(s)| + e^{\lambda s}) |X(s)|ds \int_0^{t_1} e^{-\lambda\tau} \frac{(t_1-\tau)^{\alpha-2} - (t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau \\
&+ \int_0^1 (|\varphi(s)| + e^{\lambda s}) |X(s)|ds \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau \\
&+ \int_{t_1}^{t_2} e^{\lambda s} |X(s)|ds \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau.
\end{aligned}$$

It is easy to obtain, after an integration by part, that

$$\int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau = e^{-\lambda t_1} \frac{(t_2-t_1)^{\alpha-2}}{\Gamma(\alpha)} + \lambda \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau \leq \frac{1+\lambda}{\Gamma(\alpha)}(t_2-t_1)^{\alpha-1}$$

and

$$\int_0^{t_1} e^{-\lambda\tau} \frac{(t_1-\tau)^{\alpha-2} - (t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau \leq \int_0^{t_1} \frac{(t_1-\tau)^{\alpha-2} - (t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau = \frac{(t_2-t_1)^{\alpha-1} + t_1^{\alpha-1} - t_2^{\alpha-1}}{\Gamma(\alpha)}$$

Using the inequality that $|a^p - b^p| \leq |a - b|^p$ for all $a, b \geq 0$ and $0 < p \leq 1$, we yield

$$\int_0^{t_1} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2} - (t_1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau \leq \frac{2}{\Gamma(\alpha)}(t_2-t_1)^{\alpha-1}$$

Then, since $\alpha \in [1, 2]$, we can increase $\|u_{f_n}(t_2) - u_{f_n}(t_1)\|$ by

$$\|u_{f_n}(t_2) - u_{f_n}(t_1)\| \leq K|t_2 - t_1|^{\alpha-1}$$

with $K = \int_0^1 [(3+\lambda)|\phi(s)| + (4+2\lambda)e^{\lambda s}] |X(s)|ds$. This shows that $\{u_{f_n} : n \in \mathbf{N}\}$ is equicontinuous in $C_E(I)$. Moreover, for each $t \in I$ the set $\{u_{f_n}(t) : n \in \mathbf{N}\}$, is

contained in the convex compact set $\int_0^1 G(t, s)X(s)ds$ [?, 19] so that \mathcal{X} is relatively compact in $C_E(I)$ as claimed. So, we can assume that

$$\lim_{n \rightarrow \infty} u_{f_n} = u_\infty \in C_E(I)$$

As S_X^1 is $\sigma(L_E^1, L_{E^*}^\infty)$ -compact e.g [19] we may assume that (f_n) $\sigma(L_E^1, L_{E^*}^\infty)$ -converges to $f_\infty \in S_X^1$. so that u_{f_n} weakly converges to u_{f_∞} in $C_E(I)$ where $u_{f_\infty}(t) = \int_0^1 G(t, s)f_\infty(s)ds$ and so for every $t \in I$,

$$u_\infty(t) = w\text{-}\lim_{n \rightarrow \infty} u_{f_n}(t) = w\text{-}\lim_{n \rightarrow \infty} \int_0^1 G(t, s)f_n(s)ds = \int_0^1 G(t, s)f_\infty(s)ds = u_{f_\infty}(t),$$

and

$$\begin{aligned} w\text{-}\lim_{n \rightarrow \infty} (D^{\alpha-1}u_{f_n})(t) &= w\text{-}\lim_{n \rightarrow \infty} \left[\int_0^t \exp(-\lambda(t-s))f_n(s)ds + \exp(-\lambda t) \int_0^1 \varphi(s)f_n(s)ds \right] \\ &= \int_0^t \exp(-\lambda(t-s))f_\infty(s)ds + \exp(-\lambda t) \int_0^1 \varphi(s)f_\infty(s)ds \\ &= (D^{\alpha-1}u_{f_\infty})(t), \quad t \in I. \end{aligned}$$

This means $u_\infty \in \mathcal{X}$, and the proof of the theorem is complete. \square

Theorem 3.6. *Let $I := [0, 1]$ and H be a separable Hilbert space. Let K be a convex compact subset of H . Let $\varphi : [0, T] \times K \rightarrow]-\infty, +\infty]$ be a normal lower semicontinuous convex integrand such that*

(i) $\{\varphi(\cdot, u(\cdot)), u \in S_K^1\}$ *is uniformly integrable.*

(ii) $\varphi(t, x) \leq \varphi(\tau, x) + |v(t) - v(\tau)|$ *for all $t, \tau \in [0, T], x \in K$ where $v : [0, T] \rightarrow \mathbf{R}^+$ is a positive nondecreasing continuous function with $v(0) = 0$.*

Let $A : H \rightarrow H$ be a linear continuous coercive symmetric operator and $B : H \rightarrow H$ be a linear continuous mapping.

Let $f : I \times H \rightarrow H$ be a bounded continuous mapping $\|f(t, x)\| \leq M$ for all $(t, x) \in [0, 1] \times H$. Then for any $u_0 \in H$, there exist a $W_{B, H}^{\alpha, 1}([0, 1])$ mapping $x : [0, 1] \rightarrow H$ and an absolutely continuous mappings $u : [0, 1] \rightarrow H$ satisfying

$$\begin{cases} u(0) = u_0 \in H \\ D^\alpha x(t) + \lambda D^{\alpha-1}x(t) = u(t), \quad t \in [0, 1] \\ I_{0+}^\beta x(t)|_{t=0} = 0, \quad x(1) = I_{0+}^\gamma x(1) \\ f(t, x(t)) + Bu(t) - A \frac{du}{dt}(t) \in \partial \varphi(t, \frac{du}{dt}(t)), \quad \text{a.e. } t \in [0, 1] \end{cases}$$

Proof. Let us consider the convex compact (cf Theorem 3.5) subset \mathcal{X} in the Banach space $\mathcal{C}_H([0, 1])$ defined by

$$\mathcal{X} := \{u_f : [0, 1] \rightarrow H : u_f(t) = \int_0^1 G(t, s)f(s)ds, \quad f \in S_{u_0+K}^1, \quad t \in [0, 1]\}$$

where $S_{u_0+K}^1$ denotes the set of all integrable selections of the convex compact valued constant multifunction $u_0 + K$. For each $h \in \mathcal{X}$, by Theorem 1.3 and the assumptions on f , there is a unique absolutely continuous solution v_h to the inclusion

$$\begin{cases} v_h(0) = u_0 \in H \\ f(t, h(t)) + Bv_h(t) - A \frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t)), \text{ a.e. } t \in [0, 1] \end{cases}$$

with $\frac{dv_h}{dt}(t) \in K$ a.e. so that $v_h(t) = u_0 + \int_0^t \frac{dv_h}{ds}(s)ds \in u_0 + \int_0^t K ds \subset u_0 + K, \forall t \in [0, 1]$.

Now for each $h \in \mathcal{X}$ let us consider the mapping defined by

$$\Phi(h)(t) := \int_0^t G(t, s)v_h(s)ds,$$

for $t \in [0, 1]$. Then it is clear that $\Phi(h) \in \mathcal{X}$. Now we check that Φ is continuous. It is sufficient to show that, if (h_n) uniformly converges to h in \mathcal{X} , then the absolutely continuous solution v_{h_n} associated with h_n

$$\begin{cases} v_{h_n}(0) = u_0 \in H \\ f(t, h_n(t)) + Bv_{h_n}(t) - A \frac{dv_{h_n}}{dt}(t) \in \partial\varphi(t, \frac{dv_{h_n}}{dt}(t)), \text{ a.e. } t \in [0, 1] \end{cases}$$

uniformly converges to the absolutely continuous solution v_h associated with h

$$\begin{cases} v_h(0) = u_0 \in H \\ f(t, h(t)) + Bv_h(t) - A \frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t)), \text{ a.e. } t \in [0, 1] \end{cases}$$

As (v_{h_n}) is equi-absolutely continuous with $v_{h_n}t \in u_0 + \int_0^t C(s)ds \subset u_0 + K, \forall t \in [0, 1]$ we may assume that (v_{h_n}) uniformly converges to an absolutely continuous mapping z . Since $v_{h_n}(t) = u_0 + \int_{]0,t]} \frac{dv_{h_n}}{ds}(s)ds, t \in [0, 1]$ and $\frac{dv_{h_n}}{ds}(s) \in K, \text{ a.e. } s \in [0, 1]$, we may assume that $(\frac{dv_{h_n}}{dt})$ weakly converges in $L_H^1([0, 1])$ to $w \in L_H^1[0, 1]$ with $w(t) \in K, t \in [0, 1]$ so that

$$\lim_n v_{h_n}(t) = u_0 + \int_0^t w(s)ds := u(t), t \in [0, T].$$

By identifying the limits, we get

$$u(t) = z(t) = u_0 + \int_0^t w(s)ds$$

with $\dot{u} = w$. Therefore by applying the arguments in the variational limit result as in Proposition 2.1-2.2 we get

$$f(t, h(t)) + Bu(t) - A \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)), \text{ a.e. } t \in [0, 1]$$

with $u(0) = u_0 \in H$, so that by uniqueness $u = v_h$. Since $h_n \rightarrow h$, we have

$$\begin{aligned}\Phi(h_n)(t) - \Phi(h)(t) &= \int_0^1 G(t, s)v_{h_n}(s)ds - \int_0^1 G(t, s)v_h(s)ds \\ &= \int_0^1 G(t, s)[v_{h_n}(s) - v_h(s)]ds \\ &\leq \int_0^1 M_G \|v_{h_n}(s) - v_h(s)\| ds\end{aligned}$$

As $\|v_{h_n}(\cdot) - v_h(\cdot)\| \rightarrow 0$ uniformly, by using Lemma 3.3(i) we conclude that

$$\sup_{t \in [0,1]} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 M_G \|v_{h_n}(\cdot) - v_h(\cdot)\| ds \rightarrow 0$$

so that $\Phi(h_n) \rightarrow \Phi(h)$ in $\mathcal{C}_H([0, 1])$. Since $\Phi : \mathcal{X} \rightarrow \mathcal{X}$ is continuous Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that

$$h(t) = \Phi(h)(t) = \int_0^1 G(t, s)v_h(s)ds,$$

with

$$\begin{cases} v_h(0) = u_0 \in H \\ D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = v_h(t), \quad t \in [0, 1] \\ I_{0+}^\beta h(t)|_{t=0} = 0, \quad h(1) = I_{0+}^\gamma h(1) \\ f(t, h(t)) + Bv_h(t) - A \frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t)), \quad \text{a.e. } t \in [0, 1] \end{cases}$$

The proof is complete. \square

Further variants of the above results are available. For instance, we are able to state the existence of solution to the coupled system

$$\begin{cases} D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = u(t), \quad t \in [0, 1] \\ I_{0+}^\beta h(t)|_{t=0} = 0, \quad h(1) = I_{0+}^\gamma h(1) \\ u(0) = u_0 \\ f(t, h(t)) + Bu(t) - A \frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t)), \quad \text{a.e. } t \in [0, 1] \end{cases}$$

Our tools allow to treat other variants by considering other class of FDI given in [10, 11, 12, 13, 14]. For shortness we omit the details.

References

- [1] S. Adly, T. Haddad and L. Thibault, Convex sweeping process in the framework of measure differential inclusions and evolution variational inequalities. *Math. Program*, 148(1-2, Ser. B), (2014) 5-47.

- [2] S. Adly and T. Haddad, An implicit sweeping process Approach to Quasistatic Evolution Variational inequalities, *Siam. J. Math. Anal.*, 2019
- [3] S. Adly, T. Haddad and Ba Khiet Le, State -dependent Implicit Sweeping process in the frame work of Quasi static Evolutions Quasi-Variational Inequalities, *Journal of Optimisation and Applications*, 2019.
- [4] D. Azzam-Laouir, C. Castaing and M.D.P. Monteiro Marques, Perturbed evolution problems with continuous bounded variation in time and applications. *Set-Valued Var. Anal.* DOI 10.1007/s11228-017-0432-9, (2017).
- [5] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces.* Noordhoff Int. Publ. Leyden, (1976).
- [6] V. Barbu and A. Rascanu, Parabolic variational inequalities with singular inputs. *Differential Integrals Equations* [10] (1997) 67- 83.
- [7] Aras Bacho, Etienne Emmrich, Alexander Mielke, submitted: April 2, 2018 An existence result and evolutionary γ -convergence for perturbed gradient systems, Weierstraß-Institut für Angewandte Analysis und Stochastik Leibniz-Institut im Forschungsverbund Berlin e. V. Preprint ISSN 2198-5855
- [8] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contraction dans un espace de Hilbert.* North Holland (1979).
- [9] C. Castaing. Topologie de la convergence uniforme sur les parties uniformément intégrables de L^1_E et théorèmes de compacité faible dans certains espaces du type Köthe-Orlicz. *Travaux Sémin. Anal. Convexe* , 10(1): exp. no. 5, 27, (1980).
- [10] C. Castaing, C. Godet-Thobie, L.X. Truong and F.Z Mostefai, On a fractional differential inclusion in Banach space under weak compactness condition, *Adv. Math. Econ.*, 20 (2016), 23-75.
- [11] C. Castaing, C. Godet-Thobie, T. Le Xuan, B. Satco, Optimal control problems governed by a second order ordinary differential equation with m -point boundary condition, *Adv. Math. Econ.*, 18 (2014), 1-59.
- [12] C. Castaing, C. Godet-Thobie and L.X. Truong, Fractional order of evolution inclusion coupled with a time and state dependent maximal monotone operator, *Mathematics MDPI*, (2020), 1-30.
- [13] C. Castaing, C. Godet-Thobie, F.Z. Mostefai, On a fractional differential inclusion with boundary conditions and application to subdifferential operators, *J. Nonlinear Convex Anal.*, 18(9) (2017), 1717-1752.

- [14] C. Castaing, C. Godet-Thobie, P.D. Phung, L.X. Truong, On fractional differential inclusions with nonlocal boundary conditions, *Fract. Calc. Appl. Anal.*, 22(2) 2019, 444-478.
- [15] C. Castaing, M. M. Marques and P. Raynaud de Fitte, A Skorohod problem governed by a closed convex moving set. *Journal of Convex Analysis*, 23 No 2, (2016) 387-423.
- [16] C. Castaing, M. M. Marques and S. Saidi, Evolution problems involving time dependent subdifferential operators, *Adv. Math. Econ.*, 23 (2020), 1-39.
- [17] C. Castaing, N. Marie and P. Raynaud de Fitte, Sweeping Process Perturbed by Rough Signal, 2018 hal-01738241v1
- [18] C. Castaing, P. Raynaud de Fitte and M. Valadier, Young measures on topological spaces with applications in control theory and probability theory. Kluwer Academic Publishers, Dordrecht (2004).
- [19] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics. Springer-Verlag, Berlin, 580, (1977).
- [20] P. Friz and N. Victoir. *Multidimensional Stochastic Processes as Rough Paths*. Cambridge Studies in Advanced Mathematics, 120. Cambridge University Press, 2010.
- [21] A. Grothendieck, *Espaces Vectoriels Topologiques* Mat. São Paulo, São Paulo, 3rd edn . Publ. Soc. (1964).
- [22] L.Maticiuc, A. Rascanu, L. Slominski and M.Topolewski, Cadlag Skorohod problem driven by Maximal monotone operator, *JMAA*, Article in Press
- [23] Miller, K.S., Ross, B.: *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Willey, NewYork (1993)
- [24] S. Migorski, M. Sofonea and S. Zeng, Well-Posedness of history dependent sweeping process, *Siam. J. Math. Anal.*, Vol 51, No 2, 1082-1107(2019).
- [25] M. D. P. Monteiro Marques, Differential inclusions nonsmooth mechanical problems, shocks and dry friction. *Progress in Nonlinear Differential Equations and Their Applications*, Birkhauser, Vol 9 (1993).
- [26] J. J. Moreau, Evolution problem associated with a moving convex set in a Hilbert space. *J. Differential Equations*, Vol 26 (1977) 347-374.
- [27] J. J. Moreau and M. Valadier, A chain rule involving vector functions of bounded variations, *J. Funct. Anal.*, Vol 74, No 2, (1987) 333-345.

- [28] J.C. Peralba, Équations d'évolution dans un espace de Hilbert, associées à des opérateurs sous-différentiels, Thèse de doctorat de spécialité, Montpellier, 1973.
- [29] P.H Phung and L.X Truong, *On a fractional differential inclusion with integral boundary conditions in Banach space*, Fract.Calc.Appl.Anal., Volume 16, No 3 (2013) 538-558.
- [30] A. Rascanu, Deterministic and Stochastic Differential Equations in Hilbert Spaces Involving Multivalued Maximal Monotone Operators, Panamer. Math. J. 6 (1996) no 3, 83-119.
- [31] O'Regan, D.: Fixed point theorem for weakly sequentially closed maps, Arch. Math. (Brno) **36**, 61–70 (2000)