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Paul Dorbec, Michael Antony Henning. The $1/3$ -conjectures for domination in cubic graphs. 2024. hal-04427896v3

HAL Id: hal-04427896

<https://hal.science/hal-04427896v3>

Preprint submitted on 3 Oct 2024

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The $\frac{1}{3}$ -Conjectures for Domination in Cubic Graphs

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Abstract

A set S of vertices in a graph G is a dominating set of G if every vertex not in S is adjacent to a vertex in S . The domination number of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G . In a breakthrough paper in 2008, Löwenstein and Rautenbach [17] proved that if G is a cubic graph of order n and girth at least 83, then $\gamma(G) \leq \frac{1}{3}n$. A natural question is if this girth condition can be lowered. The question gave birth to two $\frac{1}{3}$ -conjectures for domination in cubic graphs. The first conjecture, posed by Verstraete in 2010, states that if G is a cubic graph on n vertices with girth at least 6, then $\gamma(G) \leq \frac{1}{3}n$. The second conjecture, first posed as a question by Kostochka in 2009, states that if G is a cubic, bipartite graph of order n , then $\gamma(G) \leq \frac{1}{3}n$. In this paper, we prove Verstraete's conjecture when there is no 7-cycle and no 8-cycle, and we prove the Kostochka's related conjecture for bipartite graphs when there is no 4-cycle and no 8-cycle.

Keywords: Domination number; cubic graph; girth; bipartite

AMS subject classification: 05C69

1 Introduction

A *dominating set* in a graph G is a set S of vertices of G such that every vertex outside S is adjacent to at least one vertex in S . The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G . A thorough treatise on dominating sets can be found in the so-called “domination books” [3, 4, 5, 10].

Independent sets in cubic graph of given girth is very well studied, see, for example, [1, 6, 18]. The problem of determining a sharp upper bound on the domination number of a connected, cubic graph, of sufficiently large order, in terms of its order, remains one of the major outstanding problems in domination theory. A classical result due to Reed [20] states that a cubic graph G of order n satisfies $\gamma(G) \leq \frac{3}{8}n$. His proof uses ingenious counting arguments. (Another proof of Reed's $\frac{3}{8}$ -bound can be found in [2].) Kostochka and Stodolsky [14] proved that the two non-planar, connected, cubic graphs of order 8 are the only connected, cubic graphs that achieve the $\frac{3}{8}$ -bound, by proving that if G is a connected cubic graph of order $n \geq 10$,

then $\gamma(G) \leq \frac{4}{11}n$. The best general upper bound to date on the domination number of a connected cubic graph is due to Kostochka and Stocker [13].

Theorem 1 ([13]) *If G is a connected, cubic graph of order n , then $\gamma(G) \leq \frac{5}{14}n = \left(\frac{1}{3} + \frac{1}{42}\right)n$.*

However, it is not known if there are graphs of large order that achieve the $\frac{5}{14}$ -bound in Theorem 1. Reed [20] conjectured that the domination number of a connected, cubic graph of order n is $\lceil n/3 \rceil$. Kostochka and Stodolsky [14] disproved this conjecture by constructing an infinite sequence $\{G_k\}_{k=1}^{\infty}$ of connected, cubic graphs with

$$\lim_{k \rightarrow \infty} \frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{1}{3} + \frac{1}{69}.$$

Subsequently, Kelmans [11] constructed an infinite series of 2-connected, cubic graphs H_k with

$$\lim_{k \rightarrow \infty} \frac{\gamma(H_k)}{|V(H_k)|} \geq \frac{1}{3} + \frac{1}{60}.$$

Thus, there exist connected cubic graphs G of arbitrarily large order n satisfying $\gamma(G) \geq \left(\frac{1}{3} + \frac{1}{60}\right)n$. All known counterexamples to Reed's conjecture, including the above constructions of Kostochka and Stodolsky [14] and Kelmans [11], contain small cycles. Much discussion has centered on when Reed's conjecture becomes true with the additional condition that the girth, g , of the graph is sufficiently large, where the *girth* is the length of a shortest cycle in G . The first such result was presented by Kawarabayashi, Plummer, and Saito [9].

Theorem 2 ([9]) *If G is a connected cubic graph of order n and girth $g \geq 3$ has a 2-factor, then*

$$\gamma(G) \leq \left(\frac{1}{3} + \frac{1}{9\lfloor g/3 \rfloor + 3}\right)n.$$

Refining the ideas and techniques from Reed's seminal paper [20], and using intricate discharging arguments, Kostochka and Stodolsky [15] improved the upper bound in Theorem 2.

Theorem 3 ([15]) *If G is a connected cubic graph of order n and girth $g \geq 3$, then*

$$\gamma(G) \leq \left(\frac{1}{3} + \frac{8}{3g^2}\right)n.$$

Rautenbach and Reed [19] and Král, Škoda, and Volec [16] established further upper bounds on the domination number of a cubic graph in terms of its order and girth. The magic threshold of $\frac{1}{3}n$ for the domination number was first shown to hold for cubic graphs with large girth by Löwenstein and Rautenbach [17].

Theorem 4 ([17]) *If G is a cubic graph of order n and girth $g \geq 83$, then $\gamma(G) \leq \frac{1}{3}n$.*

Theorem 4 presented a breakthrough back in 2008. The Löwenstein-Rautenbach girth condition of $g \geq 83$ in Theorem 4 guaranteeing that the domination number of a cubic graph of order n is at most the magical threshold of $\frac{1}{3}n$ has yet to be improved, and has attracted considerable attention in the literature. Verstraete [21] conjectured in 2010 that the girth condition can be lowered significantly from $g \geq 83$ to $g \geq 6$ in order to guarantee that the $\frac{1}{3}$ -bound will hold.

Conjecture 5 (Verstraete) *If G is a cubic graph on n vertices with girth $g \geq 6$, then $\gamma(G) \leq \frac{1}{3}n$.*

We remark that the girth requirement in Verstraete's Conjecture 5 is essential, since the generalized Petersen graph $P(7, 2)$, shown in Figure 1, of order $n = 14$ satisfies girth $g = 5$ and $\gamma(G) = 5 > \frac{1}{3}n$. We also remark that Kostochka and Stodolsky [14] and Kelmans [11] constructed an infinite family of connected, cubic graphs of order n with girth 4 and $\gamma(G) > \frac{1}{3}n$.

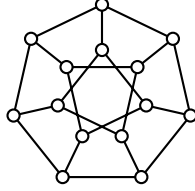


Figure 1: The generalized Petersen graph $P(7, 2)$.

Another closely related $\frac{1}{3}$ -conjecture for domination in cubic graphs can be attributed to Kostochka [12] who announced the following question in the open problem session at the Third International Conference on Combinatorics, Graph Theory and Applications, held at Elgersburg, Germany, March 2009: *Is it true that the domination number of a bipartite cubic graph is at most one-third its order?* Kostochka and Stodolsky comment in their paper in [15] that it would be interesting to answer this question. This intriguing question of Kostochka was posed seven years later as a formal conjecture in [7].

Conjecture 6 ([7]) *If G is a cubic bipartite graph of order n , then $\gamma(G) \leq \frac{1}{3}n$.*

2 Main results

Our aim in this paper is to make significant progress on both the Verstraete Conjecture 5 and the Kostochka's inspired Conjecture 6 for domination in cubic graphs. Our main result proves Verstraete's $\frac{1}{3}$ -Conjecture when the graph does not contain a 7-cycle or 8-cycle as a subgraph, that is, when the graph has girth at least 6 but is heptagon-free and octagon-free.

Theorem 7 *If G is a cubic graph of order n and girth $g \geq 6$ that does not contain a 7-cycle or 8-cycle, then $\gamma(G) \leq \frac{1}{3}n$.*

As an immediate consequence of Theorem 7, the $\frac{1}{3}$ -Conjecture inspired by Kostochka (and posed formally as a conjecture in [7]) holds when the cubic bipartite graph contains no 4-cycle and no 8-cycle.

Theorem 8 *If G is a cubic bipartite graph of order n that does not contain a 4-cycle or 8-cycle, then $\gamma(G) \leq \frac{1}{3}n$.*

As an immediate consequence of Theorem 7 we also have the following new lower bound on the girth guaranteeing that the domination number of a cubic graph is at most one-thirds its order. This result significantly lowers the previously best known girth condition, namely $g \geq 83$, due to Löwenstein-Rautenbach.

Corollary 9 *If G is a cubic graph of order n and girth $g \geq 9$, then $\gamma(G) \leq \frac{1}{3}n$.*

3 Notation

For notation and graph theory terminology we generally follow [5]. Specifically, let G be a graph with vertex set $V(G)$ of order $n(G) = |V(G)|$ and edge set $E(G)$ of size $m(G) = |E(G)|$. If G is clear from the context, we simply write V and E rather than $V(G)$ and $E(G)$. Let v be a vertex in V . We denote the *degree* of v in G by $\deg_G(v)$. The *open neighborhood* of v is $N_G(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we simply write $d(v)$, $N(v)$, $N[v]$, $N(S)$ and $N[S]$ rather than $\deg_G(v)$, $N_G(v)$, $N_G[v]$, $N_G(S)$ and $N_G[S]$, respectively.

A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n . The *girth* of G , denoted $g(G)$, is the length of a shortest cycle in G . For a set $S \subseteq V$, the subgraph of G induced by S is denoted by $G[S]$. Further if $S \neq V$, then we denote the graph obtained from G by deleting all vertices in S (as well as all incident edges) by $G - S = G[V \setminus S]$. If $S = \{v\}$, we also write $G - S$ simply as $G - v$. A graph G is said to be *subcubic* if its maximum degree is at most 3, and *cubic* if every vertex has degree 3. A vertex of degree 0 is called an *isolated vertex*. A *double star* is a tree with exactly two vertices that are not leaves. We call these two vertices the *central vertices* of the double star. Further, if the one central vertex has s leaf neighbors and the other t leaf neighbors, we denote the double star by $S(s, t)$.

4 Statement of the main theorem

In order to prove our main result, namely Theorem 7, we need to prove a much stronger result. For this purpose, we introduce the concept of a marked domination set in a graph. Let G be a graph where every vertex in G is either marked or unmarked. A *marked dominating set*, abbreviated MD-set, is a set S of vertices in G such that every vertex that is not marked belongs to the set S or is adjacent to a vertex in the set S . The *marked domination number* of G , denoted by $\dot{\gamma}(G)$, is the minimum cardinality of a MD-set in G . Thus, the marked domination number of G is the minimum number of vertices needed to dominate all the unmarked vertices, where a vertex dominates itself and all its neighbors.

Let $\dot{V}(G)$ denote the set of marked vertices in G . We note that $\dot{V}(G) \subseteq V(G)$. We let $n_i(G)$ denote the number of vertices of degree i in G that are unmarked. Further, we let $\dot{n}(G)$ denote the number of marked vertices in G , and so $\dot{n}(G) = |\dot{V}(G)|$. Thus if G is a subcubic graph, then

$$n(G) = \dot{n}(G) + \sum_{i=0}^3 n_i(G).$$

If the graph G is clear from context, we simply write V , \dot{V} , n , \dot{n} , and n_i rather than $V(G)$, $\dot{V}(G)$, $n(G)$, $\dot{n}(G)$ and $n_i(G)$, respectively. A *marked subcubic graph* is a subcubic graph in which every vertex is either marked or unmarked. We are now in a position to state our key result.

Theorem 10 *If G is a marked subcubic graph of girth at least 6 with no cycle of length 7 or 8, then*

$$12\dot{\gamma}(G) \leq 4\dot{n}(G) + 4n_3(G) + 5n_2(G) + 8n_1(G) + 12n_0(G).$$

We remark that if the set of marked vertices in a graph G is empty, then the marked domination number of G is precisely its domination number; that is, if $\dot{V} = \emptyset$, then $\dot{\gamma}(G) = \gamma(G)$. Theorem 7 follows immediately from Theorem 10 in the special case when G is a cubic graph with no marked vertices. For a vertex v in a marked subcubic graph G , we define its *weight* in G as

$$\dot{w}_G(v) = \begin{cases} 12 & \text{if } \deg_G(v) = 0 \text{ and } v \notin \dot{V}(G), \\ 8 & \text{if } \deg_G(v) = 1 \text{ and } v \notin \dot{V}(G), \\ 5 & \text{if } \deg_G(v) = 2 \text{ and } v \notin \dot{V}(G), \\ 4 & \text{if } \deg_G(v) = 3 \text{ or if } v \in \dot{V}(G). \end{cases}$$

For a subset $X \subseteq V(G)$, we define the *weight* of X in G , as

$$\dot{w}_G(X) = \sum_{v \in X} \dot{w}_G(v),$$

that is, the weight of X is the sum of the weights in G of vertices in X . We refer to the weight of $V(G)$ (the entire vertex set) in G simply as the *weight of G* , denoted $\dot{w}(G)$. Thus,

$$\dot{w}(G) = \sum_{v \in V(G)} \dot{w}_G(v) = 4\dot{n}(G) + 4n_3(G) + 5n_2(G) + 8n_1(G) + 12n_0(G).$$

We note that each marked vertex in G has weight 4, independent of its degree in G . Further, the weight of a marked vertex of degree 3 is the same as the weight of an unmarked vertex of degree 3. We now restate Theorem 10 in terms of the weight of the graph.

Theorem 10 *If G is a marked subcubic graph of girth at least 6 with no cycle of length 7 or 8, then*

$$12\dot{\gamma}(G) \leq \dot{w}(G).$$

5 Proof of the main theorem

The proof is based on a series of claims, which we distribute into thematic subsections to enhance readability of the proof. Suppose, to the contrary, that the theorem is false. Among all counterexample to Theorem 10, let G be chosen so that $|V(G)| + |E(G)|$ is a minimum and, subject to this condition, the number of marked vertices of degree 3 is a minimum. We will frequently use the following fact.

Fact 1 *If H is a proper subgraph of G , and α and β are two integers such that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, then $12\alpha > \beta$.*

Proof. Suppose, to the contrary, that $12\alpha \leq \beta$. Since G is a minimum counterexample and H is a proper subgraph of G , we note that $12\dot{\gamma}(H) \leq \dot{w}(H)$. Thus, $12\dot{\gamma}(G) \leq 12(\dot{\gamma}(H) + \alpha) = 12\dot{\gamma}(H) + 12\alpha \leq \dot{w}(H) + \beta \leq \dot{w}(G)$, a contradiction. \square

In what follows we present a series of claims which culminate in the implication of the non-existence of the counterexample graph G .

5.1 Fundamental structural properties

In this subsection, we establish some fundamental structural properties of the graph G . We define a *2-path* in G as a path all of whose vertices have degree 2 in G . The claims we prove in this section show that the graph G is a connected graph in which every vertex has degree 2 or 3. Further, we show that every marked

vertex has degree 3, and that the marked vertices are at distance at least 4 apart in G . We also establish restrictions on the length of a 2-path in G , and show that every maximal 2-path in G has length 1, 2, 4 or 5. We begin with the following property of marked vertices in the graph G .

Claim 1 *Every marked vertex in G has degree 1 or 2.*

Proof. Suppose, to the contrary, that G contains a marked vertex v of degree 0 or 3. Suppose, firstly that v is isolated in G . In this case, let $H = G - v$. Every MD-set of H is a MD-set of G , and conversely, implying that $\dot{\gamma}(G) = \dot{\gamma}(H)$. Since $\dot{w}_G(v) = 4$, we have $\dot{w}(G) = \dot{w}(H) + 4$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 0$ and $\beta = 4$, contradicting Fact 1. Suppose, next, that the vertex v has degree 3 in G . In this case, let H be the graph obtained from G by unmarking the vertex v . Since $|V(H)| + |E(H)| = |V(G)| + |E(G)|$ and H has fewer marked vertices of degree 3 than does G , our choice of G implies that the graph H is not a counterexample to our theorem. Therefore, $12\dot{\gamma}(H) \leq \dot{w}(H)$. Every MD-set of H is a MD-set of G , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H)$. Since $\dot{w}_G(v) = \dot{w}_H(v) = 4$, we note that $\dot{w}(G) = \dot{w}(H)$. Thus, $12\dot{\gamma}(G) \leq 12\dot{\gamma}(H) \leq \dot{w}(H) = \dot{w}(G)$, contradicting the fact that G is a counterexample. \square

Claim 2 *The graph G is connected.*

Proof. Suppose, to the contrary, that G is disconnected with components G_1, G_2, \dots, G_k where $k \geq 2$. By linearity,

$$\dot{\gamma}(G) = \sum_{i=1}^k \dot{\gamma}(G_i) \quad \text{and} \quad \dot{w}(G) = \sum_{i=1}^k \dot{w}(G_i).$$

By the minimality of G , all components of G satisfy Theorem 10, and so $12\dot{\gamma}(G_i) \leq \dot{w}(G_i)$ for $i \in [k]$, implying that $\dot{\gamma}(G) \leq \dot{w}(G)$, and so Theorem 10 therefore holds for G , a contradiction. \square

Claim 3 *The graph G contains at least one unmarked vertex.*

Proof. Suppose, to the contrary, that G contains only marked vertices. Then, $\dot{\gamma}(G) = 0$ and $\dot{w}(G) = 4n$, implying that Theorem 10 therefore holds for G , a contradiction. \square

Claim 4 *The graph G contains no unmarked vertex of degree 1.*

Proof. Suppose, to the contrary, that G contains an unmarked vertex, u say, of degree 1. Let v be the neighbor of u . If $n = 2$, then $\dot{\gamma}(G) = \dot{\gamma}(G) = 1$ and $\dot{w}(G) \geq 12$ noting that the vertex v may possibly be marked, contradicting the fact that G is a counterexample to the theorem. Hence, $\deg(v) \geq 2$. Let $H = G - \{u, v\}$, where we mark all neighbors of v in H . Every MD-set of H can be extended to a MD-set of G by adding to it the vertex v , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + 1$. Since $\dot{w}_G(u) = 8$ and $\dot{w}_G(v) \geq 4$ and $\dot{w}_G(w) \geq \dot{w}_H(w)$ for all vertices $w \in V(H)$, we have $\dot{w}(G) \geq \dot{w}(H) + 12$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 1$ and $\beta = 12$, contradicting Fact 1. \square

Claim 5 *The graph G does not contain adjacent marked vertices.*

Proof. Suppose, to the contrary, that G contains two adjacent marked vertices. Let H be obtained from G by removing such an edge between marked vertices. Then, $\dot{\gamma}(H) = \dot{\gamma}(G)$ and $\dot{w}(H) = \dot{w}(G)$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = \beta = 0$, contradicting Fact 1. \square

Claim 6 *Every marked vertex in G has degree 2.*

Proof. By Claim 1, every marked vertex in G has degree 1 or 2. Suppose, to the contrary, that G contains a marked vertex u of degree 1 in G adjacent to a vertex v . By Claims 4 and 5, the vertex v is an unmarked vertex of degree 2 or 3. Let $H = G - u$. Every MD-set of H is a MD-set of G , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H)$. Since $\dot{w}_G(u) = 4$ and $\dot{w}_G(v) \geq \dot{w}_H(v) - 3$, we note that $\dot{w}(G) \geq \dot{w}(H) + 4 - 3 = \dot{w}(H) + 1$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 0$ and $\beta = 1$, contradicting Fact 1. \square

Claim 7 *Every vertex in G has degree 2 or 3.*

Proof. This follows immediately from Claims 4 and Claim 6. \square

Claim 8 *The graph G does not contain a marked vertex adjacent to a vertex of degree 3.*

Proof. Suppose, to the contrary, that G contains a marked vertex u adjacent to a vertex v of degree 3. By Claim 6, the vertex u has degree 2. Let w be the second neighbor of u . By Claims 5 and 7, the vertex w is an unmarked vertex of degree 2 or 3. Let $H = G - u$. We note that $\dot{w}_G(u) = 4$, $\dot{w}_G(v) - \dot{w}_H(v) = 4 - 5 = -1$, and $\dot{w}_G(w) - \dot{w}_H(w) \geq -3$, while the weights of all other vertices remain unchanged in G and H . Thus, $\dot{w}(G) \geq \dot{w}(H) + 4 - 1 - 3 = \dot{w}(H)$. Further, every MD-set of H is a MD-set of G , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H)$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = \beta = 0$, contradicting Fact 1. \square

Claim 9 *There is no 2-path of length 3 in G that starts at a marked vertex with the remaining vertices of the path unmarked.*

Proof. Suppose, to the contrary, that G contains a 2-path $P: u_1u_2u_3u_4$ that emanates from a marked vertex u_1 , where u_2, u_3 , and u_4 are unmarked vertices of degree 2 in G . By Claim 6, the marked vertex u_1 has degree 2. Let u_0 and u_5 be the neighbors of u_1 and u_5 , respectively, not on P . The girth condition $g \geq 6$ implies that $u_0 \neq u_5$. By Claim 7, the vertices u_0 and u_5 are of degree at least 2. We now consider the graph $H = G - V(P)$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertex v_3 , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + 1$. Since $\dot{w}_G(u_1) = 4$ and $\dot{w}_G(u_i) = 5$ for $i \in \{2, 3, 4\}$, while $\dot{w}_G(u_i) - \dot{w}_H(u_i) \geq -3$ for $i \in \{0, 5\}$, we note that $\dot{w}(G) \geq \dot{w}(H) + 4 + 3 \times 5 - 2 \times 3 = \dot{w}(H) + 13$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 1$ and $\beta = 13$, contradicting Fact 1. \square

Claim 10 *The marked vertices in G are at distance at least 4 apart from each other.*

Proof. Suppose, to the contrary, that G contains two marked vertices, u and v , at distance less than 4 apart. By Claim 5, $d(u, v) \geq 2$.

Claim 10.1 $d(u, v) = 3$.

Proof. Suppose that $d(u, v) = 2$. By Claims 5, 6, 7, and 8, and the girth $g \geq 6$ condition, u and v are internal vertices of a 2-path $P: u_1u_2u_3u_4u_5$ of order 5, where $u_2 = u$ and $u_4 = v$. Recall that by definition of a 2-path, every vertex of P has degree 2 in G . Let u_0 and u_6 be the neighbors of u_1 and u_5 , respectively, not on P .

Suppose that the vertex u_0 is either marked or has degree 3. In this case, let $H = G - \{u_1, u_2, u_3\}$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertex u_2 , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + 1$.

We note that the vertex u_0 is marked or has degree 2 in H . Therefore, the weight of u_0 in H either remains unchanged from its weight in G or increases by 1, implying that $\dot{w}(G) \geq \dot{w}(H) + 4 + 2 \times 5 - 1 = \dot{w}(H) + 13$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 1$ and $\beta = 13$, contradicting Fact 1. Hence, u_0 is unmarked and has degree 2 in G . Analogously, u_6 is unmarked and has degree 2 in G .

If $u_0 = u_6$, then $G \cong C_6$, $\dot{\gamma}(G) = 2$, and $\dot{w}(G) = 4 \times 5 + 2 \times 4 > 24 = 12\dot{\gamma}(G)$, contradicting the fact that G is a counterexample to our theorem. Hence, $u_0 \neq u_6$. Since there is no cycle of length 7 in G , we note that u_0 and u_6 are not adjacent. Let u_7 be the neighbor of u_6 different from u_5 . By Claim 9, the vertex u_7 is either a marked vertex (of degree 2) or has degree 3. In this case, let $H = G - \{u_1, \dots, u_6\}$ where we mark u_7 if it is an unmarked vertex in G . We note that the weight of u_7 in H remains unchanged from its weight in G , while $\dot{w}_G(u_0) - \dot{w}_H(u_0) = -3$, implying that $\dot{w}(G) \geq \dot{w}(H) + 4 \times 5 + 2 \times 4 - 3 = \dot{w}(H) + 25$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u_2 and u_6 , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + 2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 25$, contradicting Fact 1. This completes the proof of Claim 10.1. \square

By Claim 10.1, the marked vertices in G are at distance at least 3 apart. In particular, $d(u, v) = 3$. By Claims 6, 7 and 8, and by the girth $g \geq 6$ condition, u and v are internal vertices of a 2-path $P: u_1u_2u_3u_4u_5u_6$ of order 6, where $u_2 = u$ and $u_5 = v$. If u_1 and u_6 are adjacent, then $G \cong C_6$, and $12\dot{\gamma}(G) < \dot{w}(G)$, a contradiction. Hence, u_1 and u_6 are not adjacent. Let u_0 and u_7 be the neighbors of u_1 and u_6 , respectively, not on P . As observed earlier, $u_1 \neq u_7$ and $u_0 \neq u_6$. Further, u_0 and u_7 are unmarked. Since there is no cycle of length 7 in G , we note that $u_0 \neq u_7$.

Suppose that the vertex u_7 has degree 2. Let u_8 be the neighbor of u_7 different from u_6 . By Claim 9, the vertex u_8 is either a marked vertex (of degree 2) or has degree 3. In particular, we note that $u_1 \neq u_8$. We now let $H = G - \{u_2, \dots, u_7\}$, where we mark u_8 if it is an unmarked vertex in G . We note that the weight of u_8 in H remains unchanged from its weight in G , while $\dot{w}_G(u_0) - \dot{w}_H(u_0) = -3$, implying that $\dot{w}(G) \geq \dot{w}(H) + 4 \times 5 + 2 \times 4 - 3 = \dot{w}(H) + 25$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u_3 and u_7 , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + 2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 25$, contradicting Fact 1.

Hence, the vertex u_7 has degree 3. Analogously, the vertex u_0 has degree 3. We now let $H = G - V(P)$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u_2 and u_5 , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + 2$. Since $\dot{w}_G(u_0) - \dot{w}_H(u_0) = \dot{w}_G(u_7) - \dot{w}_H(u_7) = -1$, we note that $\dot{w}(G) \geq \dot{w}(H) + 4 \times 5 + 2 \times 4 - 2 \times 1 = \dot{w}(H) + 26$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 26$, contradicting Fact 1. This completes the proof of Claim 10. \square

By Claims 9 and 10, a 2-path in G contains at most one marked vertex.

Claim 11 *The graph G is not a cycle.*

Proof. Suppose, to the contrary, that $G \cong C_n$. By the girth condition and the assumption that there is no 7-cycle in G , we note that $n \geq 6$ and $n \neq 7$. Further by Claims 9 and 10, there is no marked vertex. Thus, $\dot{w}(G) = 5n$ and $\dot{\gamma}(G) \leq \gamma(C_n) = \lceil \frac{n}{3} \rceil$. If $n \equiv 0 \pmod{3}$, then $12\dot{\gamma}(G) \leq 12 \times \lceil \frac{n}{3} \rceil = 4n < 5n \leq \dot{w}(G)$. If $n \equiv 2 \pmod{3}$, then $12\dot{\gamma}(G) \leq 12 \times \lceil \frac{n}{3} \rceil = 4n + 4 \leq 5n \leq \dot{w}(G)$. If $n \equiv 1 \pmod{3}$, then since $n \geq 6$ and $n \neq 7$, we note that $n \geq 10$, and therefore that $12\dot{\gamma}(G) \leq 12 \times \lceil \frac{n}{3} \rceil = 4n + 8 \leq 5n \leq \dot{w}(G)$. In all three cases, $12\dot{\gamma}(G) \leq \dot{w}(G)$, contradicting the fact that G is a counterexample. \square

Claim 12 *Every 2-path in G has order at most 5.*

Proof. Suppose, to the contrary, that G contains a 2-path $P: u_1u_2u_3u_4u_5u_6$ of order 6. By Claim 7, every vertex in G has degree 2 or 3. By Claim 11, G is not a cycle, and so at least one vertex in G

has degree 3. The 2-path P can therefore be chosen so that the neighbor of u_1 , say u_0 , not on P has degree 3. Let $H = G - V(P)$. Let u_7 be the neighbor of u_6 not on P . We note that $u_0 \neq u_7$, and that $\dot{w}_G(u_0) - \dot{w}_H(u_0) = -1$ and $\dot{w}_G(u_7) - \dot{w}_H(u_7) \geq -3$. Thus, since every 2-path in G contains at most one marked vertex, $\dot{w}(G) \geq \dot{w}(H) + 5 \times 5 + 1 \times 4 - 1 - 3 \geq \dot{w}(H) + 25$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u_2 and u_5 , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + 2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 25$, contradicting Fact 1. \square

Claim 13 *The graph G does not contain a maximal 2-path of order 3.*

Proof. Suppose, to the contrary, that G contains a maximal 2-path $P: u_1u_2u_3$ of order 3. Let u_0 and u_4 be the neighbor of u_1 and u_3 , respectively, not on P . By the maximality of the 2-path P , both u_0 and u_4 are vertices of degree 3. Let $H = G - V(P)$. Since every 2-path in G contains at most one marked vertex, $\dot{w}(G) \geq \dot{w}(H) + 2 \times 5 + 1 \times 4 - 2 \times 1 \geq \dot{w}(H) + 12$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertex u_2 , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + 1$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 1$ and $\beta = 12$, contradicting Fact 1. \square

5.2 The associated colored multigraph

In this subsection, we define a colored cubic multigraph associated with the graph G . For this purpose, we introduce the following notation. By Claim 12 and 13, every maximal 2-path in G has order 1, 2, 4 or 5. We call a maximal 2-path of order 2 or 5 a *red path*, and we call a maximal 2-path of order 1 or 4 a *green path*. We call a path a *colored path* if it is a *red path* or a *green path*. We call the *ends* of a colored path the vertices of degree 3 adjacent to the extremities of the corresponding 2-path.

Let u and v be the ends of a colored path $x_1 \dots x_k$, where u and v are adjacent to x_1 and x_k , respectively, and so $ux_1 \dots x_kv$ is a path in G which we denote by $[uv]$. Further, we denote by (uv) the subpath $x_1 \dots x_k$ of the path $[uv]$ consisting of its internal vertices, and we denote by $(uv)_i$ the vertex x_i for $i \in [k]$. We note that (uv) is a maximal 2-path in G , and is either a *red path* or *green path*. We call a colored path **short** if it is a *red path* of order 2 or a *green path* of order 1 and **long** if it is a *red path* of order 5 or a *green path* of order 4. By Claim 8 and 10, we note that a short colored path contains no marked vertex, and a long colored path contains at most one marked vertex.

We now 3-color the edges of G as follows. We color every edge of G joining two vertices of degree 3 with the color **black**. Every edge of G that belongs to a path $[uv]$, where the subpath (uv) is a *red path* or a *green path*, we color **red** or **green**, respectively. Thus, if u and v are vertices of degree 3 in G , and u and v are the ends of a *red path* (respectively, *green path*), say $x_1 \dots x_k$, then we color every edge on the path $[uv]: ux_1 \dots x_kv$ with the color **red** (respectively, **green**). In this way, every edge of G is colored with one of the colors **black**, **red** or **green**.

We now assimilate G with the corresponding **cubic colored multigraph** M_G , where $V(M_G)$ consists of the vertices of degree 3 in G and where the edges of M_G correspond to the **black**, **green** and **red** edges defined in G . Thus, a **black edge** uv in M_G corresponds to a **black edge** uv in G , while a **green** (respectively, **red**) edge uv in M_G corresponds to a path $[uv]$ in G associated with a **green** (respectively, **red**) path (uv) in G . We show next that the multigraph M_G is in fact a graph.

Claim 14 *The colored multigraph M_G does not contain loops or multiple edges.*

Proof. Suppose, to the contrary, that M_G contains loops or multiple edges. We are six possible cases. We consider each case in turn.

Case 1. M_G contains a loop. By the girth condition, a loop in M_G corresponds to a long **red** edge, say $x_1 \dots x_5$, both of whose ends are adjacent to a common vertex, u say, in G . Let $H = G - \{u, x_1, \dots, x_5\}$, where we mark the neighbor, v say, of u in G different from x_1 and x_5 . Since every long colored path contains at most one marked vertex, this implies that $\dot{w}(G) \geq \dot{w}(H) + 4 \times 5 + 2 \times 4 = \dot{w}(H) + 28$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u and u_3 , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + 2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 28$, contradicting Fact 1.

*Case 2. M_G contains a parallel **black** and **red** edge.* Let u and v be two (distinct) vertices in M_G that are joined by a **black** edge and a **red** edge in M_G . By the girth condition, such a **red** edge in M_G corresponds to a long **red** path, say $x_1 \dots x_5$, in G . However, $ux_1 \dots x_5vu$ is a cycle of length 7 in G , a contradiction.

*Case 3. M_G contains a parallel **black** and **green** edge.* Let u and v be two (distinct) vertices in M_G that are joined by a **black** edge and a **green** edge in M_G . By the girth condition, such a **green** edge in M_G corresponds to a long **green** path in G (with u and v as its ends). Let $P: x_1x_2x_3x_4$ denote this long **green** path in G , where u is adjacent to x_1 . Let u' be the neighbor of u different from x_1 and v , and let v' be the neighbor of v different from x_4 and u . Let $H = G - (V(P) \cup \{u, v\})$, where we mark the vertex v' in H . We note that $\dot{w}_G(u') - \dot{w}_H(u') \geq -3$. Since P contains at most one marked vertex, $\dot{w}(G) \geq \dot{w}(H) + 3 \times 5 + 3 \times 4 - 3 = \dot{w}(H) + 24$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices v and x_2 , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + 2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$, contradicting Fact 1.

*Case 4. M_G contains two **green** parallel edges.* Let u and v be two (distinct) vertices in M_G that are joined by two **green** parallel edges in M_G . By the girth condition and the non-existence of cycles of length 7, both **green** edges in M_G corresponds to a long **green** paths in G (with u and v as their ends). Let $P: x_1x_2x_3x_4$ and $Q: y_1y_2y_3y_4$ be these two long **green** paths in G , where u is adjacent to x_1 and y_1 in G . Let $H = G - (V(P) \cup V(Q) \cup \{u\})$, where we mark the neighbor, x say, of u different from x_1 and y_1 (possibly, $x = v$). We note that $\dot{w}_G(v) - \dot{w}_H(v) \geq -4$. Since every long colored path contains at most one marked vertex, this implies that $\dot{w}(G) \geq \dot{w}(H) + 6 \times 5 + 3 \times 4 - 4 = \dot{w}(H) + 38$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u , x_3 and y_3 , implying that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + 3$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 3$ and $\beta = 38$, contradicting Fact 1.

*Case 5. M_G contains two **red** parallel edges.* Let u and v be two (distinct) vertices in M_G that are joined by two **red** parallel edges in M_G . Let P and Q denote the paths in G associated with these two **red** parallel edges in M_G . By the girth condition and since G has no cycle of length 7, we note that u and v are not adjacent in G . Let ℓ be the number of long **red** paths among P and Q . By the girth condition, $\ell \geq 1$. Let $H = G - (V(P) \cup V(Q) \cup \{u, v\})$, where we mark the neighbors of u and v that do not belong to P or Q . Since each of P and Q contain at most one marked vertex, $\dot{w}(G) \geq \dot{w}(H) + (2\ell + 4) \times 5 + (\ell + 2) \times 4 = \dot{w}(H) + 14\ell + 28$. Every MD-set of H can be extended to a MD-set of G by adding to it $\ell + 2$ vertices, namely the vertices u and v , and the ℓ vertices at distance 3 from both u and v on the paths P and Q , if such. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = \ell + 2$ and $\beta = 14\ell + 28$, contradicting Fact 1.

*Case 6. M_G contains a parallel **red** and **green** edge.* Let u and v be two (distinct) vertices in M_G that are joined by a parallel **red** and **green** edge in M_G . Let P and Q denote the paths in G associated with these two edges in M_G . Let ℓ be the number of long paths among P and Q . By the girth condition and since G has no cycle of length 7, we note that u and v are not adjacent in G and $\ell \geq 1$. Let $H = G - (V(P) \cup V(Q) \cup \{u, v\})$, where we mark the neighbors of u and v that do not belong to P or Q . Since each of P and Q contain at most one marked vertex, $\dot{w}(G) \geq \dot{w}(H) + (2\ell + 3) \times 5 + (\ell + 2) \times 4 = \dot{w}(H) + 14\ell + 23 \geq \dot{w}(H) + 12\ell + 25$. Every MD-set of H can be extended to a MD-set of G by adding to it $\ell + 2$ vertices, namely the vertices u and v , and the vertex $(uv)_3$ on the path P if it is long and the vertex $(uv)_3$ on the path Q if it is long. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = \ell + 2$ and $\beta = 12\ell + 25$, contradicting Fact 1.

Since all six cases produce a contradiction, this completes the proof of Claim 14. \square

By Claim 14, the colored multigraph M_G contain no loops or multiple edges, and therefore is a colored cubic graph.

5.3 The green edges form a Matching, as do the red edges

In this section, we show that the set of **green edges** form a matching in M_G , and the set of **red edges** form a matching in M_G .

Claim 15 *The colored multigraph M_G does not contain two adjacent green edges, and so the green edges form a matching in M_G .*

Proof. Suppose, to the contrary, that M_G contains a vertex u incident with two **green edges** say uv and uw . Let H be the graph obtained from G by removing u and all vertices on the **green** colored paths (uv) and (uw) in G . Let ℓ be the number of long paths among the two **green** colored paths (uv) and (uw) in G . Since each long path contains at most one marked vertex, we note that $\dot{w}(G) \geq \dot{w}(H) + (2 + 2\ell) \times 5 + (\ell + 1) \times 4 - 2 \geq \dot{w}(H) + 14\ell + 12$. Every MD-set of H can be extended to a MD-set of G by adding to it $\ell + 1$ vertices, namely the vertex u and the vertices $(uv)_3$ and $(uw)_3$ if they exist, where we mark the third neighbor of u (that does not belong to the deleted 2-paths). Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = \ell + 1$ and $\beta = 14\ell + 12$, contradicting Fact 1. \square

Before proving that the set of **red edges** form a matching in M_G , we first show that there is no **green-red-green** path in M_G and that there is no vertex that is incident with two **black** edges and is the end of a long **red path**.

Claim 16 *The colored multigraph M_G does not contain green-red-green paths.*

Proof. Suppose, to the contrary, that M_G contains a **green-red-green** path $u_1u_2u_3u_4$, where u_1u_2 and u_3u_4 are **green edges** in M_G and u_2u_3 is a **red edge** in M_G . Thus, (u_1u_2) and (u_3u_4) are **green paths** in G , while (u_2u_3) is a **red path** in G . By Claims 14 and 15, the vertices u_1, u_2, u_3, u_4 are all distinct. Let ℓ be the number of long paths among (u_1u_2) , (u_2u_3) , and (u_3u_4) . Let H be obtained from G by removing the vertices u_2 and u_3 as well as all vertices on the (u_1u_2) , (u_2u_3) and (u_3u_4) 2-paths, where we mark the third neighbors of u_2 and u_3 (that do not belong to the deleted 2-paths). We note that $\dot{w}(G) \geq \dot{w}(H) + (2\ell + 4) \times 5 + (\ell + 2) \times 4 - 2 = \dot{w}(H) + 14\ell + 26$. Every MD-set of H can be extended to a MD-set of G by adding to it $\ell + 2$ vertices, namely the vertices u_2 and u_3 , and the vertices $(u_2u_1)_3$, $(u_2u_3)_3$, and $(u_3u_4)_3$ if they exist. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = \ell + 2$ and $\beta = 14\ell + 26$, contradicting Fact 1. \square

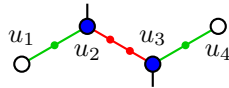


Figure 2: A colored multigraph in the proof of Lemma 16

Claim 17 *The graph G does not contain a vertex that is incident with two **black** edges and is the end of a long **red path**.*

Proof. Suppose, to the contrary, that G contains a vertex u that is incident with two **black** edges, say uv_1 and uv_2 , and is the end of a long **red path** say (uw) . By Claim 14, the vertices v, v_1 and v_2 are distinct. Let H

be obtained from G by removing the vertex u as well as all five vertices on the (uv) 2-path. We note that v, v_1 and v_2 all have degree 3 in G and degree 2 in H , implying that $\dot{w}(G) \geq \dot{w}(H) + 4 \times 5 + 2 \times 4 - 3 = \dot{w}(H) + 25$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices $(uv)_1$ and $(uv)_4$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 25$, contradicting Fact 1. \square

We are now in a position to show that the **red edges** form a matching in M_G .

Claim 18 *The colored multigraph M_G does not contain two adjacent **red edges**, and so the **red edges** form a matching in M_G .*

Proof. We proceed by a series of subclaims.

Claim 18.1 *The colored multigraph M_G does not contain **green-red-red** paths.*

Proof. Suppose, to the contrary, that M_G contains a **green-red-red** path $u_1u_2u_3u_4$, where u_1u_2 is a **green edge** in M_G and u_2u_3 and u_3u_4 are **red edges** in M_G . Thus, (u_1u_2) is a **green path** in G , and (u_2u_3) and (u_3u_4) are two **red paths** in G . By Claim 14 the vertices u_1, u_2, u_3 are all distinct. Let ℓ be the number of long paths among (u_1u_2) and (u_2u_3) . Let H be obtained from G by removing the vertices $u_2, u_3, (u_3u_4)_1$, as well as all vertices on the (u_1u_2) and (u_2u_3) 2-paths, where we mark the third neighbors of u_2 and u_3 (that do not belong to the deleted 2-paths). We note that $\dot{w}(G) \geq \dot{w}(H) + (2\ell + 4) \times 5 + (\ell + 2) \times 4 - 1 - 3 = \dot{w}(H) + 14\ell + 24$. Every MD-set of H can be extended to a MD-set of G by adding to it $\ell + 2$ vertices, namely the vertices u_2 and u_3 , and the vertices $(u_2u_1)_3$ and $(u_2u_3)_3$, if they exist. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = \ell + 2$ and $\beta = 14\ell + 24$, contradicting Fact 1. \square

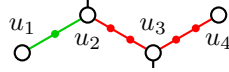


Figure 3: A colored multigraph in the proof of Lemma 18.1

Claim 18.2 *The colored multigraph M_G does not contain **red cycles**.*

Proof. Suppose, to the contrary, that M_G contains a **red cycle** $C: u_1u_2 \dots u_ku_1$. Thus, the path (u_1u_k) and the paths (u_iu_{i+1}) for $i \in [k - 1]$, are all **red paths** in G . We note that every vertex of C has degree 3 in G . Let ℓ be the number of long paths among (u_1u_k) and (u_iu_{i+1}) for $i \in [k - 1]$. Let H be obtained from G by removing the vertices in $V(C)$, as well as all vertices on the (u_1u_k) and (u_iu_{i+1}) 2-paths for $i \in [k - 1]$, where we mark the third neighbors of each vertex u_i for $i \in [k]$ (that do not belong to the deleted 2-paths). We note that $\dot{w}(G) \geq \dot{w}(H) + (2k + 2\ell) \times 5 + (k + \ell) \times 4 = \dot{w}(H) + 14(k + \ell)$. Every MD-set of H can be extended to a MD-set of G by adding to it $k + \ell$ vertices, namely the k vertices in $V(C)$, and the ℓ vertices $(u_1u_k)_3$ and $(u_iu_{i+1})_3$ for $i \in [k - 1]$, whichever exist. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = k + \ell$ and $\beta = 14(k + \ell)$, contradicting Fact 1. \square

Claim 18.3 *The colored multigraph M_G does not contain a **red-red-red-red** path.*

Proof. Suppose, to the contrary, that M_G contains a **red-red-red-red** path $u_1u_2u_3u_4u_5$, where u_iu_{i+1} is a **red edge** in M_G for $i \in [4]$. Thus, (u_1u_2) , (u_2u_3) , (u_3u_4) and (u_4u_5) are all **red paths** in G . By Claim 14, the vertices u_2, u_3 and u_4 are distinct. Let ℓ be the number of long paths among (u_2u_3) and (u_3u_4) . Let

H be obtained from G by removing the vertices $u_2, u_3, u_4, (u_2u_1)_1$ and $(u_4u_5)_1$, as well as all vertices on the (u_2u_3) and (u_3u_4) 2-paths, where we mark the third neighbors of u_2, u_3 and u_4 (that do not belong to the deleted 2-paths). We note that $\dot{w}(G) \geq \dot{w}(H) + (6 + 2\ell) \times 5 + (3 + \ell) \times 4 - 2 \times 3 = \dot{w}(H) + 14\ell + 36$. Every MD-set of H can be extended to a MD-set of G by adding to it $3 + \ell$ vertices, namely the vertices u_2, u_3 and u_4 together with the ℓ vertices $(u_2u_3)_3$ and $(u_3u_4)_3$, whichever exist. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 3 + \ell$ and $\beta = 36 + 17\ell$, contradicting Fact 1. \square

Claim 18.4 *The colored multigraph M_G does not contain **red-red-red** paths, where the central edge is a long **red** path.*

Proof. Suppose, to the contrary, that M_G contains a **red-red-red** path $u_1u_2u_3u_4$, where u_iu_{i+1} is a **red** for $i \in [3]$ and where u_2u_3 is a long **red** in M_G . Thus, (u_1u_2) , (u_2u_3) , and (u_3u_4) are all **red** paths in G . By Claims 14 and 18.2, the vertices u_1, u_2, u_3 and u_4 are distinct. Let H be obtained from G by removing the vertices $u_2, u_3, (u_2u_1)_1$ and $(u_3u_4)_1$, as well as all five vertices on the (u_2u_3) 2-path, where we mark the third neighbors of u_2 and u_3 (that do not belong to the (u_1u_2) , (u_2u_3) and (u_3u_4) paths). We note that $\dot{w}(G) \geq \dot{w}(H) + 6 \times 5 + 3 \times 4 - 2 \times 3 = \dot{w}(H) + 36$. Every MD-set of H can be extended to a MD-set of G by adding to it the three vertices u_2, u_3 and $(u_2u_3)_3$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 3$ and $\beta = 36$, contradicting Fact 1. \square

Claim 18.5 *The colored multigraph M_G does not contain **red-red-black** cycles.*

Proof. Suppose, to the contrary, that M_G contains a **red-red-black** cycle $T: uvwu$ where uv and uw are **red** edges and where vw is a **black** edge in M_G . Thus, (uv) and (uw) are all **red** paths in G , while vw is a **black** edge in G . Since G has no 7-cycle, at least one of uv and uw , say uw , is a long **red**. Let z be the third neighbor of w in M_G (not in the triangle T). Since M_G is a colored multigraph, the edge wz is colored with one of the three colors **green**, **black** or **red**. By Claim 18.1, wz is not a **green** edge. By Claim 17, wz is not a **black** edge. By Claim 18.4, wz is not a **red**. Since all three possibilities produce a contradiction, such a **red-red-black** cycle T cannot exist. \square

Claim 18.6 *The colored multigraph M_G does not contain a double star $S(2,2)$ with two **red** edges one of which is the central edge, and the remaining edges **black**.*

Proof. Suppose, to the contrary, that M_G contains a double star $S(2,2)$ with vertex set $\{u_1, u_2, \dots, u_6\}$, where u_3 and u_4 are the two central vertices and u_1 and u_2 the leaf-neighbors of u_3 , and where u_1u_3 and u_3u_4 are **red** edges and u_2u_3, u_4u_5 and u_4u_6 are **black** edges in M_G . Thus, (u_1u_3) and (u_3u_4) are **red** paths in G , and u_2u_3, u_4u_5 and u_4u_6 are **black** edges in G . By Claim 17, the **red** path (u_3u_4) is short. By the girth condition, $u_2 \notin \{u_5, u_6\}$. By Claim 18.5, $u_1 \notin \{u_5, u_6\}$. By Claim 14, the vertices u_1, u_2, \dots, u_6 are therefore distinct. By Claims 18.1, 17 and 18.4, the edge u_1u_3 is a short **red**. Let H be obtained from G by removing the vertices u_3 and u_4 , as well as the vertices on the (u_1u_3) and (u_3u_4) 2-paths. We note that $\dot{w}(G) \geq \dot{w}(H) + 4 \times 5 + 2 \times 4 - 4 \times 1 = \dot{w}(H) + 24$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices $(u_3u_1)_1$ and $(u_3u_4)_2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$, contradicting Fact 1. \square

Claim 18.7 *The colored multigraph M_G does not contain a double star $S(2,2)$ with all edges colored **red**.*

Proof. Suppose, to the contrary, that M_G contains a double star $S(2,2)$ with vertex set $\{u_1, u_2, \dots, u_6\}$, where u_3 and u_4 are the two central vertices and u_1 and u_2 the leaf-neighbors of u_3 , and where all five edges are **red** edges in M_G . By Claim 18.4, the **red** path (u_3u_4) is short. By Claim 18.2, the vertices u_1, u_2, \dots, u_6

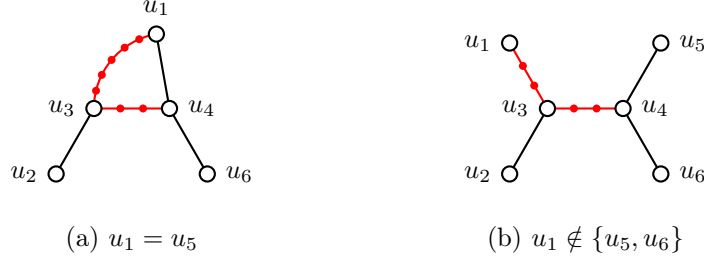


Figure 4: Colored multigraphs in the proof of Lemma 18.6

are therefore distinct. Let H be obtained from G by removing the vertices u_3 and u_4 , and all neighbors of u_3 and u_4 . We note that $\dot{w}(G) \geq \dot{w}(H) + 6 \times 5 + 2 \times 4 - 4 \times 3 = \dot{w}(H) + 26$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u_3 and u_4 . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 26$, contradicting Fact 1. \square

Claim 18.8 *The colored multigraph M_G does not contain a star $K_{1,3}$ with one edge subdivided and with all four edges colored red.*

Proof. Suppose, to the contrary, that M_G contains a star $K_{1,3}$ with one edge subdivided with vertex set $\{u, u_1, u_2, u_3, u_4\}$, where u is the vertex of degree 3, the vertices u_1, u_2 and u_3 are the three neighbors of u , the vertex u_4 is a leaf-neighbor of u_3 , and where all four edges are **red edges** in M_G . By Claims 18.1 and 18.3, the two edges incident with u_4 different from the **red** u_3u_4 are both **black** edges. Let v_3 be the third neighbor of u_3 (distinct from u and u_4). By Claim 18.1, u_3v_3 is not a **green edge**. By Claim 18.6, u_3v_3 is not a **black** edge. By Claim 18.7, u_3v_3 is not a **red**. Since all three possibilities produce a contradiction, such a star with one edge subdivided and with all four edges colored red cannot exist. \square

Claim 18.9 *The colored multigraph M_G does not contain red-red-red paths.*

Proof. Suppose, to the contrary, that M_G contains a **red-red-red** path $u_1u_2u_3u_4$, where u_iu_{i+1} is a **red** for $i \in [3]$. By Claims 18.1 and 18.3, the two edges incident with u_4 different from the **red** u_3u_4 are both **black** edges. Let v_3 be the third neighbor of u_3 (distinct from u_2 and u_4). By Claim 18.1, u_3v_3 is not a **green edge**. By Claim 18.6, u_3v_3 is not a **black** edge. By Claim 18.8, u_3v_3 is not a **red**. Since all three possibilities produce a contradiction, such a star with one edge subdivided and with all four edges colored red cannot exist. \square

Claim 18.10 *The colored multigraph M_G does not contain a star with all three edges colored red.*

Proof. Suppose, to the contrary, that M_G contains a star with vertex set $\{v, v_1, v_2, v_3\}$, where v is the vertex of degree 3 and with all three edges colored red in M_G . By Claims 18.1 and 18.9, the two edges incident with v_i different from the **red** vv_i are both **black** edges for $i \in [3]$. Further, by Claim 18.3, the three edges vv_i are short **red edges** for $i \in [3]$.

Suppose that v_1 and v_2 have no common neighbor. Let H be obtained from G by removing the vertices v, v_1 and v_2 , and all vertices on the (vv_1) , (vv_2) and (vv_3) 2-paths. We note that $\dot{w}(G) \geq \dot{w}(H) + 6 \times 5 + 3 \times 4 - 5 \times 1 = \dot{w}(H) + 37$. Every MD-set of H can be extended to a MD-set of G by adding to it the three vertices $(vv_1)_2$, $(vv_2)_2$ and $(vv_3)_1$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 3$ and $\beta = 37$, contradicting

Fact 1. Hence, v_1 and v_2 have a common neighbor. Analogously, v_1 and v_3 have a common neighbor, and v_2 and v_3 have a common neighbor

Suppose that v_1, v_2 and v_3 all have the same common neighbor, say w . As observed earlier, the edges wv_i are **black** edges for all $i \in [3]$. In this case, we let H be obtained from G by removing the vertices v, v_1, v_2, v_3 and w and all vertices on the $(vv_1), (vv_2)$ and (vv_3) 2-paths. We note that $\dot{w}(G) \geq \dot{w}(H) + 6 \times 5 + 5 \times 4 = \dot{w}(H) + 50$. Every MD-set of H can be extended to a MD-set of G by adding to it the four vertices v, v_1, v_2 and v_3 , where we mark the third neighbors of v_1, v_2 and v_3 (that were not deleted). Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 4$ and $\beta = 50$, contradicting Fact 1. Therefore, v_1, v_2 and v_3 do not all have the same common neighbor.

Let v_{ij} be the common neighbor of v_i and v_j , where $1 \leq i < j \leq 3$. By our earlier observations, the vertices v_{12}, v_{13} and v_{23} are distinct. In this case, let H be obtained from G by removing the vertices $v, v_1, v_2, v_3, v_{12}, v_{13}$ and v_{23} and all vertices on the $(vv_1), (vv_2)$ and (vv_3) 2-paths. We note that $\dot{w}(G) \geq \dot{w}(H) + 6 \times 5 + 7 \times 4 - 3 \times 3 = \dot{w}(H) + 49$. Every MD-set of H can be extended to a MD-set of G by adding to it the four vertices v, v_1, v_2 and v_3 . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 4$ and $\beta = 49$, contradicting Fact 1. Therefore, there is no star in M_G with all three edges colored red. \square

Claim 18.11 *The colored multigraph M_G does not contain a star with two red edges and one green edge.*

Proof. Suppose, to the contrary, that M_G contains a star with vertex set $\{v, v_1, v_2, v_3\}$, where v is the vertex of degree 3 and where the edge vv_1 and vv_2 are red edges and the edge vv_3 is a green edge. Let H be obtained from G by removing the vertex v and its three neighbors in G . Every MD-set of H can be extended to a MD-set of G by adding to it the vertex v . We note that $\dot{w}(G) \geq \dot{w}(H) + 3 \times 5 + 1 \times 4 - 2 \times 3 - 1 = \dot{w}(H) + 12$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 1$ and $\beta = 12$, contradicting Fact 1. \square

We now return to the proof of Claim 18 and show that the colored multigraph M_G does not contain two adjacent red edges. Suppose, to the contrary, that M_G contains a path $P: v_1v_2v_3$ where v_1v_2 and v_2v_3 are both red edges. By Claim 18.1 and 18.9, the two edges incident with v_1 (respectively, v_3) that do not belong to P are both black edges. Let e be the edge incident with v_2 that does not belong to P . By Claim 18.6, the edge e cannot be a black edge. By Claim 18.10, the edge e cannot be a red. By Claim 18.11, the edge e cannot be a green edge. Since all three possibilities produce a contradiction, the colored multigraph M_G does not contain red-red paths. Thus, the red edges form a matching in M_G . This completes the proof of Claim 18. \square

By Claim 15, the green edges form a matching in M_G . By Claim 18 the red edges form a matching in M_G .

5.4 There are no long edges

In this section, we show that there are no long edges in the colored multigraph M_G ; that is, every green path is a maximal 2-path in G of order 2 and every red path is a maximal 2-path in G of order 1. As a consequence of Claim 18, we show that there is no long red edge in M_G .

Claim 19 *The colored multigraph M_G does not contain any long red edge.*

Proof. Suppose, to the contrary, that M_G contains a long red edge. By Claim 18, such a long red edge cannot be adjacent to another red edge. Further by Claim 16, one end of the long red edge must be incident to two black edges. But this contradicts Claim 17. \square

Claim 20 *The colored multigraph M_G does not contain a green-black-red triangle.*

Proof. Suppose, to the contrary, that M_G contain a green-black-red triangle $T: uvwu$ where uv is a red edge, vw is a green edge, and wu is a black edge. By Claim 19, the red edge uv is short. Let $\ell = 0$ if the green edge is short, and let $\ell = 1$ if the green edge is long. Let H be obtained from G by removing the vertices in $V(T)$, as well as all vertices on the (uv) and (vw) 2-paths, where we mark the third neighbors of u and v that do not belong to the deleted 2-paths. We note that $\dot{w}(G) \geq \dot{w}(H) + (3 + 2\ell) \times 5 + (3 + \ell) \times 4 - 3 = \dot{w}(H) + 14\ell + 24$. Every MD-set of H can be extended to a MD-set of G by adding to it $\ell + 2$ vertices, namely the vertices u and v and, if $\ell = 1$, the vertex wu_3 . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = \ell + 2$ and $\beta = 24 + 14\ell$, contradicting Fact 1. \square

Claim 21 *No red edge is adjacent to a long green edge.*

Proof. Suppose, to the contrary, that M_G contains a red edge uv adjacent to a long green edge vw . By Claims 16 and 18, the two edges incident with u different from the red edge uv are black edges. By Claims 15 and 18, the third edge incident with v , different from the red edge uv and the long green edge vw , is a black edge.

Suppose that the long green edge vw contains no marked vertex. In this case, let H be obtained from G by removing the vertices v and w , as well as all vertices on the (uv) and (vw) 2-paths, where we mark the two neighbors of w not on these 2-paths. We note that $\dot{w}(G) \geq \dot{w}(H) + 6 \times 5 + 2 \times 4 - 2 = \dot{w}(H) + 36$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices $(uv)_2$, $(vw)_2$ and w . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 3$ and $\beta = 36$, contradicting Fact 1. Hence, the long green edge vw contains one marked vertex, namely $(vw)_2$ or $(vw)_3$.

Suppose firstly that the vertex $(vw)_2$ is marked. In this case, let H be obtained from G by removing the vertices v and w , as well the vertex $(uv)_2$ and all vertices on the (vw) 2-path, where we mark the neighbors of v not on the (uv) and (vw) 2-path. We note that at least one edge incident with w is a black edge, and so $\dot{w}(G) \geq \dot{w}(H) + 4 \times 5 + 3 \times 4 - 2 \times 3 - 1 = \dot{w}(H) + 25$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices v and $(vw)_4$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 25$, contradicting Fact 1.

Suppose secondly that the vertex $(vw)_3$ is marked. In this case, let H be obtained from G by removing the vertices u , v and w , as well as all vertices on the (uv) and (vw) 2-paths, where we mark the two neighbors of w not on these 2-paths. We note that $\dot{w}(G) \geq \dot{w}(H) + 5 \times 5 + 4 \times 4 - 3 \times 1 = \dot{w}(H) + 38$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices $(uv)_1$, $(vw)_1$ and w . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 3$ and $\beta = 38$, contradicting Fact 1. This completes the proof of Claim 21. \square

Claim 22 *The colored multigraph M_G does not contain any long green edge.*

Proof. Suppose, to the contrary, that M_G contains a long green edge uv . By Claim 21, every edge incident with u or v different from the long green edge uv is a black edge. Let H be obtained from G by removing the vertices u and v , as well as all vertices on the (uv) and (vw) 2-paths

Suppose firstly that the long green edge uv contains no marked vertex. In this case, we note that $\dot{w}(G) \geq \dot{w}(H) + 4 \times 5 + 2 \times 4 - 4 \times 1 = \dot{w}(H) + 24$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices $(uv)_1$ and $(uv)_2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$, contradicting Fact 1.

Suppose secondly that the long green edge uv contains a marked vertex. Renaming the vertices u and v if necessary, we may assume that $(vw)_3$ is the marked vertex on the long green edge uv . In this case, we mark

both neighbors of the vertex w not on the long **green edge** uv , and note that $\dot{w}(G) \geq \dot{w}(H) + 3 \times 5 + 3 \times 4 - 2 \times 1 = \dot{w}(H) + 25$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices $(uv)_1$ and v . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 25$, contradicting Fact 1. This completes the proof of Claim 22. \square

By Claims 19 and 22, there are no long **green edges** or **red edges** in the colored multigraph M_G . Thus by our earlier observations, there are no marked vertices.

5.5 Key structural claims

In this section, we prove several structural results that will be helpful when proving our main result.

Claim 23 *In the colored multigraph M_G , there is no **green-black-green** path $uu'vv'$ where u and v are adjacent to the centers u^* and v^* of black stars.*

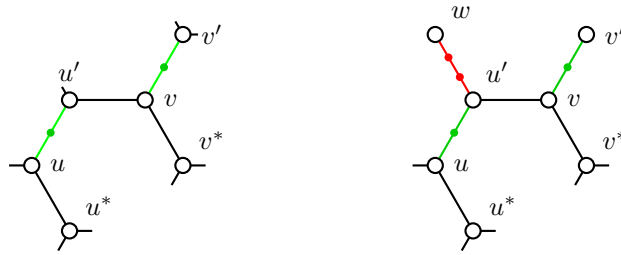


Figure 5: The two cases of Claim 23

Proof. Suppose, to the contrary, that M_G contains such a configuration. We consider two cases. The first case is when the third edge attached to u' is a black edge. In this case, let H be obtained from G by removing the vertices u, v, u^*, v^*, u' , and the vertices $(uu')_1$ and $(vv')_1$. We mark the third vertex of u and cut all the six other outgoing edges. Since the graph G does not contain a 7-cycle, the vertices u^* and v^* do not have a common neighbor. We note that $\dot{w}(G) \geq \dot{w}(H) + 5 \times 4 + 2 \times 5 - 6 = \dot{w}(H) + 24$. Every MD-set of H can be extended to a MD-set of G by adding the vertices u and v . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$, contradicting Fact 1.

Now, consider the case when u' is incident to a **red edge** $u'w$. In this case, let H be obtained from G by removing the vertices u, v, u^*, v^*, u', w , and the vertices $(uu')_1, (vv')_1, (u'w)_1$ and $(u'w)_2$. We mark the third vertex of u and cut all the seven other outgoing edges. We note that these seven edges that are cut when constructing H do not make a vertex of degree 3 drop to degree 1. Indeed since there is no 4-cycle, the vertices v' and v^* are not adjacent and since there is no 7-cycle, the vertex v' is adjacent to neither w nor u^* . Moreover since there is no 7-cycle, the vertices u^* and v^* have no common neighbor and the vertices w and v^* have no common neighbor. Since there is no 8-cycle, the vertices u^* and w have no common neighbor. Thus forbidding 7- and 8-cycles guarantees we do not create a vertex of degree 1 when constructing H . We therefore infer that $\dot{w}(G) \geq \dot{w}(H) + 6 \times 4 + 4 \times 5 - 7 = \dot{w}(H) + 37$. Every MD-set of H can be extended to a MD-set of G by adding the vertices u, v and $(u'w)_2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 3$ and $\beta = 36$, contradicting Fact 1. \square

Claim 24 *In the colored multigraph M_G , there is no **green-black-green** path $uu'vv'$ where u' and v' are incident to **red edges** uu^\dagger and vv^\dagger and where the third edge attached to v is a black edge.*

Proof. Suppose, to the contrary, that M_G contains such a configuration. By Claim 20, the edges $u'u^\dagger$ and $v'v^\dagger$ are distinct. Let H be obtained from G by removing the vertices $u', v, v', (u'u^\dagger)_1, (v'v^\dagger)_1, (uu')_1$ and $(vv')_1$. We mark the third neighbor of v' , and cut all the four other outgoing edges. We note that $\dot{w}(G) \geq \dot{w}(H) + 3 \times 4 + 4 \times 5 - 2 \times 3 - 2 \times 1 = \dot{w}(H) + 24$. Every MD-set of H can be extended to a MD-set of G by adding the vertices u' and v' . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$, contradicting Fact 1. \square

Claim 25 *In the colored multigraph M_G , there is no 6-cycle $u'vv'w^*u'$ (where vv' is a **green edge**), such that w is incident to a **green edge**, and w^* is the center of a **black star**.*

Proof. Suppose, to the contrary, that M_G contains such a configuration. Let w' be the other extremity of the **green edge** incident to w . Let H be obtained from G by removing the vertices $u', v, v', w, x, (vv')_1$ and $(ww')_1$. We mark the third neighbor of v , and cut all four other outgoing edges, namely the edge $(ww')_1w'$ and the edges incident to w^*, u' , and v' . Possibly, the third edge incident to u' is a **green edge**. Moreover, v' may possibly be incident to a **red edge**.

Suppose firstly that v' is not incident to a **red edge**. In this case, we note that $\dot{w}(G) \geq \dot{w}(H) + 5 \times 4 + 2 \times 5 - 1 \times 3 - 3 \times 1 = \dot{w}(H) + 24$. Every MD-set of H can be extended to a MD-set of G by adding the vertices v and w . (This is illustrated in the left hand figure of Figure 7 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$, contradicting Fact 1.

Hence the vertex v' is incident to a **red edge**, say vv^* . Since there is no 7-cycle, the vertex v^* has no common neighbor with w^* and is not adjacent to w' . Moreover since there is no 8-cycle, the vertex v^* is not adjacent to u' . By Claim 20, the vertex v^* is not adjacent to v . In this case, we let H be obtained from G by removing the vertices $u', v, v', w, w^*, (vv')_1, (ww')_1, (vv^*)_1, (vv^*)_2$, and v^* . By Claim 16, the two edges incident to v^* different that do not belong to the **red edge** vv^* are **black edges**. We mark the third neighbor of v , and cut all six other outgoing edges, namely the edge $(ww')_1w'$ and the edges incident to w^*, u' , and v^* . Every MD-set of H can be extended to a MD-set of G by adding the vertices v, w and v^* . (This is illustrated in the right hand figure of Figure 7 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) In this case, we note that $\dot{w}(G) \geq \dot{w}(H) + 6 \times 4 + 4 \times 5 - 4 \times 1 - 1 \times 3 = \dot{w}(H) + 37$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 3$ and $\beta = 37$, contradicting Fact 1. \square

Claim 26 *In the colored multigraph M_G , if a 6-cycle $xu'vv'wx$ (where vv' is a **green edge**) is such that the third neighbor w^* of w is the center of a **black star**, then the outgoing edge of x is a **red edge**.*

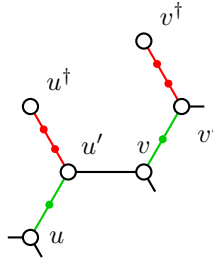


Figure 6: A colored multigraph in the proof of Claim 24

Proof. Suppose, to the contrary, that M_G contains such a configuration where the outgoing edge of x is either a **black edge** or a **green edge**. By Claim 25, the outgoing edge from u' is a black edge.

Suppose firstly that the outgoing edge from x is a **black edge**. Let H be obtained from G by removing the vertices u', v, v', w, x, w^* , and $(vv')_1$. We mark the third neighbor of v , and cut all five other outgoing edges, namely the edges incident to x, u', v' and w^* . In this case, we note that $\dot{w}(G) \geq \dot{w}(H) + 6 \times 4 + 1 \times 5 - 5 \times 1 = \dot{w}(H) + 24$. Every MD-set of H can be extended to a MD-set of G by adding the vertices v and w . (This is illustrated in the left hand figure of Figure 8 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$, contradicting Fact 1.

Hence the outgoing edge from x is a **green edge**, say xx' . In this case, let H be obtained from G by removing the vertices $u', v, v', w, x, (vv')_1$ and $(xx')_1$. Since there is no 5-cycle, the vertices x' and v' are not adjacent, and since there is no 8-cycle, they do not have a common neighbor. Moreover since there is no 4-cycle, the vertices x' and u' are not adjacent, and since there is no 5-cycle, they do not have a common neighbor. Since there is no 5-cycle, the vertices x' and v are not adjacent. Since there is no 5-cycle, the vertices u' and v' do not have a common neighbor. We cut all five other outgoing edges, namely the edge $(xx')_1x'$ and the edges incident to u', v, v' and w . In this case, we note that $\dot{w}(G) \geq \dot{w}(H) + 5 \times 4 + 2 \times 5 - 5 \times 1 = \dot{w}(H) + 25$. Every MD-set of H can be extended to a MD-set of G by adding the vertices x and $(vv')_1$. (This is illustrated in the right hand figure of Figure 8 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 25$, contradicting Fact 1. \square

Claim 27 *In the colored multigraph M_G , there is no 10-cycle $C: uu'vv'ww'w^*u$ where the following properties hold.*

- uu', vv' and ww' are **green edges**,
- u' and w are adjacent to the centers u^* and w^* , respectively, of black stars,
- w^* is the center of a black star, and
- w' is adjacent to a vertex x which is the end of a **green edge** xx' .

Proof. Suppose, to the contrary, that M_G contains such a configuration. Let x be the third neighbor of w' that does not belong to the cycle C . We note that the edge incident with v' that does not belong to the cycle C is either a **black edge** or a **red edge**. We consider the two cases in turn.

Suppose firstly that the edge incident with v' that does not belong to the cycle C is a **black edge**. By Claim 23, the vertex x is not the center of a black star. By our earlier properties and observations, the vertex x is not incident with a **red edge**. Hence, the vertex x is incident with a **green edge**, say xx' . In this case, let H be obtained from G by removing all the vertices of the cycle C , and removing the vertices w^*, x and $(xx')_1$. We mark the neighbors of u and v that do not belong to the cycle, and we mark the neighbor of x different from w' and $(xx')_1$, and we cut all other outgoing edges.

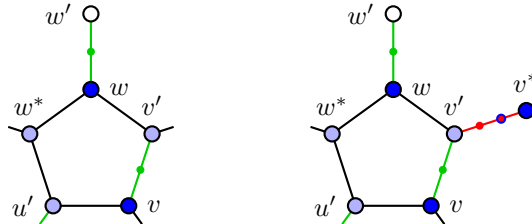


Figure 7: Colored multigraphs in the proof of Claim 25

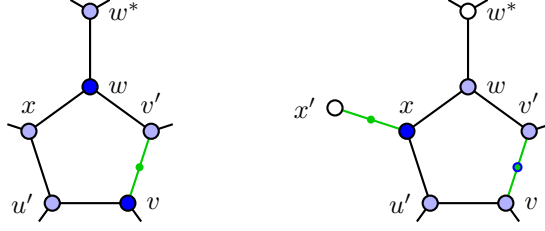


Figure 8: Colored multigraphs in the proof of Claim 26

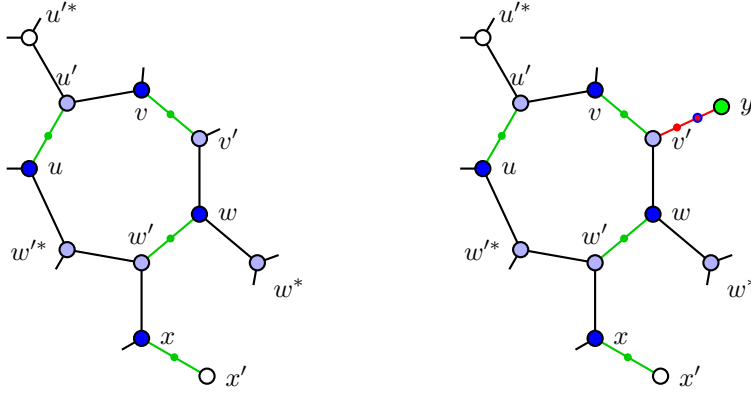


Figure 9: Colored multigraphs in the proof of Claim 27

Since there is no 5-cycle, the vertices x' and w'^* are not adjacent and the vertex u'^* is adjacent to neither v' nor w'^* . Since there is no 7-cycle, the vertices u'^* and w^* are not adjacent and the vertex x' is adjacent to neither w^* nor v' . Since u'^* is the center of a black star, we note that the vertices x' and u'^* are distinct. Since there is no 4-cycle, the vertices w^* and v' have no common neighbor. Since there is no 8-cycle, the vertices w'^* and v' have no common neighbor. We note that the vertices w'^* and w^* may possibly have a common neighbor. However if a common neighbor of w'^* and w^* exists, then by Claim 26 it is incident with a **red edge**.

From these structural properties we infer that if w'^* and w^* have no common neighbor, then $\dot{w}(G) \geq \dot{w}(H) + 9 \times 4 + 4 \times 5 - 6 \times 1 = \dot{w}(H) + 50$, while if w'^* and w^* do have a common neighbor, then $\dot{w}(G) \geq \dot{w}(H) + 9 \times 4 + 4 \times 5 - 5 \times 1 - 1 \times 3 = \dot{w}(H) + 48$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u, v, w and x . (This is illustrated in the left hand figure of Figure 9 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 4$ and $\beta \geq 48$, contradicting Fact 1.

Hence the edge incident with v' that does not belong to the cycle C is a **red edge**, say $v'y$. In this case, let H be obtained from G by removing all the vertices of the cycle C , and removing the vertices $w^*, x, y, (xx')_1, (v'y)_1$ and $(v'y)_2$. We mark the neighbors of u and v that do not belong to the cycle, we mark the neighbor of x different from w' and $(xx')_1$, we mark the vertex y , and we cut all other outgoing edges. Analogously as before, we infer that if w'^* and w^* have no common neighbor, then $\dot{w}(G) \geq \dot{w}(H) + 9 \times 4 + 6 \times 5 - 4 \times 1 = \dot{w}(H) + 61$, while if w'^* and w^* do have a common neighbor, then $\dot{w}(G) \geq \dot{w}(H) + 9 \times 4 + 6 \times 5 - 3 \times 1 - 1 \times 3 = \dot{w}(H) + 60$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u, v, w, x and $(v'y)_2$. (This is illustrated in the right hand figure

of Figure 9 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 5$ and $\beta \geq 60$, contradicting Fact 1. \square

Claim 28 *In the colored multigraph M_G , there is no 10-cycle $C: u'vv'ww'w'^*yu'^*u'$ where the following properties hold.*

- vv' and ww' are *green edges*,
- u' is incident with a *green edge* uu' ,
- u'^* and w'^* are center of black stars and have a common neighbor y ,
- w is adjacent to the center w^* of a black star, and
- the neighbor x of w' that is not on the cycle C is incident with a *green edge* xx' .

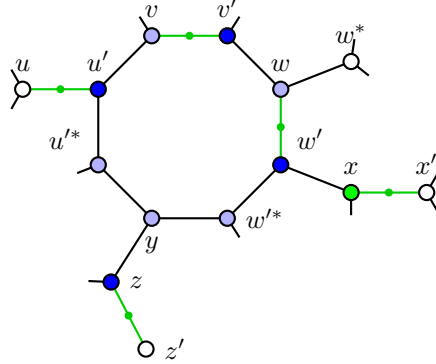


Figure 10: A colored multigraph in the first case of Claim 28

Proof. Suppose, to the contrary, that M_G contains such a configuration. Since there is no 4-cycle and no 5-cycle, the vertex v is not adjacent to u and w^* , respectively. Moreover since there is no 4-cycle, the vertices v and u'^* have no common neighbor, and since there is no 8-cycle, the vertices v and w'^* have no common neighbor. We note, however, that the vertices u and w'^* may possibly be adjacent, in which they belong to a common 6-cycle. Since there is no 4-cycle, the vertex u'^* is not adjacent to u and has no common neighbor with w'^* . Let $e = yz$ be the edge in M_G incident with y that does not belong to the cycle C .

Suppose firstly that the edge e is a **black edge** and that the vertex z is incident to a *green edge*, say zz' , as illustrated in Figure 10. Since there is no 5-cycle, the vertex z' is adjacent to neither u'^* nor w'^* . Since there is no 7-cycle, the vertices v and z' are not adjacent. Since w^* is the center of a black star, it cannot be equal to u or to z' . Moreover since there is no 5-cycle and no 7-cycle, the vertex w^* is adjacent to neither w'^* nor u'^* , respectively. We also note that the vertices u and z' are distinct since the *green edges* form a matching.

Let H be obtained from G by removing all the vertices of the cycle C , and removing the vertices z , $(zz')_1$, and $(uu')_1$. We mark the neighbors of v' and w' that do not belong to the cycle C , and we mark the neighbor of z different from y' and $(zz')_1$, and we cut all other six outgoing edges. We note that when constructing H , we create at most one vertex of degree 1, which occurs when the vertices u and w'^* are adjacent. If we create no vertex of degree 1, then $\dot{w}(G) \geq \dot{w}(H) + 9 \times 4 + 4 \times 5 - 6 \times 1 = \dot{w}(H) + 50$. On the other hand, if we create a vertex of degree 1 (in the case when u and w'^* are adjacent), then $\dot{w}(G) \geq \dot{w}(H) + 9 \times 4 + 4 \times 5 - 5 \times 1 - 1 \times 3 = \dot{w}(H) + 48$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u' , v' , w' and z . (This is illustrated in the left hand figure of Figure 10

where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 4$ and $\beta \geq 48$, contradicting Fact 1.

Hence the edge e is a **red edge** or a **green edge** or a **black edge** with the vertex z the center of a black star or a **black edge** with the vertex z incident to an additional **black edge** and to a **red edge**, say zz' . This gives rise to four cases that may occur, as illustrated in Figure 11.

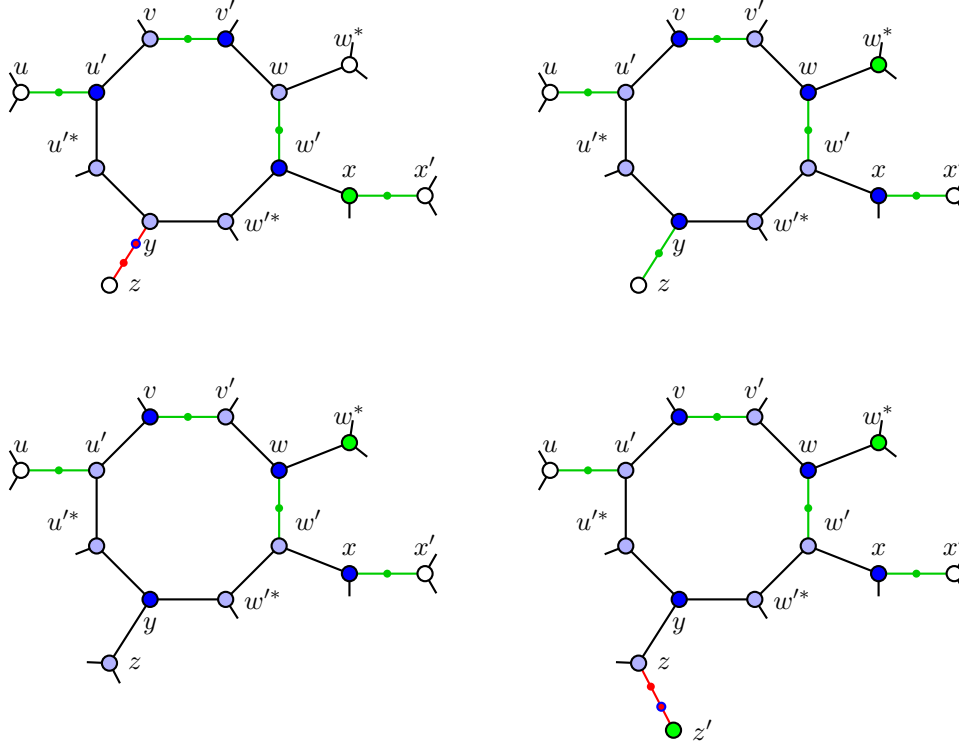


Figure 11: Colored multigraph in the second case of Claim 28

Since the vertex $(uu')_1$ is a vertex of degree 2, it is distinct from x' and is not adjacent to v' , w^* , w'^* or u'^* . Since there is no 4-cycle, the vertices u'^* and w'^* have no common neighbor different from y and the vertices v' and w^* have no common neighbor different from w . Since there is no 5-cycle, the vertex x' is not adjacent to w'^* . Since there is no 7-cycle, the vertex x' is not adjacent to u'^* , v' or w^* . Since there is no 8-cycle, the vertices v' and w'^* have no common neighbor, the vertices u'^* and w^* have no common neighbor, and the vertices v' and u'^* have no common neighbor. However, we note that the vertices w^* and w'^* may possibly have a common neighbor.

Suppose that the edge e is a **red edge**. In this case, let H be obtained from G by removing all the vertices of the cycle C , and removing the vertices x , $(uu')_1$, $(yz)_1$ and $(yz)_2$. We mark the neighbors of v' and w' that does not belong to the cycle C , and we cut all other six outgoing edges. If the vertices w^* and w'^* have no common neighbor, then $\dot{w}(G) \geq \dot{w}(H) + 8 \times 4 + 5 \times 5 - 6 \times 1 = \dot{w}(H) + 51$, while if the vertices w^* and w'^* have a common neighbor, then $\dot{w}(G) \geq \dot{w}(H) + 8 \times 4 + 5 \times 5 - 5 \times 1 + 1 \times 3 = \dot{w}(H) + 48$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u' , v' , w' and $(yz)_1$. (This is illustrated in the top left hand figure of Figure 11 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 4$ and $\beta \geq 48$, contradicting Fact 1. Hence, the edge e is not a **red edge**.

In what follows suppose that the vertices w^* and w'^* do not have a common neighbor.

Suppose that the edge e is a **green edge**. In this case, let H be obtained from G by removing all the vertices of the cycle C , and removing the vertices x , $(xx')_1$, w^* and $(yz)_1$. We mark the neighbor of v that does not belong to the cycle C , and we mark the neighbor of x different from w' and $(xx')_1$, and we cut all other eight outgoing edges. In this case, $\dot{w}(G) \geq \dot{w}(H) + 10 \times 4 + 4 \times 5 - 7 \times 1 - 1 \times 3 = \dot{w}(H) + 50$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices v , w , x and y . (This is illustrated in the top right hand figure of Figure 11 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 4$ and $\beta \geq 50$, contradicting Fact 1. Hence, the edge e is not a **green edge**.

Suppose that the edge e is a **black edge** with the vertex z the center of a black star. In this case, let H be obtained from G by removing all the vertices of the cycle C , and removing the vertices x , $(xx')_1$, w^* and z . We mark the neighbor of v that does not belong to the cycle C , and we mark the neighbor of x different from w' and $(xx')_1$, and we cut all other seven outgoing edges. In this case, $\dot{w}(G) \geq \dot{w}(H) + 11 \times 4 + 3 \times 5 - 8 \times 1 - 1 \times 3 = \dot{w}(H) + 48$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices v , w , x and y . (This is illustrated in the bottom left hand figure of Figure 11 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 4$ and $\beta \geq 48$, contradicting Fact 1.

Suppose that the edge e is a **black edge** with the vertex z incident to an additional **black edge** and to a **red edge**, say zz' . In this case, let H be obtained from G by removing all the vertices of the cycle C , and removing the vertices x , $(xx')_1$, w^* , z , z' , $(zz')_1$ and $(zz')_2$. We mark the neighbor of v that does not belong to the cycle C , and we mark the neighbor of x different from w' and $(xx')_1$, and we cut all other ten outgoing edges. In this case, $\dot{w}(G) \geq \dot{w}(H) + 12 \times 4 + 5 \times 5 - 9 \times 1 - 1 \times 3 = \dot{w}(H) + 61$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices v , w , x , y and $(zz')_2$. (This is illustrated in the bottom right hand figure of Figure 11 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 4$ and $\beta \geq 48$, contradicting Fact 1.

Hence, we may assume in what follows that the vertices w^* and w'^* have a common neighbor, say p .

By Claim 26, the outgoing edge of p from the 6-cycle C' : $w(ww')_1 w' w'^* p w^* w$ is a **red edge**, that is, the vertex p is incident with a **red edge**, say pq . By our earlier observations, there are three cases to consider, namely when the edge e is a **green edge** or a **black edge** with the vertex z the center of a black star or a **black edge** with the vertex z incident to an additional **black edge** and to a **red edge**, say zz' .

Suppose that the edge e is a **green edge**. In this case, let H be obtained from G by removing all the vertices of the cycle C , and removing the vertices x , $(xx')_1$, w^* , $(yz)_1$, p , $(pq)_1$ and $(pq)_2$. We mark the neighbor of v that does not belong to the cycle C , and we mark the neighbor of x different from w' and $(xx')_1$, and we cut all other eight outgoing edges. In this case, $\dot{w}(G) \geq \dot{w}(H) + 11 \times 4 + 6 \times 5 - 7 \times 1 - 1 \times 3 = \dot{w}(H) + 64$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices v , w , x , y and $(pq)_1$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 5$ and $\beta \geq 64$, contradicting Fact 1. Hence, the edge e is not a **green edge**. Therefore, the edge is a **black edge**.

Suppose that the edge e is a **black edge** with the vertex z the center of a black star. In this case, let H be obtained from G by removing all the vertices of the cycle C , and removing the vertices x , $(xx')_1$, w^* , z , p , $(pq)_1$ and $(pq)_2$. We mark the neighbor of v that does not belong to the cycle C , and we mark the neighbor of x different from w' and $(xx')_1$, and we cut all other nine outgoing edges. In this case, $\dot{w}(G) \geq \dot{w}(H) + 12 \times 4 + 5 \times 5 - 8 \times 1 - 1 \times 3 = \dot{w}(H) + 61$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices v , w , x , y and $(pq)_1$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 5$ and $\beta \geq 61$, contradicting Fact 1.

The final case to consider is when the edge e is a **black edge** with the vertex z incident to an additional

black edge and to a **red edge**, say zz' . In this case, let H be obtained from G by removing all the vertices of the cycle C , and removing the vertices $x, (xx')_1, w^*, z, z', (zz')_1, (zz')_2, p, (pq)_1$ and $(pq)_2$. We mark the neighbor of v that does not belong to the cycle C , and we mark the neighbor of x different from w' and $(xx')_1$, and we cut all other ten outgoing edges. In this case, $\dot{w}(G) \geq \dot{w}(H) + 13 \times 4 + 7 \times 5 - 9 \times 1 - 1 \times 3 = \dot{w}(H) + 75$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices $v, w, x, y, (zz')_2$ and $(pq)_1$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 6$ and $\beta \geq 75$, contradicting Fact 1. \square

Claim 29 *In the colored multigraph M_G , there is no **green-black-green-black-green-black-green** path $P: uu'vv'ww'xx'$ where u', w and w' are adjacent to the centers u^*, w^* and w'^* of black stars.*

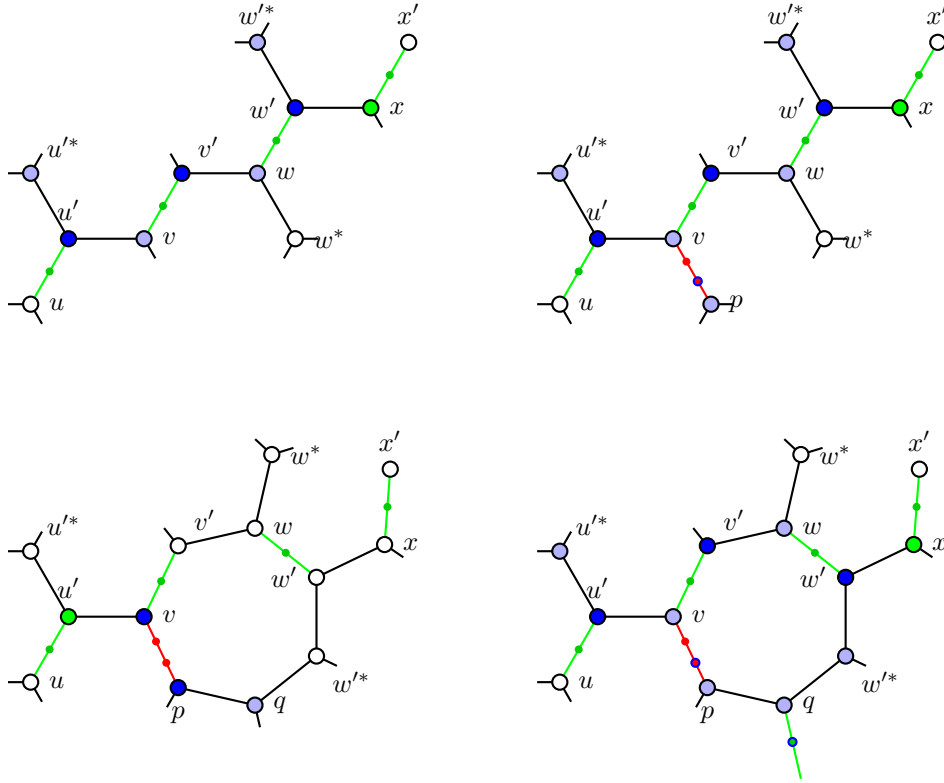


Figure 12: Colored multigraph in the proof of Claim 29

Proof. Suppose, to the contrary, that M_G contains such a path P . We first obtain structural properties of the vertices on the path P .

Since there is no 4-cycle, the vertex u is not adjacent to u^* and v . Moreover the absence of 4-cycles implies that the vertices u^* and v have no common neighbor. Since there is no 5-cycle, the vertex w^* is not adjacent to v and w' . Since there is no 7-cycle, we note that $u \neq w^*$ (this also follows from the assumption that w^* is the center of a black star, while u is incident with a **green edge**) and the vertex u^* is not adjacent to w^* . Since there is no 8-cycle, we note that $u^* \neq w'^*$. Moreover the absence of 8-cycles implies that v and w'^* have no common neighbor. If u and w'^* are adjacent, then they belong to a common 10-cycle, contradicting Claim 27. If u^* and w'^* have a common neighbor, then they belong to a common 10-cycle, contradicting Claim 28.

Suppose that the edge incident with v that does not belong to the path P is a **black edge**. In this case,

let H be obtained from G by removing the vertices $u', v, v', w, w', u^*, w^*, (uu')_1, (vv')_1$ and $(ww')_1$. We mark x the neighbor of v' that does not belong to the path P , and we cut all the seven other outgoing edges. This yields $\dot{w}(G) \geq \dot{w}(H) + 7 \times 4 + 3 \times 5 - 7 \times 1 = \dot{w}(H) + 36$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u', v' and w' . (This is illustrated in the top left hand figure of Figure 12 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 3$ and $\beta = 36$, contradicting Fact 1.

Hence the edge incident with v that does not belong to the path P is a **red edge**, say vp . By Claim 16, the vertex p is not incident with a **green edge**, and therefore p is incident with two **black edges**. Since the vertex p is incident with a **red edge**, we note that p is distinct from the vertices u, u^*, w^* , and w'^* . Since there is no 7-cycle, the vertices p and u^* have no common neighbor and p is not adjacent to u . Since there is no 8-cycle, p is not adjacent to w^* either.

Suppose that the vertices p and w'^* have no common neighbor. In this case, let H be obtained from G by removing the vertices $u', v, v', w, w', u^*, w^*, p, (uu')_1, (vv')_1, (ww')_1, (vp)_1$ and $(vp)_2$. We mark x and the neighbor of v' that does not belong to the path P , and we cut all the eight other outgoing edges. This yields $\dot{w}(G) \geq \dot{w}(H) + 8 \times 4 + 5 \times 5 - 8 \times 1 = \dot{w}(H) + 49$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u', v', w' and $(vp)_2$. (This is illustrated in the top right hand figure of Figure 12 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 4$ and $\beta = 49$, contradicting Fact 1.

Hence, the vertices p and w'^* have a common neighbor which we call q . By our earlier observations, the edges pq and qw'^* are both **black edges**. This produces a 7-cycle in the colored multigraph M_G that contains both p and w'^* , namely the **black-black-red-green-black-green-black** cycle $C_M: w'^*qpvv'ww'w'^*$. In the original graph G , this is an 11-cycle, namely the cycle $C: w'^*qp(vp)_2(vp)_1v(vv')_1v'w(ww')_1w'^*$.

Suppose that the third edge in M_G incident with q (that does not belong to the cycle C_M) is a **black edge**. In this case, let H be obtained from G by removing the vertices $v, p, q, (vv')_1, (vp)_1$ and $(vp)_2$. We mark the neighbor u' of v that was not deleted, and we mark the neighbor of p different from q and $(vp)_2$, and we cut all the three other outgoing edges. This yields $\dot{w}(G) \geq \dot{w}(H) + 3 \times 4 + 3 \times 5 - 3 \times 1 = \dot{w}(H) + 24$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices v and p . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$, contradicting Fact 1.

Hence the edge, qr say, in M_G incident with q that does not belong to the cycle C_M is either a **green edge** or a **red edge**. In this case, let H be obtained from G by removing all eleven vertices on the cycle C , removing the vertices $u', u^*, (uu')_1$ and $(qr)_1$. We mark x and the neighbor of v' that does not belong to the cycle C and we mark the neighbor of $(qr)_1$ different from q . Thus, if qr is a **green edge**, then we mark the vertex r , while if qr is a **red edge**, then we mark the vertex $(qr)_2$. Further, we cut all the six other outgoing edges. This yields $\dot{w}(G) \geq \dot{w}(H) + 9 \times 4 + 6 \times 5 - 6 \times 1 = \dot{w}(H) + 60$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices $u', v', w', (qr)_1$ and $(vp)_2$. (This is illustrated in the bottom figure of Figure 12 where the vertices in the MD-set are dark blue and the other vertices of degree 3 in G that were deleted when constructing H are colored light blue.) Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 5$ and $\beta = 60$, contradicting Fact 1. \square

5.6 Maximal green-black alternating paths

In this section, we consider maximal alternating **green-black** paths in M_G that are not cycles.

5.6.1 Structural properties of extremities of green-black alternating paths

In this subsection, we establish structural properties of the extremities of maximal alternating **green-black** paths that are not **green-black** cycles, and show that these structures can be classified into three types, which we will refer to as Type-(1), Type-(2), and Type-(3). In order to present these three different structural types, we prove three claims.

Claim 30 *The colored multigraph M_G does not contain a star with two **black** edges and one **green** edge, and with two leaves of the star incident with three **black** edges.*

Proof. Suppose, to the contrary, that M_G contains a star with vertex set $\{v, v_1, v_2, v_3\}$, where v is the vertex of degree 3 and where the edge vv_1 and vv_2 are **black** edges and the edge vv_3 is a **green** edge, and where both v_1 and v_2 are incident with three **black** edges. Let H be obtained from G by removing the vertex v and its three neighbors in G . Every MD-set of H can be extended to a MD-set of G by adding to it the vertex v . We note that $\dot{w}(G) \geq \dot{w}(H) + 1 \times 5 + 3 \times 4 - 5 \times 1 = \dot{w}(H) + 12$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 1$ and $\beta = 12$, contradicting Fact 1. \square

Claim 31 *The colored multigraph M_G does not contain a star with one **red**, one **black** edge, and one **green** edge, and with one leaf of the star incident with three **black** edges.*

Proof. Suppose, to the contrary, that M_G contains a star with vertex set $\{v, v_1, v_2, v_3\}$, where v is the vertex of degree 3 and where the edge vv_1 is **black**, the edge vv_2 is red and the edge vv_3 is **green**, and where v_1 is incident with three **black** edges. Let H be obtained from G by removing the vertex v and its three neighbors in G . Every MD-set of H can be extended to a MD-set of G by adding to it the vertex v . We note that $\dot{w}(G) \geq \dot{w}(H) + 2 \times 5 + 2 \times 4 - 1 \times 3 - 3 \times 1 = \dot{w}(H) + 12$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 1$ and $\beta = 12$, contradicting Fact 1. \square

Claim 32 *If the colored multigraph M_G contains a star centered at a vertex u with two **black** edges, uv and uw , and with one **green** edge, ux , and such that neither v nor w is incident with a **green** edge, then the following properties hold.*

- (a) *If both v and w are incident with a **red**, then the end of the **red** incident with v (respectively, w) in M_G that is different from v (respectively, w) is incident with a **green** edge.*
- (b) *If the vertex v is incident with one **red** e_v , and the vertex w is incident with no **red**, then the end of the edge e_v different from v is incident with a **green** edge.*

Proof. We first prove part (a). Suppose that there is a **red** vy incident with v in M_G and a **red** wz incident with w in M_G , where y is incident with two **black** edges. By supposition, neither v nor w is incident with a **green** edge, implying that the third edge incident with v in M_G (different from uv and vy) and the third edge incident with w in M_G (different from uw and wz) are **black** edges. This structure is illustrated by the graph in Figure 13(a). By Claim 17, both **red** edges vy and wz are short. Since G contains no 7-cycle, the vertices y and w have no common neighbor. Let H be obtained from G by removing the vertices u, v, w, x and y , and removing all vertices on the (vy) and (wz) 2-paths in G , and marking the vertex z . We note that $\dot{w}(G) \geq \dot{w}(H) + 5 \times 5 + 4 \times 4 - 5 \times 1 = \dot{w}(H) + 36$. Every MD-set of H can be extended to a MD-set of G by adding to it the three vertices $u, (vy)_2$ and $(wz)_2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 3$ and $\beta = 36$, contradicting Fact 1. This completes the proof of part (a).

We next prove part (b). Suppose, to the contrary, that the end y of the **red** $e_v = yv$ different from v is incident with two **black** edge. This structure is illustrated by the graph in Figure 13(b). Let H be obtained from G by removing the vertices u, v, w, x and y , and removing both vertices on the (vy) 2-path in G . We

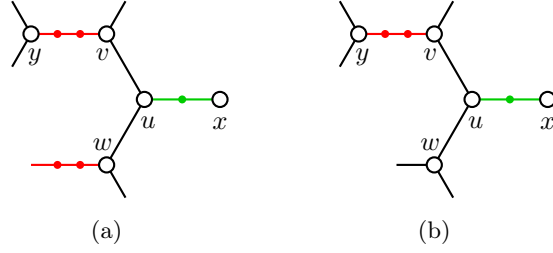


Figure 13: Forbidden structures in the proof of Lemma 32

note that $\dot{w}(G) \geq \dot{w}(H) + 3 \times 5 + 4 \times 4 - 6 \times 1 = \dot{w}(H) + 25$. Every MD-set of H can be extended to a MD-set of G by adding to it the two vertices u and $(vy)_2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 25$, contradicting Fact 1. \square

We are now in a position to present three structural properties of the extremities of maximal alternating **green-black** paths. These properties result in a classification of the extremities into three types, which we will refer to as Type-(1), Type-(2), and Type-(3).

Consider a maximal **green-black** path P in M_G that starts at a vertex u . Let x be the neighbor of u on P . We note that the ux is a **green edge**. Let v and w be the neighbors of u in M_G that do not belong to the path P . We note that both uv and uw are **black edge**, or exactly one of uv and uw is a **red edge**. By the maximality of the path P , neither v nor w is incident with a **green edge**. There are two cases for us to consider.

Suppose firstly that both uv and uw are **black edge**. By Claim 30, at least one of v and w is incident with a **red edge**. Renaming v and w if necessary, we may assume that v is incident with a **red edge**, say vy . We note that the edge incident with v , different from uv and vy , is a **black edge**.

Suppose that w is incident with a **red edge**, say wz . We note that the edge incident with w , different from uw and wz , is a **black edge**. By Claim 32(b), the end y (respectively, z) of the **red** incident with v (respectively, w) in M_G that is different from v (respectively, w) is incident with a **green edge**. We call the subgraph of M_G induced by the vertices $\{u, v, w, x, y, z\}$ and their neighbors a Type-(1) structure of M_G from the extremity u of a maximal alternating **green-black** path. This structure is illustrated in Figure 14(a).

Suppose that w is incident only with **black edges**. By Claim 32(b), the vertex y is incident with a **green edge**. We call the subgraph of M_G induced by the vertices $\{u, v, w, x, y\}$ and their neighbors a Type-(2) structure of M_G from the extremity u of a maximal alternating **green-black** path. This structure is illustrated in Figure 14(b).

Suppose next that one of uv and uw is a **red edge**. Renaming vertices if necessary, we may assume that uw is a **red edge**. Thus, the edge uv is a **black edge**. By the maximality of the path P , the vertex v is not incident with a **green edge**. By Claim 30, the vertex v is incident with a **red edge**, say vy . We call the subgraph of M_G induced by the vertices $\{u, v, w, x, y\}$ and their neighbors a Type-(3) structure of M_G from the extremity u of a maximal alternating **green-black** path. This structure is illustrated in Figure 14(c).

Using our structural classification of the extremities of maximal alternating **green-black** paths, we now give two counting strategies that we will apply to each of these three structures. Let

$$P: u_1, u'_1, u_2, u'_2, \dots, u_k, u'_k$$

be a maximal alternating **green-black** path in M_G , and so the edges $u_i u'_i$ are **green edges** for $i \in [k]$ and edges $u'_i u_{i+1}$ are **black edges** for $i \in [k-1]$. We now define a strategy for handling the extremities u_1

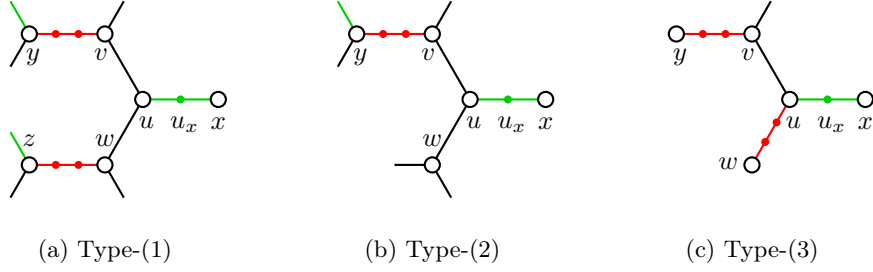


Figure 14: Structures for the extremity u of a maximal alternating **green-black** path

and u'_k of such a maximal alternating **green-black** path P . Let the **green edge** $u_i u'_i$ correspond to the path u_i, v_i, u'_i in the graph G , and so v_i has degree 2 in G with u_i and u'_i as its ends for $i \in [k]$.

In Strategy 1, we analyse the structure of the extremity of u when we delete the vertex u (and selected other vertices) but we do not delete the vertex v_1 . In the resulting graph we do not count the weight change of the vertex v_1 , since the idea is to subsequently dominate the vertex v_1 by carefully selecting vertices u'_i for $i \in [k]$ on the maximal alternating **green-black** path P .

In Strategy 2, we analyse the structure of the extremity of u when we delete both vertices u and v_1 (and selected other vertices) but we do not delete the vertex u'_1 . In the resulting graph we do not count the weight change of the vertex u'_1 , since the idea is to subsequently dominate the vertex u'_1 by carefully selecting vertices u_i for $i \in [k]$ on the maximal alternating **green-black** path P . More precisely, we explain next these two strategies applied to each of the three structural types in turn.

Strategies 1 and 2 applied to a Type-(1) structure. Suppose that the extremity u belongs to a structure of Type-(1), as illustrated in Figure 14(a). Let the vertices in such a structure be named as in Section 5.6.1, where $x = u'_1$ and $u_x = v_1$. Let y' be the neighbor of y on the **green edge** incident with y .

In Strategy 1, let H be obtained from G by removing the vertices u, v, w , and y , and removing all vertices on the (vy) , (yy') , and (wz) 2-paths in G , and marking the neighbors of v and y that are not deleted. Note that we remove four vertices of degree 3 and five vertices of degree 2 when constructing H . Further, five vertices in H decrease their degrees from 3 to 2, and two of these vertices are marked in H . In Strategy 1, we do not count the cost of the vertex v_1 , whose degree has decreased from 2 to 1 when constructing H . With this assumption, where we exclude the weight change of the vertex v_1 , we have $\dot{w}(G) = \dot{w}(H) + 4 \times 4 + 5 \times 5 - 3 \times 1 = (\dot{w}(H) + 36) + 2$. Every MD-set of H can be extended to a MD-set of G by adding to it the three vertices v, y , and $(wz)_1$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta + 2$, where $\alpha = 3$ and $\beta = 36$. This results in a surplus weight of 2, which we denote by $+2$.

In Strategy 2, let H be obtained from G by removing the vertices u, v, v_1 , and w , and removing all vertices on the (vy) and (wz) 2-paths in G , and marking the vertices y and z . Note that we remove three vertices of degree 3 and five vertices of degree 2 when constructing H . Further, four vertices in H (different from u'_1) decrease their degrees from 3 to 2, and two of these vertices are marked in H . In Strategy 2, we do not count the weight change of the vertex u'_1 , whose degree has decreased from 3 to 2 when constructing H . With this assumption, where we exclude the weight change of the vertex u'_1 , we have $\dot{w}(G) = \dot{w}(H) + 3 \times 4 + 5 \times 5 - 2 \times 1 = (\dot{w}(H) + 36) - 1$. Every MD-set of H can be extended to a MD-set of G by adding to it the three vertices $u, (vy)_2$, and $(wz)_2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta - 1$, where $\alpha = 3$ and $\beta = 36$. This results in a weight shortage of 1, which we denote by -1 .

Strategies 1 and 2 applied to a Type-(2) structure. Suppose that the extremity u belongs to a structure of

Type-(2), as illustrated in Figure 14(b). Let the vertices in such a structure be named as in Section 5.6.1, where $x = u'_1$ and $u_x = v_1$. Let y' be the neighbor of y on the **green edge** incident with y .

In Strategy 1, let H be obtained from G by removing the vertices u , v , and y , and removing all vertices on the (vy) and (yy') 2-paths in G , and marking the neighbors of v and y that are not deleted. Note that we remove three vertices of degree 3 and two vertices of degree 2 when constructing H . Further, four vertices in H decrease their degrees from 3 to 2, and two of these vertices are marked in H . As before, in Strategy 1 we do not count the cost of the vertex v_1 , whose degree has decreased from 2 to 1 when constructing H . With this assumption, we have $\dot{w}(G) = \dot{w}(H) + 3 \times 4 + 3 \times 5 - 2 \times 1 = (\dot{w}(H) + 24) + 1$. Every MD-set of H can be extended to a MD-set of G by adding to it the two vertices v and y . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta + 1$, where $\alpha = 2$ and $\beta = 24$. This results in a surplus weight of 1, which we denote by +1.

In Strategy 2, let H be obtained from G by removing the vertices u , v_1 , v , and w , and removing both vertices on the (vy) 2-path in G , and marking the vertex y . Note that we remove three vertices of degree 3 and three vertices of degree 2 when constructing H . Further, four vertices in H (different from u'_1) decrease their degrees from 3 to 2, and one of these vertices is marked in H . In Strategy 2, we do not count the weight change of the vertex u'_1 , whose degree has decreased from 3 to 2 when constructing H . With this assumption, we have $\dot{w}(G) = \dot{w}(H) + 3 \times 4 + 3 \times 5 - 3 \times 1 = \dot{w}(H) + 24$. Every MD-set of H can be extended to a MD-set of G by adding to it the two vertices u and $(vy)_2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$. This results in neither a surplus weight nor a weight shortage. In this case, we say that the weight difference between G and H is *balanced*, denoted by 0.

Strategies 1 and 2 applied to a Type-(3) structure. Suppose that the extremity u belongs to a structure of Type-(3), as illustrated in Figure 14(c). Let the vertices in such a structure be named as in Section 5.6.1, where $x = u'_1$ and $u_x = v_1$. Let y' be the neighbor of y on the **green edge** incident with y .

In Strategy 1, let H be obtained from G by removing the vertices u and v , and removing all vertices on the (vy) and (uw) 2-paths in G , and marking the neighbors of v and y that are not deleted. Note that we remove two vertices of degree 3 and four vertices of degree 2 when constructing H . Further, three vertices in H decrease their degrees from 3 to 2. As before, in Strategy 1 we do not count the weight change of the vertex v_1 , whose degree has decreased from 2 to 1 when constructing H . With this assumption, we have $\dot{w}(G) = \dot{w}(H) + 2 \times 4 + 4 \times 5 - 3 \times 1 = (\dot{w}(H) + 24) + 1$. Every MD-set of H can be extended to a MD-set of G by adding to it the two vertices $(uw)_1$ and $(vy)_2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta + 1$, where $\alpha = 2$ and $\beta = 24$. This results in a surplus weight of 1, which we denote by +1.

In Strategy 2, let H be obtained from G by removing the vertices u , v_1 , $(uw)_1$, and v , and removing all vertices on the (vy) 2-path in G , and marking the vertex y . Note that we remove two vertices of degree 3 and four vertices of degree 2 when constructing H . Further, two vertices in H (different from u'_1) decrease their degrees from 3 to 2, and one of these vertices is marked in H , while one vertex decreases its degree from 2 to 1. In Strategy 2, we do not count the weight change of the vertex u'_1 . With this assumption, we have $\dot{w}(G) = \dot{w}(H) + 2 \times 4 + 4 \times 5 - 1 - 3 = \dot{w}(H) + 24$. Every MD-set of H can be extended to a MD-set of G by adding to it the two vertices u and $(vy)_2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$. This results in neither a surplus weight nor a weight shortage. In this case, we say that the weight difference between G and H is *balanced*, denoted by 0. The surplus or deficit weight resulting from applying our two strategies to the three structural types of extremities are summarized in Table 1.

	Type-(1)	Type-(2)	Type-(3)
Strategy 1	+2	+1	+1
Strategy 2	-1	0	0

Table 1: The two strategies and their corresponding surplus or deficit weights

5.6.2 Structural properties of internal vertices of black-green alternating paths

In the previous section, Section 5.6.1, we studied the structure of extremities of maximal alternating **green-black** path in M_G that start and end with **green edges**, and gave two counting strategies that we will apply to each of these three structures. In this section, we study the structure of internal vertices of a maximal alternating **green-black** path. We adopt our notation in Section 5.6.1. In particular,

$$P: u_1, u'_1, u_2, u'_2, \dots, u_k, u'_k$$

is a maximal alternating **green-black** path in M_G that starts and ends with **green edges**. We say that a vertex v is a *neighbor* of a path P if it does not belong to the path but is a neighbor of some internal vertex on the path P (different from one the extremities of P). Further, if v has two neighbors or more neighbors on P , we say that v is a *close neighbor* of the path P . We call an edge e a *neighbor* of the path P if it does not belong to the path, but is incident with some internal vertex on the path P .

Recall that in Section 5.6.1 we applied two strategies, namely Strategy 1 and Strategy 2, which yielded counting arguments to handle the extremities of the path P depending on the type of structure, namely Type-(1), -(2) or -(3) structure of the extremity and on structure of the graph G which we analyse in this section. Our initial aim is to apply Strategy 1 to one extremity of the path P and Strategy 2 to the other extremity of the path P , and to remove from G the vertices associated with the path P .

If we apply Strategy 1 to the extremity u_1 of the path P , then we wish to extend a MD-set in the resulting graph to a MD-set in the graph G by adding to it, among possibly other vertices, the $k - 1$ vertices u'_i of the path P for all $i \in [k - 1]$. In this case, the vertices u_j for all $j \in [k] \setminus \{1\}$ and the vertices of degree 2 associated with the internal vertices of **green edge** in the path P , are not added to the dominating set.

If we apply Strategy 2 to the extremity u'_k of the path P , then we wish to extend a MD-set in the resulting graph to a MD-set in the graph G by adding to it, among possibly other vertices, the $k - 1$ vertices u_i of the path P for all $i \in [k] \setminus \{k\}$. In this case, the vertices u'_j for all $j \in [k - 1]$ and the vertices of degree 2 associated with the internal vertices of **green edge** in the path P , are not added to the dominating set.

As mentioned earlier, we focus next on the structure of the internal vertices of the maximal alternating **green-black** path P . We call an edge that does not belong to the path P but joins two vertices of the path P a *well-behaved edge* with respect to the path P . Such edges will not trouble us in anyway, since when we remove the vertices of G associated with the path P , all well-behaved edges with respect to P are also removed. Hence, we may assume there are no well-behaved edges with respect to P .

In what follows, we call a vertex u'_i for some $i \in [k - 1]$ an *upper vertex*, and we call a vertex u_i for some $i \in [k] \setminus \{1\}$ a *lower vertex*. Every neighboring edge of P that is incident with an upper vertex we call an *upper edge*, while every neighboring edge of P that is incident with a lower vertex we call a *lower edge*. By our assumption that there are no well-behaved edges with respect to P , every neighboring edge of P is either upper or lower (but not both).

We consider the contribution of upper and lower edges when we remove all vertices from the path P . For this purpose, we define a set S_P^{upper} (resp., S_P^{lower}) that associates with an upper (resp., lower) neighboring edge of P a set of vertices that will be removed when we remove the vertices of the path P . We also associate a function f_P^{upper} (resp., f_P^{lower}) that determines the additional cost incurred by the removal of such vertices in S_P^{upper} or S_P^{lower} , where this function does not count the weight of the vertices of the path P (which will be counted separately when the remove the vertices of P). We consider several types of neighboring edges of the path P , and analyse each edge in turn. By Claim 16, we note that no neighboring **red edge** of the path P has both its ends belonging to the path P .

We consider first several types of neighboring edges of the path P , and analyse each edge in turn.

Red islands. We call a neighboring edge $e = uv$ of the path P a **red island** if e is a **red edge** such that one end, say u , of the edge e belongs to the path P , and the other end v of the edge e is adjacent to two additional vertices on the path P . Let e' and e'' denote the two edges incident with v that are different from the **red edge** e . By supposition, the ends of e' and e'' different from the vertex v belongs to the path P . Necessarily, both edges e' and e'' are **black edges** by Claims 16 and 18. An example of a **red island** is illustrated in Figure 15. We call the three edges e, e' and e'' the edges associated with a **red island**.

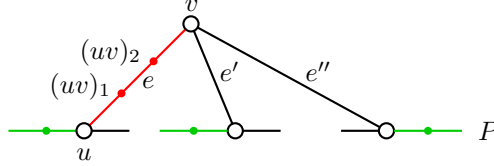


Figure 15: A red island

For each edge $e_v \in \{e, e', e''\}$ incident with v , we define both $S_P^{\text{upper}}(e_v)$ and $S_P^{\text{lower}}(e_v)$ to consist of the vertex v and both internal vertices of the path in G corresponding to the **red edge** e in M_G , that is

$$S_P^{\text{upper}}(e_v) = S_P^{\text{lower}}(e_v) = \{v, (uv)_1, (uv)_2\}.$$

Let H be obtained from G by removing the vertices $v, (uv)_1$, and $(uv)_2$ that belong to the set $S_P^{\text{upper}}(e_v)$ and $S_P^{\text{lower}}(e_v)$. Note that we remove one vertex of degree 3 and two vertices of degree 2. Not counting the weight change of the three vertices on the path P that are adjacent with v in M_G , we have $\dot{w}(G) = \dot{w}(H) + 1 \times 4 + 2 \times 5 = \dot{w}(H) + 14$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertex $(uv)_2$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta + 2$, where $\alpha = 1$ and $\beta = 12$. This results in an excess weight of 2, which we share equally among the three edges e, e' and e'' , that is, each edge is assigned a weight of $2/3$. Thus for each edge e_v incident with v in the multigraph M_G , we define

$$f_P^{\text{upper}}(e_v) = f_P^{\text{lower}}(e_v) = \frac{2}{3}.$$

Green islands. We call a **green edge** $e = uv$ with both ends adjacent to two vertices on the path P a **green island**. Let e_1, e_2, e_3 and e_4 be the four edges joining the **green island** to the path P . By Claims 15 and 16, these four edges are all **black edges**. An example of a **green island** is illustrated in Figure 16. We call the four edges e_1, e_2, e_3 and e_4 the edges associated with a **green island**.

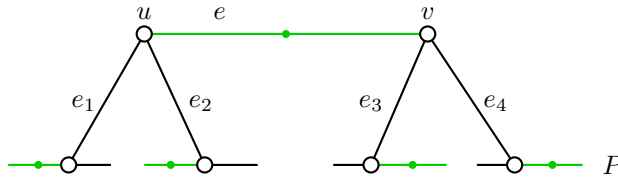


Figure 16: A green island

For each edge $e_v \in \{e_1, e_2, e_3, e_4\}$ adjacent with the **green edge** e , we define both $S_P^{\text{upper}}(e_v)$ and $S_P^{\text{lower}}(e_v)$ to consist of the vertices u, v , and the internal vertex of the path in G corresponding to the **green edge** e in M_G , that is

$$S_P^{\text{upper}}(e_v) = S_P^{\text{lower}}(e_v) = \{u, v, (uv)_1\}.$$

Let H be obtained from G by removing the vertices u , v , and $(uv)_1$ that belong to the set $S_P^{\text{upper}}(e_v)$ and $S_P^{\text{lower}}(e_v)$. Note that we remove two vertices of degree 3 and one vertex of degree 2. Not counting the weight change of the four vertices on the path P that are adjacent to u or v in M_G , we have $\dot{w}(G) = \dot{w}(H) + 2 \times 4 + 1 \times 5 = \dot{w}(H) + 13$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertex $(uv)_1$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta + 1$, where $\alpha = 1$ and $\beta = 12$. This results in an excess weight of 1, which we share equally among the four edges e_1 , e_2 , e_3 and e_4 , that is, each edge is assigned a weight of $1/4$. Thus for each edge e_v incident with v in the multigraph M_G , we define

$$f_P^{\text{upper}}(e_v) = f_P^{\text{lower}}(e_v) = \frac{1}{4}.$$

Green isthmus. We call a **green edge** $e = uv$ that is not a **green island** a **green isthmus** if one end, say u , of the edge e is adjacent to two vertices on the path P . Since e is not a **green island**, the vertex v is adjacent to at most one vertex of the path P . An example of a **green isthmus** is illustrated in Figure 17.

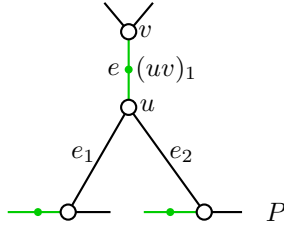


Figure 17: A **green isthmus**

Suppose that at least one of e_1 and e_2 is an upper edge, say e_1 . Thus, $e_1 = u'_i$ for some $u \in [k - 1]$. In this case, we let $S_P^{\text{upper}}(e_1) = \{u\}$, and we let $S_P^{\text{upper}}(e_2) = S_P^{\text{lower}}(e_2) = \{u\}$. Further, we let H be obtained from G by removing the vertex u of degree 3. We note that the degree of vertex $(uv)_1$ decreases from 2 to 1 when constructing H , and hence its weight increases from 5 to 8. Therefore, not counting the weight change of the vertices on the path P adjacent to u , we have $\dot{w}(G) = \dot{w}(H) + 4 - 3 = \dot{w}(H) + 1$. Every MD-set of H is a MD-set of G (noting that the vertex u is subsequently dominated when we added the vertex u_i to a MD-set of H). Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + 1$, where $\alpha = \beta = 0$. This results in an excess weight of 1, which we share among the two edges e_1 and e_2 , that is, we define

$$f_P^{\text{upper}}(e_1) = \frac{1}{2} \quad \text{and} \quad f_P^{\text{upper}}(e_2) = f_P^{\text{lower}}(e_2) = \frac{1}{2}.$$

Suppose that both e_1 and e_2 are lower edges. In this case, we let $S_P^{\text{lower}}(e_1) = S_P^{\text{lower}}(e_2) = \{u, (uv)_1\}$. Further, we let H be obtained from G by removing the vertices u and $(uv)_1$ and marking the vertex v . Note that we remove one vertex of degree 3 and one vertex of degree 2. As before, not counting the weight change of the vertices on the path P adjacent to u , we have $\dot{w}(G) = \dot{w}(H) + 5 + 4 = \dot{w}(H) + 9$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertex $(uv)_1$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta - 3$, where $\alpha = 1$ and $\beta = 12$. This results in a deficit weight of 3, which we share among the two edges e_1 and e_2 , that is, we define

$$f_P^{\text{lower}}(e_1) = f_P^{\text{lower}}(e_2) = -\frac{3}{2}.$$

Red isthmus. We call a **red edge** $e = uv$ a **red isthmus** if neither u nor v belongs to the path P , but at least one end, say u , of the edge e is adjacent to two vertices on the path P . We note that the vertex v is adjacent to zero, one or two vertices of the path P . An example of a **red isthmus** is illustrated in Figure 18.

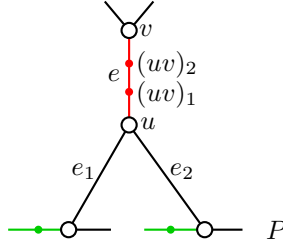


Figure 18: A **red isthmus**

We proceed now in an analogous way as in the previous case of **green isthmus**. Thus if at least one of e_1 and e_2 is an upper edge, say e_1 , then we let $S_P^{\text{upper}}(e_1) = \{u\}$, and we let $S_P^{\text{upper}}(e_2) = S_P^{\text{lower}}(e_2) = \{u\}$, and we define

$$f_P^{\text{upper}}(e_1) = \frac{1}{2} \quad \text{and} \quad f_P^{\text{upper}}(e_2) = f_P^{\text{lower}}(e_2) = \frac{1}{2}.$$

If both e_1 and e_2 are lower edges, then we let $S_P^{\text{lower}}(e_1) = S_P^{\text{lower}}(e_2) = \{u, (uv)_1\}$. In this case, we let H be obtained from G by removing the vertices u and $(uv)_1$ and marking the vertex $(uv)_2$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertex $(uv)_1$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta - 3$, where $\alpha = 1$ and $\beta = 12$. This results in a deficit weight of 3, which we share among the two edges e_1 and e_2 , that is, we define

$$f_P^{\text{lower}}(e_1) = f_P^{\text{lower}}(e_2) = -\frac{3}{2}.$$

Black detour. We call a neighboring **black edge** $e = uv$ of the path P a **black detour** if one end, say u , of the edge e belongs to the path P and the other end, v , is not adjacent to a vertex of the path, except for the vertex u . Further, the vertex v is incident with a **green edge** or a **red edge** (or both a **green edge** or a **red edge**). An example of a **black detour** is illustrated in Figure 19.

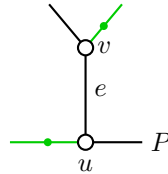


Figure 19: A **black detour**

In this case, we let $S_P^{\text{upper}}(e) = S_P^{\text{lower}}(e) = \emptyset$, and we let H be obtained from G by removing the edge e . Further, if e is an upper edge, we mark the vertex v in H . Suppose that e is an upper edge. Not counting the weight change of the vertex u on the path P , we have $\dot{w}(G) = \dot{w}(H)$ noting that in this case the vertex u is added to a MD-set of H . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H)$, where $\alpha = \beta = 0$. Hence, we define $f_P^{\text{upper}}(e) = 0$. Suppose next that e is a lower edge. In this case, noting that the degree of vertex v decreases from 3 to 2 when constructing H , we have $\dot{w}(G) = \dot{w}(H) - 1$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) - 1$, where $\alpha = \beta = 0$. This results in a deficit weight of -1 , and we define $f_P^{\text{lower}}(e) = -1$. To summarize, we have

$$S_P^{\text{upper}}(e) = S_P^{\text{lower}}(e) = \emptyset, \quad f_P^{\text{upper}}(e) = 0, \quad \text{and} \quad f_P^{\text{lower}}(e) = -1.$$

Special red edge. We call a neighboring **red edge** $e = uv$ of the path P that is not a **red island** a **special red edge** if one end, say u , of the edge e belongs to the path P . Since e is not a **red island** the vertex v is adjacent to at most one vertex of the path P in addition to the vertex u . (Possibly, the vertex v is incident with a **green edge**.) An example of a **special red edge** is illustrated in Figure 21. As mentioned earlier, our aim is to remove from G the vertices associated with the path P , and to extend a MD-set in the resulting graph to a MD-set in the original graph G by adding to it all $k - 1$ vertices u'_i (incident with upper edges) for all $i \in [k - 1]$.

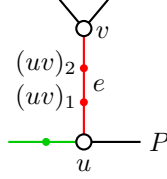


Figure 20: A **special red edge**

Special black edge. We call a neighboring **black edge** $e = uv$ of the path P a **special black edge** if one end, say u , of the edge e belongs to the path P and the other end, v , is the center of a black star. An example of a **special black edge** is illustrated in Figure 21.

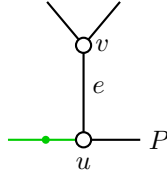


Figure 21: A **special black edge**

5.6.3 The score of a maximal alternating green-black path and its reverse

Adopting our earlier notation, consider a maximal alternating **green-black** path in M_G

$$P: u_1, u'_1, u_2, u'_2, \dots, u_k, u'_k$$

that starts and ends with **green edges**. In particular, the **green edge** $u_i u'_i$ correspond to the path u_i, v_i, u'_i in the graph G , and so v_i has degree 2 in G with u_i and u'_i as its ends for $i \in [k]$. Recall that a neighbor edge of the path P is an edge that does not belong to the path, but is incident with some internal vertex on P . By our earlier assumption, every such edge is incident with exactly one internal vertex of the path P .

Claim 33 *The path P contains a neighboring edge that is a **special red edge** or a **special black edge**.*

Proof. Suppose, to the contrary, that no neighboring edge of the path P is a **special red edge** or a **special black edge**. Thus, every neighboring edge of P belongs to exactly one of the following structures: a **red island**, a **green island**, a **green isthmus**, a **red isthmus**, or a **black detour**.

In Section 5.6.1, we analysed the structure of the extremities of maximal alternating **green-black** paths, and showed that there are three types of structures, namely Type-(1), -(2), and -(3). We presented two

strategies to apply on each of these three structural types, and we summarized in Table 1 the surplus or deficit weight resulting from applying each strategy. In order to define the score of the path P , which we denote by $\text{score}(P)$, we introduce some additional notation.

- Let t_1 be the indicator variable to indicate whether the left extremity u_1 of the path P belongs to a structure of type-(1), that is, $t_1 = 1$ if the structure containing u_1 is of Type-(1), and $t_1 = 0$ otherwise.
- Let t'_1 be the indicator variable to indicate whether the right extremity u'_k of the path P belongs to a structure of type-(1), that is, $t'_1 = 1$ if the structure containing u'_k is of Type-(1), and $t'_1 = 0$ otherwise.
- Let r_{island} be the number of **red islands**, and let g_{island} be the number of **green islands**.
- Let n_{isthmus} be the number of **red isthmus** or **green isthmus** incident with two upper edges, and let n'_{isthmus} be the number of **red isthmus** or **green isthmus** incident with two lower edges.
- Let m_{isthmus} be the number of **red isthmus** or **green isthmus** incident with one upper edge and one lower edge.
- Let b_{detour} be the number of upper edges that are **black detour** edges, and let b'_{detour} be the number of lower edges that are **black detour** edges.

We are now in a position to define the score of the path P , which recall is denoted by $\text{score}(P)$.

- Initially, we let $\text{score}(P) = 0$.
- Applying Strategy 1 to the extremity u_1 of the path P results in a surplus weight 2, 1 or 1 depending on whether u_1 belongs to a structure of Type-(1), -(2), and -(3), respectively (see Table 1.) This adds $1 + t_1$ to $\text{score}(P)$.
- Applying Strategy 2 to the extremity u'_k of the path P results in a deficit weight of 1 if u'_k belongs to a Type-(1) structure, and otherwise it results in a balanced weight of 0 (see Table 1). This removes t'_1 from $\text{score}(P)$.
- Removing all vertices of the path P , we add to a MD-set of the resulting graph all the upper vertices of P , that is, the $k - 1$ vertices u'_i where $i \in [k - 1]$. For each such vertex u'_i we can uniquely associate its two neighbors v_i and u_{i+1} on the path P . The sum of the weights of these three vertices v_i, u'_i, u_{i+1} is $2 \times 4 + 1 \times 5 = 13$, which results in a surplus weight of 1 for each $i \in [k - 1]$. This adds $k - 1$ to $\text{score}(P)$.
- For each **red island**, we add 2 to $\text{score}(P)$, while for each **green island**, we add 1 to $\text{score}(P)$. (See the earlier discussion on the counting associated with **red islands** and **green islands**.) This adds $2r_{\text{island}} + g_{\text{island}}$ to $\text{score}(P)$.
- For each **red isthmus** and **green isthmus** incident with two upper edges, we add 1 to $\text{score}(P)$, while for each **red isthmus** and **green isthmus** incident with two lower edges we remove 3 from $\text{score}(P)$. Further, for each **red isthmus** and **green isthmus** incident with one upper edge and one lower edge, we add 1 to $\text{score}(P)$. This adds $n_{\text{isthmus}} + m_{\text{isthmus}} - 3n'_{\text{isthmus}}$ to $\text{score}(P)$.
- For each **black detour** incident with a lower edge, we remove 1 from $\text{score}(P)$.

We note that $r_{\text{island}} \geq 0$, $g_{\text{island}} \geq 0$, and $m_{\text{isthmus}} \geq 0$. Thus, the score of the path P is given by

$$\begin{aligned} \text{score}(P) &= (1 + t_1) - t'_1 + (k - 1) + 2r_{\text{island}} + 2g_{\text{island}} \\ &\quad + (n_{\text{isthmus}} + m_{\text{isthmus}} - 3n'_{\text{isthmus}}) - b'_{\text{detour}} \\ &\geq t_1 - t'_1 + k + (n_{\text{isthmus}} - 3n'_{\text{isthmus}}) - b'_{\text{detour}}. \end{aligned} \tag{1}$$

By the *reverse* of the path P , denoted \overleftarrow{P} , we mean the path P in the reverse direction that starts at u'_k and ends at u_1 , that is,

$$\overleftarrow{P}: u'_k, u_k, u'_{k-1}, u_{k-1}, \dots, u'_2, u_2, u'_1, u_1.$$

We note that the roles of the vertices u_i and u'_i are reversed when we consider the reverse, \overleftarrow{P} , of the path P , as are the roles of the upper and lower edges, noting that in this case we apply Strategy 1 to the extremity u'_k of \overleftarrow{P} , and Strategy 2 to the extremity u_1 of \overleftarrow{P} . Thus when we remove all vertices of the path \overleftarrow{P} , we now add to a MD-set of the resulting graph the $k-1$ vertices u_i where $i \in [k] \setminus \{1\}$, etc. The score of the path \overleftarrow{P} is given by

$$\begin{aligned} \text{score}(\overleftarrow{P}) &= (1 + t'_1) - t_1 + (k-1) + 2r_{\text{island}} + 2g_{\text{island}} \\ &\quad + (n'_{\text{isthmus}} + m_{\text{isthmus}} - 3n_{\text{isthmus}}) - b_{\text{detour}} \\ &\geq t'_1 - t_1 + k + (n'_{\text{isthmus}} - 3n_{\text{isthmus}}) - b_{\text{detour}}. \end{aligned} \tag{2}$$

Claim 33.1 $\text{score}(P) + \text{score}(\overleftarrow{P}) \geq 2$.

Proof. By Equation (1) and (2), we have

$$\text{score}(P) + \text{score}(\overleftarrow{P}) \geq 2k - 2n_{\text{isthmus}} - 2n'_{\text{isthmus}} - b_{\text{detour}} - b'_{\text{detour}}. \tag{3}$$

Recall that an upper vertex of the path P is a vertex u'_i for some $i \in [k-1]$, and a lower vertex of P is a vertex u_i for some $i \in [k] \setminus \{1\}$. Let $m_{\text{upper}}(P)$ and $m_{\text{lower}}(P)$ denote the number of upper and lower edges, respectively, of P . We note that

$$k-1 = m_{\text{upper}}(P) \geq 2n_{\text{isthmus}} + b_{\text{detour}}$$

and

$$k-1 = m_{\text{lower}}(P) \geq 2n'_{\text{isthmus}} + b'_{\text{detour}}.$$

Thus,

$$2k \geq 2 + 2n_{\text{isthmus}} + b_{\text{detour}} + 2n'_{\text{isthmus}} + b'_{\text{detour}}. \tag{4}$$

Hence, by Equations (3) and (4), we have $\text{score}(P) + \text{score}(\overleftarrow{P}) \geq 2$. \square

As an immediate consequence of Claim 33.1, we have the following result.

Claim 33.2 *If the score of the path P is negative, then the score of \overleftarrow{P} is positive.*

Interchanging the names of the path P and the reverse of \overleftarrow{P} if necessary, we may assume by Claim 33.2 that $\text{score}(P) \geq 1$. This implies that there exists a set R of vertices of G whose removal from G produces a graph $H = G - R$ such that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta + 1$, where $\beta = 12\alpha$, contradicting Fact 1. This completes the proof of Claim 33. \square

By Claim 33, the path P contains a neighboring edge that is a **special red edge** or a **special black edge**. Let v_e be the first vertex of the path P that is incident with a special edge, and let e be the special edge incident with v_e . Thus, $v_e = u'_q$ for some $q \in [k-1]$ or $v_e = u_q$ for some $q \in [k] \setminus \{1\}$. We consider the two cases in turn.

Case 1. $v_e = u'_q$ for some $q \in [k-1]$. In this case, we consider the path

$$Q: u_1, u'_1, u_2, u'_2, \dots, u_q, u'_q.$$

We proceed as in the proof of Claim 33, except that the score of Q , which we denote by $\text{score}(Q)$, differs only on the extremity u'_q of the path Q . We apply Strategy 1 to the extremity u_1 of the path Q , and we remove all vertices of the path Q , and add to a MD-set of the resulting graph all the upper internal vertices of Q , that is, the $q-1$ vertices u'_i where $i \in [q-1]$.

By the minimality of the vertex $v_e = u'_q$, no upper or lower edge incident with an upper or lower internal vertex of the path Q is a **special red edge** or a **special black edge**. Our scoring associated with internal vertices of the path Q therefore proceeds exactly as in the proof of Claim 33, except for the extremity u'_q of the path Q which we discuss next. Let v'_e be the neighbor of v_e in M_G that is incident with the special edge e .

Case 1.1 The edge e is a **special red edge**. In this case, we remove the vertex $(v_e v'_e)_1$, and we mark the vertex u_{q+1} . We can uniquely associate the vertices v_q and $(v_e v'_e)_1$ with the vertex u'_q . The sum of the weights of the three vertices $v_q, u'_q, (v_e v'_e)_1$ is $1 \times 4 + 2 \times 5 = 14$, which results in a surplus weight of 2 since we will add the vertex u'_q to the MD-set (which decreases the weight by 12). However, the degree of the vertex $(v_e v'_e)_2$ decreases from 2 to 1 with the removal of the vertex $(v_e v'_e)_1$, and this decreases the weight by 3. The net gain/loss to the score of Q is $+2 - 3 = -1$. Thus, the score of the path Q is given by

$$\begin{aligned} \text{score}(Q) &= (1 + t_1) - 1 + (q - 1) + 2r_{\text{island}} + 2g_{\text{island}} \\ &\quad + (n_{\text{isthmus}} + m_{\text{isthmus}} - 3n'_{\text{isthmus}}) - b'_{\text{detour}} \\ &\geq t_1 + q - 1 + (n_{\text{isthmus}} - 3n'_{\text{isthmus}}) - b'_{\text{detour}}, \end{aligned} \tag{5}$$

where t_1 is as defined as before, and where the other parameters above, namely $r_{\text{island}}, g_{\text{island}}, n_{\text{isthmus}}, n'_{\text{isthmus}}$, etc., are defined analogously as before but restricted in their definition to the path Q . We now consider the reverse of the path Q , denoted \overleftarrow{Q} , that is,

$$\overleftarrow{Q}: u'_q, u_q, u'_{q-1}, u_{q-1}, \dots, u'_1, u_1.$$

As before, we apply Strategy 2 to the extremity u_1 , and we remove all vertices from the path Q , and in this case add to a MD-set of the resulting graph all the lower internal vertices of Q , that is, the $q-1$ vertices u'_i where $i \in [q] \setminus \{1\}$. We note that in Strategy 2 the vertex u_1 is added to the MD-set. Once again, our scoring associated with internal vertices of the path Q therefore proceeds exactly as in the proof of Claim 33, except for the extremity u'_q of the path Q which we discuss next.

We remove the vertices $(v_e v'_e)_1$ and $(v_e v'_e)_2$, and add the vertex $(v_e v'_e)_1$ to the MD-set. We can uniquely associate the vertices $(v_e v'_e)_2$ and u'_q with the vertex $(v_e v'_e)_1$. The sum of the weights of the three vertices $u'_q, (v_e v'_e)_1, (v_e v'_e)_2$ is $1 \times 4 + 2 \times 5 = 14$, which results in a surplus weight of 2, since we will add the vertex $(v_e v'_e)_1$ to the MD-set (which decreases the weight by 12). However, the degree of the vertices v'_e and u_{q+1} both decrease from 3 to 1 with the removal of the vertices $(v_e v'_e)_1$ and $(v_e v'_e)_2$, and this decreases the weight by 2. The net gain/loss to the score of Q is $+2 - 2 = 0$, and so the extremity u'_q contributes 0 to the score of \overleftarrow{Q} . The score of the path \overleftarrow{Q} is given by

$$\begin{aligned} \text{score}(\overleftarrow{Q}) &= -t_1 + (q - 1) + 2r_{\text{island}} + 2g_{\text{island}} \\ &\quad + (n'_{\text{isthmus}} + m_{\text{isthmus}} - 3n_{\text{isthmus}}) - b_{\text{detour}} \\ &\geq -t_1 + q - 1 + (n'_{\text{isthmus}} - 3n_{\text{isthmus}}) - b_{\text{detour}}. \end{aligned} \tag{6}$$

By Inequalities (5) and (6), we have

$$\text{score}(Q) + \text{score}(\overleftarrow{Q}) \geq 2q - 2 - 2n_{\text{isthmus}} - 2n'_{\text{isthmus}} - b_{\text{detour}} - b'_{\text{detour}}. \quad (7)$$

Let $m_{\text{upper}}(Q)$ and $m_{\text{lower}}(Q)$ denote the number of upper and lower (neighboring) edges, respectively, of internal vertices of Q . We note that $q - 1 = m_{\text{upper}}(Q) \geq 2n_{\text{isthmus}} + b_{\text{detour}}$ and $q - 1 = m_{\text{lower}}(Q) \geq 2n'_{\text{isthmus}} + b'_{\text{detour}}$, and so

$$2q - 2 \geq 2n_{\text{isthmus}} + b_{\text{detour}} + 2n'_{\text{isthmus}} + b'_{\text{detour}}. \quad (8)$$

Hence, by Inequalities (7) and (8), we have $\text{score}(Q) + \text{score}(\overleftarrow{Q}) \geq 0$. Thus, $\text{score}(Q) \geq 0$ or $\text{score}(\overleftarrow{Q}) \geq 0$. This implies that there exists a set R of vertices of G whose removal from G produces a graph $H = G - R$ such that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\beta = 12\alpha$, contradicting Fact 1.

*Case 1.2 The edge e is a **special black edge**.* In this case, we remove the vertex v'_e , and we mark the vertex u_{q+1} . We can uniquely associate the vertices v_q and v'_e with the vertex v_e (where recall that here $v_e = u'_q$). The sum of the weights of these three vertices is $1 \times 5 + 2 \times 4 = 13$, which results in a surplus weight of 1 since we will add the vertex u'_q to the MD-set (which decreases the weight by 12). However, the removal of the vertex v'_e decreases the weight of its two neighbors different from v_e by $2 \times -1 = -2$. The net gain/loss to the score of Q is $+1 - 2 = -1$. Thus, the extremity u'_q contributes -1 to the score of Q , which is the same as the previous case when e is a **special red edge**. Hence, the score of the path Q again satisfies Inequality (5), that is,

$$\text{score}(Q) \geq t_1 + q - 1 + (n_{\text{isthmus}} - 3n'_{\text{isthmus}}) - b'_{\text{detour}}. \quad (9)$$

The score for the reverse \overleftarrow{Q} of the path Q is as in Case 1.1 when e is a **special red edge**, except for the extremity u'_q of the path Q . In this case, we do not remove the vertex u'_q from the path Q . The degree of u'_q decreases from 3 to 2, and results in the removal of -1 from the score of \overleftarrow{Q} . Thus, the score of \overleftarrow{Q} is one smaller than in Case 1.1, implying that

$$\text{score}(\overleftarrow{Q}) \geq -t_1 + q - 2 + (n'_{\text{isthmus}} - 3n_{\text{isthmus}}) - b_{\text{detour}}. \quad (10)$$

By Inequalities (9) and (10), we have

$$\text{score}(Q) + \text{score}(\overleftarrow{Q}) \geq 2q - 3 - 2n_{\text{isthmus}} - 2n'_{\text{isthmus}} - b_{\text{detour}} - b'_{\text{detour}}. \quad (11)$$

Hence, by Inequalities (11) and (8), we have $\text{score}(Q) + \text{score}(\overleftarrow{Q}) \geq -1$. Since $\text{score}(Q)$ and $\text{score}(\overleftarrow{Q})$ are integer values, we deduce that $\text{score}(Q) \geq 0$ or $\text{score}(\overleftarrow{Q}) \geq 0$. This implies that there exists a set R of vertices of G whose removal from G produces a graph $H = G - R$ such that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\beta = 12\alpha$, contradicting Fact 1.

Case 2. $v_e = u_q$ for some $q \in [k] \setminus \{1\}$. In this case, we consider the path

$$Q: u_1, u'_1, u_2, u'_2, \dots, u_{q-1}, u'_{q-1}, u_q.$$

By the minimality of the vertex $v_e = u_q$, no upper or lower edge incident with an upper or lower internal vertex of the path Q is a **special red edge** or a **special black edge**. We proceed as in the proof of Claim 33, except that the score of Q differs on the extremity u'_q of the path Q . We apply Strategy 2 to the extremity u_1

of the path Q , and we remove all vertices of the path Q , and add to a MD-set of the resulting graph all the lower internal vertices of Q , that is, the $q - 2$ vertices u_i where $i \in [q - 1] \setminus \{1\}$. We note that in Strategy 2 the vertex u_1 is added to the MD-set. Let v'_e be the neighbor of v_e in M_G that is incident with the special edge e . We consider two cases, depending on whether the edge e is a **special red edge** or a **special black edge**.

*Case 2.1 The edge e is a **special red edge**.* To take care of the extremity u_q of the path Q , we remove the vertices $(v_e v'_e)_1$ and $(u_q u'_q)$ (where recall that $v_e = u_q$). We can uniquely associate the vertices u'_{q-1} , $(v_e v'_e)_1$ and $(u_q u'_q)$ with the vertex u_q . The sum of the weights of these four vertices is $2 \times 4 + 2 \times 5 = 18$, which results in a surplus weight of 6 since we will add the vertex u_q to the MD-set (which decreases the weight by 12). However, removing these four vertices decrease the degree of $(v_e v'_e)_2$ from 2 to 1, and decreases the degrees of u'_q and the neighbor of u'_{q-1} not on the path Q from 3 to 2. This decreases the weight by $-1 - 1 - 3 = -5$. The net gain to the score of Q is $6 - 5 = 1$. Thus, the score of the path Q is given by

$$\begin{aligned} \text{score}(Q) &= -t_1 + 1 + (q - 2) + 2r_{\text{island}} + 2g_{\text{island}} \\ &\quad + (n'_{\text{isthmus}} + m_{\text{isthmus}} - 3n_{\text{isthmus}}) - b_{\text{detour}} \\ &\geq -t_1 + q - 1 + (n'_{\text{isthmus}} - 3n_{\text{isthmus}}) - b_{\text{detour}}. \end{aligned} \tag{12}$$

We now consider the reverse of the path Q , namely the path

$$\overleftarrow{Q} : u_q, u'_{q-1}, u_{q-1}, \dots, u'_1, u_1.$$

We now apply Strategy 1 to the extremity u_1 , and we remove all vertices from the path Q , except for the vertex u_q . In this case add to a MD-set of the resulting graph all the upper internal vertices of Q , that is, the $q - 1$ vertices u'_i where $i \in [q] \setminus \{1\}$. The degree of u_q decreases from 3 to 2, and results in the removal of -1 from the score of \overleftarrow{Q} . The score of the path \overleftarrow{Q} is given by

$$\begin{aligned} \text{score}(\overleftarrow{Q}) &= (1 + t_1) - 1 + (q - 1) + 2r_{\text{island}} + 2g_{\text{island}} \\ &\quad + (n_{\text{isthmus}} + m_{\text{isthmus}} - 3n'_{\text{isthmus}}) - b'_{\text{detour}} \\ &\geq t_1 + q - 1 + (n_{\text{isthmus}} - 3n'_{\text{isthmus}}) - b'_{\text{detour}}. \end{aligned} \tag{13}$$

By Inequalities (12) and (13), we have

$$\text{score}(Q) + \text{score}(\overleftarrow{Q}) \geq 2q - 2 - 2n_{\text{isthmus}} - 2n'_{\text{isthmus}} - b_{\text{detour}} - b'_{\text{detour}}. \tag{14}$$

Let $m_{\text{upper}}(Q)$ and $m_{\text{lower}}(Q)$ denote the number of upper and lower (neighboring) edges, respectively, of internal vertices of Q . We note that $q - 1 = m_{\text{upper}}(Q) \geq 2n_{\text{isthmus}} + b_{\text{detour}}$ and $q - 2 = m_{\text{lower}}(Q) \geq 2n'_{\text{isthmus}} + b'_{\text{detour}}$, and so

$$2q - 2 \geq 1 + 2n_{\text{isthmus}} + b_{\text{detour}} + 2n'_{\text{isthmus}} + b'_{\text{detour}}. \tag{15}$$

Hence, by Inequalities (14) and (15), we have $\text{score}(Q) + \text{score}(\overleftarrow{Q}) \geq 1$. Thus, $\text{score}(Q) \geq 1$ or $\text{score}(\overleftarrow{Q}) \geq 1$. This implies that there exists a set R of vertices of G whose removal from G produces a graph $H = G - R$ such that $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\beta = 12\alpha$, contradicting Fact 1.

*Case 2.2 The edge e is a **special black edge**.* In this case, to take care of the extremity u_q of the path Q , we remove the vertices v'_e and $(u_q u'_q)$ (where recall that $v_e = u_q$). We can uniquely associate the vertices u'_{q-1} , v'_e and $(u_q u'_q)$ with the vertex u_q . The sum of the weights of these four vertices is $3 \times 4 + 1 \times 5 = 17$,

which results in a surplus weight of 5 since we will add the vertex u_q to the MD-set (which decreases the weight by 12). However, removing these four vertices decrease the degrees of four neighboring vertices (the vertex u'_q , the neighbor of u'_{q-1} not on the path Q , and the two neighbors of v'_e different from v_e) from 3 to 2. This decreases the weight by -4 . The net gain to the score of Q is $5 - 4 = 1$. Thus, the score of the path Q is again given by Inequality (12).

The score for the reverse \overleftarrow{Q} of the path Q is exactly as in Case 2.1 when e is a **special red edge**, except for the extremity u_q of the path Q . Thus, we apply Strategy 1 to the extremity u_1 , and we remove all vertices from the path Q , except for the vertex u_q , and we add to a MD-set of the resulting graph all the upper internal vertices of Q , that is, the $q - 1$ vertices u'_i where $i \in [q] \setminus \{1\}$. As before, Inequalities (13), (14) and (15) hold, and we contradict contradicting Fact 1.

We have therefore proven the following result.

Claim 34 *Every maximal alternating **green-black** path in M_G is a cycle.*

5.7 Maximal alternating green-black cycles

In this section, we consider alternating **green-black** cycles in M_G . Let

$$C: u_1, u'_1, u_2, u'_2, \dots, u_k, u'_k$$

be an alternating **green-black** cycle in M_G , and so the edges $u_i u'_i$ are **green edges** for $i \in [k]$ and edges $u'_i u_{i+1}$ are **black edges** for $i \in [k]$ where addition is taken modulo k . Adopting our earlier notation, we let the **green edge** $u_i u'_i$ correspond to the path u_i, v_i, u'_i in the graph G , and so v_i has degree 2 in G with u_i and u'_i as its ends for $i \in [k]$. We note that $v_i = (u_i u'_i)_1$ for $i \in [k]$. The cycle C when $k = 4$ is illustrated in Figure 22.

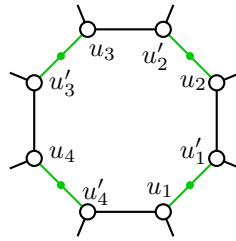


Figure 22: An alternating **green-black** cycle

We adopt our notation employed in the proof of Claim 33, except that we apply the notation to the cycle C rather than the path P . In particular, a vertex u'_i is an *upper vertex* and a vertex u_i is a *lower vertex* for all $i \in [k]$. Every neighboring edge of C that is incident with an upper vertex we call an *upper edge*, while every neighboring edge of C that is incident with a lower vertex we call a *lower edge*. An edge e is a *neighbor* of the cycle C if it does not belong to the cycle C , but is incident with a vertex on the cycle. Recall that a neighboring **black edge** $e = uv$ of the cycle C is a **special black edge** if one end, say u , of the edge e belongs to the cycle C and the other end, v , is the center of a black star.

In the following, we consider a coloring of the **green edges** in the **green-black** cycle C . This coloring uses two colors, **Amber** and **Blue**, and will be decided in Section 5.7.2. We now define the sets A and B of vertices in the **green-black** cycle C as follows.

Definition 11 We define the set A to consist of the lower vertices in C that belong to Amber edges and the upper vertices in C that belong to Blue edges, and, analogously, we define the set B to consist of the upper vertices in C that belong to Amber edges and the lower vertices in C that belong to Blue edges. We will also refer to vertices in the set A as *amber vertices* and vertices in the set B as *blue vertices*, thereby producing an amber-blue coloring of the vertices of C .

We note that the sets A and B partition the vertices of the green-black cycle C , and so $A \cup B$ is the set of all upper and lower vertices of C . Further, one end of every green edge in the cycle C belongs to the set A and the other end belongs to the set B . However, a black edge of C may possibly have both its ends in the set A (resp., B), and the amber-blue coloring of the vertices of C is not necessarily a proper coloring.

Definition 12 When a black edge is incident to two green edges of different colors (one Amber and one Blue), we say the black edge is a *color change*. The extremities of this black edge are said to be *dotted vertices*.

Given a black edge corresponding to a change of color $u'_i u_{i+1}$, if the green edge incident to the lower vertex u_{i+1} is Amber and the green edge incident to the upper vertex u'_i is Blue, then both extremities of the black edge belong to A , and neither is dominated when choosing the set B . Conversely, if the green edge incident to the lower vertex u_{i+1} is Blue and the green edge incident to the upper vertex u'_i is Amber, then both extremities of the black edge belong to B , and neither is dominated when choosing the set A . Note that there necessarily are the same number of dotted amber and dotted blue vertices, which is equal to the number of change of colors.

When the set A is chosen, we delete the vertices in A and the blue vertices dominated by A . However, we do not delete the dotted blue vertices. Analogously, when the set B is chosen, we delete the vertices in B and the amber vertices dominated by B . However, we do not delete the dotted amber vertices. In both cases, since all dotted vertices are initially of degree 3, we lose a contribution of 4 for each change of color.

We are now in a position to explain the updated counting associated with the amber-blue coloring of the vertices of the green-black cycle C . Our strategy is to remove the set A of vertices (colored amber) and all vertices in B (colored blue) on C dominated by the set A , together with the internal vertex of degree 2 from every green edge of C , noting that every such vertex is dominated by the set A in the graph G . However, we do not delete the dotted blue vertices (noting that these vertices are not dominated by the set A). In addition, we delete or mark all neighboring vertices of the cycle C that are dominated by the set A . Therefore in the graph that results from choosing the set A and applying the above rules, every dotted blue vertex, which originally had degree 3 in G , is now a vertex of degree 2, and this decrease in its degree will contribute -1 to the overall cost of selecting the set A . However if in this resulting graph, a dotted blue vertex becomes marked (which occurs if one of its neighbors outside the cycle C is added to the dominating set A), then the contribution of the dotted blue vertex in the resulting graph is 0 noting that the weight of a degree 3 vertex and a marked vertex is the same, that is, we save 1 in our counting for each dotted blue vertex in the resulting graph that becomes marked. We will frequently use this fact in our counting arguments.

Definition 13 The *Green cycle graph* is a subgraph of G consisting of an alternating green-black cycle, together with some extra artifacts referred to as *links* or *fibers*, respectively, joining two vertices (called the extremities of the link) on the cycle or attached to a single extremity. Each added link corresponds to a path of length 2 between its extremities in the original graph, where the central vertex is the center of a black star. Each fiber corresponds to a black star attached to the fiber's extremity (as in Claim 23).

A link is said to be *well colored* if it has one end in A and one end in B . It is said to be *badly colored* if both ends are in A or both in B . If any extremity of a link is dotted, then we say the link is dotted. Similarly, a fiber is dotted if its extremity is dotted.

5.7.1 Counting arguments associated with the green-black cycle

In this section, we perform counting arguments, according to the different structures associated with the green-black cycle C . The strategy is then to show that the counting when choosing at least one of the sets A and B is nonnegative. Since the graph G is a minimum counterexample, we apply the desired result to the graph resulting from choosing one of the sets A or B that yields the maximum weight gain (which is nonnegative).

We consider the different possible structures associated with the green-black cycle C . We usually make an assumption that some vertex in the cycle is Amber, the other case being symmetrical. Then, we represent the resulting scoring by $[a \mid b]$, where the first entry a is the score when A is played, and the second entry b is the score when B is played. We say $[a \mid b]$ is at least $[c \mid d]$ if $a \geq c$ and $b \geq d$.

At the end, we use an overall argument stating that since the sum of the scores when A is played and when B is played is nonnegative, one of these choices must be nonnegative. Anticipating this argument, we use an *average score* for a and b , and we simplify the notation $[a \mid b]$ into $[\frac{a+b}{2}]$. We also use addition and comparison and average scores, since these operations are consistent in regards of the sign of the final score.

In the following, results are stated for the average score, but the proof details use the scores $[a \mid b]$.

Claim 35 *The average score of a red island is at least $[\frac{1}{2}]$.*

Proof. Consider a red island shown in Figure 23, where v' and v'' are the ends of the edge e' and e'' , respectively, different from the vertex v .

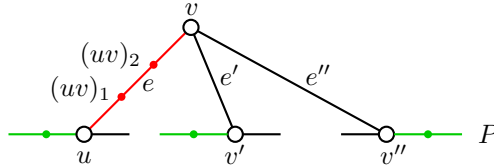


Figure 23: A red island

For counting purposes, suppose that the vertex u is colored amber (possibly, dotted), and so $u \in A$. We consider two possibilities, depending on the color of v' and v'' .

Claim 35.1 *If v' or v'' is colored blue, then the average score is at least $[1]$.*

Proof. Suppose that v' or v'' , say v' , is colored blue. Let H be obtained from G by removing the vertices v , $(uw)_1$, and $(uw)_2$. If the set A is chosen, then we add the vertex v to the dominating set A , and mark any dotted blue neighbor of v , if it exists. The score we obtain is $+5 + 5 + 4 - 12 + \ell = +2 + \ell$ where ℓ stands for the number of dotted blue neighbors of v . If the set B is chosen, then we add the vertex $(uw)_1$ to the dominating set B , and mark the vertex u if it is a dotted amber vertex. We remove the vertices v , $(uw)_1$, and $(uw)_2$. If v'' is a dotted amber vertex, we also cut the edge vv'' , resulting in the vertex v'' having degree 1, which costs an extra -3 . This yields a score of at least $+4 + 5 + 5 + 1 - 3 - 12 = 0$. Thus, the score if v' or v'' is in B is at least $[2 \mid 0]$. \square

Claim 35.2 *If both v' and v'' are colored amber, then the average score is at least $[\frac{1}{2}]$.*

Proof. Suppose that both v' and v'' are colored amber. If the set A is chosen, then we let H be obtained from G by removing the vertices v , $(uv)_1$, and $(uv)_2$, and adding the vertex v to the dominating set A , resulting in a score of 2. Suppose the set B is chosen. We consider three possible subcases.

Suppose that the vertex u is a dotted amber vertex, and that at least one of v' and v'' is a dotted amber vertex. In this case, we do nothing in the sense that we do not delete any of the vertices v , $(uv)_1$, and $(uv)_2$. This contributes 0 to the count, either because all three vertices are dotted, or if only one of v' and v'' is not dotted, then the cutting of the corresponding edge was already taken into account. In this case, the score is $[2 \mid 0]$.

Suppose that the vertex u is not a dotted amber vertex. In this case, we let H be obtained from G by removing the vertices $(uv)_1$, and $(uv)_2$, and adding the vertex $(uv)_2$ to the dominating set B , and marking the vertex v . We gain $2 \times 5 = 10$ from deleting the vertices $(uv)_1$ and $(uv)_2$, and we lose 12 from adding the vertex $(uv)_2$ to the dominating set. Moreover, we save the cutting of the outgoing edge joining u and $(uv)_1$, gaining an extra 1. The overall contribution is $10 - 12 + 1 = -1$, yielding the score $[2 \mid -1]$.

Suppose that the vertex u is a dotted amber vertex, and none of v' and v'' is a dotted amber vertex. In this case, we let H be obtained from G by removing the vertices v and $(uv)_2$, and adding the vertex $(uv)_2$ to the dominating set B , and marking the vertex $(uv)_1$. We gain 9 from deleting the vertices $(uv)_2$ and v , and we lose 12 from adding the vertex $(uv)_2$ to the dominating set. Moreover, we save the cutting of the outgoing edges joining v to v' and v'' , gaining an extra 2. We also gain 1 for marking the vertex $(uv)_1$ which was a vertex of degree 2. The overall contribution is $9 - 12 + 2 + 1 = 0$, yielding the score $[2 \mid 0]$.

Thus when both v' and v'' are colored amber, the score is at least $[2 \mid -1]$, yielding an average score of a **red island** of at least $[\frac{1}{2}]$. \square

In summary, by Claims 35.1 and 35.2, the average score of a **red island** is at least $[\frac{1}{2}]$. \square

Claim 36 *The average score of a **green island** is at least $[0]$.*

Proof. Consider a **green island** shown in Figure 24, where u_1 and u_2 are the ends of the edge e_1 and e_2 , respectively, different from the vertex u , and where v_3 and v_4 are the ends of the edge e_3 and e_4 , respectively, different from the vertex v . Let $W = \{u_1, u_2, v_3, v_4\}$.

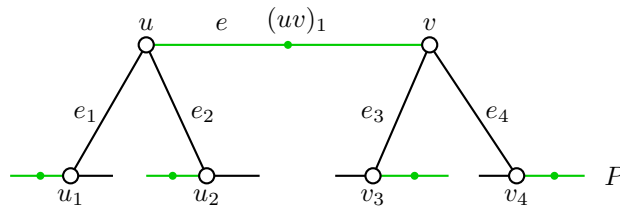


Figure 24: A green island

Suppose that no vertex in W is a dotted amber or dotted blue vertex. In this case, let H be obtained from G by removing the vertices u , v , and $(uv)_1$. If the set A (resp., B) is chosen, we add the vertex $(uv)_1$ to the dominating set A (resp., B), resulting in an excess weight of at least $2 \times 4 + 5 - 12 = 1$. We also increase the score by saving from cutting the edges e_1 , e_2 , e_3 and e_4 , but in all cases, the score is always at least $[1 \mid 1]$. In view of these observations, we may assume that at least one vertex in W is a dotted vertex. By symmetry and for counting arguments, we may assume that the vertex u_1 is a dotted amber vertex. In particular, $u_1 \in A$. We now consider a few possibilities that may occur.

Claim 36.1 *If at least one of v_3 and v_4 is a dotted amber vertex, then the average score is at least $[0]$.*

Proof. Suppose that at least one of v_3 and v_4 , say v_3 , is a dotted amber vertex. Suppose firstly that the set A is chosen. Suppose u_2 or v_4 , say v_4 by symmetry, is amber. In this case, let H be obtained from G by removing the vertices u , v , and $(uv)_1$, adding the vertex u to the set A , and marking the vertex u_2 if it is a dotted blue vertex. This results in an excess weight of $13 - 12 = 1$ if u_2 is not a blue vertex, and an excess weight of $13 - 12 + 1 = 2$ if u_2 is a blue vertex. Suppose u_2 and v_4 are both blue. (Possibly, both u_2 and v_4 are dotted blue vertices.) In this case, we do not delete any of the vertices u , v , and $(uv)_1$, but we mark both u and v . This contributes 0 to the count.

Suppose next that the set B is chosen. In this case, we do not remove any of the vertices u , v , and $(uv)_1$. We mark u if u_2 is blue and we mark v if v_4 is blue. If the vertex u_2 is an amber vertex that is not dotted, then we gain +1 for the deletion of the outgoing edge uu_2 and we lose 1 for the degree of u dropping from 3 to 2. The net contribution balances out to 0. Hence in this case, the contribution to the count is 0. As observed earlier, the contribution to the count when A is chosen is 0. Thus, the score is at least $[0 \mid 0]$. \square

Claim 36.2 *If neither v_3 nor v_4 is a dotted amber vertex, and at least one of v_3 and v_4 is blue, then the average score is at least $[1]$.*

Proof. Suppose that neither v_3 nor v_4 is a dotted amber vertex, and at least one of v_3 and v_4 is blue. Suppose firstly that the set B is chosen. In this case, let H be obtained from G by removing the vertices u , v , and $(uv)_1$, adding the vertex u to the set B , and marking its amber dotted neighbors on C . We gain $2 \times 4 + 5 = 13$ by removing the vertices u , v , and $(uv)_1$, and we lose -12 by adding the vertex u to the set B , resulting in nett gain of +1. Further, for each amber neighbor of u on C that is dotted, we gain +1 from such a dotted amber neighbor noting that the degree drops from 3 to 2, while for each amber neighbor of u on C that is not dotted, we gain +1 from the outgoing edge from u to that vertex. Hence we have an excess weight of at least 2 if u_2 is a blue vertex, and an excess weight of at least 3 if u_2 is an amber vertex (from marking it if it is dotted, or deleting the outgoing edge uu_2 if it is not dotted). Hence, the contribution to the count when the set B is chosen is at least 2.

Suppose next that the set A is chosen. If u_2 is not a dotted blue vertex, then let H be obtained from G by removing the vertices u , v , and $(uv)_1$, adding the vertex v to the set A , and marking its blue dotted neighbors on C , if any. This results in an excess weight of at least +1. If u_2 is a dotted blue vertex and at least one of v_3 and v_4 is an amber vertex, then we mark the vertices u and v , yielding a contribution of 0 to the count. If u_2 is a dotted blue vertex and both v_3 and v_4 are blue vertices, then let H be obtained from G by removing the vertices u , v , and $(uv)_1$, adding the vertex v to the set A , and marking its blue dotted neighbors on C , if any. The vertices v_3 and v_4 both contribute 1 to the count, either by being marked dotted vertices, or by saving the cost of cutting the outgoing edges to v . Cutting the edge e_2 cost 3 extra since it makes of the blue dotted vertex u_2 a degree 1 vertex. This results in a contribution of $2 \times 4 + 5 + 2 \times 1 - 12 - 3 = 0$, which counts 0 to the overall count. Hence, the contribution to the count when the set A is chosen is at least 0. As observed earlier, the contribution to the count when B is chosen is at least 2. Thus, the score is at least $[0 \mid 2]$. \square

Claim 36.3 *If neither v_3 nor v_4 is a dotted amber vertex, but both v_3 and v_4 are amber, then the average score is at least $[0]$.*

Proof. Suppose that neither v_3 nor v_4 is a dotted amber vertex, but both v_3 and v_4 are amber. Suppose firstly that the set A is chosen. In this case, we let H be obtained from G by removing the vertices u , v , and $(uv)_1$, adding the vertex u to the set A , and marking the vertex u_2 if it is a dotted blue vertex. This results

in an excess weight of 2 if u_2 is a blue vertex, and an excess weight of 1 if u_2 is an amber vertex. Hence, the contribution to the count when the set A is chosen is at least 1.

Suppose next that the set B is chosen. In this case, let H be obtained from G by removing the vertices v and $(uv)_1$, adding the vertex $(uv)_1$ to the set B , and marking the vertex u . Both amber vertices v_3 and v_4 contribute 1 to the count for saving the cost of cutting the outgoing edges e_3 and e_4 . This results in a contribution of $5 + 4 - 12 + 2 \times 1 = -1$ to the overall count. Hence, the contribution to the count when the set B is chosen is at least -1 . As observed earlier, the contribution to the count when A is chosen is at least 1. Thus, the score is at least $[1 \mid -1]$. \square

In summary, by Claims 36.1, 36.2 and 36.3, the average score of a **green island** is at least 0. \square

Claim 37 *The average score of a **green isthmus** is at least $[0]$.*

Proof. Consider a **green isthmus** shown in Figure 25, where u_1 and u_2 are the ends of the edge e_1 and e_2 , respectively, different from the vertex u . Suppose that u_1 and u_2 are of different colors. Whether A or B is chosen, we do not delete the vertex u , and instead mark it. This contributes 0 to the count, possibly more since we save the cutting of the edge e_1 or e_2 , yielding a score of $[1 \mid 1]$ if none of u_1, u_2 is marked and a score of at least $[0, 0]$ if some are dotted.

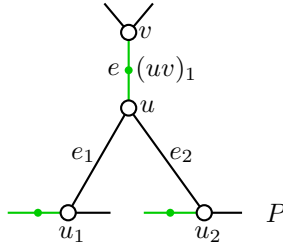


Figure 25: A **green isthmus**

Suppose u_1 and u_2 are of the same color, say amber. If the set A is chosen, then we let H be obtained from G by removing the vertex u , resulting in an excess weight of $+1$. Suppose now the set B is chosen. If neither u_1 nor u_2 is dotted, then we let H be obtained from G by removing the vertices u and $(uv)_1$, adding the vertex $(uv)_1$ to the set B , and marking the vertex v . We get an extra contribution of $2 \times 1 = 2$ for not cutting the edges e_1 and e_2 . This results in a contribution of $5 + 4 - 12 + 2 = -1$ to the overall count. If at least one of u_1 and u_2 is a dotted amber vertex, say u_1 , then we do not delete the vertex u . The overall contribution is then 0, noting that the possible cut of the outgoing edge e_2 is already taken care of in the counting in the case when u_2 is not dotted. Hence, the contribution to the count when the set B is chosen is at least -1 . As observed earlier, the contribution to the count when A is chosen (and u_1 and u_2 are of the same color) is at least 1. Thus, the score is at least $[1 \mid -1]$.

In summary, if u_1 and u_2 are of different colors, then the score is at least $[0, 0]$, and if u_1 and u_2 are of the same color, then the score is at least $[1 \mid -1]$. Thus, the average score of a **green isthmus** is at least $[0]$. \square

Claim 38 *The average score of a **red isthmus** is at least $[0]$.*

Proof. The proof is analogous to the proof of Claim 37 with the only difference that the vertex $(uv)_2$ may be marked instead of v , which is now a vertex of degree 2. We gain an extra $+1$ in that case and all contributions are now nonnegative. This yields a scoring of at least $[0 \mid 0]$. \square

Claim 39 *The average score of a **black detour** or a **special black edge** is at least $[0]$.*

Proof. Consider a **black detour** or a **special black edge** as illustrated in Figure 26. We may assume that the vertex u is an amber vertex. If the set A is chosen, then we do not delete the vertex v , and instead mark it. This contributes 0 to the count. If the set B is chosen, then we do not delete the vertex v . This contributes 0 to the overall count if the vertex u is a dotted amber vertex, and also contributes 0 to the overall count if the vertex u is not a dotted vertex since in this case we gain +1 from deleting the vertex u but lose 1 from the degree of v dropping from 3 to 2. This yields a score of at least $[0 \mid 0]$. \square

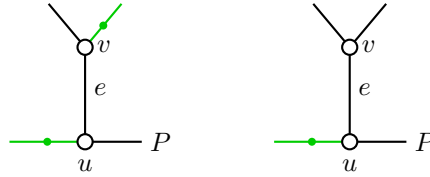


Figure 26: A **black detour** or a **special black edge**

Claim 40 *The average score of a **special red edge** is at least $[\frac{1}{2}]$.*

Proof. Consider a **special red edge**, as illustrated in Figure 27. We may assume that the vertex u is an amber vertex. If the set A is chosen, then we delete $(uv)_1$ (and cut the edge e). We gain 5 from deleting the vertex $(uv)_1$ and we lose 3 from the degree of $(uv)_2$ dropping from 2 to 1. The overall contribution is +2. Suppose next that the set B is chosen. If the vertex u is dotted, then we do nothing (that is, we do not delete a vertex on the **special red edge**), which contributes 0. If the vertex u is not dotted, then we delete $(uv)_1$ and $(uv)_2$, and we add the vertex $(uv)_2$ to the dominating set B , and mark the vertex v . We save the cutting of the outgoing edge joining u and $(uv)_1$, gaining an extra 1. The overall contribution is $2 \times 5 - 12 + 1 = -1$ to the total counting. This yields a score of $[+2 \mid -1]$. \square

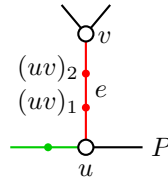


Figure 27: A **special red edge**

From here on, we start using the special artifacts on the **Green cycle graph** as defined in Definition 13, namely **links** and **fibers**. Recall that **links** have two extremities on the **Green cycle graph**, and we say a **link** is *well colored* if its extremities belong to different sets A and B , while a **link** is *badly colored* if its extremities are in the same set. We next introduce two types of **links**, namely *broad links* and *narrow links*, that satisfy the following properties.

Definition 14 (Average scores for links)

- *Broad links* will have an average score of at least $[+5]$ when the **link** is well colored, $[-1]$ when badly colored and when dotted, $[+\frac{1}{2}]$ if only one extremity is dotted, and $[+1]$ if both extremities are dotted.
- *Narrow links* will have an average score of $[+4]$ when the **link** is well colored, $[-\frac{1}{2}]$ when badly colored, and $[0]$ when at least one extremity is dotted.

Moreover, we introduce an artifact called a *fiber* that we attach to a single vertex on the **Green cycle graph** with the incentive to avoid dotting this vertex. A fiber will have the following property.

Definition 15 (Scores for fibers) • *Fibers will have a nonnegative score on average, except that they are worth $[-\frac{1}{2}]$ when the associated single vertex on the **Green cycle graph** is dotted.*

Definition 16 If the center vertex v of a black 3-star is adjacent to three vertices v_1 , v_2 and v_3 on the **green-black-cycle** C as illustrated in Figure 28, then we call the 3-star an *isolated 3-star* with respect to the cycle C . Further, we introduce a narrow **link** and a fiber in the **Green cycle graph**, where a narrow **link** is added between v_1 and v_2 and a fiber is added to v_3 .

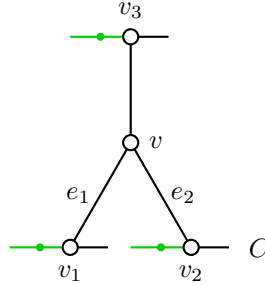


Figure 28: An isolated 3-star

In the following claim, we show that a broad **link** and a narrow **link** both have an average score at least as large as what the **links** and fiber induce. Moreover, the average score is nonnegative if there are no **links** or fibers.

Claim 41 *The following properties hold for a narrow **link** and fiber associated with an isolated 3-star in the **Green cycle graph**.*

- (a) *If the **link** is well colored and the fiber is not dotted, then the average score is at least $[5] \geq [4] + [0]$.*
- (b) *If the **link** is well colored and the fiber dotted, then the score is at least $[\frac{7}{2}] \geq [4] + [-\frac{1}{2}]$.*
- (c) *If the **link** is dotted, then the score is at least $[\frac{1}{2}]$, whether the fiber is dotted or not.*
- (d) *If the **link** is badly colored, the score is at least $[-\frac{1}{2}]$, whether the fiber is dotted or not.*

Proof. Let v be the center of an isolated 3-star S_v with respect to a **green-black-cycle** C . Let v_1 , v_2 and v_3 be the three neighbors of v on the cycle C .

(a) If the **link** is well colored and the fiber is not dotted, then we can simply remove v in all cases, and save from cutting the edges from v to its neighbors. In this case, we may assume by symmetry that $v_1, v_3 \in A$ and $v_2 \in B$, yielding a score of $[5 \mid 6]$. This proves (a).

(b) Suppose the **link** is well colored and the fiber is dotted. We may assume that v_1 is Amber, v_2 is blue, and v_3 is dotted Amber. We once again remove v . If the set **A** is chosen, we save one by cutting the edge vv_2 , while if the set **B** is chosen, we save one by cutting the edge vv_1 . However, when choosing the set **B**, we pay 3 for cutting the edge vv_3 . This yields a score of $[5 \mid 2]$, with an average score of $[\frac{7}{2}]$, which proves (b).

(c) Suppose the **link** is dotted, say v_1 is dotted Amber. Suppose firstly that at least one of v_2 and v_3 is not dotted blue. In this case, when choosing the dominating set **A**, we remove v and get a score of at least 4 for removing v , and we pay 3 at most once (when at most one of v_2 and v_3 is dotted blue). When choosing the

dominating set **B**, we mark v to obtain a score of 0. This yields a score of at least $[1 \mid 0]$, with an average score of $[\frac{1}{2}]$. Suppose next that both v_2 and v_3 are dotted blue. In this case, interchanging the roles of v_1 and v_2 we remove v when choosing the dominating set **A**, and we mark v when choosing the dominating set **A** to obtain a score of at least $[0 \mid 1]$. In both cases, this yields an average score of $[\frac{1}{2}]$, whether the fiber is dotted or not. This proves (c).

(d) Suppose the link is badly colored. We may assume that v_1 and v_2 both are Amber and are not dotted. Suppose firstly that v_3 is not dotted. In this case, when choosing the dominating set **A**, we remove v and get a score of at least 4 for removing v . When choosing the dominating set **B**, we once again remove v and get a score of 4 for removing v . If v_3 is Amber, then we save 3 for by cutting the three edges incident with v to obtain a final score of $4 + 3 - 12 = -5$ for removing v , saving the three edges cut, but add v to the dominating set. If v_3 is blue, then the vertex v is not added to the dominating set to obtain a final score of $4 + 2 = 6$. This yields a score of at least $[4 \mid -5]$, with an average score of $[-\frac{1}{2}]$.

Suppose next that v_3 is dotted. Suppose firstly that v_3 is dotted blue. In this case when choosing the dominating set **A**, we remove v and pay 3 for cutting the edge vv_3 to obtain a score of $4 - 3 = 1$. When choosing the dominating set **B**, we remove v and save from cutting the two edges vv_1 and vv_2 to obtain a score of $4 + 2 = 6$. This yields a score of $[1 \mid 6]$, with an average score of $[\frac{7}{2}]$. Suppose next that v_3 is dotted amber. In this case when choosing the dominating set **A**, we remove v and obtain a score of 4. When choosing the dominating set **B**, we do not remove v and save the two edges vv_1 and vv_2 cut. However, the degree of v drops from 3 to 1, which costs 4. Thus we obtain a score of $2 - 4 = -2$. This yields a score of $[4 \mid -2]$, with an average score of $[1]$. In summary, if the link is badly colored, we obtain an average score of at least $[-\frac{1}{2}]$. This proves (d). \square

We define next an isolated 2-star with respect to a **green-black-cycle**.

Definition 17 If the center vertex u of a black 3-star is adjacent to exactly two vertices on the **green-black-cycle** C as illustrated in Figure 29, then we call the 3-star an *isolated 2-star* with respect to the cycle C .

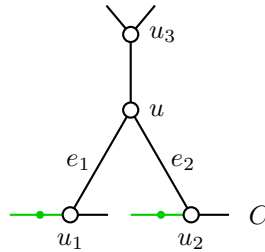


Figure 29: An isolated 2-star

Claim 42 Let C be an isolated 2-star with respect to the **green-black-cycle** C . If the neighbor of the center of the star not on the cycle C is not adjacent to the center of any other 2-star associated with the cycle C , then the average score of the isolated 2-star is at least $[0]$.

Proof. Consider an isolated 2-star with center u that is adjacent to the vertices u_1 and u_2 on the **green-black-cycle** C , as illustrated in Figure 29, where the third neighbor, u_3 say, of u does not belong to the cycle C and is not adjacent to the center of any other 2-star associated with the cycle C . We may assume that the vertex u_1 is an amber vertex.

Suppose that the vertex u_2 is an amber vertex. When choosing the dominating set **A**, we remove u and get a score of at least 4 for removing u . The degree of u_3 drops from 3 to 2 which costs 1 to yield a score of $4 - 1 = 3$. When choosing the dominating set **B**, we do not delete the vertex u . If both u_1 and u_2 are dotted amber, then the degree of u is unchanged, and this contributes 0 to the score. If exactly one of u_1 and u_2 are dotted amber, then the degree of u drops by 1 which costs 1 but we regain the loss since we gain 1 for cutting the edge from u to the amber vertex that is not dotted, once again resulting in a score of $1 - 1 = 0$. If neither u_1 nor u_2 is dotted, then the degree of u decreases by 2, resulting in a weight loss of -4 . However, in our counting we have already attributed a cost of 1 to the cutting of each of the outgoing edges uu_1 and uu_2 , which contributes 2 to the counting. Hence, the overall contribution in this case is $2 - 4 = -2$. Hence when the vertex u_2 is amber, we obtain a score of at least $\lceil 3 \rceil - 2$, with an average score of at least $\lceil \frac{1}{2} \rceil$.

Suppose that the vertex u_2 is a blue vertex. Suppose that the vertex u_2 is not dotted blue. In this case when choosing the dominating set **A**, we remove u and get a score of at least 4 for removing u . The degree of u_3 drops from 3 to 2 which costs 1. We have already counted a cost of 1 for the cutting of the outgoing edge uu_2 . This yields a score of $4 - 1 + 1 = 4$. Suppose that the vertex u_2 is dotted blue. In this case when choosing the dominating set **A**, we do not remove the vertex u but mark it, yielding a score of 0. A symmetrical argument (noting that the vertex u_1 is amber) yields a score of 4 or 0 when choosing the dominating set **B** in this case when the vertex u_2 is a blue vertex. Hence when the vertex u_2 is blue, we obtain a score of at least $\lceil 0 \rceil - 0$, with an average score of at least $\lceil 0 \rceil$. \square

We proceed further by formally defining a broad link in the **Green cycle graph**.

Definition 18 Let u and v be the center vertices of isolated 2-stars S_u and S_v , respectively, with respect to the **green-black-cycle** C . Let u_1 and u_2 be the two neighbors of u that belong to the cycle C , and let v_1 and v_2 be the two neighbors of v that belong to the cycle C . If u and v are adjacent as illustrated in Figure 30, then we say that the isolated 2-stars S_u and S_v are adjacent. In this case, we introduce two broad links in the **Green cycle graph**, where we add a broad link between u_1 and u_2 and a broad link between v_1 and v_2 .

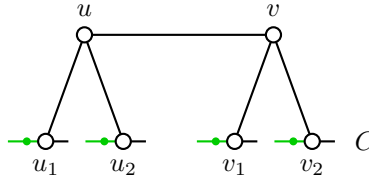


Figure 30: Adjacent 2stars

Claim 43 *The following properties hold for the two broad links associated with adjacent isolated 2-stars in the **Green cycle graph**.*

- (a) *If a broad link is well colored, then the average score of the link is at least $\lceil 5 \rceil$.*
- (b) *If a broad link contains at least one dotted extremity, then the average score of the link is at least $\lceil \frac{1}{2} \rceil$.*
- (c) *If a broad link is badly colored, then the average score of the link is at least $\lceil -1 \rceil$.*

Proof. Consider two adjacent 2-stars S_u and S_v with centers u and v , where u is adjacent to the vertices u_1 and u_2 on the **green-black-cycle** C and v is adjacent to the vertices v_1 and v_2 on C , as illustrated in Figure 30. Let $W = \{u_1, u_2, v_1, v_2\}$. We proceed further with three subclaims.

Claim 43.1 *If no vertex in W is a dotted vertex, then the following properties hold for the average score of the two adjacent 2-stars.*

- (a) If both *links* are badly colored, then the average score is at least $[-2] = [-1] + [-1]$.
- (b) If one *link* is badly colored and the other well colored, then the average score is at least $[4] = [-1] + [5]$.
- (c) If both *links* are well colored, then the average score is at least $[10] = [5] + [5]$.

Proof. Suppose that no vertex in W is a dotted vertex. Renaming vertices if necessary, we may assume that if v_1 and v_2 have the same color, then u_1 and u_2 have the same color. Further, we may assume that u_1 is an amber vertex.

(a) Suppose that both *links* are badly colored, and thus that u_1 and u_2 are amber. Suppose firstly that both v_1 and v_2 are amber. If the set A is chosen, then we delete both vertices u and v , which yields $2 \times 4 = 8$ to the score. If the set B is chosen, then we delete both vertices u and v , and we add the vertex v to the dominating set. This yields 2×4 for the deletion of u and v , -12 for adding the vertex v to the dominating set, and $+4$ for the cost of cutting the four outgoing edges uu_1, uu_2, vv_1 and vv_2 , which are already included in the count. The overall contribution if the set B is chosen is therefore $8 - 12 + 4 = 0$. This yields a score of $[8 \mid 0]$ in this case when both v_1 and v_2 are amber. Suppose next that both v_1 and v_2 are blue. If the set A is chosen, then we delete both vertices u and v , and we add the vertex v to the dominating set. This yields 2×4 for the deletion of u and v , -12 for adding the vertex v to the dominating set, and $+2$ for the cost of cutting the two outgoing edges vv_1 and vv_2 , which are already included in the count. The overall contribution if the set A is chosen is therefore $8 - 12 + 2 = -2$. If the set B is chosen, then we delete both vertices u and v , and we add the vertex u to the dominating set. By symmetry to the previous case, the overall contribution in this case is again -2 . The above yields a score of $[-2 \mid -2]$. This proves (a).

(b) Suppose that one *link* is badly colored and the other well colored. Without loss of generality, we may assume u_1, u_2 and v_1 are amber, and v_2 is blue. If the set A is chosen, then we delete both vertices u and v , which yields $2 \times 4 = 8$ to the count. Moreover, we gain $+1$ for the outgoing edge vv_2 , which are already included in the count. The overall contribution if the set A is chosen is therefore $8 + 1 = 9$. If the set B is chosen, then we delete both vertices u and v , and we add the vertex u to the dominating set. This yields 2×4 for the deletion of u and v , -12 for adding the vertex u to the dominating set, and $+3$ for the cost of cutting the three outgoing edges uu_1, uu_2 , and vv_1 , which are already included in the count. The overall contribution if the set B is chosen is therefore $8 - 12 + 3 = -1$. The above yields a score of $[9 \mid -1]$. This proves (b).

(c) Suppose that both *links* are well colored. Without loss of generality, we may assume u_1 and v_1 are amber, and u_2 and v_2 are blue. If the set A is chosen, then we delete both vertices u and v , yielding $2 \times 4 = 8$ to the count. Further, we count $+2$ for the cost of cutting the two outgoing edges uu_2 and vv_2 , which are already included in the count. The overall contribution if the set A is chosen is therefore $8 + 2 = 10$. By symmetry, the overall contribution if the set B is chosen is 10. The above yields a score of $[10 \mid 10]$. This proves (c). \square

Claim 43.2 *If both *links* contain at least one dotted extremity, then the following properties hold for the average score of the two adjacent 2-stars.*

- (a) *If W contains at least three vertices of the same color, then the average score is at least $[\frac{5}{2}]$.*
- (b) *If W contains exactly two vertices of the same color and both u and v have a neighbor of each color, then the average score is at least $[2]$.*
- (c) *If W contains exactly two vertices of the same color and both u and v have neighbors of different colors, then the average score is at least $[1]$.*

Proof. Suppose both *links* contain at least one dotted extremity. In particular, this implies that at least two vertices in W are dotted.

(a) Suppose that W contains at least three vertices of the same color. Suppose firstly that all four vertices of W are colored the same, say with color Amber. If the set A is chosen, then we delete both u and v , which

yields $2 \times 4 = 8$ to the score. If the set **B** is chosen, then neither u nor v is deleted, which yields 0 to the score. Thus in this case, the score is at least $[8 \mid 0]$. Suppose next that exactly three vertices in W have the same color. We may assume that u_1, u_2 and v_1 are Amber. If the set **A** is chosen, then as before we delete both u and v , which yields 8 to the score. However in this case, if the blue vertex v_2 is dotted, then we pay 3 for cutting the edge vv_2 , yielding an overall score of at least $8 - 3 = 5$ when the set **A** is chosen. If the set **B** is chosen, then we do not delete u and v but we mark the vertex v . We note that since at least one of the amber vertices u_1 and u_2 is dotted, the degree of the vertex u is at least 2. This yields a score of at least 0 when the set **B** is chosen. In summary, this yields a score of at least $[5 \mid 0]$. This proves (a).

(b) Suppose that W contains exactly two vertices of the same color and both u and v have a neighbor of each color. If the set **A** or the set **B** is chosen, then we delete both u and v , which yields $2 \times 4 = 8$ to the score. When choosing the set **A**, if the blue neighbor of u (resp., v) is dotted, then we pay 3 for cutting the edge from u to that vertex. Analogously, when choosing the set **B**, if the amber neighbor of u (resp., v) is dotted, then we pay 3 for cutting the edge from u (resp., v) to that vertex. This yields an overall score of at least $8 - 3 - 3 = 2$ when each set **A** and **B** is chosen. In summary, this yields a score of at least $[2 \mid 2]$. This proves (b).

(c) Suppose that W contains exactly two vertices of the same color and both u and v have neighbors of different colors. We may assume that both neighbors of u are Amber, and therefore both neighbors of v are Blue. If the set **A** is chosen, then we delete the vertex u but not the vertex v , and we cut the edge uv . The deletion of u contributes 4 to the score. If both v_1 and v_2 are dotted, then the degree of v drops from 3 to 2, and so cutting the edge uv costs 1. If exactly one of v_1 and v_2 is dotted, then the degree of v drops from 3 to 1, which costs 4, but we gain 1 from deleting the outgoing edge from v to its blue neighbor that is not dotted, and so cutting the edge uv costs 3. Hence, the score is at least $4 - 3 = 1$ if the set **A** is chosen. Identical arguments show that the score is at least 1 if the set **B** is chosen. In summary, this yields a score of at least $[1 \mid 1]$. This proves (c). \square

Claim 43.3 *If only one link contains at least one dotted extremity, then the following properties hold for the average score of the two adjacent 2-stars.*

- (a) *If only one extremity of the dotted link is dotted and the other link is well colored, then the average score is at least $[\frac{11}{2}] = [\frac{1}{2}] + [5]$.*
- (b) *If both extremities of the dotted link are dotted and the other link is well colored, then the average score is at least $[6] = [1] + [5]$.*
- (c) *If only one extremity of the dotted link is dotted and the other link is badly colored, then the average score is at least $[-\frac{1}{2}] = [\frac{1}{2}] + [-1]$.*
- (d) *If both extremities of the dotted link are dotted and the other link is badly colored, then the average score is at least $[\frac{1}{2}] = [\frac{3}{2}] + [-1]$.*

Proof. (a) Suppose that only one extremity of the dotted link is dotted and the other link is well colored. We may assume that u_1 is dotted amber, v_1 is amber (and not dotted) and v_2 is blue (and not dotted). Suppose firstly that u_2 is colored Amber. By supposition, u_2 is not dotted. If the set **A** is chosen, then we delete both u and v , which contributes $2 \times 4 = 8$ to the score. Since we gain 1 by cutting the outgoing edge from v to the blue vertex v_2 , this yields a score of at least 9. If the set **B** is chosen, then we delete the vertex v , which contributes 4 to the score, but we do not delete the vertex u . We gain 1 by cutting the outgoing edges from u to the amber vertex u_2 and from v to the amber vertex v_1 . However, the degree of u drops from 3 to 1, which costs 4. Hence, the score when the set **B** is chosen is at least $4 + 2 - 4 = 2$. Thus, the overall score is at least $[9 \mid 2]$ if u_2 is colored Amber.

Suppose secondly that u_2 is colored blue. If the set **A** is chosen, then we delete both u and v , which contributes 8 to the score. Since we gain 1 by cutting the outgoing edges from u to the blue vertex u_2 and from v to the blue vertex v_2 , this yields a score of at least 10. If the set **B** is chosen, then we delete both u

and v , which contributes 8 to the score. We gain 1 by cutting the outgoing edge from v to the amber vertex v_1 . However, cutting the edge from u to the dotted amber vertex u_1 costs 3. Hence, the score when the set **B** is chosen is at least $8 + 1 - 3 = 6$. Thus, the overall score is at least $[10 \mid 6]$ if u_2 is colored blue. This proves (a).

(b) Suppose that both extremities of the dotted link are dotted and the other link is well colored. We may assume that u_1 and u_2 are dotted. Further, we may assume that u_1 is dotted amber, v_1 is amber (and not dotted) and v_2 is blue (and not dotted). Suppose firstly that the dotted vertex u_2 is colored amber. If the set **A** is chosen, then as before this yields a score of at least 9. If the set **B** is chosen, then we delete the vertex v , which contributes 4 to the score, but we do not delete the vertex u . We gain 1 by cutting the outgoing edges from v to the amber vertex v_1 . However the degree of u drops from 3 to 2, which costs 1. Hence, the score when the set **B** is chosen is at least $4 + 1 - 1 = 4$. Thus, the overall score is at least $[9 \mid 4]$ if the dotted vertex u_2 is colored amber.

Suppose secondly that u_2 is colored blue. If the set **A** is chosen, then we delete both u and v , which contributes 8 to the score. We gain 1 by cutting the outgoing edges from v to the blue vertex v_2 . However, cutting the edge from u to the dotted blue vertex u_2 costs 3. Hence, the score when the set **A** is chosen is at least $8 + 1 - 3 = 6$. If the set **B** is chosen, then we delete both u and v , which contributes 8 to the score. We gain 1 by cutting the outgoing edge from v to the amber vertex v_1 . However, cutting the edge from u to the dotted amber vertex u_1 costs 3. Hence, the score when the set **B** is chosen is at least $8 + 1 - 3 = 6$. Thus, the overall score is at least $[6 \mid 6]$ if the dotted vertex u_2 is colored blue. This proves (b).

(c) Suppose that only one extremity of the dotted link is dotted and the other link is badly colored. We may assume that u_1 and u_2 are colored amber and that v_1 is dotted. By supposition, v_2 is not dotted.

Suppose firstly that v_1 or v_2 is amber. If the set **A** is chosen, then we delete both u and v , which contributes 8 to the score. If the dotted vertex v_1 is blue, then cutting the edge vv_1 costs 3. If the dotted vertex v_1 is amber, then cutting the outgoing edges vv_1 and vv_2 contributes at least 0 to the score. Thus, the score when the set **A** is chosen is at least $8 - 3 = 5$. If the set **B** is chosen, then we delete both u and v , and we add the vertex v to the dominating set. This yields 2×4 for the deletion of u and v , -12 for adding the vertex v to the dominating set, and $+2$ for the cost of cutting the outgoing edges uu_1 and uu_2 , which are already included in the count. If the dotted vertex v_1 is amber, then we mark the vertex v_1 (which is dominated by the deleted vertex v) which yields an additional score of $+1$. If the dotted vertex v_1 is blue, then the vertex v_2 is amber and we gain $+1$ for the cost of cutting the outgoing edge vv_2 . The overall contribution if the set **B** is chosen is therefore at least $8 - 12 + 2 + 1 = -1$. Thus, the overall score is at least $[5 \mid -1]$ if v_1 or v_2 is amber.

Suppose next that both v_1 and v_2 are blue. Recall that v_1 is dotted and v_2 is not dotted. If the set **A** is chosen, then we delete u , which contributes 4 to the score, but do not delete v . We gain 1 for the deletion of the outgoing edge vv_2 , but the cost of cutting the edges uv and vv_1 is 4 since the degree of v decreases from 3 to 1. Thus, the score when the set **A** is chosen is at least $4 + 1 - 4 = 1$. If the set **B** is chosen, then we delete both u and v , which contributes 8 to the score, and we add either u or v to the dominating set, which costs -12 . We gain 2 by cutting the outgoing edge from u to the amber vertices u_1 and u_2 . Hence, the score when the set **B** is chosen is at least $8 - 12 + 2 = -2$. Thus, the overall score is at least $[1 \mid -2]$ if both v_1 and v_2 are blue. This proves (c).

(d) Suppose that both extremities of the dotted link are dotted and the other link is badly colored. We may assume that u_1 and u_2 are colored amber, and that both v_1 and v_2 are dotted. Suppose v_1 or v_2 is amber. Proceeding exactly as in the part (c) above, the overall score in this case is at least $[5 \mid -1]$. Suppose that both v_1 and v_2 are (dotted) blue. If the set **A** is chosen, then we delete u , which contributes 4 to the score, but do not delete v . The cutting of the edge uv only costs 1 since the degree of v decreases from 3 to 2 (noting that the edges from v to the dotted blue vertices v_1 and v_2 are not cut). Thus, the score when the set **A** is chosen is at least $4 - 1 = 3$. If the set **B** is chosen, then we proceed exactly in part (c) above

and delete both u and v and add either u or v to the dominating set, to yield a score of at least -2 . Thus, the overall score is at least $\lfloor 3 \rfloor - 2$ if both v_1 and v_2 are blue. This proves (d). \square

The result of Claim 43 now follows from Claims 43.1, 43.2, and 43.3. \square

Note that until now, every vertex adjacent to a vertex not on the **green-black** cycle that gets removed in the graph is marked. This is not true for the two following claims, and an extra care is taken to avoid creating isolated vertices.

Claim 44 Consider two isolated 2-stars with centers u and v , respectively, with respect to the **green-black**-cycle C , where u is adjacent to the vertices u_1 and u_2 on C and v is adjacent to the vertices v_1 and v_2 on C , and where u and v have a common neighbor w that has no neighbor on the cycle C . The structure of the resulting configuration is illustrated in Figure 31. Then the average score is at least $\lfloor 0 \rfloor$.

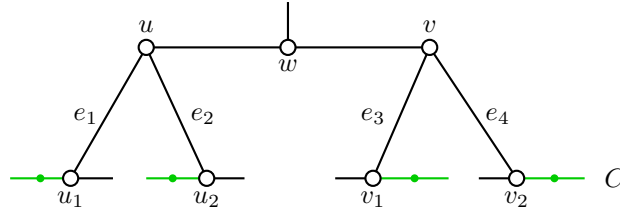


Figure 31: A common neighbor of two 2stars

Proof. Consider the above configuration and denote $W = \{u_1, u_2, v_1, v_2\}$. Let x be the third neighbor of the vertex w different from u and v . We proceed further with a series of six subclaims that analyse all possible cases that can occur.

Claim 44.1 If no vertex in W is dotted, then the average score is at least $\lfloor 2 \rfloor$.

Proof. Suppose that no vertex in W is dotted. In this case, we remove the vertices u , v and w , we add the vertex w to the set **A** (resp., **B**) when the set **A** (resp., **B**) is chosen. Further, we mark the vertex x . The deletion of the three vertices contributes 12 to the score, and the addition of the vertex w to the dominating set contributes -12 to the score. If a and b are the number of vertices in W that are amber and blue, respectively, then the overall contribution to the cost is b if the set **A** is chosen, and a if the set **B** is chosen. Thus, the overall score of the configuration is at least $\lfloor b \rfloor - a$. We note that $a, b \geq 0$ and $a + b = 4$. Hence, the average score of the configuration is at least $\lfloor 2 \rfloor$. \square

Claim 44.2 If at least one vertex in W is dotted and all vertices in W have the same color, then the average score is at least $\lfloor 1 \rfloor$.

Proof. Suppose that at least one vertex in W is dotted and all vertices in W have the same color. By symmetry and renaming vertices if necessary, we may assume that all vertices in W are amber and that the vertex u_1 is dotted. If the set **A** is chosen, then we delete the vertices u and v . The deletion of the two vertices contributes $2 \times 4 = 8$ to the score, and since w does not become an isolate vertex, cutting the edges uw and vw contributes -4 to the score. Hence, the score when the set **A** is chosen is at least $8 - 4 = 4$.

Suppose the set **B** is chosen. If at least one of v_1 or v_2 is dotted, then we do not remove any vertex, yielding a contribution to the score of at least 0. If neither v_1 nor v_2 is dotted and u_2 is dotted, then we

delete the vertices v and w , add the vertex w to the dominating set \mathbf{B} , and mark the vertices u and x , yielding a contribution to the cost of at least $2 \times 4 - 12 + 2 = -2$ noting that we gain 2 from deleting the outgoing edges from v to v_1 and v_2 . If u_1 is the only vertex in W that is dotted, then we delete the vertices u , v and w , we add the vertex w to the dominating set \mathbf{B} , and we mark the vertex x , yielding a contribution to the score of at least $3 \times 4 - 12 + 3 - 3 = 0$, noting that the dotted vertex u_1 decreases the score by 3 and the three outgoing edges uu_2 , vv_1 and vv_2 each contribute 1 to the score. Hence, the score when the set \mathbf{B} is chosen is at least -2 . Thus, the overall score of the configuration is at least $[4 \mid -2]$, and so the average score of the configuration is at least $[1]$. \square

Claim 44.3 *If at least one vertex in W is dotted and the vertices in $\{u_1, u_2\}$ are colored the same, and the vertices in $\{v_1, v_2\}$ are colored the same but with a different color, then the average score is at least $[\frac{5}{2}]$.*

Proof. By symmetry and renaming vertices if necessary, suppose that the vertices u_1 and u_2 are amber, where u_1 is dotted, and the vertices v_1 and v_2 are blue. Suppose the set \mathbf{A} is chosen. If at least one of v_1 and v_2 is dotted, then we remove the vertex u , but remove neither v nor w . The deletion of u contributes 4 to the score, while the cutting of the edge uw decreases the degree of the vertex w by 1, which contributes -1 to the score, yielding an overall contribution to the score of $4 - 1 = 3$ in this case. If neither v_1 nor v_2 is dotted, then we remove the vertices u , v and w , we add the vertex w to the dominating set \mathbf{A} , and we mark the vertex x . This yields an overall contribution to the score of $3 \times 4 - 12 + 2 = 2$, noting that the two outgoing edges vv_1 and vv_2 each contribute 1 to the score. Hence, the score when the set \mathbf{A} is chosen is at least 2. If the set \mathbf{B} is chosen, then we remove the vertex v , but remove neither v nor w . The deletion of v contributes 4 to the score, while the cutting of the edge vw decreases the degree of the vertex w by 1, which contributes -1 to the score, yielding an overall contribution to the score of $4 - 1 = 3$ when the set \mathbf{B} is chosen. Thus, the overall score of the configuration is at least $[2 \mid 3]$, and so the average score of the configuration is at least $[\frac{5}{2}]$. \square

Claim 44.4 *If exactly three vertices in W have the same color, at least one of which is dotted, then the average score is at least $[\frac{1}{2}]$.*

Proof. Suppose that three vertices in W are amber, say u_1 , u_2 and v_1 . Suppose firstly that u_1 is dotted. If the set \mathbf{A} is chosen, we remove the vertices u and v . The deleting of the two vertices u and v contributes $2 \times 4 = 8$ to the score, and since w does not become an isolated vertex, cutting the edges uw and vw contributes -4 to the score. If the vertex v_2 is not a dotted (blue) vertex, then its contribution to the score is 1, while if the vertex v_2 is a dotted (blue) vertex, then its contribution to the score is -3 . Hence, the score when the set \mathbf{A} is chosen is at least $8 - 4 - 3 = 1$. If the set \mathbf{B} is chosen, then we do not remove any of the vertices u , v or w , but we mark the vertex v , yielding a score of at least 0. Thus, the overall score of the configuration is at least $[1 \mid 0]$, and so the average score of the configuration is at least $[\frac{1}{2}]$.

Suppose now that neither u_1 nor u_2 is dotted, but v_1 is dotted. If the set \mathbf{A} is chosen, then we delete the vertex u and mark the vertex v . The deleting of the vertex u contributes 4 to the score. The cutting of the edge uw contributes -1 to the score, and the marked vertex v contributes 0 to the score. Hence, the score when the set \mathbf{A} is chosen is at least $4 - 1 = 3$. If the set \mathbf{B} is chosen, then we remove the vertices u , v and w , we add the vertex w to the dominating set \mathbf{B} , and we mark the vertex x . The deleting of the three vertices contributes 12 to the score, and the addition of the vertex w to the dominating set contributes -12 to the score. The cutting of the outgoing edge from v to the dotted vertex v_1 contributes -3 to the score, and the cutting of the two outgoing edges from u to the amber vertices u_1 and u_2 contributes 2 to the score. Hence, the score when the set \mathbf{B} is chosen is at least $12 - 12 - 3 + 2 = -1$, yielding an overall score of at least $[3 \mid -1]$, and so the average score of the configuration is at least $[\frac{1}{2}]$. \square

Claim 44.5 *If exactly three vertices in W have the same color, none of which is dotted, but the fourth vertex is of a different color and dotted, then the average score is at least $[0]$.*

Proof. Suppose that u_1 is dotted amber and all of u_2 , v_1 and v_2 are blue and not dotted. If the set **A** is chosen, then we delete the vertices u , v and w , we add the vertex w to the dominating set **A**, and we mark the vertex x . The deleting of the three vertices contributes 12 to the score, and the addition of the vertex w to the dominating set contributes -12 to the score. The cutting of the outgoing edge from u to the vertex u_2 contributes 1 to the score if u_2 is not dotted (blue) and contributes -3 to the score if u_2 is dotted (blue). The cutting of the two outgoing edges from v to the blue vertices v_1 and v_2 contributes 2 to the score. Hence, the score when the set **B** is chosen is at least $12 - 12 - 3 + 2 = -1$. If the set **B** is chosen, then we remove the vertices u and v , which contributes 8 to the score. The cutting of the edge uw and vw contributes -4 to the score since the degree of the vertex w decreases from 3 to 1. The cutting of the outgoing edge from u to the dotted (amber) vertex u_1 contributes -3 to the score. Hence, the score when the set **B** is chosen is at least $8 - 4 - 3 = 1$, yielding an overall score of at least $[-1 \mid 1]$, and so the average score of the configuration is at least $[0]$. \square

Claim 44.6 *If each of u and v have neighbors of both colors, and at least one vertex is dotted, then the average score is at least $[0]$.*

Proof. Suppose that both u and v have neighbors of different colors. Then in both cases when the set **A** is chosen and when the set **B** is chosen, we do not remove any vertices, but we mark both u and v , yielding a score of at least $[0 \mid 0]$. \square

The desired result in the statement of Claim 44 now follows from Claims 44.1 to 44.6. \square

We define next an isolated 1-star with respect to a **green-black**-cycle.

Definition 19 If the center vertex of a black 3-star is adjacent to exactly one vertex on the **green-black**-cycle C , then we call the 3-star an *isolated 1-star* with respect to the cycle C .

We consider next the cases where the center u of an isolated 2-star is adjacent to an isolated 1-star centered at v with respect to the **green-black**-cycle C , as shown in Figure 32. Let w be the neighbor of v not on the cycle C and different from the vertex u . If the vertex w belongs to the **green-black**-cycle C , then this is covered by Claim 43. Hence, we may assume in what follows that the vertex w does not belong to C . We consider two cases, depending on the status of the third neighbor w of v . We first consider the case when the vertex w is the center u of an isolated 2-star, that is, w is adjacent to two vertices of the **green-black** cycle C .

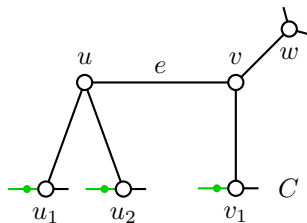


Figure 32: A 2-star adjacent to a 1-star

Definition 20 Let u and w be the center vertices of isolated 2-stars S_u and S_w , respectively, with respect to the **green-black**-cycle C , and let v be the center vertex of an isolated 1-star S_v with respect to C . Let u_1 and u_2 be the two neighbors of u that belong to the cycle C , and let w_1 and w_2 be the two neighbors of

w that belong to the cycle C . If S_v is adjacent to both S_u and S_w as illustrated in Figure 33, then we place a fiber on each of the vertices u_1, u_2, w_1, w_2 in the **Green cycle graph**. We refer to the subgraph induced by the vertices of S_u, S_v and S_w as a 2-star-1-star-2-star configuration.

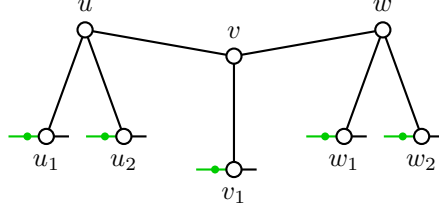


Figure 33: A 2-star-1-star-2-star configuration

Claim 45 *In a 2-star-1-star-2-star configuration, the average score is at least $[0]$, unless two fibers associated with the configuration are dotted in which case the average score is at least $[-\frac{1}{2}]$.*

Proof. Consider a 2-star-1-star-2-star configuration as defined in Definition 20. Thus, u and w are the centers of isolated 2-stars with respect to the **green-black-cycle** C , and v is the center of an isolated 1-star with respect to C . In our 2-star-1-star-2-star configuration, the vertex v is adjacent to both u and w . Moreover, u_1, u_2, v_1, w_1 and w_2 are the neighbors of u, v and w on the **green-black-cycle** C , as illustrated in Figure 33. Let W denote the set $\{u_1, u_2, w_1, w_2\}$.

Suppose firstly that no vertex in W is dotted. In this case, we delete all three vertices u, v and w when each of the sets **A** and **B** is chosen, and we add the vertex v to the dominating set. Further, we mark the vertex v_1 . The deletion of the three vertices u, v and w contributes 12 to the score, while the addition of the vertex v to the dominating set contributes -12 to the score. We save 1 from cutting each of the outgoing edges from u and w to vertices of W of the opposite color to the chosen set. Further, if v_1 is not dotted, then we save 1 from cutting the outgoing edge vv_1 in the case when v_1 has the opposite color to the chosen set, while if v_1 is dotted, then we save 1 from marking the vertex. Thus the score of the configuration is of the form $[a \mid b]$, where $a + b = 5$, implying that the average score is at least $[\frac{5}{2}]$.

Suppose that only one vertex in W is dotted. By symmetry, we may assume that u_1 is dotted Amber. We proceed as in the previous case. That is, we delete all three vertices u, v and w when each of the sets **A** and **B** is chosen, we add the vertex v to the dominating set, and we mark the vertex v_1 . The contribution to the count is as before, except that by cutting the outgoing edge from u to the dotted amber vertex u_1 when the set **B** is chosen, there is a cost of 3, implying that in this case the score of the configuration is of the form $[a \mid b]$, where $a + b = 1$, yielding an average score of at least $[\frac{1}{2}]$.

Suppose that exactly two vertices in W are dotted, and these two vertices belong to the same isolated 2-star. By symmetry, we may assume that u_1 and u_2 are dotted. Suppose that u_1 and u_2 are both amber. Let b be the number of blue vertices among v_1, w_1 and w_2 . If the set **A** is chosen, then as before we delete all three vertices u, v and w , we add the vertex v to the dominating set, and we mark the vertex v_1 . In this case, we gain an additional b by deleting the outgoing edge from v to v_1 and from w to w_1 and w_2 for the blue vertices among v_1, w_1 and w_2 . This yields a score of $12 - 12 + b$ when the set **A** is chosen. If the set **B** is chosen, we delete the vertices v and w , but we do not delete the vertex u . Further, we add the vertex v to the dominating set, and we mark the vertices u and v_1 . In this case, we gain $3 - b$ for the amber vertices among v_1, w_1 and w_2 . This yields a score of $8 - 12 + 3 - b = -1 - b$ when the set **B** is chosen. Thus, the score of the configuration is at least $[b \mid -1 - b]$, yielding an average score of at least $[-\frac{1}{2}]$.

Suppose finally that at least one of u_1 and u_2 is dotted, and at least one of w_1 and w_2 is dotted. In this case, we do not delete any of the three vertices u , v and w when each of the sets **A** and **B** is chosen. However, whenever such a vertex is adjacent to a vertex of the opposite color to the chosen set, we mark the vertex. Suppose, for example, that the set **A** is chosen. If one of u_1 and u_2 is amber, then the vertex u is marked. If both u_1 and u_2 are blue vertices, then the vertex u is not marked. Further if both u_1 and u_2 are dotted (blue), then the degree of u remains unchanged. If exactly one of u_1 and u_2 is dotted (blue), then the degree of u drops from 3 to 2, which costs 1, but we gain 1 by deleting the outgoing edge from u to its (blue) neighbor on the cycle C that is not dotted. Since at least one of u_1 and u_2 is dotted, we conclude that in all cases, the contribution of u to the count is 0. More generally, we note that if one of the vertices u , v and w is not marked when a particular set is chosen, say **A**, then such a vertex has degree at least 2 and its contribution to the count is 0. Thus, the score of the configuration in this case is at least $[0 \mid 0]$, yielding an average score of at least $[0]$. \square

We proceed further with the following definition.

Definition 21 Let u be the center vertex of an isolated 2-star S_u with respect to the **green-black-cycle** C , and let v be the center vertex of an isolated 1-star S_v with respect to C . Let S_u and S_v be adjacent, and so, uv is an edge. Let w be the third neighbor of v that does not belong to the cycle C and is different from u . If w does not belong to the cycle C and w is not the center of an isolated 2-star with respect to the cycle C , then we place a narrow **link** between u_1 and u_2 in the **Green cycle graph**. In this case, we refer to the subgraph induced by the vertices of S_u and S_v as a 2-star-1-star-not-2-star configuration.

Claim 46 *The following properties hold for the average score of a 2-star-1-star-not-2-star configuration in the **Green cycle graph**.*

- (a) *If the **link** is badly colored, then the average score is at least $[-\frac{1}{2}]$.*
- (b) *If the **link** contains at least one dotted vertex, then the average score is at least $[0]$.*
- (c) *If the **link** is well colored, then the average score is at least $[4]$.*

Proof. Consider a 2-star-1-star-not-2-star configuration as defined in Definition 21. Thus, u is the center vertex of an isolated 2-star with respect to the **green-black-cycle** C , and v is the center vertex of an isolated 1-star with respect to C , and u and v are adjacent. Let w be defined as in Definition 21. We note that in our 2-star-1-star-not-2-star configuration, the vertex v is adjacent to both u and w . Let u_1 and u_2 be the neighbors of u on the **green-black-cycle** C , and let v_1 be the neighbor of v on the cycle C . The structure of the configuration is illustrated in Figure 32.

(a) Suppose firstly that the **link** in the configuration (that we added between u_1 and u_2) is badly colored. We may assume that both u_1 and u_2 are amber (and not dotted). If the set **A** is chosen, then we delete u but not v , and we mark v if v_1 is amber. The deletion of u contributes 4 to the score. If v_1 is dotted blue, then the degree of v drops from 3 to 2, and the contribution of cutting the edge uv is -1 in this case. If v_1 is blue and not dotted, then the degree of v drops from 3 to 1, and the contribution of cutting the edge uv is $-4 + 1 = -3$ in this case noting that we gain 1 by cutting the outgoing edge from v to the vertex v_1 . Hence, the score when the set **A** is chosen is at least $4 - 3 = 1$. If the set **B** is chosen, then we delete neither u nor v , and we mark v . We gain 2 from cutting the two outgoing edges from u to the amber vertices u_1 and u_2 . However, the degree of v drops from 3 to 1, which costs 4. Hence, the score when the set **B** is chosen is at least $2 - 4 = -2$. Thus, the overall score of the configuration is at least $[1 \mid -2]$ if the **link** is badly colored, which yields an average score of at least $[-\frac{1}{2}]$.

(b) Suppose that the **link** contains at least one dotted vertex. We may assume that u_1 is dotted amber. If the set **A** is chosen, then we delete neither u nor v , and we mark u , yielding a score of at least 0. If the set **B** is chosen, we do not delete u or v . Since u_1 is dotted and the edge uu_1 is not cut, we note that the

degree of the vertex u is at least 2. Thus, the overall score of the configuration is at least $[0 \mid 0]$ if the link contains at least one dotted vertex, which yields an average score of at least $[0]$.

(c) Suppose that the link is well colored. We may assume that u_1 is blue and u_2 is amber (and neither u_1 nor u_2 is dotted). Further by symmetry on the roles of the colors, we may further assume that the vertex v_1 is blue. If the set **A** is chosen, then we delete u but not v . The deletion of u contributes 4 to the score, and we gain 1 by cutting the outgoing edge from u to the blue vertex u_1 . If v_1 is dotted blue, then the degree of v drops from 3 to 2, and the contribution of cutting the edge wv is -1 . If v_1 is blue and not dotted, then the degree of v drops from 3 to 1, and the contribution of cutting the edge wv is $-4 + 1 = -3$ noting that in this case we gain 1 by cutting the outgoing edge from v to the vertex v_1 . Hence, the score when the set **A** is chosen is at least 4 if the blue vertex v_1 is dotted and at least 2 if the blue vertex v_1 is not dotted.

If the set **B** is chosen, then we delete both u and v , which contributes 8 to the score. Moreover, we gain 1 by cutting the outgoing edge from u to the amber vertex u_2 . If cutting the edge wv results in the vertex w having degree at least 1, then the contribution of cutting the edge wv is at most -3 , yielding a score when the set **B** is chosen of at least $8 + 1 - 3 = 6$. Combined with the score when the set **A** is chosen, the overall score of the configuration is at least $[2 \mid 6]$ in this case, which yields an average score of at least $[4]$, as desired. Hence we may assume that the vertex w becomes isolated upon the deletion of u and v .

Suppose that w is a vertex of degree 2 in G . In that case, its second neighbor v' is not on the green-black-cycle C by supposition. Since by assumption the vertex w is isolated when the set **B** is chosen, the edge wv' was cut and we infer that v' is the center of an isolated 1-star that is adjacent to the center u' of an isolated 2-star with respect to the green-black-cycle C . Further, the neighbors of u' on the cycle C are of different colors and the neighbor v'_1 of v' on C is colored blue, that is, a mirror situation occurs with u' and v' as with u and v . Thus, if v'_1 is the neighbor of v' on the cycle C , and if u'_1 and u'_2 are the neighbors of u' on C , then the link between u'_1 and u'_2 is well colored. Further, we may assume u'_1 is blue and u'_2 is amber (and neither u'_1 nor u'_2 is dotted), while v'_1 is blue, as illustrated in Figure 34 where in this case the vertex w has degree 2 in G .

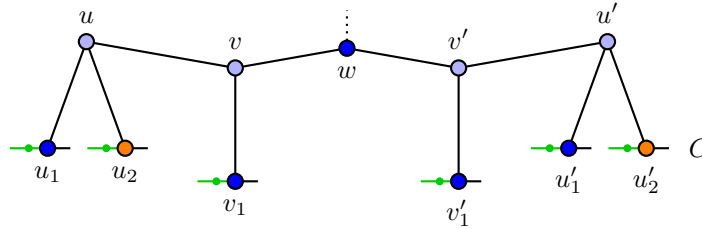


Figure 34: A special case in the proof of Claim 46(c)

We now combine the scores of the two configurations, where the one configuration contains u and v and the other configuration contains u' and v' . In this special case, we delete u, v, w, u' and v' , and we add the vertex w to the dominating set. The deletion of these five vertices contributes $4 \times 4 + 5 = 21$ to the score. We gain 2 by cutting the outgoing edge from u to the amber vertex u_2 and the outgoing edge from u' to the amber vertex u'_2 . However, adding the vertex w to the dominating set contributes -12 to the score. Hence, the score when the set **B** is chosen is at least $21 + 2 - 12 = 11$.

Recall that the score when the set **A** is chosen is at least 4 if the blue vertex v_1 is dotted and at least 2 if the blue vertex v_1 is not dotted. By symmetry, the score when the set **A** is chosen is at least 4 if the blue vertex v'_1 is dotted and at least 2 if the blue vertex v'_1 is not dotted. If at least one of v_1 and v'_1 is dotted, then the overall score of combining the two configurations is at least $[6 \mid 11]$. Sharing this score equally among the two configurations yields a score of at least $[3 \mid \frac{11}{2}]$ assigned to each configuration. This in turn

yields an average score of at least $\lceil \frac{17}{4} \rceil > [4]$ for each configuration. Hence, we may assume that neither v_1 nor v'_1 is dotted. In this special case, when choosing the set \mathbf{A} , we delete u, v, w, u' and v' , and we add the vertex w to the dominating set. The deletion of these five vertices contributes 21 to the score. We gain 4 by cutting the outgoing edges to the blue vertices, namely $uu_1, vv_1, u'u'_1$, and $v'v'_1$ noting that neither v_1 nor v'_1 is dotted. Hence, the score when the set \mathbf{A} is chosen is at least $21 + 4 - 12 = 13$. Thus, the overall score of combining the two configurations is at least $\lceil \frac{13}{2} \rceil + \lceil \frac{11}{2} \rceil$ assigned to each configuration. This in turn yields once again an average score of at least $\lceil \frac{24}{4} \rceil > [4]$ for each configuration.

Suppose next that w is a vertex of degree 3 in G . Recall that w is not the center of an isolated 2-star with respect to the **green-black**-cycle C . Thus, either w is the center of a configuration of Claim 44, or w has at least one neighbor v' that is the center of an isolated 1-star that is adjacent to the center u' of an isolated 2-star with respect to the **green-black**-cycle C . In the latter, as before, we infer that the neighbors of u' on the cycle C are of different colors and the neighbor v'_1 of v' on C is colored blue, as illustrated in Figure 34 where in this case the vertex w has degree 3 in G . We combine the scores of the two configurations and use analogous arguments to yield as before an average score of at least $\lceil \frac{17}{4} \rceil > [4]$ for each configuration. We note that in this case when the vertex w is added to the dominating set, then we mark its third neighbor different from v and v' .

Suppose finally that w is the center of a configuration of Claim 44, i.e., w is adjacent to two centers of 2-stars, namely u' and v' as illustrated in Figure 35. Let W' be the set of neighbors of u' and v' on the alternating **green-black** cycle C , namely $W' = \{u'_1, u'_2, v'_1, v'_2\}$. In this case, associated with the entire subgraph, we need a nonnegative contribution for the two isolated 2-stars with w as a common neighbor, and an average of $[4]$ for the link, so the average contribution needs to be at least $[4]$ on the whole subgraph.

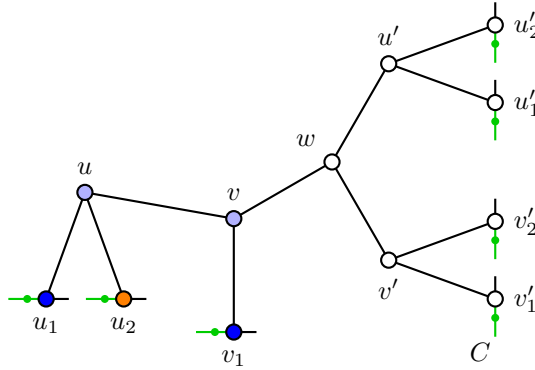


Figure 35: Another special case in the proof of Claim 46(c)

Assume first that one vertex in W' is blue, say u'_1 . In this case, we remove also u, v, w, u' and v' , and we add v' to the set. The contribution of removing the vertices is 20, and the contribution of adding the vertex v' to the dominating set is -12 . We may also lose -3 for cutting the edge $u'u'_2$ if u'_2 is dotted amber, and we gain 1 for deleting the edge uu_2 . We mark any dotted amber neighbor of v' . The total contribution to the score is therefore at least $\lceil 2 \rceil + 6$, and thus the average score is at least $[4]$ as required.

Suppose now that all vertices in W' are amber, and none is dotted. In that case, we put w in the set and remove the vertices u, v, w, u' and v' . The contribution is 20 for deleting the vertices, -12 for adding the vertex w , and we gain 5 for deleting the outgoing edges. The total contribution to the score is therefore at least $\lceil 2 \rceil + 13$, yielding an average score of at least $\lceil \frac{15}{2} \rceil$, which is more than enough for the desired average score of at least $[4]$.

Suppose now that all vertices in W' are amber, and at least one is dotted but all dotted vertices are on only one side, say only u'_1 and u'_2 are dotted. In this case, we mark u' , delete u, v, w and v' , and add w to the set. The contribution to the score is 16 for deleting the vertices, -12 for adding vertex w . Moreover, we obtain 3 for deleting the outgoing edges. The total contribution to the score is therefore at least $[2 \mid 7]$, once again yielding an average score of at least $[4]$ as required.

Finally, suppose all vertices in W are amber and there are dotted vertices on both sides, say at least u'_1 and v'_1 are dotted. In this case, we remove u and v , with a contribution of 8 to the score, and cut the edge vw with a contribution of -1 to the score. We also gain 1 for cutting the edge uu_2 . The total contribution to the score is therefore at least $[2 \mid 8]$, once again yielding an average score of at least $[4]$, as required. This completes the proof of Claim 46. \square

5.7.2 Assigning colors Amber and Blue to the green edges

We now define the coloring of the green edge in the green-black cycle to ensure that selecting one set among **A** or **B** ends up with a nonnegative score. Recall that our Green cycle graph consists now of one alternating green-black cycle, to which are added some links joining two vertices in the green-black cycle, and fibers attached to only one vertex in the cycle, as in Definition 13. The green edges are given colors among Amber and Blue, from which are deduced the colors of vertices as from Definition 11.

Initially, we let all green edges of the green-black cycle be colored Amber. Let w_B denote the number of well colored broad links in the Green cycle graph, w_N the number of well colored narrow links, b_B the number of badly colored broad links, and b_N the number of badly colored narrow links. By definition of our scoring, if

$$5w_B + 4w_N - b_B - \frac{1}{2}b_N \geq 0.$$

then the average score of all configurations together is nonnegative. This necessarily implies that either the set **A** has a nonnegative score, or the set **B** does.

If the average score of all configurations is less than zero, then we apply a set of rules (to be defined shortly) iteratively to modify the green edges coloring, as long as one of them apply. After proving that this process necessarily ends, we prove important properties of the resulting Green cycle graph, and use a discharging procedure to guarantee that the resulting average score is nonnegative, including the contributions of changes of colors and of dotted edges and fibers. From these arguments, we infer that the conclusion is the same, i.e., that one of **A** and **B** must have a nonnegative score.

The underlying idea of each of the following rules is the following. We define a *cut* that is a portion of the alternating green-black cycle that goes from one black edge to another. These two extremal black edges are called the *ends* of the cut. The central idea is to switch the colors of all Amber and Blue green edges in that cut (and thus also switching the colors of the vertices inside the cut) whenever that increases the total average score of the Green cycle graph. During this process, we introduce in the Green cycle graph changes of colors and dotted vertices as defined in Definition 12. Recall that each change of color contributes -4 to the score of the resulting removed vertices.

We now formally define three rules, each corresponding to a different situation of a cut. The first rule applies when neither black edge at the end of the cut corresponds to a change of color, the second applies when both ends correspond to a change of color, the third when precisely one end corresponds to a change of color. Each rule simply describes necessary and sufficient condition such that changing the colors on one side of the cut increases the average score, or does not change the average score but reduces the total number of changes of color in the cycle, or changes neither of these two parameters but decreases the number of dotted fibers.

For this, we identify now in variables all **links** and fibers whose score is modified by such an operation. In particular this implies all **links** and fibers that are *hit by a cut*, i.e., that are incident to an extremity of a **black edge** of the cut. It also implies **links** that are *cut*, i.e., **links** that have one extremity inside the cut and the other outside. Note that **links** with a dotted extremity do not change score when cut, and thus are not considered below.

For Rule 1, we use the following variables:

- wc is the number of extremities of cut, not hit, well colored **links**.
- bc is the number of extremities of cut, not hit, badly colored **links**.
- wch is the number of extremities of cut, hit, well colored **links**.
- bch is the number of extremities of cut, hit, badly colored **links**.
- wh is the number of extremities of not cut, hit, well colored **links**.
- bh is the number of extremities of not cut, hit, badly colored **links**.
- oh is the number of link extremities hit whose opposite extremity is dotted.
- f is the number of hit fibers.

Moreover, each of these **link** variables may be restricted to broad or narrow links, by the addition of a subscript $*_B$ or $*_N$, respectively. We remark that a **link** that is hit on both extremities should be counted twice, within two variables.

Rule 1: For cuts whose ends do not correspond to color changes, switching the colors in the cut will increase the score whenever:

$$6(bc_B - wc_B) + \frac{9}{2}(bc_N - wc_N - wch_B - wh_B) - 4(wh_N + wch_N) + \frac{3}{2}(bch_B + bh_B) + \frac{1}{2}(bch_N + bh_N) + \frac{1}{2}(oh_B - f) > 8. \quad (16)$$

A schematized green cycle graph coloring and application of Rule 1 is illustrated in Figure 36.

Proof that Rule 1 increases the score: Based on Definitions 14 and 15, the score of each **link** and fiber after implementation of Rule 1 changes as follows:

- A well colored **link** that is cut but not hit has a score changing from +5 to -1 if it is broad, from +4 to $-\frac{1}{2}$ if it is narrow. The resulting contribution to the score is $-6wc_B - \frac{9}{2}wc_N$.
- A badly colored **link** that is cut but not hit has a score changing from -1 to +5 if it is broad, from $-\frac{1}{2}$ to +4 if it is narrow. The resulting contribution to the score is $6bc_B + \frac{9}{2}bc_N$.
- A well colored link that is hit, whether cut or not, has a score that changes from +5 to $+\frac{1}{2}$ if it is broad, from +4 to 0 if it is narrow. The resulting contribution to the score is $-\frac{9}{2}(wch_B + wh_B) - 4(wch_N + wh_N)$.
- A badly colored link that is hit, whether cut or not, has a score that changes from -1 to $+\frac{1}{2}$ if it is broad, from $-\frac{1}{2}$ to 0 if it is narrow. The resulting contribution to the score is $\frac{3}{2}(bch_B + bh_B) + \frac{1}{2}(bch_N + bh_N)$.
- A link whose other extremity is dotted see its score changing from $\frac{1}{2}$ to 1 if it is broad, but its score remains 0 if it is narrow. The resulting contribution to the score is $\frac{1}{2}oh_B$.
- A fiber that is hit has its score dropping from 0 to $-\frac{1}{2}$. The resulting contribution to the score is $-\frac{1}{2}f$.
- Finally, the rule introduces two new changes of color. The resulting contribution to the score is -8.

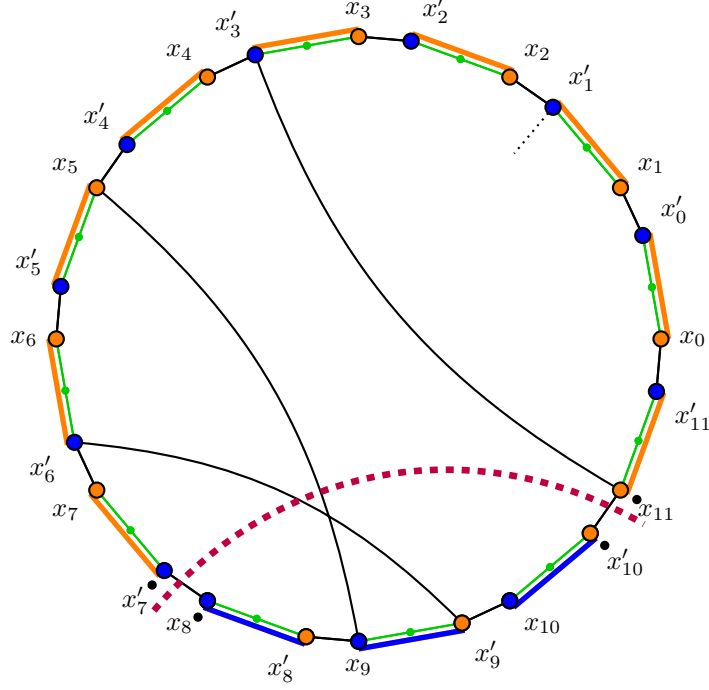


Figure 36: Schematized green cycle graph coloring and application of Rule 1.

Summing it all up, yields the formula given in (16). This formula can be simplified when ignoring whether the links are broad or narrow, taking the worse scenario. This provides the following simplified formula:

Simplified Rule 1:

$$-6wc + \frac{9}{2}(bc - wch - wh) + \frac{1}{2}(bch + bh - f) > 8. \quad (17)$$

For Rule 2, we use similar definitions as above, but all links and fibers that are hit are incident to color changes, and so were dotted.

Before formally stating Rule 2, we add the following definitions of variables:

- $wchd$ is the number of extremities of cut, hit, well colored dotted links.
- $bchd$ is the number of extremities of cut, hit, badly colored dotted links.
- whd is the number of extremities of not cut, hit, well colored dotted links.
- bhd is the number of extremities of not cut, hit, badly colored dotted links.
- ohd is the number of dotted link extremities hit whose opposite extremity is also dotted.
- fd is the number of dotted hit fibers.

Rule 2: For cuts whose both ends correspond to color changes, switching the colors in the cut will not decrease the score (and reduce the number of color changes) whenever:

$$\begin{aligned} 6(bc_B - wc_B) + \frac{9}{2}(bc_N - wc_N + whd_B + bchd_B) + 4(whd_N + bchd_N) \\ - \frac{3}{2}(bhd_B + wchd_B) + \frac{1}{2}(fd - bhd_N - wchd_N - ohd_B) \geq -8 \end{aligned} \quad (18)$$

Proof that Rule 2 does not decrease the score: Based on Definitions 14 and 15, the score of each link and fiber after implementation of Rule 2 changes as follows:

- A well colored link that is cut but not hit has a score changing from +5 to -1 if it is broad, from +4 to $-\frac{1}{2}$ if it is narrow. The resulting contribution to the score is $-6wc_B - \frac{9}{2}wc_N$.
- A badly colored link that is cut but not hit has a score changing from -1 to +5 if it is broad, from $-\frac{1}{2}$ to +4 if it is narrow. The resulting contribution to the score is $6bc_B + \frac{9}{2}bc_N$.
- A well colored dotted link that is hit but not cut, or a badly colored dotted link that is hit and cut, has a score that changes from $+\frac{1}{2}$ to +5 if it is broad, and from 0 to +4 if it is narrow. The resulting contribution to the score is $\frac{9}{2}(whd_B + bchd_B) + 4(whd_N + bchd_N)$.
- A badly colored dotted link that is hit but not cut or a well colored dotted link that is hit and cut, has a score that changes from $+\frac{1}{2}$ to -1 if it is broad, from 0 to $-\frac{1}{2}$ if it is narrow. The resulting contribution to the score is $-\frac{3}{2}(bhd_B + wchd_B) - \frac{1}{2}(bhd_N + wchd_N)$.
- A link whose other extremity is dotted see its score changing from 1 to $\frac{1}{2}$ if it is broad, but its score remains 0 if it is narrow. The resulting contribution to the score is $-\frac{1}{2}ohd_B$.
- A fiber that is hit has its score changing from $-\frac{1}{2}$ to 0. The resulting contribution to the score is $+\frac{1}{2}fd$
- Finally, the rule removes two changes of color. The resulting contribution to the score is +8.

Summing it all up, yields the formula given in (18). We also apply Rule 2 when there is equality since it will reduce the number of color changes, which we want to minimize if it does not increase the score. This formula can be simplified when ignoring whether the links are broad or narrow, taking the worse scenario. This provides the following simplified formula:

Simplified Rule 2:

$$-6wc + \frac{9}{2}bc + 4(whd + bchd) - \frac{3}{2}(bhd + wchd) + \frac{1}{2}(fd - ohd) \geq -8. \quad (19)$$

We present next Rule 3 which combines all the changes of the above rules.

Rule 3: For cuts with exactly one end corresponding to a color change, switching the colors in the cut will not decrease the score (and reduce the number of color changes) whenever:

$$\begin{aligned} & 6(bc_B - wc_B) + \frac{9}{2}(bc_N - wc_N - wch_B - wh_B + whd_B + bchd_B) \\ & + 4(whd_N + bchd_N - wh_N - wch_N) + \frac{3}{2}(bch_B + bh_B - bhd_B - wchd_B) \\ & + \frac{1}{2}(bch_N + bh_N - bhd_N - wchd_N + oh_B - ohd_B - f + fd) > 0 \end{aligned} \quad (20)$$

or when the left term is equal to 0, but $fd > 0$.

Proof that Rule 3 does not decrease the score: The proof is simply a combination of the two above proofs.

We also apply Rule 3 when there is equality since it will reduce the number of dotted fibers, which we want to minimize. This formula can be simplified when ignoring whether the links are broad or narrow, taking the worse scenario. This provides the following simplified formula:

Simplified Rule 3:

$$\begin{aligned}
& -6wc + \frac{9}{2}(bc - wh - wch) + 4(whd + bchd) - \frac{3}{2}(wchd + bhd) \\
& \quad + \frac{1}{2}(bh + bch + fd - f - ohd_B) > 0.
\end{aligned} \tag{21}$$

Suppose we apply these rules whenever such a rule can be applied. We show that the result has a positive overall score. We first note that this process is finite. Indeed, every application of a rule either increase the score (Rule 1, 2 or 3), or does not change the score but reduces the number of color changes (Rule 2), or does not change the score or the number of color changes, but reduces the number of dotted fibers (Rule 3).

In the following, we prove some properties of the coloring resulting by applying the rules, until none of the three rules can be applied. We generally consider a configuration with **black edges** on the cycle and **links** of fiber extremities, implicitly using an orientation of the cycle (clockwise or counterclockwise, it does not matter provided it is consistent within the configuration). For a **black edge** e , we call the *first black edge after e* , denoted e^{+1} , the **black edge** that is further by the orientation but adjacent to the same **green edge**. We similarly define the *first edge before e* that we denote e^{-1} . The first **black edge** before or after a vertex x is the first **black edge** before or after the **black edge** incident to x .

For a **black edge** e or a vertex x , we say it is *followed by* (or *preceded by*) a **link** extremity or a color change when there is neither color changes nor extremities of **link** whose other extremity is not dotted in between. In other words, a **black edge** e is followed by a vertex x if all **black edge** in between may be incident only to fibers or extremities of dotted **links**, which will not contribute to the score in the application of one of the above rules. Implicitly, the **link** extremity has its other extremity not dotted.

A **black edge** e or a vertex x is followed by two **link** extremities x_2 and x_3 when it is followed by the **link** extremity x_2 itself followed by the other **link** extremity x_3 , and so on.

Observe that when x is the extremity of a **link** or a fiber, it is adjacent to the center of a black star, and so by Claim 23, the first **black edge** before or after x is incident to at most one vertex itself adjacent to the center of a black star (which could then be the extremity of a **link** or a fiber). A similar property holds for three consecutive **black edges** as stated by the following claim.

Claim 47 *If x_1, x_2 and x_3 are three extremities of links or fibers, then there is a **black edge** hitting nothing between x_1 and x_3 .*

Proof. Suppose firstly that two extremities (say x_1 and x_2) are incident to the same **black edge** e . Then by Claim 23, e^{-1} and e^{+1} are incident to no **link** or fiber and thus e^{+1} is a **black edge** hitting nothing between x_1 and x_3 . Suppose now that x_1, x_2 and x_3 belong to three consecutive **black edges**, say u'_1u_2, u'_2u_3 and u'_3u_4 . By symmetry, suppose $x_2 = u_3$ is incident to a **link** or a fiber, and thus that u_3 is adjacent to the center of a black star. We illustrate this configuration in Figure 37. Then, by Claim 23, neither u_2 nor u_4 is adjacent to the center of a black star. By our assumption, this means that both $x_1 = u'_1$ and $x_3 = u'_3$ are incident to a fiber or a **link**, and thus adjacent to the center of a black star. But this is in contradiction with Claim 29. \square

Claim 48 *For every bad link whose both extremities are in the same color region (that we call an arch), there are at least two **black edges** not hitting either extremities within the arch. Moreover, one **black edge** is incident to no **link** or fiber.*

Proof. Since a bad **link** in a color area has both extremities of same parity, the path between the extremities is composed of identical numbers of **green edges** and **black edges**. Thus if the path has k **black edges**, the

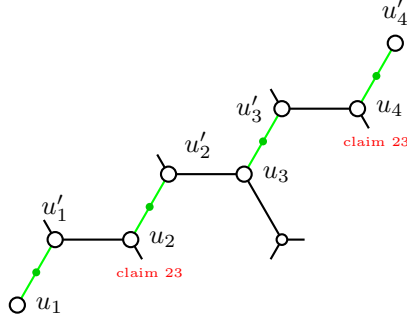


Figure 37: An illustration of the proof of Claim 47

cycle composed of this path and the arch is of length $3k + 2$. By the initial assumption, no cycle of length 5 or 8 belong to the graph, so there are at least three **black edges** in the path, and two of them do not hit the extremities of the arch. The second part of the claim is now a direct consequence of Claim 47. \square

Claim 49 *No **black edge** on the **Green cycle graph** is incident to two badly colored links.*

Proof. Suppose, to the contrary, that a **black edge** e is incident to two badly colored links. By Claim 23, e^{-1} and e^{+1} are incident to no **link** or fiber. Thus we apply Rule 1 that match the number of color change on e^{-1} and e^{+1} . We get that $bc = 2$ and all other values are zero, and so the counting is at worse $2 \times \frac{9}{2} > 8$, which is a contradiction. \square

Claim 50 *If a **black edge** e that hits nothing is followed by three badly colored **link** extremities x_1, x_2 and x_3 , then x_1x_2 is an arch.*

Proof. Let the **black edge** e and extremities x_1, x_2 and x_3 be defined as in the claim statement, and suppose, to the contrary, that x_1x_2 is not an arch. Let e' be the first **black edge** after x_2 . By Claim 23, e' is incident to at most one extremity of a **link** or a fiber. We now apply simplified Rule 1 from e to e' . We note that $wh + wch = 0$ or x_3 would not follow x_2 . This yields $bc = 2, f + bh + bch + oh \leq 1$, with all other terms in the simplified Rule 1 zero. The counting gives us at least $2 \times \frac{9}{2} - \frac{1}{2} > 8$, and simplified Rule 1 applies, which is a contradiction. \square

Claim 51 *If an extremity x_1 of a badly colored **link** is followed by three other badly colored **link** extremities, namely x_2, x_3 and x_4 , then x_2x_3 is an arch.*

Proof. Let x_1, x_2, x_3 and x_4 are defined as in the claim, and suppose, to the contrary, that x_2x_3 is not an arch. We consider the first **black edge** e_ℓ before x_2 and the first **black edge** e_r after x_3 , and try to apply simplified Rule 1 on the cut from e_ℓ to e_r . By Claim 23, each of e_ℓ and e_r is incident to at most one **link** or fiber. Moreover, if e_ℓ (resp., e_r) hits anything, then by Claim 47, e_ℓ^{-1} (resp. e_r^{+1}) hits nothing. We note that by hypothesis, if e_ℓ (resp., e_r) is incident to the extremity of a **link**, either it has its other extremity dotted, or it is x_1 (resp., x_4).

Suppose that e_ℓ is incident to x_1 , and that x_1 is the other extremity of an arch ending in x_2 or x_3 . Note that by Claim 48, the arch must be x_1x_3 and there is a **black edge** e hitting nothing between x_2 and x_3 . Applying Rule 1 from e_ℓ^{-1} to e , we get that $bc = 2$ (corresponding to the links ending in x_1 and x_2) and

nothing else is cut or hit. The counting gives us at least $2 \times \frac{9}{2} - \frac{1}{2} > 8$, and simplified Rule 1 applies, which is a contradiction.

By a symmetric argument, e_r is not incident to x_4 . Further, x_4 does not form an arch with x_2 or x_3 . Now, if e_ℓ (resp., e_r) hits anything, we replace it with e_ℓ^{-1} (resp., e_r^{+1}). The same counting applies except possibly $bc = 3$ or 4 , which is even better. Thus once again, simplified Rule 1 applies, which is a contradiction. \square

As a corollary of Claim 51, we obtain the following property of badly colored **link** extremities.

Claim 52 *No badly colored **link** extremity is followed by four badly colored **link** extremities.*

Proof. Suppose, to the contrary, that x_1, x_2, x_3, x_4 and x_5 are four badly colored **link** extremities following a badly colored **link** extremity. By Claim 51, x_2x_3 and x_3x_4 are both arches, a contradiction.

We now consider fiber and **link** extremities hit by a color change.

Claim 53 *In the **Green cycle graph**, no color change hits a fiber. Moreover, if a color change hits two **link** extremities, both also have their other extremity dotted.*

Proof. Consider a **black edge** e on the **Green cycle graph** corresponding to a color change. We note that both extremities of e are Amber or both are Blue. Suppose firstly that the color change $e = u'_i u_{i+1}$ hits two extremities, either of some **links** or fibers. By Claim 23, the **black edges** e^{-1} and e^{+1} hit no **link** or fiber.

Suppose, to the contrary, that one extremity is of a **link** and one of a fiber. By symmetry, say u'_i hits the extremity of a **link** and u_{i+1} hits the extremity of a fiber. If the **link** incident to u'_i is badly colored, we use Rule 3 (or Rule 2 if e^{-1} is also a change of color) on the cut from e^{-1} to e . In this case, $bchd = 1$ and $fd = 1$, with all other terms in the associated formula for Rule 3 equal to zero. This yields $+4 + \frac{1}{2} > 0$, and simplified Rule 3 applies, a contradiction. If the **link** incident to u' is well colored, we use Rule 3 (or Rule 2 if e^{+1} is also a change of color) on the cut from e to e^{+1} . In this case, $whd = 1$ and $df = 1$, with all other terms in the associated formula for Rule 3 equal to zero. This yields $+4 + \frac{1}{2} > 0$ and Rule 3 applies, a contradiction. If the **link** incident to u' has its other extremity dotted, we use Rule 3 (or Rule 2 if e^{+1} is also a change of color) on the cut from e to e^{+1} . In this case, $ohd = 1$ and $df = 1$, with all other terms in the associated formula for Rule 3 equal to zero. If e^{+1} is also a color change, we get $\frac{1}{2}(+1 - 1) > -8$ and Rule 2 applies, otherwise we get $\frac{1}{2}(+1 - 1) = 0$ and Rule 3 applies since its application reduces the number of dotted fibers. In both cases, we obtain a contradiction.

Suppose next that the color change e hits two fiber extremities. We use Rule 3 on the cut from e^{-1} to e . In this case, $df = 2$, with all other terms in the associated formula for Rule 3 equal to zero. This yields $2\frac{1}{2} > 0$, and Rule 3 applies, a contradiction.

Suppose now that the color change e hits two **link** extremities. If both **links** are well colored or both are badly colored, then we use Rule 3 on the cut from e^{-1} to e . We have either $bhd = bchd = 1$ or $whd = wchd = 1$, with all other terms in the associated formula for Rule 3 equal to zero. This yields $4 - \frac{3}{2} > 0$, and Rule 3 applies, a contradiction. If one **link** is well colored and the other badly, say u'_i is well colored, then we use Rule 3 on the cut from e to e^{+1} . In this case, $whd = 1$ and $bchd = 1$, with all other terms in the associated formula for Rule 3 equal to zero. This yields $4 + 4 > 0$, and Rule 3 applies, a contradiction.

Suppose now that one **link** has its other extremity dotted, say u'_i . If the other **link** is well colored, then we apply Rule 3 from e^{-1} to e , where $whd = 1$ and $ohd = 1$, with all other terms in the associated formula for Rule 3 equal to zero. The counting gives $+4 - \frac{1}{2} > 0$, and Rule 3 applies, a contradiction. If the other **link** is badly colored, then we apply Rule 3 from e to e^{+1} . In this case, $bchd = 1$ and $ohd = 1$, with all other

terms in the associated formula for Rule 3 equal to zero. The counting is the same, and so Rule 3 applies, a contradiction. Therefore, only the case when the other **link** also has its other extremity dotted remains, which may occur according to the claim.

Suppose now that the color hits only one extremity, of a fiber. By Claim 47, at least one of the **black edges** e^{-1} and e^{+1} hit no **link** or fiber, name it e' . Applying Rule 3 from e to e' , we get $df = 1$, with all other terms in the associated formula for Rule 3 equal to zero. This yields $\frac{1}{2} > 0$, and Rule 3 applies, a contradiction. \square

Claim 54 *The contribution of the links that are hit by one color change when applying Rule 2 or 3 is at least $-\frac{3}{2}$. Moreover, if the color change does not hit a broad **link** extremity, then the contribution is at least $-\frac{1}{2}$.*

Proof. Using Claim 53, we know that for a color change when applying Rule 2 or 3, either $ohd \leq 2$ and $wch + wh + bch + bh = 0$, or $ohd = 0$ and $wch + wh + bch + bh \leq 1$. In both cases, $fd = 0$.

First assume that the **links** hit may be broad. The contribution of ohd is then $-\frac{1}{2}$ and the contribution of other links that are hit (from $wch + wh + bch + bh$) is then at least $-\frac{3}{2}$. In the first situation, the contribution of **links** that are hit by this color change is then at least $2 \times -\frac{1}{2} = -1$ and in the second situation it is at least $-\frac{3}{2}$. This yields a minimum contribution of at least $-\frac{3}{2}$ for each color change.

If we add the assumption that the color change hits no broad **link**, then ohd contributes 0 and the other **links** that are hit contribute at least $-\frac{1}{2}$. In the first situation, the contribution is 0 and in the second situation, the contribution is at least $-\frac{1}{2}$. This yields a minimum contribution of at least $-\frac{1}{2}$ for each color change. \square

In view of Claim 54, we can therefore reformulate Rules 2 and 3, introducing cc_B for counting the ends of the cut corresponding to color changes that hit at least one broad **link** ($0 \leq cc_B \leq 2$ for Rule 2, $0 \leq cc_B \leq 1$ for Rule 3). In Rule 2, one color change contributes at least $-\frac{1}{2} - cc_B$, while in Rule 3, the two color changes contribute $2 \times -\frac{1}{2} - cc_B$.

Reformulated Rule 2:

$$6(bc_B - wc_B) + \frac{9}{2}(bc_N - wc_N) - cc_B \geq -7 \quad (22)$$

Reformulated simplified Rule 2:

$$-6wc + \frac{9}{2}bc - cc_B \geq -7 \quad (23)$$

Reformulated Rule 3:

$$\begin{aligned} 6(bc_B - wc_B) + \frac{9}{2}(bc_N - wc_N - wch_B - wh_B) + 4(-wh_N - wch_N) \\ + \frac{3}{2}(bch_B + bh_B) - cc_B + \frac{1}{2}(bch_N + bh_N + oh_B - f) > \frac{1}{2} \end{aligned} \quad (24)$$

Reformulated simplified Rule 3:

$$-6wc + \frac{9}{2}(bc - wch - wh) - cc_B + \frac{1}{2}(bch + bh - f) > \frac{1}{2} \quad (25)$$

Claim 55 *No color change is followed by another color change.*

Proof. Suppose, to the contrary, that e is a color change followed by another color change e' . In this case, we apply the reformulated Rule 2 from e to e' , where $wc + bc = 0$ and $cc_B \leq 2$. The counting gives at least $-2 \geq -7$, and the reformulated simplified Rule 2 applies, a contradiction.

Claim 56 *If a color change e is followed by one **link** extremity and then another color change e' , then the **link** is well colored, broad, and both color changes hit a broad **link** extremity.*

Proof. Let the color changes e and e' be defined as above. We want to apply reformulated Rule 2 from e to e' . Suppose firstly that the **link** is not a broad well colored link. In this case, we get $wc_B = 0$, $wc_N + bc_N + bc_B = 1$, and $cc_B \leq 2$. The counting gives at least $-\frac{9}{2} - 2 = -\frac{13}{2} \geq -7$, and the reformulated simplified Rule 2 applies, a contradiction. This proves the first part of the claim, that the **link** is a well colored broad **link**. Suppose next that $wc_B = 1$. Suppose that at most one of the color changes hits a broad link, i.e., that is, $cc_B \leq 1$. In this case, the reformulated Rule 2 gives at least $-6 - 1 = -7$, and again reformulated Rule 2 applies, a contradiction. \square

Claim 57 *If there is a bad **link** extremity x_1 preceded by a color change, then it is immediately followed by a well colored **link** extremity x_2 , i.e., either the two **link** extremities are on the same **black edge**, or the well colored **link** extremity x_2 is on the first **black edge** after the badly colored **link** extremity x_1 .*

Proof. Suppose that after a color change on some **black edge** e , there is an extremity x_1 of a badly colored **link** not immediately followed by a well colored **link** extremity. In particular, the first **black edge** e' after x_1 does not hit a well colored **link**. We want to apply (reformulated simplified) Rule 3 from e to e' . By Claim 48, e' does not hit the other extremity of the badly colored **link** ending on x_1 , and so $bc = 1$. By supposition, $wh + wch = 0$ and $wc = 0$. By Claim 23, e' hits at most one fiber, and so $f \leq 1$. Applying the reformulated counting where $cc_B \leq 1$ gives at least $\frac{9}{2} - \frac{1}{2} - 1 > \frac{1}{2}$, and reformulated Rule 3 applies, a contradiction. \square

Claim 58 *If a color change e is followed by two **link** extremities and then another color change, then both **links** are distinct and well colored.*

Proof. Suppose at least one of the **links** is badly colored. In this case, we apply (reformulated) Rule 2 from one color change to the other. By supposition, we have $wc \leq 1$, $bc \geq 1$, and $cc_B \leq 2$. The counting thus gives at least $-6 + \frac{9}{2} - 2 = -\frac{7}{2} \geq -7$, and reformulated Rule 2 applies, a contradiction. If the two **link** extremities are those of a same **link**, then $wc = 0$, $bc = 0$, and $cc_B \leq 2$, and the counting yields $-2 \geq -7$, implying once again that reformulated Rule 2 applies, a contradiction.

Claim 59 *Suppose a color change is followed by three **link** extremities, then another color change. Then at least two extremities are those of well colored **links**.. If only two are well colored, then either both well colored **links** are broad, or both changes of color hit a broad **link**, or the badly colored **link** is narrow and at least one color change hits a broad **link**.*

Proof. Suppose that a color change e is followed by three **link** extremities, x_1 , x_2 and x_3 , and then another color change e' . First suppose only one **link** extremity is of a well-colored **link**. By Claim 57, x_2 is well colored, and x_1 and x_3 are badly colored. Moreover, x_2 is immediately after x_1 . Applying Claim 57 in the other direction, x_3 is also immediately after x_2 . But this contradicts Claim 47.

Assume now only two **link** extremities are well colored. We apply Rule 2 from e to e' . If neither color change hit a broad **link** then at least one of the well colored **link** is narrow. In this case we have $wc_N \geq 1$,

$wc \leq 2$, $bc = 1$, and $cc_B = 0$. The counting gives at least $-6 - \frac{9}{2} + \frac{9}{2} = -6 \geq -7$, and reformulated Rule 2 applies, a contradiction.

Suppose now the third **link** extremity is badly colored and broad, and at least one color change does not hit a broad **link**. In this case, we have $wc = 2$, $bc_B = 1$ and $cc_B \leq 1$. The counting gives at least $-2 \times 6 + 6 - 1 = -7$, and the reformulated Rule 2 applies, a contradiction. \square

Claim 60 *If a color change is followed by four **link** extremities, and then another color change, then at most two **link** extremities are badly colored, and if there are two, they form an arch.*

Proof. Suppose that two color changes e and e' are separated by four **link** extremities. Suppose, to the contrary, that there are at least two badly colored **link** extremities which do not form an arch. In this case we apply reformulated Rule 2 from e to e' , and we get $bc \geq 2$, $wc \leq 2$, and $cc_B \leq 2$. The counting gives at least $2\frac{9}{2} - 12 - 2 = -5 \geq -7$ and reformulated Rule 2 applies, a contradiction. \square

Claim 61 *If a color change is followed by at least four **link** extremities, then either there are two well colored **link** extremities among the first four extremities, or the color change is followed by five **link** extremities x_1, x_2, \dots, x_5 such that the following properties hold in the configuration.*

- (a) Among x_1 and x_2 , one is the extremity of a badly colored **link** (name it x_b), and the other is the extremity of a well colored **link** (name it x_w).
- (b) If $x_b = x_1$, then x_b is immediately followed by x_2 .
- (c) The edge $x_b x_3$ is a badly colored arch.
- (d) The badly colored **link** extremity x_4 is immediately followed by a well colored **link** extremity x_5 .
- (e) If $x_b x_3$ is broad and x_w is the end of a narrow **link**, then the color change hits a broad **link**.

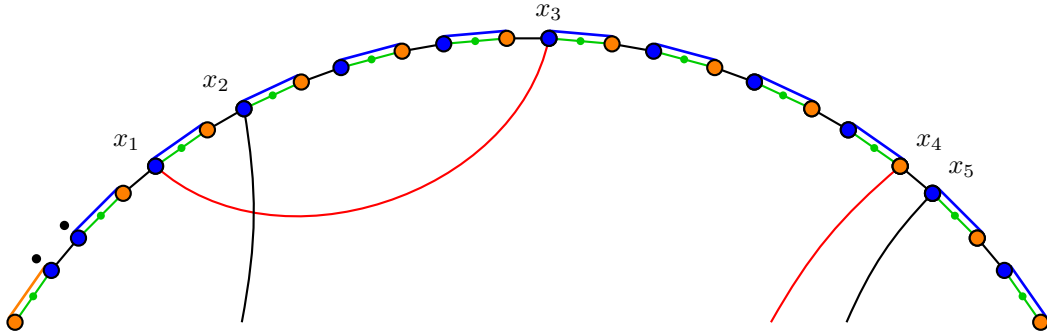


Figure 38: Illustration of the special case of Claim 61 when $x_b = x_1$

Proof. Let e be a color change, followed by five **link** extremities x_1, x_2, \dots, x_5 . Suppose that no two of the first four **link** extremities are well colored, i.e., only one of x_1, x_2, x_3, x_4 is well colored. We show that properties (a) to (e) hold. By Claim 57, the well colored **link** extremity must be x_1 or x_2 , name it x_w and the other x_b . Furthermore by Claim 57, if x_b is x_1 , then it is immediately followed by $x_w = x_2$ (and so, properties (a) and (b) hold). Moreover, x_3 and x_4 are the extremities of badly colored links.

Suppose firstly that $x_b x_3$ is not an arch. Let e_3 be the first **black edge** after x_3 . Our intention is to apply Rule 3 from e to e_3 . In this case, $bc = 2$ and $wc = 1$. By Claim 23, e' hits at most one fiber or link, which

may be x_4 or a **link** with the other extremity dotted. Thus in the counting, we get $bh + bch + f + oh \leq 1$ and $wh + wch = 0$. Using the reformulated simplified Rule 3 with $cc_B \leq 1$, we get $2 \times \frac{9}{2} - 6 - \frac{1}{2} - 1 = \frac{3}{2} > \frac{1}{2}$, and reformulated Rule 3 applies, a contradiction. Hence, $x_b x_3$ is a badly colored arch, which proves part (c).

By Claim 48, there is a **black edge** e_2 between x_b and x_3 hitting nothing. If $x_b = x_1$, since x_2 is immediately after x_1 , this **black edge** is also between x_2 and x_3 . Let e_4 be the first **black edge** after x_4 not hitting it. If there is no well colored **link** extremity immediately after x_4 , then e_4 does not hit the extremity of a well colored **link**. In this case, we wish to apply Rule 1 from e_2 to e_4 , with $bc = 2$ and $wc = 0$. We also have by supposition that $wh = wch = 0$. Since e_2 hits nothing, by Claim 23 on e_4 , we have $bch + bh + f + oh \leq 1$. The computation thus gives us at least $2\frac{9}{2} - \frac{1}{2} > 8$, and Rule 1 applies, a contradiction. Hence, there must be a well colored **link** extremity x_5 immediately after x_4 , which proves part (d).

Finally, let $x_b x_3$ be broad and let x_w be the end of a narrow **link**. Suppose, to the contrary, that the color change does not hit the end of a broad link. We wish to apply reformulated Rule 3 from e to e_2 . In this case, $bc_B = 1$, $wc_N = 1$, $wch_B + bch_B + wch_N + bch_N + oh_B = 0$, and $cc_B = 0$. The computation gives $6 - \frac{9}{2} \geq \frac{1}{2}$, and reformulated Rule 3 applies, a contradiction. This proves part (e). \square

Claim 62 *If a color change is followed by at least five **link** extremities and then a color change, then at least three of the **link** extremities are well colored.*

Proof. First, suppose the area between the two color changes contains six or more **link** extremities. By Claim 57, one of the first two extremities in both directions is of a well colored **link**, and those two are necessarily distinct. By Claim 61, there must also be two well colored **link** extremities among the first five. If the second well colored **link** extremity is the same as the first one in the other direction, then there are exactly six **link** extremities between the two color changes, and we are in the special situation of Claim 61. But then x_5 is immediately after x_4 , and from the other direction, x_5 is immediately after x_6 , contradicting Claim 47.

Suppose now the area between the two color changes contains exactly five **link** extremities. We denote the vertices that belong to these extremities by x_1, x_2, x_3, x_4 and x_5 . Suppose two or fewer of these **links** are well colored. By Claim 57, one of x_1 and x_2 is the extremity of a well colored **link** and also one of x_4 and x_5 . By supposition, all three other extremities are extremities of badly colored **links**. In particular, the **link** ending in x_3 is the extremity of a badly colored **link**. This link can form an arch with only one of the two badly colored **link** extremities. We may assume it does not form an arch with x_1 or x_2 . Let e^\dagger be the **black edge** immediately after x_3 . The edge e^\dagger may hit x_4 but in that case, x_4 is the extremity of a badly colored **link**. Indeed, if x_4 is well colored, then by Claim 57 (considered in the opposite direction), it is immediately followed by x_5 , and this contradicts Claim 47. Hence, e^\dagger hits no well colored **link**. Applying the reformulated simplified Rule 3 between e and e^\dagger , we have $bc = 2$, $wc = 1$, $0 \leq bh + bch + f \leq 1$, $wh = wch = 0$, $cc_B \leq 1$. The counting therefore yields at least $2\frac{9}{2} - 6 - \frac{1}{2} - 1 = \frac{3}{2} > \frac{1}{2}$, and so reformulated Rule 3 applies, a contradiction. \square

We proceed further by defining what we have coined a “heavy configuration.”

Definition 22 (Heavy configurations) *Consider the extremity of a well colored **link**. We say it is heavy if it is in one of the following configurations:*

- (a) *preceded by a color change and followed by a bad **link** extremity then a **black edge** hitting no fiber or well colored **link** extremity,*
- (b) *followed by a bad **link** extremity then a color change,*
- (c) *preceded by a color change and followed by a well colored narrow **link** then a color change,*
- (d) *followed by three badly colored **link** extremities, the first two of which do not form an arch.*

We also have two special heavy configurations for broad links:

- **super heavy:** preceded and followed by color changes.
- **slightly heavy:** preceded by a color change and followed by a well colored **link** then a color change or preceded by a color change and followed by a **black edge** hitting nothing.

Remark 23 From Claim 51, if a well colored **link** extremity is followed by four badly colored **link** extremities, then it is necessarily heavy from configuration (d).

Claim 63 No well colored **link** has both extremities heavy or super heavy, Moreover, no well colored **link** has one extremity super heavy and the other one slightly heavy.

Proof. Suppose one **link** is well colored and has both extremities heavy or super heavy. In this case we apply some rule that cuts the **link** on each side, and the **link** remains well colored. We thus proceed by checking that if we don't have to pay for the cut of that **link**, then we can apply one of the three rules on each of the described configuration.

Heavy configuration (a): Suppose the **link** extremity x_1 is preceded by a color change e , and followed by a bad **link** extremity x_2 , itself followed by a **black edge** e' hitting nothing else except possibly a badly colored **link**. In this case, if we apply reformulated simplified Rule 3 from e to e' and ignore the cut of the well colored **link** ending in x_1 , then we get $bc = 1$, $cc_B \leq 1$, $wc = 0$ (since we ignore the cut of the **link** incident to x_1), $wh + wch = 0$ and $bh + bch + f \leq 1$. The counting gives $\frac{9}{2} - 1 - \frac{1}{2} > \frac{1}{2}$, and reformulated simplified Rule 3 applies, a contradiction.

Heavy configuration (b): This case is similar to the above one. Denote by e the **black edge** of the color change, x_1 and x_2 the **link** extremities, where x_2 is the heavy extremity of the well colored **link** (and x_1 is the extremity of a badly colored **link**). From Claim 57, x_2 is immediately after x_1 , and thus by Claim 47, there is a **black edge** e' hitting nothing immediately after x_2 . We apply reformulated simplified Rule 3 from e to e' , and ignore the cut of the well colored **link** ending in x_2 . This yields $bc = 1$, $cc_B \leq 1$, $wc = 0$ (since we ignore the cut of the **link** incident to x_2), and $wh + wch + bh + bch + f = 0$. The counting gives $\frac{9}{2} - 1 > \frac{1}{2}$, and reformulated simplified Rule 3 applies, a contradiction.

Heavy configuration (c): Let e and e' be the two color changes. We apply reformulated Rule 2 from e to e' , where in this case we ignore the cut of one **link** (or have no cut at all if the two extremities are those of an arch). This yields $wc_N \leq 1$, $cc_B \leq 2$, and the contribution of all other terms in the reformulated Rule 2 is zero. The counting gives $-\frac{9}{2} - 2 > -7$, and reformulated Rule 2 applies, a contradiction.

Heavy configuration (d): Let x_1, x_2, x_3 and x_4 be the extremities of the **links**, here x_1 is the well colored **link** extremity. We note that by Claim 50, since x_2x_3 is not an arch, there is no **black edge** hitting nothing between x_1 and x_2 . Thus, x_2 must be immediately after x_1 , and by Claim 47, there is a **black edge** e hitting nothing just before x_1 . Suppose there is no **black edge** hitting nothing between x_3 and x_4 (and thus x_3x_4 is not an arch by Claim 48). In this case by Claim 47, the first edge after x_4 hits nothing, which contradicts Claim 50. Hence let e' be a **black edge** between x_3 and x_4 hitting nothing. We now apply simplified Rule 1 from e to e' . This yields $bc = 2$, and the contribution of all other terms in the reformulated Rule 1 is zero. The counting gives us $2\frac{9}{2} > 8$, and simplified Rule 1 applies, a contradiction.

We also need to show that a super heavy configuration satisfies the same, i.e., that we can apply one of the three rules if we ignore the cost of cutting the link. We actually show that when ignoring the cost of the cut, we can get a positive score with an extra saving of 2.

Super heavy configuration: Suppose the **link** extremity x_1 is surrounded by two color changes e and e' . In this case if we apply reformulated simplified Rule 2 from e to e' and ignore the cut of the well colored **link** ending in x_1 , we get $bc = 0 = wc$ (since we ignore the cut of the **link** incident to x_1) and $cc_B \leq 2$. The counting gives $-2 \geq -7 + 2$ and reformulated simplified Rule 2 applies, with an extra saving of 2.

Finally, we show thanks to this saving, that a link cannot have one extremity super heavy and one slightly heavy.

Slightly heavy configuration : Suppose the link extremity x_1 is preceded by a color change e and followed by a link extremity x_2 then a color change e' . We apply reformulated simplified Rule 2 from e to e' . We ignore the cost of cutting the link ending in x_1 , and obtain $wc \leq 1$, $cc_B \leq 2$, and the contribution of all other terms in the simplified Rule 2 is zero. The counting gives $-6 - 2 = -8$ which gives -6 thanks to the saving of 2 from the super heavy other extremity, and reformulated simplified Rule 2 applies, a contradiction.

Suppose now the link extremity is next to a color change and a **black edge** hitting nothing. We apply reformulated simplified Rule 3 from the color change to the **black edge**, we get $cc_B \leq 1$, and the contribution of all other terms in the reformulated simplified Rule 3 is zero. Thus the counting gives -1 , which yields $+1$ thanks to the extra saving of 2, and so reformulated simplified Rule 3 applies, a contradiction. This completes the proof of Claim 63. \square

5.7.3 Discharging procedure

Now, we want to show that at least one of the colorings Amber and Blue gives a positive score. For that purpose, we show that the sum of the scoring for Amber and Blue is positive, and therefore infer that at least one is positive.

We say a well colored link extremity is *close* to a color change if they are separated by nothing else than badly colored link extremities (or fibers or extremities of links whose other extremity is dotted).

To prove the sum is positive, we will use some discharging, so that no negative weight remains in the graph. The process is the following:

- Well colored links receive 4 or 5 depending whether they are broad or narrow. Those are shared among the extremities of the link, giving $\frac{5}{2}$ to any extremity of a link in a heavy configuration or of a broad link on a slightly heavy configuration, $\frac{7}{2}$ to any extremity of a (necessarily broad) link in a super heavy configuration, and at least $\frac{3}{2}$ to other extremities. From Claim 63, no well colored link has two heavy extremities, or one super heavy and one slightly heavy or heavy, and thus all links have enough for this transfer.
- Well colored link extremities close to color changes give $\frac{7}{4}$ to any close color change hitting a broad link, and 2 to any close color change hitting no broad link.
- Well colored link extremities transfer weight to surrounding badly colored link extremities according to the following rules:
 - The transferred weight is $\frac{1}{2}$ if the badly colored link extremity is broad, and $\frac{1}{4}$ if it is narrow.
 - Well colored heavy link extremities always transfer weights to badly colored link extremities at distance 1.
 - Well colored heavy link extremities close to no color change transfer weights to badly colored link extremities at distance 2 when necessary.
 - Well colored link extremities close to no color change transfer weights to badly colored link extremities at distance 1.
- Every link near a link extremity close to a color change transfer to it a weight of $\frac{1}{2}$ as a backup if required.
- Every broad link extremity hit by a color transfers a weight of $\frac{1}{2}$ to the color change.

We need to prove that the resulting discharging results in nonnegative weights in all configurations. We discuss the configurations in turn.

color changes: Every color change must get 4: By Claim 55, every color change gets at least $\frac{7}{4}$ on each side when it hits a broad link (from which it receives $\frac{1}{2}$), or at least 2 from each side otherwise.

fibers hit by a color change: By Claim 53, no color change hits a fiber.

badly colored link extremity: We need to show that each such extremity receives a weight of $\frac{1}{2}$:

- Consider first a bad **link** extremity *far* from any color change, i.e., such that none of the closest well colored **links** extremities is close to a color change. In this case we consider *nearby* well colored **link** extremities (i.e., closest in each direction) and the *area* of badly colored **link** extremities in between. If in this area there are at most two badly colored **link** extremity, then each receives a half from its closest well colored **link** extremity. If the area contains three badly colored **link** extremities, say x_1 , x_2 and x_3 , then one of x_1x_2 and x_2x_3 is not an arch, implying that x_1 or x_3 is next to a heavy well colored **link** extremity (by configuration (d)). In this case, x_1 and x_3 receive $\frac{1}{2}$ from their closest well colored **link** extremity, and x_2 receive $\frac{1}{2}$ from the heavy extremity. Finally, the area may not contain more than five bad **link** extremities (Claim 52), and if it contains four, then by Claim 51, the two middle ones form an arch. We therefore infer that the well colored **link** extremities nearby are both heavy (by configuration (d) again) and all four bad **link** extremity receives $\frac{1}{2}$.
- Suppose that a badly colored **link** extremity is next to a color change. It cannot be next to two color changes by Claim 56. Therefore by Claim 57, it is also next to a well colored **link** extremity, which is heavy from configuration (c). Thus it receives a weight of $\frac{1}{2}$ from this heavy link extremity.
- Suppose that both nearby well colored **link** extremities are next to a color change. In this case by Claim 61 and 62, the area contains at most two badly colored **link** extremities. If there are two, then both nearby well colored **link** extremities are heavy from configuration (a). If there is only one, then at least one is heavy from configuration (a) and Claim 47. Therefore each receives some weight from a nearby heavy extremity.
- Suppose finally that one nearby well colored **link** extremity is next to a color change. In this case, we call the extremities $x_1, x_2, x_3 \dots$ where x_1 is close to the color change. By Claim 61, the area contains at most three badly colored **link** extremities. If there is only one, then x_2 receives the necessary weight from x_3 (which is well colored). If there are two, x_1 is heavy by configuration (a), and thus covers x_2 , and x_3 received weight from x_4 . If there are three, then we are in the special configuration of Claim 61 and x_2x_3 form an arch. In this case, both x_1 and x_5 are heavy, x_1 covers x_2 and x_5 covers the two others.

well colored link extremity close to two color changes: We note that by Claim 57, this case may occur only if there are at most three **link** extremities between the two color changes. If there are two or three extremities, Claims 58 and 59, respectively, show that two of the **links** are well colored. Therefore this situation occurs only when the two color changes are separated by exactly one **link** extremity, in the condition of Claim 56. However, then both color changes hit a broad link extremity, and the **link** extremity is a super heavy extremity, and so it receives $\frac{7}{2}$ and gives twice $\frac{7}{4}$ which makes it nonnegative.

well colored link extremity close to a color change and possibly giving backup: Suppose that a **link** extremity close to a color change needs to give a backup. Then it is close to another **link** extremity itself next to a color change, and thus there are exactly two well colored **link** extremities between the two color changes. By Claim 62, there are at most four **link** extremities between the two color changes. If there are four extremities only two of which are well colored, we denote them x_1, x_2, x_3 and x_4 , and show the two well colored extremities are both heavy, from which we infer that neither needs a

backup. By Claim 60, the two badly colored **link** extremities are those of an arch. By Claim 57, only one extremity of x_1 and x_2 is badly colored, and also one of x_3 and x_4 . If x_1 (resp., x_4) is badly colored, then x_2 (resp., x_3) is well colored and a heavy configuration (b). If x_1 is well colored, then x_2 is badly colored. In this case, x_1 is a heavy configuration (a), unless x_3 is well colored and immediately follows x_2 . However then x_4 is badly colored and immediately follows x_3 by Claim 57, which contradicts Claim 47.

Suppose there are three extremities and only two are well colored. If the badly colored **link** extremity is x_1 or x_3 , then x_2 is a heavy configuration (b). Otherwise, by Claim 47, there is a **black edge** hitting nothing between x_1 and x_3 , say between x_1 and x_2 . In this case, x_1 is a heavy configuration (a). Hence at least one well colored **link** extremity is heavy. Now following Claim 59, we consider three cases:

Case 1. Both color changes hit a broad link: In this case, each well colored link extremity gives $\frac{7}{4}$ to the change of color next to it, and the heavy link extremity gives $\frac{1}{2}$ to the badly colored **link** extremity in the area. Thus, the heavy link extremity receives $\frac{5}{2}$, and gives $\frac{7}{4} + \frac{1}{2}$, and so it has sufficient weight to discharge $\frac{1}{4}$ for a backup. The other **link** extremity receives at least $\frac{3}{2}$, and needs to give only $\frac{7}{4}$, and so it requires a weight of at most $\frac{1}{4}$ as a backup, which the heavy **link** extremity can discharge to it. Therefore, in this case both extremities end up with a nonnegative weight.

Case 2. Only one color change hits a broad link but the badly colored link is narrow: The total share received by the two **link** extremities is $\frac{3}{2} + \frac{5}{2}$ and it should cover $2 + \frac{7}{4}$ for the color changes and $\frac{1}{4}$ for the badly colored **link** extremity, implying that the total share it receives is what is needed. However for clarity, we include the details as follows. If the broad well colored **link** extremity is next to the color change hitting a broad **link**, then the extremity gives $\frac{7}{4}$ to the color change, $\frac{1}{4}$ to the badly colored **link** extremity, and $\frac{1}{2}$ to the narrow well colored **link** extremity as backup, and therefore it can transfer a weight of 2 to the color change. If the broad well colored **link** extremity is next to the color change not hitting a broad **link**, then the extremity gives 2 to the color change, $\frac{1}{4}$ to the badly colored **link** extremity, and $\frac{1}{4}$ to the narrow well colored **link** extremity as backup, which can thus give $\frac{7}{4}$ to the color change hitting no broad **link** extremity.

Case 3. Both extremities are those of broad links: In this case, both well colored **link** extremity are heavy or slightly heavy. Indeed, let x_1 , x_2 and x_3 be the three link extremities. By Claim 47, there is a **black edge** hitting nothing between x_1 and x_3 , say between x_2 and x_3 . We note that by Claim 57, the badly colored **link** extremity is not x_3 . Thus among x_1 and x_2 , there is one badly colored **link** extremity and a heavy **link** extremity (from configuration (a) or (b)). Moreover, since x_3 is the extremity of a broad **link** and is surrounded by a color change and a **black edge** hitting nothing, it is slightly heavy. Thus both receive $\frac{5}{2}$ and neither need to give backup.

Suppose now there are only two extremities. In this case, by Claim 58 we infer that both extremities are well colored. Moreover either one is a heavy configuration (c), or both are broad and slightly heavy. In the first case, the heavy extremity has enough to give 2 to the color change and $\frac{1}{2}$ as backup, while in the second case, both extremities receive enough weight, and so neither needs backup.

well colored link extremity close to a color change but not giving backup: If the extremity is heavy, it may give at most 2 to the color change and twice $\frac{1}{2}$ to adjacent badly colored **link** extremities. Therefore it ends up with a positive weight with a backup of at most $\frac{1}{2}$. If the extremity is narrow, then it may give at most 2 to the adjacent color change, and so it requires at most $\frac{1}{2}$ as a backup.

well colored link extremities not adjacent to color changes : A heavy well colored **link** extremity not next to a color change may have to give $\frac{1}{2}$ to badly colored **link** extremities at distance at most 2, so it may be required to transfer to at most such extremities. It may also give backup up to $\frac{1}{2}$. Suppose this sums up to more than $\frac{5}{2}$, implying that it gives backup and to two badly colored links on both sides. Since the next well colored link extremity on each direction needs a backup, it is close to a color change. Since there are two badly colored **link** extremities on each side, the next well colored links are heavy, and so there are actually three badly colored **link** extremities on each sides, and both

must have the properties of the special configurations in Claim 61. However in that configuration, x_5 is immediately after x_4 and this property occurs on both sides, which contradicts Claim 47. Therefore a heavy well colored **link** extremity not adjacent to color changes ends up with a nonnegative weight.

We next consider a light well colored **link** extremity x_0 . Suppose again that it ends up with a negative weight, that is, it gives weight to badly colored **link** extremities close by on each side x_{-1} and x_1 , and gives backup on each direction to the closest well colored **link** extremity. We note that by Claim 47, there is a **black edge** hitting nothing between x_{-1} and x_1 , say between x_{-1} and x_0 . Thus the closest well colored **link** extremity on that direction x_{-3} is heavy, and if it requires backup, it must be next to two other badly colored **link** extremities x_{-4} and x_{-2} . However in this case, there are three bad **link** extremities following the change of color before x_{-4} , meaning that this correspond to the special configuration of Claim 61, and thus x_0 is immediately after x_{-1} , which contradicts our assumption that a **black edge** hitting nothing was between x_{-1} and x_0 .

The above arguments and discussion show that the discharging procedure is such that upon completion of the discharging no negative weight remains in the graph. Therefore, at least one of the colorings Amber and Blue gives a positive score.

5.8 Red edges

By our results in Section 5.6, there are no **green edges** in the colored multigraph. In this section, we show that there are no **red edges**. Recall that by Claim 18 the **red edges** form a matching in the colored multigraph M_G . Further recall that by Claim 22, there are no long **red edges** in the colored multigraph M_G . Thus by our earlier observations, there are no marked vertices.

Claim 64 *No adjacent vertices are incident to distinct red edges.*

Proof. Suppose that there are adjacent vertices v and w incident with distinct **red edges**. Let vy and wz be **red edges** incident with v and w , respectively, where the edge vw is a **black edge**. This structure is illustrated in Figure 39. Let H be obtained from G by removing the vertices v and w , and removing the vertices $(vy)_1, (vy)_2, (wz)_1,$ and $(wz)_2$. We note that $\dot{w}(G) = \dot{w}(H) + 2 \times 4 + 4 \times 5 - 4 \times 1 = 24$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices $(v, y)_1$ and $(w, z)_1$. Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$, contradicting Fact 2. \square

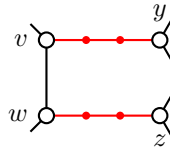


Figure 39: Adjacent vertices incident with distinct **red edges**

Claim 65 *There is no red edge.*

Proof. Suppose that there is a **red edge** uv . Let u_1 and u_2 be the neighbors of u not on the **red edge** uv , and let v_1 and v_2 be the neighbors of v not on the **red edge** uv . Thus, $uu_1, uu_2, vv_1,$ and vv_2 are all **black edges**. By Claim 64, none of vertices u_1, u_2, v_1 and v_2 is incident with a **red edge**.

Suppose firstly that u_1 or u_2 is adjacent to v_1 or v_2 . Renaming vertices if necessary, we may assume that u_1v_1 is an edge. As observed earlier, u_1v_1 is a **black edge**. In this case, let H be obtained from G by removing the vertices u, v, u_1 and v_1 , removing the vertices $(uv)_1$ and $(uv)_2$, and marking the vertices u_2 and v_2 . We note that $\dot{w}(G) = \dot{w}(H) + 4 \times 4 + 2 \times 5 - 2 \times 1 = 24$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u and v . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$, contradicting Fact 2.

Hence, the set $\{u_1, u_2, v_1, v_2\}$ is an independent set. Let H be obtained from G by removing the vertices u, v, u_1, u_2 and v_1 , removing the vertices $(uv)_1$ and $(uv)_2$, and marking the vertex v_2 . Since the graph G contains no 4- or 7-cycles, every vertex of H is adjacent in G to at most one vertex in $V(G) \setminus V(H)$, implying that $\dot{w}(G) = \dot{w}(H) + 5 \times 4 + 2 \times 5 - 6 \times 1 = 24$. Every MD-set of H can be extended to a MD-set of G by adding to it the vertices u and v . Thus, $\dot{\gamma}(G) \leq \dot{\gamma}(H) + \alpha$ and $\dot{w}(G) \geq \dot{w}(H) + \beta$, where $\alpha = 2$ and $\beta = 24$, contradicting Fact 2. \square

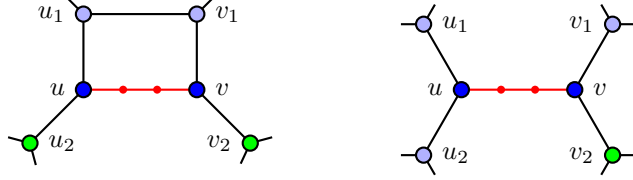


Figure 40: The two cases of the proof of Claim 65

5.9 The cubic graph G

By our earlier observations, there are no **green edges** and no **red edges** in the colored multigraph, implying that the graph G is a cubic graph and contains no marked vertex. Since there are no marked vertices in G , we note that $\dot{\gamma}(G) = \gamma(G)$ and $\dot{w}(G) = w(G)$. As a consequence of Fact 1, we may infer the following fact.

Fact 2 *If H is a proper subgraph of G , and α and β are two integers such that $\gamma(G) \leq \gamma(H) + \alpha$ and $w(G) \geq w(H) + \beta$, then $12\alpha > \beta$.*

Recall that by supposition, the girth of G is at least 6 with no cycles of length 7 or 8. Let g denote the girth of G , and so $g = 6$ or $g \geq 9$. Let $C: v_1v_2 \dots v_gv_1$ be a cycle in G of smallest length, namely g . Let u_i be the neighbor of v_i not on C for all $i \in [g]$. Since the girth of G is equal to $g \geq 6$, the vertices u_1, u_2, \dots, u_g are distinct and do not belong to the cycle C . We define the *boundary* of the cycle C in G , denoted $\partial(C)$, to be the set of all vertices in G that do not belong to the cycle C but are adjacent in G to at least one vertex on the cycle. Thus, $\partial(C) = \{u_1, u_2, \dots, u_g\}$. We show firstly that the graph G contains no 6-cycle.

Claim 66 *The graph G contains no 6-cycle.*

Proof. Suppose, to the contrary, that the girth of G is equal to 6. Thus, $g = 6$ and in this case $C: v_1v_2 \dots v_6v_1$ is a 6-cycle in G . Suppose that the set $\partial(C)$ is an independent set in G . Since the graph G contains no 7-cycle, the vertices u_1 and u_4 have no common neighbor. We now consider the graph $H - (V(C) \cup \{u_1, u_4\})$. By our earlier observations, the graph H has minimum degree 2. Further, $n_3(H) = n_3(G) - 8$ and $n_2(H) = 8$. Thus, $w(G) = w(H) + 8 \times 4 - 8 = 24$. Every γ -set of H can be extended to a γ -set of G by adding to it the vertices v_1 and v_4 , and so $\gamma(G) \leq \gamma(H) + 2$. Hence, $\gamma(G) \leq \gamma(H) + \alpha$ and $w(G) \geq w(H) + \beta$, where $\alpha = 2$ and $\beta = 24$, contradicting Fact 2. Hence, the set $\partial(C)$ is not an independent set.

Since the graph G contains no 4-cycle and no 5-cycle, the only possible edges in the subgraph $G[\partial(C)]$ induced by $\partial(C)$ are the edges $u_i u_{i+3}$ where $i \in [3]$. Renaming vertices if necessary, we may assume that $u_1 u_4$ is an edge of G . We note that $N_G[v_1] \cup N_G[v_4] = V(C) \cup \{u_1, u_4\}$. In this case, we consider the graph $H = G - (V(C) \cup \{u_1, u_4\})$. Since the girth of G is equal to 6, every vertex in H is adjacent in G to at most one vertex in $V(C) \cup \{u_1, u_4\}$, implying that the graph H has minimum degree 2. Further, $n_3(H) = n_3(G) - 8$ and $n_2(H) = 6$. Thus, $w(G) = w(H) + 8 \times 4 - 6 = 26$. Every γ -set of H can be extended to a γ -set of G by adding to it the vertices v_1 and v_4 , and so $\gamma(G) \leq \gamma(H) + 2$. Hence, $\gamma(G) \leq \gamma(H) + \alpha$ and $w(G) \geq w(H) + \beta$, where $\alpha = 2$ and $\beta = 26$, contradicting Fact 2. \square

By Claim 66, the graph G contains no 6-cycle. By supposition, the girth of G is at least 6 with no cycles of length 7 or 8. Therefore, the girth of G is at least 9, that is, $g \geq 9$. This implies that every vertex in $V(G) \setminus (V(C) \cup \partial(C))$ is adjacent in G to at most one vertex in the boundary, $\partial(C)$, of C .

Claim 67 $g \not\equiv 0 \pmod{3}$.

Proof. Suppose, to the contrary, that $g \equiv 0 \pmod{3}$. Thus, $g = 3k$ for some integer $k \geq 3$. Hence, $|V(C)| = g = 3k$. In this case, we let

$$S = \bigcup_{i=1}^k \{v_{3i-1}\} \quad \text{and} \quad U = \bigcup_{i=1}^k \{u_{3i-1}\}.$$

We note that $U \subset \partial(C)$ and $|S| = |U| = k$. The subgraph of G that contains the set $V(C) \cup U$ is illustrated in Figure 41. In this case, we consider the graph $H = G - (V(C) \cup U)$ of order $n - 4k$. Every vertex in H is adjacent in G to at most one vertex in $V(C) \cup U$. Thus, the graph H has minimum degree 2. We note that each vertex in U is adjacent in G to two vertices in $V(H)$, and each vertex in $V(C) \setminus S$ is adjacent in G to exactly one vertex in $V(H)$, while each vertex in S has no neighbor in G that belongs to $V(H)$. These observations imply that $w(G) = w(H) + 4 \times 4k - 4k = 12k$. Every γ -set of H can be extended to a γ -set of G by adding to it the set S , and so $\gamma(G) \leq \gamma(H) + |S| = \gamma(H) + k$. Hence, $\gamma(G) \leq \gamma(H) + \alpha$ and $w(G) \geq w(H) + \beta$, where $\alpha = k$ and $\beta = 12k$, contradicting Fact 2. \square

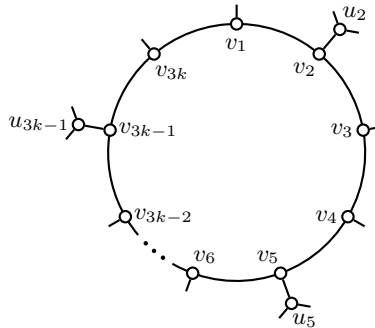


Figure 41: The case when $g \equiv 0 \pmod{3}$ and $g \geq 9$

By Claim 67, $g \not\equiv 0 \pmod{3}$. Thus by our earlier observations, $g \geq 10$. Let $N_G(u_i) = \{v_i, x_i, w_i\}$ for all $i \in [g]$. By the girth condition, the sets $N_G(u_i)$ and $N_G(u_j)$ are vertex disjoint for $i, j \in [g]$ and $i \neq j$, that is, $N_G(u_i) \cap N_G(u_j) = \emptyset$ for $1 \leq i < j \leq g$.

Claim 68 $g \not\equiv 2 \pmod{3}$.

Proof. Suppose, to the contrary, that $g \equiv 2 \pmod{3}$. Thus, $g = 3k + 2$ for some integer $k \geq 3$. Hence, $|V(C)| = g = 3k + 2$. Recall that $N_G(u_1) = \{v_1, x_1, w_1\}$ and $N_G(u_2) = \{v_2, x_2, w_2\}$. In this case, we let

$$S = \{u_1, u_2\} \cup \left(\bigcup_{i=1}^k \{v_{3i+1}\} \right) \quad \text{and} \quad U = \bigcup_{i=1}^k \{u_{3i+1}\}.$$

We note that $|S| = k + 2$ and $|U| = k$. Let $U_{12} = \{u_1, u_2, x_1, x_2, w_1, w_2\}$. The girth condition implies that $U \cap U_{12} = \emptyset$. The subgraph of G that contains the set $V(C) \cup U \cup U_{12}$ is illustrated in Figure 42. We now consider the graph $H = G - (V(C) \cup U \cup U_{12})$ of order $n - (3k + 2) - k - 6 = n - 4k - 8$. Since the girth $g = 3k + 2 \geq 11$, every vertex in H is adjacent in G to at most one vertex in $V(C) \cup U \cup U_{12}$. Thus, the graph H has minimum degree 2. We note that each vertex in $U \cup \{x_1, x_2, w_1, w_2\}$ is adjacent in G to two vertices in $V(H)$, each vertex in $V(C) \setminus (S \cup \{v_1, v_2\})$ is adjacent in G to exactly one vertex in $V(H)$, while each vertex in $S \cup \{u_1, u_2, v_1, v_2\}$ has no neighbor in G that belongs to $V(H)$. These observations imply that $w(G) = w(H) + 4 \times (4k + 8) - (4k + 8) = 12(k + 2)$. Every γ -set of H can be extended to a γ -set of G by adding to it the set S , and so $\gamma(G) \leq \gamma(H) + |S| = \gamma(H) + k + 2$. Hence, $\gamma(G) \leq \gamma(H) + \alpha$ and $w(G) \geq w(H) + \beta$, where $\alpha = k + 2$ and $\beta = 12(k + 2)$, contradicting Fact 2. \square

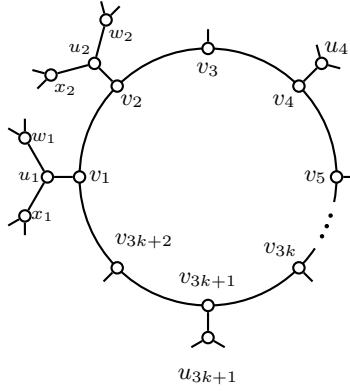


Figure 42: The case when $g \equiv 2 \pmod{3}$ and $g \geq 11$

By Claim 67 and 68, we have $g \equiv 1 \pmod{3}$. Thus, $g = 3k + 1$ for some integer $k \geq 3$.

Claim 69 $k = 3$.

Proof. Suppose, to the contrary, that $k \geq 4$, and so $g = 3k + 1 \geq 13$. Hence, $|V(C)| = g = 3k + 1$. Recall that $N_G(u_1) = \{v_1, x_1, w_1\}$. In this case, we let

$$S = \{u_1\} \cup \left(\bigcup_{i=1}^k \{v_{3i}\} \right) \quad \text{and} \quad U = \bigcup_{i=1}^k \{u_{3i}\}.$$

We note that $|S| = k + 1$ and $|U| = k$. Let $U_1 = \{u_1, x_1, w_1\}$. The girth condition implies that $U \cap U_1 = \emptyset$. The subgraph of G that contains the set $V(C) \cup U \cup U_1$ is illustrated in Figure 43. We now consider the graph $H = G - (V(C) \cup U \cup U_1)$ of order $n - (3k + 1) - k - 3 = n - 4k - 4$. Since the girth $g = 3k + 1 \geq 13$, every vertex in H is adjacent in G to at most one vertex in $V(C) \cup U \cup U_1$. Thus, the graph H has minimum degree 2. We note that each vertex in $U \cup \{x_1, w_1\}$ is adjacent in G to two vertices in $V(H)$, each vertex in $V(C) \setminus (S \cup \{v_1\})$ is adjacent in G to exactly one vertex in $V(H)$, while each vertex in $S \cup \{u_1, v_1\}$ has no neighbor in G that

belongs to $V(H)$. These observations imply that $w(G) = w(H) + 4 \times (4k + 4) - (4k + 4) = 12(k + 1)$. Every γ -set of H can be extended to a γ -set of G by adding to it the set S , and so $\gamma(G) \leq \gamma(H) + |S| = \gamma(H) + k + 1$. Hence, $\gamma(G) \leq \gamma(H) + \alpha$ and $w(G) \geq w(H) + \beta$, where $\alpha = k + 1$ and $\beta = 12(k + 1)$, contradicting Fact 2. \square

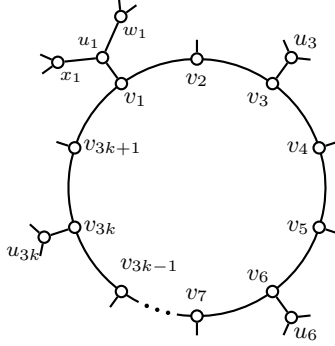


Figure 43: The case when $g \equiv 1 \pmod{3}$ and $g \geq 13$

By Claim 69, the girth of G is $g = 10$. Hence, $|V(C)| = 10$. We proceed now as in the proof of Claim 69. In this case when $g = 10$, we have $S = \{u_1, v_3, v_6, v_9\}$, $U = \{u_3, u_6, u_9\}$, and $U_1 = \{u_1, x_1, w_1\}$. The girth condition once again implies that $U \cap U_1 = \emptyset$. The subgraph of G that contains the set $V(C) \cup U \cup U_1$ is illustrated in Figure 44. We now consider the graph $H = G - (V(C) \cup U \cup U_1)$ of order $n(H) = n - 16$. If every vertex in H is adjacent in G to at most one vertex in $V(C) \cup U \cup U_1$, then proceeding exactly as in the proof of Claim 69 we produce a contradiction.

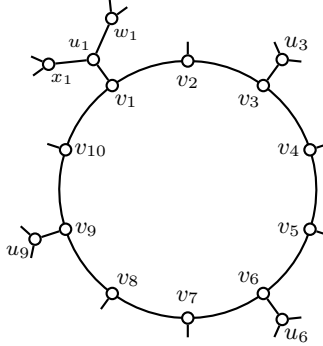


Figure 44: The case when $g = 10$

Hence, we may assume that there is a vertex z in H that is adjacent in G to two or more vertices in $V(C) \cup U \cup U_1$. Since the girth of G is $g = 10$, the vertex z is adjacent in G to exactly two vertices in $V(C) \cup U \cup U_1$, namely to one of the neighbors of u_1 different from v_1 and to the vertex u_6 . Thus, z is adjacent to either x_1 or w_1 , and $z \in \{x_6, w_6\}$. Renaming the vertices x_1, w_1, x_6 and w_6 if necessary, we may assume that $z = w_6$ and that z is adjacent to w_1 . Thus, $Q_1: v_1 u_1 w_1 w_6 u_6 v_6$ is a path in G . By symmetry, renaming the vertices u_i and w_i if necessary, we may assume that $Q_i: v_i u_i w_i w_{5+i} u_{5+i} v_{5+i}$ is a path in G for all $i \in [5]$. We now consider the cycle C and the paths Q_1 and Q_2 , and let

$$S = \{u_1, u_2, u_6, u_7, v_4, v_9\} \quad \text{and} \quad Q = V(Q_1) \cup V(Q_2) \cup \{u_4, u_9, x_1, x_2, x_6, x_7\}.$$

The subgraph of G that contains the set $V(C) \cup Q$ is illustrated in Figure 45.

We now consider the graph $H = G - (V(C) \cup Q)$ of order $n(H) = n - 24$. Since the girth of G is $g = 10$, every vertex in H is adjacent in G to at most one vertex in $V(C) \cup Q$, except possibly for two vertices, namely

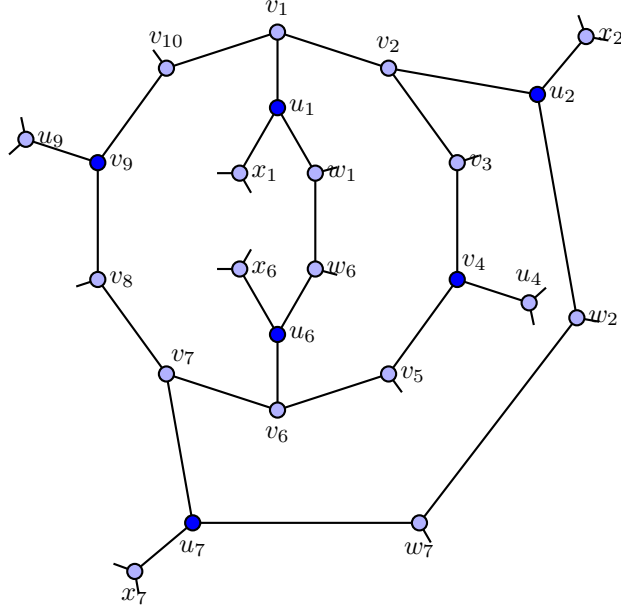


Figure 45: A subgraph of G induced by the set $V(C) \cup Q$

a vertex adjacent to both x_1 and x_7 , and a vertex adjacent to both x_2 and x_6 . Thus every vertex in H has degree at least 2, except possibly for two vertices which have degree 1 in H . If H has no vertex of degree 1, then $n_1(H) = 0$, $n_2(H) = 20$, and $w(G) = w(H) + 4 \times 24 - 1 \times 20 = w(H) + 6 \times 12 + 4$. If H has one vertex of degree 1, then $n_1(H) = 1$, $n_2(H) = 18$, and $w(G) = w(H) + 4 \times 24 - 1 \times 18 - 4 \times 1 = w(H) + 6 \times 12 + 2$. If H has two vertices of degree 1, then $n_1(H) = 2$, $n_2(H) = 16$, and $w(G) = w(H) + 4 \times 24 - 1 \times 16 - 4 \times 2 = w(H) + 6 \times 12$. In all cases, $w(G) \geq w(H) + 6 \times 12$. Every γ -set of H can be extended to a γ -set of G by adding to it the set S , and so $\gamma(G) \leq \gamma(H) + |S| = \gamma(H) + 6$. Hence, $\gamma(G) \leq \gamma(H) + \alpha$ and $w(G) \geq w(H) + \beta$, where $\alpha = 6$ and $\beta = 6 \times 12$, contradicting Fact 2. This completes the proof of Theorem 10. \square

6 Concluding remarks

The best general upper bound to date on the domination number of a connected cubic graph G of order $n \geq 10$ is due to Kostochka and Stoker [13] (see, Theorem 1) in 2009 who showed that $\gamma(G) \leq \frac{5}{14}n = (\frac{1}{3} + \frac{1}{42})n$.

In 2010 Verstraëte [21] posed a most intriguing conjecture that if the girth, $g(G)$, of a cubic graph G of order n is at least 6, then $\gamma(G) \leq \frac{1}{3}n$. The girth requirement in Verstraëte's Conjecture is essential, since there are connected cubic graphs G of arbitrarily large order n that contain 4-cycles and 5-cycles and satisfy $\gamma(G) > \frac{1}{3}n$. The best known girth condition (prior to this paper) guaranteeing that the domination number of a cubic graph G of order n is at most the magical threshold of $\frac{1}{3}n$ is due to Löwenstein and Rautenbach [17] in 2008 who showed that if $g(G) \geq 83$, then $\gamma(G) \leq \frac{1}{3}n$ holds. In this paper, we prove Verstraëte's conjecture when there is no 7-cycle and no 8-cycle, that is, we show that if G is a cubic graph of order n and girth $g(G) \geq 6$ that does not contain a 7-cycle or 8-cycle, then $\gamma(G) \leq \frac{1}{3}n$.

Equally appealing to Verstraëte's conjecture is a conjecture inspired by Kostochka in 2009 that if G is a bipartite graph of order n , then $\gamma(G) \leq \frac{1}{3}n$. In this paper, we prove the Kostochka's related conjecture when there is no 4-cycle and no 8-cycle, that is, we show that if G is a bipartite cubic graph of order n that

contains no 4-cycle and no 8-cycle, then $\gamma(G) \leq \frac{1}{3}n$.

It would be extremely interesting to prove Verstraete's conjecture in general when 7-cycles and 8-cycles are allowed in the mix, and to prove the Kostochka inspired conjecture in general when 4-cycles and 8-cycles are present. The considerable effort made in this paper suggests that this may be difficult. With considerably more work, the methods employed in the paper (using the new concept of marked domination in graphs, and using colored multigraphs, matchings, and intricate discharging methods) may have some hope of proving these two $\frac{1}{3}$ -domination conjectures if we allow 8-cycles back into the mix. However, completely new methods and ideas are needed to prove these conjectures in general (when we allow 7-cycles in the case of Verstraete's conjecture and when we allow 4-cycles in the case of the Kostochka inspired conjecture).

7 Acknowledgements

The authors wish to thank Justin Southey and Mickael Montassier for helpful discussions.

M. A. Henning research was supported in part by the South African National Research Foundation (grants 132588, 129265) and the University of Johannesburg.

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