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Mathematical modelling as a research field: Transposition challenges and future directions

Berta Barquero

¹Universitat de Barcelona, Faculty of Education, Barcelona, Spain; bbarquero@ub.edu

The field of research on mathematical modelling has contributed with significant advances to mathematics education research and educational practice over the past decades. This paper focuses on the evolution of this research field based on the experiences of the CERME working groups. Through the selection of some modelling approaches and particular investigations, the kinds of research questions addressed, as well as the processes of transposing modelling from research to practice, are examined. To discuss the foundations and developments of the existing theoretical approaches to modelling, the paper emphasises the epistemological and didactic dimensions of research problems. The ecological view or, more specifically, the analysis of the conditions facilitating and the constraints hindering the long-term dissemination of modelling is addressed to point out the present and future directions of this research field.

Keywords: Mathematical modelling, didactic transposition, systems-models, epistemological dimension, didactic ecology.

Introduction: Research field on mathematical modelling

The field of research on mathematical modelling —more broadly called *applications and modelling*— has made substantial contributions to the realm of mathematics education research and practice over the past decades. The promotion of modelling competencies, both amongst students and teachers, is internationally recognised as a central goal for mathematics education and has gained significant support in the past decades through curricular reforms.

The origins of this field of research can be traced back to the pioneering work of Freudenthal (1968) and Pollak (1968, 1979), both of whom participated in the symposium dedicated to “Why to teach mathematics so as to be useful.” This initial impetus was followed by the establishment of the *International Conferences on the Teaching of Mathematical Modelling and Applications* (ICTMA) in 1983. Furthermore, working groups dedicated to research on mathematical modelling were created within various international conferences, such as in the International Congress on Mathematical Education (ICME) and in the Congress of the European Society for Research in Mathematics Education. Of particular relevance to this paper, and to the plenary in CERME13, is the thematic working group on “Applications and Modelling.”

Understanding the trajectory of the European community in the research field of mathematical modelling entails situating this community in a broader and intricately interconnected research network (Carreira et al., 2019). This should undoubtedly consider the knowledge that has been generated through the *14th ICMI Study about modelling and applications* on Modelling and Applications (Blum et al., 2007). Furthermore, insights can be obtained from the publications within the ICTMA book series on *International perspectives on the teaching and learning of mathematical modelling*.

In addition, numerous research works, some of which feature in special issues in international journals (i.e, Kaiser & Sriraman, 2006; Blum, 2015; Schukajlow et al., 2018) and book publications (Niss & Blum, 2020; Blomhøj & Ärlebäck, 2018, in Dreyfus et al., 2018), aim to make an outline of the origin and development of this research field. These works have also delved into the theoretical and empirical research advances, and their implications for teachers and students' practice.

The progression of this research field is closely intertwined with the didactic transposition of mathematical modelling, that is, its consideration and elaboration as part of the knowledge that is to be taught at school (Chevallard, 1985). It is imperative to underscore the substantial interaction of research in mathematical modelling with all those institutions tasked with shaping the knowledge to be taught. A noteworthy example of this interface can be found within the context of the PISA Programme for International Student Assessment) framework, as elucidated by the OECD:

The notion of mathematical modelling has been a cornerstone of the PISA framework for mathematics [...] The modelling cycle is a central aspect of the PISA conception of students as active problem solvers; however, it is often not necessary to engage in every stage of the modelling cycle, especially in the context of an assessment [...]. (OECD, 2019, pp. 75–76).

Moreover, in the discourse of competencies, which currently takes a dominant position in European curriculum reforms, modelling competencies – encompassing the competencies to address and solve real-world problems through mathematics – have gained global recognition as a pivotal goal for mathematics education (Kaiser, 2020). Modelling competency has been included into the various skills attributed to the discipline of mathematics, drawing inspiration from the Danish KOM project (Niss, 2003; Blomhøj & Jensen, 2003). Some colleagues in this field of research, that have taken part in several CERME congresses, were asked to provide information on *when mathematical modelling first became part of their curriculum*. What seems clear is that the transposition work to integrate modelling has unfolded over varied timeframes and under distinct paradigms. For example, in Denmark and Portugal, the first references to modelling in curricula appeared in the 1960s and 1970s. After the year 2000, German and Swedish curricula reforms introduced modelling competency, initially at the lower secondary level, and subsequently extending it to other educational levels. Spain seems to be an exception, with sporadic mentions of modelling in curricula between 2007 and 2014, gaining explicit recognition only in the most recent curriculum reforms in 2020.

In all the responses received from participating researchers from different countries, mathematical modelling has gained a defined position as one of the specific competencies within the subject of school mathematics. However, persistent tensions exist, particularly concerning other approaches to mathematics education, such as problem-solving or, more recently, the emphasis on interdisciplinarity. While this aspect will not be explored further in this paper, it is anticipated to be the focal point of forthcoming research. Consistent with previous studies (such as, Jessen & Kjeldsen, 2021), it is deemed critical to address the systemic conditions that either facilitate or hinder the sustained integration of modelling, partially driven by curricula reforms.

A brief overview of the thematic working group on Applications and Modelling

Since CERME4, which took place in 2005 in Sant Feliu de Guíxols (Spain), the thematic working group on “Applications and modelling” has been consolidated, and it has been more active than ever,

up until the last CERME13 (July 2023). Since CERME4 (my first CERME in the working group), there has been a considerable increase in both the number of contributions, approximately 240, and the diversity of participants, gathering participants of over 25 countries. A significant characteristic of this working group is the rich diversity of themes, theoretical perspectives, and research questions addressed. These aspects align with the diversity of views on mathematical modelling and applications, reflecting the various viewpoints within the group. After the analysis of the contributions in this thematic working group (TWG) from the past ten CERME congresses, it became pertinent to distinguish the evolution of the TWG across three distinct periods.

The initial period, from CERME 4 to CERME 7, was marked by the emergence and initial dialogues between different approaches, with no uniform understanding or conceptualisation of mathematical modelling. This was the reason why the need to understand and classify these diverse approaches arose. Some approaches were specifically “domain-specific frameworks” to mathematical modelling (in the sense of Kieran et al., 2015) within mathematics education, such as the modelling cycle approach; while others corresponded to more “intermediate-level frameworks”, like the realistic mathematics education (RME). Notably, during CERME 6 and CERME 7, specific salient themes and theoretical approaches began to crystallise: the modelling cycle and competencies approach, the anthropological theory of the didactic (ATD), the cultural-historical activity theory (CHAT), the models and modelling perspective (MMP), and the theory of realistic mathematics education (RME). It is worth mentioning that, among the papers that explicitly refer to certain school levels, more than 80% focus on secondary or university level. Conversely, research pertaining to pre-school or primary level is a mere 9%, as are contributions related to in-service teacher education (9%).

During the second period, spanning from CERME 8 to CERME 10, the diversity of approaches continues to be considered as a touchstone of the group. However, this phase saw a significant increase in contributions using the modelling cycle and competencies approach. The discussion about the relationship and specificity of mathematical modelling with problem-solving, project-based, and inquiry-based approaches emerged within the domain of mathematics education. Some approaches began to explicitly address the tools required for the design and analysis of modelling tasks, and researchers engaged in discussions concerning the authenticity of modelling tasks. Moreover, there was a growing emphasis on examining the theory and practice of modelling within teacher education and the broader mathematics educators’ community. While the majority of research on modelling continued to be focused on secondary school level (64%) or university level contexts (14%), the representation of research in primary school (4%), vocational training (3%), and in-service teacher education (9%) remained comparatively limited.

In the third and last period, from CERME11 to CERME 13, a large increase in the number of submissions to the group was observed. This increment can be attributed, in part, to the growing research community in mathematics education and to external societal needs. The significance of mathematical modelling and its applications became apparent as our society underwent significant transformations. The emergence or recognition of fields such as data science, forecasting and simulations, along with the scientific community’s work on modelling for decision-making in response to the COVID pandemic, stressed the vital importance of debates regarding the educational use of models and modelling.

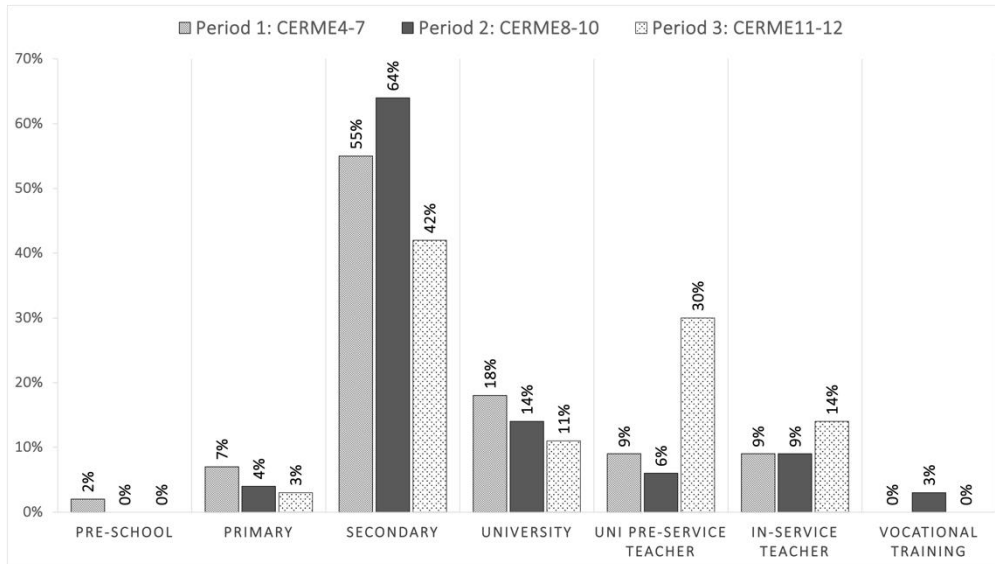


Figure 1: Frequency distribution of papers in the TWG per school level and period

During the most recent period, another relevant fact observed is the increase of the research focus on teacher education for mathematical modelling. This has led to tripling the number of papers addressing pre-service education, and nearly doubling those dealing with in-service teacher education. This change can be interpreted as an indicator of our field maturity, as it raises questions and engages a broader educational community. On the one hand, it reflects the awareness that teachers play a crucial role in facilitating a shift towards a school paradigm that fosters favourable conditions for modelling. On the other hand, it means an acknowledgement of important obstacles and constraints for modelling, fostering greater collaboration among various communities and institutions (of students, teachers, and educators).

Systems, models, and the ecology of modelling: divergent and convergent encounters

Several attempts have been made in the modelling and applications research community to analyse various theoretical approaches and establish connections between frameworks for designing and analysing mathematical modelling activities (Kaiser & Sriraman, 2006; Cai et al., 2014). When addressing challenges related to the teaching and learning of mathematical modelling, diverse conceptions of mathematical modelling emerge. Furthermore, the way these difficulties are interpreted, the types of entities under consideration, and the empirical domains selected as units of analysis can vary significantly based on the chosen research framework. Building on previous work by Barquero et al. (2019) and Barquero and Jessen (2019), we can consider two complementary dimensions, the epistemological and the ecological ones, to explore the divergent and convergent encounters between different modelling approaches.

The epistemological dimension of modelling: how is the dialectics between systems and models interpreted?

When tackling a research problem in mathematics education that pertains to a specific mathematical content or knowledge, such as mathematical modelling, it is important for researchers to consider their own perspective on this content, that is, their *reference epistemological model* (Bosch & Gascón,

2006). These reference epistemological models can take various forms depending on the research framework adopted. In other words, researchers develop them by using specific epistemological notions and relationships that align with their research framework.

To explain how different approaches interpret and conceptualise mathematical modelling, I will start with a simple and flexible perspective, initially proposed by Chevallard (1989). This perspective will contribute to bringing to light the often-implicit underlying hypothesis of the different frameworks. Within any modelling process, two fundamental elements can be considered: the notion of the *system* and that of the *model*. The systems and models do not possess inherent, predetermined qualities, nor do they stand as independent entities by themselves. Instead, they function in accordance with the role(s) they assume within the modelling process. In a sense, this perspective aims to offer a unifying framework for interpreting modelling in any scientific activity, including *didactics* as one possible case. In fact, in didactics, we examine systems (didactic systems), and create models to gain insight and knowledge from the systems under consideration.

Regarding the first element, a *system* (or anything that can be mathematically modelled) is regarded as a piece of (tangible and intangible) reality that can, at least hypothetically, be isolated from its surroundings. As far as *models* are concerned, their value and richness stem from their capacity to generate knowledge of the system being modelled, knowledge that would be challenging to acquire through other means. The questions related to the adaptation, contrast, and validation of models serve as the driving force propelling the modelling process forward. These conceptualisations of systems and models have significant consequences for the analysis of the modelling process. Following Chevallard (1989), I would like to stress two main properties that are central for understanding the potential relationships or *dialectics between the systems and models*.

The first property is the *recursivity of the modelling process*, the fact that a model can be considered as the system of a further modelling process. This property entails that working on a model may involve the construction of successive models, each one of them better adapted to the system under study at its respective step. This recursive process implies a continuous redefinition of the systems to be modelled, that can integrate the systems, models, and knowledge previously generated. In essence, the initial system undergoes a progressive “mathematisation” based on the creation of “models of models” of the initially considered system.

The second property refers to the *reversibility between the system-model relationship*, which means that the link between system and model can be inverted. The system can appear as a model of its model. To illustrate this, let us consider a simple example adapted from (Chevallard, 1989): a system formed by a class consisting of 11 boys and 15 girls. We can model the proportion of girls or boys in the class, with the fraction $15/26$ or $11/26$, respectively. Reciprocally, when discussing the inequality $15/26 < 16/27$ (where the inequality now serves as our system), the class with boys and girls can be used as a model, and the following can be stated: “if in a class with 15 girls out of a total of 26 children, one more girl joins, the proportion of girls clearly increases”. In this scenario, it is now the extra-mathematical model that helps us acquire new insights on the mathematical system, particularly in the context of comparing fractions.

How do different research frameworks of modelling interpret both properties, *recursivity* and *reversibility*, is a crucial question. It serves to shed light on the distinct perspectives and ideas that various frameworks put forth as their reference epistemological models for mathematical modelling.

The ecological dimension of mathematical modelling

The ecological dimension is used to refer to the institutional conditions enabling and the constraints hindering the way a given activity or piece of knowledge is produced, transposed, taught, and learnt in a given educational setting (Artigue, 2009). Several researchers have underscored the existence of strong challenges impinging on the large-scale dissemination of mathematics as a modelling activity in current educational systems at all school levels. For instance, the “counterarguments” of students to modelling (Blum, 1991), teachers’ beliefs or dilemmas as obstacles for modelling (Kaiser & Maaß, 2007; Blomhøj & Kjeldsen, 2006), the barriers and levers from curricula reforms or educational policies (Burkhardt, 2006), or the tensions inherent in the discourse surrounding modelling within the mathematician’s community (Galleguillos & Borba, 2018).

While the research literature reveals various types of constraints for modelling, they are not always structured as research questions, nor approached with specific analysis tools. In this regard, the theory of didactic transposition (Chevallard, 1985; Chevallard & Bosch, 2020) offers valuable contributions through defining the different stages of didactic transposition. The fundamental assumption here is that it is not possible to interpret what occurs in the school, classrooms, and among individuals, without considering the influence of other institutions involved in the reconstruction of mathematical knowledge, from the moment mathematical knowledge is produced until it is taught and learnt (Bosch & Gascón, 2006, pp. 55–56). Figure 2 below illustrates the steps of the didactic transposition process.

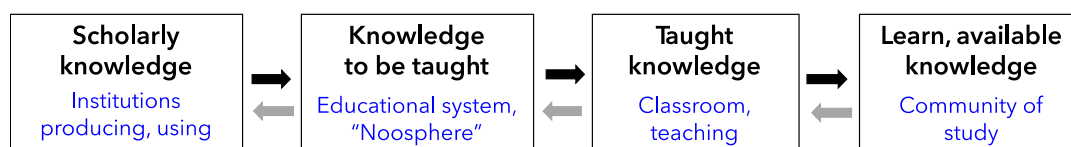


Figure 2: Process of didactic transposition

Using the various steps and back-and-forth “transposition” of the knowledge at stake, in our case, of mathematical modelling, can serve on the one hand, as an asset for placing and characterising the extensive number of constraints our community is currently examining. On the other hand, it may serve to be aware that the choice of institutions or individuals pertaining to them, comprising the unit of analysis, can lead to the formulation of distinct research questions and give rise to different, even incommensurable, results.

Considering how theoretical frameworks encompass both the epistemological and ecological dimensions is viewed as a strategy to advance in the comparison and contrast of different approaches within the research field of mathematical modelling. To this end, I have selected three approaches to modelling, which correspond to the three most prevalent frameworks within the CERME working group, using a selection of specific CERME papers. These papers are used as illustrative examples to analyse how the papers address these two dimensions and how decisions are made. On the one hand,

with respect to the epistemological dimension, we enquire into the epistemological conceptions of mathematical modelling and how they interpret the relationship between systems and models. On the other hand, corresponding to the ecological dimension, we examine the conditions and constraints discussed, and identify the agents and institutions that are observed and analysed.

Case 1: The modelling cycle and competency-based approach

In the research field of modelling and applications (Blum, 2002; Blum et al., 2007), the prevailing interpretation of modelling often revolves around the concept of the *modelling cycle*. Several modelling cycles have been introduced in the literature. The chapter by Greefrath and Vorhölter (2016) in the ICME-13 survey, provides a comprehensive analysis of the origin and development of modelling cycles. One of the earliest proposals, dating back to 1976 at ICME 3, was put forth by Pollak, who described modelling as a cyclic interaction between reality and mathematics. The entire modelling process is often represented as a cycle. The best-known modelling cycle in Germany was created by Blum (1985), featuring various intervening components and the processes that interconnect them. Subsequent proposals, such as the work of Maaß (2005), introduced the concept of interpreting the solution as a step between the mathematical solution and reality, along with processes about interpreting and validating.

The modelling cycle, along with its variations, appears to be particularly valuable as a reference epistemological model for analysing the *cognitive processes* undertaken by students and teachers (Borromeo Ferri, 2007, 2010; Blum and Leiß (2005)). For example, Borromeo Ferri used the modelling cycle to analyse “mathematical thinking styles of learners” or the “individual modelling routes” followed by students confronted with modelling tasks:

What influences do the mathematical thinking styles of the learners’ and teachers’ have on modelling processes in contextual mathematics lessons? [...] (Borromeo Ferri, 2007, p. 2082).

How do grade 10 pupils solve modeling tasks, and what influences do the mathematical thinking styles of the learners have on the modelling processes in mathematics lessons? (Borromeo Ferri, 2010, p. 100).

It can be observed that the notions of the modelling cycle offer methodological tools for examining specific features of modelling activities and of cognitive process followed when solving modelling tasks, which a more conceptualist approach may struggle to describe. For example, these cycles are used to analyse different phases in the modelling process of students to characterise the individual modelling routes, as illustrated in Figure 3. The *recursivity* property, according to this approach, can be interpreted as the non-linearity inherent in the back-and-forth movement along the modelling cycle.

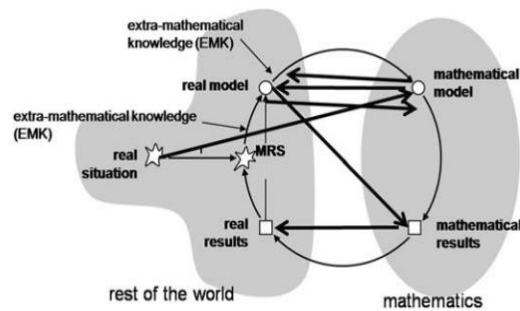


Figure 3: Max's (grade 10 student) modelling route (Borromeo Ferri, 2010, p. 113)

The modelling activities conceived in this approach are generally short teaching processes (a few sessions) allowing observation of the learning and teaching processes of students and teachers. In the case presented by Borromeo Ferri (2010), the sample consisted of 64 pupils and three teachers in three classes of grade 10. The students were given three different modelling tasks (the “Lighthouse”, the “Bales of straw” and, the “jungle” task) one per class, in three 90-minute sessions. The modelling tasks used are well-known tasks in this field, developed in previous projects such as the DISUM project (Blum, 2011). The “Lighthouse” task is proposed as one of them:

In the bay of Bremen, directly on the coast, a lighthouse called “Roter Sand” was built in 1884, measuring 30.7 m in height. Its beacon was meant to warn ships that they were approaching the coast. How far, approximately, was a ship from the coast when it saw the lighthouse for the first time? Explain your solution. (Borromeo Ferri, 2010, p. 109)

As the author justifies, the choice was made to take account of the different mathematical thinking styles of the students. The aim was not to analyse one group in three lessons, but to involve as many students as possible in individual modelling processes. In this case, the unit of analysis includes new ways of describing students' and teachers' processes when solving modelling activities.

There has been also some variation of the modelling cycle, for instance, to integrate the use and role of technology in modelling processes. It had led to an extension of the modelling cycle to include the “world of technology” (as can be found in the contributions of Siller & Greefrath, 2011; Greefrath, 2011) where this “technological world” is describing the “world” where problems are solved with the help of technology.

The conditions and constraints with regard to students or teachers reacting in classroom settings or training contexts are discussed. Many of the mentioned investigations focus on the conditions and constraints that teachers and students can establish in their field of competency. For example, Galbraith and Stillman (2006) analyse what kinds of *student blockages* occur when progressing through the modelling cycle. Kaiser and Maaß (2007, p. 100), for their part, focus on how students' mathematical *beliefs* change when modelling problems are included in instruction, and Siller & Greefrath (2007) address teachers' *beliefs* of the use of modelling in modelling.

Case 2: Models and modelling perspective

Another perspective on modelling is the *models and modelling perspective* (MMP), which provides a coherent framework where modelling is described as a sequence of model developments, involving

different types of activities: model creation, model exploration, and model adaptation (Lesh & Doerr, 2003). It is through engaging in learning activities that students develop, modify, extend, and revise models through “multiple cycles of interpretations, descriptions, conjectures, explanations, and justifications that are iteratively refined and reconstructed by the learner” (Doerr & English, 2003, p. 112). As explained in Ärleback & Doerr (2015), connecting, coordinating, and integrating models is proposed to capture the dialectic and complex nature of creating and developing models. Connecting, coordinating, and integrating models in or through the model eliciting (MEA), model exploration (MXA), and model application (MAA) activities are supposed to provide a dynamic conceptualisation of mathematical modelling (see Figure 4). Modelling is conceived as a continuous activity comprising a chain of models, model sequences, model construction, and refinement. Consequently, modelling activities are designed to help students and teachers progress through a sequence of models: “eliciting”, “exploring”, “applying”, and “developing” models.

In the MMP approach, the *recursivity* property (the evolution of models, comprising sequence of models that are more complex and coordinated at each step) appears to be a central trait in the conceptualisation of modelling. In the papers selected, there is no clear reference to the *reversibility* property between systems-models. Although the consideration of “model application activities” to new contexts, which are always planned at the end of the tasks’ sequence, could be interpreted as this *reversibility* models-systems. The model previously built is now considered as the system to study, that is, to discuss its range of validity and applicability to other contexts.

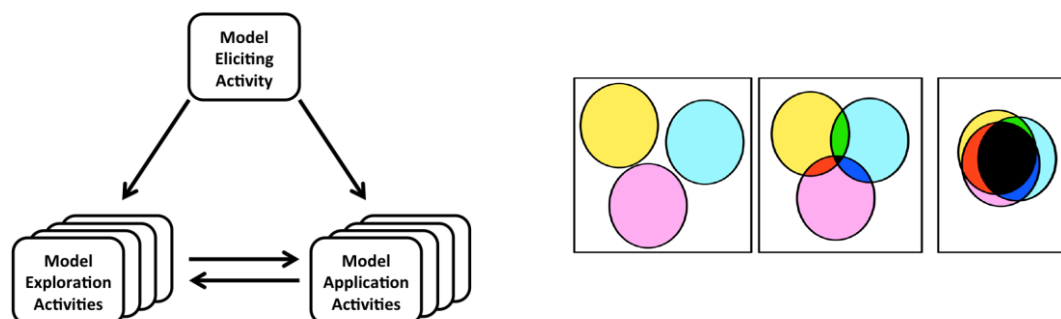


Figure 4: Representation of the different activities in MMP (Lesh et al., 2003) and that of connecting, coordinating, and integrating models (Ärleback & Doerr, 2015)

Ärleback et al. (2013) and Ärleback & Doerr (2018) provide an example of a course that follows a sequence of model eliciting, exploration and application activities, focusing on understanding negative rates of change in various physical phenomena drawn from different fields of physics. This sequence was design and implemented for a six-week entrance course on mathematics for prospective engineering students. The research questions addressed in these papers primarily revolve around the creation and interpretation of models related to negative rates of change across diverse contexts of physical phenomena, as well as the development of a coherent sequence for modelling across these different contexts.

How do students interpret average rates of change related to decreasing functions? How can the design of a model development sequence about negative rates of change be developed?

In the model eliciting activity (MEA) of this sequence, students examined their own body motion along a straight line. They conducted experiments using motion detectors attached to graphing calculators to generate position vs. time graphs, constructed linear graphs based on written instructions, replicated the motion behind given positions vs. time graphs, and provided written descriptions of their movements. In this context, the students' initial concepts and models about function values (position), average rate of change (average velocity), sequences of varying average rate of change (sequences of differing average velocities), and the interplay among these quantities were elicited.

In the model exploration activity (MXA), the students analysed various representations to describe and interpret changing phenomena using their emerging model of average rate of change. They employed two different computer environments to create animations of characters by generating velocity information into position graphs. The model application activity (MAA) allowed the students to use their models to make explicit interpretations, descriptions, and predictions concerning two new scenarios. First, they examined the relationship between the intensity of light with respect to the distance from a light source. Second, they investigated the voltage drop over a fully charged discharging capacitor in a simple resistor-capacitor circuit.

This approach to understanding mathematical modelling has prompted some authors to propose alternative ways of describing mathematical content. For instance, in the research conducted by Carlson et al. (2002), the authors describe the modelling activities in terms of “reasoning about change” or “covariational reasoning”, rather than exclusively referring to concepts like derivatives or change. What seems clear in much of the research in the MMP is the need to construct alternative epistemological models to describe the mathematical knowledge to be taught. In the research of Ärlebäck et al. (2013), the negative rates of change are defined within a broader framework related to covariational reasoning (as defined by Carlson et al., 2002). The unit of analysis here considered is broader than in the previous case, encompassing more extensive and broader sequences of model construction, use, and adaptation of models. Furthermore, if necessary, it even involves the development of new “knowledge to be taught”, particularly in cases where such knowledge diverges from existing content, or is altogether absent.

As far as the *ecological dimension* is concerned, although this approach does not explicitly address it, several investigations within this framework have revealed significant constraints hindering mathematical modelling. For instance, in the selected papers, the authors highlight the absence of appropriate terminology to refer to variation and co-variation. Suitable terminology needs to be introduced and studied to see how students use and adopt it. This constraint is closely related to students' difficulties in communicating the context of changing phenomena.

Case 3: Modelling in the anthropological theory of the didactic

Since the initial works by Chevallard (1989, p. 53), he described modelling processes in different steps: (1) delimitation of the *system* to be studied, specifying its relevant aspects, (2) construction of *model(s)* and work *within* the model; (3) work *with* the model to generate knowledge about the system. As far as models are concerned, Chevallard (1989) makes a distinction between “working *on/within* the model” and “working *with* the model.” Working *with* the model consists in producing

knowledge about the system under study. The interest and value of a model lies in its ability to produce knowledge about the system being modelled that another approach would not provide as easily. Work *on/within* the model may involve the construction of successive models, better adapted to the study, and which imply a redefinition of the systems to be modelled, so that the systems, models, and knowledge generated are part of the new systems to be considered. In the interaction or dialectics between system definition and model construction, the *reversibility* of the system-model relationship and the *recursivity* of the modelling process are two key properties for understanding the conceptualisation of modelling proposed in this theoretical framework.

Recent works on modelling from the ATD pay particular attention to all the questions and answers around these main steps in modelling, and its properties. For instance, initial questions that can start with a modelling process, system-related questions, model-related questions, questions about the adequacy and productivity of models and systems, and/or new questions derived from the whole modelling process.

The relationship between mathematical modelling and the construction of mathematical or extra-mathematical knowledge is addressed through the notion of praxeology (p), which is the main tool proposed by the ATD to describe knowledge and activities in institutional contexts (Chevallard, 2019). The notion of praxeology links the conceptual and procedural aspects of human activities by including, as inseparable entities, the *praxis*, made up of types of tasks and techniques for solving them, and the *logos*, made up of discourses and theoretical tools for describing, explaining, justifying, and nourishing the praxis. Modelling a given situation to obtain new information or knowledge about it can be described in terms of *praxeologies*: we start with a task we want to solve; we use a technique to produce a model of the situation or system underlying the task, and we support this praxis with notions, tools, and justifications provided by the theory (or *logos*). Furthermore, once a given system has been modelled, a new praxeology can be developed by integrating the model produced into new techniques for solving new tasks within a more developed *logos*.

Previous works on the ATD, like García et al. (2006) and Barquero et al. (2019), reformulated modelling as a process of constructing and articulating mathematical praxeologies to answer some initial questions Q_0 that start appearing in the initial system S_0 (as represented in Figure 5). This modelling process is intrinsically *recursive*, since each model (or praxeology) proposed can, in turn, be called into question and become a system for a new modelling process. This enables connecting and coordinating this *dialectic of systems and mathematical models* (or mathematical praxeologies) into broader and more complete knowledge organisations.

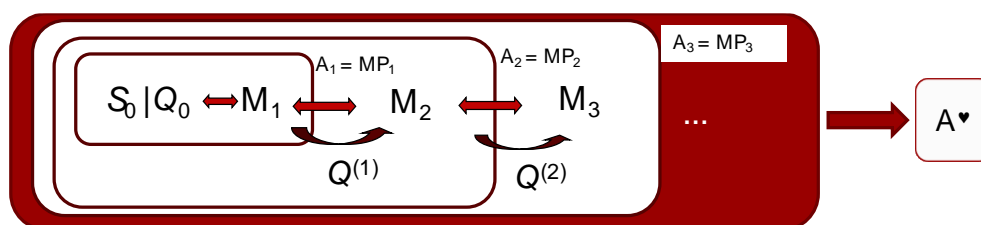


Figure 5: Representation of the modelling process in the ATD

In the following sections, two examples will be employed to illustrate the fundamental traits of the reference epistemological model for modelling, according to the ATD. These examples were chosen for two reasons. Firstly, they represent experiences in different school levels (pre-school, and university). Secondly, they allow illustrating the teaching devices proposed in the framework of the ATD for the teaching of modelling, the so-called *study and research paths* (SRPs) (Chevallard, 2006, 2015; Bosch, 2018). SRPs emerge within the transition of pedagogical paradigms, moving away from the prevailing “paradigm of visiting works” where curricula tend to be described in terms of contents or works to “visit” and to learn. In this context, modelling often takes a back seat. SRPs are proposed as teaching devices towards the paradigm of “questioning the world” with the aim of recovering the dialectics between questions to enquire and answers to be built. In this paradigm, mathematics is assumed to play a prominent role as a modelling tool to enquire into questions.

The study of the conditions facilitating and the constraints hindering mathematical modelling and, in particular, all those that emerge when implementing SRPs for the teaching and learning of modelling, is at the core of the institutional perspective, fundamental to the ATD. Therefore, the *unit of analysis* considered in the ATD includes empirical data from all the steps and institutions involved in the process of didactic transposition (Figure 2) to analyse what can be transposed, and what could encounter resistance to be transposed to a particular teaching and learning context, and why.

Observing all those institutions, investigations on the ATD show important didactic phenomena and associated constraints for mathematical modelling. For instance, when observing school institutions and how modelling is taught and learnt, a prevalence of “fake” enquiries (Bosch et al., 2020), or “fake” *modelling activities*, as a means to the visit of concepts has been perceived. Those fake enquiries and modelling activities lack justification and validation as far as the modelling *praxis* (Wozniak, 2012) is concerned. When we look at curricula, the *isolation of modelling* (as a competency, or extra thematic block) is observed, that hardly transforms and helps articulate the conceptual structure of mathematical knowledge in the curriculum (García et al., 2006). When studying the initial steps of didactic transposition and exploring how mathematics and mathematical modelling is conceived in scholarly institutions (by mathematicians and natural scientists), Barquero et al. (2013) characterise and empirically contrast the prevalence of “*applicationism*” as the dominant epistemological model under which the relation between mathematics and natural sciences is interpreted, described, and conceptualised. This perspective tends to act with the implicit assumption that mathematics might be introduced before any contact with natural sciences takes place. This view tends to reduce modelling to a mere “application” of previously introduced knowledge, limiting its full potential.

An SRP in pre-school education: much more than collecting silkworms

The first example of an SRP is the one discussed in García and Ruiz-Higueras (2010), designed and implemented in pre-school with 4-year-old students. This SRP centred around the topic of silkworms and the transformation process into butterflies. The authors sought to investigate how theoretical frameworks not explicitly developed for mathematical modelling, here in particular the ATD, can facilitate and enhance the design of modelling activities. Additionally, their focus on pre-school

education is noteworthy, as this area has been somewhat overlooked in existing research on mathematical modelling.

It was spring and the pupils were used to gathering silkworms and feeding them with mulberry leaves. Therefore, it was easy to bring a box with silkworms into the classroom and observe their life cycle. At that time of the school year, the students were in the process of developing their understanding of cardinal numbers. However, not all of them consistently used numbers as the most effective way to answer questions related to “how many.” For some students, numerals were known (up to 9 or even higher numbers), but numbers were meaningless to them. A lot of students struggled to use numbers in contexts such as measuring collections, creating new collections, or comparing collections. However, the modelling work within this SRP extended beyond the use of cardinal numbers for counting collections. It also involved questions related to time measurement and data handling. The context was authentic and real: silkworms were present in the classroom, requiring care and feeding. Moreover, the collection of silkworms would soon transform. The silkworms would become cocoons and, ultimately, the silk moths (referred to as butterflies by the students) would live and die.

When the children had their box with silkworms, the initial question (Q_0) was “*If we have N silkworms, how many leaves do we need to feed them?*” In order to answer this question, certain decisions had to be made by both the students and the teacher. It was decided that each silkworm would require one leaf per day, and these leaves might need to be replenished daily. Additionally, the students had to consult with the gardener to determine the daily leaf requirements. This led to a quantification activity that involved working with cardinals and comparing different collections.

From this question, the students began exploring various methods of counting. They used drawing models to help them count the leaves and tabular models to organise information about time, the number of silkworms ($s(t)$), and the number of leaves ($L(s(t))$). However, the system soon underwent a change, as the silkworms began their transformation into cocoons. This prompted new questions, such as: “*If the silkworms are turning into cocoons, how does it change the counting of our collection and the amount of leaves we have to ask for?*” (Q_1).

This situation caused a significant evolution (or transformation) of both the *system* and the *models* under consideration. On the one hand, the students decided to separate the silkworms from the cocoons by putting them in another box because they were concerned that the cocoons could be damaged when the box was clean and when the silkworms were being fed. That decision caused the division of the original collection. Furthermore, the concept of time became increasingly important, as the students needed to keep track of *how long it would take for a cocoon to become a butterfly*. The system, initially assumed to be *static*, had now evolved into a *dynamical* one. In response to these new requirements, a new tabular model became necessary that gathered information about the date, the number of silkworms, the number of cocoons, and the leaves.

The teacher had to introduce certain tools in the classroom to enable the students to simultaneously monitor quantity and time. The system was divided into different sub-collections, each represented by different boxes containing silkworms or cocoons. Additionally, a tabular model was used to record the evolution of the system (see Figure 6, central image).

From the day the first moth emerged, new questions arose, such as “What happens when the butterflies are born, how does it change the counting of our (sub)collections?” (Q_2). This prompted the need to calculate the time that had elapsed from the moment the cocoon appeared, including questions like: “How long did it take for the moths to be born?” and “How many days do they live?” This development meant an important evolution of the system. To accommodate all this additional modelling work, a new variable, the butterflies, had to be considered, even though they unfortunately had a short lifespan. The students felt the need to create a new tabular table to, on a daily basis, keep track of the time, and record information about the birth of new butterflies, the number of cocoons, new moths, dead moths, and moths that were still alive. The teacher decided to introduce a new tabular model to quantify time and collections of cocoons, new moths, deceased moths, and moths that were still alive (see Figure 6, right side).

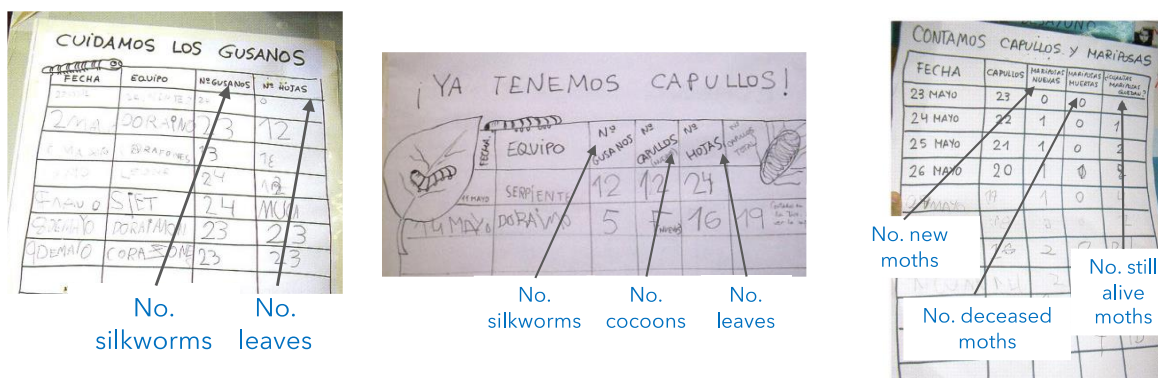


Figure 6: Kinds of tabular models (1, 2 and 3) considered in the SRP about silkworms

It is intriguing to observe the recursive nature of the modelling process, and how, partly due to the changes in the collection, the mathematical models (comprising counting and tabular models, denoted as Model 1, 2 and 3, in Figure 7) are transformed and completed at each step, evolving into more comprehensive and complex structures. Moreover, it is noteworthy to see how the systems also evolve and, at each step, the systems (referred to as system 1, 2 and 3) encompass all the modelling work conducted thus far. Figure 7 aims to summarise this evolution and dialectics between the system and the models, serving as an illustrative example of the modelling process according to the ATD (as introduced earlier in Figure 5).

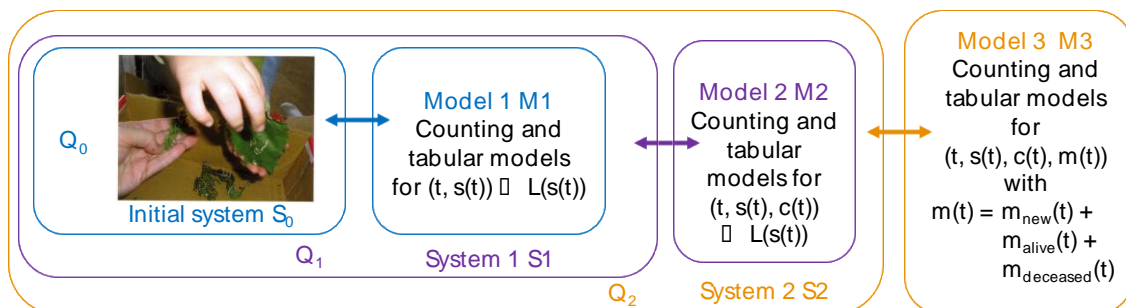


Figure 7: Recursivity of the modelling process both of systems and models

When all the butterflies died, the system was over, and the activity finished. However, the class had a lot of information about the system and its evolution. This system was not easy at all and was in

constant evolution. The models constructed during the enquiry process recorded this evolution. The teacher then planned an activity to make these tabular models useful to retrieve information about the system(s) that had been analysed. The students were asked to use all the information registered (that is, models 1, 2 and 3, Figure 6) to reconstruct the evolution of the system. It is important to note the exchange, the *reversibility* between models and system, now considering the models as systems to address Q_3 . Figure 8 shows the reconstruction of the systems, produced by the students (from García & Ruiz-Higueras, 2010, Ruiz-Higueras & García, 2011).



Figure 8: Reconstructing the system from the model

An SRP at university and teacher education: Evolution of Facebook users

The second SRP centres on modelling the evolution of the number of Facebook users. Specifically, it studied a noteworthy event in 2014 when certain researchers, affiliated with Princeton University, made the prediction that Facebook would lose 80% of its users by 2017 (pre-print by Cannarella & Spechler, 2014). This prediction attained substantial media coverage, and initiated a somewhat strong reaction from Facebook, which expressed scepticism about the accuracy of the underlying models used by the researchers.

The generating question Q_0 of this SRP refers to “*Can the forecasts published by Princeton in 2014 about the future evolution of Facebook users be true? How can we model and fit real data from Facebook users to provide our own forecasts?*”. When designing this SRP, the opportunity arose to implement it with first-year students of Business Administration and Innovation Management degrees. The type of modelling work that emerged with the students will be summarised below, and further details are available in Barquero et al. (2017 and 2019). It is worth noting that this SRP has transitioned between various institutions over the past year. It has been implemented with secondary school students, with in-service secondary school teachers, and for the professional development of university lecturers. The versatility of this SRP across these different settings has provided us with the opportunity to analyse various conditions and constraints that have arisen because of its adaptation to different institutional settings.

The initial design for first-year students consisted of three distinct phases, each corresponding to the main derived questions that were intended to be addressed. The first phase was dedicated to addressing Q_1 , which involved searching and determining what data on Facebook users used, looking for existing models, and formulating initial hypothesis about the evolution of the number of users. During this stage, the students began by selecting and organising data about Facebook users ($Q_{1.1}$). This process required them to make decisions regarding which variables to consider and explore, and study existing analyses concerning the historical trends of Facebook ($Q_{1.2}$). From this work, the

students identified *pre-existing models* ($Models_1$) that presented fitting models and/or trends analysis of Facebook users (often sourced from *Statista* or *Datareportal* or Facebook's quarterly reports). This first phase concluded with questions pertaining to what models to use, in alignment with the specific hypotheses (Q_2).

The second phase focused on addressing Q_2 , and when working with first-year university students, the mathematical models primarily revolved around elementary functions. Lecturers took advantage of this, as it was one of the topics to be covered in the regular course syllabus, thus serving as the starting point for the introduction of this course topic. Within this context, the students identified which elementary function to use, and which hypothesis to formulate in support of their selection ($Q_{2.1}$). They also enquired into how the coefficient of these functions could be interpreted within the specific context of Facebook that was under consideration ($Q_{2.2}$). This second phase finished with some decision-making on what could be considered as the "best" *models*, based on elementary functions or combinations thereof ($Models_2$). These decisions were justified by the hypothesis and the interpretation of the coefficients they had been considering.

The third phase focused on Q_3 , what a "good" model means, how to collectively agree on the choice of the "best" model, and to do what. Usually, the students agree with the teachers on distinguishing models "to fit data" and models "to forecast" the short-, medium- and long-term evolution of the number of Facebook users. Concerning the first, the central questions related to how to calculate and interpret the errors made when comparing simulation of models with data ($Q_{3.1}$), that is, the adequacy of the model. With respect to the second, the students used to select the more recent data and analyse their variation. At this stage, the teachers opened new questions about what models to use to study the rates of data variation ($Q_{3.2}$). This led to the consideration of a new enriched "system": the one formed by the original data and the variation of the number of Facebook users. It is the starting point of double modelling of the number of Facebook users and the variation in the number of users. This leads to considering the derivative as a model of the variation that constitutes a further mathematisation step.

This work was inspired by Serrano et al. (2013), which analysed the back-and-forth movements between the initial system – a time-series of the sales of a firm – and the different models proposed to do the forecasting. The analysis of these movements, that are at the core of the "mathematising steps", shows how the initial empirical system (at the beginning with data) is enlarged and progressively enriched with new variables and mathematical objects (such as one-variable elementary functions, their simulation and forecasting). Progressing in the modelling activity, initiated with a real situation of Facebook, soon led to a process where mathematising affected both the system and the model.

The question about how to validate short- and long-term model forecasting (Q_4) remained open in the different implementations of this SRP and continues to be open today. Facebook data are constantly updated, often providing new information to assist in the validation of the proposed models, as well as in their revision (obtaining new data to rethink the hypothesis). What is certain today is that Facebook did not lose all its users in 2017. In fact, in February 2022, it was announced that Facebook

was losing users for the first time in its history, but new forecasts do not predict a quick decay. The SRP continues to be as open and lively as it was in its first implementations.

Final comments on the epistemological and ecological analysis of the SRP

Designing and implementing SRPs, while aligning them as close as possible to the paradigm of “questioning the world”, open an infinite process of *recursivity* and *reflexivity* of systems, models and their dialectics. They also offer new ways of organising mathematical knowledge in response to the question we aim to address and the models we want (or are institutionally able) to consider. When this same SRP has been used into teacher education (Barquero et al., 2018; Florensa et al., 2020) and in-service teachers have had the opportunity to experience this SRP, before analyzing it and planning possible adaptation for secondary school and university, the range of possibilities expands indefinitely. Figure 9 provides a condensed overview of a *question-answer map* (in the sense of Winsløw et al., 2013), with particular focus on the types of models that could be explored. This map was collaboratively developed by researcher and teachers, starting from the initial design as explained earlier ($Q_0, Q_1, Q_2, Q_3, \dots$), to include various potential modelling paths. However, it will be during the redesign and managing of new implementation of this SRP that these branches can be further refined.

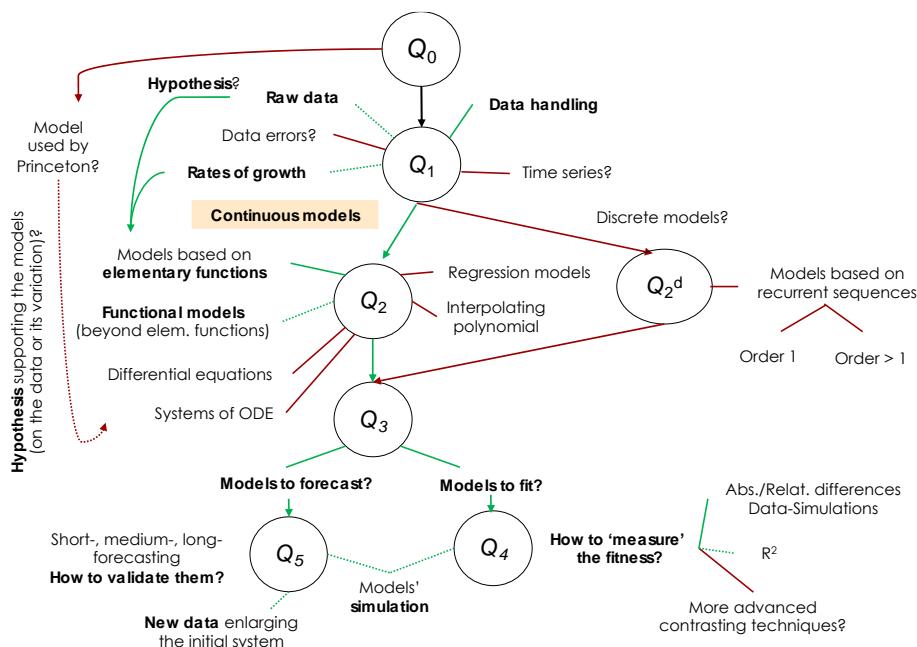


Figure 9: Condensed question-answer map of possible modelling paths

Working with SRPs contributes to the emergence of, and enables studying numerous conditions created and many constraints limiting the “life” of modelling. These different constraints appear at different levels of mathematics specificity or beyond the mathematics discipline scope. To carry out this ecological analysis, we use the *levels of didactic co-determinacy* (introduced by Chevallard (2002), see Figure 10) as a framework to identify different kinds of conditions and constraints that affect the transposition processes and evolution of teaching and learning modelling. This hierarchy of levels goes from the most generic level, *civilisations*, to the most particular one, the specific *tasks* considered. The lower levels refer to the way a discipline is organised (in domains, sectors, themes,

and particular tasks), and that varies depending on the discipline, and on the teaching institution considered. The *upper levels of codeterminacy* refer to the more general constraints coming from the way our *civilisations* and *societies*, through *schools*, organise the teaching of any kind of knowledge (*pedagogy* level). We use these levels to place the conditions and constraints appearing at different levels, including the specific ones related to how mathematical content is proposed to be taught and learnt at school, with the general ones regarding the general school and pedagogical organisation of disciplines, and the role assigned to schools in our societies.

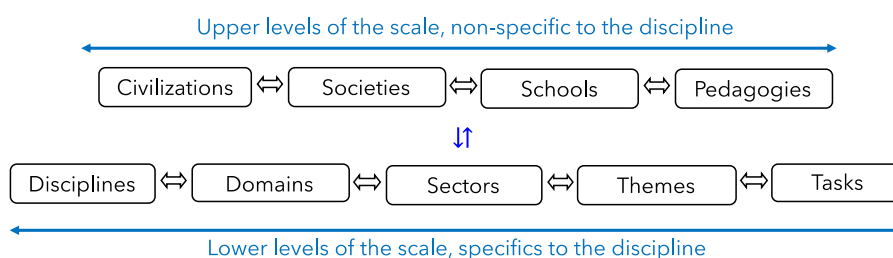


Figure 10: Scale of levels of didactic codeterminacy (adapted from Chevallard, 2002)

Complementarily to the ones mentioned before, we have identified some *generic constraints* to modelling, such as the already commented one of *applicationism* as the prevailing epistemological model, which is supported by other constraints. For instance, strong boundaries exist between mathematics –the *disciplinary confinement*–, as if these disciplines could evolve (at least, in schools) independently, and consequently be taught without too many interactions. Another constraint is that of having curricula defined as *set of “works”* rather than as *questions to enquire*, deeply rooted in our schools and society. Those constraints appearing at the general level certainly have an impact at the level of the *pedagogy*, where the lack of routines and teaching devices to support *enquiry gestures* is evident. There are also difficulties for teachers and students to manage the *didactic time* more in terms of the modelling work done and questions addressed than by disciplinary contents transferred. At the level of the *discipline* (or at the didactic-disciplinary level), the need for introducing changes into the prevailing didactic contracts to assign new responsibilities to students (i.e., in formulating questions, looking for models, validating the models and modelling processes, reporting on the work done) and for teachers (i.e., guiding, holding debates, asking new questions to understand different proposals) was obvious.

Concluding remarks: Parallel and complementary questions to address jointly

Throughout this paper, my focus has been on two major questions that have evolved and may continue evolving together, complementing each other. These questions refer to the epistemological, didactic, and ecological needs and questioning that extend not only to the research community, but also the broader mathematics education community.

On the one hand, we have explored the question of *what mathematical modelling is* and *how it is characterised or conceptualised*. This conceptualisation profoundly influences the way we design and analyse the teaching and learning of modelling. On the other hand, we have studied the question of *how to disseminate mathematical modelling as a normalised activity in the classroom* (at all school

levels); and, what *conditions* need to be established, what *constraints* we can foresee to foster a self-sustaining and long-term development of mathematical modelling.

Regarding the first question, we can assume or define different educational aims for our object of knowledge, mathematical modelling: it can be assumed as an object to be taught and learnt, or as a means for teaching mathematics, amongst others. Moreover, we can approach problems related to modelling from different positions: from research, from the curriculum or curricular developers, from classroom practice, amongst others. By comparing different approaches with the selection of the modelling cycle approach, the models and modelling perspective and the ATD (along with the selection of the associated CERME papers and derived ones), we can illustrate the varying ways in which mathematical modelling is conceptualised in research. This diversity in conceptualisation influences how researchers formulate different research questions and adopt diverse strategies when it comes to designing and analysing teaching and learning practices. It is important to ensure that these reference epistemological models are made visible to others, and to progress in a *collective construction of an epistemological understanding* of modelling which does not contradict particular models, but rather facilitates the comparison of their particularities. In this paper, to explain how different approaches interpret and conceptualise mathematical modelling, I used a *simple and flexible perspective* of systems, models, and their dialectics as initially proposed by Chevallard (1989), along with the *recursivity* and *reversibility* properties of the modelling processes. A crucial question is how different research frameworks interpret the dialectics between systems and models and incorporate the properties of *recursivity* and *reversibility* into their epistemological models.

As for the second question, a new dimension has been introduced: the *ecological* one, related to the conditions and constraints that enable or hinder the existence of activities as self-sustaining and long-term development of mathematical modelling. This *ecological analysis* may vary depending on the educational settings in which the research takes place: at pre-school, primary school, secondary school, or university (the first two being significantly underrepresented).

The ecological dimension is rarely considered in mathematics education, even if some implicit references can be found in research on mathematical modelling and applications, in terms of “barriers”, “obstacles”, “beliefs”, amongst other. However, their consideration is not often structured in the form of research questions, nor is it approached with specific analysis tools. Moreover, it does not entail a substantial modification of the unit of analysis considered by the different frameworks, which often remain focused on the activities carried by students and/or teachers within the teaching institution (and under concrete conceptions about modelling and teaching or learning). In contrast, in the institutional perspective adopted by the ATD, the two notions of didactic transposition process and scale of levels of codeterminacy involve an important extension of the unit of analysis considered. They also provide methodological tools to systematically approach the analysis of conditions and constraints for modelling to exist as a normalised activity at school.

The ecological analysis is essential. It can help us to collaborate effectively to address the important challenge of detecting and locating constraints limiting the long-term dissemination of modelling in our classrooms, for our society. It can be also useful to compare, contrast and combine different proposals to overcome these obstacles. It may serve not only for research to detect and study the

origin, scope, and implications of constraints for modelling, but it is also crucial to make them visible for teachers (through teacher education) to deal with them. Most, if not all, of these institutional constraints are anchored in deep-rooted practices and are difficult to notice for teachers and for researchers, since they appear as “the natural way of doing things”. This underscores the importance of collaboration not only among teachers and researchers in mathematics education, but also with scholars of different fields—including mathematicians—, to work together in creating and setting up favourable conditions for modelling. Now more than ever, there is the need to build the necessary epistemological and didactic infrastructure for collaboration: to share instructional designs for modelling, to facilitate their “migration” and adaptation across different school setting and levels, and to empirically identify conditions, constraints and their institutional relativity (what can vary and what remains unaltered although our collective efforts).

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