Unique continuation estimates for Baouendi–Grushin equations on cylinders

Paul Alphonse, Albrecht Seelmann

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PAUL ALPHONSE AND ALBRECHT SEELMANN

Abstract. We prove time-pointwise quantitative unique continuation estimates for the evolution operators associated to (fractional powers of) the Baouendi–Grushin operators on the cylinder $\mathbb{R}^d \times \mathbb{T}^d$. Corresponding spectral inequalities, relating for functions from spectral subspaces associated to finite energy intervals their L^2-norm on the whole cylinder to the L^2-norm on a suitable subset, and results on exact and approximate null-controllabilty are deduced. This extends and complements results obtained recently by the authors and by Jaming and Wang.

1. Introduction

The present paper continues our considerations from [6] on spectral inequalities for Shubin operators on $\mathbb{R}^d$ and extends them to Baouendi–Grushin operators on the cylinder $\mathbb{R}^d \times \mathbb{T}^d$, that is, to the operators

$$\Delta_\gamma = \Delta_x + |x|^{2\gamma}\Delta_y, \quad (x,y) \in \mathbb{R}^d \times \mathbb{T}^d, \tag{1.1}$$

where $\gamma \geq 1$ is a positive integer. Here, we consistently associate with x the coordinates in $\mathbb{R}^d$ and with y the coordinates in the d-dimensional torus $\mathbb{T}^d$, and Δ_y denotes the Laplacian on $\mathbb{T}^d$. Analogously to [6], we aim at establishing so-called *spectral inequalities* of the form

$$\forall \lambda \geq 0,\ \forall f \in \mathcal{E}_\lambda(-\Delta_\gamma), \quad \|f\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq c_{\lambda,\omega}\|f\|_{L^2(\omega)},$$

where $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ is a measurable subset with positive measure whose geometry is to be understood and $c_{\lambda,\omega} > 0$ is a positive constant whose dependence on ω and $\lambda > 0$ is to be tracked. We refer to [6] and the references cited therein for an overview on the notion of a spectral inequality and corresponding works on related models.

The core of our considerations here are *time-pointwise quantitative unique continuation estimates* for the evolution operators associated to (a fractional power of) the Baouendi–Grushin operators (1.1), which are of the form

$$\|e^{-t(-\Delta_\gamma)^{\frac{1+\gamma}{2}}} g\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq C_{\varepsilon,\omega,t}\|e^{-t(-\Delta_\gamma)^{\frac{1+\gamma}{2}}} g\|^2_{L^2(\omega)} + \varepsilon\|g\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \tag{1.2}$$

for $t > 0$, $\varepsilon > 0$ and $g \in L^2(\mathbb{R}^d\times\mathbb{T}^d)$. Such estimates allow to deduce spectral inequalities of the form we are looking for and, as a consequence, imply positive exact null-controllability results for the heat-like equations associated to the fractional power $(-\Delta_\gamma)^s$ in the regime $s > (1+\gamma)/2$. We also deduce from the estimates (1.2) approximate null-controllability results in the critical case $s = (1+\gamma)/2$ that would otherwise not be available. Our proof of such estimates (1.2) essentially follows the technique established in [33], which is in turn based on [29, 30], and relies on so-called smoothing estimates for the evolution operators. Here, we can exploit the fact that via a Fourier expansion with respect to the y variable the operators Δ_γ are linked to the anharmonic oscillators

$$-\Delta_x + |n|^2|x|^{2\gamma}, \quad x \in \mathbb{R}^d,\ n \in \mathbb{Z}^d\,.$$

An analogous relation has been used in the recent work [27], which among others studies exact null-controllability for the heat-like equations associated to operators of the same form as (1.1), but with $|x|^{2\gamma}$ replaced by a more general potential $V(x)$ exhibiting a certain

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power growth. In our case with the specific potential $|x|^{2\gamma}$ we have much more regularity available and can make use of smoothing estimates for the evolution operators associated to fractional powers of the anharmonic oscillators $-\Delta_x + |x|^{2\gamma}$ established in [3]. As a consequence, compared with [27], our sensor sets $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ are more general in two respects: (i) we are not restricted to sets of the form $\omega' \times \mathbb{T}^d$ with suitable $\omega' \subset \mathbb{R}^d$, and (ii) our sets ω do not need to have suitably distributed inner points.

Outline of the work. In Section 2, we present in detail the main results in the current work and discuss applications in the context of control theory. Section 3 then provides a general scheme towards unique continuation estimates that is of its own interest and hopefully proves useful also in future work on the subject. This is complemented by corresponding smoothing estimates for the evolution operators in Section 4, which serve as an input for the general scheme from Section 3. The proofs of the main results stated in Section 2 are then collected in Section 5. Section 6 demonstrates the usefulness of the general scheme from Section 3 by complementing our work [6] on Shubin operators $(-\Delta)^m + |x|^{2k}$ on $\mathbb{R}^d$, $k, m \in \mathbb{N}^*$, with corresponding unique continuation estimates. In Appendices A and B we finally discuss some technical results used for the proofs of our statements on (linear combinations of) eigenfunctions and the optimality of the smoothing estimates, respectively.

Notations. The following notations and conventions will be used throughout this work:

1. $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and nonnegative integers, respectively. Moreover, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.
2. $\mathbb{T} \cong \mathbb{R}/(2\pi\mathbb{Z})$ denotes the torus.
3. The canonical Euclidean scalar product of $\mathbb{R}^d$ is denoted by $\cdot$, and $|\cdot|$ stands for the associated canonical Euclidean norm. We will also use the notation $|\cdot|_1$ for the 1-norm of vectors in $\mathbb{R}^d$.
4. The length of a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ is denoted $|\alpha|$ and defined by
$$|\alpha| = |\alpha|_1 = \alpha_1 + \cdots + \alpha_d.$$
5. The Lebesgue measure of a measurable set $\omega \subset \mathbb{R}^d$ is denoted $|\omega|$.
6. 1_ω denotes the characteristic function of any subset $\omega \subset \mathbb{R}^d$.
7. For every measurable subset $\omega \subset \mathbb{R}^d$, the inner product of $L^2(\omega)$ is denoted $\langle \cdot, \cdot \rangle_{L^2(\omega)}$, while $\| \cdot \|_{L^2(\omega)}$ stands for the associated norm.
8. The Fourier transform of a function $f \in L^2(\mathbb{R}^d)$ is denoted $\mathcal{F}f$ or $\widehat{f}$, depending on the situation, and is given for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by
$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} \, \mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$
With this convention, Plancherel's theorem states that
$$\forall f \in L^2(\mathbb{R}^d), \quad \|\widehat{f}\|_{L^2(\mathbb{R}^d)} = (2\pi)^{d/2} \|f\|_{L^2(\mathbb{R}^d)}.$$
9. For a nonnegative selfadjoint operator A on $L^2(\mathbb{R}^d)$, $\mathcal{E}_\lambda(A) = 1_{(-\infty,\lambda]}(A)$ with $\lambda \geq 0$ denotes the spectral subspace for A associated with the interval $(-\infty, \lambda]$.

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2. Statement of the main results

This section is devoted to present the main results contained in this paper.

2.1. **Thickness in the cylinder.** First of all, let us define the notion of thickness in the cylinder $\mathbb{R}^d\times\mathbb{T}^d$, which is central in this work. To that end, given $L_x, L_y>0$, we introduce in $\mathbb{R}^{2d}=\mathbb{R}^d\times\mathbb{R}^d$ the hyperrectangle $\Lambda_{L_x,L_y}:=(0,L_x)^d\times(0,L_y)^d$. The notion of thickness is then defined as follows.

Definition 2.1. A measurable set $\omega\subset\mathbb{R}^d\times\mathbb{T}^d$ is said to be (θ,L_x,L_y)-*thick in* $\mathbb{R}^d\times\mathbb{T}^d$ with some $\theta\in(0,1]$, $L_x>0$, and $L_y\in(0,2\pi]$ if

$$\forall z\in\mathbb{R}^{2d},\quad |\omega\cap(z+\Lambda_{L_x,L_y})|\geq\theta L_x^dL_y^d.$$

Remark 2.2. This is a common extension of the classical notion of thickness in $\mathbb{R}^{2d}$, cf., for instance, [22, 23, 24]. In this regard, let us recall that a measurable set $\omega\subset\mathbb{R}^{2d}$ is called (θ,L)-*thick* with some $\theta\in(0,1]$ and $L>0$ if

$$\forall z\in\mathbb{R}^{2d},\quad |\omega\cap(z+(0,L)^{2d})|\geq\theta L^{2d}. \tag{2.1}$$

The main result in this paper is the following statement, which provides quantitative unique continuation estimates for the evolution operators generated by the fractional Baouendi–Grushin operator $(-\Delta_\gamma)^s$ in the critical regime $s=(1+\gamma)/2$.

Theorem 2.3. *There exist constants $K>0$ and $t_0\in(0,1)$, depending only on γ and the dimension d, such that for every (θ,L_x,L_y)-thick set ω in $\mathbb{R}^d\times\mathbb{T}^d$ and all $t\in(0,t_0)$, $\varepsilon>0$, and $g\in L^2(\mathbb{R}^d\times\mathbb{T}^d)$, we have*

$$\|e^{-t(-\Delta_\gamma)^{\frac{1+\gamma}{2}}}g\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}\leq\left(\frac{K}{\theta}\right)^{KC_{\varepsilon,\gamma,t}}\|e^{-t(-\Delta_\gamma)^{\frac{1+\gamma}{2}}}g\|^2_{L^2(\omega)}+\varepsilon\|g\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}$$

with

$$C_{\varepsilon,\gamma,t}=(1-\log\varepsilon-\log t)\exp\left(\frac{KL_x}{t^{\frac{\gamma}{\gamma+1}}}+\frac{KL_y}{t}\right).$$

In the particular case of $\gamma=1$, the same holds even for all positive times $t>0$, and the term $-\log t$ in the constant $C_{\varepsilon,\gamma,t}$ can be skipped.

As a corollary to Theorem 2.3 and its proof, we obtain the following spectral inequality.

Corollary 2.4. *Given $L_x>0$ and $L_y\in(0,1]$, there exists a constant $K>0$, depending only on d, γ, L_x, and L_y, such that for every (θ,L_x,L_y)-thick set ω in $\mathbb{R}^d\times\mathbb{T}^d$, every $\lambda\geq0$, and all $f\in\mathcal{E}_\lambda(-\Delta_\gamma)$ we have*

$$\|f\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}\leq\left(\frac{K}{\theta}\right)^{K(1+\lambda^{\frac{1+\gamma}{2}})}\|f\|^2_{L^2(\omega)}.$$

In the particular case of $\gamma=1$, this estimate can be strengthened as

$$\|f\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}\leq\left(\frac{K}{\theta}\right)^{K(1+L_y\lambda+L_x^2\lambda)}\|f\|^2_{L^2(\omega)}$$

with a constant $K>0$ depending only on the dimension d.

Remark 2.5. (1) Considering formally $\gamma=0$ in Corollary 2.4, we are in the situation of the pure Laplacian acting on $\mathbb{R}^d\times\mathbb{T}^d$. Our spectral inequality is then consistent with established results for that case, see [22] and also [23, Corollary 1.6], but without a quantitative description of the dependence on the parameters L_x and L_y.

(2) To the best of the authors knowledge, the above spectral inequalities are the first ones for the Baouendi–Grushin operator Δ_γ on $\mathbb{R}^d\times\mathbb{T}^d$. However, uncertainty relations for single eigenfunctions have been obtained in the literature for the very same operator considered on compact manifolds. Corresponding results from the work [31], which actually deal with more general type I Hörmander operators, are briefly discussed in Remark 2.17 below.

Let us now discuss the optimality of the spectral inequalities for the general Baouendi–Grushin operators Δ_γ, beginning with the geometry of the sensor set $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$. We first prove that this sensor set has to be thick in order for a spectral inequality to hold.

Proposition 2.6. *Let $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ be a measurable set with positive measure and $\lambda \geq 0$ be a non-negative energy level. If for some constant $c_{\lambda,\omega} > 0$ we have the spectral inequality*

$$\forall f \in \mathcal{E}_\lambda(-\Delta_\gamma), \quad \|f\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq c_{\lambda,\omega}\|f\|_{L^2(\omega)},$$

then the set ω is thick in $\mathbb{R}^d \times \mathbb{T}^d$.

Let us now focus on the power in the spectral inequalities stated in Corollary 2.4. The following result, which is exactly the counterpart of [31, Proposition 1.12] for Baouendi–Grushin operators acting on the cylinder $\mathbb{R}^d \times \mathbb{T}^d$ (and not on a compact manifold as in the work [31]), shows that the power $(1+\gamma)/2$ appearing in Corollary 2.4 is optimal for sensor sets $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ avoiding the critical line $\{x = 0\}$.

Proposition 2.7. *Let $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ be a measurable set satisfying the geometric condition $\overline{\omega} \cap \{x = 0\} = \emptyset$. Then, there exist positive constants $c_0, c_1 > 0$ and a sequence $((\lambda_n, \psi_n))_n$ of eigenpairs of the high-order Baouendi–Grushin operator $-\Delta_\gamma$ with $\lambda_n \to +\infty$ such that for each n we have*

$$\|\psi_n\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \geq c_0 e^{c_1 \lambda_n^{\frac{1+\gamma}{2}}} \|\psi_n\|_{L^2(\omega)}.$$

2.2. **Applications to control theory.** We now apply the unique continuation estimates from Theorem 2.3 and the spectral inequalities from Corollary 2.4 to the study of the null-controllability of the evolution equations associated the fractional Baouendi–Grushin operators. More precisely, given a positive real number $s > 0$, we consider the heat-like evolution equation

$$(E_{\gamma,s}) \qquad \begin{cases} \partial_t f(t,x,y) + (-\Delta_\gamma)^s f(t,x,y) = h(t,x,y)1_\omega(x,y), & t > 0,\ (x,y) \in \mathbb{R}^d \times \mathbb{T}^d, \\ f(0,\cdot,\cdot) = f_0 \in L^2(\mathbb{R}^d \times \mathbb{T}^d), \end{cases}$$

where $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ is a control support and $h \in L^2((0,T)\times\omega)$ is a control. We focus on two notions of null-controllability and present a result for each of them. The results presented in this subsection are in line with articles devoted to the study of the null-controllability of Baouendi–Grushin heat equations. A pioneering article in this theory is [7], which paved the way for a numerous series of articles of which we can cite [2, 8, 10, 15, 20, 27, 28].

2.2.1. *Exact null-controllability.* We first study the exact null-controllability properties of the equation $(E_{\gamma,s})$ and begin with recalling the corresponding notion.

Definition 2.8. The evolution equation $(E_{\gamma,s})$ is said to be *exactly null-controllable from the control support ω in time $T > 0$* if for every initial datum $f_0 \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ there exists a control function $h \in L^2((0,T) \times \mathbb{R}^d \times \mathbb{T}^d)$ such that the mild solution to $(E_{\gamma,s})$ satisfies $f(T,\cdot) = 0$.

According to the Hilbert Uniqueness Method, the exact null-controllability of the equation $(E_{\gamma,s})$ from $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ at time $T > 0$ is equivalent to the existence of a positive constant $C_{\omega,T} > 0$ such that for all $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$,

$$\left\|e^{-T(-\Delta_\gamma)^s} g\right\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq C_{\omega,T} \int_0^T \left\|e^{-t(-\Delta_\gamma)^s} g\right\|^2_{L^2(\omega)} \mathrm{d}t. \tag{2.2}$$

The latter is called an exact observability estimate for the semigroup $(e^{-t(-\Delta_\gamma)^s})_{t\geq 0}$. Several results in the literature allow to derive exact observability estimates from spectral inequalities. Most of them are proven by using the well-known Lebeau–Robbiano strategy. We do not give a precise statement of one of those results here, but just mention that the estimate (2.2) holds once a spectral inequality of the following form has been established, where $d_0 > 0$, $d_1 \geq 0$ and $\eta \in (0, s)$:

$$\forall f \in \mathcal{E}_\lambda(-\Delta_\gamma), \quad \|f\|_{L^2(\mathbb{R}^d)} \leq d_0 e^{d_1 \lambda^\eta} \|f\|_{L^2(\omega)}.$$

We refer the reader to [37, Theorem 2.8] for a precise statement particularly well adapted towards an explicit form of the constant $C_{\omega,T}$ in terms of the parameters d_0, d_1, and η. We thus deduce from Corollary 2.4 the following exact null-controllability result in the high dissipation regime $s > (1+\gamma)/2$.

Theorem 2.9. *Suppose that $s > (1+\gamma)/2$, and let $T > 0$ and $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ be a measurable set with positive measure. The following assertions are equivalent:*

(i) The equation $(E_{\gamma,s})$ is exactly null-controllable from ω in time T.
(ii) The set ω is thick in $\mathbb{R}^d \times \mathbb{T}^d$.

Remark 2.10. (1) The implication $(i) \Rightarrow (ii)$ has already been proven in [6, Proposition 2.14]. Moreover, the converse $(ii) \Rightarrow (i)$ strengthens Theorem 2.15 in [6], which states that for every thick set $\omega \subset \mathbb{R}^d$ (cf. Remark 2.19), the equation (2.2) is exactly null-controllable in every positive time $T > 0$ from the sensor set $\omega \times \mathbb{T}^d$. Note that the set $\omega \times \mathbb{T}^d$ is then thick in $\mathbb{R}^d \times \mathbb{T}^d$ in the sense of Definition 2.1.

(2) The exact null-controllability properties of the equation $(E_{\gamma,s})$ when $0 < s \leq (1+\gamma)/2$ were investigated in [6, Theorem 2.17, Theorem 2.19]. Let us briefly recall the content of those results, starting with the case where a control support $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ satisfying the geometric condition $\overline{\omega} \cap \{x = 0\} = \emptyset$ is considered. On the one hand, the equation $(E_{\gamma,s})$ is then never exactly null-controllable from ω when $0 < s < (1+\gamma)/2$. On the other hand, in the critical case $s = (1+\gamma)/2$, the equation $(E_{\gamma,s})$ is not exactly null-controllable from ω at any time $T < T_*$, where $T_* > 0$ is a positive time explicitly given in terms of γ and the distance from the origin to ω. Moreover, still in the situation $s = (1+\gamma)/2$, the equation $(E_{\gamma,s})$ is exactly null-controllable in every large enough positive time $T > T_*$ from every control support of the form $\omega \times \mathbb{T}^d$ with a thick set $\omega \subset \mathbb{R}^d$.

2.2.2. *Cost-uniform approximate null-controllability.* We are now interested in the concept of approximate null-controllability with uniform cost for the equation $(E_{\gamma,s})$, which is defined as follows.

Definition 2.11. The evolution equation $(E_{\gamma,s})$ is said to be *approximately null-controllable with uniform cost* from the control support ω in time $T > 0$ if for all $\varepsilon > 0$, there exists a positive constant $C_{\varepsilon,s,\omega,T} > 0$ such that for all $f_0 \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$, there exists a control $h \in L^2((0,T) \times \omega)$ such that the mild solution of $(E_{\gamma,s})$ satisfies

$$\|f(T,\cdot)\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq \varepsilon \|f_0\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)},$$

with moreover

$$\frac{1}{T} \int_0^T \|h(t,\cdot)\|^2_{L^2(\omega)} \, \mathrm{d}t \leq C_{\varepsilon,s,\omega,T} \|f_0\|^2_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}. \tag{2.3}$$

Just as the Hilbert Uniqueness Method gives a dual interpretation of the notion of exact null-controllability in terms of exact observability inequalities, there is also a dual interpretation of the notion of cost-uniform approximate null-controllability in terms of weak observability inequalities. This is the purpose of the following result, taken from the work [41].

Proposition 2.12 (Proposition 6 in [41])**.** *The evolution equation $(E_{\gamma,s})$ is cost-uniformly approximately null-controllable from the control support ω in time $T > 0$ if for all $\varepsilon \in (0,1)$, there exists a positive constant $C_{\varepsilon,s,\omega,T} > 0$ such that for all $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$,*

$$\left\| e^{-T(-\Delta_\gamma)^s} g \right\|^2_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq \frac{C_{\varepsilon,s,\omega,T}}{T} \int_0^T \left\| e^{-t(-\Delta_\gamma)^s} g \right\|^2_{L^2(\omega)} \mathrm{d}t + \varepsilon \|g\|^2_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}. \tag{2.4}$$

Moreover, every constant $C_{\varepsilon,s,\omega,T} > 0$ satisfying (2.4) also satisfies (2.3) and vice versa.

Our result of cost-uniform approximate null-controllability for the equation $(E_{\gamma,s})$ is the following one.

Theorem 2.13. *Suppose that $s \geq (1+\gamma)/2$, and let $T > 0$ and $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ be a measurable set with positive measure. The following assertions are equivalent:*

(i) The equation ($E_{\gamma,s}$) is cost-uniformly approximately null-controllable from ω in time T.

(ii) The set ω is thick in $\mathbb{R}^d \times \mathbb{T}^d$.

Moreover, in the critical dissipation setting $s = (1+\gamma)/2$, for every (θ, L_x, L_y)-thick set $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ with $\theta \in (0,1]$ and $L > 0$, the control cost $C_{\varepsilon,s,\omega,T} > 0$ appearing in (2.3) satisfies

$$C_{\varepsilon,\gamma,(1+\gamma)/2,\omega,T} \leq \left(\frac{K}{\theta}\right)^{K(1-\log\varepsilon-\log T)\exp(KL_x/T^{\frac{\gamma}{1+\gamma}}+KL_y/T)},$$

where $K > 0$ is a positive constant only depending on γ and the dimension d. In the particular case of $\gamma = 1$, the term $-\log T$ in the exponent of the right-hand side of the latter inequality can be skipped.

Remark 2.14. Notice that the critical case $s = s_\gamma = (1+\gamma)/2$ is allowed in Theorem 2.13, whereas it is excluded in Theorem 2.9, which is not surprising given the results recalled in Remark 2.10. More precisely, it is known from [6, Theorem 2.17] that the equation (E_{γ,s_γ}) is not exactly null-controllable in every positive time $T > 0$ from thick control supports $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ in general. However, Theorem 2.13 shows that the equation (E_{γ,s_γ}) is cost-uniformly approximately null-controllable from every thick control support $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ and in every positive time $T > 0$, with a control cost $C_{\varepsilon,s_\gamma,\omega,T}$ enjoying a polynomial behavior with respect to the parameter $\varepsilon > 0$.

Investigating the cost-uniform approximate null-controllability properties of the equation ($E_{\gamma,s}$) in the regime $0 < s < (1+\gamma)/2$ is an open and very interesting problem that will not be tackled in the present paper.

Remark 2.15. Let us briefly explain how Theorem 2.13 can be deduced from all the previous results.

(1) The implication $(ii) \Rightarrow (i)$ in the case $s > (1+\gamma)/2$ is a consequence of Theorem 2.9 since the notion of exact null-controllability is stronger than the notion of cost-uniform approximate null-controllability.

(2) The implication $(ii) \Rightarrow (i)$ in the case $s = (1+\gamma)/2$ is obtained from the above Proposition 2.12 by integrating the quantitative unique continuation properties in Theorem 2.3 with respect to t. Indeed, since the evolution operators $e^{-t(-\Delta_\gamma)^s}$ are contractions on $L^2(\mathbb{R}^d \times \mathbb{T}^d)$, we have

$$\begin{aligned}\left\|e^{-T(-\Delta_\gamma)^s} g\right\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} &= \frac{1}{T}\int_0^T \left\|e^{-T(-\Delta_\gamma)^s} g\right\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \mathrm{d}t \\ &\leq \frac{1}{T}\int_0^T \left\|e^{-t(-\Delta_\gamma)^s} g\right\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \mathrm{d}t.\end{aligned}$$

(3) The implication $(i) \Rightarrow (ii)$ can be proved by proceeding similarly as in [5, Theorem 2.1 (i)]. We omit the details here.

2.3. **Eigenfunctions.** Let us finally present uncertainty relations for (linear combinations of) eigenfunctions of the operator Δ_γ. We refer to Section 4.1.3 where a precise description of this countable family of eigenfunctions and the associated eigenvalues is given. The first result concerns single eigenfunctions.

Theorem 2.16. *Let $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ be a measurable set with positive measure. Then, there exists a positive constant $K > 0$, depending only on ω, γ, and the dimension d, such that for every eigenfunction $\psi \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ of the operator Δ_γ with corresponding eigenvalue $\lambda > 0$, we have*

$$\|\psi\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq K e^{K\lambda^{\frac{1+\gamma}{2}}\log(1+\lambda)} \|\psi\|^2_{L^2(\omega)}.$$

Remark 2.17. As already mentioned in Remark 2.5, uncertainty relations were obtained for the eigenfunctions of the Baouendi–Grushin operator Δ_γ acting on a compact manifold $\mathcal{M}$ (rigorously, of a variant of this operator with analytic coefficients). More precisely, given an open subset $\omega \subset \mathcal{M}$, we get from [31, Theorem 1.10] the uncertainty relation

$$\|\psi\|^2_{L^2(\mathcal{M})} \leq Ke^{K\lambda^{\frac{1+\gamma}{2}}} \|\psi\|^2_{L^2(\omega)},$$

where ψ is an eigenfunction of the operator Δ_γ associated with the eigenvalue $\lambda > 0$. In fact, the result [31, Theorem 1.10] holds more generally with type I Hörmander operators. It is worth to note that the power $(1+\gamma)/2$ here is essentially consistent with the one stated in Theorem 2.16. An interpretation of the additional $\log(1+\lambda)$ term in Theorem 2.16 is given in Remark 2.19 below.

Our second result in this section concerns linear combinations of eigenfunctions of the operator Δ_γ. For any non-negative energy level $\lambda \geq 0$, let us consider the subspace

$$E_\lambda(-\Delta_\gamma) = \operatorname{span}\left\{\psi \in \mathcal{E}_\lambda(-\Delta_\gamma) \colon\ -\Delta_\gamma \psi = \mu\psi,\ \mu \in [0,\lambda]\right\} \subset \mathcal{E}_\lambda(-\Delta_\gamma). \tag{2.5}$$

Notice that the spectral subspace $\mathcal{E}_\lambda(-\Delta_\gamma)$ does not coincide with the above subspace $E_\lambda(-\Delta_\gamma)$ since the spectrum of the operator Δ_γ is not purely discrete. We obtain the following uncertainty relations for functions in $E_\lambda(-\Delta_\gamma)$.

Theorem 2.18. *Let $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ be a measurable set with positive measure. Then, there exists a positive constant $K > 0$, depending only on ω, γ, and the dimension d, such that for all $\lambda \geq 0$ and all $f \in E_\lambda(-\Delta_\gamma)$,*

$$\|f\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq Ke^{K\lambda^{(\frac{1}{2}+\frac{1}{2\gamma})(1+\gamma)}\log(1+\lambda)} \|f\|^2_{L^2(\omega)}.$$

Remark 2.19. Notice that Theorem 2.16 and Theorem 2.18 allow to consider sensor sets $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ with merely positive measure, whereas Corollary 2.4 requires the sensor set to be thick in the cylinder $\mathbb{R}^d \times \mathbb{T}^d$. This is due to the fact that the eigenfunctions of the operator Δ_γ enjoy localizing properties, in contrast to general functions in the spectral subspaces $\mathcal{E}_\lambda(-\Delta_\gamma)$. More precisely, as becomes clear in the proof of Theorem 2.18, there is a constant $c > 0$, depending only on γ and the dimension d, such that the functions $f \in E_\lambda(-\Delta_\gamma)$ are localized in the sense

$$\|f\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq \|f\|_{L^2(B(0,c\lambda^{1/2\gamma})\times\mathbb{T}^d)}.$$

It is therefore sufficient to prove for functions in $E_\lambda(-\Delta_\gamma)$ estimates on the finite cylinder $B(0, c\lambda^{1/2\gamma}) \times \mathbb{T}^d$ in order to obtain estimates on the whole cylinder $\mathbb{R}^d \times \mathbb{T}^d$. Moreover, as discussed in Section 5.3, the $\log(1+\lambda)$ term appearing in Theorems 2.16 and 2.18 is linked to the volume of this finite cylinder $B(0, c\lambda^{1/2\gamma}) \times \mathbb{T}^d$.

Remark 2.20. (1) Notice that the two powers in Theorems 2.16 and 2.18 are different, and that the one for single eigenfunctions in Theorem 2.16 is stronger. This suggests that there are cancellation phenomena, just as we already know for linear combinations of eigenfunctions of the Laplacian on the torus, see, e.g., Turan's lemma in [25, 36].

(2) Let us now analyse the power in the spectral inequalities of Theorem 2.18. On the one hand, the term $1/2 + 1/(2\gamma)$ is familiar from the literature since it also plays a role in the spectral inequalities for linear combinations of eigenfunctions of the anharmonic oscillator $H_\gamma = -\Delta + |x|^{2\gamma}$, which according to [35, Theorem 2.1 (ii)] read

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq Ke^{K\lambda^{\frac{1}{2}+\frac{1}{2\gamma}}\log(1+\lambda)} \|f\|^2_{L^2(\omega)}, \quad f \in \mathcal{E}_\lambda(H_\gamma),\ \lambda > 0. \tag{2.6}$$

Here, the operator H_γ is linked to the Baouendi–Grushin operator by a diagonalization with respect to the y variable, see the similarity relation (4.4) below. Moreover, the term $1/2+1/(2\gamma)$ can be interpreted as the (inverse of the) order of the anharmonic oscillator H_γ in a class of pseudo-differential operators associated with a Hörmander metric, as proved in the paper [14, Corollary 3.8]. On the other hand, the term $1+\gamma$ is the hypoellipticity index of the Baouendi–Grushin operator Δ_γ, see, e.g., [31, Example 1.7].

Remark 2.21. Let us consider the Baouendi–Grushin operator

$$\Delta_\gamma^{bd} = \Delta_x + |x|^{2\gamma}\Delta_y, \quad (x,y) \in \mathbb{R}^d \times (0,2\pi)^d,$$

where Δ_y denotes in this case the Laplacian on the hypercube $(0,2\pi)^d$ with Dirichlet boundary conditions. The spectrum of the operator $-\Delta_\gamma^{bd}$ is now purely discrete, see Section 4.4. As a consequence, the spectral subspaces $\mathcal{E}_\lambda(-\Delta_\gamma^{bd})$ consist exclusively of linear combinations of eigenfunctions of the operator Δ_γ^{bd}, that is, the spaces $\mathcal{E}_\lambda(-\Delta_\gamma^{bd})$ and $E_\lambda(-\Delta_\gamma^{bd})$ coincide. Corresponding variants of Theorems 2.16 and 2.18, which are still valid for the operator Δ_γ^{bd} as explained in Section 4.4, therefore deal with the complete spectral subspaces $\mathcal{E}_\lambda(-\Delta_\gamma^{bd})$. This has ramifications on the null-controllability properties of the evolution equations associated to the operator $-\Delta_\gamma^{bd}$, but we do not pursue this further here.

3. General unique continuation estimates

In this section, we provide a general framework towards unique continuation estimates for functions satisfying suitable smoothing estimates. Here, we follow the general strategy from the classical approach by Kovrijkine [29, 30], which has been exploited, adapted, and generalized in several recent works such as [6, 9, 17, 18, 23, 24, 33, 34, 42]. We focus specifically on [33] and aim for a ready-made tool box to be applied in various different situations. This tool box will be formulated explicitly for domains in $\mathbb{R}^d$, but it applies, of course, also to domains in $\mathbb{R}^{2d}$ like the strip $\mathbb{R}^d \times (0,2\pi)^d$, which corresponds to the cylinder $\mathbb{R}^d \times \mathbb{T}^d$ from Section 2. The strategy itself relies on a suitable covering of the underlying domain and a local estimate on a sufficiently large subclass of the covering sets, so-called good elements of the covering. The local estimate is presented in Section 3.1, whereas the covering strategy is discussed in Sections 3.2 and 3.3. The general scheme is summarized in Corollary 3.6 and demonstrated in several examples in Section 3.3.

3.1. A local estimate.

Let us begin with the discussion of the local estimate. To this end, we use the following result, which is a straightforward adaptation of Example 5.11 in [33]; cf. also Lemma 5.4 and Proposition 5.10 there.

Proposition 3.1. *Let $\mathcal{Q} \subset \mathbb{R}^d$ be a non-empty bounded convex open set, and let $\mathcal{E} \subset \mathcal{Q}$ be measurable with positive measure. Suppose that $g \in C^\infty(\mathcal{Q}) \setminus \{0\}$ satisfies*

$$\forall \alpha \in \mathbb{N}^d, \quad \|\partial^\alpha g\|_{L^\infty(\mathcal{Q})} \leq AB^\alpha(|\alpha|!)^\mu$$

with some $A > 0$, $B \in (0,\infty)^d$, and $\mu \in [0,1]$. Then, there is a constant $K = K(d,\mu) > 0$, only depending on μ and the dimension d, such that

$$\|g\|_{L^2(\mathcal{Q})} \leq \left(\frac{K|\mathcal{Q}|}{|\mathcal{E}|}\right)^{KC_\mu} \|g\|_{L^2(\mathcal{E})}$$

with

$$C_\mu = \begin{cases} 1 - \log\|g\|_{L^\infty} + \log(A) + (|B|_1 \operatorname{diam}\mathcal{Q})^{\frac{1}{1-\mu}}, & 0 \leq \mu < 1, \\ \big(1 - \log\|g\|_{L^\infty} + \log(A)\big) e^{K|B|_1 \operatorname{diam}\mathcal{Q}}, & \mu = 1. \end{cases}$$

The following consequence of Proposition 3.1 is tailored towards a combination with the covering strategy described in Section 3.2 below. Here and from now on, we adopt the notation $\ell x \in \mathbb{R}^d$ for the coordinatewise product of $\ell \in (0,\infty)^d$ and $x \in \mathbb{R}^d$.

Corollary 3.2. *Let $\mathcal{Q} = (0,1)^d$ be the unit hypercube in $\mathbb{R}^d$, and let $Q = \Psi(\mathcal{Q})$, where $\Psi\colon \mathbb{R}^d \to \mathbb{R}^d$ is given by $\Psi(x) = x_0 + \ell x$, $x \in \mathbb{R}^d$, with some $x_0 \in \mathbb{R}^d$ and $\ell \in (0,\infty)^d$. Moreover, suppose that $f \in C^\infty(Q)$ satisfies*

$$\forall \alpha \in \mathbb{N}^d, \quad \|\partial^\alpha f\|_{L^2(Q)} \leq AB^\alpha(|\alpha|!)^\mu \|f\|_{L^2(Q)}$$

with some $A > 0$, $B \in (0,\infty)^d$, and $\mu \in [0,1]$. Then, there is a constant $K = K(d,\mu) > 0$, only depending on μ and the dimension d, such that for every measurable set $E \subset Q$ of positive measure we have

$$\|f\|^2_{L^2(Q)} \leq \left(\frac{K|Q|}{|E|}\right)^{KC_\mu} \|f\|^2_{L^2(E)}$$

with

$$C_\mu = \begin{cases} 1 + \log(A) + (|\ell B|_1)^{\frac{1}{1-\mu}}, & 0 \leq \mu < 1, \\ \big(1 + \log(A)\big) e^{K|\ell B|_1}, & \mu = 1. \end{cases}$$

Proof. There is nothing to prove if $f = 0$, so we may suppose that $f \neq 0$. Define the function $g \in C^\infty(\mathcal{Q})$ by

$$g(x) := \frac{\sqrt{|\mathcal{Q}|}}{\|f \circ \Psi\|_{L^2(\mathcal{Q})}} (f \circ \Psi)(x) = \frac{\sqrt{|\mathcal{Q}|}}{\|f \circ \Psi\|_{L^2(\mathcal{Q})}} f(x_0 + \ell x), \quad x \in \mathcal{Q},$$

which obviously satisfies

$$\|g\|_{L^\infty(\mathcal{Q})} = \frac{\sqrt{|\mathcal{Q}|}}{\|f \circ \Psi\|_{L^2(\mathcal{Q})}} \|f \circ \Psi\|_{L^\infty(\mathcal{Q})} \geq 1, \tag{3.1}$$

and also

$$\forall \alpha \in \mathbb{N}^d, \quad \|\partial^\alpha g\|_{L^2(\mathcal{Q})} \leq (A\sqrt{|\mathcal{Q}|})(\ell B)^\alpha (|\alpha|!)^\mu \tag{3.2}$$

by the hypothesis on f.

Now, $\mathcal{Q}$ satisfies the cone condition, so that by [1, Theorem 4.12] we have the Sobolev embedding $W^{d,2}(\mathcal{Q}) \hookrightarrow L^\infty(\mathcal{Q})$, that is, there is a constant C, depending only on the dimension d, such that

$$\forall u \in W^{d,2}(\mathcal{Q}), \quad \|u\|_{L^\infty}(\mathcal{Q}) \leq C\|u\|_{W^{d,2}(\mathcal{Q})}.$$

Taking into account that for multiindices $\alpha, \beta \in \mathbb{N}^d$ with $|\beta| \leq d$ we have

$$|\alpha + \beta|! \leq 2^{|\alpha|+|\beta|} |\alpha|!|\beta|! \leq 2^d d! 2^{|\alpha|} |\alpha|!,$$

we then deduce from (3.2) for every $\alpha \in \mathbb{N}^d$ that

$$\begin{aligned} \|\partial^\alpha g\|^2_{L^\infty(\mathcal{Q})} &\leq C^2 \|\partial^\alpha g\|_{W^{d,2}(\mathcal{Q})} = C^2 \sum_{|\beta| \leq d} \|\partial^{\alpha+\beta} g\|^2_{L^2(\mathcal{Q})} \\ &\leq \big(CA\sqrt{|\mathcal{Q}|}\big)^2 (\ell B)^{2\alpha} \sum_{|\beta| \leq d} (\ell B)^{2\beta} (|\alpha + \beta|!)^{2\mu} \\ &\leq \big(C 2^{d\mu} (d!)^\mu A \sqrt{|\mathcal{Q}|}\big)^2 (2^\mu \ell B)^{2\alpha} (|\alpha|!)^{2\mu} \sum_{|\beta| \leq d} (\ell B)^{2\beta} \\ &\leq \big(C 2^{d\mu} (d!)^\mu A \sqrt{|\mathcal{Q}|}\big)^2 (1 + |\ell B|^2)^d (2^\mu \ell B)^{2\alpha} (|\alpha|!)^{2\mu}, \end{aligned}$$

and, thus,

$$\forall \alpha \in \mathbb{N}^d, \quad \|\partial^\alpha g\|_{L^\infty(\mathcal{Q})} \leq \big(C 2^d d! A \sqrt{|\mathcal{Q}|}\big)(1 + |\ell B|^2)^{d/2} (2\ell B)^\alpha (|\alpha|!)^\mu. \tag{3.3}$$

In light of (3.1) and (3.3), the function g on $\mathcal{Q}$ satisfies the hypotheses of Proposition 3.1 with A and B replaced by $C 2^d d! A \sqrt{|\mathcal{Q}|}(1 + |\ell B|^2)^{d/2}$ and $2\ell B$, respectively. Applying Proposition 3.1 with the measurable subset $\mathcal{E} := \Psi^{-1}(E) \subset \mathcal{Q}$, we therefore conclude that

$$\frac{\|f\|^2_{L^2(Q)}}{\|f\|^2_{L^2(E)}} = \frac{\|g\|^2_{L^2(\mathcal{Q})}}{\|g\|^2_{L^2(\mathcal{E})}} \leq \left(\frac{K|\mathcal{Q}|}{|\mathcal{E}|}\right)^{KC_{\mu,\ell}} = \left(\frac{K|Q|}{|E|}\right)^{KC_{\mu,\ell}},$$

upon a suitable adaptation of the constant K, especially in order to subsume the term $\log(1 + |\ell B|^2)$ into $(|\ell B|_1)^{\frac{1}{1-\mu}}$ and $e^{K|\ell B|_1}$ in $C_{\mu,\ell}$, respectively. This completes the proof. □

Remark 3.3. (1) The restriction to the very specific type of sets Q in Corollary 3.2 is due to the Sobolev embedding used in the proof as it is imperative that the constant $K > 0$ in the statement can be chosen uniformly with respect to the corresponding Sobolev constants. This is accomplished exactly via the change of variables. The specific restriction to the hypercube for $\mathcal{Q}$, however, is not essential and could simply be extended to a larger class of sets, as long as a Sobolev embedding with a uniform constant remains available.

(2) It is worth to note that the case $\mu < 1$ in Corollary 3.2 could alternatively also be handled using the more classical analytic approach by Kovrijkine (instead of Proposition 3.1), cf. [23] and [16] and also Appendix A below, which would not require the use of the Sobolev embedding. This does, however, not seem to work for the critical case of $\mu = 1$ because then the bounds on the partial derivatives of f are not strong enough to control the domain of convergence of a suitable Taylor expansion as is possible for $\mu < 1$.

3.2. **Covering with good and bad elements.** The following result formulates the general scheme to obtain global uncertainty estimates by means of the local estimate provided by Corollary 3.2 via a covering strategy. Here, $\Gamma \subset \mathbb{R}^d$ could be the whole domain under consideration, say $\mathbb{R}^d$ or a strip as in the context of Section 2, but could also be a proper subdomain thereof, depending on the situation at hand.

Lemma 3.4. *Let $\Gamma \subset \mathbb{R}^d$ be open, and let $\{Q_j\}_{j\in J}$ be a finite or countably infinite family of open sets $Q_j \subset \mathbb{R}^d$ such that almost everywhere we have*

$$1_\Gamma \leq \sum_{j\in J} 1_{Q_j} \leq \kappa 1_\Gamma$$

with some $\kappa \geq 1$. Moreover, suppose that $f \in C^\infty(\Gamma)$ satisfies

$$\forall \alpha \in \mathbb{N}^d, \quad \|\partial^\alpha f\|_{L^2(\Gamma)} \leq D B^\alpha (|\alpha|!)^\mu$$

with some $D > 0$, $B \in (0,\infty)^d$, and $\mu \in [0,1]$. Then, for every $\varepsilon > 0$ there is a subset $J_{\mathrm{gd}} = J_{\mathrm{gd}}(\varepsilon) \subset J$ of indices such that

$$\|f\|^2_{L^2(\Gamma)} \leq \sum_{j\in J_{\mathrm{gd}}} \|f\|^2_{L^2(Q_j)} + \varepsilon D^2,$$

while for each $j \in J_{\mathrm{gd}}$ we have

$$\forall \alpha \in \mathbb{N}^d, \quad \|\partial^\alpha f\|_{L^2(Q_j)} \leq \Big(\frac{2^d\kappa}{\varepsilon}\Big)^{1/2} (2B)^\alpha (|\alpha|!)^\mu \|f\|_{L^2(Q_j)}. \tag{3.4}$$

Proof. Given $\varepsilon > 0$, we define $J_{\mathrm{gd}} \subset J$ as the subset of indices $j \in J$ for which (3.4) is satisfied; note that this subset might a priori be empty. In any case, taking into account that

$$\|f\|^2_{L^2(\Gamma)} \leq \sum_{j\in J_{\mathrm{gd}}} \|f\|^2_{L^2(Q_j)} + \sum_{j\in J\setminus J_{\mathrm{gd}}} \|f\|^2_{L^2(Q_j)},$$

we only have to show that

$$\sum_{j\in J\setminus J_{\mathrm{gd}}} \|f\|^2_{L^2(Q_j)} \leq \varepsilon D^2.$$

To this end, we observe that for each $j \in J \setminus J_{\mathrm{gd}}$ there is by definition some $\beta \in \mathbb{N}^d$ such that

$$\begin{aligned}\|f\|^2_{L^2(Q_j)} &< \frac{\varepsilon}{2^d\kappa} \frac{1}{(2B)^{2\beta}(|\beta|!)^{2\mu}} \|\partial^\beta f\|^2_{L^2(Q_j)} \\ &\leq \frac{\varepsilon}{2^d\kappa} \sum_{\alpha\in\mathbb{N}^d} \frac{1}{(2B)^{2\alpha}(|\alpha|!)^{2\mu}} \|\partial^\alpha f\|^2_{L^2(Q_j)},\end{aligned}$$

and summing over $j \in J \setminus J_{\mathrm{gd}}$ together with the hypotheses on f and the family $\{Q_j\}_{j\in J}$ gives

$$\sum_{j\in J\setminus J_{\mathrm{gd}}} \|f\|^2_{L^2(Q_j)} \le \frac{\varepsilon}{2^d} \sum_{\alpha\in\mathbb{N}^d} \frac{1}{(2B)^{2\alpha}(|\alpha|!)^{2\mu}} \|\partial^\alpha f\|^2_{L^2(\Gamma)}$$
$$\le \frac{\varepsilon}{2^d} D^2 \sum_{\alpha\in\mathbb{N}^d} \frac{1}{4^{|\alpha|}} = \varepsilon D^2$$

since

$$\sum_{\alpha\in\mathbb{N}^d} \frac{1}{4^{|\alpha|}} = \sum_{m=0}^{\infty} \frac{1}{4^m} \binom{m+d-1}{d-1} \le \sum_{m=0}^{\infty} \frac{1}{4^m} 2^{m+d-1} = 2^d. \qquad \square$$

Remark 3.5. The quantity κ in Lemma 3.4 can be interpreted as the maximal essential overlap between the covering sets. The elements Q_j with $j \in J_{\mathrm{gd}}$ are traditionally referred to as *good*, which explains the subscript in J_{gd}. It is worth to note that the extreme cases for J_{gd} are, of course, $J_{\mathrm{gd}} = J$ and $J_{\mathrm{gd}} = \emptyset$, in which case we have $\|f\|^2_{L^2(\Gamma)} \le \sum_{j\in J_{\mathrm{gd}}} \|f\|^2_{L^2(Q_j)}$ and $\|f\|^2_{L^2(\Gamma)} \le \varepsilon D^2$, respectively. The latter case is usually not interesting as it trivially leads to a unique continuation estimate of the form we are looking for (with any measurable subset ω). In this sense, we may always assume that J_{gd} is not empty.

On the good covering elements Q_j in Lemma 3.4 (i.e. $j \in J_{\mathrm{gd}}$), the bound provided in (3.4) is designed to guarantee applicability of the local estimate from Corollary 3.2 with $A = (2^d\kappa/\varepsilon)^{1/2}$ and with B replaced by $2B$. Together with the bound from Lemma 3.4, this then leads to a global (on Γ) uncertainty estimate of the desired form, as long as the constant relating the L^2-norm of f on Q_j, $j \in J$, to the one on the subset of Q_j in Corollary 3.2 is uniform over $j \in J$; it would, in fact, suffice it to be uniform over $j \in J_{\mathrm{gd}}$, but J_{gd} is usually not known explicitly. An instance of this is summarized in the following result.

Corollary 3.6. *In the situation of Lemma 3.4, suppose, in addition, that each Q_j is as in Corollary 3.2 the affine image of the hypercube $(0,1)^d$ with corresponding scaling parameters $\ell_j \in (0,\infty)^d$ such that the coordinatewise supremum $\ell := \sup_{j\in J} \ell_j$ belongs to $(0,\infty)^d$. Then, there is a constant $K = K(d,\mu,\kappa) > 0$ such that for every $\varepsilon > 0$ and every measurable subset $\omega \subset \Gamma$ with*

$$\vartheta := \inf_{j\in J} \frac{|\omega \cap Q_j|}{|Q_j|} > 0,$$

we have

$$\|f\|^2_{L^2(\Gamma)} \le \left(\frac{K}{\vartheta}\right)^{KC_{\varepsilon,\mu}} \|f\|^2_{L^2(\omega)} + \varepsilon D^2, \tag{3.5}$$

where

$$C_{\varepsilon,\mu} = \begin{cases} 1 - \log\varepsilon + (|\ell B|_1)^{\frac{1}{1-\mu}}, & 0 \le \mu < 1, \\ (1-\log\varepsilon) e^{K|\ell B|_1}, & \mu = 1. \end{cases}$$

3.3. **Choosing suitable coverings.** In order to apply Corollary 3.6, the specific choices of the subdomain Γ and a corresponding essential covering $\{Q_j\}_{j\in J}$ highly depend on the situation at hand, especially with respect to having a good control over the quantities $\ell \in (0,\infty)^d$ and $\vartheta > 0$. We describe here two examples with the focus on the case where the underlying domain is

$$M = \mathbb{R}^d \times (0,2\pi)^d \subset \mathbb{R}^{2d},$$

which addresses upon a suitable identification the cylinder $\mathbb{R}^d \times \mathbb{T}^d$ encountered in Section 2 for the Baouendi–Grushin operators. Other cases like $\mathbb{R}^d$ or $\mathbb{R}^{2d}$ are completely analogous and are briefly commented on in Remarks 3.8 and 3.11 below.

We consider functions $f \in C^\infty(M)$ satisfying

$$\forall \alpha, \beta \in \mathbb{N}^d, \quad \|\partial_x^\alpha \partial_y^\beta f\|_{L^2(M)} \leq D c_x^{|\alpha|} c_y^{|\beta|} (\alpha!)^\mu (\beta!)^\mu \tag{3.6}$$

with some constants $D, c_x, c_y > 0$ and $\mu \in [0,1]$, which fit into the framework of Lemma 3.4 and Corollary 3.6 with $B = (c_x \mathbf{1}, c_y \mathbf{1}) \in (0,\infty)^{2d}$, where we used the short-hand notation $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^d$.

Our first example now addresses the case of thick subsets of M, which applies when no additional information on the decay of f in (3.6) is available. This situation is encountered in Theorem 2.3, and in a variant on $\mathbb{R}^d$ also in Example 3.9 below.

Example 3.7. Suppose that $f \in C^\infty(M)$ satisfies (3.6) and that $\omega \subset M$ is (θ, L_x, L_y)-thick in M in the sense of Definition 2.1. We may then cover $\Gamma = M$, up to a set of measure zero, with a countable number of translates Q_j, $j \in J$, of $\Lambda_{L_x,L_y} = (0,L_x)^d \times (0,L_y)^d$ with essential overlap $\kappa \leq 2^d$. Each of these hyperrectangles Q_j is the affine image of the hypercube $\mathcal{Q} = (0,1)^d \times (0,1)^d = (0,1)^{2d}$ with corresponding scaling parameters $\ell = \ell_j = (L_x \mathbf{1}, L_y \mathbf{1}) \in (0,\infty)^{2d}$, so that

$$|\ell B|_1 = dL_x c_x + dL_y c_y.$$

Applying Corollary 3.6 and suitably adapting the constant $K = K(d,\mu)$, we conclude that (3.5) holds with $\vartheta \geq \theta > 0$ and

$$C_{\varepsilon,\mu} = \begin{cases} 1 - \log \varepsilon + (L_x c_x + L_y c_y)^{\frac{1}{1-\mu}}, & 0 \leq \mu < 1, \\ (1 - \log \varepsilon) \exp(K L_x c_x + K L_y c_y), & \mu = 1. \end{cases}$$

Remark 3.8. An analogue to Example 3.7 is available if M is replaced by $\mathbb{R}^d$ or $\mathbb{R}^{2d}$, where $\mathbb{R}^{2d}$ is, in fact, just another instance of $\mathbb{R}^d$ with d replaced by $2d$. The case of $\mathbb{R}^d$ differs from the above essentially only in the way that all terms corresponding to the y coordinates can be removed; the situation is then even a bit simpler since the essential covering of $\mathbb{R}^d$ with translates of $(0,L_x)^d$ can be achieved without any overlap, that is, $\kappa = 1$.

Example 3.9. As an illustration of Remark 3.8, let us consider the fractional heat semigroup $(e^{-t(-\Delta)^s})_{t\geq 0}$ acting on $L^2(\mathbb{R}^d)$, with $s > 0$ a positive real number. We prove in Lemma 4.3 below that this semigroup enjoys for all $t > 0$, $\alpha \in \mathbb{N}^d$, and $g \in L^2(\mathbb{R}^d)$, the smoothing estimates

$$\|\partial_x^\alpha (e^{-t(-\Delta)^s} g)\|_{L^2(\mathbb{R}^d)} \leq \left(\frac{d}{2st}\right)^{\frac{|\alpha|}{2s}} (\alpha!)^{\frac{1}{2s}} \|g\|_{L^2(\mathbb{R}^d)}.$$

Therefore, when $s \geq 1/2$, every function $f = e^{-t(-\Delta)^s} g$ satisfies the $\mathbb{R}^d$-variant of (3.6) with $D = \|g\|_{L^2(\mathbb{R}^d)}$, $c_x = (\frac{d}{2st})^{\frac{1}{2s}}$, and $\mu = \frac{1}{2s} \in (0,1]$. We therefore conclude from Remark 3.8 that there exists a positive constant $K > 0$, depending only on s and the dimension d, such that for every (θ, L)-thick set $\omega \subset \mathbb{R}^d$, with $\theta \in (0,1]$ and $L > 0$, and all $t > 0$, $\varepsilon > 0$ and $g \in L^2(\mathbb{R}^d)$, we have

$$\|e^{-t(-\Delta)^s} g\|^2_{L^2(\mathbb{R}^d)} \leq \left(\frac{K}{\theta}\right)^{K C_{\varepsilon,s,t}} \|e^{-t(-\Delta)^s} g\|^2_{L^2(\omega)} + \varepsilon \|g\|^2_{L^2(\mathbb{R}^d)} \tag{3.7}$$

with

$$C_{\varepsilon,s,t} = \begin{cases} 1 - \log \varepsilon + \left(\dfrac{L}{t^{\frac{1}{2s}}}\right)^{\frac{2s}{2s-1}}, & s > 1/2, \\ (1 - \log \varepsilon) \exp\left(\dfrac{KL}{t}\right), & s = 1/2. \end{cases}$$

Let us make some bibliographical comments:

(1) Non-quantitative estimates of the form

$$\|e^{-t(-\Delta)^s} g\|^2_{L^2(\mathbb{R}^d)} \leq C_{\varepsilon,s,t,\theta,L} \|e^{-t(-\Delta)^s} g\|^2_{L^2(\omega)} + \varepsilon \|g\|^2_{L^2(\mathbb{R}^d)}$$

have already been stated in [5, Proposition 5.2]. In fact, this result states non-quantitative unique continuation estimates for evolution operators associated with more general Fourier multipliers of the form $F(|D_x|)$, where $F\colon [0,+\infty) \to \mathbb{R}$ is a continuous function bounded from below, generating a quasi-analytic sequence $\mathcal{M}^F$, see [5, Section 2.2] for the definition of this notion. The merit of the estimates (3.7) is a precise description of the constant $C_{\varepsilon,s,t,\theta,L}$ in the case of the fractional heat equations, which correspond to the functions $F(t) = t^{2s}$.

(2) Similar unique continuation estimates have also been obtained for the heat equation (case $s = 1$) considered in bounded domains of $\mathbb{R}^d$, see, e.g., [39, Section 2.1].

In some situations, the function f under consideration exhibits also a strong decay in L^2-sense with respect to the x coordinates, say

$$\|e^{c_0|x|^{1/\nu}} f\|_{L^2(M)} \leq D \tag{3.8}$$

with some $c_0, \nu > 0$ and the same $D > 0$ as in (3.6). In this case, for every $R > 0$ and every choice of measurable $\Gamma \subset M$ satisfying $\Gamma \supset (-R,R)^d \times (0,2\pi)^d$ we have

$$\|f\|_{L^2(M\setminus\Gamma)} \leq e^{-c_0 R^{1/\nu}} \|e^{c_0|x|^{1/\nu}} f\|_{L^2(M\setminus\Gamma)} \leq e^{-c_0 R^{1/\nu}} D$$

and, therefore,

$$\|f\|^2_{L^2(M)} = \|f\|^2_{L^2(\Gamma)} + e^{-2c_0 R^{1/\nu}} D^2. \tag{3.9}$$

Here, upon choosing R large enough, the term $e^{-2c_0 R^{1/\nu}}$ can be hidden inside ε in (3.5), while the term $\|f\|^2_{L^2(\Gamma)}$ is to be addressed via Corollary 3.6; note that always $\|\partial_x^\alpha \partial_y^\beta f\|_{L^2(\Gamma)} \leq \|\partial_x^\alpha \partial_y^\beta f\|_{L^2(M)}$, so that we can use (3.6) on Γ instead of M with the same constants. In the context of Corollary 3.6, this has the benefit that only the part $\Gamma \cap \omega$ of the sensor set ω plays a role and the remainder $\omega \setminus \Gamma$ can be discarded. The following example demonstrates this for sensor sets $\omega \subset M$ with merely positive measure, but no other additional information. This situation is encountered in Theorem 6.1, as well as in Example 6.4.

Example 3.10. Suppose that $f \in C^\infty(M)$ satisfies (3.6) and (3.8) and that $\omega \subset M$ has positive measure. We choose $R > 0$ such that the intersection of ω with the hyperrectangle $\Gamma = (-R,R)^d \times (0,2\pi)^d$ has positive measure, that is,

$$\theta_R := \frac{|\omega \cap \Gamma|}{|\Gamma|} > 0.$$

Here, Γ itself is the affine image of the hypercube $(0,1)^{2d}$ with corresponding scaling parameters $\ell = (R\mathbf{1}, 2\pi\mathbf{1})$, so that

$$|\ell B|_1 = dRc_x + 2\pi d c_y.$$

We may then cover Γ with just the one element family $\{\Gamma\}$ with $\kappa = 1$ and apply Corollary 3.6 with the particular choice $\varepsilon = e^{-2c_0 R^{1/\nu}}$. Upon suitably adapting the constant $K = K(d,\mu)$, and taking into account (3.9), this gives

$$\|f\|^2_{L^2(M)} \leq \Big(\frac{K}{\theta_R}\Big)^{KC_{\varepsilon,\mu}} + 2e^{-2c_0 R^{1/\nu}} D^2$$

with

$$C_{\varepsilon,\mu} = \begin{cases} 1 + c_0 R^{1/\nu} + (Rc_x + c_y)^{\frac{1}{1-\mu}}, & 0 \leq \mu < 1, \\ (1 + c_0 R^{1/\nu}) \exp(KRc_x + Kc_y), & \mu = 1. \end{cases}$$

Remark 3.11. (1) In the situation of (3.6) and (3.8), we could also consider sensor sets that have not only positive measure but are even thick in M with a decaying density and with respect to a variable local scale in the x coordinates, cf. [6, 17, 18, 19]. The strategy of proof would combine the above scheme with the well-known Besicovitch covering theorem, just as in the mentioned references. We decided not to pursue this here.

(2) An analogue to Example 3.10 is available if M is replaced by $\mathbb{R}^d$. The difference is only that just as in Remark 3.8 all terms in Example 3.10 corresponding to the y coordinates can be removed.

4. Smoothing properties of the fractional Baouendi–Grushin equations

This section is devoted to studying the smoothing properties of the evolution equations associated by the fractional Baouendi–Grushin operator $(-\Delta_\gamma)^{(1+\gamma)/2}$. This is a crucial step for obtaining the unique continuation estimates stated in Theorem 2.3 via the framework from Section 3. We also discuss the case of eigenfunctions of the operator Δ_γ, or finite linear combinations thereof, which additionally enjoy localization properties.

4.1. Prolegomena.

First of all, we introduce some notations, and recall some very well-known facts of spectral analysis.

4.1.1. *Diagonalization.* After diagonalizing the Laplace operator Δ_y on the torus $\mathbb{T}^d$, the Baouendi–Grushin operator Δ_γ is transformed as

$$\Delta_x + |x|^{2\gamma}\Delta_y \rightsquigarrow \Delta_x - |n|^2|x|^{2\gamma}.$$

This motivates to introduce, for each frequency $n \in \mathbb{Z}^d$, the anharmonic oscillator $H_{\gamma,n}$ with variably scaled potential, defined by

$$H_{\gamma,n} = -\Delta_x + |n|^2|x|^{2\gamma}, \quad x \in \mathbb{R}^d.$$

Recall that when $n \neq 0$, each of these operators, equipped with the respective domain

$$D(H_{\gamma,n}) = \big\{g \in L^2(\mathbb{R}^d) : H_{\gamma,n}g \in L^2(\mathbb{R}^d)\big\},$$

has purely discrete spectrum consisting of a sequence of positive eigenvalues diverging to $+\infty$. Moreover, the negatives of each $H_{\gamma,n}$ and of its fractional powers $(H_{\gamma,n})^s$ with $s > 0$ (understood via the standard functional calculus) generate strongly continuous contraction semigroups on $L^2(\mathbb{R}^d)$. When $n = 0$, the operator $H_{\gamma,0}$ reduces to the negative Laplacian $-\Delta_x$, and $-H_{\gamma,0}$ likewise generates a strongly continuous contraction semigroups on $L^2(\mathbb{R}^d)$ (as do the operators $-(H_{\gamma,0})^s$). Consequently, we have for all $t \geq 0$, $s > 0$ and $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$,

$$e^{-t(-\Delta_\gamma)^s} g = \sum_{n\in\mathbb{Z}^d} (e^{-t(H_{\gamma,n})^s} g_n) \otimes \varphi_n, \tag{4.1}$$

with the partial Fourier coefficients

$$g_n = \int_M \overline{\varphi_n(y)} g(\cdot, y)\, \mathrm{d}y \quad \text{where} \quad \varphi_n(y) = e^{in\cdot y}. \tag{4.2}$$

4.1.2. *Unitary transform.* For every non-zero frequency $n \in \mathbb{Z}^d \setminus \{0\}$, let us introduce the isometry $M_{\gamma,n}$ on $L^2(\mathbb{R}^d)$ by

$$M_{\gamma,n}g = |n|^{\frac{d}{2(1+\gamma)}} g(|n|^{\frac{1}{1+\gamma}}\cdot), \quad g \in L^2(\mathbb{R}^d). \tag{4.3}$$

A straightforward computation shows that

$$(M_{\gamma,n})^*(H_{\gamma,n})^s M_{\gamma,n} = |n|^{\frac{2s}{1+\gamma}} (H_\gamma)^s, \tag{4.4}$$

where H_γ denotes the standard anharmonic oscillator

$$H_\gamma = -\Delta_x + |x|^{2\gamma}, \quad x \in \mathbb{R}^d.$$

In the following, we denote by $(\lambda_{\gamma,m})_m$ the sequence of eigenvalues of the maximal realization of the operator H_γ on $L^2(\mathbb{R}^d)$ in non-decreasing order and counting multiplicities,

and by $(\phi_{\gamma,m})_m$ with $\phi_{\gamma,m} \in L^2(\mathbb{R}^d)$ a corresponding orthonormal basis of eigenfunctions for H_γ such that each $\phi_{\gamma,m}$ is associated to $\lambda_{\gamma,m} > 0$.

4.1.3. *Eigenfunctions.* In view of the similarity relation (4.4), it is clear that the eigenvalues of the Baouendi–Grushin operator $-\Delta_\gamma$ are precisely given by

$$\lambda_{\gamma,n,m} = |n|^{\frac{2}{1+\gamma}}\lambda_{\gamma,m}, \quad n \in \mathbb{Z}^d \setminus \{0\},\ m \in \mathbb{N},$$

and the function

$$\psi_{\gamma,n,m} = (M_{\gamma,n}\phi_{\gamma,m}) \otimes \varphi_n, \tag{4.5}$$

in $L^2(\mathbb{R}^d \times \mathbb{T}^d)$ is an associated normalized eigenfunction of the operator $-\Delta_\gamma$. Moreover, the subspace $E_\lambda(-\Delta_\gamma)$ defined in (2.5) can be rewritten as

$$E_\lambda(-\Delta_\gamma) = \operatorname{span}\big\{\psi_{\gamma,n,m} \colon n \in \mathbb{Z}^d \setminus \{0\},\ m \in \mathbb{N},\ \lambda_{\gamma,n,m} \leq \lambda\big\}.$$

4.2. **Smoothing properties of the evolution operators.** Let us now focus on the smoothing properties of the semigroups generated by Baouendi–Grushin operators. We are first interested in the smoothing properties with respect to the variable $y \in \mathbb{T}^d$.

Lemma 4.1. *For all $s > 0$, $t > 0$ and $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$, we have*

$$\big\|\partial_y^\alpha (e^{-t(-\Delta_\gamma)^s} g)\big\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq \left(\frac{1+\gamma}{2s\lambda_\gamma^s t}\right)^{\frac{|\alpha|(1+\gamma)}{2s}} (\alpha!)^{\frac{1+\gamma}{2s}} \|g\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}, \tag{4.6}$$

where $\lambda_\gamma > 0$ stands for the smallest eigenvalue of the anharmonic oscillator H_γ.

Remark 4.2. It is worthing noticing from the above result that the evolution operators generated by the fractional Baouendi–Grushin operator $(-\Delta_\gamma)^s$ enjoy (at least) analytic smoothing properties in the variable y whenever $s \geq (1+\gamma)/2$. This point is clear from the estimates (4.7) since the analytic smoothness is equivalent to the exponential decrease of the Fourier transform. As a consequence, we are able to prove unique continuation estimates for the functions $e^{-t(-\Delta_\gamma)^s} g$ only in this regime, and in fact, we will focus only on the case $s = (1+\gamma)/2$.

Proof of Lemma 4.1. Let us begin by proving that for all $t \geq 0$ and $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$,

$$\big\|e^{\lambda_\gamma^s t|D_y|^{\frac{2s}{1+\gamma}}} (e^{-t(-\Delta_\gamma)^s} g)\big\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq \|g\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}. \tag{4.7}$$

Let $t \geq 0$ and $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ be fixed. First, we get from the diagonalization formula (4.1) and Parseval's theorem that

$$\big\|e^{\lambda_\gamma^s t|D_y|^{\frac{2s}{1+\gamma}}} (e^{-t(-\Delta_\gamma)^s} g)\big\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} = \sum_{n\in\mathbb{Z}^d} e^{2\lambda_\gamma^s t|n|^{\frac{2s}{1+\gamma}}} \big\|e^{-t(H_{\gamma,n})^s} g_n\big\|^2_{L^2(\mathbb{R}^d)},$$

where g_n denotes the partial Fourier coefficient defined in (4.2). Let $n \in \mathbb{Z}^d$ be fixed. On the one hand, when $n = 0$, the operator $H_{\gamma,n}$ reduces to the negative Laplacian $-\Delta_x$ on $\mathbb{R}^d$. Since the evolution operators $e^{-t(-\Delta_x)^s}$ are contractions on $L^2(\mathbb{R}^d)$, we deduce that

$$\big\|e^{-t(H_{\gamma,0})^s} g_0\big\|^2_{L^2(\mathbb{R}^d)} = \big\|e^{-t(-\Delta_x)^s} g_0\big\|^2_{L^2(\mathbb{R}^d)} \leq \|g_0\|^2_{L^2(\mathbb{R}^d)}.$$

On the other hand, when $n \in \mathbb{Z}^d \setminus \{0\}$, we deduce from the relation (4.4) that

$$\big\|e^{-t(H_{\gamma,n})^s} g_n\big\|_{L^2(\mathbb{R}^d)} = \big\|e^{-|n|^{\frac{2s}{1+\gamma}} t H_\gamma^s} (M_{\gamma,n})^* g_n\big\|_{L^2(\mathbb{R}^d)}.$$

Moreover, we get from Plancherel's theorem that

$$\big\|e^{-tH_\gamma^s}\big\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq e^{-\lambda_\gamma^s t}, \tag{4.8}$$

which implies

$$e^{2\lambda_\gamma^s t|n|^{\frac{2s}{1+\gamma}}} \big\|e^{-t(H_{\gamma,n})^s} g_n\big\|^2_{L^2(\mathbb{R}^d)} \leq e^{2\lambda_\gamma^s t|n|^{\frac{2s}{1+\gamma}}} e^{-2\lambda_\gamma^s t|n|^{\frac{2s}{1+\gamma}}} \|g_n\|^2_{L^2(\mathbb{R}^d)} = \|g_n\|^2_{L^2(\mathbb{R}^d)}.$$

In a nutshell, we obtained that for all $t \geq 0$ and $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$,

$$\left\| e^{\lambda_\gamma^s t |D_y|^{\frac{2s}{1+\gamma}}} (e^{-t(-\Delta_\gamma)^s} g) \right\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2 \leq \sum_{n=0}^{+\infty} \|g_n\|_{L^2(\mathbb{R}^d)}^2 = \|g\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2,$$

which proves estimate (4.7).

Let us now explain how to derive the estimate (4.6) from (4.7). Notice first from the decomposition (4.1) and Parseval's theorem that for all $\alpha \in \mathbb{N}^d$, $t > 0$, and $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$,

$$\begin{aligned} \|\partial_y^\alpha (e^{-t(-\Delta_\gamma)^s} g)\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2 &= \sum_{n \in \mathbb{Z}^d} |n^\alpha|^2 \|e^{-t(H_{\gamma,n})^s} g_n\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq \sum_{n \in \mathbb{Z}^d} |n|^{2|\alpha|} \|e^{-t(H_{\gamma,n})^s} g_n\|_{L^2(\mathbb{R}^d)}^2 \\ &= \||D_y|^{|\alpha|} (e^{-t(-\Delta_\gamma)^s} g)\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2. \end{aligned} \tag{4.9}$$

Since for all $p, q > 0$, $c > 0$ and $x \geq 0$, we have the elementary inequality

$$x^p e^{-cx^q} \leq \left(\frac{p}{ecq} \right)^{\frac{p}{q}}, \tag{4.10}$$

which follows from a straightforward study of function, and taking into account the estimate $|\alpha|^{|\alpha|} \leq e^{|\alpha|} |\alpha|!$, we deduce from (4.9) for all $\alpha \in \mathbb{N}^d$, $t > 0$ and $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$,

$$\begin{aligned} \left\| \partial_y^\alpha (e^{-t(-\Delta_\gamma)^s} g) \right\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} &\leq \left\| |D_y|^{|\alpha|} e^{-\lambda_\gamma^s t |D_y|^{\frac{2s}{1+\gamma}}} e^{\lambda_\gamma^s t |D_y|^{\frac{2s}{1+\gamma}}} (e^{-t(-\Delta_\gamma)^s} g) \right\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \\ &\leq \left(\frac{|\alpha|(1+\gamma)}{2es\lambda_\gamma^s t} \right)^{\frac{|\alpha|(1+\gamma)}{2s}} \|g\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \\ &\leq \left(\frac{1+\gamma}{2s\lambda_\gamma^s t} \right)^{\frac{|\alpha|(1+\gamma)}{2s}} (\alpha!)^{\frac{1+\gamma}{2s}} \|g\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}. \end{aligned}$$

This ends the proof of Lemma 4.1. □

Before treating the case of the variable $x \in \mathbb{R}^d$, let us recall and give a proof of the smoothing properties of the fractional heat semigroups acting on the space $L^2(\mathbb{R}^d)$.

Lemma 4.3. *For all $s > 0$, $t > 0$, $\alpha \in \mathbb{N}^d$ and $g \in L^2(\mathbb{R}^d)$, we have*

$$\left\| \partial_x^\alpha (e^{-t(-\Delta)^s} g) \right\|_{L^2(\mathbb{R}^d)} \leq \left(\frac{d}{2st} \right)^{\frac{|\alpha|}{2s}} (\alpha!)^{\frac{1}{2s}} \|g\|_{L^2(\mathbb{R}^d)}.$$

Proof. Let $t > 0$, $\alpha \in \mathbb{N}^d$ and $g \in L^2(\mathbb{R}^d)$ be fixed. We first deduce from Plancherel's theorem that

$$\left\| \partial_x^\alpha (e^{-t(-\Delta)^s} g) \right\|_{L^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{d/2}} \left\| (i\xi)^\alpha e^{-t|\xi|^{2s}} \widehat{g} \right\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{d/2}} \left\| |\xi|^{|\alpha|} e^{-t|\xi|^{2s}} \widehat{g} \right\|_{L^2(\mathbb{R}^d)}.$$

Moreover, we deduce from (4.10) and the estimates $|\alpha|^{|\alpha|} \leq e^{|\alpha|} |\alpha|!$ and $|\alpha|! \leq d^{|\alpha|} \alpha!$ that for all $\xi \in \mathbb{R}^d$,

$$|\xi|^{|\alpha|} e^{-t|\xi|^{2s}} \leq \left(\frac{|\alpha|}{2est} \right)^{\frac{|\alpha|}{2s}} \leq \left(\frac{d}{2st} \right)^{\frac{|\alpha|}{2s}} (\alpha!)^{\frac{1}{2s}}.$$

This ends the proof of Lemma 4.3 by using anew Plancherel's theorem. □

We are now in position to derive smoothing estimates in the variable $x \in \mathbb{R}^d$ for the fractional Baouendi–Grushin evolution equation in the critical regime $s = (1+\gamma)/2$.

Lemma 4.4. *There exist some positive constants $c>0$ and $t_0\in(0,1)$ such that for all $t\in(0,t_0)$ and $g\in L^2(\mathbb{R}^d\times\mathbb{T}^d)$,*

$$\big\|\partial_x^\alpha(e^{-t(-\Delta_\gamma)^{\frac{1+\gamma}{2}}}g)\big\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}\leq\frac{c^{1+|\alpha|}}{t^{\frac{\gamma|\alpha|}{1+\gamma}+\frac{d}{\gamma}}}\,\alpha!\,\|g\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}.\tag{4.11}$$

In the particular case of $\gamma=1$, the same holds even for all positive times $t>0$ and the term $t^{d/\gamma}$ on the right-hand side of (4.11) can be skipped.

Proof. Let us first treat the case $\gamma\geq 2$. In order to alleviate the writing, we denote by $s_\gamma=(1+\gamma)/2$ the dissipation index. Let $t>0$ and $g\in L^2(\mathbb{R}^d\times\mathbb{T}^d)$ be fixed. We first deduce from Plancherel's theorem and the diagonalization formula (4.1) that

$$\big\|\partial_x^\alpha(e^{-t(-\Delta_\gamma)^{s_\gamma}}g)\big\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}=\sum_{n\in\mathbb{Z}^d}\big\|\partial_x^\alpha(e^{-t(H_{\gamma,n})^{s_\gamma}}g_n)\big\|^2_{L^2(\mathbb{R}^d)},\tag{4.12}$$

where g_n is still defined as in (4.2). We now need to estimate each term in the above sum. When $n=0$, we deduce from Lemma 4.3 that there exists a positive constant $c_0>0$ only depending on s and the dimension d such that

$$\big\|\partial_x^\alpha(e^{-t(H_{\gamma,0})^{s_\gamma}}g_0)\big\|_{L^2(\mathbb{R}^d)}=\big\|\partial_x^\alpha(e^{-t(-\Delta_x)^{s_\gamma}}g_0)\big\|_{L^2(\mathbb{R}^d)}\leq\frac{c_0^{|\alpha|}}{t^{\frac{|\alpha|}{1+\gamma}}}\,(\alpha!)^{\frac{1}{1+\gamma}}\,\|g_0\|_{L^2(\mathbb{R}^d)}.\tag{4.13}$$

Until the end of the proof, we consider $n\in\mathbb{Z}^d\setminus\{0\}$. We deduce from the relation (4.4) that

$$\begin{aligned}\big\|\partial_x^\alpha(e^{-t(H_{\gamma,n})^{s_\gamma}}g_n)\big\|_{L^2(\mathbb{R}^d)}&=\big\|\partial_x^\alpha(M_{\gamma,n}e^{-|n|tH_\gamma^{s_\gamma}}(M_{\gamma,n})^*g_n)\big\|_{L^2(\mathbb{R}^d)}\\&=|n|^{\frac{|\alpha|}{1+\gamma}}\big\|\partial_x^\alpha(e^{-|n|tH_\gamma^{s_\gamma}}(M_{\gamma,n})^*g_n)\big\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

In order to control the above term, we use Corollary 2.2 in [3], which states that there exist positive constants $c_1>1$ and $0<t_0<1$ such that for all $0<t\leq t_0$, $\alpha\in\mathbb{N}^d$ and $g\in L^2(\mathbb{R}^d)$,

$$\big\|\partial_x^\alpha(e^{-tH_\gamma^{s_\gamma}}g)\big\|_{L^2(\mathbb{R}^d)}\leq\frac{c_1^{|\alpha|}}{t^{\frac{\gamma|\alpha|}{1+\gamma}+\frac{d}{\gamma}}}\,(\alpha!)^{\frac{\gamma}{\gamma+1}}\,\|g\|_{L^2(\mathbb{R}^d)}.\tag{4.14}$$

By using (4.8) anew, we also deduce that there exists another positive constant $c_2>1$ such that for all $t>t_0$, $\alpha\in\mathbb{N}^d$ and $g\in L^2(\mathbb{R}^d)$,

$$\big\|\partial_x^\alpha(e^{-tH_\gamma^{s_\gamma}}g)\big\|_{L^2(\mathbb{R}^d)}\leq c_2^{1+|\alpha|}\,e^{-\lambda_\gamma^{s_\gamma}t}\,(\alpha!)^{\frac{\gamma}{\gamma+1}}\,\|g\|_{L^2(\mathbb{R}^d)}.\tag{4.15}$$

There are two cases to consider.

▷ *Case 1:* $|n|t\leq t_0$. In this situation, we deduce from the estimate (4.14) that

$$|n|^{\frac{|\alpha|}{1+\gamma}}\big\|\partial_x^\alpha(e^{-|n|tH_\gamma^{s_\gamma}}(M_{\gamma,n})^*g_n)\big\|_{L^2(\mathbb{R}^d)}\leq\frac{c_1^{|\alpha|}|n|^{\frac{|\alpha|}{1+\gamma}}}{(|n|t)^{\frac{\gamma|\alpha|}{1+\gamma}+\frac{d}{\gamma}}}\,(\alpha!)^{\frac{\gamma}{\gamma+1}}\,\|g_n\|_{L^2(\mathbb{R}^d)}.$$

Noticing that $\gamma-1\geq 0$ and using the fact that $|n|\geq 1$, we get that

$$|n|^{\frac{|\alpha|}{1+\gamma}}\big\|\partial_x^\alpha(e^{-|n|tH_\gamma^{s_\gamma}}(M_{\gamma,n})^*g_n)\big\|_{L^2(\mathbb{R}^d)}\leq\frac{c_1^{|\alpha|}}{t^{\frac{\gamma|\alpha|}{1+\gamma}+\frac{d}{\gamma}}}\,(\alpha!)^{\frac{\gamma}{\gamma+1}}\,\|g_n\|_{L^2(\mathbb{R}^d)}.\tag{4.16}$$

▷ *Case 2:* $|n|t>t_0$. In this case, we use (4.15) to obtain that

$$|n|^{\frac{|\alpha|}{1+\gamma}}\big\|\partial_x^\alpha(e^{-|n|tH_\gamma^{s_\gamma}}(M_{\gamma,n})^*g_n)\big\|_{L^2(\mathbb{R}^d)}\leq c_2^{1+|\alpha|}\,|n|^{\frac{|\alpha|}{1+\gamma}}e^{-\lambda_\gamma^{s_\gamma}|n|t}\,(\alpha!)^{\frac{\gamma}{\gamma+1}}\,\|g_n\|_{L^2(\mathbb{R}^d)}.$$

Combining (4.10) and the estimates $|\alpha|^{|\alpha|}\leq e^{|\alpha|}|\alpha|!$ and $|\alpha|!\leq d^{|\alpha|}\alpha!$ implies that

$$|n|^{\frac{|\alpha|}{1+\gamma}}e^{-\lambda_\gamma^{s_\gamma}|n|t}\leq\left(\frac{|\alpha|}{e\lambda_\gamma^{s_\gamma}(1+\gamma)t}\right)^{\frac{|\alpha|}{1+\gamma}}\leq\frac{c_3^{|\alpha|}}{t^{\frac{|\alpha|}{1+\gamma}}}\,(\alpha!)^{\frac{1}{1+\gamma}}.$$

In a nutshell, we proved that

$$|n|^{\frac{|\alpha|}{1+\gamma}}\big\|\partial_x^\alpha(e^{-|n|tH_\gamma^{s_\gamma}}(M_{\gamma,n})^*g_n)\big\|_{L^2(\mathbb{R}^d)} \leq \frac{(c_2c_3)^{1+|\alpha|}}{t^{\frac{|\alpha|}{1+\gamma}}}\,\alpha!\,\|g_n\|_{L^2(\mathbb{R}^d)}. \tag{4.17}$$

▷ *Summary:* Gathering (4.12) and the estimates (4.13), (4.16) and (4.17), and using the fact that $0<t<t_0<1$, we get that

$$\big\|\partial_x^\alpha(e^{-t(-\Delta_\gamma)^{s_\gamma}}g)\big\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq \frac{(c_1c_2c_3)^{1+|\alpha|}}{t^{\frac{\gamma|\alpha|}{1+\gamma}+\frac{d}{\gamma}}}\,\alpha!\,\|g\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}.$$

The proof is ended when $\gamma\geq 2$. In the case $\gamma=1$, instead of (4.14), we use the estimates

$$\big\|\partial_x^\alpha(e^{-tH_1}g)\big\|_{L^2(\mathbb{R}^d)} \leq \frac{c_1^{|\alpha|}}{t^{\frac{|\alpha|}{2}}}\sqrt{\alpha!}\,\|g\|_{L^2(\mathbb{R}^d)},$$

taken from Theorem 2.3 in [3], which hold for all $t>0$ and differ from (4.14) by a lack of the factor $t^{-d/\gamma}$. The rest of the proof is then analogous to the one for $\gamma\geq 2$. □

As a consequence, we obtain the following global smoothing estimates on $\mathbb{R}^d\times\mathbb{T}^d$.

Corollary 4.5. *There exist constants $c>0$ and $t_0\in(0,1)$, depending only on γ and the dimension d, such that for all $t_x\in(0,t_0)$, $t_y>0$, $\alpha,\beta\in\mathbb{N}^d$, and $g\in L^2(\mathbb{R}^d\times\mathbb{T}^d)$,*

$$\|\partial_x^\alpha\partial_y^\beta(e^{-(t_x+t_y)(-\Delta_\gamma)^{\frac{1+\gamma}{2}}}g)\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq \frac{c^{1+|\alpha|+|\beta|}}{(t_x)^{\frac{\gamma|\alpha|}{\gamma+1}+\frac{d}{\gamma}}(t_y)^{|\beta|}}\,\alpha!\,\beta!\,\|g\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}. \tag{4.18}$$

If $\gamma=1$, then (4.18) holds even for all $t_x>0$ and the term $t_x^{-\frac{d}{\gamma}}$ on the right-hand side of (4.18) can be skipped.

Proof. Observe that the operators ∂_y^β and Δ_γ commute, and the same property holds for the operators ∂_y^β and $e^{-t(-\Delta_\gamma)^{(1+\gamma)/2}}$ for all $t>0$. Consequently, for all $(\alpha,\beta)\in\mathbb{N}^{2d}$ and $t_x,t_y\geq 0$, we have

$$\partial_x^\alpha\partial_y^\beta e^{-(t_x+t_y)(-\Delta_\gamma)^{\frac{1+\gamma}{2}}} = (\partial_x^\alpha e^{-t_x(-\Delta_\gamma)^{\frac{1+\gamma}{2}}})(\partial_y^\beta e^{-t_y(-\Delta_\gamma)^{\frac{1+\gamma}{2}}}).$$

The claim then follows from Lemmas 4.1 and 4.4. □

Let us now discuss the optimality of the result stated in Lemma 4.4 in the particular case $\gamma=1$. Notice from Lemma B.1 that the estimates (4.11) can be rewritten for all $t\in[0,1]$ and $g\in L^2(\mathbb{R}^d\times\mathbb{T}^d)$ as

$$\big\|e^{c_1\sqrt{t}|D_x|}(e^{t\Delta_1}g)\big\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq c_2\|g\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}. \tag{4.19}$$

We prove that this exponential decrease cannot be improved.

Lemma 4.6. *Let $\alpha>1$ and $t>0$. Then, there are no positive constants $c,c_1>0$ such that for all $g\in L^2(\mathbb{R}^d\times\mathbb{T}^d)$,*

$$\big\|e^{c|D_x|^\alpha}(e^{t\Delta_1}g)\big\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq c_1\|g\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}. \tag{4.20}$$

Proof. Assume to the contrary that there exists positive constants $c,c_1>0$ such that the estimate (4.20) holds. For each $n\in\mathbb{Z}^d\setminus\{0\}$, let us consider the function $\psi_n:=\psi_{1,n,0}$ defined in (4.5), which is a normalized eigenfunction of the operator $-\Delta_1$ associated with the eigenvalue $d|n|$ (since the first eigenvalue of the harmonic oscillator acting on $L^2(\mathbb{R}^d)$ is d). Recall that the function ψ_n is given by

$$\psi_n(x,y)=\left(\frac{|n|}{\pi}\right)^{d/4}e^{in\cdot y}e^{-|n||x|^2/2},\quad (x,y)\in\mathbb{R}^d\times\mathbb{T}^d.$$

Let us notice from Plancherel's theorem that the left-hand side of the inequality (4.20) applied with the function ψ_n reads as

$$\left\|e^{c|D_x|^\alpha}(e^{t\Delta_1}\psi_n)\right\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} = \left(\frac{|n|}{\pi}\right)^{d/2} e^{-2d|n|t}\|\varphi_n\|^2_{L^2(\mathbb{T}^d)}\left\|e^{c|D_x|^\alpha}(e^{-|n||x|^2/2})\right\|^2_{L^2(\mathbb{R}^d)}$$
$$= \frac{e^{-2d|n|t}}{\pi^{d/2}}\int_{\mathbb{R}^d} e^{c|\xi|^\alpha}e^{-|\xi|^2/(2|n|)}\,\mathrm{d}\xi.$$

Let us check that the above integral diverges to $+\infty$ as n goes to $+\infty$. Since the function ψ_n is normalized, this then contradicts the inequality (4.20). Notice that this result is immediate in the case where $\alpha \geq 2$ since the integral is equal to $+\infty$ when $|n| \gg 1$ is large enough. We therefore only consider the case $\alpha \in (1,2)$. Moreover, since the involved functions are radial, we can assume that $n \geq 1$ is a positive integer and focus on the integral

$$e^{-2dnt}\int_0^{+\infty} e^{c\xi^\alpha}e^{-\xi^2/(2n)}\xi^{d-1}\,\mathrm{d}\xi.$$

Let us consider the integrand

$$g_n(\xi) = e^{c\xi^\alpha}e^{-\xi^2/(2n)}, \quad \xi \geq 0.$$

Notice that this function attains its maximum on $[0,+\infty)$ at the point

$$\xi_n = (\alpha c n)^{\frac{1}{2-\alpha}} > 0,$$

and that this maximum is given by

$$\|g_n\|_\infty = g_n(\xi_n) = \exp\left(c^{\frac{2}{2-\alpha}}\left(\alpha^{\frac{\alpha}{2-\alpha}} - \frac{1}{2}\alpha^{\frac{2}{2-\alpha}}\right)n^{\frac{\alpha}{2-\alpha}}\right) = e^{C_\alpha n^{\frac{\alpha}{2-\alpha}}},$$

where we set

$$C_\alpha = c^{\frac{2}{2-\alpha}}\left(\alpha^{\frac{\alpha}{2-\alpha}} - \frac{1}{2}\alpha^{\frac{2}{2-\alpha}}\right) > 0.$$

Moreover, let $V_n \subset (0,+\infty)$ be a bounded neighborhood of ξ_n satisfying the property

$$g_n(\xi) \geq g_n(\xi_n)e^{-dnt}, \quad \xi \in V_n.$$

Since the function g_n is non-negative, we therefore get that

$$e^{-2dnt}\int_0^{+\infty} e^{c\xi^\alpha}e^{-\xi^2/(2n)}\xi^{d-1}\,\mathrm{d}\xi \geq c_n|V_n|g_n(\xi_n)e^{-3dnt} = c_n|V_n|e^{C_\alpha n^{\frac{\alpha}{2-\alpha}}-3dnt},$$

where we set $c_n = \min_{\xi\in V_n}\xi^{d-1} > 0$. Taking into account that ξ_n grows only polynomially in n, it is not hard to show that the neighborhood V_n can be chosen such that $|V_n|$ is uniformly bounded in n and enjoys at most a polynomial decay with respect to n. In turn, the constant c_n then enjoys polynomial growth. Moreover, since $\alpha \in (1,2)$, we have $\alpha/(2-\alpha) > 1$ and this implies that

$$c_n|V_n|e^{C_\alpha n^{\frac{\alpha}{2-\alpha}}-3dnt} \to +\infty \quad \text{as } n \to +\infty,$$

which contradicts (4.20) and, thus, proves the claim. □

4.3. **Case of the eigenfunctions.** Let us now focus on the (linear combinations of) eigenfunctions $\psi_{\gamma,n,m}$ of the Baouendi–Grushin operator Δ_γ, defined in (4.5). Recall that for any non-negative energy level $\lambda \geq 0$, the subspace $E_\lambda(-\Delta_\gamma)$ is defined as in (2.5).

We first need to derive estimates for the eigenvalue counting function for the purely discrete part of $-\Delta_\gamma$, that is, for

$$N(\lambda) = \#\{(n,m) \in \mathbb{Z}^d\setminus\{0\}\times\mathbb{N}\colon \lambda_{\gamma,n,m} \leq \lambda\},$$

where # denotes the cardinality of the corresponding set.

Lemma 4.7. *There exists a positive constant $c' > 0$ such that for all $\lambda \geq 0$,*

$$N(\lambda) \leq c' \lambda^{c_\gamma} \quad \text{with} \quad c_\gamma = d\left(\frac{1}{4} + \frac{1}{2\gamma}\right)(1+\gamma).$$

Proof. Let us recall that each eigenvalue $\lambda_{\gamma,n,m}$ is given by $\lambda_{\gamma,n,m} = |n|^{\frac{2}{1+\gamma}} \lambda_{\gamma,m}$. As a consequence, $\lambda_{\gamma,n,m} \leq \lambda$ implies the inequalities $\lambda_{\gamma,m} \leq \lambda$ and $|n| \leq (\lambda/\lambda_{\gamma,0})^{(1+\gamma)/2}$. On the one hand, it is classical that the cardinality of $n \in \mathbb{Z}^d \setminus \{0\}$ satisfying the latter is asymptotically bounded by a constant multiple of $(\lambda^{(1+\gamma)/2})^{d/2} = \lambda^{(1+\gamma)d/4}$, see, e.g., [40, Proposition XIII.15.2]. On the other hand, the cardinality of $m \in \mathbb{N}$ satisfying the former is bounded by a constant multiple of $\lambda^{(1+\gamma)d/(2\gamma)}$, see, e.g., [14, Remark 5.7] and also [13, Theorem 2.3.2]. In light of $\lambda^{(1+\gamma)d/4} \lambda^{(1+\gamma)d/(2\gamma)} = \lambda^{c_\gamma}$, this proves the claim. □

We can now derive very precise quantitative smoothing estimates for (linear combinations of) eigenfunctions.

Lemma 4.8. *There exist some positive constants $c, c_1 > 0$ such that for all $(\alpha, \beta) \in \mathbb{N}^{2d}$, $n \in \mathbb{Z}^d \setminus \{0\}$ and $m \geq 0$,*

$$\|\partial_x^\alpha \partial_y^\beta \psi_{\gamma,n,m}\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq c^{1+|\alpha|+|\beta|} |n|^{\frac{|\alpha|}{1+\gamma}+|\beta|} (\alpha!)^{\frac{\gamma}{1+\gamma}} e^{c_1 \lambda_{\gamma,m}^{\frac{1}{2}+\frac{1}{2\gamma}}}.$$

Moreover, c can be chosen such that for all $(\alpha, \beta) \in \mathbb{N}^{2d}$, $\lambda \geq 0$, and $f \in E_\lambda(-\Delta_\gamma)$, we also have

$$\|\partial_x^\alpha \partial_y^\beta f\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq c^{1+|\alpha|+|\beta|} \lambda^{c_\gamma + \frac{|\alpha|+(1+\gamma)|\beta|}{2}} (\alpha!)^{\frac{\gamma}{1+\gamma}} e^{c_1 \lambda^{\frac{1}{2}+\frac{1}{2\gamma}}} \|f\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)},$$

where $c_\gamma > 0$ is the constant appearing in Lemma 4.7.

Proof. It follows from the tensorized structure (4.5) of each eigenfunction $\psi_{\gamma,n,m}$ that for all $(\alpha, \beta) \in \mathbb{N}^{2d}$, $n \in \mathbb{Z}^d \setminus \{0\}$, and $m \geq 0$, we have

$$\|\partial_x^\alpha \partial_y^\beta \psi_{\gamma,n,m}\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} = \|\partial_x^\alpha (M_{\gamma,n} \phi_{\gamma,m})\|_{L^2(\mathbb{R}^d)} \|\partial_y^\beta \varphi_n\|_{L^2(\mathbb{T}^d)}.$$

We have to consider the two above norms. On the one hand, it is known from the work [3, Theorem 2.1] that there exist positive constants $c, c_1 > 0$ such that for all $m \geq 0$,

$$\|\partial_x^\alpha \phi_{\gamma,m}\|_{L^2(\mathbb{R}^d)} \leq c^{1+|\alpha|} (\alpha!)^{\frac{\gamma}{1+\gamma}} e^{c_1 \lambda_{\gamma,m}^{\frac{1}{2}+\frac{1}{2\gamma}}}.$$

We therefore deduce from these estimates and the definition (4.3) of the isometry $M_{\gamma,n}$ that for all $n \in \mathbb{Z}^d \setminus \{0\}$ and $m \geq 0$,

$$\begin{aligned} \|\partial_x^\alpha (M_{\gamma,n} \phi_{\gamma,m})\|_{L^2(\mathbb{R}^d)} &= |n|^{\frac{|\alpha|}{1+\gamma}} \|\partial_x^\alpha \phi_{\gamma,m}\|_{L^2(\mathbb{R}^d)} \\ &\leq c^{1+|\alpha|+|\beta|} |n|^{\frac{|\alpha|}{1+\gamma}} (\alpha!)^{\frac{\gamma}{1+\gamma}} e^{c_1 \lambda_{\gamma,m}^{\frac{1}{2}+\frac{1}{2\gamma}}}. \end{aligned}$$

On the other hand, it immediately follows from the definition (4.2) of the functions φ_n that for all $\beta \in \mathbb{N}^d$ and all $n \in \mathbb{Z}^d \setminus \{0\}$, we have

$$\|\partial_y^\beta \varphi_n\|_{L^2(\mathbb{T}^d)} = n^\beta \leq |n|^{|\beta|}. \tag{4.21}$$

This ends the proof of the estimate for the eigenfunctions $\psi_{\gamma,n,m}$.

Let us now consider $\lambda \geq 0$ and $f \in E_\lambda(-\Delta_\gamma)$. We expand f as a linear combination of eigenfunctions,

$$f = \sum_{\lambda_{\gamma,n,m} \leq \lambda} a_{n,m} \psi_{\gamma,n,m}$$

with coefficients $a_{n,m}$. Using Hölder's inequality, the partial derivatives of the function f can then be bounded as

$$\|\partial_x^\alpha \partial_y^\beta f\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2 \leq N(\lambda) \sum_{\lambda_{\gamma,n,m} \leq \lambda} |a_{n,m}|^2 \|\partial_x^\alpha \partial_y^\beta \psi_{\gamma,n,m}\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2.$$

Recalling that $\lambda_{\gamma,n,m} \leq \lambda$ implies the inequalities $\lambda_{\gamma,m} \leq \lambda$ and $|n| \leq (\lambda/\lambda_{\gamma,0})^{(1+\gamma)/2}$, the claimed estimates for the function f therefore easily follow from the ones for the single eigenfunctions $\psi_{\gamma,n,m}$ and Lemma 4.7. □

Unlike the evolution operators generated by the operator $-\Delta_\gamma$ and its fractional powers, the eigenfunctions $\psi_{\gamma,n,m}$ and their linear combinations enjoy localization properties in the variable $x \in \mathbb{R}^d$, which are quantified in the following lemma.

Lemma 4.9. *There exist positive constants $c, c_1, c_2 > 0$ such that for all $n \in \mathbb{Z}^d \setminus \{0\}$ and $m \geq 0$,*

$$\|e^{c_2|n||x|^{1+\gamma}}\psi_{\gamma,n,m}\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq ce^{c_1\lambda_{\gamma,m}^{\frac{1}{2}+\frac{1}{2\gamma}}}.$$

Moreover, c can be chosen such that for all $\lambda \geq 0$ and $f \in E_\lambda(-\Delta_\gamma)$, we have

$$\|e^{c_2|x|^{1+\gamma}}f\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq c\lambda^{c_\gamma}e^{c_1\lambda^{\frac{1}{2}+\frac{1}{2\gamma}}},$$

where $c_\gamma > 0$ is the constant appearing in Lemma 4.7.

Proof. Taking into account again the tensorized structure (4.5) of each eigenfunction $\psi_{\gamma,n,m}$ and the definition (4.3) of the isometry $M_{\gamma,n}$, it follows that for all $c_2 > 0$, $n \in \mathbb{Z}^d \setminus \{0\}$, and $m \geq 0$,

$$\begin{aligned}\|e^{c_2|n||x|^{1+\gamma}}\psi_{\gamma,n,m}\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} &= \|e^{c_2|n||x|^{1+\gamma}}M_{\gamma,n}\phi_{\gamma,m}\|_{L^2(\mathbb{R}^d)}\|\varphi_n\|_{L^2(\mathbb{T}^d)}\\ &= \|e^{c_2|x|^{1+\gamma}}\phi_{\gamma,m}\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

where we have used that the functions φ_n are normalized in $L^2(\mathbb{T}^d)$. Moreover, it is known from the work [3, Theorem 2.1] that c_2 can be chosen such that with some positive constants $c, c_1 > 0$ for all $m \geq 0$ we have

$$\|e^{c_2|x|^{1+\gamma}}\phi_{\gamma,m}\|_{L^2(\mathbb{R}^d)} \leq ce^{c_1\lambda_{\gamma,m}^{\frac{1}{2}+\frac{1}{2\gamma}}}.$$

This proves the claim for the single eigenfunctions $\psi_{\gamma,n,m}$. Taking into account the bound $e^{c_2|x|^{1+\gamma}} \leq e^{c_2|n||x|^{1+\gamma}}$ for all $n \in \mathbb{Z}^d \setminus \{0\}$, the case of the functions in $E_\lambda(-\Delta_\gamma)$ is finally treated analogously as in the proof of Lemma 4.8. □

4.4. **Boundary case.** To end this section, let us consider the Baouendi–Grushin operator

$$\Delta_\gamma^{bd} = \Delta_x + |x|^{2\gamma}\Delta_y, \quad (x,y) \in \mathbb{R}^d \times (0,2\pi)^d,$$

where Δ_y denotes in this case the Laplacian on the hypercube $(0,2\pi)^d$ with Dirichlet boundary conditions. In the following, we explain how the results of this section fit for this operator.

First of all, recall that the spectrum of the operator $-\Delta_y$ is given by

$$\sigma(-\Delta_y) = \left\{\frac{|n|^2}{4} : n \in (\mathbb{N}^*)^d\right\}.$$

Moreover, for each $n \in (\mathbb{N}^*)^d$, let $\varphi_n \in L^2((0,2\pi)^d)$ be given by

$$\varphi_n(y) = \pi^{-\frac{d}{2}}\prod_{j=1}^d \sin\left(\frac{n_j y_j}{2}\right), \quad y \in (0,2\pi)^d.$$

Clearly, each such function φ_n is a normalized eigenfunction of the operator $-\Delta_y$ associated with the eigenvalue $|n|^2/4$, and the family $(\varphi_n)_n$ forms a Hilbert basis of the space $L^2((0,2\pi)^d)$. Correspondingly, the operator Δ_γ^{bd} is transformed as

$$\Delta_x + |x|^{2\gamma}\Delta_y \rightsquigarrow \Delta_x - \frac{|n|^2}{4}|x|^{2\gamma},$$

and respective variants of the diagonalization formula (4.1), the eigenvalues $\lambda_{\gamma,n,m}$ of Δ_γ^{bd}, and of the associated eigenfunctions $\psi_{\gamma,n,m}$ in (4.5) hold. Since also

$$\forall n \in (\mathbb{N}^*)^d,\ \alpha \in \mathbb{N}^d, \quad \|\partial_y^\alpha \varphi_n\|_{L^2((0,2\pi)^d)} = \frac{n^\alpha}{2^{|\alpha|}} \leq \left(\frac{|n|}{2}\right)^{|\alpha|},$$

corresponding variants of Lemmas 4.1, 4.4 and 4.6 for the evolution operators generated by Δ_γ^{bd} and of Lemmas 4.8 and 4.9 for its eigenfunctions can be proved in exactly the same fashion.

5. Unique continuation estimates for the Baouendi–Grushin equations

In this section, we prove unique continuation estimates for the semigroups generated by (fractional) Baouendi–Grushin operators by combining the abstract framework from Section 3 with the smoothing and localizing properties established in Section 4. As a byproduct, we also obtain spectral inequalities for the Baouendi–Grushin operators from thick sensor sets, and prove two results regarding their optimality. Finally, we derive spectral inequalities for the (linear combinations of) eigenfunctions of the Baouendi–Grushin operators.

5.1. Unique continuation from thick sets.

In this first subsection, we prove Theorem 2.3 and Corollary 2.4.

Proof of Theorem 2.3 and Corollary 2.4. It follows from Corollary 4.5 that the function $f = e^{-(t_x+t_y)(-\Delta_\gamma)^{(1+\gamma)/2}} g$ for $t_x \in (0,t_0)$, $t_y > 0$ satisfies the smoothing estimate (3.6) with $D = c\|g\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}/(t_x)^{d/\gamma}$, $c_x = c/(t_x)^{\gamma/(\gamma+1)}$, $c_y = c/t_y$, and $\mu = 1$. In light of the hypothesis that ω is (θ, L_x, L_y)-thick in $\mathbb{R}^d \times \mathbb{T}^d$, we are therefore in the setting of Example 3.7. Working with $\varepsilon t_x^{2d/\gamma}/c^2$ instead of ε there, we conclude that there exists a constant $K = K(d,\gamma) > 0$ such that for all $t_x \in (0,t_0)$, $t_y > 0$, $\varepsilon > 0$, and $g \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$, we have

$$\|e^{-(t_x+t_y)(-\Delta_\gamma)^{\frac{1+\gamma}{2}}} g\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq \left(\frac{K}{\theta}\right)^{KC_{\varepsilon,\gamma,t_x,t_y}} \|e^{-(t_x+t_y)(-\Delta_\gamma)^{\frac{1+\gamma}{2}}} g\|^2_{L^2(\omega)} + \varepsilon \|g\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}$$

with

$$C_{\varepsilon,\gamma,t_x,t_y} = (1 - \log\varepsilon - \log t_x) \exp\left(\frac{KL_x}{(t_x)^{\frac{\gamma}{\gamma+1}}} + \frac{KL_y}{t_y}\right),$$

upon a suitable adaptation of the constant K, especially in order to absorb the constant c. The claim of Theorem 2.3 with $t_0/2$ instead of t_0 then follows with the particular choice $t_x = t_y = t \in (0, t_0/2)$.

In order to derive Corollary 2.4, we take specifically $g = e^{(t_x+t_y)(-\Delta_\gamma)^{\frac{1+\gamma}{2}}} f \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ with $f \in \mathcal{E}_\lambda(-\Delta_\gamma) = \mathcal{E}_{\lambda^{\frac{1+\gamma}{2}}}((-\Delta_\gamma)^{\frac{1+\gamma}{2}})$ and any choices of $t_x \in (0,t_0)$ and $t_y > 0$. We then have by functional calculus that

$$\|g\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq e^{2(t_x+t_y)\lambda^{\frac{1+\gamma}{2}}} \|f\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}.$$

Choosing $\varepsilon = \frac{1}{2} e^{-2(t_x+t_y)\lambda^{\frac{1+\gamma}{2}}}$, the claim of Corollary 2.4 follows from the above after reordering the terms.

The stronger statements in the case $\gamma = 1$ are an easy modification of the above since then Corollary 4.5 guarantees that $f = e^{(t_x+t_y)\Delta_\gamma} g$ satisfies (3.6) for all $t_x, t_y > 0$ and with instead $D = c\|g\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}$. As a consequence, the term $\log t_x$ is removed from $C_{\varepsilon,1,t_x,t_y}$. In the context of Corollary 2.4, we then finally choose $t_x = L_x^2$ and $t_y = L_y$. This completes the proof. □

5.2. **Optimality.** In this subsection, we tackle the proof of the optimality results stated in Section 2.1, namely, Propositions 2.6 and 2.7. First of all, we check that the thickness is a necessary geometric condition in order to get spectral inequalities for the Baouendi–Grushin operators.

Proof of Proposition 2.6. Consider a measurable subset $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ with positive measure and a non-negative energy level $\lambda \geq 0$. Suppose that for some constant $c_{\lambda,\omega} > 0$ we have the spectral inequality

$$\forall f \in \mathcal{E}_\lambda(-\Delta_\gamma), \quad \|f\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq c_{\lambda,\omega}\|f\|_{L^2(\omega)}. \tag{5.1}$$

We aim at proving that the set ω is thick in the sense of Definition 2.1. Since 0 is an eigenvalue of the Laplace operator Δ_y on the d-dimensional torus $\mathbb{T}^d$, we have the inclusion

$$\mathcal{E}_\lambda(-\Delta_x) \subset \mathcal{E}_\lambda(-\Delta_\gamma), \tag{5.2}$$

where $\mathcal{E}_\lambda(-\Delta_x)$ denotes the spectral subspace of the operator $-\Delta_x$ on the whole Euclidean space $\mathbb{R}^d$ associated with the energy level λ, that is,

$$\mathcal{E}_\lambda(-\Delta_x) = 1_{[0,\lambda]}(-\Delta_x) = \Big\{ f \in L^2(\mathbb{R}^d) : \mathrm{Supp}(\mathscr{F} f) \subset \overline{B(0,\sqrt{\lambda})} \Big\},$$

where $\mathscr{F}$ denotes the Fourier transform. The following is inspired by [26, p.113]. Let us consider the function $f \in L^2(\mathbb{R}^d)$ defined by $f = \mathscr{F}^{-1}(1_{[-c\sqrt{\lambda},c\sqrt{\lambda}]^d})$, where $c > 0$ is chosen such that $[-c\sqrt{\lambda}, c\sqrt{\lambda}]^d \subset B(0,\sqrt{\lambda})$ and where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. Fixing $x_0 \in \mathbb{R}^d$, we also consider the translate $f_{x_0} \in L^2(\mathbb{R}^d)$ of f defined by

$$f_{x_0}(x) = f(x - x_0), \quad x \in \mathbb{R}^d.$$

Since the Fourier transform of the function f_{x_0} coincides with the one of the function f, up to a complex factor of modulus 1, we also have $f_{x_0} \in \mathcal{E}_\lambda(-\Delta_x)$. Notice that every function $g \in L^2(\mathbb{R}^d)$ can be treated as a function in $L^2(\mathbb{R}^d \times \mathbb{T}^d)$ that is constant with respect to the $\mathbb{T}^d$-variable. In this sense, recalling the inclusion (5.2) of the spectral subspaces, we deduce that $f_{x_0} \in \mathcal{E}_\lambda(-\Delta_\gamma)$. As a consequence of the spectral inequality (5.1), we then obtain

$$\|f\|^2_{L^2(\mathbb{R}^d)} = \|f_{x_0}\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \leq c^2_{\lambda,\omega}\|f_{x_0}\|^2_{L^2(\omega)} = c^2_{\lambda,\omega}\int_{\mathbb{T}^d} \|f_{x_0}\|^2_{L^2(\omega_y)}\,\mathrm{d}y,$$

where we set

$$\omega_y = \big\{x \in \mathbb{R}^d \colon (x,y) \in \omega\big\}, \quad y \in \mathbb{T}^d.$$

Given some $y \in \mathbb{T}^d$, we now have to control the norm $\|f_{x_0}\|_{L^2(\omega_y)}$. To this end, let $L = L_{\lambda,\omega} > 0$ be a radius whose value will be adjusted later. We split the norm into two parts,

$$\begin{aligned}\|f_{x_0}\|^2_{L^2(\omega_y)} &= \int_{(\omega_y - x_0)\cap[-L,L]^d} |f(x)|^2\,\mathrm{d}x + \int_{(\omega_y-x_0)\cap([-L,L]^d)^c} |f(x)|^2\,\mathrm{d}x \\ &\leq \int_{(\omega_y-x_0)\cap[-L,L]^d} |f(x)|^2\,\mathrm{d}x + \int_{|x|>L} |f(x)|^2\,\mathrm{d}x.\end{aligned}$$

On the one hand, since $f \in L^2(\mathbb{R}^d)$, the dominated convergence theorem implies that the radius $L \gg 1$ can be chosen large enough such that

$$c^2_{\lambda,\omega}\int_{|x|>L} |f(x)|^2\,\mathrm{d}x \leq \frac{1}{2}\|f\|^2_{L^2(\mathbb{R}^d)}.$$

On the other hand, the Fourier inversion formula, the Cauchy-Schwarz inequality, and Plancherel's theorem imply that

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^d}\|\mathscr{F} f\|_{L^1(\mathbb{R}^d)} \leq \frac{(2c\sqrt{\lambda})^{\frac{d}{2}}}{(2\pi)^d}\|\mathscr{F} f\|_{L^2(\mathbb{R}^d)} = \left(\frac{c\sqrt{\lambda}}{\pi}\right)^{\frac{d}{2}}\|f\|_{L^2(\mathbb{R}^d)}.$$

We therefore deduce that

$$\int_{(\omega_y-x_0)\cap[-L,L]^d} |f(x)|^2\,\mathrm{d}x \leq |(\omega_y - x_0)\cap[-L,L]^d| \|f\|^2_{L^\infty(\mathbb{R}^d)}$$
$$\leq |\omega_y \cap (x_0+[-L,L]^d)| \left(\frac{c\sqrt{\lambda}}{\pi}\right)^d \|f\|^2_{L^2(\mathbb{R}^d)}$$

since the Lebesgue measure is invariant by translation. Gathering the above estimates, we conclude that there exists a ratio $\theta = \theta_{\lambda,\omega} \in (0,1]$ such that for all $x_0 \in \mathbb{R}^d$,

$$|\omega \cap ((x_0+[-L,L]^d)\times\mathbb{T}^d)| = \int_{\mathbb{T}^d} |\omega_y \cap (x_0+[-L,L]^d)|\,\mathrm{d}y \geq \theta(2L)^d,$$

that is, the set ω is thick in the sense of Definition 2.1. □

We now check that the power $\lambda^{(1+\gamma)/2}$ appearing in the spectral inequalities stated in Corollary 2.4 is optimal. The following proof follows the same strategy as the one of [6, Theorem 2.17 (i) and Theorem 2.19].

Proof of Proposition 2.7. Let $\omega \subset \mathbb{R}^d \times \mathbb{T}^d$ be a measurable set with positive measure satisfying the geometric condition $\overline{\omega} \cap \{x = 0\} = \emptyset$. Recall that we aim at constructing a sequence $((\lambda_n, \psi_n))_n$ of eigenpairs of the operator $-\Delta_\gamma$, with $\lambda_n \to +\infty$, such that

$$\|\psi_n\|_{L^2(\mathbb{R}^d\times\mathbb{T}^d)} \geq c_0 e^{c_1 \lambda_n^{\frac{1+\gamma}{2}}} \|\psi_n\|_{L^2(\omega)}. \tag{5.3}$$

To this end, let $\lambda_\gamma > 0$ be the smallest eigenvalue of the anharmonic oscillator H_γ, and let $\phi_\gamma \in L^2(\mathbb{R}^d)$ be an associated normalized eigenfunction. For each $n \in \mathbb{Z}^d \setminus \{0\}$, let us also consider the function $\psi_{\gamma,n,0} \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$ defined as in (4.5) by

$$\psi_{\gamma,n,0}(x,y) = e^{in\cdot y}(M_{\gamma,n}\phi_\gamma)(x), \quad (x,y) \in \mathbb{R}^d \times \mathbb{T}^d,$$

where $M_{\gamma,n}$ denotes the unitary transform on $L^2(\mathbb{R}^d)$ given by (4.3). Recall from Section 4.1.3 that the function $\psi_{\gamma,n,0}$ is a normalized eigenfunction of the operator $-\Delta_\gamma$, associated with the eigenvalue $\lambda_n := |n|^{2/(1+\gamma)}\lambda_{\gamma,0}$. Moreover, it is clear that

$$\|\psi_{\gamma,n,0}\|_{L^2(\omega)} = \|\phi_\gamma\|_{L^2(\omega_n)} \quad \text{where} \quad \omega_n = \big\{(|n|^{1/(1+\gamma)}x, y) : (x,y) \in \omega\big\}, \tag{5.4}$$

and where ϕ_γ is interpreted as a function in $L^2(\mathbb{R}^d \times \mathbb{T}^d)$ that is constant with respect to the y-variable. Recall from [12, Theorem 3.3] that the eigenfunction ϕ_γ satisfies the decay property

$$\big\|e^{\varepsilon|x|^{1+\gamma}}\phi_\gamma\big\|_{L^2(\mathbb{R}^d)} \leq c_{\varepsilon,\gamma},$$

where $\varepsilon > 0$ and $c_{\varepsilon,\gamma} > 0$ are positive constants. Thus, setting $L := d(0,\omega) > 0$, we deduce that for all $n \in \mathbb{Z}^d \setminus \{0\}$,

$$\|\phi_\gamma\|_{L^2(\omega_n)} = \big\|e^{-\varepsilon|x|^{1+\gamma}} e^{\varepsilon|x|^{1+\gamma}}\phi_\gamma\big\|_{L^2(\omega_n)} \leq c_{\varepsilon,\gamma} e^{-\varepsilon|n|L^{1+\gamma}}.$$

Taking into account that $\psi_n = \psi_{\gamma,n,0} \in L^2(\mathbb{R}^d\times\mathbb{T}^d)$ is normalized, combining this estimate with (5.4) and writing

$$|n|L^{1+\gamma} = \left(\frac{L}{\sqrt{\lambda_{\gamma,0}}}\right)^{1+\gamma} (|n|^{\frac{2}{1+\gamma}}\lambda_{\gamma,0})^{\frac{1+\gamma}{2}} = \left(\frac{L}{\sqrt{\lambda_{\gamma,0}}}\right)^{1+\gamma} \lambda_n^{\frac{1+\gamma}{2}},$$

we deduce that (5.3) holds, which proves the claim. □

5.3. **Eigenfunctions.** In this subsection, we give the proofs of Theorems 2.16 and 2.18 regarding (linear combinations of) eigenfunctions of the operator Δ_γ.

Proof of Theorem 2.16. Take $R_\omega > 0$ such that $\theta_\omega := |\omega \cap (-R_\omega, R_\omega)^d \times \mathbb{T}^d| > 0$, and fix $n \in \mathbb{Z}^d \setminus \{0\}$ and $m \geq 0$. It follows from Lemmas 4.8 and 4.9 that the eigenfunction $\psi_{\gamma,n,m}$ satisfies

$$\forall \alpha, \beta \in \mathbb{N}^d, \quad \|\partial_x^\alpha \partial_y^\beta \psi_{\gamma,n,m}\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq D c_x^{|\alpha|} c_y^{|\beta|} (\alpha!)^{\mu_x} (\beta!)^{\mu_y} \tag{5.5}$$

where

$$D = c e^{c_1 \lambda_{\gamma,m}^{\frac{1+\gamma}{2\gamma}}}, \quad c_x = c|n|^{\frac{1}{1+\gamma}}, \quad \mu_x = \frac{\gamma}{1+\gamma}, \quad c_y = c|n|, \quad \mu_y = 0,$$

and also (3.8) with

$$c_0 = c_2|n|, \quad \nu = \frac{1}{1+\gamma}.$$

Note that (5.5) is a variant of (3.6) that takes into account the different smoothing properties with respect to the x and y coordinates. In turn, upon replacing Corollary 3.2 in the general framework by Proposition A.1 and adapting Lemma 3.4 and Corollary 3.6 accordingly, we conclude as in Example 3.10 that for every $R \geq R_\omega$ we have

$$\|\psi_{\gamma,n,m}\|^2_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq \left(\frac{K}{\theta_R}\right)^{KC_R} \|\psi_{\gamma,n,m}\|^2_{L^2(\omega)} + 2c^2 e^{2c_1 \lambda_{\gamma,m}^{\frac{1+\gamma}{2\gamma}}} e^{-2c_2|n|R^{1+\gamma}} \tag{5.6}$$

with

$$\theta_R = \frac{|\omega \cap (-R,R)^d \times \mathbb{T}^d|}{|(-R,R)^d \times \mathbb{T}^d|} \geq \frac{\theta_\omega}{(2R)^d} > 0, \tag{5.7}$$

and

$$C_R = 1 + |n|R^{1+\gamma} + |n|R^{1+\gamma} + |n|,$$

and where $K > 0$ depends only on γ and the dimension d.

Let us now consider $p > 1$ satisfying

$$c^2 e^{2c_1(1-p)\lambda_{\gamma,0}^{\frac{1+\gamma}{2\gamma}}} \leq \frac{1}{2}.$$

Note that p depends only on γ, c, and c_1. We distinguish two cases and write $a \lesssim b$ for $a, b > 0$ if the quotient a/b is bounded by a constant depending at most on ω, γ, and the dimension d.

▷ *Case 1:* $c_2|n|R_\omega^{1+\gamma} > pc_1 \lambda_{\gamma,m}^{\frac{1+\gamma}{2\gamma}}$. In this situation, we take $R = R_\omega$ in (5.6). We then have

$$c^2 e^{2c_1 \lambda_{\gamma,m}^{\frac{1+\gamma}{2\gamma}}} e^{-2c_2|n|R^{1+\gamma}} \leq c^2 e^{2c_1(1-p)\lambda_{\gamma,m}^{\frac{1+\gamma}{2\gamma}}} \leq c^2 e^{2c_1(1-p)\lambda_{\gamma,0}^{\frac{1+\gamma}{2\gamma}}} \leq \frac{1}{2}, \tag{5.8}$$

by the choice of p. Moreoever, we have $|n| \lesssim |n|R_\omega^{1+\gamma} \lesssim \lambda_{\gamma,n,m}^{\frac{1+\gamma}{2}}$ and, thus,

$$C_{R_\omega} \lesssim \lambda_{\gamma,n,m}^{\frac{1+\gamma}{2}}.$$

Since also

$$\log\left(\frac{K}{\theta_{R_\omega}}\right) \lesssim \log(1 + R_\omega) \lesssim 1 \lesssim \log(1 + \lambda_{\gamma,0,0}) \leq \log(1 + \lambda_{\gamma,n,m}),$$

inequality (5.6) is in this case consistent with the claimed inequality for $\psi_{\gamma,n,m}$, upon a suitable adaptation of the constant K.

▷ *Case 2:* $c_2|n|R_\omega^{1+\gamma} \leq pc_1 \lambda_{\gamma,m}^{\frac{1+\gamma}{2\gamma}}$. In this case, there exists some $R_0 \geq R_\omega$ with

$$c_2|n|R_0^{1+\gamma} = pc_1 \lambda_{\gamma,m}^{\frac{1+\gamma}{2\gamma}}.$$

Choosing $R = R_0$ in (5.6), the inequality (5.8) is still valid, and we have

$$|n| \lesssim |n|R_\omega^{1+\gamma} \leq |n|R_0^{1+\gamma} = \frac{pc_1}{c_2} \lambda_{\gamma,m}^{\frac{1+\gamma}{2\gamma}} \leq \frac{pc_1}{c_2} \lambda_{\gamma,0}^{\frac{1+\gamma}{2\gamma}} \left(\frac{\lambda_{\gamma,n,m}}{\lambda_{\gamma,0}}\right)^{\frac{1+\gamma}{2\gamma}} \lesssim \lambda_{\gamma,n,m}^{\frac{1+\gamma}{2}},$$

since $\gamma \geq 1$, so that again

$$C_{R_0} \lesssim \lambda_{\gamma,n,m}^{\frac{1+\gamma}{2}}.$$

At the same time, R_0 satisfies

$$R_0^{1+\gamma} \leq |n| R_0^{1+\gamma} \lesssim \lambda_{\gamma,n,m}^{\frac{1+\gamma}{2}},$$

so that

$$\log\left(\frac{K}{\theta_{R_0}}\right) \lesssim \log(1+\lambda_{\gamma,n,m}).$$

We therefore conclude that also in this case (5.6) is consistent with the claim for $\psi_{\gamma,n,m}$, again with a suitable adaptation of the constant K. This ends the proof of Theorem 2.16. □

Let us finally turn to linear combinations of eigenfunctions of the operator Δ_γ.

Proof of Theorem 2.18. Let R_ω be as in the proof of Theorem 2.16 above, and fix some $f \in E_\lambda(-\Delta_\gamma)$. It then follows from Lemmas 4.8 and 4.9 that f satisfies (5.5) and (3.8) with instead

$$D = c\lambda^{c_\gamma} e^{c_1 \lambda^{\frac{1+\gamma}{2\gamma}}} \|f\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}, \quad c_x = c\sqrt{\lambda}, \quad \mu_x = \frac{\gamma}{1+\gamma}, \quad c_y = c\lambda^{\frac{1+\gamma}{2}}, \quad \mu_y = 0$$

and

$$c_0 = c_2, \quad \nu = \frac{1}{1+\gamma}.$$

We then conclude in the same way as in the proof of Theorem 2.16 that for every $R \geq R_\omega$ we have

$$\|f\|^2_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq \left(\frac{K}{\theta_R}\right)^{KC_R} \|f\|^2_{L^2(\omega)} + c^2 \lambda^{2c_\gamma} e^{2c_1 \lambda^{\frac{1+\gamma}{2\gamma}}} e^{-2c_2 R^{1+\gamma}} \|f\|^2_{L^2(\mathbb{R}^d \times \mathbb{T}^d)},$$

with θ_R given by (5.7) and

$$C_R = 1 + R^{1+\gamma} + \lambda^{\frac{1+\gamma}{2}} R^{1+\gamma} + \lambda^{\frac{1+\gamma}{2}},$$

and where $K > 0$ depends only on γ and the dimension d. Now, we choose R such that

$$c^2 \lambda^{2c_\gamma} e^{2c_1 \lambda^{\frac{1+\gamma}{2\gamma}}} e^{-2c_2 R^{1+\gamma}} \leq \frac{1}{2}.$$

We obviously have the bound

$$R^{1+\gamma} \lesssim \lambda^{\frac{1+\gamma}{2\gamma}},$$

so that

$$C_R \lesssim \lambda^{(\frac{1}{2}+\frac{1}{2\gamma})(1+\gamma)}$$

and

$$\log(K/\theta_R) \lesssim \log(1+\lambda).$$

This proves the claim. □

6. The anisotropic Shubin operators

In this last section, we apply the general framework from Section 3 to the evolution equations associated to the anisotropic Shubin operators, given by

$$H_{k,m} = (-\Delta)^m + |x|^{2k}, \quad x \in \mathbb{R}^d,$$

where $k, m \geq 1$ are positive integers. Recall (e.g. from [3, Section 2]) that the operators $H_{k,m}$ equipped with the domains

$$D(H_{k,m}) = \left\{ g \in L^2(\mathbb{R}^d) : H_{k,m} g \in L^2(\mathbb{R}^d) \right\},$$

generate strongly continuous semigroups $(e^{-tH_{k,m}})_{t\geq 0}$ on $L^2(\mathbb{R}^n)$. It is also known from the work [3] that the associated evolution operators $e^{-tH_{k,m}}$ enjoy very strong smoothing and localizing properties in Gelfand–Shilov spaces, for precise estimates see (6.1) and (6.2)

below, which allow us to derive quantitative unique continuation estimates for them. As a byproduct, we obtain positive cost-uniform approximate null-controllability results for the evolution equations associated to the operators $H_{k,m}$, which are qualitatively new in the particular case of $k = m = 1$.

6.1. **Unique continuation estimates.** The aforementioned smoothing and localizing result for the evolution operators generated by $H_{k,m}$ is precisely [3, Theorem 2.3]. It states that there exist positive constants $c, c_1 > 0$ and $t_0 \in (0,1)$ such that for all $\alpha \in \mathbb{N}^d$, $t \in (0,t_0)$, and $g \in L^2(\mathbb{R}^d)$,

$$\big\|\partial_x^\alpha (e^{-tH_{k,m}} g)\big\|_{L^2(\mathbb{R}^d)} \leq \frac{c^{1+|\alpha|}}{t^{\frac{k|\alpha|}{k+m}}} (\alpha!)^{\frac{k}{k+m}} \|g\|_{L^2(\mathbb{R}^d)}, \tag{6.1}$$

and

$$\big\|e^{c_1 t|x|^{1+\frac{k}{m}}} (e^{-tH_{k,m}} g)\big\|_{L^2(\mathbb{R}^d)} \leq c\|g\|_{L^2(\mathbb{R}^d)}. \tag{6.2}$$

These enable us to obtain the following quantitative unique continuation estimates.

Theorem 6.1. *Let $\omega \subset \mathbb{R}^d$ be measurable with positive measure, and let $R_\omega \in [0,+\infty)$ be the radius defined by*

$$R_\omega = \inf\big\{R > 0 : |\omega \cap (-R,R)^d| > 0\big\}. \tag{6.3}$$

There exist some positive constants $c, c_1, K > 0$ and $t_0 \in (0,1)$ such that for all $t \in (0,t_0)$, $R > R_\omega$, and $g \in L^2(\mathbb{R}^d)$,

$$\big\|e^{-tH_{k,m}} g\big\|^2_{L^2(\mathbb{R}^d)} \leq \left(\frac{K}{\theta_R}\right)^{K(1+tR^{1+\frac{k}{m}}+R^{1+\frac{k}{m}}/t^{\frac{k}{m}})} \big\|e^{-tH_{k,m}} g\big\|^2_{L^2(\omega)} + ce^{-c_1 tR^{1+\frac{k}{m}}} \|g\|^2_{L^2(\mathbb{R}^d)},$$

where the ratio $\theta_R \in (0,1]$ is given by

$$\theta_R = \frac{|\omega \cap (-R,R)^d|}{(2R)^d}. \tag{6.4}$$

Proof. In line of Remark 3.11 (2), we infer from (6.1) and (6.2) that each $f = e^{-tH_{k,m}} g$ satisfies the $\mathbb{R}^d$ variants of (3.6) and (3.8) with

$$D = c\|g\|_{L^2(\mathbb{R}^d)}, \quad c_x = \frac{c}{t^{\frac{k}{k+m}}}, \quad \mu = \frac{k}{k+m}$$

and

$$c_0 = c_1 t, \quad \nu = \frac{m}{k+m}.$$

The claim with c and c_1 replaced by $2c^2$ and $2c_1$, respectively, therefore follows just as in Example 3.10, and it only remains to relabel the constants accordingly. □

6.2. **Approximate null-controllability.** The quantitative estimates stated in Theorem 6.1 have consequences on the cost-uniform approximate null-controllability properties of the evolution equations associated with the operators $H_{k,m}$, given by

$$(E_{k,m}) \qquad \begin{cases} \partial_t f(t,x) + H_{k,m} f(t,x) = h(t,x) 1_\omega(x), & t > 0,\ x \in \mathbb{R}^d, \\ f(0,\cdot) = f_0 \in L^2(\mathbb{R}^d), \end{cases}$$

where $\omega \subset \mathbb{R}^d$ is a measurable set with positive measure and $h \in L^2((0,T)\times\omega)$ is a control. The notions of exact and cost-uniform approximate null-controllability for the equation $(E_{\gamma,s})$ from Definitions 2.8 and 2.11 and their interpretation in terms of (weak) observability estimates in (2.2) and Proposition 2.12, respectively, carry over to the equation $(E_{k,m})$ verbatim. With this in mind, a straightforward consequence of Theorem 6.1 is the following null-controllability result.

Corollary 6.2. *The evolution equation* $(E_{k,m})$ *is cost-uniformly approximately null-controllable from every measurable set* $\omega \subset \mathbb{R}^d$ *with positive measure and in every positive time* $T > 0$.

Remark 6.3. The exact null-controllability properties of the equation $(E_{k,m})$ (and its fractional counterparts) have been widely studied, see, e.g., [9, 11, 18, 34, 35] for the specific case $k = m = 1$, and [6, 16, 17, 32, 33] where the general case $k, m \geq 1$ is considered. In this regard, Corollary 6.2 does not give a qualitatively new result when $(k, m) \neq (1, 1)$. In fact, we know from [32, Theorem 2.5] that when $(k, m) \neq (1, 1)$, then the evolution equation $(E_{k,m})$ is even exactly null-controllable in every positive time $T > 0$ and from every measurable control support $\omega \subset \mathbb{R}^d$ with positive measure. In the case $k = m = 1$, however, the exact null-controllability properties of the harmonic heat equation $(E_{1,1})$ are not yet fully understood. On the one hand, it is known from [35, Theorem 1.10] (see also [21, Proposition 5.1]) that the equation $(E_{1,1})$ is not exactly null-controllable in any positive time whenever the control support $\omega \subset \mathbb{R}^d$ is contained in a half space. In fact, it can be readily checked that a half space satisfies a geometric condition of the form

$$\forall x \in \mathbb{R}^d, \quad |\omega \cap B(x, \rho(x))| \geq \theta |B(x, \rho(x))|, \tag{6.5}$$

with a function $\rho : \mathbb{R}^d \to \mathbb{R}_+$ taking the form

$$\rho(x) = L\langle x \rangle, \quad x \in \mathbb{R}^d,$$

with some $L > 0$. This raises the question whether local scales ρ can be allowed that exhibit an arbitrary sublinear growth. This question has not been answered yet, but a first step in this direction is made by the result [6, Corollary 2.13] stating that when the control support $\omega \subset \mathbb{R}^d$ satisfies the geometric condition (6.5) with a function $\rho : \mathbb{R}^d \to \mathbb{R}_+$ satisfying

$$\rho(x) \leq \frac{L\langle x \rangle}{(g \circ g)^\alpha(|x|) g(|x|)} \quad \text{where} \quad g(r) = \log(e + r), \quad r \geq 0,$$

with some $L > 0$ and $\alpha > 2$, then the equation $(E_{1,1})$ is exactly null-controllable from the control support ω in every positive time $T > 0$. By contrast, Corollary 6.2 shows that the cost-uniform approximate null-controllability properties of the equation $(E_{1,1})$ are far more simple since they hold in every positive time $T > 0$ and from every measurable control support $\omega \subset \mathbb{R}^d$ with positive measure.

6.3. **Other models.** The results presented in this section for the anisotropic Shubin semigroups can be generalized to any other strongly continuous semigroup $(e^{-tA})_{t \geq 0}$ acting on $L^2(\mathbb{R}^d)$ and also satisfying Gelfand–Shilov smoothing and localizing properties of the form (6.1) and (6.2). Rather than writing a very general result, we just present an example different from the anisotropic Shubin operator.

Example 6.4. Let us consider the anharmonic oscillator

$$H = -\Delta + i|x|^2, \quad x \in \mathbb{R}^d.$$

The operator H equipped with the domain

$$D(H) = \{ g \in L^2(\mathbb{R}^d) : Hg \in L^2(\mathbb{R}^d) \}$$

is known to generate a strongly continuous semigroup $(e^{-tH})_{t \geq 0}$ on $L^2(\mathbb{R}^d)$. Moreover, it follows from the work [4, Theorem 2.6] that there exist positive constants $c, c_1 > 0$ and $t_0 \in (0, 1)$ such that for all $\alpha \in \mathbb{N}^d$, $t \in (0, t_0)$, and $g \in L^2(\mathbb{R}^d)$,

$$\left\| \partial_x^\alpha (e^{-tH} g) \right\|_{L^2(\mathbb{R}^d)} \leq \frac{c^{1+|\alpha|}}{t^{\frac{|\alpha|}{2}}} \sqrt{\alpha!}\, \|g\|_{L^2(\mathbb{R}^d)},$$

and

$$\left\| e^{c_1 t^3 |x|^2} (e^{-tH} g) \right\|_{L^2(\mathbb{R}^d)} \leq c \|g\|_{L^2(\mathbb{R}^d)}.$$

We thus deduce exactly as in the proof of Theorem 6.1, but with

$$D = c\|g\|_{L^2(\mathbb{R}^d)}, \quad c_x = \frac{c}{\sqrt{t}}, \quad \mu = \frac{1}{2}$$

and

$$c_0 = c_1 t^3, \quad \nu = \frac{1}{2},$$

that for every measurable set $\omega \subset \mathbb{R}^d$ with positive measure, there exists some positive constant $K > 0$ such that for all $t \in (0, t_0)$, $R > R_\omega$, and $g \in L^2(\mathbb{R}^d)$,

$$\|e^{-tH} g\|^2_{L^2(\mathbb{R}^d)} \leq \left(\frac{K}{\theta_R}\right)^{K(1+t^3R^2+R^2/t)} \|e^{-tH} g\|^2_{L^2(\omega)} + 2c^2 e^{-2c_1 t^3 R^2} \|g\|^2_{L^2(\mathbb{R}^d)},$$

where the radius $R_\omega \in [0, +\infty)$ is as in (6.3) and the ratio $\theta_R \in (0, 1]$ is as in (6.4).

Appendix A. A local estimate with anisotropic smoothing

We have the following variant of Corollary 3.2.

Proposition A.1. *Let $Q = \Psi((0,1)^{2d})$, where the mapping $\Psi\colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is given by $\Psi(z) = z_0 + (\ell_x, \ell_y)z$, $z \in \mathbb{R}^{2d}$, with some $z_0 \in \mathbb{R}^{2d}$ and $\ell_x, \ell_y \in (0,\infty)^d$. Moreover, suppose that $f \in C^\infty(Q)$ satisfies*

$$\forall \alpha, \beta \in \mathbb{N}^d, \quad \|\partial_x^\alpha \partial_y^\beta f\|_{L^2(Q)} \leq A B_x^\alpha B_y^\beta (|\alpha|!)^{\mu_x} (|\beta|!)^{\mu_y} \|f\|_{L^2(Q)}$$

with some $A > 0$, $B_x, B_y \in (0,\infty)^d$, and $\mu_x, \mu_y \in [0, 1)$. Then, there is a constant $K = K(d, \mu_x, \mu_y) > 0$, depending only on μ_x, μ_y, and the dimension d, such that for every measurable set $E \subset Q$ of positive measure we have

$$\|f\|^2_{L^2(Q)} \leq \left(\frac{K|Q|}{|E|}\right)^{KC} \|f\|^2_{L^2(E)}$$

with

$$C = 1 + \log(A) + (|\ell_x B_x|)^{\frac{1}{1-\mu_x}} + (|\ell_y B_y|)^{\frac{1}{1-\mu_y}}.$$

For the proof of Proposition A.1, we rely on the following particular case of [23, Proposition 31] tailored to our situation.

Proposition A.2. *Suppose that $g \in W^{\infty,2}((0,1)^d) = \bigcap_{k\in\mathbb{N}} W^{k,2}((0,1)^d)$ satisfies*

$$\forall m \in \mathbb{N}, \quad \sum_{|\alpha|=m} \frac{1}{\alpha!} \|\partial^\alpha g\|^2_{L^2((0,1)^d)} \leq \frac{C(m)}{m!} \|g\|^2_{L^2((0,1)^d)}$$

with constants $C(m) > 0$ such that

$$h := \sum_{m\in\mathbb{N}} \sqrt{C(m)} \frac{(10d)^m}{m!} < \infty.$$

Then, there is a constant $K = K(d) > 0$, only depending on the dimension d, such that for every measurable set $\mathcal{E} \subset (0,1)^d$ of positive measure we have

$$\|g\|^2_{L^2((0,1)^d)} \leq \left(\frac{K}{|\mathcal{E}|}\right)^{K(1+\log h)} \|g\|^2_{L^2(\mathcal{E})}.$$

Proof of Proposition A.1. We may assume that f is not identically equal to zero. Moreover, it is easy to see that the function $g := f \circ \Psi \in W^{\infty,2}((0,1)^{2d})$ satisfies

$$\|\partial_x^\alpha \partial_y^\beta g\|_{L^2((0,1)^{2d})} \leq A(\ell_x B_x)^\alpha (\ell_y B_y)^\beta (|\alpha|!)^{\mu_x} (|\beta|!)^{\mu_y} \|g\|_{L^2((0,1)^{2d})}.$$

Taking into account that

$$\sum_{|\alpha|=k} \frac{(\ell_x B_x)^{2\alpha}}{\alpha!} = \frac{|\ell_x B_x|^{2k}}{k!} \quad \text{and} \quad \sum_{|\alpha|=m-k} \frac{(\ell_y B_y)^{2\alpha}}{\alpha!} = \frac{|\ell_y B_y|^{2(m-k)}}{(m-k)!}$$

for $k \in \mathbb{N}$ with $0 \leq k \leq m$, we observe

$$m!A^2 \sum_{|(\alpha,\beta)|=m} \frac{1}{\alpha!\beta!}(\ell_x B_x)^{2\alpha}(\ell_y B_y)^{2\beta}(|\alpha|!)^{2\mu_x}(|\beta|!)^{2\mu_y}$$
$$\leq A^2 \sum_{k=0}^{m} \binom{m}{k} (k!)^{2\mu_x} |\ell_x B_x|^{2k} ((m-k)!)^{2\mu_y} |\ell_y B_y|^{2(m-k)} \leq C(m)$$

with

$$C(m) := \left(A \sum_{k=0}^{m} \binom{m}{k} (k!)^{\mu_x} |\ell_x B_x|^k ((m-k)!)^{\mu_y} |\ell_y B_y|^{(m-k)} \right)^2 .$$

Thus, the function g satisfies

$$\forall m \in \mathbb{N}, \quad \sum_{|(\alpha,\beta)|=m} \frac{1}{\alpha!\beta!} \|\partial_x^\alpha \partial_y^\beta g\|^2_{L^2((0,1)^{2d})} \leq \frac{C(m)}{m!} \|g\|^2_{L^2((0,1)^d)}. \tag{A.1}$$

Now, the asymptotics

$$\sum_{m=0}^{\infty} \frac{t^m}{(m!)^p} = \frac{e^{pt^{1/p}}}{p^{1/2}(2\pi t^{1/p})^{(p-1)/2}} \Big\{ 1 + O\Big(\frac{1}{t^{1/p}} \Big) \Big\} \qquad (p \in (0,4],\ t \to \infty)$$

derived in [38, Chapter 8, Eq. (8.07)] imply that for every $\mu \in (0,1]$ there is a constant $K_\mu > 0$ such that

$$\forall t \geq 0, \quad \sum_{m=0}^{\infty} \frac{t^m}{(m!)^{1-\mu}} \leq K_\mu e^{t^{1/(1-\mu)}}.$$

Setting $K' := \max\{K_{\mu_x}, K_{\mu_y}\}$, by Cauchy product formula this yields

$$h := \sum_{m=0}^{\infty} \sqrt{C(m)} \frac{(10d)^m}{m!} = A \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(k!)^{\mu_x} |10d\ell_x B_x|^k}{k!} \frac{((m-k)!)^{\mu_y} |10d\ell_y B_y|^{(m-k)}}{(m-k)!}$$
$$= A \left(\sum_{k=0}^{\infty} \frac{|10d\ell_x B_x|^k}{(k!)^{1-\mu_x}} \right) \left(\sum_{l=0}^{\infty} \frac{|10d\ell_y B_y|^l}{(l!)^{1-\mu_y}} \right)$$
$$\leq A(K')^2 \exp\Big(|10d\ell_x B_x|^{\frac{1}{1-\mu_x}} + |10d\ell_y B_y|^{\frac{1}{1-\mu_y}} \Big) < \infty.$$

In light of (A.1) and the latter, we may apply Proposition A.2 to g in dimension $2d$ instead of d and with the set $\mathcal{E} := \Psi^{-1}(E) \subset (0,1)^{2d}$ to obtain

$$\frac{\|f\|^2_{L^2(Q)}}{\|f\|^2_{L^2(E)}} = \frac{\|g\|^2_{L^2((0,1)^{2d})}}{\|g\|^2_{L^2(\mathcal{E})}} \leq \left(\frac{K}{|\mathcal{E}|} \right)^{K(1+\log h)} = \left(\frac{K|Q|}{|E|} \right)^{K(1+\log h)},$$

which proves the claim upon a suitable adaptation of the constant K. □

Appendix B. Exponential form of the ultra-analytic regularity

In this second and last section of the appendix, we give the proof of the following lemma which was used in Section 4.2 to deduce the estimates (4.19) from Lemma 4.4.

Lemma B.1. *Let $\Lambda_1 > 0$, $\Lambda_2 > 1$ be some positive constants and $m \geq 1$ be a positive integer. For every function $u \in L^2(\mathbb{R}^d)$ satisfying*

$$\forall \alpha \in \mathbb{N}^d, \quad \|\partial_x^\alpha u\|_{L^2(\mathbb{R}^d)} \leq \Lambda_1 \Lambda_2^{|\alpha|} \, (\alpha!)^{\frac{1}{m}}, \tag{B.1}$$

the following estimate holds

$$\big\| e^{C_{m,\Lambda_2} |D_x|^m} u \big\|_{L^2(\mathbb{R}^d)} \leq 2^d \Lambda_1,$$

where the positive constant $C_{m,\Lambda_2} > 0$ is given by

$$C_{m,\Lambda_2} = \frac{1}{2em(2d\Lambda_2)^m}. \tag{B.2}$$

Proof. Let $u \in L^2(\mathbb{R}^d)$ satisfy (B.1). By using the estimate

$$\forall N \geq 0, \forall \xi \in \mathbb{R}^d, \quad |\xi|^N \leq d^N \sum_{|\alpha|=N} |\xi^\alpha|,$$

we get that for all $N \geq 0$,

$$\begin{aligned} \big\| |D_x|^N u \big\|_{L^2(\mathbb{R}^d)} &\leq d^N \sum_{|\alpha|=N} \big\| \partial_x^\alpha u \big\|_{L^2(\mathbb{R}^d)} \\ &\leq d^N \sum_{|\alpha|=N} \Lambda_1 \Lambda_2^{|\alpha|} (\alpha!)^{\frac{1}{m}} \\ &\leq d^N 2^{N+d-1} \Lambda_1 \Lambda_2^N (N!)^{\frac{1}{m}}, \end{aligned} \tag{B.3}$$

where we used the fact that

$$\#\big\{\alpha \in \mathbb{N}^d : |\alpha| = N\big\} = \binom{N+d-1}{N} \leq 2^{N+d-1}.$$

Considering the positive constant $C_{m,\Lambda_2} > 0$ defined in (B.2), we deduce from (B.3) that

$$\begin{aligned} \big\| e^{C_{m,\Lambda_2} |D_x|^m} u \big\|_{L^2(\mathbb{R}^d)} &\leq \sum_{N=0}^{+\infty} \frac{1}{2^N} \frac{1}{(2d\Lambda_2)^{mN}} \frac{1}{(em)^N N!} \big\| |D_x|^{mN} u \big\|_{L^2(\mathbb{R}^d)} \\ &\leq 2^{d-1} \Lambda_1 \sum_{N=0}^{+\infty} \frac{1}{2^N} \frac{((mN)!)^{\frac{1}{m}}}{(em)^N N!}. \end{aligned}$$

Moreover, by using the fact that $N^N \leq e^N N!$ for every $N \geq 0$, we get that

$$((mN)!)^{\frac{1}{m}} \leq (mN)^N \leq (me)^N N!.$$

The proof of Lemma B.1 is now ended. □

References

[1] R.A. Adams & J.J.F. Fournier, *Sobolev Spaces*, Second edition, Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam (2003).

[2] D. Allonsius, F. Boyer & M. Morancey, *Analysis of the null controllability of degenerate parabolic systems of Grushin type via the moments method*, J. Evol. Equ. **21** (2021), 4799–4843.

[3] P. Alphonse, *Null-controllability of evolution equations associated with fractional Shubin operators through quantitative Agmon estimates*, to appear in Ann. Inst. Fourier, arXiv:2012.04374.

[4] P. Alphonse & J. Bernier, *Polar decomposition of semigroups generated by non-selfadjoint quadratic differential operators and regularizing effects*, Ann. Sci. Éc. Norm. Supér. (4) **56** (2023), 323–382.

[5] P. Alphonse & J. Martin, *Stabilization and approximate null-controllability for a large class of diffusive equations from thick control supports*, ESAIM Control Optim. Calc. Var. **28** (2022), 30 pp.

[6] P. Alphonse & A. Seelmann, *Quantitative spectral inequalities for the anisotropic Shubin operators and applications to null-controllability*, preprint (2023), arXiv:2212.10842.

[7] K. Beauchard, P. Cannarsa & R. Guglielmi, *Null controllability of Grushin-type operators in dimension two*, J. Eur. Math. Soc. (JEMS) **16** (2014), 67–101.

[8] K. Beauchard, J. Dardé & S. Ervedoza, *Minimal time issues for the observability of Grushin-type equations*, Ann. Inst. Fourier **70** (2020), 247–312.

[9] K. Beauchard, P. Jaming & K. Pravda-Starov, *Spectral estimates for finite combinations of Hermite functions and null-controllability of hypoelliptic quadratic equations*, Studia Math. **260** (2021), 1–43.

[10] K. Beauchard, L. Miller & M. Morancey, *2D Grushin-type equations: Minimal time and null controllable data*, J. Differ. Equations **259** (2015), 5813–5845.

[11] K. Beauchard & K. Pravda-Starov, *Null-controllability of hypoelliptic quadratic differential equations*, J. Éc. polytech. Math. **5** (2018), 1–43.

[12] F. A. Berezin & M. A. Shubin, *The Schrödinger Equation*, Kluwer, Dordrecht (1991).

[13] P. Boggiatto, E. Buzano & L. Rodino, *Global Hypoellipticity and Spectral Theory*, Mathematical Research, Vol. 92, Akademie Verlag, Berlin (1996).

[14] M. Chatzakou, J. Delgado & M. Ruzhansky, *On a class of anharmonic oscillators*, J. Math. Pures Appl. **153** (2021), 1–29.

[15] J. Dardé, A. Koenig & J. Royer, *Null-controllability properties of the generalized two-dimensional Baouendi-Grushin equation with non-rectangular control sets*, Ann. H. Lebesgue **6** (2023), 1479–1522.
[16] A. Dicke & A. Seelmann, *Uncertainty principles with error term in Gelfand-Shilov spaces*, Arch. Math. **119** (2022), 413–425.
[17] A. Dicke, A. Seelmann & I. Veselić, *Spectral inequality with sensor sets of decaying density for Schrödinger operators with power growth potentials*, preprint (2022), arXiv:2206.08682.
[18] A. Dicke, A. Seelmann & I. Veselić, *Uncertainty principle for Hermite functions and null-controllability with sensor sets of decaying density*, J. Fourier Anal. Appl. 29 (2023), 11, 19 pp.
[19] A. Dicke, A. Seelmann & I. Veselić, *Control problem for quadratic parabolic differential equations with sensor sets of finite volume or anisotropically decaying density*, ESAIM Control Optim. Calc. Var. **29** (2023), 80, 35 pp.
[20] M. Duprez & A. Koenig, *Control of the Grushin equation: non-rectangular control region and minimal time*, ESAIM Control Optim. Calc. Var. **26** (2020), 3, 18 pp.
[21] T. Duyckaerts & L. Miller, *Resolvent conditions for the control of parabolic equations*, J. Funct. Anal. **11** (2012), 3641–3673.
[22] M. Egidi, *On null-controllability of the heat equation on infinite strips and control cost estimate*, Math. Nachr. **294** (2021), 843–861.
[23] M. Egidi & A. Seelmann, *An abstract Logvinenko-Sereda type theorem for spectral subspaces*, J. Math. Anal. Appl. **500** (2021), 125149, 32 pp.
[24] M. Egidi & I. Veselić, *Scale-free unique continuation estimates and Logvinenko-Sereda Theorems on the torus*, Ann. Henri Poincaré **21** (2020), 3757–3790.
[25] N. Fontes-Merz, *A multidimensional version of Turán's Lemma*, J. Approx. Theory **140**, 27–30.
[26] V. Havin & B. Jöricke, *The Uncertainty Principle in Harmonic Analysis*, Ergeb. Math. Grenzgeb. (3), Vol. 28, Springer, Berlin (1994).
[27] P. Jaming & Y. Wang, *Null-controllability of the Generalized Baouendi-Grushin heat like equations*, preprint (2023), arXiv:2310.11215.
[28] A. Koenig, *Non-null-controllability of the Grushin operator in 2D*, C. R. Math. Acad. Sci. Paris **355** (2017), 1215–1235.
[29] O. Kovrijkine, *Some estimates of Fourier transforms*, ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–California Institute of Technology.
[30] O. Kovrijkine, *Some results related to the Logvinenko-Sereda Theorem*, Proc. Amer. Math. Soc. **129** (2001), 3037–3047.
[31] C. Laurent & M. Léautaud, *Tunneling estimates and approximate controllability for hypoelliptic equations*, Mem. Amer. Math. Soc. **276** (2022), 1357, 95pp.
[32] J. Martin, *Spectral inequalities for anisotropic Shubin operators*, preprint (2022), arXiv:2205.11868.
[33] J. Martin, *Uncertainty principles in Gelfand-Shilov spaces and null-controllability*, J. Funct. Anal. **283** (2022), 109619.
[34] J. Martin & K. Pravda-Starov, *Spectral inequalities for combinations of Hermite functions and null-controllability for evolution equations enjoying Gelfand-Shilov smoothing effects*, J. Inst. Math. Jussieu (2022), 1–50.
[35] L. Miller, *Unique continuation estimates for sums of semiclassical eigenfunctions and null-controllability from cones*, preprint (2008), hal-00411840.
[36] F. L. Nazarov, *Local estimates of exponential polynomials and their applications to inequalities of the uncertainty principle type*, St. Petersbg. Math. J. **5** (1993), 3–66.
[37] I. Nakić, M. Täufer, M. Tautenhan & I. Veselić, *Sharp estimates and homogenization of the control cost of the heat equation on large domains*, ESAIM Control Optim. Calc. Var. **26** (2020), 54, 26 pp.
[38] F. W. J. Olver, *Asymptotics and Special Functions*, Reprint of the 1974 original. AKP Classics. A K Peters, Ltd., Wellesley, MA (1997).
[39] K. D. Phung, *Carleman commutator approach in logarithmic convexity for parabolic equations*, Math. Control Relat. Fields **8** (2018), 899–933.
[40] M. Reed & B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of Operators*, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York (1978).
[41] E. Trélat, G. Wang & Y. Xu, *Characterization by observability inequalities of controllability and stabilization properties*, Pure Appl. Anal. **2** (2020), 93–122.
[42] G. Wang, M. Wang, C. Zhang & Y. Zhang, *Observable set, observability, interpolation inequality and spectral inequality for the heat equation in* $\mathbb{R}^n$, J. Math. Pures Appl. (9) **126** (2019), 144–194.

(Paul Alphonse) Université de Lyon, ENSL, UMPA - UMR 5669, F-69364 Lyon
Email address: paul.alphonse@ens-lyon.fr

(Albrecht Seelmann) Technische Universität Dortmund, Fakultät für Mathematik, D-44221 Dortmund, Germany
Email address: albrecht.seelmann@tu-dortmund.de