COLIMITS OF LOCAL ORDERS

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ÉCOLE POLYTECHNIQUE

IP PARIS

What is a local order ?

Partially ordered spaces

Topology and Order, L. Nachbin, 1965

Partially ordered spaces

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A partially ordered space (or pospace) is $(X, \sqsubseteq)$ such that:

- X : topological space
- $\sqsubseteq$: partial order on X
- $\{(a, b) \in X \times X \mid a \sqsubseteq b\}$ closed in $X \times X$

A pospace morphism is an order-preserving continuous map.

Pospaces and their morphisms form the category $\mathcal{PoSp}$.

The underlying space of a pospace is Hausdorff.

Local order

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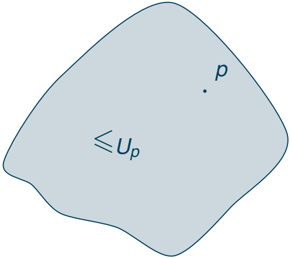

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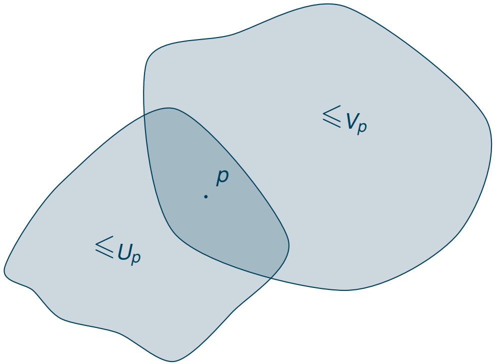

Local order

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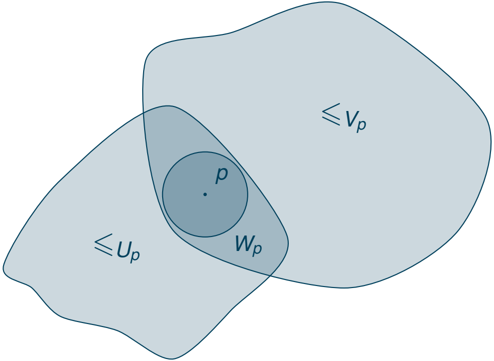
for all points $p \in |\mathcal{U}|$, for all $U_p \in \mathcal{U}$, for all $V_p \in \mathcal{U}$,
there exists W_p open in both U_p and V_p on which $\leqslant_{U_p}$ and $\leqslant_{V_p}$ match



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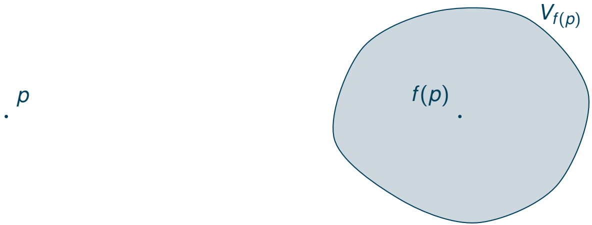

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A map $f : |\mathcal{U}| \to |\mathcal{V}|$ such that:

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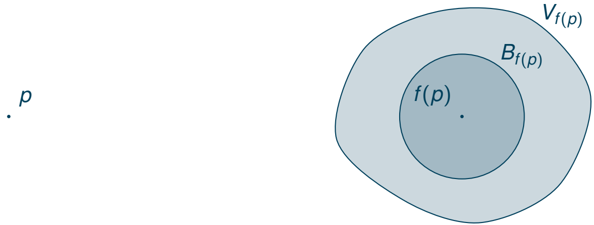

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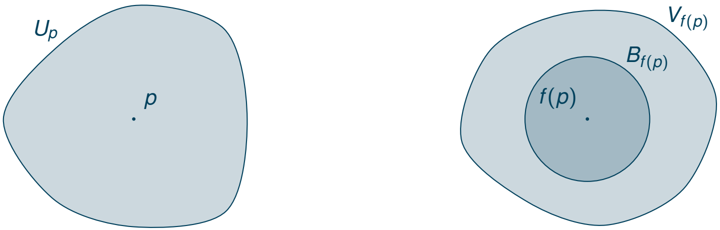

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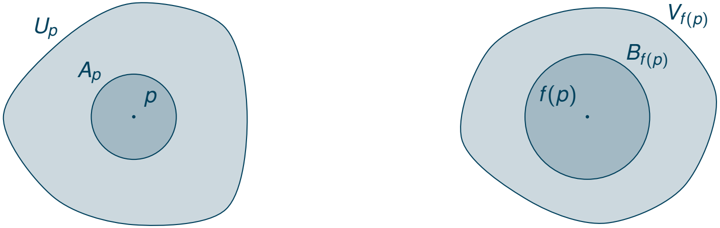

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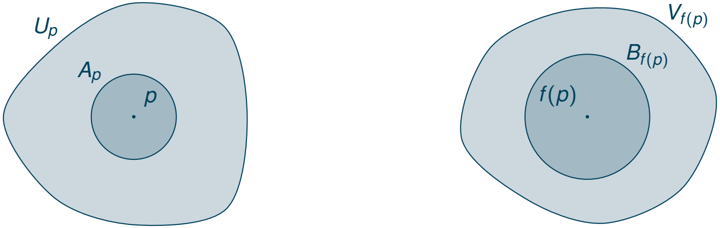

Where do local orders come from ?

Local order

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Mathematical Cosmology and Extragalactic Astronomy,
I. E. Segal, 1976, p.23.

Ordered Manifolds, Invariant Conal Fields, and Semigroups,
J. D. Lawson, Forum Math. 1989, p.280-1.

Algebraic topology and concurrency,
L. Fajstrup, É. Goubault, and M. Raussen, TCS 2006, p.245. (also MFCS 1998)

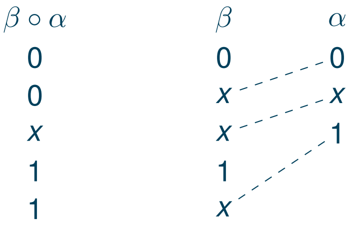
What local orders have to do with concurrency ?

The box category $\square$

$\{\text{Objects of } \square\} = \mathbb{N}$ $\quad\quad$ $\square(a,b) = \big\{ w \in \{0,1,x\}^b \;\big|\; \#_x w = a \big\}$

$\#_x w$: number of occurrences of x in $w \in \{0,1,x\}^*$

Given $\alpha \in \square(a,b)$ and $\beta \in \square(b,c)$, the composite $\beta \circ \alpha$ is obtained by replacing the i^{th} occurrence of x in β by the i^{th} occurrence of α (for $i \in \{1,\ldots,b\}$).



Precubical sets

A precubical set K is a functor form $\square^{op}$ to $\mathcal{S}et$

The precubical set morphisms are the natural transformations.

Precubical sets and their morphisms form the category $\mathcal{PCS}$

If $K_n = \emptyset$ for $n \geqslant 2$ then K is a graph.

A bouquet is a graph with a single vertex.

An automaton is a graph morphism whose target is a bouquet.

Higher Dimensional Automata (HDA)

R. van Glabbeek and V. Pratt (early 90's)

A higher dimensional bouquet (HDB) is a precubical set B whose n-dimensional elements are words of length n.

For any $w \in \square(n, m)$, $B_w(a_1 \cdots a_m) = (a_{i_1} \cdots a_{i_n})$
with $\{i_1 < \cdots < i_n\} \;=\; \{i \in \{1, \ldots, m\} \mid \text{the } i^{th} \text{ letter of } w \text{ is } x\}$

e.g. : $\quad B_{0xx1x}(abcde) \;=\; (bce)$

HDA : precubical set morphism whose target is a bouquet

Precubical realization

From any precubical set K we deduce the diagram D_K

$$\begin{array}{ccc} \{K_w(x)\} \times [0,1]^n & \to & \{x\} \times [0,1]^m \\ (K_w(x), t) & \mapsto & (x, w(t)) \end{array}$$

with n, $m \in \mathbb{N}$, $x \in K_m$, and $w \in \square(n, m)$.

The realization of K is the colimit of D_K in the category in which $[0, 1]$ is interpreted.

Which category to choose ?

d-spaces

Directed homotopy theory, I, M. Grandis, 2003

X : topological space

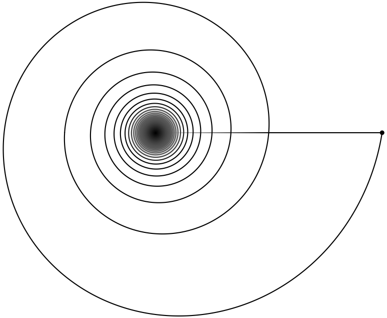
dX : collection of paths on X containing constant paths and stable under concatenation, reparametrization, and *subpaths*.

A morphism from (X, dX) to (Y, dY) is a continuous map $f : X \to Y$ such that $f \circ dX \subseteq dY$.

d-spaces and morphisms form the category $\mathcal{dSpc}$

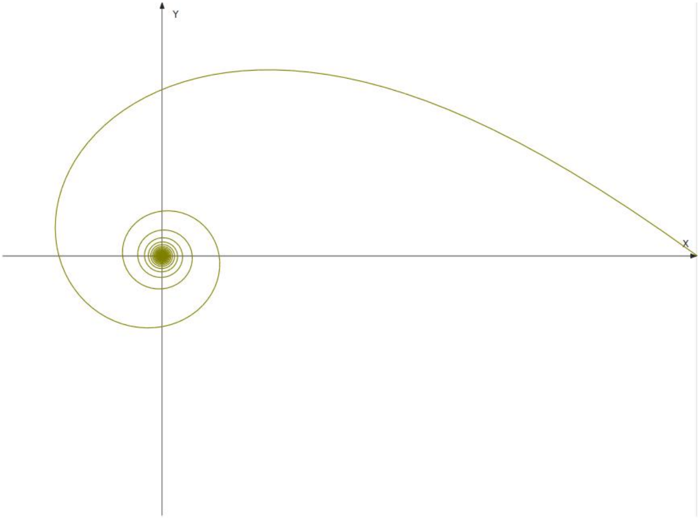
Vortex from the computer science viewpoint

allow infinitely many turns in finite time



Vortex from the computer science viewpoint

allow loop deletion



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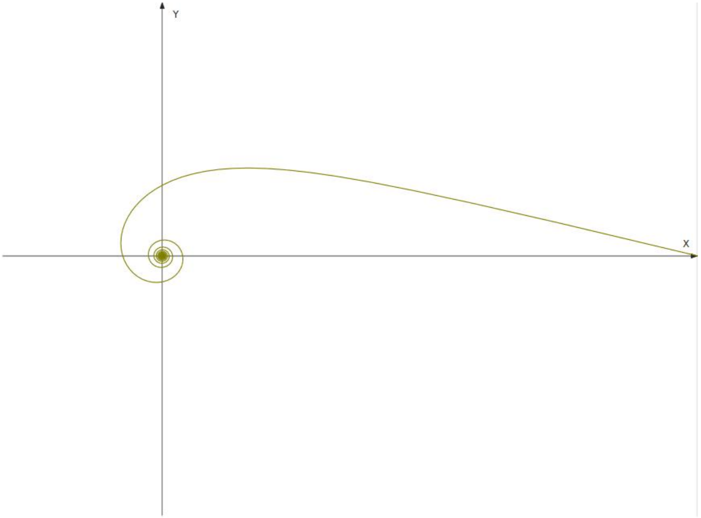

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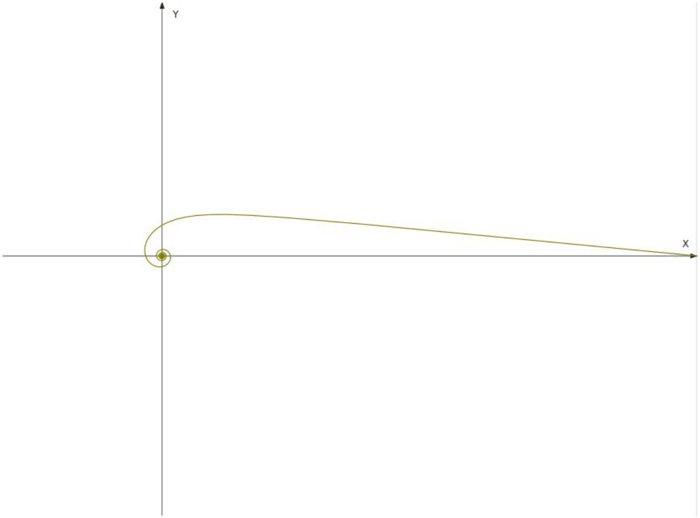

Local orders and d-spaces

Introducing vortex

U : open subset of X

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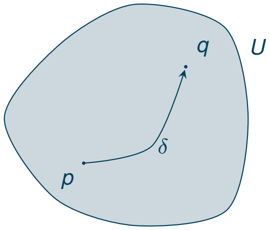
Introducing vortex

U : open subset of X

$p \preccurlyeq_U q$: there exists $\delta \in dX$ from p to q with $img(\delta) \subseteq U$



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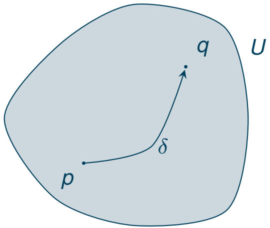
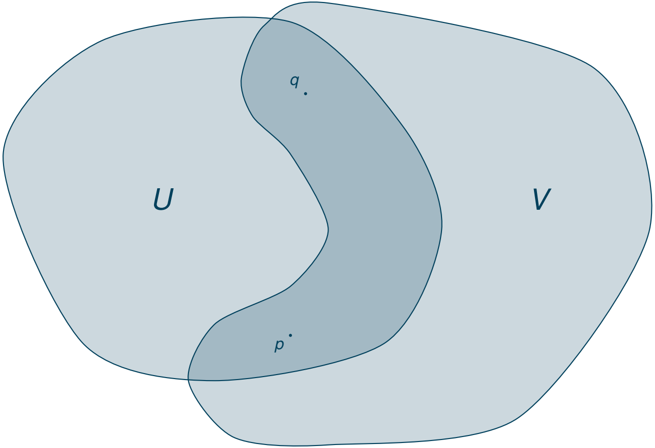
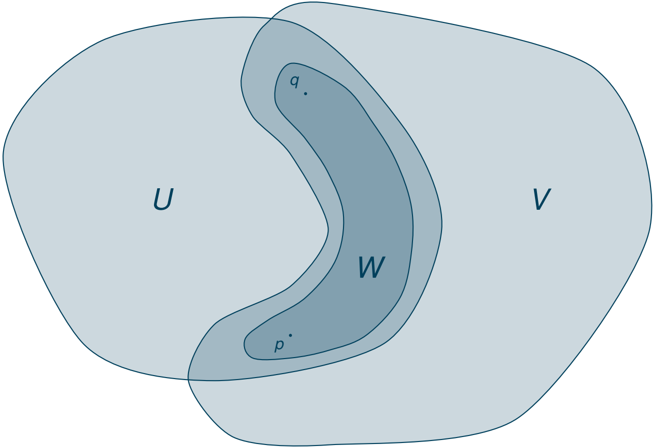


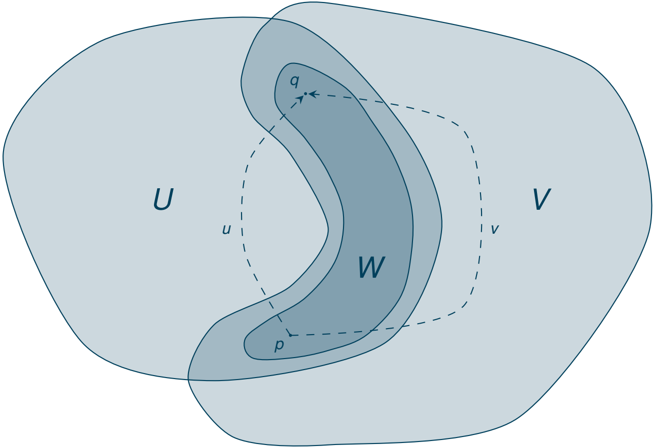
Is there an open covering $\mathcal{U}$ of X such that $\{(U, \preccurlyeq_U) \mid U \in \mathcal{U}\}$ is a local order ?


q
p


q
U
V
p



q
U
V
W
p


q
U
u
v
V
W
p

Lawson correspondence

Wedges and cones

E : real vector space

$W \subseteq E$ is a wedge when $\lambda W + W \subseteq W$ for all $\lambda \in \mathbb{R}_+$

A wedge C is a cone when $C \cap (-C) = \{0\}$

Cone fields

$\mathcal{M}$: manifold

A cone field C is a map $p \in \mathcal{M} \mapsto (C_p : \text{cone of } T_p\mathcal{M})$

[Wedge fields are defined the same way]

$(\mathcal{M}, C)$ is a conal manifold

A curve on $\mathcal{M}$ is conal when $\dot{\gamma}(t) \in C(\gamma(t))$ for all $t \in \text{dom}(\gamma)$

Cone fields $\mapsto$ d-spaces and local orders

$(\mathcal{M}, dC)$ is a d-space with $dC = \{\text{conal curves on } (\mathcal{M}, C)\}$

If C is upper semicontinuous then there is an open covering $\mathcal{U}$ of $\mathcal{M}$ such that:

- $\{(U, \preccurlyeq_U) \mid U \in \mathcal{U}\}$ is a local order, and
- each $(U, \preccurlyeq_U)$ is locally convex (i.e. the order convex open subsets of U form a basis of topology)

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E.g. The local order on the unit circle comes from the vector field $z \mapsto iz$

Local orders $\mapsto$ cone fields

$\mathcal{U}$: local order on $\mathcal{M}$ whose elements are locally convex

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For all $p \in \mathcal{M}$, the least wedge of $T_p\mathcal{M}$ containing $\dot{\gamma}(0)$ for all curves γ with $\gamma(0) = p$ is a cone $C(p)$, and ...

Local orders $\mapsto$ cone fields

$\mathcal{U}$: local order on $\mathcal{M}$ whose elements are locally convex

For all $p \in \mathcal{M}$, the least wedge of $T_p\mathcal{M}$ containing $\dot{\gamma}(0)$ for all curves γ with $\gamma(0) = p$ is a cone $C(p)$, and

the mapping $p \mapsto C_p$ is upper semicontinuous.

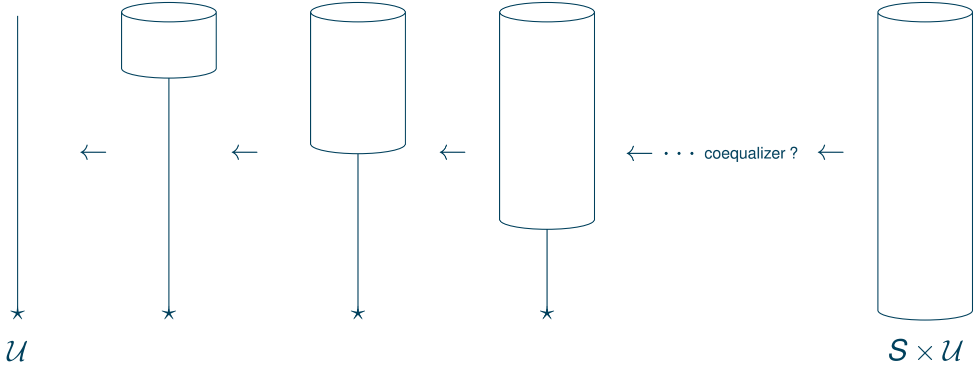
Lawson correspondence

upper semicontinuous cone fields on $\mathcal{M}$

$$\updownarrow$$

locally convex local orders on $\mathcal{M}$

How local order colimits behave ?



$\mathcal{U}$
· · · coequalizer ?
$S \times \mathcal{U}$

Pinching a loop

Setting the problem

S : the directed unit circle

$\mathcal{U}$: local order with a distinguished point $\star$

$i_\star \quad : \quad z \in S \quad \mapsto \quad (z, \star) \in S \times \mathcal{U}$

$c_\star \quad : \quad z \in S \quad \mapsto \quad (1, \star) \in S \times \mathcal{U}$

The *coequalizer* of $i_\star$ and $c_\star$ exists iff the family of clopens containing $\star$ has a *least* element O

[NB: connected component of $\star \subseteq \bigcap\{\text{clopens containing } \star\}$, with equality if the topology of $\mathcal{U}$ is *locally connected*.]

Pinching a loop

Description of the colimit

For $A \in \mathcal{U}$ and α proper arc of S define

$$U_{\alpha,A} \quad = \quad (O \cap A) \quad \sqcup \quad \alpha \times (O^c \cap A)$$

$$p(u) \quad = \quad \begin{cases} u & \text{if } u \in O \cap A \\ x & \text{if } u = (z,x) \in \alpha \times (O^c \cap A) \end{cases}$$

$$u \sqsubseteq u' \qquad \text{when} \qquad p(u) \leqslant_A p(u') \quad \text{and} \quad \begin{cases} \{u, u'\} \cap O \neq \emptyset \\ \text{or} \\ u = (z,x),\ u' = (z',x'), \text{ and } z \leqslant_\alpha z' \end{cases}$$

Pinching a loop

Description of the colimit

The transitive closure $\sqsubseteq^*$ is antisymetric, and
we have $u \sqsubseteq^* u'$ iff there exists $u'' \in O \cap A$ such that $u \sqsubseteq u'' \sqsubseteq u'$.

The collection $\left\{ U_{\alpha,A} \;\middle|\; \alpha : \text{proper arc}\,;\ A \in \mathcal{U} \right\}$ is a local order we denote by $\mathcal{U}_O$

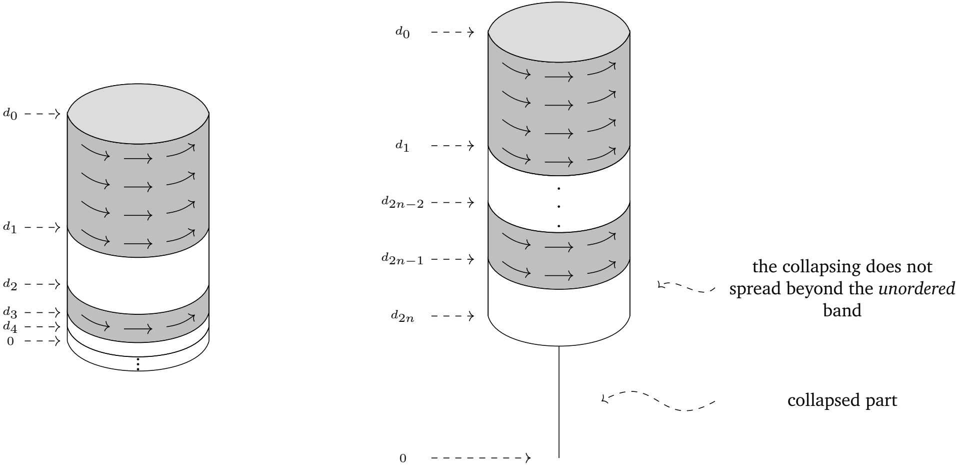
The quotient map $q_O \ :\ S \times \mathcal{U} \to \mathcal{U}_O$ is the coequalizer of $i_\star$ and $c_\star$.

If $\mathcal{U}$ is $\mathbb{R}$ or S (with the obvious local order), then the coequalizer is $\mathcal{U}$.

If $\mathcal{U}$ is $\mathbb{Q}$ (sub local order of $\mathbb{R}$) or $\mathbb{R}_\star$ (with the least topology containing all the usual open subsets and all the single elements set but $\{\star\}$) the coequalizer does not exist.

The zebra cylinder (no more topological trick)

The underlying topology is the standard cylinder $S \times [0, 1]$

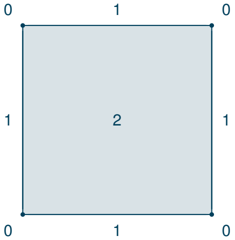


Locally ordered realization

of a *non-geometric* precubical set

A non-geometric precubical set

$$K_n = \begin{cases} \{n\} & \text{if } n \leqslant 2 \\ \emptyset & \text{if } n > 2 \end{cases}$$



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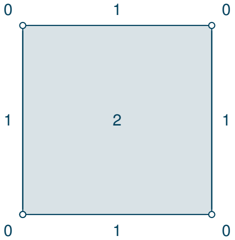

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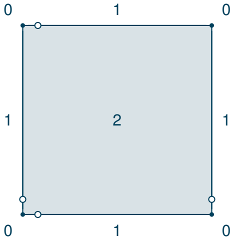

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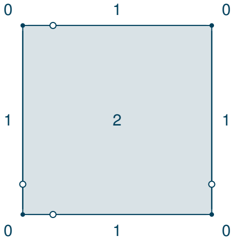

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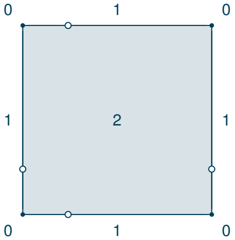

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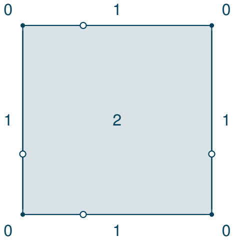

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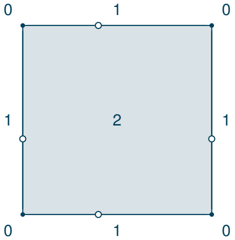

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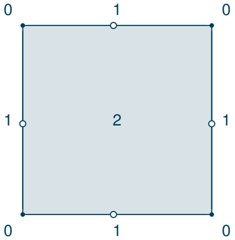

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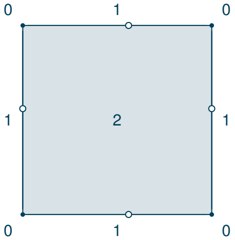

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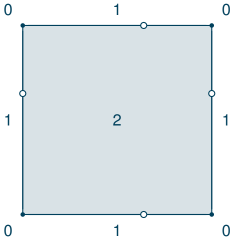

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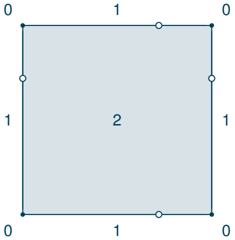

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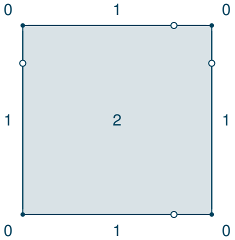

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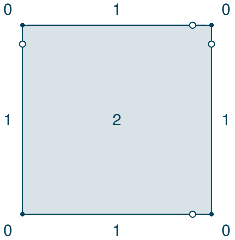

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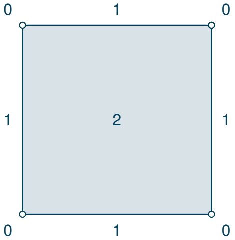
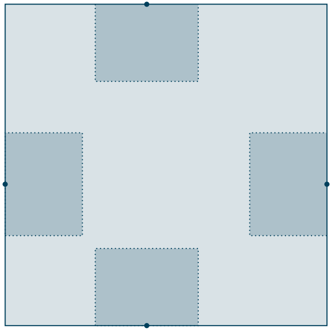
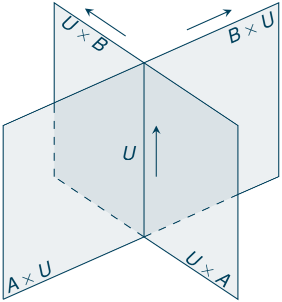
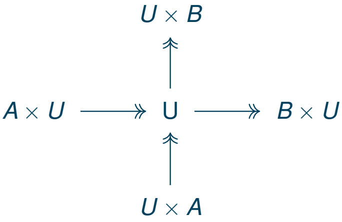


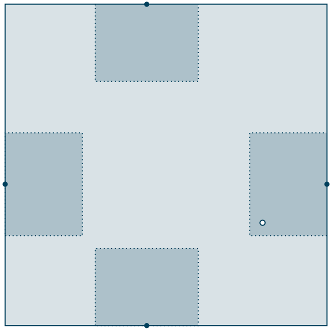
Neighborhood of a point of the edge

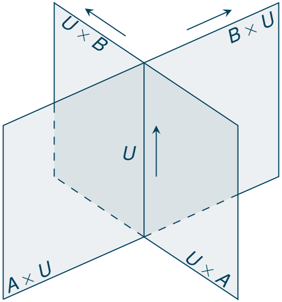






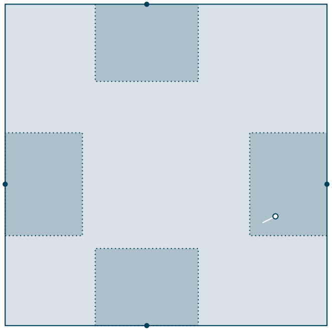
Neighborhood of a point of the edge

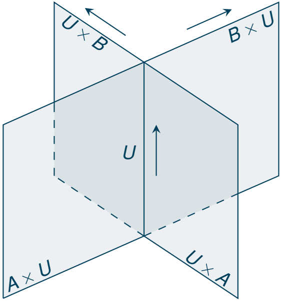


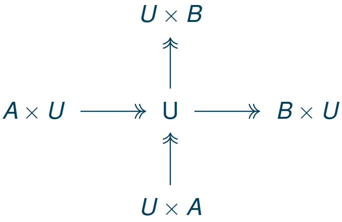


$$\begin{array}{ccccc} & & U \times B & & \\ & & \uparrow\!\!\uparrow & & \\ A \times U & \twoheadrightarrow & \mathrm{U} & \twoheadrightarrow & B \times U \\ & & \uparrow\!\!\uparrow & & \\ & & U \times A & & \end{array}$$

Neighborhood of a point of the edge







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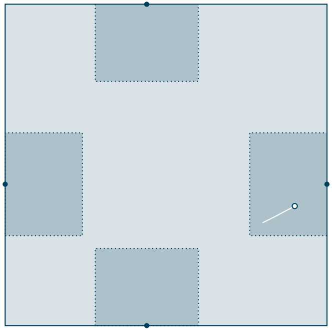

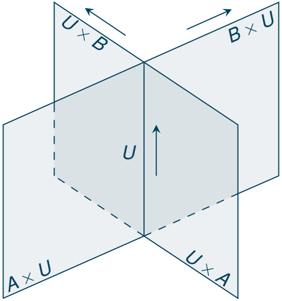

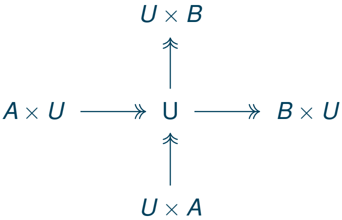

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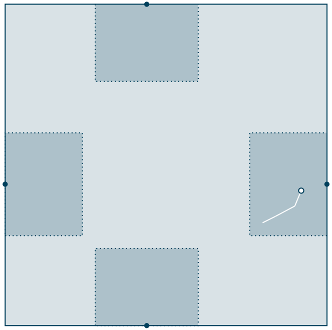

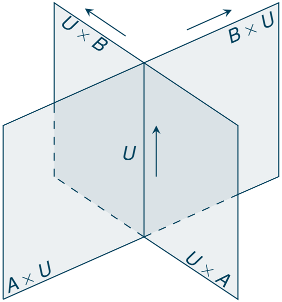

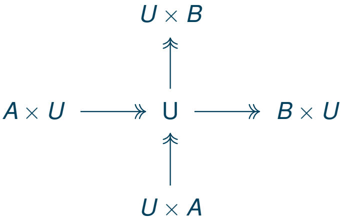

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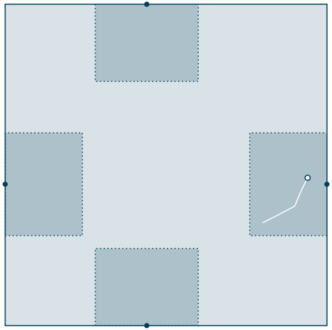

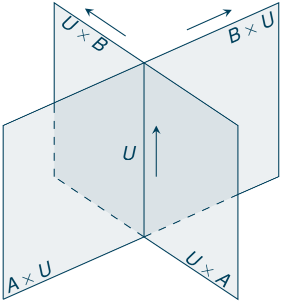

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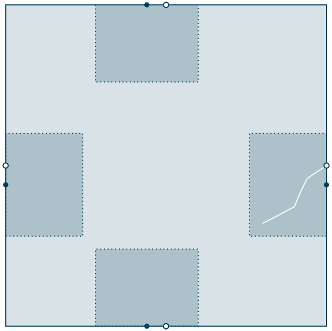
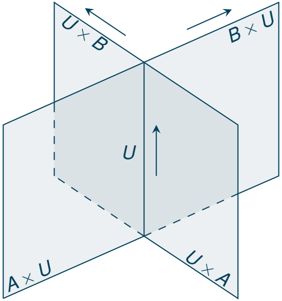
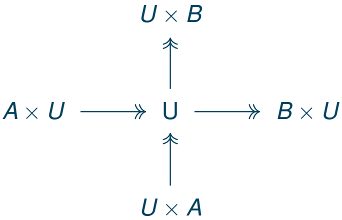
Neighborhood of a point of the edge







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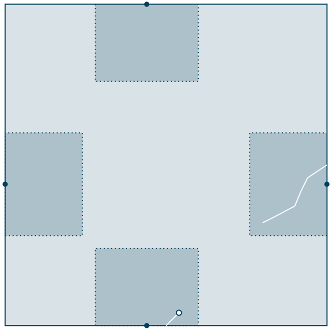

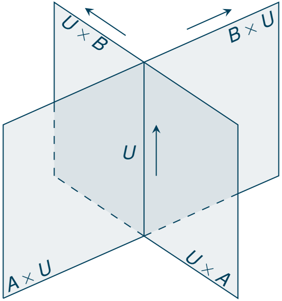

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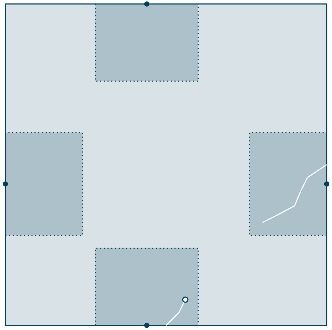

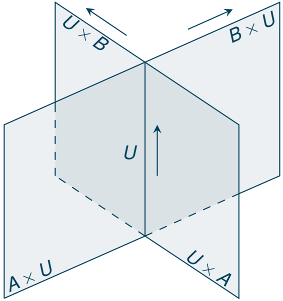

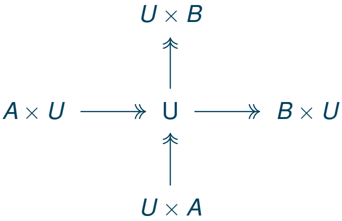

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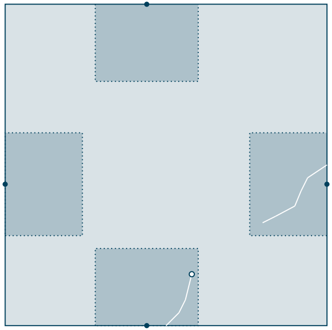

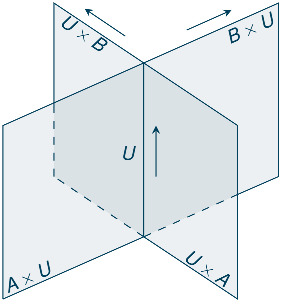

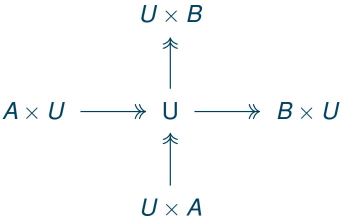

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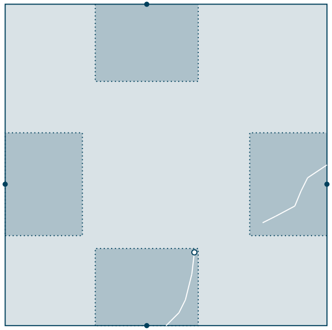
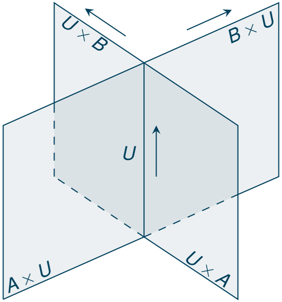
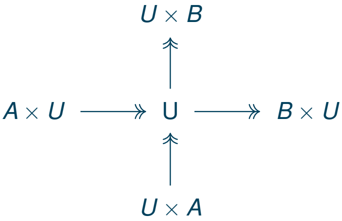
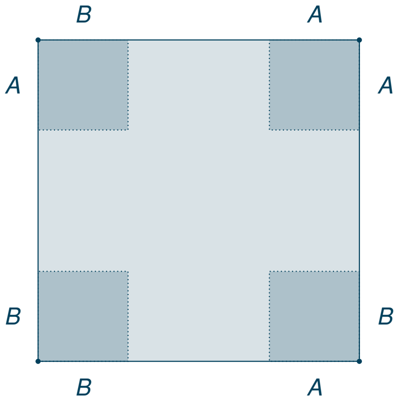
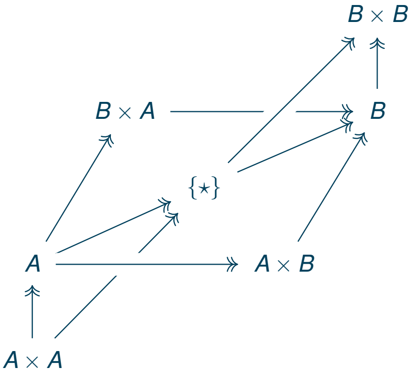




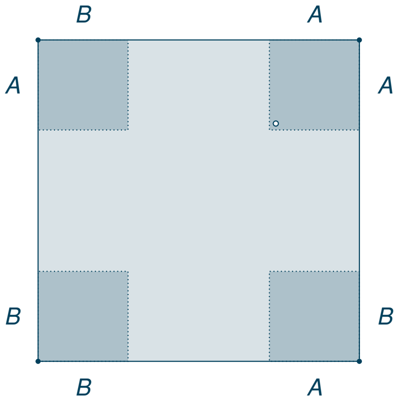


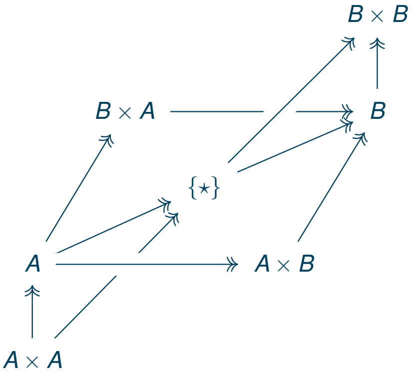
Neighborhood of the vertex





Neighborhood of the vertex





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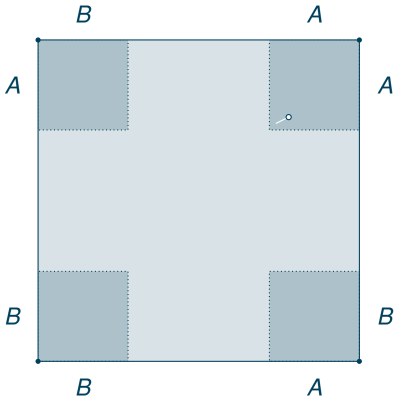

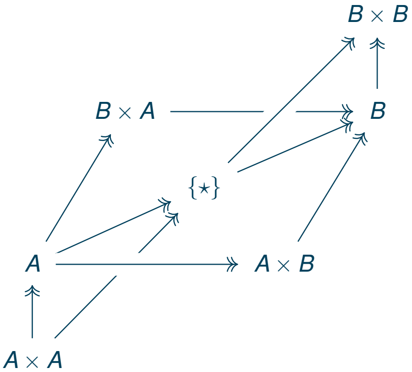

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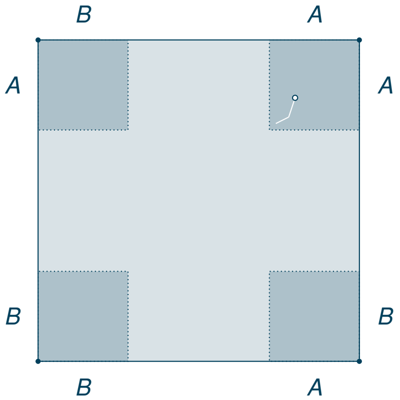
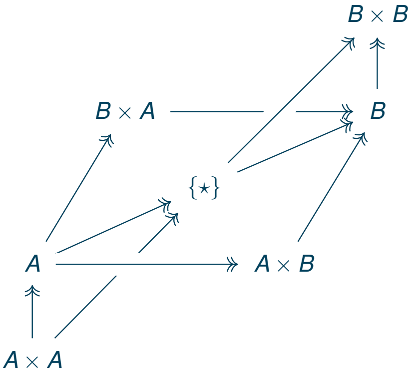
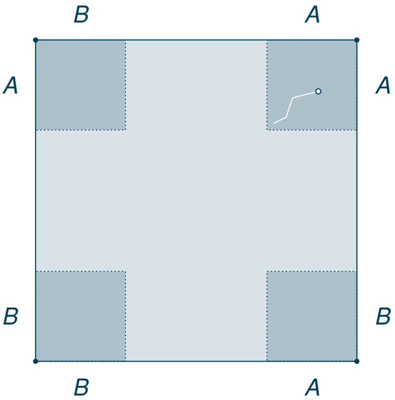
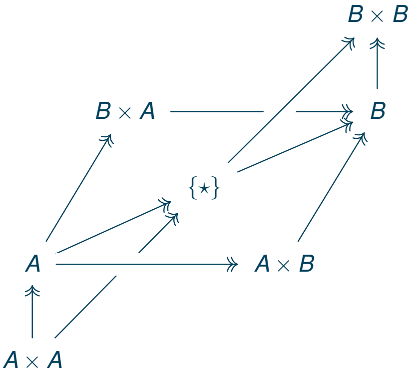


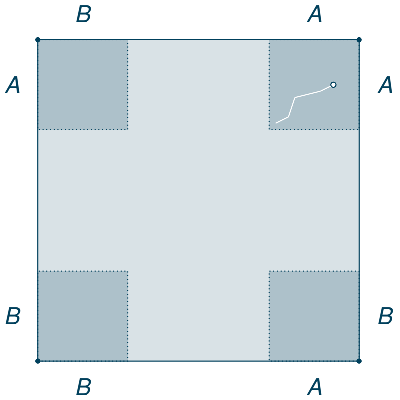


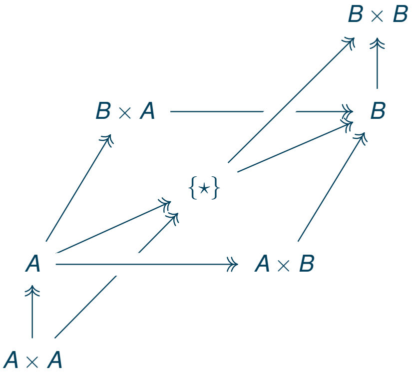
Neighborhood of the vertex



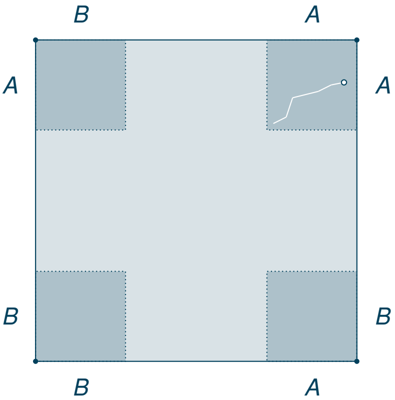


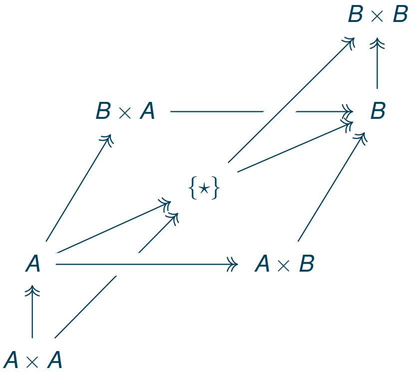
Neighborhood of the vertex





Neighborhood of the vertex





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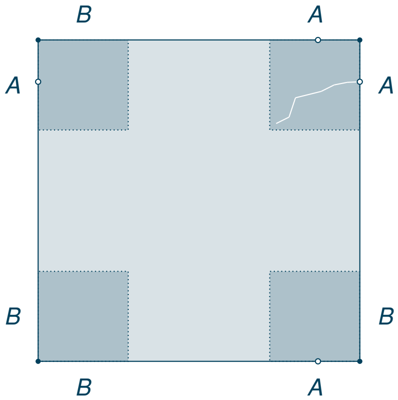

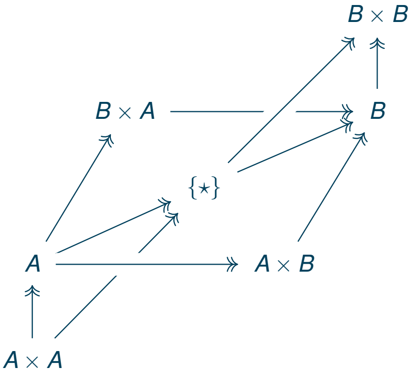

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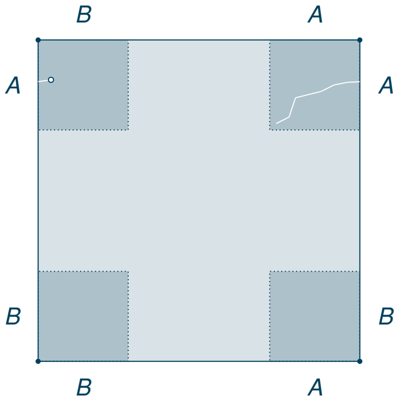

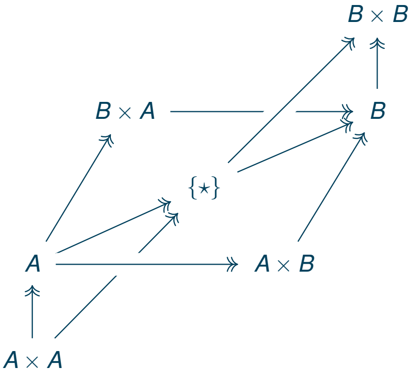

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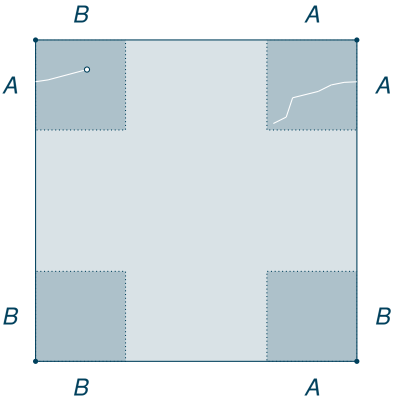

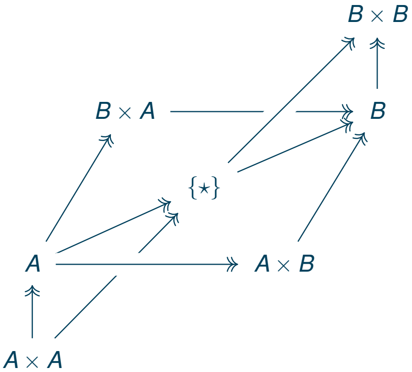

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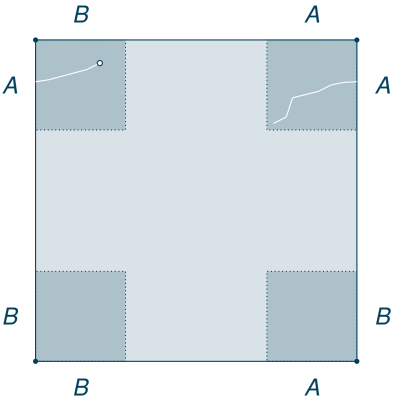

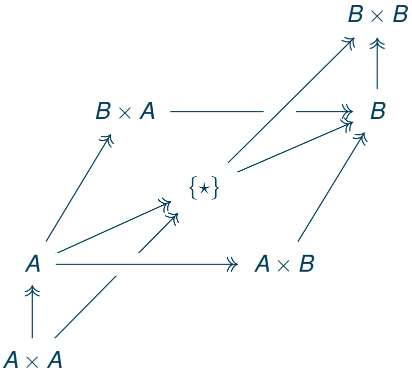

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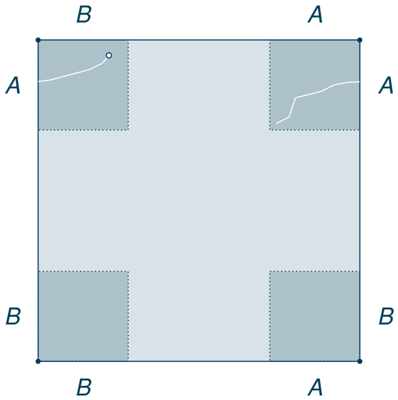

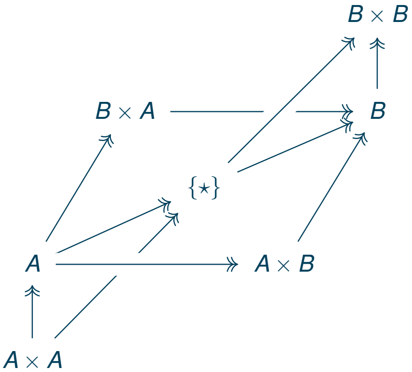

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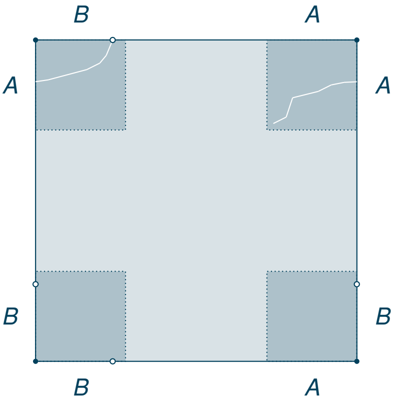

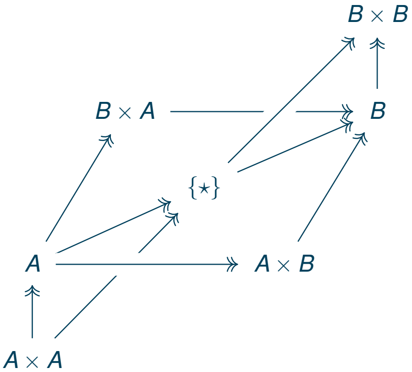

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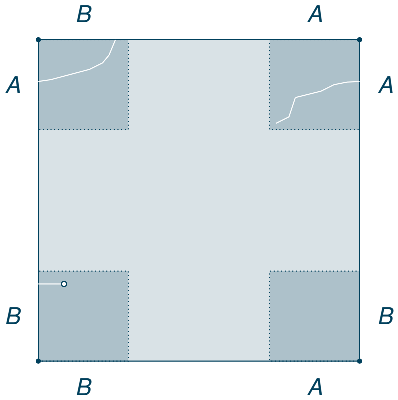
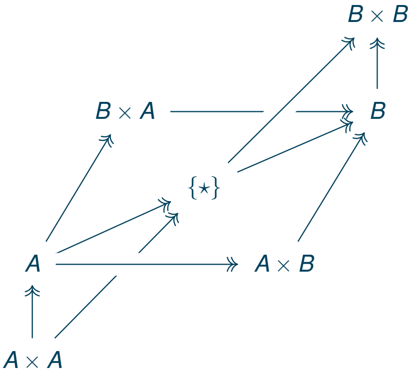
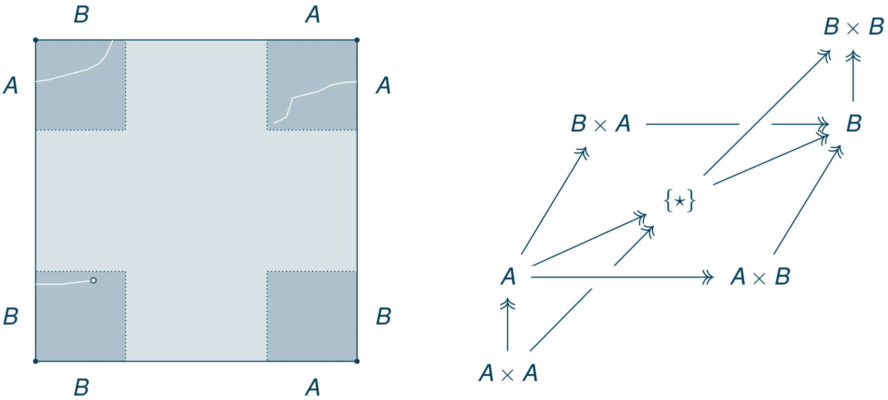
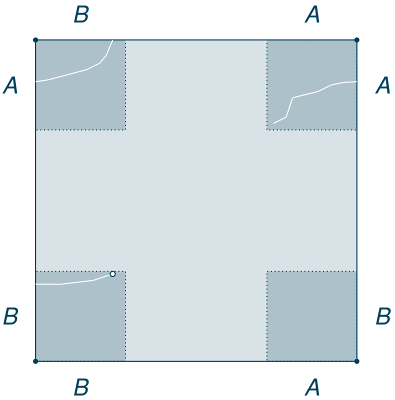


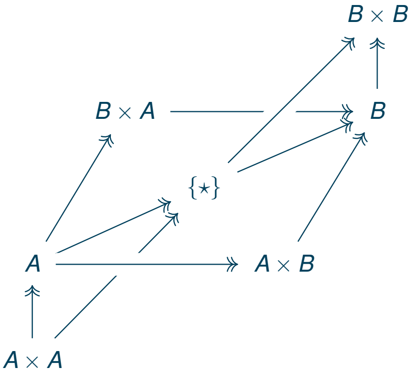


Neighborhood of the vertex

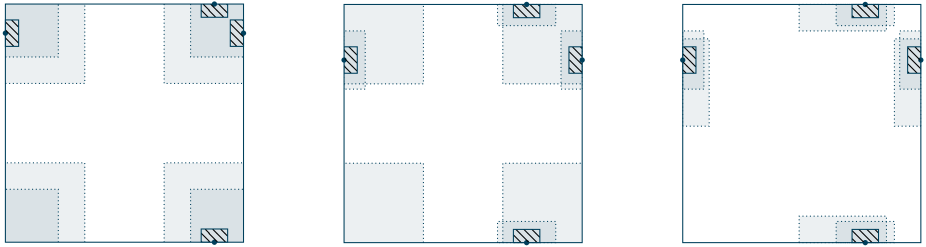


Neighborhood of the vertex





Coherence



$p \in W_\star \cap W'_\star$ $p \in W_u \cap W_\star$ $p \in W_u \cap W_{u'}$