Robust switching control design using Lur'e Lyapunov functions

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Abstract

This article addresses the problem of robust output regulation for uncertain switched systems. A general result is first proposed for the case of uncertain and nonlinear systems. In order to cope with model uncertainties, the proposed approach implements the internal model principle. Local asymptotic stability of the closed-loop system as well as zero steady state error on the controlled output are proven using Lur'e Lyapunov functions. Then, this result is specialized to the linear case. Sufficient conditions ensuring global or local stability of the closed loop system are provided. Furthermore, a constructive method based on LMI is presented. When only local stability is achieved, outcomes of this procedure are not only the controller gains, but also an estimation of the robust domain of attraction. Finally, illustrative examples are proposed, showing the efficiency of the proposed methods.

KEYWORDS

integral action, LMI, output regulation, parametric uncertainties, robust control, switched systems

1 | INTRODUCTION

Switched systems can be conceived as particular hybrid systems for which the active subsystem is selected according to some switching rule.[1] The study of switched systems has attracted a lot of attention from the scientific community over the past few years.[2–5] This line of research is indeed fully justified by the adequation of this class of model with many practical applications in various fields such as power electronics, electromechanical, and aerospace applications.[6–11]

From a theoretical point of view, analysis and control design devoted to switched systems are non trivial tasks. To date, several important contributions in this research field can be found in the literature, using either quadratic[12–14] or Lur'e Lyapunov functions[15] as the key tool. Yet, most of those contributions are developed in an uncertainty free context. This prevents real implementation, since uncertainties must be handled in most of practical cases.

Few papers have investigated the robust case. Contributions taking up this challenge can be gathered in two groups. The first one focuses on specific applications, see for example, References 16–19 and references therein, whereas the second one considers the stabilization problem in a broader context.[20–22] In both cases, the most widely used control strategy consists in implementing an adaptive controller fed by a perturbation observer. This allows to handle perfectly

Abbreviation: LMI, linear matrix inequalities.

modeled uncertainties that are estimated and compensated by the control law. However, it is well-known that this strategy is fragile with respect to unmodeled uncertainties and, therefore, is not suitable for most practical applications.

For continuous systems, the internal model principle can be conceived as an alternative control strategy to tackle robustness. In stark contrast with the adaptive strategy, control design via internal model allows to guarantee structural stability, that is, output regulation and closed-loop stability are preserved for any system parameters variation of small enough magnitude.[23] For this reason, this article aims tackling the output regulation problem following an internal model approach. Note that this strategy has been already implemented in References 24 and 25 for hybrid systems. However, those studies focus on the particular case where the exosystem, the plant and the controller switch periodically in time, independently on the instantaneous value of the state, which does not fall within the scope of this article focusing on systems for which inputs can switch at any instant time.

The key ingredient to implement robust output regulation using the internal model approach is a robust stabilizer for the augmented model. Inspection of the current literature on switched systems shows that the design of this stabilizer is far from being trivial. Indeed, in the context of switched systems, most of the existing constructive results rely on the following assumption:

(H) *There exists a Hurwitz convex combination of the subsystems.*

Clearly, the augmented dynamics resulting from the internal model approach does not have this property since the marginally stable dynamics of the exosystem is embedded in each augmented subsystem, so that (H) is not satisfied. For this reason, results in Reference 26 on robust stabilization of switched systems cannot be used. Perhaps, this technical difficulty is the reason why very few papers tackle the output regulation problem head-on in the switched system context. Besides, note that beyond the problem of output regulation, literature contains example of real switched systems for which (H) is not satisfied. One of them is the parallel interconnection of buck converters, see References 27 and 28. This makes general methodology for robust stabilizer design, that is, without (H), even more desirable.

Remarkably, there are some attempts in the literature to avoid (H). In Reference 29, a design procedure based on a max-type Lyapunov function and on nonlinear inequalities is proposed. In Reference 30, a control design that locally stabilizes switched affine systems is offered. In Reference 31, a stabilizer is developed for an open loop dynamics satisfying (H) and augmented by integrators. Very recently, the global stabilization of rank-deficient switched affine systems is addressed in Reference 32. As an illustration of the potentiality of this study, result of Reference 31 is somehow recovered as a particular case. The key point here is that robustness is never treated a priori in all those papers.

To the best of our knowledge, Reference 33 is the only paper that proposes a robust control law design embedding an internal model, in the context of switched systems. Unlike previously cited papers, (H) is not required, robustness is ensured a priori *at the stabilizer design step* and, finally, LMI-based design procedure allowing to enlarge an inner estimation of the basin of attraction is proposed. However, the results presented in Reference 33 rely on some restrictive assumptions, namely: (i) the approach is limited to invertible linear systems, which implies that the number of inputs must be equal to number of regulated outputs; (ii) by using quadratic Lyapunov functions, not only achievable closed-loop stability is merely local but also estimates of the domain of attraction can only be ellipsoidal sets which may be conservative.

In a nutshell, this article aims tackling the robust output regulation problem for uncertain switched systems by adopting an internal approach. To this end, one has to overcome the obstacle of robust stabilizer design. Even without uncertainties, this is a non trivial task due to the fact that (H) does not hold for the augmented switched dynamics. In contrast with existing results, a design procedure handling robustness a priori is proposed in Reference 33. Its applicability is limited to invertible linear systems and achievable stability is only local, though. Therefore, the general considered problem is still widely open.

In this article, we present a methodology that dramatically enlarges the scope of Reference 33 and offers less conservative results. The contributions of this article are threefold.

1. The design of a robust switching controller is considered in the general context of uncertain *nonlinear* switched systems, whereas only linear dynamics is handled in Reference 33. Sufficient conditions to ensure stability are given, under the assumption of the existence of a control Lyapunov function for the related continuous system and another specific assumption which is extensively discussed (see forthcoming Assumption 2).
2. Unlike Reference 33, the study is not confined to the case of square and invertible systems. Indeed, we are able to consider the broadest class of systems for which the number of inputs and outputs is arbitrary. Note that by removing

this assumption, the core of the technical challenge taken up in Reference 33 for invertible systems is unveiled and exhibited via the peculiar Assumption 2.

3. By using Lur'e Lyapunov functions, the proposed approach allows global asymptotic stability to be achievable, even in the case where the open loop state matrix of the plan is not Hurwitz. This is in stark contrast with the results presented in Reference 33. Furthermore, whenever only local asymptotic stability is achievable, it also gives rise to non ellipsoidal inner estimates of the basin of attraction, hence possibly larger as compared to the one used in Reference 33.

This article is organized as follows. Section 2 formalizes the problem under study. Section 3 presents some preliminaries, including the exposition of the previously mentioned Assumption 2, and gives a general result for the asymptotic stability of uncertain nonlinear systems with switching control. This result is specialized to the linear case in Section 4 where we provide constructive sufficient conditions to ensure global or local stabilization. A LMI based approach is employed for the computation of the control parameters and of the non ellipsoidal inner estimation of the basin of attraction for the local stabilization. Numerical examples that illustrate the efficiency of the proposed approaches are provided in Section 5. The article is ended with some concluding remarks and perspectives. Note that two distinct sufficient conditions for Assumption 2 to hold are provided. The first one is given in Section 3.5 and applies for the general nonlinear case. The second one is exposed in Section 4.1 for the particular case where the dynamics of the plant is linear. In the same subsection, Remark 5 discussed the differences between the two sufficient conditions.

Notations: By $\|\cdot\|_2$, we denote the Euclidean vector norm of a vector. The identity matrix of dimension $n\times n$ is denoted by $\mathbf{I}_n$. $\mathbf{0}_{m\times n}$ is the $m\times n$ matrix of zeros. $\mathbf{1}_m$ ($\mathbf{0}_m$) denotes a m dimensional column vector with 1 (0) entries. $\mathcal{I}_N$ represents the set $\{1,2,\ldots,N\}$. We denote by $\mathbb{S}^n_+ := \{P = P^\intercal \in \mathbb{R}^{n\times n} : \forall x\neq \mathbf{0}, x^\intercal P x > 0\}$. For a symmetric positive definite matrix $P\in\mathbb{S}^n_+$ and a positive scalar c, we denote by $\mathcal{E}(P,x_0,c)$ the open ellipsoid centered on x_0 with radius $c>0$ that is, $\{x\in\mathbb{R}^n : (x-x_0)^\intercal P(x-x_0) < c\}$. For a given set $\mathcal{U}$, $\mathrm{Int}\{\mathcal{U}\}$ denotes its interior. Let e_i be the ith unit vector. For a given matrix M (vector v), let $M_i := e_i^\intercal M$ ($v_i := e_i^\intercal v$) and $M_{i,j} := M_i e_j$. If M is symmetric, the symmetric elements are denoted by $*$. Define $\mathbb{R}^n_{>0} := \{x\in\mathbb{R}^n : x_i > 0\}$ as the positive orthant, and $\mathbb{R}^n_{\geqslant 0} := \{x\in\mathbb{R}^n : x_i \geqslant 0\}$ as the nonnegative orthant. By Λ_N we denote the unit simplex:

$$\Lambda_N := \left\{ [\lambda_1, \ \ldots\ , \ \lambda_N]^\intercal \in \mathbb{R}^N_{\geqslant 0} : \sum_{j=1}^N \lambda_j = 1 \right\}.$$

Given scalar $\alpha>0$, define $\mathrm{sat}_\alpha(\cdot) : \mathbb{R}\to\mathbb{R}$ as follows:

$$\mathrm{sat}_\alpha(x) := \begin{cases} -\alpha & \text{if} \quad x < -\alpha, \\ x & \text{if} \quad -\alpha \leqslant x \leqslant \alpha, \\ \alpha & \text{if} \quad x > \alpha. \end{cases}$$

For a vector $a\in\mathbb{R}^m_{>0}$, let $\mathrm{sat}_a(\cdot) : \mathbb{R}^m\to\mathbb{R}^m$ reads:

$$\mathrm{sat}_a(x) := \begin{bmatrix} \mathrm{sat}_{a_1}(x_1) \\ \vdots \\ \mathrm{sat}_{a_m}(x_m) \end{bmatrix}.$$

2 | PROBLEM FORMULATION

We consider the class of uncertain continuous time systems with switching control given by:

$$\dot{x}(t) = f^0_\theta(x(t)) + B_0 u(t), \tag{1a}$$
$$y(t) = h(x(t)), \tag{1b}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $y(t)\in\mathbb{R}^p$ the controlled output and $u(t)\in\mathbb{R}^m$ is the control input vector which takes value in a finite set of constant vectors $\mathcal{U} = \{u^{[1]}, u^{[2]}, \ldots, u^{[N]}\}\subset\mathbb{R}^m$, where N is a strictly positive integer. Mapping f^0_θ

is parametrized by time-invariant vector $\theta \in \Theta$ which gathers uncertainties of the system, where Θ is a known set.* Note that the non connected nature of $\mathcal{U}$ induces discontinuity in the right hand side of (1a), so that the closed-loop system is hybrid, in the sense that it exhibits both continuous and discrete dynamics.

Our objective is to design a state-dependent switching law that guarantees constant and robust output regulation, that is, for all uncertainty θ, controlled output $y(t)$ must asymptotically converge to a given constant reference y_{ref}, while $x(t)$ remains bounded at all time.

Problem 1. Given system (1), vector $y_{\text{ref}} \in \mathbb{R}^p$ and sets $\mathcal{U}$ and Θ. Design $n_z \in \mathbb{N}_{>0}$, $\phi : \mathbb{R}^{n+n_z} \to \mathbb{R}^{n_z}$ and $\psi : \mathbb{R}^{n+n_z} \to \mathbb{R}^m$ characterizing dynamical state-feedback relay controller

$$\dot{z}(t) = \phi(x(t), z(t)), \tag{2a}$$

$$u(t) \in \arg\min_{v \in \mathcal{U}} \psi^\intercal(x(t), z(t))v, \tag{2b}$$

such that, for all $\theta \in \Theta$, resulting closed-loop admits an asymptotically stable equilibrium (x_θ^*, z_θ^*) for which $y = y_{\text{ref}}$.

Among solutions of Problem 1, we are interested in the ones enlarging the basin of attraction of the resulting closed-loop.

Let us already make the following standing assumption, which is valid *throughout this article*.

Assumption 1. For all $\theta \in \Theta$, there exists $(x_\theta^*, u_\theta^*) \in \mathbb{R}^n \times \mathbb{R}^m$ such that following equations admit solution:

$$f_\theta^0(x_\theta^*) + B_0 u_\theta^* = \mathbf{0}, \tag{3a}$$

$$h(x_\theta^*) = y_{\text{ref}}. \tag{3b}$$

Under this assumption, it is possible to define a mapping $\theta \mapsto (x_\theta^*, u_\theta^*)$ from Θ to $\mathbb{R}^n \times \mathbb{R}^m$, which relates θ to a solution of (3). Obviously, this mapping is not unique when (3) admits multiple solutions.

Remark 1. Here, controller input $u(t)$ must belong to non connected set $\mathcal{U}$. Thus, closed loop (1) with (2) is governed by a differential equation with discontinuous right hand side (unless $u(t)$ is constant). In this article, solutions of the closed loop are considered in the sense of Filippov, see for example, References 34 and 35. Given a locally bounded discontinuous vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, recall that a Filippov solution of the following differential equation

$$\dot{x}(t) = f(x(t)), \tag{4}$$

on the interval $[t_a, t_b] \subset [0, \infty)$ is an absolutely continuous map $\varphi : [t_a, t_b] \to \mathbb{R}^n$ such that differential inclusion $\dot{\varphi}(t) \in \mathcal{F}(\varphi(t))$ is satisfied for almost every $t \in [t_a, t_b]$, with $\mathcal{F}(\varphi) = \cap_{c>0} \cap_{\mu(S)=0} \overline{\text{conv}}\{f(\tilde{x}) : \tilde{x} \in \mathcal{B}(\varphi, c) \setminus S\}$, where μ represents the usual Lebesgue measure, $\mathcal{B}(\varphi, c)$ denotes the open ball centered on φ with radius c and $\overline{\text{conv}}\{\mathcal{A}\}$ stands for the closed convex hull of the set $\mathcal{A}$.

3 | A GENERAL SOLUTION

In this section, we provide a general framework leading to solutions of Problem 1.

3.1 | Integral action in the continuous case

Let us start with a preliminary discussion considering the continuous framework for which the input $u(t)$ is free to evolve in $\mathbb{R}^m$, instead of being constrained in $\mathcal{U}$.

Following the internal model principle, the error induced by parameter uncertainties is cancelled by adding an integral action in the control scheme and design a robust stabilizer for the augmented model. Thus, state z of the controller satisfies $z = \int (y - y_{\text{ref}})$. Denoting $\zeta := \begin{bmatrix} x \\ z \end{bmatrix}$, the augmented model reads:

$$\dot{\zeta}(t) = \bar{f}_\theta(\zeta(t)) + Bu(t), \tag{5}$$

with $\zeta(t) \in \mathbb{R}^{n+p}$ and

$$\bar{f}_\theta\left(\begin{bmatrix} x \\ z \end{bmatrix}\right) := \begin{bmatrix} f_\theta^0(x) \\ h(x) - y_{\text{ref}} \end{bmatrix}, B := \begin{bmatrix} B_0 \\ \mathbf{0}_{p\times m} \end{bmatrix}. \tag{6}$$

3.2 | The pitfall of stabilizer design of Reference 15 in the uncertain context

To solve Problem 1, we follow the path outlined in the previous subsection: we first add a continuous integrator, and then we design a stabilizing control law for the augmented system. To implement this last task, a naive strategy is to directly adapt the methodology exposed in Reference 15 to the uncertain context. In this case, one arrives to the following statement: given a *continuous* controller $\kappa : \mathbb{R}^{n+p} \to \text{conv}\{\mathcal{U}\}$ such that some positive definite Lyapunov function $V_\theta : \mathbb{R}^{n+p} \to \mathbb{R}$ is decreasing along the trajectories of closed-loop (5) with $u = \kappa(\zeta)$, one gets the following controller as a candidate to solve Problem 1:

$$\begin{aligned} \dot{z}(t) &= h(x(t)) - y_{\text{ref}}, \\ u(\zeta) &\in \arg\min_{v\in\mathcal{U}} \frac{\partial V_\theta}{\partial \zeta}(\zeta - \zeta_\theta^*)Bv, \end{aligned}$$

where ζ_θ^* define an equilibrium of (5), that is, $\bar{f}_\theta(\zeta_\theta^*) + Bu_\theta^* = \mathbf{0}$ holds for some u_θ^*. This controller cannot be implemented, though, due to its dependency with respect to the *unknown* parameter vector θ, via ζ_θ^* and V_θ. The rest of this section can be conceived as a workaround to this issue.

3.3 | A particular equilibrium

To tackle the issue exposed in the previous subsection, the first step is to exploit the degrees of freedom in the definition of the steady state of (5). Indeed, pick any mapping $\theta \mapsto (x_\theta^*, u_\theta^*)$ and define $\zeta_\theta^* = \begin{bmatrix} x_\theta^* \\ z_\theta^* \end{bmatrix}$ with $z_\theta^* : \Theta \to \mathbb{R}^p$. By definition of (x_θ^*, u_θ^*), the following equation

$$\bar{f}_\theta(\zeta_\theta^*) + Bu_\theta^* = \mathbf{0}, \tag{7a}$$

is solved for *any* $\theta \mapsto z_\theta^*$.[†] From this observation, we seek for some pair $(\zeta_\theta^*, u_\theta^*)$ solving (7a) and, at the same time, the following equation:

$$K\left(\zeta_\theta^* - \begin{bmatrix} x_{\theta_n}^* \\ \mathbf{0}_p \end{bmatrix}\right) = \mathbf{0}_m, \tag{7b}$$

for some matrix K, some vector $\theta_n \in \Theta$ and for all $\theta \in \Theta$. As shown in the sequel, the peculiar equality (7b) is the key to arrive at a controller which does not depend on the uncertainty θ. Let us now formalize the assumption on the equilibrium solving (7).

Assumption 2. There exist a vector $\theta_n \in \Theta$, a scalar $\lambda \in \mathbb{R}_{>0}$ and a matrix $P \in \mathbb{S}_+^{n+p}$ defining $K \in \mathbb{R}^{m\times(n+p)}$ as follows

$$K := -\frac{\lambda}{2}B^\intercal P, \tag{8}$$

such that, for all $\theta \in \Theta$, there exists $\theta \mapsto \left(\zeta_\theta^*, u_\theta^*\right) \in \mathbb{R}^{n+p} \times \mathbb{R}^m$ satisfying (7) for all $\theta \in \Theta$.

In the next subsection, a general solution to Problem 1 is proposed, by relying on the Assumption 2. Then, Section 3.5 is dedicated to commenting Assumption 2 and constructing the conditions under which this assumption holds.

3.4 | A sufficient condition for robust stabilization

Given any $\theta \in \Theta$. Assume that Assumption 2 holds. Define the following relative coordinates:

$$\tilde{\zeta} := \zeta - \zeta_\theta^*, \quad \tilde{u} := u - u_\theta^*.$$

In view of (3), it follows that (5) with $u(t) \in \mathcal{U}$ can be rewritten as follows:

$$\dot{\tilde{\zeta}}(t) = f_\theta(\tilde{\zeta}(t)) + B\tilde{u}(t), \tag{9}$$

with $\tilde{u}(t) \in \mathcal{U} - u_\theta^*$ and

$$f_\theta(\tilde{\zeta}) := \bar{f}_\theta(\tilde{\zeta} + \zeta_\theta^*) - \bar{f}_\theta(\zeta_\theta^*). \tag{10}$$

In the next assumption, we suppose that there exists a continuous controller delivering bounded inputs and such that closed-loop system admits a Lur'e-type Lyapunov function.

Assumption 3. Assumption 2 is valid for some $\theta_n \in \Theta$, $\lambda \in \mathbb{R}_{>0}$ and $P \in \mathbb{S}_+^{n+p}$. In addition to that, there exist a vector $c \in \mathbb{R}_{>0}^m$, a domain $\mathcal{D} \subseteq \mathbb{R}^{n+p}$ containing $\mathbf{0}$, a diagonal positive semi definite matrix $\Omega \in \mathbb{R}^{m\times m}$, a scalar $\gamma \in \mathbb{R}_{>0}$ and a continuous application $\kappa : \mathbb{R}^{n+p} \to \mathbb{R}^m$ satisfying $\kappa(\mathcal{D}) \subseteq \text{conv}\{\mathcal{U}\} - u_\theta^*$ for all $\theta \in \Theta$, such that

$$\frac{\partial V}{\partial \tilde{\zeta}}(\tilde{\zeta})\big(f_\theta(\tilde{\zeta}) + B\kappa(\tilde{\zeta})\big) < 0, \tag{11}$$

holds for all $\tilde{\zeta} \in \mathcal{D} \setminus \{\mathbf{0}\}$ and for all $\theta \in \Theta$, where $V : \mathbb{R}^{n+p} \to \mathbb{R}$ reads

$$V(\tilde{\zeta}) := \gamma \tilde{\zeta}^\intercal P \tilde{\zeta} - 2\sum_{k=1}^{m} \Omega_{k,k} \int_0^{K_k\tilde{\zeta}} \phi_{c_k}(s)\mathrm{d}s, \tag{12}$$

and $\phi_c : \mathbb{R}^m \to \mathbb{R}^m$ is the following dead-zone

$$\phi_c(x) = \text{sat}_c(x) - x. \tag{13}$$

and K is given by (8).

Theorem 1. *Assume that Assumption 3 holds. Then, $n_z = p$ and the following mappings solve Problem 1:*

$$\phi(x, z) = h(x) - y_{\text{ref}}, \tag{14a}$$

$$\psi^\intercal(x, z) = \left(\gamma \begin{bmatrix} x - x_{\theta_n}^* \\ z \end{bmatrix}^\intercal P - \phi_c^\intercal(K \begin{bmatrix} x - x_{\theta_n}^* \\ z \end{bmatrix})\Omega K\right) B. \tag{14b}$$

Besides, the set $\Phi(\zeta_\theta^*, r)$ defined as follows:

$$\Phi(\zeta_\theta^*, r) := \{\zeta \in \mathbb{R}^{n+p} : V(\zeta - \zeta_\theta^*) \le r\}, \tag{15}$$

belongs to the closed-loop domain of attraction for any r such that the following inclusion holds:

$$\Phi(\mathbf{0}, r) \subseteq \mathcal{D}. \tag{16}$$

As a result, for any r satisfying (16), for all $\theta \in \Theta$ and for all $\begin{bmatrix} x(0) \\ z(0) \end{bmatrix} \in \Phi(\zeta_\theta^*, r)$, $\zeta(t)$ asymptotically converges to an equilibrium for which $y(t) = y_{\text{ref}}$.

Proof. First recall that function V is positive definite, since $P \succ \mathbf{0}$ and $\Omega \succeq \mathbf{0}$ hold, see (A2) in the appendix.

Then, select $\theta \in \Theta$ arbitrarily. Assumption 3 ensures that $\kappa(\mathcal{D}) \subseteq \text{conv}\{\mathcal{U}\} - u_\theta^*$, so that, for all $\tilde{\zeta} \in \mathcal{D}$, there exists $\rho(\tilde{\zeta}) \in \Lambda_N$ such that:

$$\kappa(\tilde{\zeta}) = \sum_{i=1}^{N} \rho_i(\tilde{\zeta}) \tilde{u}^{[i]}, \tag{17}$$

where $\tilde{u}^{[i]} := u^{[i]} - u_\theta^*$. Bearing in mind that $\sum_{i=1}^{N} \rho_i(\tilde{\zeta}) = 1$ and replacing (17) in (11), it follows that:

$$\sum_{i=1}^{N} \rho_i(\tilde{\zeta}) \frac{\partial V}{\partial \tilde{\zeta}}(\tilde{\zeta}) \big(f_\theta(\tilde{\zeta}) + B\tilde{u}^{[i]} \big) < 0, \forall \tilde{\zeta} \in \mathcal{D} \setminus \{\mathbf{0}\}. \tag{18}$$

Since $\rho_i(\tilde{\zeta}) \geqslant 0$, $(i \in \mathcal{I}_N)$, there exists at least one $i \in \mathcal{I}_N$ such that:

$$\frac{\partial V}{\partial \tilde{\zeta}}(\tilde{\zeta}) \big(f_\theta(\tilde{\zeta}) + B\tilde{u}^{[i]} \big) < 0. \tag{19}$$

Define

$$\zeta_{\theta_n}^* := \begin{bmatrix} x_{\theta_n}^* \\ \mathbf{0}_p \end{bmatrix}. \tag{20}$$

From the definition of V and the fact that (7) holds, one has:

$$\begin{aligned} \frac{\partial V}{\partial \tilde{\zeta}}(\tilde{\zeta}) B &= -\frac{4\gamma}{\lambda} (K\tilde{\zeta})^\intercal - 2\phi_c^\intercal (K\tilde{\zeta}) \Omega K B \\ &= -\frac{4\gamma}{\lambda} \Big(K(\zeta - \zeta_{\theta_n}^*) + K(\zeta_{\theta_n}^* - \zeta_\theta^*) \Big)^\intercal - 2\phi_c^\intercal \Big(K(\zeta - \zeta_{\theta_n}^*) + K(\zeta_{\theta_n}^* - \zeta_\theta^*) \Big) \Omega K B \\ &\overset{(7)}{=} \frac{-4\gamma}{\lambda} \Big(K(\zeta - \zeta_{\theta_n}^*) \Big)^\intercal - 2\phi_c^\intercal \Big(K(\zeta - \zeta_{\theta_n}^*) \Big) \Omega K B. \end{aligned}$$

Using this relation and noticing that (8) is equivalent to $(-2/\lambda) K^\intercal = PB$, (19) reads:

$$\frac{\partial V}{\partial \tilde{\zeta}}(\tilde{\zeta}) f_\theta(\tilde{\zeta}) + 2\psi^\intercal(x, z) \tilde{u}^{[i]} < 0. \tag{21}$$

As a result trajectories of closed-loop (9) with a controller satisfying

$$\tilde{u}(\zeta) \in \arg\min_{v \in (\mathcal{U} - u_\theta^*)} \psi^\intercal(x, z) v, \tag{22}$$

converge asymptotically to $\tilde{\zeta} = \mathbf{0}$, where $y = y_{\text{ref}}$. Then, observe that closed-loop (9) with (22) coincides with the one made of (1) and (14), due to linearity of $\chi : v \mapsto \psi^\intercal(x, z) v$. Indeed, $\chi(\tilde{u}(\zeta)) \leq \chi(v)$ holds for all $v \in \mathcal{U} - u_\theta^*$ iff $\chi(\tilde{u}(\zeta) + u_\theta^*) \leq \chi(w)$ for all $w := v + u_\theta^* \in \mathcal{U}$, so that $u(\zeta) = \tilde{u}(\zeta) + u_\theta^* \in \arg\min_{w \in \mathcal{U}} \psi^\intercal(x, z) w$ holds.

The proof ends by showing that $\tilde{\zeta}(t)$ belongs to $\mathcal{D}$ for all $t \geq 0$. Since $\{\tilde{\zeta} : V(\tilde{\zeta}) \leq r\} = \Phi(\mathbf{0}, r)$ is forward invariant, this inclusion is indeed valid by imposing $\tilde{\zeta}(0) \in \Phi(\mathbf{0}, r) \subseteq \mathcal{D}$, which is equivalent to $\zeta(0) \in \Phi(\zeta_\theta^*, r)$ with $\Phi(\mathbf{0}, r) \subseteq \mathcal{D}$. ∎

Schematic of the resulting closed-loop system is depicted in Figure 1.

Example 1. Consider the following system:

$$\dot{x}_1(t) = -\theta x_1(t)^3 + 10\, u(t), \tag{23}$$

$$\dot{x}_2(t) = x_1(t) - \theta x_2(t), \tag{24}$$

$$y(t) = x_2(t), \tag{25}$$

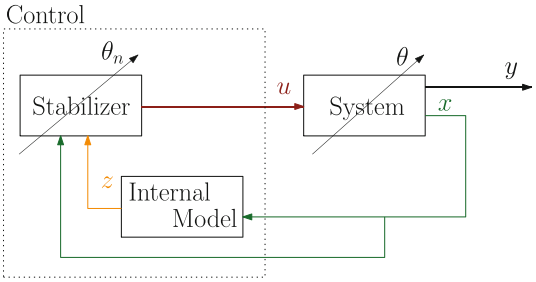

FIGURE 1 Block diagram of the closed-loop system.

with $\theta \in \Theta = [-1, 1]$ and $u(t) \in \mathcal{U} = \{-3, 3\}$. Pick any y_{ref} in $[-1, 1]$. For this system, one can easily prove that Assumption 1 is satisfied and (x_θ^*, u_θ^*) is given by:

$$(x_\theta^*, u_\theta^*) = \left(\begin{bmatrix} \theta y_{\text{ref}} \\ y_{\text{ref}} \end{bmatrix}, \frac{\theta^4 y_{\text{ref}}^3}{10} \right). \tag{26}$$

The augmented system described by (5) and (6) is characterized by:

$$\bar{f}_\theta \left(\begin{bmatrix} x \\ z \end{bmatrix} \right) = \begin{bmatrix} -\theta x_1^3 \\ x_1 - \theta x_2 \\ x_2 - y_{\text{ref}} \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}.$$

Then, selecting $\lambda = 1/5$ and positive definite matrix

$$P = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 17 & 9 \\ 2 & 9 & 8 \end{bmatrix},$$

one gets $K = -\begin{bmatrix} 1 & 4 & 2 \end{bmatrix}$ by (8). It can be easily verified that Assumption 2 holds with

$$(\zeta_\theta^*, u_\theta^*) = \left(\begin{bmatrix} x_\theta^* \\ -\frac{1}{2} y_{\text{ref}} (\theta - \theta_n) \end{bmatrix}, u_\theta^* \right),$$

and for any $\theta_n \in \mathbb{R}$. As pointed out in Section 3.2, dependency of ζ_θ^* with respect to θ prevents direct implementation of the strategy proposed in Reference 15. Now, select $\gamma = 1/2$, $\Omega = 0$ and

$$\kappa(\tilde{\zeta}) = \frac{1}{10} \left(-\theta^4 y_{\text{ref}}^3 + \theta \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tilde{\zeta} + \theta y_{\text{ref}} \right)^3 + \begin{bmatrix} -5 & 4\theta - 7 & -3 \end{bmatrix} \tilde{\zeta} \right).$$

Choose also any $c \in \mathbb{R}_{>0}^m$. Bearing in mind (26) and the definition of f_θ, observe that the left-hand side of (11) reads

$$\frac{\partial V}{\partial \tilde{\zeta}}(\tilde{\zeta}) \left(f_\theta(\tilde{\zeta}) + B\kappa(\tilde{\zeta}) \right) = \frac{1}{2} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix}^\intercal P$$

$$\left(\begin{bmatrix} -\theta(\tilde{x}_1 + \theta y_{\text{ref}})^3 \\ \tilde{x}_1 + \theta y_{\text{ref}} - \theta(\tilde{x}_2 + y_{\text{ref}}) \\ \tilde{x}_2 + y_{\text{ref}} - y_{\text{ref}} \end{bmatrix} - \begin{bmatrix} -\theta(\theta y_{\text{ref}})^3 \\ \theta y_{\text{ref}} - \theta y_{\text{ref}} \\ y_{\text{ref}} - y_{\text{ref}} \end{bmatrix} + \begin{bmatrix} 10\kappa(\tilde{\zeta}) \\ 0 \\ 0 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} \tilde{x}_1 + 4\tilde{x}_2 + 2\tilde{z} \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix}^\intercal \begin{bmatrix} -1 & \mathbf{0} \\ \mathbf{0} & Q(\theta) \end{bmatrix} \begin{bmatrix} \tilde{x}_1 + 4\tilde{x}_2 + 2\tilde{z} \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix}$$

with

$$Q(\theta) = \begin{bmatrix} -\theta - 3 & -\theta/2 - 1 \\ -\theta/2 - 1 & -2 \end{bmatrix}.$$

This expression is negative for all $\theta \in [-1, 1]$ and for all $\tilde{\zeta} \neq \mathbf{0}$ since $Q(\theta)$ is negative definite for all $\theta \in \{-1, 1\}$ and, in turn, for all $\theta \in [-1, 1]$, by convexity. Define $\mathcal{D} = [-1, 1]^3$. Observe that

$$\kappa(\tilde{x}_1, \tilde{x}_2, \tilde{z}) = \frac{1}{10}(r(\tilde{x}_1) + (4\theta - 7)\tilde{x}_2 - 3\tilde{z}) - u^*_\theta$$

with $r(\tilde{x}_1) = \theta(\tilde{x}_1 + \theta y_{\text{ref}})^3 - 5\tilde{x}_1$. Exploiting the fact that $r(\tilde{x}_1) \in [-13, 13]$ holds for all $\tilde{x}_1 \in [-1, 1]$ and for all $(\theta, y_{\text{ref}}) \in [-1, 1]^2$, one gets

$$\kappa(\mathcal{D}) \subset \frac{1}{10}[-27, 27] - u^*_\theta \subset \text{conv}\{\mathcal{U}\} - u^*_\theta$$

is valid for all $(\theta, y_{\text{ref}}) \in [-1, 1]^2$. This proves that Assumption 3 is satisfied.

As a result, Theorem 1 applies and guaranties that any controller like the following one:

$$\begin{aligned} &\dot{z}(t) = x_2(t) - y_{\text{ref}}, \\ &u(t) \in \arg\min_{v \in \{-3,3\}} 5(x_1(t) + 4x_2(t) + 2z(t) - y_{\text{ref}}(\theta_n + 4))v, \end{aligned}$$

solves Problem 1, for example, with $\theta_n = 0$, one gets

$$\begin{aligned} &\dot{z}(t) = x_2(t) - y_{\text{ref}}, \\ &u(t) = \begin{cases} -3, & (\text{if } x_1(t) + 4x_2(t) + 2z(t) \geq 4y_{\text{ref}}), \\ 3, & (\text{otherwise}). \end{cases} \end{aligned}$$

Remark 2. Two comments can be made on the selection of κ, introduced in Assumption 3. Each one is illustrated by the previous example. (i) Application κ can be dependent from the uncertain vector θ. Indeed, in Assumption 3, it is merely require that the controller κ exists, but its precise expression is useless for the computation of the controller (2) proposed by Theorem 1. This is in contrast with the Lyapunov function V, which needs to be independent from θ for being evaluated by the controller (2). Observe that the exact opposite conclusion is drawn in the continuous case, where the control law must be independent from the uncertain vector, unlike the Lyapunov function, see for example, Reference 36. (ii) If (12) suggests that $\tilde{\zeta} \mapsto \text{sat}_c(K\tilde{\zeta})$ is a relevant way to select κ, alternative choice can be made, provided that (11) is valid.

Remark 3 (On $\Phi(\zeta^*_\theta, r)$). Set $\Phi(\zeta^*_\theta, r)$ is an inner estimate of the basin of attraction if r satisfies (16). Note that it is always possible to select $r > 0$ sufficiently small for (16) to hold, since V is continuous and $\mathcal{D}$ is open and contains the origin.

Then, a crucial question is the following: how to exploit this estimate $\Phi(\zeta^*_\theta, r)$ since it depends on the unknown vector θ, via ζ^*_θ? The idea is to make sure that the initial condition $(x(0), z(0))$ belongs to $\Phi(\zeta^*_\theta, r)$, for any $\theta \in \Theta$ and for some ζ^*_θ satisfying (7), that is for some $\zeta^*_\theta \in \Psi(K, \theta_n; \theta)$ where

$$\Psi(K, \theta_n; \theta) := \{\zeta^*_\theta | \exists (\theta \mapsto u^*_\theta) : (7)\ \forall \theta \in \Theta\}.$$

This can be achieved via the following set, which is *independent* of θ:

$$\mathcal{G} := \cap_{\theta \in \Theta} \cup_{\zeta^*_\theta \in \Psi(K, \theta_n; \theta)} \Phi(\zeta^*_\theta, r).$$

Indeed, by construction, observe that for all $\theta \in \Theta$, there exists $\zeta^*_\theta \in \Psi(K, \theta_n; \theta)$ such that $\mathcal{G} \subseteq \Phi(\zeta^*_\theta, r)$ holds. As a result, if the initial condition $(x(0), z(0))$ belongs to $\mathcal{G}$, then $y(t)$ converges to y_{ref}. This suggests that one may

have to adapt the value of $z(0)$ to the initial condition $x(0)$ of the system in order to ensure $(x(0), z(0)) \in \mathcal{G}$. Thus, instead of initializing the integrator at $z(0) = \mathbf{0}$, one should rather compute $z(0)$ in order to move $(x(0), z(0))$ away from the boundaries of $\mathcal{G}$. This is expected to enhance robustness with respect to $x(0)$.

Finally, it is worth noting that the set $\Psi(K, \theta_n; \theta)$ (i) is not empty if Assumption 2 holds and (ii) is an affine subspace which can be explicitly computed in the linear case, see the proof of Lemma 2.

3.5 | A first comment on Assumption 2

A first condition under which Assumption 2 holds is now offered. It is shown that validity of this assumption is strongly related to the existence of an equilibrium of a perturbed version of (1) constructed by adding a constant vector $d_{\theta,\theta_n} \in \mathbb{R}^n$ on the right-hand side of (1a). Indeed, in this context, augmented system (5) becomes:

$$\dot{\zeta}^d(t) = \bar{f}_\theta(\zeta^d(t)) + Bu^d(t) + \begin{bmatrix} d_{\theta,\theta_n} \\ \mathbf{0} \end{bmatrix}. \tag{27}$$

Expression of this virtual perturbation will be defined later on. But let us already mention that it depends on the actual unknown vector θ as well as some vector $\theta_n \in \Theta$ to be selected. Implementing the linear controller $u^d = K\zeta^d$ leads to the following closed-loop system:

$$\dot{\zeta}^d(t) = \bar{f}_\theta(\zeta^d(t)) + BK\zeta^d(t) + \begin{bmatrix} d_{\theta,\theta_n} \\ \mathbf{0} \end{bmatrix}. \tag{28}$$

Lemma 1. *Assume that (3) admits a unique solution. Assume further that there exist a vector $\theta_n \in \Theta$, a scalar $\lambda \in \mathbb{R}_{>0}$ and a matrix $P \in \mathbb{S}_+^{n+p}$ defining $K \in \mathbb{R}^{m\times(n+p)}$ as in (8), such that, for all $\theta \in \Theta$, closed-loop system (28) with*

$$d_{\theta,\theta_n} = B_0 \left(u_\theta^* - K \begin{bmatrix} x_{\theta_n}^* \\ \mathbf{0} \end{bmatrix} \right), \tag{29}$$

admits an equilibrium, that is, there exists $\theta \mapsto \zeta_\theta^{d*} \in \mathbb{R}^{n+p}$ such that

$$\bar{f}_\theta(\zeta_\theta^{d*}) + BK\zeta_\theta^{d*} + \begin{bmatrix} d_{\theta,\theta_n} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}, \tag{30}$$

holds for all $\theta \in \Theta$. Then, Assumption 2 holds.

Proof. From (6) and (29), relationship (30) can be equivalently rewritten as:

$$f_\theta^0(x^{d*}) + B_0 \left(u_\theta^{d*} + K \left(\begin{bmatrix} x^{d*} \\ z^{d*} \end{bmatrix} - \begin{bmatrix} x_{\theta_n}^{d*} \\ \mathbf{0} \end{bmatrix} \right) \right) = \mathbf{0}, \ h(x^{d*}) = y_{\text{ref}},$$

where ζ_θ^{d*} has been split as $\begin{bmatrix} x_\theta^{d*} \\ z_\theta^{d*} \end{bmatrix}$ with $z_\theta^{d*} \in \mathbb{R}^p$. Then, it suffices to invoke Assumption 1 and unicity of solution of (3) to arrive at (7) with $(\zeta_\theta^*, u_\theta^*) = (\zeta_\theta^{d*}, u_\theta^{d*})$. ∎

The next section deals with the particular case where both f_θ^0 and h are linear. In the context, additional comments on Assumption 2 can be made: it is shown that Assumption 2 is satisfied for all $\theta_n \in \Theta$, if linear map $\zeta \mapsto \bar{f}_\theta(\zeta) + BK\zeta$, with K defined as in (8), is invertible for all $\theta \in \Theta$, see Lemma 2.

4 | THE LINEAR AND CONVEX POLYTOPIC CASE

This section specializes the study via the following assumptions.

Assumption 4. Applications $x \mapsto f_\theta^0(x)$ and $x \mapsto h(x)$ are linear, that is, (1) can be written as follows

$$\begin{cases} \dot{x}(t) = A(\theta)x(t) + B_0 u(t), \\ y(t) = Cx(t), \end{cases} \tag{31}$$

for some matrices $A(\theta) \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$.

Assumption 5. Set Θ is a known convex polytope and $\theta \mapsto A(\theta)$ is affine, that is, there exist a matrix A_0 and a linear map $\theta \mapsto A_1(\theta)$ such that $A(\theta) = A_0 + A_1(\theta)$ for all $\theta \in \Theta$.

In this context, constructive conditions are offered, under which Assumption 3 and, in turn, Theorem 1 apply.

4.1 | Preliminaries

Under Assumption 4, (5) can be re-expressed as follows:

$$\dot{\zeta}(t) = \bar{A}(\theta)\zeta(t) + Bu(t) + h, \tag{32}$$

with

$$\bar{A}(\theta) := \begin{bmatrix} A(\theta) & \mathbf{0}_{n\times p} \\ C & \mathbf{0}_{p\times p} \end{bmatrix}, h := \begin{bmatrix} \mathbf{0}_n \\ -y_{\text{ref}} \end{bmatrix}. \tag{33}$$

In addition to that, if Assumption 5 holds, then $\bar{A}(\Theta)$ is a convex polytope of N_s vertices $\bar{A}^{[i]} := \bar{A}(\theta^{[i]})$ where $\theta^{[i]}$ are vertices of Θ:

$$\bar{A}(\Theta) = \text{conv}\{\bar{A}^{[1]}, \dots, \bar{A}^{[N_s]}\}. \tag{34}$$

Finally, let us define system matrix $P_\Sigma(\theta, s)$ as follows:

$$P_\Sigma(\theta, s) := \begin{bmatrix} A(\theta) - s\mathbf{I}_n & B_0 \\ C & \mathbf{0}_{p\times m} \end{bmatrix}.$$

Focusing on the linear context, next lemma provides new conditions under which Assumption 2 is valid.

Lemma 2. *Assume that Assumption 4 holds. Given a scalar $\lambda \in \mathbb{R}_{>0}$ and a matrix $P \in \mathbb{S}_+^{n+p}$ defining $K \in \mathbb{R}^{m\times(n+p)}$ as in (8). Consider the two following statements*:

- *S1*: *There exists a vector $\theta_n \in \Theta$ such that, for all $\theta \in \Theta$, system (28) with (29) admits a unique equilibrium, that is, (30) admits a unique solution $\theta \mapsto \zeta_\theta^{d*}$;*
- *S2*: *Matrix $\bar{A}(\theta) + BK$ is invertible for all $\theta \in \Theta$.*

Then, S1 and S2 are equivalent. Further, if S1 or S2 holds, then for any $\theta_n \in \Theta$, there exists a unique mapping $\theta \mapsto (\zeta_\theta^*, u_\theta^*)$ satisfying (7) for all $\theta \in \Theta$, so that Assumption 2 is valid.

Proof. Given any $\theta \in \Theta$. Define $[K_x, K_z] := K$ with $K_x \in \mathbb{R}^{m\times n}$.

S1⇔S2: Under Assumption 4, (30) reads:

$$(\bar{A}(\theta) + BK)\zeta_\theta^{d*} + \begin{bmatrix} d_{\theta,\theta_n} \\ -y_{\text{ref}} \end{bmatrix} = \mathbf{0}. \tag{35}$$

Thus S1⇐S2 trivially holds. Furthermore, from (6), (29), and (33) and denoting $\zeta_\theta^{d*} = \begin{bmatrix} x^{d*} \\ z^{d*} \end{bmatrix}$, (35) can be rewritten as follows

$$P_\Sigma(\theta;0)\begin{bmatrix} x^{d*} \\ u_\theta^{d*} + K\begin{bmatrix} x^{d*} - x_{\theta_n}^* \\ z^{d*} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ y_{\text{ref}} \end{bmatrix},$$

or, equivalently,

$$P_\Sigma(\theta;0)\left(N\begin{bmatrix} x^{d*} - x_{\theta_n}^* \\ z^{d*} \end{bmatrix} + \begin{bmatrix} x_{\theta_n}^* - x_\theta^{d*} \\ \mathbf{0} \end{bmatrix} \right) = \mathbf{0},$$

by using Assumption 1 and with

$$N := \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ K_x & K_z \end{bmatrix}.$$

From S1, (x^{d*}, z^{d*}) is unique. This implies that $\ker(P_\Sigma(\theta;0)N) = \{\mathbf{0}\}$. Since it holds

$$P_\Sigma(\theta;0)N = \begin{bmatrix} A(\theta) + B_0K_x & B_0K_z \\ C & \mathbf{0} \end{bmatrix} = \bar{A}(\theta) + BK,$$

this proves S1⇒S2.

S2⇒Assumption 2: S2 implies that K_z is full rank and, in turn, that the following two matrices are invertible:

$$M_2 := \begin{bmatrix} \mathbf{I}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p & \mathbf{0} \\ K_x & \mathbf{0} & K_z \end{bmatrix}, M_3(\theta) := \begin{bmatrix} A(\theta) + B_0K_x & \mathbf{0} & B_0K_z \\ C & \mathbf{0} & \mathbf{0} \\ K_x & K_z & \mathbf{0} \end{bmatrix},$$

for all $\theta \in \Theta$. From the observation that (7) can be written as follows

$$M_1(\theta)\begin{bmatrix} \zeta_\theta^* \\ u_\theta^* \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ y_{\text{ref}} \\ K_x x_{\theta_n}^* \end{bmatrix} \quad \text{with} \quad M_1(\theta) := \begin{bmatrix} A(\theta) & \mathbf{0} & B_0 \\ C & \mathbf{0} & \mathbf{0} \\ K_x & K_z & \mathbf{0} \end{bmatrix},$$

and the fact that $M_1(\theta)M_2 = M_3(\theta)$ holds, this proves that $M_1(\theta)$ is invertible. As a result, for any $\theta_n \in \Theta$, (7) is satisfied for a unique $(\zeta_\theta^*, u_\theta^*)$. ∎

Remark 4 (On Assumption 1). If Assumption 4 holds, then Assumption 1 reads: $\begin{bmatrix} \mathbf{0} \\ y_{\text{ref}} \end{bmatrix} \in \text{Im}P_\Sigma(\theta, 0)$ for all $\theta \in \Theta$. This condition is satisfied if, for all $\theta \in \Theta$, the quadruple $\{A(\theta), B_0, C, \mathbf{0}\}$ is right-invertible and 0 is not an invariant zero, that is, if $rank P_\Sigma(\theta, 0)$ equals $n + p$ for all $\theta \in \Theta$, which corresponds to the well-known non-resonance condition.

Remark 5 (Lemma 1 versus 2). Both Lemmas 1 and 2 offer conditions for Assumption 2 to hold. Those conditions are related to (3) and (30). In contrast with Lemma 2, unicity of the solution of (3) is required by Lemma 1. This suggests that Lemma 2 benefits from a larger applicability than Lemma 1, whenever the dynamics is linear. The fact that unicity of solution of (30) is needed by S1 of Lemma 2, unlike Lemma 1, weaken this conclusion only apparently. Indeed, this last unicity is implied by quadratic stabilizability, which is necessary for the synthesis conditions of this section to be satisfied, see the proof of forthcoming propositions.

In other words, unicity of solution of (30) is given for free in the context of the constructive results of this section.

4.2 | Global stability

Define the following hyperrectangle:

$$\mathcal{P}(c) := \left\{u \in \mathbb{R}^m \,:\, |u_k| \leqslant c_k,\ \forall k \in \mathcal{I}_m\right\}, \tag{36}$$

where $c \in \mathbb{R}^m_{>0}$.

Proposition 1. *Assume that Assumptions 4 and 5 hold. Assume further that there exist*:

A1: *A scalar* $\lambda \in \mathbb{R}_{>0}$ *and a matrix* $P \in \mathbb{S}^{n+p}_+$ *defining* $K \in \mathbb{R}^{m\times(n+p)}$ *as in (8)*;
A2: *A diagonal positive semi-definite matrix* $\Omega \in \mathbb{R}^{m\times m}$, *symmetric matrices with non-negative entries* $T^+, T^- \in \mathbb{R}^{m\times m}$, *such that* $T^-_{j,j} = 0$ *holds for all* $j \in \mathcal{I}_m$;
A3: *A scalar* $\gamma \in \mathbb{R}_{>0}$ *and a vector* $c \in \mathbb{R}^m_{>0}$;

such that the following relationships are satisfied with $\Gamma = TK$ *and* $T = T^+ - T^-$:

$$\mathcal{P}(c) \subseteq \mathcal{V} := \cap_{\theta\in\Theta}\left(\mathrm{conv}(\mathcal{U}) - u^*_\theta\right), \tag{37}$$

$$\sum_{j\in\mathcal{I}_m\setminus\{k\}} c_j\left(T^+_{k,j} + T^-_{j,k}\right) \leq c_k T^+_{k,k},\ \forall k \in \mathcal{I}_m, \tag{38}$$

$$\begin{bmatrix} (\bar{A}^{[i]} + BK)^\intercal\gamma P + \gamma P(\bar{A}^{[i]} + BK) & \gamma PB - \Gamma^\intercal - (\bar{A}^{[i]} + BK)^\intercal K^\intercal\Omega \\ * & -2T - \Omega KB - (\Omega KB)^\intercal \end{bmatrix} \prec \mathbf{0}, \forall i \in \mathcal{I}_{N_s}. \tag{39}$$

Then, Assumption 3 is satisfied for any vector $\theta_n \in \Theta$ *and for* $\mathcal{D} = \mathbb{R}^{n+p}$, *so that (14) solves Problem 1 globally, that is, for any* $\theta \in \Theta$ *and any initial condition* $(x(0), z(0)) \in \mathbb{R}^{n+p}$, *output* $y(t)$ *converges to* y_{ref}.

Proof. Condition (39) implies that:

$$\begin{bmatrix} (\bar{A}(\theta) + BK)^\intercal\gamma P + \gamma P(\bar{A}(\theta) + BK) & \gamma PB - \Gamma^\intercal - (\bar{A}(\theta) + BK)^\intercal K^\intercal\Omega \\ * & -2T - \Omega KB - (\Omega KB)^\intercal \end{bmatrix} \prec \mathbf{0}, \tag{40}$$

holds for all $\theta \in \Theta$. Since $\gamma \in \mathbb{R}_{>0}$, it follows that:

$$(\bar{A}(\theta) + BK)^\intercal P + P(\bar{A}(\theta) + BK) \prec \mathbf{0},$$

is valid, so that $\bar{A}(\theta) + BK$ is Hurwitz for all $\theta \in \Theta$. From Lemma 2, this implies that Assumption 2 is valid for any $\theta_n \in \Theta$.

To prove that Assumption 3 holds, it suffices (i) to invoke Lemma 4 on (40) and (38) and (ii) to remark that, for any $\theta \in \Theta$ and by definition of $\mathcal{V}$, (37) proves that $\kappa(\tilde{\zeta}) = \mathrm{sat}_c(K\tilde{\zeta}) \in \mathrm{conv}(\mathcal{U}) - u^*_\theta$ since $\mathcal{P}(c) \subseteq \mathcal{V} \subseteq \mathrm{conv}(\mathcal{U}) - u^*_\theta$. ∎

4.3 | Local stabilization

For Assumption 3 to hold, one needs to find a stabilizing continuous controller delivering signal in $\mathrm{conv}(\mathcal{U}) - u^*_\theta$. In the continuous context, recall that the global stabilization of the linear system $\dot{x} = Ax + Bu$ subject to bounded controls is possible if and only if the pair (A, B) is stabilizable and none of the eigenvalues of A has strictly positive real part[37(p. 19)]. As a result, for Assumption 3 to hold, it is necessary that, for all $\theta \in \Theta$, no eigenvalue of $A(\theta)$ has strictly positive real part, since the spectrum $\sigma(\bar{A}(\theta))$ of $\bar{A}(\theta)$ equals $\sigma(A(\theta)) \cup \{0\}$, see (33).

If this condition is not satisfied, *local* stabilization could still be achievable, though. In this subsection, we provide a constructive approach to tackle Problem 1 locally but for *any* state matrix $A(\theta)$, that is, possibly having eigenvalues of $A(\theta)$ with positive real part. Besides, a constructive procedure to compute r satisfying (16) is offered. Recall that this scalar r is related to the dimension of the robust domain of attraction.

Proposition 2. *Assume that Assumptions 4 and 5 hold. Assume further that there exist*:

A1: *A scalar* $\lambda \in \mathbb{R}_{>0}$ *and a matrix* $P \in \mathbb{S}_+^{n+p}$ *defining* $K \in \mathbb{R}^{m\times(n+p)}$ *as in (8)*;
A2: *A diagonal positive semi-definite matrix* $\Omega \in \mathbb{R}^{m\times m}$, *a diagonal positive definite matrix* $T \in \mathbb{S}_+^m$, *a matrix* $\Gamma \in \mathbb{R}^{m\times(n+p)}$ *and a vector* $\tau \in \mathbb{R}_{>0}^m$;
A3: *A scalar* $\gamma \in \mathbb{R}_{>0}$ *and a vector* $c \in \mathbb{R}_{>0}^m$;

such that (37), (39) and

$$\begin{bmatrix} \gamma P & T_{jj}K_j^\intercal - \Gamma_j^\intercal \\ * & \tau_j c_j^2 \end{bmatrix} \succeq \mathbf{0}, \forall j \in \mathcal{I}_m, \tag{41}$$

are satisfied. Then, Assumption 3 is satisfied for any vector $\theta_n \in \Theta$ *and for*

$$\mathcal{D} = \{\tilde{\zeta} : (K - T^{-1}\Gamma)\tilde{\zeta} \in \mathcal{P}(c)\}. \tag{42}$$

Furthermore, (16) is satisfied for

$$r = \left(\max_{j\in\mathcal{I}_m}\left\{\frac{\tau_j}{T_{j,j}^2}\right\}\right)^{-1}. \tag{43}$$

As a result, (14) solves Problem 1 locally, that is, for any $\theta \in \Theta$ *and any initial condition* $(x(0), z(0)) \in \Phi(\zeta_\theta^*, r)$, *output* $y(t)$ *converges to* y_{ref}.

Proof. As in the proof of Proposition 1, (39) implies that Assumption 2 is valid for any $\theta_n \in \Theta$. By virtue of Lemma 3, (39) also implies that (11) with $\kappa(\tilde{\zeta}) = \text{sat}_c(K\tilde{\zeta})$ holds for all $\tilde{\zeta} \in \mathcal{D} \setminus \{\mathbf{0}\}$. Furthermore, $\kappa(\tilde{\zeta}) \in \text{conv}(\mathcal{U}) - u_\theta^*$ can be proved as in the proof of Proposition 1. As a result, Assumption 3 is satisfied for all $\theta_n \in \Theta$. Finally, Lemma 3 also proves that (16) is valid. ∎

In the linear case, local stabilization certificate provided by Proposition 2 can be related to structural properties of the open loop system. This allows to prove that conditions associated with Proposition 2 are not only sufficient but also necessary for Assumption 3 to hold, if S1 and S2 of the following lemma are valid.

Proposition 3. *Assume that Assumptions 4 and 5 are satisfied. Assume further that*:

S1: $u_\theta^* \in \text{Int}\{\text{conv}(\mathcal{U})\}$ *for all* $\theta \in \Theta$;
S2: $(\bar{A}(\theta), B)$ *is quadratically stabilizable via linear control for all* $\theta \in \Theta$, *that is, there exist scalars* $\delta, \lambda \in \mathbb{R}_{>0}$ *and a matrix* $Q \in \mathbb{S}_+^{n+p}$ *such that*:

$$\bar{A}^{[i]}Q + Q(\bar{A}^{[i]})^\intercal - \lambda BB^\intercal \prec -2\delta Q, \ \forall i \in \mathcal{I}_{N_s}. \tag{44}$$

Then, there exists $c \in \mathbb{R}_{>0}^m$ *sufficiently closed to* $\mathbf{0}_m$ *such that (37) holds. Choose any of such a vector* c. *Select* $P = Q^{-1}$, $\lambda = 2$, K *defined as in (8)*, $\Omega = \mathbf{0}$, $\Gamma = -\gamma K$ *and* $T = \mathbf{I}_m$. *In this case, set* $\mathcal{D}$ *defined in (42) is not empty and reads*:

$$\mathcal{D} = \{\tilde{\zeta} : (1+\gamma)K\tilde{\zeta} \in \mathcal{P}(c)\}.$$

Besides, (39) and (41) are satisfied for any $\gamma \in \mathbb{R}_{>0}$ *and any* $\tau \in \mathbb{R}^m$ *satisfying*:

$$\tau_j > \frac{(1+\gamma)^2 K_j K_j^\intercal}{\gamma \lambda_{\min}(P) c_j^2}, \tag{45}$$

for all $j \in \mathcal{I}_m$, *where* $\lambda_{\min}(P)$ *is the smallest eigenvalue of* P.

Proof. First note that S1 is equivalent to $\mathbf{0} \in \mathrm{Int}\{\mathcal{V}\}$ which guarantees that (37) is satisfied for some $c \in \mathbb{R}^m_{>0}$. Besides, for any $c \in \mathbb{R}^m_{>0}$, $\mathbf{0}$ belongs in $\mathcal{P}(c)$ and, in turn, in $\mathcal{D}$. This proves that $\mathcal{D}$ is non empty.

Then, select any $i \in \mathcal{I}_{N_s}$. If S2 is satisfied, then the following condition holds for $P = Q^{-1}$, K defined as in (8), any $T \succ \mathbf{0}$ and any $\gamma \in \mathbb{R}_{>0}$:

$$\begin{bmatrix} (\bar{A}^{[i]} + BK)^\intercal \gamma P + \gamma P (\bar{A}^{[i]} + BK) & \mathbf{0} \\ \mathbf{0} & -2T \end{bmatrix} \prec \mathbf{0}. \tag{46}$$

Observe that this inequality is nothing but (39) with $\Gamma = -\gamma K$, $\Omega = \mathbf{0}$ and $\lambda = 2$.

Let us prove that (41) is satisfied. Select any $j \in \mathcal{I}_m$. First note that (45) implies that there exists $\epsilon_j \in \mathbb{R}_{>0}$ such that

$$\tau_j = \frac{(1+\gamma)^2 (K_j K_j^\intercal + \epsilon_j)}{\gamma \lambda_{\min}(P) c_j^2}, \tag{47}$$

since $(1+\gamma)^2/(\gamma \lambda_{\min}(P) c_j^2)$ is strictly positive. Thus, the following inequalities hold:

$$\begin{aligned} P &\succeq \lambda_{\min}(P) \mathbf{I}_{n+p}, \\ (K_j K_j^\intercal + \epsilon_j) \mathbf{I}_{n+p} &\succ K_j K_j^\intercal \mathbf{I}_{n+p} \succeq K_j^\intercal K_j. \end{aligned}$$

Together with (47), theses inequalities imply that:

$$\gamma P \succ \frac{\gamma \lambda_{\min}(P)}{(K_j K_j^\intercal + \epsilon_j)} K_j^\intercal K_j \Leftrightarrow \gamma P \succ \frac{(1+\gamma)^2}{\tau_j c_j^2} K_j^\intercal K_j.$$

Using Schur complements, one can see that this last inequality implies that:

$$\begin{bmatrix} \gamma P & (1+\gamma) K_j^\intercal \\ * & \tau_j c_j^2 \end{bmatrix} \succ \mathbf{0}. \tag{48}$$

which is nothing but (41) with a strictly inequality and for $\Gamma = -\gamma K$ and $T = \mathbf{I}_m$. ∎

4.4 | Comparison with Reference 33

Let us now recall the main result of Reference 33, which is reformulated via the following proposition to ease the comparison.

Proposition 4. *([33, Proposition 11 and Remark 12]) Assume that Assumptions 4, 5 and equality* $m = p$ *hold. Assume further that conditions S1 and S2 of Proposition 3 are satisfied. Then, Assumption 3 is satisfied for any vector* $\theta_n \in \Theta$, *for* $\Omega = \mathbf{0}$, $\gamma = 1$ *and*

$$\mathcal{D} = \{\tilde{\xi} \,:\, K\tilde{\xi} \in \mathcal{V}\}, \tag{49}$$

where $\mathcal{V}$ *reads as in (37).*

In addition to restrict the study to linear systems, let us emphasize that Proposition 4 only consider square system, that is, $m = p$. Such a constraint does not exist in this article.

Besides, Propositions 4 and 3 lead to distinct expressions of $\mathcal{D}$, see (42) and (49). On the one side, Proposition 3 offers additional degrees of freedom, namely T and Γ, so that $\mathcal{D}$ computed as in (42) is expected to be larger than that of (49). On the other side, (42) uses the subset $\mathcal{P}_c$ of $\mathcal{V}$, which suggests that $\mathcal{D}$ given by (42) might actually be the smallest set.

The key point here is that $\Phi(\mathbf{0}, r)$ is constrained to be an ellipsoid in Proposition 4, by setting Ω to $\mathbf{0}$. However, by allowing this matrix to be non zero, the geometry of $\Phi(\mathbf{0}, r)$ obtained via Proposition 3 can be more complex, so that r satisfying (16) can be larger.

4.5 | A constructive algorithm using LMIs

Consider Proposition 1. Matrices inequalities (38) and (39) depend on the decision variables P, λ, Ω, T_+, T_-, Γ, γ, and c, in a bilinear way, which makes the solution difficult to compute. To overcome this problem, it is proposed to implement the following procedure, where c and δ act as design parameters:

1. Pick a vector $c \in \mathbb{R}^m_{>0}$ such that (37) holds. Note that selecting the largest value of c that satisfies (37) is desirable since the approach is based on a stabilizing continuous controller that delivers signal in $\text{conv}(\mathcal{U}) - u^*_\theta$.
2. Solve (44) parametrized by δ to obtain numerical value of P and λ. This parameter is related to the rate of decrease of the Lyapunov function, therefore the larger δ is selected, the faster V converges to zero.
3. Compute K by way of (8).
4. Solve (38) and (39) to compute the remaining variables which parametrized those inequalities in a linear way.

In the case where global stability cannot be achieved, one should consider Proposition 2. It is suggested to implement the procedure outline above and exploit the degrees of freedom existing at the second and fourth steps in order to enlarge the basin of attraction. Specifically, we aims at maximizing $\Phi(\mathbf{0}, r)$ (see Theorem 1) by implementing the second step as in Reference 33 (Proposition 11) and the fourth step as in Reference 37 (p. 134). In proceeding this way, the design parameters list now includes the positive scalars β_1, β_{2i} and β_{3i}, $(i \in \mathcal{I}_m)$, in addition to c and δ as in Proposition 1. For completeness, let us outline the resulting procedure:

1. Compute integer $n_g \in \mathbb{N}$ and vectors $g_j \in \mathbb{R}^m$, $(j \in \mathcal{I}_{n_g})$ defined as follows:

$$\mathcal{V} = \{v \in \mathbb{R}^m \,:\, g_j^\intercal v \leqslant 1, (j \in \mathcal{I}_{n_g})\}. \tag{50}$$

   The existence of such vectors is ensured under the assumption that $\mathbf{0} \in \text{Int}\{\mathcal{V}\}$, which is nothing but statement S1 of Proposition 3, see Reference 38 (p. 87);
2. Select a scalar $\delta \in \mathbb{R}_{>0}$ and solve the following program to compute Q and λ:

$$\min_{Q \in \mathbb{S}^{n+p}_+, \lambda \in \mathbb{R}_{>0}, \rho \in \mathbb{R}_{>0}} \rho \text{ s.t.}$$

$$(44), \tag{51a}$$

$$\begin{bmatrix} 1 & \frac{\lambda}{2} g_j^\intercal B^\intercal \\ * & Q \end{bmatrix} \succ \mathbf{0}, \ (j \in \mathcal{I}_{n_g}), \tag{51b}$$

$$\begin{bmatrix} \rho \mathbf{I}_{n+p} & \mathbf{I}_{n+p} \\ * & Q \end{bmatrix} \succ \mathbf{0}. \tag{51c}$$

3. Get value of K by way of (8) by setting $P = Q^{-1}$;
4. Solve the following program to compute Ω, T, Γ, τ, and γ:
   $\min_{\Omega \in \mathbb{R}^{m \times m}, T \in \mathbb{S}^m_+, \Gamma \in \mathbb{R}^{m \times (n+p)}, \tau \in \mathbb{R}^m, \gamma \in \mathbb{R}_{>0}} f(T, \Omega, \tau, \gamma)$

s.t. (41) and (39) with:

$$f(T,\Omega,\tau,\gamma) := \beta_1 \text{trace}(\gamma P + K^\intercal \Omega K) + \sum_{i=1}^{m} (\beta_{2i}\tau_i - \beta_{3i} T_{i,i}), \tag{52}$$

where the positive scalars β_1, β_{2i}, and β_{3i}, $(i \in \mathcal{I}_m)$ are given weighting parameters.

Three comments can be made here. (i) Parameter δ plays the same role as for global stability. Generally speaking, the following observation is expected: the larger δ is selected, the faster V converges to zero, but the smaller is the basin of attraction. (ii) Step 4 actually maximizes $\mathcal{E}(\gamma P + K^\intercal \Omega K, \mathbf{0}, r)$. This set is included in $\Phi(\mathbf{0}, r)$, see (A2), so that the strategy enlarges $\Phi(\mathbf{0}, r)$ indirectly. (iii) In the optimization problem defined in Step 4, β_1 is linked to the size of $\mathcal{E}(\gamma P + K^\intercal \Omega K, \mathbf{0}, 1)$ via $\text{trace}(\gamma P + K^\intercal \Omega K)$ and β_{2i} and β_{3i}, $(i \in \mathcal{I}_m)$ are related to r respectively via τ_i and $T_{i,i}$, $(i \in \mathcal{I}_m)$. Based on this, there is a trade-off between maximizing either $\mathcal{E}(\gamma P + K^\intercal \Omega K, \mathbf{0}, 1)$ or r. However, since they are both associated to the domain of attraction, an in-depth analysis needs to be carried out to better understand their impact on the size of $\Phi(\mathbf{0}, r)$. This will be a topic for future work.

5 | ILLUSTRATIVE EXAMPLES

5.1 | Global stability

Consider the system described by (31) with:

$$A(\theta) = \begin{bmatrix} 0 & \theta \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \Theta = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad \mathcal{U} := \{0, 10\}, \quad y_{\text{ref}} = -4, \tag{53}$$

and satisfying Assumptions 4 and 5. Note that the spectrum of the open loop matrix $A(\theta)$ equals $\left\{ \pm i\sqrt{\theta} \right\}$. Since it lies on the imaginary axis, to the best of our knowledge, the results presented in the literature cannot be used to solve globally Problem 1 for this system.

This system satisfies Assumption 1 and admits a unique equilibrium satisfying $y = y_{\text{ref}}$, that is, (3) admits the following unique solution:

$$(x^*_\theta, u^*_\theta) = \left(\begin{bmatrix} 4\theta \\ -4 \end{bmatrix}, 4\theta \right). \tag{54}$$

Then, one gets $\mathcal{V} = [-4,\ 2]$, so that $c = 2$ is the largest value satisfying (37).

Let us apply Proposition 1. Vector c being already chosen, we now pick $\delta = 0.1$ for which (44) admits the following solution:

$$Q = \begin{bmatrix} 4.26 & 0.56 & 5.9 \\ 0.56 & 9.038 & -3.28 \\ 5.9 & -3.28 & 11.36 \end{bmatrix}, \ \lambda = 12.07,$$

which, via (8), leads to

$$K = \begin{bmatrix} -7.65 & 1.39 & 4.37 \end{bmatrix}. \tag{55}$$

Then, observe that (38) and (39) with $P = Q^{-1}$ are solved for

$$\gamma = 3.18.10^{-9},\ T = 2.7.10^{-10},\ \Omega = 1.9.10^{-11}.$$

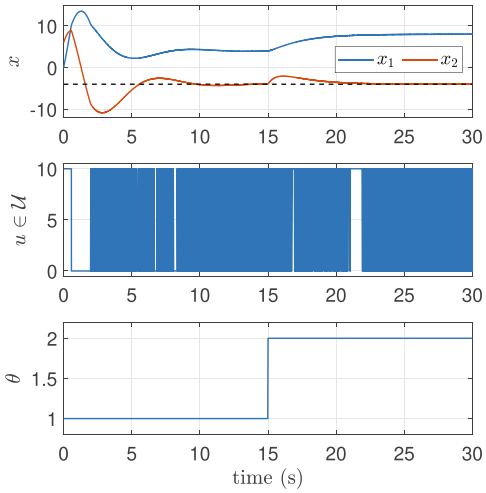

FIGURE 2 Evolution of the state variables x, the control u, and the parameter θ for system (31) defined by (53).

This proves that Assumption 3 is satisfied with $\mathcal{D} = \mathbb{R}^3$ and for any $\theta_n \in \Theta$, so that Theorem 1 applies and gives a global solution to Problem 1.

As an illustration of the effectiveness of the resulting closed-loop system to robustly regulate the output to its reference, we first pick $\theta_n = 1$ so that:

$$\zeta^*_{\theta_n} = \begin{bmatrix} 4 & -4 & 0 \end{bmatrix}^\intercal,$$

and we set the initial condition to $x(0) = \begin{bmatrix} 0 & 6 \end{bmatrix}^\intercal$ and $z(0) = 0$. Besides, we apply a step on θ at time $t = 15$ s. Simulation results are depicted on Figure 2. It can be verified that the desired steady state is achieved since $y(t) = x_2(t)$ asymptotically converges to $y_{\text{ref}} = -4$, after each of the two transients.

5.2 | Local stability

Consider the system described by (31) with:

$$A(\theta) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -\theta \\ \theta & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Theta = \begin{bmatrix} 1, 1.1 \end{bmatrix},$$
$$\mathcal{U} := \left\{ \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\}, y_{\text{ref}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \tag{56}$$

and satisfying Assumptions 4 and 5. Observe that some of the eigenvalues of the open loop matrix $A(\theta)$ have positive real part for some θ: for example, the eigenvalues of $A(1)$ are 0.715 and $0.142 \pm 1.67i$.

The equilibrium satisfying $y = y_{\text{ref}}$ is given by:

$$(x^*_\theta, u^*_\theta) = \left(\begin{bmatrix} \frac{2}{\theta} \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -\frac{2}{\theta} + 3 \\ 3\theta \end{bmatrix} \right). \tag{57}$$

Therefore, Assumption 1 is satisfied. In this case, one can verify that statements S1 and S2 of Proposition 3 are valid, so that assumptions related to Proposition 2 hold true.

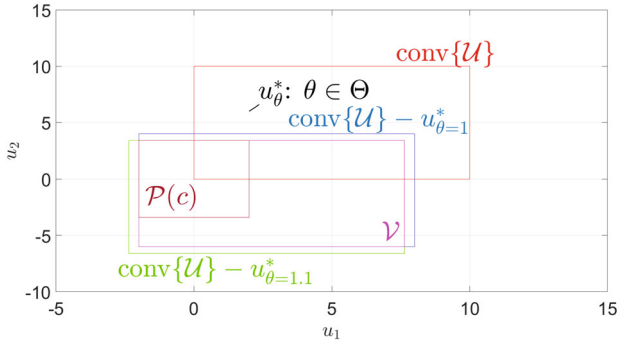

FIGURE 3 Illustration of construction of $\mathcal{V}$ and $\mathcal{P}(c)$.

In order to apply Proposition 2, we compute $\mathcal{V}$ from (37):

$$\begin{aligned}\mathcal{V} &= (\text{conv}\mathcal{U} - u^*_{\theta=1}) \cap (\text{conv}\mathcal{U} - u^*_{\theta=1.1}),\\ &= \Big[-1, 3.8182\Big] \times \Big[-3, 1.7\Big],\end{aligned} \tag{58}$$

so that $c = [1 \quad 1.7]^\intercal$ satisfies (37). The sets $\mathcal{V}$ and $\mathcal{P}(c)$ are depicted by Figure 3.

Let us now implement the procedure proposed in Section 4.5:

1. The set $\mathcal{V}$ defined in (58) can be characterized as in (50) with $n_g = 4$ and vectors g_j, $(j \in \mathcal{I}_4)$ defined as follows:

$$g_1 = [-1 \quad 0], \quad g_2 = [1/3.8182 \quad 0], g_3 = [0 \quad -1/3], \quad g_4 = [0 \quad 1/1.7];$$

2. Now, we choose $\delta = 1$ for which the optimization problem (51) admits $(Q^*, \lambda^*, \gamma^*)$ as a solution;
3. This allows us to compute K via (8) by setting $P = (Q^*)^{-1}$;
4. Finally, we solve (52) with the weighting parameters $\beta_1 = 8$, $\beta_{21} = \beta_{22} = 0.1$ and $\beta_{31} = \beta_{32} = 2$, which yields to the following solution:

$$\Omega = 10^{-4}\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}, \tag{59}$$

$$\gamma = 1.10^{-5}, \; r = 2.1.10^{-5}. \tag{60}$$

This proves in particular that Assumption 3 is satisfied for any $\theta_n \in \Theta$, so that Theorem 1 applies and gives a local solution to Problem 1.

As an illustration, we select $\theta_n = 1$ so that:

$$\zeta^*_{\theta_n} = \Big[2 \quad -1 \quad 3 \quad \mathbf{0}_2^\intercal\Big]^\intercal,$$

and we set the initial condition to $x(0) = \Big[1.6 \quad -0.8 \quad 2.9\Big]^\intercal$ and $z(0) = \Big[-0.07 \quad 0.04\Big]^\intercal$. From Remark 3, one need to make sure that $\zeta(0) \in \mathcal{G}$ holds. Note that the methodology implemented here ensures that S2 of Lemma 2 is satisfied, so that $\Psi(K, \theta_n; \theta) = \{\zeta^*_\theta\}$ holds. As a result, $\mathcal{G} = \cap_{\theta\in\Theta}\Phi(\zeta^*_\theta, r)$ is the intersection of all the translations by ζ^*_θ of the sublevel set of V of magnitude r. Here, r is given in (59) and ζ^*_θ is obtained from (57) and (7).

One can check that $\zeta(0) \in \mathcal{G}$ holds with this initial condition, then the control law (14) stabilizes the system. Observe also that $\zeta(0) \in \mathcal{G}$ is *not* satisfied for $z(0) = \mathbf{0}_2$ and with the same $x(0)$. This illustrates that careful initialization of the integrator allows to enlarge admissible excursions of the initial state $x(0)$ from the equilibrium, see Remark 3.

Again, to assess robustness with respect to θ, a step is applied on its value at time $t = 5$ s which leads to simulation results presented in Figure 4. For the two values of θ, one observes that after a relatively short transient, the desired steady state is achieved with zero steady state error on the controlled output x_2 and x_3. This proves the effectiveness of our approach.

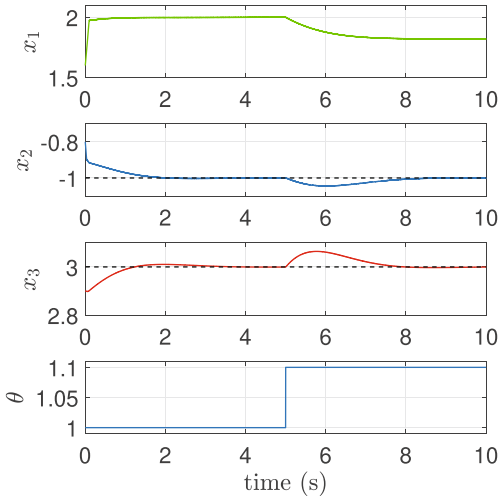

FIGURE 4 Chronograph of the state variables x, and the parameter θ for system (1) defined by (56).

Finally, note that with the approach proposed in Reference 33, $\zeta(0) \in \mathcal{G}$ is not satisfied for both $z(0) = \mathbf{0}$ and $z(0) = \begin{bmatrix} -0.07 & 0.04 \end{bmatrix}^{\mathrm{T}}$. Therefore this approach cannot be used for those initial conditions.

6 | CONCLUSION

In this article, a robust switching controller for nonlinear switched system with uncertain equilibrium has been proposed. The control strategy is based on the internal model approach and on the use of Lur'e Lyapunov functions. Unlike Reference 33, non square systems can be handled.

A first sufficient condition to ensure local stability is given, relying on the assumption of the existence of a control Lyapunov function. Remarkably, it comes out that the associated controller can depend on the uncertain parameters. Exploiting further this peculiarity is a topic of future works.

Then, the study is particularized to the linear case. The employed methodology allows to avoid the classical obstruction on the existence of Hurwitz convex combinations. It guarantees either global asymptotic stability or local asymptotic stability with a possible enlargement of the robust basin of attraction with respect to Reference 33. Simulation results show the effectiveness and the usefulness of the proposed control design approach. Analyzing the impact of the β parameters in the enlargement of the domain of attraction and generalizing the LMI conditions for a larger class of switched system are relevant new research directions. Another interesting point is to limit the switching frequency at the steady state for some practical issues.

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CONFLICT OF INTEREST STATEMENT
The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

ENDNOTES
*Applications h and f_θ^0 are assumed to be locally Lipschitz on $\mathbb{R}^n$.

†Note that other degrees of freedom might come from non unicity of solutions of (3).

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APPENDIX A

Let us recall a few important results, given in Reference 37, about matrix inequalities and Lur'e-type Lyapunov function for saturated system. Notations used in this self-contained appendix are independent of the rest of the article, unless explicit references to definitions already introduced.

Consider the following closed-loop composed of a linear system with a saturated linear controller:

$$\dot{x}(t) = Ax(t) + B\mathrm{sat}_c(Kx(t)), \tag{A1}$$

where $x(t) \in \mathbb{R}^n$ and $c \in \mathbb{R}^m_{>0}$.

Define Lur'e-type Lyapunov function $x \mapsto V(x)$, where V reads as in (12) with $\gamma = 1$. From Reference 39, if $P \succ \mathbf{0}$ and $\Omega \succeq \mathbf{0}$ hold, function V satisfies the inequalities:

$$x^\intercal Px \leqslant V(x) \leqslant x^\intercal (P + K^\intercal \Omega K)x, \tag{A2}$$

for all $x \in \mathbb{R}^n$. As a result, V is positive definite.

In Reference 37, conditions are given under which V is decreasing along trajectories of (A1) for any initial condition $x(0)$ in a domain $\mathcal{D} \subset \mathbb{R}^n$ containing the origin, that is,

$$\frac{\partial V}{\partial x}(x)(Ax + B\mathrm{sat}_c(Kx)) < 0, \tag{A3}$$

holds for all $x(0) \in \mathcal{D} \setminus \{\mathbf{0}\}$.

Lemma 3. *Assume that there exist*:

A1: *Two matrices* $P \in \mathbb{S}^n_+$ *and* $K \in \mathbb{R}^{m\times n}$;
A2: *A diagonal positive semi-definite matrix* $\Omega \in \mathbb{R}^{m\times m}$, *a diagonal positive definite matrix* $T \in \mathbb{S}^m_+$ *and a matrix* $\Gamma \in \mathbb{R}^{m\times n}$;

such that the following condition holds:

$$\begin{bmatrix} (A+BK)^\intercal P + P(A+BK) & PB - \Gamma^\intercal - (A+BK)^\intercal K^\intercal \Omega \\ * & -2T - \Omega KB - (\Omega KB)^\intercal \end{bmatrix} \prec \mathbf{0}, \tag{A4}$$

then V *is positive definite and (A3) holds for all* $x \in \mathcal{D} \setminus \{\mathbf{0}\}$ *with*

$$\mathcal{D} := \{x : (K - T^{-1}\Gamma)x \in \mathcal{P}(c)\}, \tag{A5}$$

where $\mathcal{P}(c)$ *is defined by (36).*

Besides, if there exists

A3: *A vector* $\tau \in \mathbb{R}^m_{>0}$,

such that (41) holds with $\gamma = 1$ then, one has

$$\Phi(r) := \{x : V(x) \leqslant r\} \subset \mathcal{D}, \tag{A6}$$

with

$$r^{-1} := \max_{j \in \mathcal{I}_m} \left\{ \frac{\tau_j}{T_{j,j}^2} \right\}.$$

Proof. The proof follows from careful inspection of statement and proof of Reference 37 (Proposition 3.4). For completeness of this article, let us recall the main steps on the reasoning.

Select any $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. On the first hand, pre post multiply (A4) by $[x^\intercal, \phi_c^\intercal(Kx)]$ and its transpose:

$$\begin{aligned} &x^\intercal((A+BK)^\intercal P + P(A+BK))x + \phi_c^\intercal(Kx)(PB - \Gamma^\intercal - (A+BK)^\intercal K^\intercal \Omega)^\intercal x \\ &\quad + x^\intercal(PB - \Gamma^\intercal - (A+BK)^\intercal K^\intercal \Omega)\phi_c(Kx) + \phi_c^\intercal(Kx)(-2T - \Omega KB - (\Omega KB)^\intercal)\phi_c(Kx) < 0, \end{aligned}$$

or, equivalently,

$$2x^\intercal P(Ax + BKx) + 2\phi_c^\intercal(Kx)(B^\intercal P - \Gamma - \Omega K(A+BK))x + 2\phi_c^\intercal(Kx)(-T - \Omega KB)\phi_c(Kx) < 0,$$

which can be rewritten as follows

$$2x^\intercal P(Ax + B\mathrm{sat}_c(Kx)) - 2\phi_c^\intercal(Kx)\Omega K(Ax + B\mathrm{sat}_c(Kx)) - 2\phi_c^\intercal(Kx)T(\phi_c(Kx) + Gx) < 0, \tag{A7}$$

since $Kx + \phi_c(Kx) = \mathrm{sat}_c(Kx)$ and by denoting $G = T^{-1}\Gamma$. On the other hand, using Reference 37 (Lemma 1.6) and the definition of $\mathcal{P}(c)$ in (36), one proves that:

$$\phi_c^\intercal(Kx)T(\phi_c(Kx) + Gx) \leqslant 0, \forall x \in \mathcal{D}. \tag{A8}$$

This proves that (A3) holds for all $x \in \mathcal{D} \setminus \{\mathbf{0}\}$ since, by definition of V, one has:

$$\frac{\partial V}{\partial x}(x) = 2x^\intercal P - 2\phi_c^\intercal(Kx)\Omega K. \tag{A9}$$

Let us now prove that $\Phi(r) \subseteq \mathcal{D}$. Select any $j \in \mathcal{I}_m$. Multiply (41) with $\gamma = 1$ from both sides by $\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & T_{j,j}^{-1} \end{bmatrix}$:

$$\begin{bmatrix} P & K_j^\intercal - G_j^\intercal \\ * & \dfrac{\tau_j}{T_{j,j}^2} c_j^2 \end{bmatrix} \succeq \mathbf{0}. \tag{A10}$$

Since $r^{-1} > 0$, this gives:

$$\begin{bmatrix} P & K_j^\intercal - G_j^\intercal \\ * & r^{-1} c_j^2 \end{bmatrix} \succeq \mathbf{0}.$$

This relation leads to

$$x^\intercal(K_j - G_j)^\intercal c_j^{-2}(K_j - G_j)x \leqslant r^{-1} x^\intercal P x,$$

for all $x \in \mathbb{R}^n$. Therefore for any $x \in \mathcal{E}(r^{-1}P, \mathbf{0}, 1)$, it holds:

$$x^\intercal(K_j - G_j)^\intercal c_j^{-2}(K_j - G_j)x \leqslant 1 \Leftrightarrow \|(K_j - G_j)x\|_2 \leqslant c_j.$$

Since j is arbitrary, this proves that $x \in \mathcal{D}$, so that one has the following inclusion:

$$\mathcal{E}(r^{-1}P, \mathbf{0}, 1) = \mathcal{E}(P, \mathbf{0}, r) \subseteq \mathcal{D}. \tag{A11}$$

Besides, from (A2), it holds $\Phi(r) \subseteq \mathcal{E}(P, \mathbf{0}, r)$, which, in turn, implies that $\Phi(r) \subseteq \mathcal{D}$. ∎

Lemma 4. *Assume that there exist*:

A1: *Two matrices* $P \in \mathbb{S}_+^n$ *and* $K \in \mathbb{R}^{m\times n}$;
A2: *A diagonal positive semi-definite matrix* $\Omega \in \mathbb{R}^{m\times m}$, *symmetric matrices with non-negative entries* $T^+, T^- \in \mathbb{R}^{m\times m}$, *such that* $T^-_{jj} = 0$ *holds for all* $j \in \mathcal{I}_m$;

such that (38) and (A4) hold with $\Gamma = TK$ *and* $T = T^+ - T^-$, *then* V *is positive definite and (A3) holds for all* $x \in \mathbb{R}^n$.

Proof. See Proposition 3.5, Remarks 3.7, and 3.13 of Reference 37. ∎