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# Rate of Convergence in the Functional Central Limit Theorem for Stable Processes

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## Abstract

In this article, we quantify the functional convergence of the rescaled random walk with heavy tails to a stable process. This generalizes the Generalized Central Limit Theorem for stable random variables in finite dimension. We show that provided we have a control between the random walk or the limiting stable process and their respective affine interpolation, we can lift the rate of convergence obtained for multivariate distributions to a rate of convergence in some functional spaces.

**Keywords:** functional convergence, Stein's method, stable distribution, central limit theorem

# 1 Introduction

## 1.1 Motivations

The Stein's method is one way to bound the Wasserstein-1 ( $\rho^1$  for short) distance between two probability measures on a metric space  $(E, d)$ , defined by

$$\rho_E^1(\mu, \nu) = \sup_{F \in \text{Lip}_1(E)} \left( \int_E F \, d\mu - \int_E F \, d\nu \right) \quad (1)$$

where

$$\text{Lip}_1(E) = \left\{ F : E \rightarrow \mathbb{R}, |F(x) - F(y)| \leq d(x, y), \forall x, y \in E \right\}.$$

It is known [1] that convergence in the  $\rho^1$  distance implies the convergence in distribution and the convergence of first moments. The Stein's method initiated in the seventies by C. Stein, was originally built to assess rate of  $\rho_{\mathbb{R}}^1$  convergence towards the Gaussian distribution. It was soon extended to the Poisson limit by L. Chen (see [2] and references therein). Since then, the majority of the tremendous number of papers dealing with this approach has been concentrated on Gaussian limits and in a tinier percentage to Poisson limits. It is only very recently that some attention were devoted to some other limiting regimes, like convergence to a stable distribution [3–8]. There is here a dramatic difference between the cases  $\alpha > 1$  and  $\alpha \leq 1$ , since in the latter situation the distribution has no longer a finite first moment. We refer to monographs such as [9–13] for an overview of non-Gaussian CLT. In view of the preceding remark, this means that we cannot expect a convergence in the sense of the Wasserstein-1 distance but we have to restrict the set of test functions taken in the supremum of (1) as in [8]. We will not deal with this problem here but it is the object a paper in preparation [14]. We thus assume throughout this work that  $\alpha$  belongs to  $(1, 2)$ . Another extension of the Stein techniques is to consider functional convergence theorems, i.e. CLT-like theorems in spaces of functions. The first paper to handle such a situation was [15]. More recently, in a series of paper [16–20], the rate of convergence of some classical theorems like the Donsker Theorem or the Brownian approximation of a normalized Poisson process were established. There are two remarks which can be made here. First, the rate of convergence depends on the functional space into which we are envisioning the processes under study. For instance, the sample-paths of a Brownian motion can be seen as continuous functions or as Hölder continuous functions: The rate of convergence of the random walk towards the Brownian motion has been shown to be  $n^{-1/6} \log n$  (respectively  $n^{-1/6+\gamma/3}$ ) in the space of continuous functions (respectively the space of  $\gamma$ -Hölder continuous functions for  $\gamma < 1/2$ ). The second remark is that we can get rid of the calculations in infinite dimension which appeared in [20] by considering affine interpolations of the different processes involved and then reduce the computations to estimates in some finite dimensional spaces.

The main contribution of this paper is to show how we can convert a rate of convergence for multivariate distributions into a rate of convergence in some Banach spaces provided we have a control on the error made by replacing some processes by their affine interpolation. In passing, we greatly simplify one of our former proof developed for the Donsker theorem in [17]. We treat here the case of a random walk with increments in the domain of attraction of a stable distribution with finite mean but infinite variance. As a

key element of our proofs, we derive a moment bound for sums of stable random variables which seems to be new and interesting by itself.

## 1.2 Main result

A random variable  $Y$  is said to be in the domain of attraction of a stable distribution if for a collection of random variables  $Y_1, \dots, Y_n$  independent and with the same distribution as  $Y$ , there exists two sequences  $(a_n)_{n \geq 0}$  and  $(b_n)_{n \geq 0}$  such that

$$a_n(Y_1 + \dots + Y_n) - b_n \xrightarrow[n \rightarrow \infty]{\text{law}} S,$$

where  $S$  is an  $\alpha$ -stable random variable, for  $\alpha \in (0, 2]$ . In the literature, the appartenance to the domain of attraction has been linked to the tail of the distribution, see e.g. [12] or [11] for the review of the literature, and complete statement of the theorems.

One of the first paper to use Stein's method to derive rates of convergence for the stable central limit is [7]. In that paper, the authors introduce the normal domain of attraction of an  $\alpha$ -stable distribution: random variables whose cumulative distribution functions is of the form

$$1 - F_Y(t) = \mathbf{P}(Y \geq t) = \frac{A + \varepsilon(t)}{t^\alpha} \text{ and } F_Y(-t) = \frac{A + \varepsilon(-t)}{(-t)^\alpha}, \quad (2)$$

whenever  $t \geq 1$ , and where  $\varepsilon$  is a function with a specified decay at infinity. For  $(Y_n)_{n \geq 1}$  a sequence of IID random variables of this sort, define

$$S_n = \frac{1}{n^{1/\alpha}} \sum_{i=1}^n Y_i.$$

It is shown in [7] that:

$$\rho_{\mathbb{R}}^1(S_n, S) \leq \frac{c}{n^{1-2/\alpha}}.$$

Building on that paper, and their subsequent work, we aim at establishing the rate of convergence of the continuous time analog of  $S_n$ , namely the random walk  $(X_n(t))_{t \geq 0}$  defined by

$$X_n(t) = \frac{1}{2^{n/\alpha}} \sum_{i=1}^{\lfloor 2^n t \rfloor} Y_i, \quad (3)$$

to an  $\alpha$  stable process  $(S_t)_{t \geq 0}$  in a space of functions. As in some of our previous works, given  $p \geq 1$  and  $\eta \in [0, 1]$ , we introduce the fractional Sobolev space

$$W_{\eta, p} = \left\{ f \in L^p([0, 1], \mathbb{R}^d); \int_0^1 \int_0^1 \frac{|f(s) - f(t)|^p}{|s - t|^{1+\eta p}} ds dt < +\infty \right\},$$

equipped with the norm

$$\|f\|_{W_{\eta, p}}^p = \int_0^1 |f(t)|^p dt + \int_0^1 \int_0^1 \frac{|f(s) - f(t)|^p}{|s - t|^{1+\eta p}} ds dt.$$

Note that for  $\eta - 1/p > 0$ ,  $W_{\eta,p}$  is embedded in the space  $(\eta - 1/p)$ -Hölder continuous functions whereas for  $\eta - 1/p < 0$ ,  $W_{\eta,p}$  can be embedded into the space of  $p(1 - \eta p)^{-1}$  integrable functions over  $[0, 1]$ . Since  $(X_n(t))_{t \geq 0}$  and  $(S_t)_{t \geq 0}$  induce probability distributions on some of these functional spaces  $W_{\eta,p}$  (for  $\eta < \alpha^{-1} < 1 \leq p < \alpha$ ), their distance can be quantified by the Wasserstein-1 distance on any of these spaces defined as

$$\rho_{W_{\eta,p}}^1(\text{law}(X_n), \text{law}(S)) = \sup_{F \in \text{Lip}_1(W_{\eta,p})} (\mathbf{E}[F(X_n)] - \mathbf{E}[F(S)]).$$

Our main result is an upper-bound of this rate of convergence:

**Theorem 1.** *Let  $(Y_i)_{i \geq 1}$  be a sequence of IID random variables of cdf  $F_Y$ , in the normal domain of attraction of an  $\alpha$ -stable distribution with  $\alpha \in (1, 2)$ . We assume that there exists  $A > 0$ ,  $\gamma > 0$  and a function  $\varepsilon$  such that*

$$\begin{aligned} \mathbb{P}[Y_1 > t] &= \frac{A}{2} \frac{1}{t^\alpha} \mathbf{1}_{[1, +\infty[}(t) + (1 - A) \frac{\varepsilon(t)}{t^\alpha} \quad \text{for } t > 0, \\ \mathbb{P}[Y_1 \leq t] &= \frac{\eta}{2} \frac{1}{|\min(t, 1)|^\alpha} + (1 - A) \frac{\varepsilon(t)}{t^\alpha} \quad \text{for } t \leq 0, \end{aligned}$$

where  $\varepsilon$  is a bounded function on  $[-1, 1]$  such that  $\sup_t |t^\gamma \varepsilon(t)| < +\infty$ . Let  $X_n$  be defined as in (3). For  $(\eta, p) \in (0, 1/\alpha) \times [1, \alpha)$ , there exist  $v > 0$  and  $c > 0$  both depending on  $(\eta, p, \alpha, \gamma)$  such that we have

$$\rho_{W_{\eta,p}}^1(X_n, S) \leq c 2^{-nv},$$

where

$$v = \left( \frac{1}{\alpha} - \eta \right) p \frac{\min\left(\frac{2}{\alpha} - 1, \frac{\gamma}{\alpha}\right)}{1 + \left(\frac{1}{\alpha} - \eta\right) p} > 0. \quad (4)$$

The strategy we use here is inspired by [17]. First, in Section 2, we consider a projection of  $(X_n(t))_{t \geq 0}$  and  $(S_t)_{t \geq 0}$  on a certain finite dimensional space. This reduction to the finite dimension allows us to rely on [7, 21, 22] to control the distance of the two projected processes. Then, the control the sample-paths distances between  $(X_n(t))_{t \geq 0}$  and  $(S_t)_{t \geq 0}$  and their respective projections is obtained via Lemma 5 of Section 3. Finally, we summarize our results and state the final control in Section 3.3. As a key element of our approach, we establish a moment bound in Section 4 that, up to our knowledge, was not known, even for Pareto distributions. We prove that when  $Y_1, \dots, Y_n$  are in the normal domain of attraction of a stable distribution, for all  $p < \alpha$ ,

$$\sup_n \mathbf{E} \left[ \left| \frac{Y_1 + \dots + Y_n}{n^{1/\alpha}} \right|^p \right] < +\infty.$$

## 2 The Interpolated Random Walk, and Reduction to Finite Dimension

### 2.1 Notations and the Interpolated Random Walk

For the sake of simplicity, we choose the dyadic partition to define the affine interpolations of the random walk. The general situation, for which we have non nested intervals and

thus side effects, can be handled as in [17]. We consider the sequence of functions:

$$h_n^i(t) = \sqrt{2^n} \int_0^t \mathbf{1}_{[\frac{i}{2^n}, \frac{i+1}{2^n})}(s) ds.$$

Notice that this sequence actually forms an orthonormal family with respect to the inner product

$$\langle f, g \rangle = \int_0^1 f'(s)g'(s) ds.$$

We then form the interpolated random walk:

$$\begin{aligned} X_n(t) &= \frac{\sqrt{2^n}}{2^{n/\alpha}} \sum_{i=1}^{2^n} Y_i h_n^{i-1}(t) \\ &= \frac{1}{2^{n/\alpha}} \sum_{i=1}^{\lfloor 2^n t \rfloor - 1} Y_i + \frac{Y_{\lfloor 2^n t \rfloor}}{2^{n/\alpha}} (2^n t - \lfloor 2^n t \rfloor). \end{aligned}$$

To obtain a convergence rate, we introduce for  $m \leq n$  (to be determined later) the projection operator on  $\text{span}\{h_m^j, j = 0, \dots, 2^m - 1\}$ :

$$\pi_m(f) = \sum_{j=0}^{2^m-1} \langle f, h_m^j \rangle h_m^j.$$

The introduction of this projection operation is the tool that allows us to use results available in finite dimension. We split the difference as follows:

$$X_n - S = (X_n - \pi_m(X_n)) + (\pi_m(X_n) - \pi_m(S)) + (\pi_m(S) - S).$$

The first and third terms are of the same nature, they represent the gap between a process and its projection. In Section 3, we prove the Lemma 6 that deduces a control of these distances from Hölder sample-path regularity. For the second term above, we will see that this term is controlled using the finite-dimensional distribution convergence, which is estimated using Stein's method in finite dimension. For that term, we use the control given by [7], but we also provide new approaches below.

## 2.2 Reduction to finite dimension

In this section, we show how the the projection operator allows one to reduce the analysis on finite dimensional spaces, from where we can cite existing work in the literature. This adapts arguments developed initially in [17].

**Lemma 2.** *The projection of the partial sums write:*

$$\pi_m(X_n)(t) = \sum_{j=1}^{2^m} \langle X_n, h_m^j \rangle h_m^j(t) = \sum_{j=1}^{2^m} \frac{\sqrt{2^m}}{2^{\frac{n-m}{\alpha}}} \left( \frac{1}{2^{\frac{m}{\alpha}}} \sum_{i=j2^{n-m}}^{(j+1)2^{j+1}} Y_i \right) h_m^{j-1}(t).$$

*Proof.* Recall that  $X_n(t) = \frac{\sqrt{2^n}}{2^{n/\alpha}} \sum_{i=1}^{2^n} Y_i h_n^{i-1}(t)$ . Taking the projection leads to

$$\pi_m(X_n)(t) = \frac{\sqrt{2^n}}{2^{n/\alpha}} \sum_{j=1}^{2^m} \sum_{i=1}^{2^n} \langle h_n^{i-1}, h_m^{j-1} \rangle h_m^{j-1}(t) Y_i.$$

Now, the scalar product  $\langle h_n^i, h_m^j \rangle$  is zero except when the intervals  $[\frac{i}{2^n}, \frac{i+1}{2^n}]$  and  $[\frac{j}{2^m}, \frac{j+1}{2^m}]$  are nested. In that case, the scalar product yields the length of the resulting interval, hence  $\langle h_n^i, h_m^j \rangle = \frac{\sqrt{2^n} \sqrt{2^m}}{2^n}$ . Thus, the projection becomes

$$\pi_m(X_n)(t) = \frac{\sqrt{2^n}}{2^{n/\alpha}} \sum_{j=1}^{2^m} \sum_{i=j2^{n-m}}^{(j+1)2^{n-m}-1} \frac{\sqrt{2^n} \sqrt{2^m}}{2^n} h_m^{j-1}(t) Y_i.$$

We simplify the terms and normalise the inner sum by  $2^{\frac{n-m}{\alpha}}$  in the above expression to get:

$$\pi_m(X_n)(t) = \frac{\sqrt{2^m}}{2^{\frac{m}{\alpha}}} \sum_{j=1}^{2^m} \left( \frac{1}{2^{\frac{n-m}{\alpha}}} \sum_{i=j2^{n-m}}^{(j+1)2^{n-m}-1} Y_i \right) h_m^j(t).$$

The proof is thus complete.  $\square$

**Remark 1.** In the proof above, we see that considering the dyadic partition simplifies the discussion, since the intervals  $[\frac{j}{2^n}, \frac{j+1}{2^n}]$  and  $[\frac{i}{2^m}, \frac{i+1}{2^m}]$  are nested. In the case of an arbitrary partition, we would have boundary terms that needs to be handled. We refer the reader to [17], where such a discussion is performed.

Meanwhile, we have that the projection  $\pi_m(S)$  is:

$$\pi_m(S)(t) = \sum_{j=1}^{2^m} \langle S, h_m^{j-1} \rangle h_m^{j-1}(t).$$

This scalar product can be computed with an integration by parts:

$$\langle S, h_m^j \rangle = \sqrt{2^m} \int_{[0,1]} S'(t) \mathbf{1}_{[\frac{j}{2^m}, \frac{j+1}{2^m}]}(t) dt = \sqrt{2^m} \left( S\left(\frac{j+1}{2^m}\right) - S\left(\frac{j}{2^m}\right) \right).$$

Thus, we have:

$$\pi_m(S)(t) = \sum_{j=0}^{2^m-1} h_m^j(t) \sqrt{2^m} \left( S\left(\frac{j+1}{2^m}\right) - S\left(\frac{j}{2^m}\right) \right)$$

and each term  $S\left(\frac{j+1}{2^m}\right) - S\left(\frac{j}{2^m}\right) \stackrel{(d)}{=} \frac{1}{2^{m/\alpha}} S_1$ . Hence, for all  $t \in [0, 1]$ , we have established that

$$\begin{aligned} & F\left(\pi_m(X_n)(t)\right) - F\left(\pi_m(S)(t)\right) = \\ & F\left(\frac{\sqrt{2^m}}{2^{\frac{m}{\alpha}}} \sum_{j=0}^{2^m-1} \left( \frac{1}{2^{\frac{n-m}{\alpha}}} \sum_{i=j2^{n-m}}^{(j+1)2^{n-m}-1} Y_i \right) h_m^j(t)\right) - F\left(\sqrt{2^m} \sum_{j=0}^{2^m-1} \left( S\left(\frac{j+1}{2^m}\right) - S\left(\frac{j}{2^m}\right) \right) h_m^j(t)\right). \end{aligned}$$

Let

$$\frac{1}{2^{\frac{n-m}{\alpha}}} \sum_{i=j2^{n-m}}^{(j+1)2^{n-m}-1} Y_i = U_{m,j}^n, \text{ and } S\left(\frac{j+1}{2^m}\right) - S\left(\frac{j}{2^m}\right) = S_{m,j},$$

we have obtained:

$$\mathbf{E}\left[F\left(\pi_m(X_n)\right)\right] - \mathbf{E}\left[F\left(\pi_m(S)\right)\right] = \mathbf{E}\left[F\left(\sum_{j=0}^{2^m-1} U_{m,j}^n h_j^m\right)\right] - \mathbf{E}\left[F\left(\sum_{j=0}^{2^m-1} S_{m,j} h_j^m\right)\right], \quad (5)$$

where  $U_{m,j}^n$  and  $S_{m,j}$  are IID. Using the equivalent characterizations of the Wasserstein-1 distance, we know that there exists a coupling (not necessarily unique) between the random variables  $U_{m,j}^n$  and  $S_{m,j}$  that realizes the distance  $\rho_{\mathbb{R}^d}^1(U_{m,j}^n, S_{m,j})$  for any  $j \in \{0, \dots, 2^m - 1\}$ . Let  $(\hat{U}_{m,j}^n, \hat{S}_{m,j})$  be independent random variables such that for any  $j \in \{0, \dots, 2^m - 1\}$

$$\rho_W^1(U_{m,j}^n, S_{m,j}) = \mathbf{E}\left[\left|\hat{U}_{m,j}^n - \hat{S}_{m,j}\right|\right].$$

The difference (5) now gives:

$$\begin{aligned} \rho_{W_{\eta,p}}^1\left(\pi_m(X_n), \pi_m(S)\right) &= \sup_{F \in \text{Lip}_1(W_{\eta,p})} \mathbf{E}\left[F\left(\pi_m(X_n)\right)\right] - \mathbf{E}\left[F\left(\pi_m(S)\right)\right] \\ &= \sup_{F \in \text{Lip}_1(W_{\eta,p})} \mathbf{E}\left[F\left(\sum_{j=1}^{2^m} \hat{U}_{m,j}^n h_j^m\right)\right] - \mathbf{E}\left[F\left(\sum_{j=1}^{2^m} \hat{S}_{m,j} h_j^m\right)\right] \\ &\leq \sum_{j=0}^{2^m-1} \rho_{\mathbb{R}}^1(U_{m,j}^n, S_{m,j}) \|h_j^m\|_{W_{\eta,p}} \\ &\leq c 2^m 2^{-m(1/2-\eta)} \rho_{\mathbb{R}}^1(U_{m,1}^n, S_{m,1}) \\ &\leq c 2^{m(1/2+\eta)} \rho_{\mathbb{R}}^1(U_{m,1}^n, S_{m,1}). \end{aligned}$$

The last part  $\rho_{\mathbb{R}^d}^1(U_{m,1}^n, S_{m,1})$  is now bounded using the estimate of [7, 21]:

$$\rho_{\mathbb{R}}^1(U_{m,1}^n, S_{m,1}) = \rho_{\mathbb{R}^d}^1\left(\frac{1}{2^{\frac{n-m}{\alpha}}} \sum_{i=j2^{n-m}}^{(j+1)2^{n-m}-1} Y_i, \frac{1}{2^{m/\alpha}} Y_i, j = 1, \dots, 2^m, \mathbf{S}\right) \leq C_{\alpha,m} 2^{n(1-\frac{2}{\alpha})}.$$

We explain how to use [7, 21] in the next paragraph.

**Remark 2.** *It should be noted that the characterization of the Wasserstein-1 distance as a coupling is new in this setting. It simplifies greatly the proof of the analog result in [17, Theorem 4.7].*

### 2.3 Control in Finite Dimension

The purpose of this section is to explain how to derive the functional convergence from existing results in the literature. Many authors have investigated the stable central limit using Stein's method, starting from Barbour [15], and more recently, a series of paper [7], [21], [22]. We will rely on their result in the multidimensional setting.



The rate of convergence for heavy-tailed random walk converging to stable distributions is obtained in [7]. In a subsequent work [21], this result is extended from one dimension to the multivariate case. We copy their result here keeping the notations used in [21].

**Theorem 3.** *Set*

$$S_n = (\zeta_{n,1} - \mathbf{E}[\zeta_{n,1}]) + (\zeta_{n,2} - \mathbf{E}[\zeta_{n,2}]) + \cdots + (\zeta_{n,n} - \mathbf{E}[\zeta_{n,n}]);$$

Let  $\mu$  be an  $\alpha$ -stable distribution  $\alpha \in (1, 2)$  with Lévy measure

$$\nu(d\theta) = a g(\theta) d\theta + b \sum_{i=1}^d (\sigma_i \delta_{e_i} + \sigma'_i \delta_{-e_i}) + c \nu_\gamma(d\theta).$$

Then we have:

$$\rho_W^1(\mathcal{L}(S_n), \mu) \leq C_{\alpha,d} \left( n^{-\frac{1}{\alpha}} \sum_{i=1}^n \mathbf{E} \left[ |\zeta_{n,i}|^{2-\alpha} \right] + n^{-\frac{1}{\alpha}} \sum_{i=1}^n \mathbf{E} [|\zeta_{n,i}|^2] + \mathbf{E} \left[ \sum_{i=1}^n |\mathcal{R}_{n,i}| \right] \right),$$

where the remainder  $\mathcal{R}_{n,i}$  is related to the distribution function of the  $\zeta_{n,i}$  and the Lévy measure of the stable distribution.

They proceed to give an estimation for the remainder in the case where  $\zeta_{n,i}$  are symmetrized Pareto. The following upper bound can be found in [21] section 5:

**Corollary 4.** *Let  $\tau_i = \varepsilon_i \xi_i$ , where  $\varepsilon_i$  is a random unit vector such that  $\mathbf{P}(\varepsilon_i \in A) = \nu(A)$ , and  $\xi$  is a Pareto random variable. Define*

$$S_n = \left( \frac{\alpha}{\rho_\alpha^1} \right)^{1/\alpha} \frac{1}{n^{1/\alpha}} \left( (\tau_1 - \mathbf{E}[\tau_1]) + \cdots + (\tau_n - \mathbf{E}[\tau_n]) \right).$$

Then,

$$\rho_W^1(\mathcal{L}(S_n), \mu) \leq C_{\alpha,d} n^{\frac{\alpha-2}{\alpha}}.$$

Recall we need a convergence rate for each  $j = 1, \dots, 2^m$ :

$$\frac{1}{2^{\frac{n-m}{\alpha}}} \sum_{i=j2^{n-m}}^{(j+1)2^{n-m}} \frac{1}{2^{m/\alpha}} Y_i \Rightarrow S\left(\frac{j+1}{2^m}\right) - S\left(\frac{j}{2^m}\right).$$

Since each of the  $2^m$  component are independent, the vector:

$$\left( S\left(\frac{1}{2^m}\right), S\left(\frac{2}{2^m}\right) - S\left(\frac{1}{2^m}\right), \dots, S(1) - S\left(\frac{2^m-1}{2^m}\right) \right) \stackrel{(d)}{=} \mathbf{S},$$

has a stable distribution, whose Lévy measure is

$$\sum_{j=1}^{2^m} \frac{1}{2} (\delta_{e_j} + \delta_{-e_j}),$$

Now, to use Theorem 3, we need to compute each term in the right hand side, and  $\zeta_{n,i}$  in terms of our to our  $Y_i$ . Fortunately, in [21], Paragraph 5.2, Example 2 (refer to the arXiv version 1), is a section devoted to the calculation we need. Using the same notations as in [21], for a distribution function of the form

$$F_\xi(drd\theta) = \frac{A}{r^{\alpha+1}} drv(d\theta) + \frac{B(r\theta)}{r^{\beta+1}} drd\theta,$$

the convergence rate towards  $\pi$ , the multidimensional stable distribution, is:

$$\rho_W^1(\mathcal{L}(S_n), \pi) \leq C_{\alpha,d,A,B} \left( n^{\frac{\alpha-2}{\alpha}} + n^{\frac{\alpha-\beta}{\alpha}} \right),$$

where  $d$  is the dimension, that is in our case,  $2^m$ . Recall that in our setting, we consider  $Y_i$  in the normal domain of attraction (see Equation (2)), we now actually have to prescribe a decay rate for  $\varepsilon$ . From now on, assume that

$$\varepsilon(x) \leq \frac{K_\varepsilon}{|x|^\gamma}.$$

Note that this is the decay rate prescribed in [7], this relate to the Example discussed in [21] by taking  $\gamma = \beta - \alpha \geq 0$ . Hence, in our case, the convergence rate is

$$\rho_W^1(\mathcal{L}(S_n), \pi) \leq C_{\alpha,d,A,B} \left( 2^{\frac{\alpha-2}{\alpha}n} + 2^{\frac{-\gamma}{\alpha}n} \right),$$

where  $\gamma$  relate to the choice of  $\varepsilon$ . At this point, we need to be very careful with the constant  $C_{\alpha,d,A,B}$ , as a compromise between  $n$  and  $m$  is to be chosen. In [21], this constant comes from the density estimation for the stable distribution. Since we deal with symmetric stable distribution (that are unimodal with a mode at zero), it is enough to upper bound the density at  $x = 0$  to get a good estimate on that constant. Let us denote  $\mathbf{q}(\mathbf{x})d\mathbf{x}$  the density of  $\mathbf{S}$ , we have:

$$\begin{aligned} \mathbf{q}(0) &= \frac{1}{(2\pi)^{2^m}} \int_{\mathbb{R}^{2^m}} \mathbf{E} \left[ e^{i\langle \xi, \mathbf{S} \rangle} \right] d\xi \\ &= \frac{1}{(2\pi)^{2^m}} \int_{\mathbb{R}^{2^m}} \exp \left( \left( e^{i\langle \xi, \theta \rangle} - 1 - i\langle \xi, \theta \rangle \mathbf{1}_{\{|\theta| \leq 1\}} \right) \right. \\ &\quad \left. \times \left( a g(\theta) d\theta + b \sum_{i=1}^d (\sigma_i \delta_{e_i} + \sigma'_i \delta_{-e_i}) + c v_\gamma(d\theta) \right) \right) d\xi. \end{aligned}$$

In general, one would proceed by change of variable looking for Gamma functions to compute this integral. We point out that if the Lévy measure had had a density with respect to Lebesgue measure, the volume of the sphere  $S^{2^m-1}$  would have come into consideration. However, because stable distribution  $\mathbf{S}$  we consider has independent increments, this constant becomes linear in the dimension  $2^m$ :

We therefore obtain the rate in Wasserstein distance:

$$\rho_{\mathbb{R}^d}^1 \left( \frac{1}{2^{\frac{n-m}{\alpha}}} \sum_{i=j2^{n-m}}^{(j+1)2^{n-m}} \frac{1}{2^{m/\alpha}} Y_i, j = 1, \dots, 2^m, \mathbf{S} \right) \leq C_{\alpha, A, \varepsilon} 2^m \left( 2^{n(1-\frac{2}{\alpha})} + 2^{-n\frac{\gamma}{\alpha}} \right).$$

### 3 Approximation and Final Estimate

In what follows, the constants may change from line to line.

#### 3.1 Approximation Lemma

Let  $\tau_m = (t_k^m)$  where  $t_k^m = k2^{-m}$ ,  $k = 0, \dots, 2^m$  be the dyadic partition of  $[0, 1]$ . Let  $F$  be a stochastic process and  $\pi^m(F)$  be its linear interpolation along  $\tau^m$ , that is

$$\pi_m(F)_t = F_{t_k^m} + [t - t_k^m]2^m [F_{t_{k+1}^m} - F_{t_k^m}], \quad t \in [t_k^m, t_{k+1}^m], \quad k = 0, \dots, 2^m - 1.$$

**Lemma 5.** *Let  $1 < p < \alpha < 2$ , let  $F$  be a stochastic process such that there exists  $C > 0$  such that*

$$\mathbf{E}[|F_t - F_s|^p] \leq C |t - s|^{p/\alpha}, \quad \forall s, t \in [0, 1]^2. \quad (6)$$

*Then for  $\eta < 1/\alpha$ , there exists a constant  $C > 0$  (depending on  $\alpha, p, \eta$ ) such that, for all  $m \geq 1$ :*

$$I^m = \int \int_{[0,1]^2} \frac{\mathbf{E}[|\pi_m(F)_t - \pi_m(F)_s - F_t + F_s|^p]}{|t - s|^{1+p\eta}} ds dt \leq C 2^{-m(1/\alpha-\eta)p}.$$

$$J^m = \int_0^1 \mathbf{E}[|\pi_m(F)_t - F_t|^p] dt \leq C 2^{-m\frac{p}{\alpha}}.$$

**Remark 3.** *The stable process satisfies the assumption of Lemma 6, since for any  $0 \leq s < t \leq 1$ , we have*

$$S(t) - S(s) \stackrel{d}{=} (t - s)^{1/\alpha} S(1)$$

*and  $S(1) \in L^p$  for any  $p < \alpha$ .*

**Remark 4.** *Since  $p/\alpha < 1$ , (6) does not entail that  $F$  has continuous sample-paths. Moreover, note that we have  $I^m + J^m = \mathbf{E}[\|F - \pi_m(F)\|_{W_{\eta,p}}^p]$ . Hence, this Lemma gives a control of the norm, in  $W_{\eta,p}$ , of the difference between a process and its affine interpolation provided that we have a bound on the moments of its increments.*

*Proof.* Without loss of generality, let  $s < t$ . First, let us assume that  $s$  and  $t$  are elements of  $\tau_m$ , that is  $s = k2^{-m}$ ,  $t = l2^{-m}$ . Then  $F_t - F_s = \pi_m(F)_t - \pi_m(F)_s$ , in other words, the projection agrees with the process on the points on the partition.

Let  $T_k^m = [t_k^m, t_{k+1}^m]$ , for  $k = 0, \dots, 2^m - 1$ . Let  $i, j \in \{0, \dots, 2^m\}$   $i \leq j$  then for  $s \in T_j^m$  and  $t \in T_i^m$  we have

$$\begin{aligned} & \pi_m(F)_t - \pi_m(F)_s - F_t + F_s \\ = & \begin{cases} (t - s)2^m [F_{t_{i+1}^m} - F_{t_i^m}] - [F_t - F_s], & \text{if } i = j \\ [t - t_j^m]2^m [F_{t_{j+1}^m} - F_{t_j^m}] - [t_{i+1}^m - s]2^m [F_{t_{i+1}^m} - F_{t_i^m}] - [F_t - F_{t_i^m}] + [F_{t_{i+1}^m} - F_s] & \text{otherwise.} \end{cases} \end{aligned}$$

Then,

$$|\pi_m(F)_t - \pi_m(F)_s - F_t + F_s|^p \leq 4^{p-1} \times \begin{cases} |t-s|^p 2^{mp} |F_{t_{i+1}^m} - F_{t_i^m}|^p + |F_t - F_s|^p, & \text{if } i = j \\ [t-t_j^m]^p 2^{mp} |F_{t_{j+1}^m} - F_{t_j^m}|^p + |t_{i+1}^m - s|^p 2^{mp} |F_{t_{i+1}^m} - F_{t_i^m}|^p + |F_t - F_{t_i^m}|^p + |F_{t_{i+1}^m} - F_s|^p \end{cases} \text{ otw.}$$

Using Fubini to exchange the integration order, we have:

$$I^m = 2 \int_0^1 \int_0^t \frac{\mathbf{E}[|\pi_m(F)_t - \pi_m(F)_s - F_t + F_s|^p]}{|t-s|^{1+p\eta}} ds dt = I_1^m + I_2^m$$

where

$$I_1^m = \sum_{i=0}^{2^m-1} \int_{T_i^m} \int_{T_i^m, s < t} \frac{\mathbf{E}[|\pi_m(F)_t - \pi_m(F)_s - F_t + F_s|^p]}{|t-s|^{1+p\eta}} ds dt, \\ I_2^m = 2 \sum_{j=1}^{2^m-1} \sum_{i=0}^{j-1} \int_{T_j^m} \int_{T_i^m} \frac{\mathbf{E}[|\pi_m(F)_t - \pi_m(F)_s - F_t + F_s|^p]}{|t-s|^{1+p\eta}} ds dt.$$

In the integral  $I_1^m$ ,  $t$  and  $s$  are in the same interval  $T_k^m$ , thus are at a distance at most  $2^{-m}$ , where as in  $I_2^m$ ,  $s$  and  $t$  are not in the same interval  $T_k^m$ . Note crucially that for  $I_2^m$ , the denominator in  $|s-t|$  is not singular.

For  $I_1^m$ , since  $t$  and  $s$  are in the same  $T_i^m$ , we can split the expectation as:

$$\mathbf{E}\left[|\pi_m(F)_t - \pi_m(F)_s - F_t + F_s|^p\right] \leq 4^{p-1} \mathbf{E}\left[|t-s|^p 2^{mp} |F_{t_{i+1}^m} - F_{t_i^m}|^p + |F_t - F_s|^p\right].$$

Using the fact that  $\mathbf{E}[|F_t - F_s|^p] \leq |t-s|^{p/\alpha}$ , we can bound  $I_1^m \leq 4^{p-1} [I_{1,1}^m + I_{1,2}^m]$ , where:

$$I_{1,1}^m = \sum_{i=0}^{2^m-1} \int \int_{T_i^m \times T_i^m, s < t} \frac{|t-s|^p 2^{-m(\frac{p}{\alpha}-p)}}{|t-s|^{1+p\eta}} ds dt = 2 \sum_{i=0}^{2^m-1} 2^{-m(\frac{p}{\alpha}-p)} \int_{T_i^m} \int_{t_i^m}^t (t-s)^{p-p\eta-1} ds dt \\ I_{1,2}^m = \sum_{i=0}^{2^m-1} \int \int_{T_i^m \times T_i^m, s < t} \frac{|t-s|^{\frac{p}{\alpha}}}{|t-s|^{1+p\eta}} ds dt = 2 \sum_{i=0}^{2^m-1} \int_{T_i^m} \int_{t_i^m}^t |t-s|^{\frac{p}{\alpha}-p\eta-1} ds dt.$$

Integrating in  $s$ , we get:

$$I_{1,1}^m = 2 \sum_{i=0}^{2^m-1} 2^{-m(\frac{p}{\alpha}-p)} \frac{1}{p(1-\eta)} \int_{T_i^m} (t-t_i^m)^{p-p\eta} dt, \\ I_{1,2}^m = 2 \sum_{i=0}^{2^m-1} \frac{1}{p(1/\alpha-\eta)} \int_{T_i^m} |t-t_i^m|^{\frac{p}{\alpha}-p\eta} dt.$$

Recall that  $T_i^m = [i2^{-m}, (i+1)2^{-m}]$ , we get:

$$I_{1,1}^m = 2 \sum_{i=0}^{2^m-1} 2^{-m(\frac{p}{\alpha}-p)} \frac{1}{p(1-\eta)(p-p\eta+1)} 2^{-m(p-p\eta+1)},$$

$$I_{1,2}^m = 2 \sum_{i=0}^{2^m-1} \frac{1}{p(1/\alpha-\eta)(1-p\eta+p/\alpha)} 2^{-m(\frac{p}{\alpha}-p\eta+1)}$$

and

$$I_{1,1}^m = 2^{-m[\frac{p}{\alpha}-p]} \frac{2}{p(1-\eta)(p-p\eta+1)} 2^{-m(p-p\eta)},$$

$$I_{1,2}^m = \frac{2}{p[-\eta+1/\alpha](1-p\eta+p/\alpha)} 2^{-m[\frac{p}{\alpha}-p\eta]}.$$

Note that the crucial part is that we assumed  $\alpha > 1$  and  $\eta < 1/\alpha$ , thus, the exponent is indeed negative. Now, we turn to the study of  $I_2^m$ . In this case,  $s$  and  $t$  are not in the same interval  $T_k^m$ , we can split the expectation four ways:

$$\mathbf{E} \left[ \left| \pi_m(F)_t - \pi_m(F)_s - F_t + F_s \right|^p \right] \leq 4^{p-1} \left[ |t - t_j^m|^p 2^{mp} \mathbf{E} \left[ |F_{t_{j+1}^m} - F_{t_j^m}|^p \right] + \mathbf{E} \left[ |F_t - F_{t_i^m}|^p \right] \right. \\ \left. + \mathbf{E} \left[ |F_{t_{i+1}^m} - F_s|^p \right] + |t_{i+1}^m - s|^p 2^{mp} \mathbf{E} \left[ |F_{t_{i+1}^m} - F_{t_i^m}|^p \right] \right].$$

This prompts us to split  $I_2^m$  four ways according to the above decomposition.

$$I_2^m \leq 2 \sum_{l=1}^4 I_{2,l}^m$$

For the first contribution above, and since  $p > 1$ , we can use the Lemma 6. We get

$$I_{2,1}^m = 2 \sum_{j=1}^{2^m-1} \sum_{i=0}^{j-1} \int \int_{T_j^m \times T_i^m} \frac{|t - t_j^m|^p 2^{mp} \mathbf{E} \left[ |F_{t_{j+1}^m} - F_{t_j^m}|^p \right]}{|t - s|^{1+p\eta}} ds dt \\ \leq 2 \sum_{j=1}^{2^m-1} \sum_{i=0}^{j-1} \int \int_{T_j^m \times T_i^m} 2^{-m[\frac{p}{\alpha}-p]} \frac{|t - t_j^m|^p}{|t - s|^{1+p\eta}} ds dt.$$

We can now simply compute the remaining integrals:

$$I_{2,1}^m \leq 2 \sum_{j=1}^{2^m-1} \int \int_{T_j^m \times [0, t_j^m]} 2^{-m[\frac{p}{\alpha}-p]} \frac{|t - t_j^m|^p}{|t - s|^{1+p\eta}} ds dt \\ \leq 2 \frac{2^{-m[\frac{p}{\alpha}-p]}}{p\eta} \sum_{j=1}^{2^m-1} \int_{T_j^m} |t - t_j^m|^{p-p\eta} dt \\ \leq 2 \frac{2^{-m[\frac{p}{\alpha}-p+p-p\eta+1-1]}}{p\eta(p-p\eta+1)} \leq \frac{2^{1-m[\frac{p}{\alpha}-p\eta]}}{p\eta(p-p\eta+1)}.$$

Again, we note that since  $p \geq 1$  and  $1/\alpha > \eta$ , this exponent is indeed negative. For the second contribution, we proceed similarly, using (6) to estimate:

$$\begin{aligned} I_{2,2}^m &= 2 \sum_{j=1}^{2^m-1} \sum_{i=0}^{j-1} \int \int_{T_j^m \times T_i^m} \frac{\mathbf{E} \left[ |F_t - F_{t_i^m}|^p \right]}{|t-s|^{1+p\eta}} \, ds \, dt \\ &\leq 2 \sum_{j=1}^{2^m-1} \sum_{i=0}^{j-1} \int \int_{T_j^m \times T_i^m} \frac{|t-t_j^m|^{\frac{p}{\alpha}}}{|t-s|^{1+p\eta}} \, ds \, dt. \end{aligned}$$

Here the sum of integrals in  $d$  on  $T_i^m$  yields an integral on  $s$  between 0 and  $t_j^m$ . We can therefore estimate

$$I_{2,2}^m \leq \frac{2}{p\eta} \sum_{j=1}^{2^m-1} \int_{T_j^m} |t-t_j^m|^{\frac{p}{\alpha}-p\eta} \, dt \leq \frac{2^{1-m(\frac{p}{\alpha}-\eta)}}{p\eta(1-p\eta+p/\alpha)}.$$

Now, for the terms involving  $s \in T_i^m$ , we use (6) to estimate:

$$\begin{aligned} I_{2,3}^m &= 2 \sum_{j=1}^{2^m-1} \sum_{i=0}^{j-1} \int \int_{T_j^m \times T_i^m} \frac{\mathbf{E} \left[ |F_{t_{i+1}^m} - F_s|^p \right]}{|t-s|^{1+p\eta}} \, ds \, dt \\ &\leq 2 \sum_{j=1}^{2^m-1} \sum_{i=0}^{j-1} \int \int_{T_j^m \times T_i^m} \frac{|s-t_{i+1}^m|^{\frac{p}{\alpha}}}{|t-s|^{1+p\eta}} \, ds \, dt. \end{aligned}$$

Which gives after integration:

$$\begin{aligned} I_{2,3}^m &\leq 2 \sum_{i=0}^{2^m-2} \sum_{j=i+1}^{2^m-1} \int \int_{T_j^m \times T_i^m} \frac{|s-t_{i+1}^m|^{\frac{p}{\alpha}}}{|t-s|^{1+p\eta}} \, ds \, dt \\ &= 2 \sum_{i=0}^{2^m-2} \int_{T_i^m} \int_{t_{i+1}^m}^1 \frac{|s-t_{i+1}^m|^{\frac{p}{\alpha}}}{|t-s|^{1+p\eta}} \, ds \, dt \\ &= \frac{2}{p\eta} \sum_{i=0}^{2^m-2} \int_{T_i^m} |s-t_{i+1}^m|^{\frac{p}{\alpha}-p\eta} \, ds \leq \frac{2^{1-m(\frac{p}{\alpha}-p\eta)}}{p\eta(1-p\eta+p/\alpha)}. \end{aligned}$$

The last term handled similarly:

$$\begin{aligned} I_{2,4}^m &= 2 \sum_{j=1}^{2^m-1} \sum_{i=0}^{j-1} \int \int_{T_j^m \times T_i^m} \frac{|t_{i+1}^m - s|^{p/2^m} \mathbf{E} \left[ |F_{t_{i+1}^m} - F_{t_i^m}|^p \right]}{|t-s|^{1+p\eta}} \, ds \, dt \\ &= 2^{1-m(\frac{p}{\alpha}-p)} \sum_{j=1}^{2^m-1} \sum_{i=0}^{j-1} \int \int_{T_j^m \times T_i^m} \frac{|s-t_{i+1}^m|^p}{|t-s|^{1+p\eta}} \, ds \, dt \\ &= 2^{1-m(\frac{p}{\alpha}-p)} \sum_{i=0}^{2^m-1} \sum_{j=i+1}^{2^m-1} \int \int_{T_j^m \times T_i^m} \frac{|s-t_{i+1}^m|^p}{|t-s|^{1+p\eta}} \, ds \, dt \end{aligned}$$

$$\begin{aligned}
&= 2^{1-m\lfloor \frac{p}{\alpha} - p \rfloor} \sum_{i=0}^{2^m-1} \int \int_{[t_{i+1}^m, 1] \times T_i^m} \frac{|s - t_{i+1}^m|^p}{|t - s|^{1+p\eta}} ds dt \\
&\leq 2^{-m\lfloor \frac{p}{\alpha} - p \rfloor} \sum_{i=0}^{2^m-1} \int_{[t_{i+1}^m, 1]} \frac{|s - t_{i+1}^m|^{p-p\eta}}{p\eta} ds \\
&= \frac{2^{1-m\lfloor \frac{p}{\alpha} - p\eta \rfloor}}{p\eta(p - p\eta + 1)}.
\end{aligned}$$

Then, there exists a constant  $C$  depending on  $p, \eta, \alpha$ , such that

$$|I_{k,l}|^m \leq C 2^{-m\lfloor \frac{p}{\alpha} - 1 \rfloor}, \text{ for } (k, l) \in \{(1, 2), (1, 1)\} \cup \{(2, i), i = 1, \dots, 4\}, \forall m$$

The case of  $J_n$  is simpler since

$$\mathbf{E}[\sup_t |\pi_m(F)_t - F_t|^p] \leq C 2^{-m\frac{p}{\alpha}}.$$

The proof is thus complete.  $\square$

**Remark 5.** Since  $I_m + J_m = \mathbf{E}[\|F - \pi_m(F)\|_{W_{\eta,p}}^p]$ , we get the estimate:

$$\mathbf{E}[\|F - \pi_m(F)\|_{W_{\eta,p}}^p] \leq C \left( 2^{-m\frac{p}{\alpha}} + 2^{-m(1/\alpha - \eta)p} \right) \leq C 2^{-m(\frac{1}{\alpha} - \eta)p}.$$

### 3.2 The Interpolated Process Fits the setting of Lemma 5

In this section, we show how the interpolated random walk satisfies the assumptions of Lemma 5.

**Lemma 6.** Assume that for some  $p < \alpha$ ,

$$\sup_n \mathbf{E} \left[ \left| \frac{Y_1 + \dots + Y_n}{n^{1/\alpha}} \right|^p \right] < +\infty. \quad (7)$$

Then, for some  $C > 0$ , for all  $n \geq 1$ , we have:

$$\mathbf{E} \left[ |X_n(t) - X_n(s)|^p \right] \leq C |t - s|^{p/\alpha}.$$

*Proof.* We start with the case where  $t$  and  $s$  are on the grid:  $t = k2^{-n}$  and  $s = j2^{-n}$ . Without loss of generality, assume  $t > s$ . In this case, the increment of the interpolated process writes:

$$X_n(t) - X_n(s) = \frac{1}{2^{n/\alpha}} \sum_{i=j}^{k-1} Y_i.$$

From (7), we deduce that for all  $n \in \mathbb{N}$ , there exists some constant  $C > 0$  such that :

$$\mathbf{E} \left[ |Y_1 + \dots + Y_n|^p \right] \leq C n^{p/\alpha}.$$

Hence, we directly get:

$$\begin{aligned}\mathbf{E}[|X_n(t) - X_n(s)|^p] &= \mathbf{E}\left[\left|\frac{1}{2^{n/\alpha}} \sum_{i=j}^{k-1} Y_i\right|^p\right] \\ &\leq C \frac{1}{2^{np/\alpha}} |k-j|^{p/\alpha} \\ &= C \left|\frac{k}{2^n} - \frac{j}{2^n}\right|^{p/\alpha} = C|t-s|^{p/\alpha},\end{aligned}$$

thus, the moment condition in lemma 5 is satisfied for points on the grid.

Now, let us discuss the case where  $t$  and  $s$  are not on the grid, but in the same sub-interval  $[j2^{-n}, (j+1)2^{-n}]$ . In that case, the increment of the interpolated process reduces to

$$X_n(t) - X_n(s) = \frac{1}{2^{n/\alpha}} Y_j \sqrt{2^n}(t-s).$$

In that case, we get:

$$\mathbf{E}[|X_n(t) - X_n(s)|^p] = \frac{\sqrt{2^{pn}}(t-s)^p}{2^{np/\alpha}} \mathbf{E}[|Y_j|^p].$$

Since  $\alpha \leq 2$ , we have  $\frac{p}{2} \leq \frac{p}{\alpha}$ , thus  $(2^n)^{p/2} \leq (2^n)^{p/\alpha}$ . Hence, we get:

$$\mathbf{E}[|X_n(t) - X_n(s)|^p] \leq C(t-s)^p \leq C(t-s)^{p/\alpha},$$

the last inequality being true since  $t$  and  $s$  are in the same sub-interval, hence  $t-s \leq 1$ .

It remains us to discuss the case where  $t$  or  $s$  are not on the grid, and in separate sub-intervals. In that case, we introduce  $s^+$  and  $t^-$ , on the grid  $\{k2^{-n}, k=1, \dots, 2^n\}$  such that  $s \leq s^+ \leq t^- \leq t$ . We can always split the difference into:

$$X_n(t) - X_n(s) = (X_n(t) - X_n(t^-)) + (X_n(t^-) - X_n(s^+)) + (X_n(s^+) - X_n(s)).$$

Now,  $t$  and  $t^-$  are in the same sub-interval, and so are  $s$  and  $s^+$ . Moreover,  $s^+$  and  $t^-$  are on the grid, hence, we can directly use the controls explained above to get:

$$\mathbf{E}[|X_n(t) - X_n(s)|^p] \leq C\left((t-t^-)^{p/\alpha} + (t^- - s^+)^{p/\alpha} + (s^+ - s)^{p/\alpha}\right) \leq C_p(t-s)^{p/\alpha},$$

where for the last inequality, we used the fact that  $a^p + b^p \leq C_p(a+b)^p$ , that holds with  $C_p = 2^{1-p}$  if  $p \leq 1$  and  $C_p = 1$  if  $p \geq 1$ . Hence, Lemma 5 is applicable and the norm in  $W_{\eta,p}$  is controlled. □



### 3.3 Derivation of the Final Rate of convergence

In this section, we derive the final rate of convergence of the interpolated random walk  $(X_n(t))_{t \in [0,1], n \geq 0}$  towards the stable process  $S$  in  $W_{\eta,p}$ . Recall:

$$\rho_{W_{\eta,p}}^1(X_n, S) = \sup_{F \in \text{Lip}_1} \left( \mathbf{E}[F(X_n)] - \mathbf{E}[F(S)] \right).$$

Let  $F$  be a Lipschitz function from  $W_{\eta,p}$  to  $\mathbb{R}$ . We have:

$$\begin{aligned} \mathbf{E}[F(X_n) - F(S)] &= \left( \mathbf{E}[F(X_n)] - \mathbf{E}[F(\pi_m(X_n))] \right) + \left( \mathbf{E}[F(\pi_m(X_n))] - \mathbf{E}[F(\pi_m(S))] \right) \\ &\quad + \left( \mathbf{E}[F(\pi_m(S))] - \mathbf{E}[F(S)] \right) \\ &\leq \mathbf{E} \left[ \|X_n - \pi_m(X_n)\|_{W_{\eta,p}} \right] + \rho_{W_{\eta,p}}^1(\pi_m(X_n), \pi_m(S)) + \mathbf{E} \left[ \|S - \pi_m(S)\|_{W_{\eta,p}} \right]. \end{aligned}$$

Thus,

$$\begin{aligned} \rho_{W_{\eta,p}}^1(X_n, S) &= \sup_{F \in \text{Lip}(W_{\eta,p})} \mathbf{E}[F(X_n)] - \mathbf{E}[F(S)] \\ &\leq \mathbf{E} \left[ \|X_n - \pi_m(X_n)\|_{W_{\eta,p}} \right] + \rho_{W_{\eta,p}}^1(\pi_m(X_n), \pi_m(S)) + \mathbf{E} \left[ \|S - \pi_m(S)\|_{W_{\eta,p}} \right]. \end{aligned}$$

We plug-in each the controls we obtained for each term, to get:

$$\rho_{W_{\eta,p}}^1(X_n, S) \leq C \left( 2^{-m(\frac{1}{\alpha}-\eta)p} + 2^m 2^{n\frac{\alpha-2}{\alpha}} + 2^m 2^{-n\frac{\gamma}{\alpha}} + 2^{-m(\frac{1}{\alpha}-\eta)p} \right)$$

Crucially, we can already notice we can choose  $m$  and  $n$  such that every exponent is actually negative. Up to a modification of the constant, we can group the first and fourth term together, to get an estimate:

$$\rho_{W_{\eta,p}}^1(X_n, S) \leq C \left( 2^{-m(\frac{1}{\alpha}-\eta)p} + 2^m 2^{-n(\frac{2}{\alpha}-1)} + 2^m 2^{-n\frac{\gamma}{\alpha}} \right). \quad (8)$$

We see that we need to choose  $m$  in terms of  $n$  to ensure that:

- $2^{-m(\frac{1}{\alpha}-\eta)p}$  tends to 0 as fast as possible, that is  $m$  as large as possible,
- $2^m 2^{-n(\frac{2}{\alpha}-1)}$  tends to 0 as fast as possible, that is  $m \leq n(\frac{2}{\alpha}-1)$ .
- $2^m 2^{-n\frac{\gamma}{\alpha}}$  tends to 0 as fast as possible, that is  $m \leq n\frac{\gamma}{\alpha}$ .

One can solve this optimization problem to find the optimal  $m$  in terms of  $n$ , but the calculations are quite heavy. Hence, we provide a more tractable calculation giving a sub-optimal rate. First, note that

$$\rho_{W_{\eta,p}}^1(X_n, S) \leq C \left( 2^{-m(\frac{1}{\alpha}-\eta)p} + 2^{m-n \times \min(\frac{2}{\alpha}-1, \frac{\gamma}{\alpha})} \right)$$

We set  $m = \kappa n$ , and find  $\kappa$  such that the exponent match:

$$-\kappa \left( \frac{1}{\alpha} - \eta \right) p = \kappa - \min \left( \frac{2}{\alpha} - 1, \frac{\gamma}{\alpha} \right) \Rightarrow \kappa = \frac{\min \left( \frac{2}{\alpha} - 1, \frac{\gamma}{\alpha} \right)}{1 + \left( \frac{1}{\alpha} - \eta \right) p}.$$

Our final bound comes out to be:

$$\rho_W^1(X_n, S) \leq C 2^{-nv},$$

with

$$v = \left( \frac{1}{\alpha} - \eta \right) p \frac{\min \left( \frac{2}{\alpha} - 1, \frac{\gamma}{\alpha} \right)}{1 + \left( \frac{1}{\alpha} - \eta \right) p}.$$

## 4 Proof of the Moment Condition

In this section, we prove that the moment condition 7 actually holds for all  $p < \alpha$ . Recall that a random variable  $Y$  is in the normal domain of attraction of a stable distribution if its distribution function  $F_Y$  is of the form:

$$1 - F_Y(t) = \mathbf{P}(Y \geq t) = \frac{A + \varepsilon(t)}{t^\alpha} \text{ and } F_Y(-t) = \frac{A + \varepsilon(-t)}{(-t)^\alpha},$$

whenever  $|t| \geq 1$ . We split this paragraph into two parts in order to isolate the main ideas. We first deal with the symmetrized Pareto case in subsection 4.1, then deal with the full case in subsection 4.2. In both case, the proof uses a technique very reminiscent of Stein's method. The key is to perform an integration by parts with respect to the Pareto distribution.

### 4.1 The case of Symmetrized Pareto

**Theorem 7.** *Let  $Y_1, \dots, Y_n$  independent and identically distributed with probability density function:*

$$\mathbb{P}(Y \in dy) = \frac{\alpha}{2} \frac{dy}{|y|^{\alpha+1}} \mathbf{1}_{\{|y|>1\}}.$$

*Then, it holds that for all  $p < \alpha$ :*

$$\sup_n \mathbb{E} \left[ \left| \frac{1}{n^{1/\alpha}} \sum_{i=1}^n Y_i \right|^p \right] < +\infty.$$

This result is of independent interest in the sense that, to the best of our knowledge, this estimate is not present in the literature. This result is not surprising however, the Pareto distributions being in the domain of attraction of the stable distribution, similar estimate are expected to hold.

Similarly to the non-integrable case, the idea is to relate the moment of  $\frac{1}{n^{1/\alpha}} \sum Y_i$  to the moment of a stable random variable. However, the lack of uniform Lipschitz property of

$x \mapsto |x|^p$  at 0 prevents us from using the results of [7] directly. We introduce the function  $\phi_p$ :

$$\phi_p(x) = \begin{cases} |x|^p & \text{if } |x| > 1 \\ \frac{p}{2}x^2 + (1 - \frac{p}{2}) & \text{else.} \end{cases} \quad (9)$$

This function interpolates a parabola close to the origin and the function  $x \mapsto |x|^p$ . By construction, it holds that

$$|x|^p \leq 1 + \phi_p(x), \quad (10)$$

and  $\phi_p \in \mathcal{C}_b^{2,p}$ , meaning twice differentiable with  $p$ -Hölder first derivative and bounded second derivative. We will use this function to establish the moment condition. Namely, we prove:

$$\mathbb{E} \left[ \phi_p \left( \frac{1}{n^{1/\alpha}} \sum_{j=1}^n Y_j \right) \right] < +\infty. \quad (11)$$

Estimate (11) coupled with inequality (10) on  $\phi_p$  yield the moment condition of Theorem 7. We now focus of establishing (11).

**Lemma 8.** *Let  $G \in \mathcal{C}_b^{2,p}(\mathbb{R})$ . Let  $Y$  with probability density function*

$$\mathbb{P}(Y \in dy) = \frac{\alpha}{2} \frac{dy}{|y|^{\alpha+1}} \mathbf{1}_{\{|y|>1\}}.$$

*It holds that*

$$\frac{2}{\alpha} \mathbf{E}[G'(Y)Y] = LG(0) - (G(1) + G(-1) - 2G(0)) - \int_{|y| \leq 1} (G(y) - G(0) - G'(0)y) \frac{\alpha dy}{|y|^{\alpha+1}},$$

*where  $L$  is the non local operator:*

$$L\varphi(x) = \int_{-\infty}^{+\infty} (\varphi(x+y) - \varphi(x) - \varphi'(x)y) \frac{\alpha dy}{|y|^{1+\alpha}}.$$

*Proof.* We start by expressing the left hand side using the pdf of  $Y$ :

$$\begin{aligned} \mathbf{E}[G'(Y)Y] &= \int_{|y|>1} G'(y)y \frac{\alpha}{2} \frac{dy}{|y|^{\alpha+1}} \mathbf{1}_{\{|y|>1\}} \\ &= \int_1^{+\infty} G'(y)y \frac{\alpha}{2} \frac{dy}{y^{\alpha+1}} + \int_{-\infty}^{-1} G'(y)y \frac{\alpha}{2} \frac{dy}{(-y)^{\alpha+1}} \\ &= E_+ + E_-. \end{aligned}$$

By integration by parts, we have:

$$\begin{aligned} E_+ &= \int_1^{+\infty} G'(y) \frac{\alpha}{2} \frac{dy}{y^\alpha} = \frac{\alpha}{2} \left[ \frac{G(y) - G(0)}{y^\alpha} \right]_1^{+\infty} - \frac{\alpha}{2} \int_1^{+\infty} (G(y) - G(0)) \frac{(-\alpha)}{y^{\alpha+1}} dy \\ &= \frac{\alpha}{2} \left( \alpha \int_1^{+\infty} (G(y) - G(0)) \frac{1}{y^{\alpha+1}} dy - (G(1) - G(0)) \right). \end{aligned}$$

Similarly, for the negative side, we have:

$$\begin{aligned} E_- &= - \int_{-\infty}^{-1} G'(y) \frac{\alpha}{2} \frac{dy}{(-y)^\alpha} = - \frac{\alpha}{2} \left[ \frac{G(y) - G(0)}{(-y)^\alpha} \right]_{-\infty}^{-1} + \frac{\alpha}{2} \int_{-\infty}^{-1} (G(y) - G(0)) \frac{(-1)(-\alpha)}{(-y)^{\alpha+1}} dy \\ &= \frac{\alpha}{2} \left( \alpha \int_{-\infty}^{-1} (G(y) - G(0)) \frac{1}{(-y)^{\alpha+1}} dy - (G(-1) - G(0)) \right). \end{aligned}$$

We can group the two integrals together writing:

$$\int_{-\infty}^{-1} (G(y) - G(0)) \frac{(-1)(-\alpha)}{(-y)^{\alpha+1}} dy + \int_{-\infty}^{-1} (G(y) - G(0)) \frac{1}{(-y)^{\alpha+1}} dy = \int_{|y|>1} (G(y) - G(0)) \frac{dy}{|y|^{\alpha+1}}.$$

Besides, since the distribution is symmetric, we can add  $G'(0)y$  under the integral without changing its value:

$$\alpha \int_{|y|>1} (G(y) - G(0)) \frac{dy}{|y|^{\alpha+1}} = \int_{|y|>1} (G(y) - G(0) - G'(0)y) \frac{\alpha dy}{|y|^{\alpha+1}}.$$

Adding the integral for  $|y| \leq 1$ , we get:

$$\alpha \int_{|y|>1} (G(y) - G(0)) \frac{dy}{|y|^{\alpha+1}} = LG(0) - \int_{|y|\leq 1} (G(y) - G(0) - G'(0)y) \frac{\alpha dy}{|y|^{\alpha+1}}.$$

Consequently, we finally have:

$$\mathbf{E}[G'(Y)Y] = \frac{\alpha}{2} \left( LG(0) - (G(1) + G(-1) - 2G(0)) - \int_{|y|\leq 1} (G(y) - G(0) - G'(0)y) \frac{\alpha dy}{|y|^{\alpha+1}} \right).$$

□

*Proof of Theorem 7.* Consequently, we see that we can bound  $\mathbf{E}[G'(Y)Y]$  with the second derivative of  $G$ . Let us now define some notations. Recall the function  $\phi_p$  defined in (9) above. We use bold letters to denote vectors in  $\mathbb{R}^n$ . Besides, capital letters will denote random variables.

Let  $\mathbf{Y} = (Y_1, \dots, Y_n)$  be a vector with IID entries, such that

$$\mathbb{P}(Y_i \in dy) = \frac{\alpha}{2} \frac{dy}{|y|^{\alpha+1}} \mathbf{1}_{\{|y|>1\}}.$$

Let also  $\mathbf{S} = (S_1, \dots, S_n)$  be a stable process in  $\mathbb{R}^n$  with independent coordinates. For any function  $F \in \mathcal{C}_b^{2,p}$ , it holds that

$$\mathbf{E}[F(\mathbf{Y})] - \mathbf{E}[F(\mathbf{S})] = - \int_0^{+\infty} \mathbf{E}[\mathcal{L}P_t F(\mathbf{Y})] dt, \quad (12)$$

where  $\mathcal{L}$  is the generator of the stable driven Ornstein-Uhlenbeck process:

$$d\mathbf{X}_t = -\frac{1}{\alpha}\mathbf{X}_t dt + d\mathbf{S}_t,$$

We note here that the generator  $\mathcal{L}$  has the expression:

$$\mathcal{L}\varphi(\mathbf{x}) = -\frac{1}{\alpha}\nabla\varphi(\mathbf{x})\cdot\mathbf{x} + \sum_{i=1}^n \int_{\mathbb{R}} \left( F(\mathbf{x} + u\mathbf{e}_i) - F(\mathbf{x}) - \nabla F(\mathbf{x})\cdot u \right) \frac{1}{|u|^{\alpha+1}} du,$$

where  $\mathbf{e}_i$  is the canonical basis element of  $\mathbb{R}^n$ .

We use equation 12 with

$$F(\mathbf{x}) = \phi_p \left( \frac{x_1 + \dots + x_n}{n^{1/\alpha}} \right).$$

This leads us to the identity:

$$\mathbf{E} \left[ \phi_p \left( \frac{Y_1 + \dots + Y_n}{n^{1/\alpha}} \right) \right] - \mathbf{E} \left[ \phi_p \left( \frac{S_1 + \dots + S_n}{n^{1/\alpha}} \right) \right] = - \int_0^{+\infty} \mathbf{E} [\mathcal{L}P_t F(\mathbf{Y})] dt. \quad (13)$$

Now, since  $\mathbf{S} = (S_1, \dots, S_n)$  be a stable process in  $\mathbb{R}^n$ , it has finite moments of order  $p < \alpha$ . Thus, equation (11) will hold as soon as we control the right hand side of (13). Hence, we need to control:

$$E = \sum_{i=1}^n \mathbf{E} \left[ -\frac{1}{\alpha} \partial_{y_i} G(\mathbf{Y}) Y_i + \int_{\mathbb{R}} \left( G(\mathbf{Y} + u\mathbf{e}_i) - G(\mathbf{Y}) - \nabla G(\mathbf{Y})\cdot u \right) \frac{1}{|u|^{\alpha+1}} du \right],$$

where  $G = P_t F$ . Let us denote for  $i = 1, \dots, n$

$$G_{\setminus Y_i}(y) = G(Y_1, \dots, Y_{i-1}, y, Y_{i+1}, \dots, Y_n).$$

In other words,  $G_{\setminus Y_i}$  is the (random) function  $G$  where we replace the  $i$ -th component by the variable  $y \in \mathbb{R}$ . By conditioning in the sum by  $Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n$ , it holds that

$$E = \sum_{i=1}^n \mathbf{E} \left[ -\frac{1}{\alpha} G'_{\setminus Y_i}(Y_i) Y_i + \int_{\mathbb{R}} \left( G_{\setminus Y_i}(Y_i + u) - G_{\setminus Y_i}(Y_i) - G'_{\setminus Y_i}(Y_i)\cdot u \right) \frac{1}{|u|^{\alpha+1}} du \right].$$

These notations allows us to reduce to the once dimensional case, and use Lemma 8 above. Namely, we deduce that:

$$\begin{aligned} \mathbf{E} \left[ G'_{\setminus Y_i}(Y_i) Y_i \right] &= \mathbf{E} \left[ \frac{\alpha}{2} \left( L G_{\setminus Y_i}(0) - \left( G_{\setminus Y_i}(1) + G_{\setminus Y_i}(-1) - 2G_{\setminus Y_i}(0) \right) \right. \right. \\ &\quad \left. \left. - \int_{|y|\leq 1} \left( G_{\setminus Y_i}(y) - G_{\setminus Y_i}(0) - G'_{\setminus Y_i}(0)y \right) \frac{\alpha dy}{|y|^{\alpha+1}} \right) \right]. \end{aligned}$$

Recall  $G = P_t F$ , we need to control

$$\begin{aligned}
\mathbf{E}[\mathcal{L}P_t F(\mathbf{Y})] &= \sum_{i=1}^n \mathbf{E} \left[ -\frac{1}{\alpha} G'_{\setminus Y_i}(Y_i) Y_i + LG(Y_i) \right] \\
&= \sum_{i=1}^n \mathbb{E} \left( \frac{1}{2} \left( -LG_{\setminus Y_i}(0) + \left( G_{\setminus Y_i}(1) + G_{\setminus Y_i}(-1) - 2G_{\setminus Y_i}(0) \right) \right. \right. \\
&\quad \left. \left. + \int_{|y| \leq 1} \left( G_{\setminus Y_i}(y) - G_{\setminus Y_i}(0) - G'_{\setminus Y_i}(0)y \right) \frac{\alpha \, dy}{|y|^{\alpha+1}} \right) + LG(Y_i) \right).
\end{aligned}$$

Looking at this identity, it appears that the quantity we need to focus on is  $LG_{\setminus Y_i}(Y_i) - LG_{\setminus Y_i}(0)$ , all other terms can be controlled by  $\|G''\|_\infty$  from a Taylor's expansion. We have:

$$\begin{aligned}
LG_{\setminus Y_i}(Y_i) - LG_{\setminus Y_i}(0) &= \int_{-\infty}^{+\infty} \left( G_{\setminus Y_i}(Y_i + y) - G_{\setminus Y_i}(Y_i) - G'_{\setminus Y_i}(Y_i)y \right) \frac{\alpha \, dy}{|y|^{\alpha+1}} \\
&\quad - \int_{-\infty}^{+\infty} \left( G_{\setminus Y_i}(y) - G_{\setminus Y_i}(0) - G'_{\setminus Y_i}(0)y \right) \frac{\alpha \, dy}{|y|^{\alpha+1}}.
\end{aligned}$$

We split this integral between  $\{|y| \leq 1\}$ , where we can do a Taylor's expansion, and  $\{|y| > 1\}$ :

$$\begin{aligned}
&LG_{\setminus Y_i}(Y_i) - LG_{\setminus Y_i}(0) \\
&= \int_{\{|y| \leq 1\}} \left( (G_{\setminus Y_i}(Y_i + y) - G_{\setminus Y_i}(y)) - (G_{\setminus Y_i}(Y_i) - G_{\setminus Y_i}(0)) - (G'_{\setminus Y_i}(Y_i) - G'_{\setminus Y_i}(0))y \right) \frac{\alpha \, dy}{|y|^{\alpha+1}} \\
&\quad + \int_{\{|y| > 1\}} \left( (G_{\setminus Y_i}(Y_i + y) - G_{\setminus Y_i}(y)) - (G_{\setminus Y_i}(Y_i) - G_{\setminus Y_i}(0)) - (G'_{\setminus Y_i}(Y_i) - G'_{\setminus Y_i}(0))y \right) \frac{\alpha \, dy}{|y|^{\alpha+1}} \\
&\hspace{15em} = I + II.
\end{aligned}$$

We now turn to the second integral above. First, noticing that on  $\{|y| > 1\}$ , the measure  $\frac{\alpha \, dy}{|y|^{\alpha+1}}$  is finite and symmetric, we can cancel the compensation term. Next, we have:

$$\begin{aligned}
G_{\setminus Y_i}(Y_i + y) - G_{\setminus Y_i}(y) &= \int_0^1 G'_{\setminus Y_i}(y + \theta Y_i) \cdot Y_i \, d\theta, \\
G_{\setminus Y_i}(Y_i) - G_{\setminus Y_i}(0) &= \int_0^1 G'_{\setminus Y_i}(0 + \theta Y_i) \cdot Y_i \, d\theta.
\end{aligned}$$

This gives us:

$$II = \int_{\{|y| > 1\}} \int_0^1 \left( G'_{\setminus Y_i}(y + \theta Y_i) - G'_{\setminus Y_i}(\theta Y_i) \right) Y_i \, d\theta \frac{\alpha \, dy}{|y|^{\alpha+1}}.$$

Now, we do an additional Taylor's expansion, writing that

$$G'_{\setminus Y_i}(y + \theta Y_i) - G'_{\setminus Y_i}(\theta Y_i) = \int_0^1 G''_{\setminus Y_i}(\theta Y_i + \mu y) y \, d\mu,$$

which can be bounded, since  $\mathbf{1}_{\{|y|>1\}} \frac{\alpha dy}{|y|^{\alpha+1}}$  integrates  $|y|$  at infinity. This gives us the following estimate:

$$\mathbf{E}[II] \leq \|G''\|_\infty \mathbf{E}[|Y_1|].$$

**Remark 6.** We point out that this last estimate is rather tricky. Doing bluntly a second order Taylor estimation on  $G_{\setminus Y_i}$  would not work here because  $y^2$  is not integrable at infinity against  $\frac{\alpha dy}{|y|^{\alpha+1}}$ . Instead, here, the second increment  $y$  is replaced by  $Y_i$ , which can be estimated.

For the sake of completeness, let us write the estimate we obtain for  $I$ :

$$\mathbf{E}[I] \leq \|G''\|_\infty \frac{1}{2-\alpha}.$$

Hence, we managed to relate every terms in (12) to the derivatives of  $G$ . We have the following lemma:

**Lemma 9.** For any  $F \in \mathcal{C}_b^{2,p}$ , it holds that

- $\nabla P_t F(x) \leq e^{-t/\alpha} P_t(\nabla F)(x)$ .
- If  $\nabla F$  is Lipschitz, then  $\nabla P_t F$  is Lipschitz, with

$$|\nabla P_t F(x) - \nabla P_t F(y)| \leq C e^{-t/\alpha} |x - y|^{\beta-1}.$$

Recalling  $G = P_t F$ , derivatives of  $G$  actually yields an additional  $1/n^{1/\alpha}$  factor. Hence, our final estimate is the following:

$$\begin{aligned} \mathbf{E}[\mathcal{L} P_t F(\mathbf{Y})] &= \sum_{i=1}^n \mathbf{E} \left[ -\frac{1}{\alpha} G'_{\setminus Y_i}(Y_i) Y_i + LG(Y_i) \right] \\ &= \sum_{i=1}^n \mathbf{E} \left[ \frac{1}{2} \left( -LG_{\setminus Y_i}(0) + \left( G_{\setminus Y_i}(1) + G_{\setminus Y_i}(-1) - 2G_{\setminus Y_i}(0) \right) \right. \right. \\ &\quad \left. \left. + \int_{|y| \leq 1} \left( G_{\setminus Y_i}(y) - G_{\setminus Y_i}(0) - G'_{\setminus Y_i}(0)y \right) \frac{\alpha dy}{|y|^{\alpha+1}} \right) + LG(Y_i) \right]. \end{aligned}$$

Finally, in this sum, we bound each term by  $\|G''\|_\infty = \frac{1}{n^{2/\alpha}} e^{-t/\alpha} \|\phi_p''\|_\infty$ . Now recall that  $\phi_p$  interpolates between a 2nd order polynomial and  $x \mapsto x^p$ , with  $p \leq 2$ , hence, its second derivative is bounded.

This gives us the upper bound:

$$|E| \leq C \sum_{i=1}^n \frac{1}{n^{2/\alpha}} e^{-t/\alpha} \|\phi_p''\|_\infty = C n^{1-\frac{2}{\alpha}} e^{-t/\alpha}. \quad (14)$$

The presence of the exponential term allows us to integrate in  $t$  from 0 to  $\infty$ , and finally, the proof is complete.  $\square$

Note that we also established the following lemma:

**Lemma 10.** Let  $G$  be a  $C_b^{2,p}$  function with bounded derivative then,

$$\left| \int_{|u| \leq 1} [G(u+y) - G(y) - uG'(y)] \frac{1}{|u|^{\alpha+1}} du \right| \leq \frac{2}{2-\alpha} \|G''\|_\infty, \quad (15)$$

$$|L(G)(y) - L(G)(0)| \leq C_\alpha \|G''\|_\infty [1 + |y|], \quad \forall y \in \mathbb{R}. \quad (16)$$

## 4.2 In the Normal domain of attraction

In this paragraph, we demonstrate how to establish the moment condition (7) for a general random variable in a Normal Stable domain of attraction. We recall that for a random variable  $Y$  to be in the normal domain of attraction of a stable distribution, we require its distribution function  $F_Y$  to satisfy:

$$1 - F_Y(t) = \mathbf{P}(Y \geq t) = \frac{A + \varepsilon(t)}{t^\alpha} \text{ and } F_Y(-t) = \frac{A + \varepsilon(-t)}{(-t)^\alpha},$$

whenever  $|t| \geq 1$ . From that decomposition, we see that we can extract a Pareto component from  $F_Y$ . We let  $\mathbf{P}_Y(dy) = \eta \mathbf{P}_Z(dy) + (1 - \eta)\mu(dy)$ , where  $P_Z$  is the distribution of a symmetrized Pareto.

**Remark 7.** The constant  $\eta$  is chosen according to  $A$  and  $\alpha$  to ensure that we indeed define a probability distribution. In the case where the constant  $\eta = 1$ , notice that we recover the previous symmetrized Pareto distribution. This constant  $\eta$  will also play a role in the generator below.

Observe now that  $\mu$  is simply a signed measure, for which we have the following information:

- $\mu$  is centered, signed and finite
- $\mu(t; +\infty) = \frac{\varepsilon(t)}{t^\alpha}$ , if  $t \geq 1$ , and  $\mu(-\infty; -t] = \frac{\varepsilon(-t)}{(-t)^\alpha}$ , if  $t \leq -1$
- $|\mu| \leq \mathbb{P}_Y + \mathbb{P}_Z \leq (1 + \|\varepsilon\|_\infty) \mathbb{P}_Z$

**Proposition 11.** For all  $1 < \alpha < \beta$  there exists a constant  $c_{\alpha, \beta, \eta}$  such that for all  $Y$  a random variable with distribution in  $\mathcal{D}^{\alpha, \eta}$  and  $G \in \mathcal{F}^\beta$  and  $A > 1$

$$\mathbf{E}[\mathcal{L}^{\alpha, \eta} G(Y)] \leq c_{\alpha, \beta, \eta} \left[ \|G''\|_\infty + (1 - \eta)(1 + \|\varepsilon\|_\infty) \left\{ A^{\beta-\alpha} \|G'\|_{\beta-1, Hol} + A^{2-\alpha} \|G''\|_\infty \right\} \right].$$

*Proof of Proposition 11.* The main lines of the proof is similar, we first derive from Stein's equation:

$$\mathbf{E}[F(Y)] - \mathbf{E}[F(S)] = \int_0^t \mathbf{E}[\mathcal{L} P_t F(Y)] dt,$$

where  $\mathcal{L}$  is the generator of the Orstein Uhlenbeck process:

$$\begin{aligned} \mathcal{L}\varphi(x) &= -\frac{1}{\alpha} \varphi'(x) \cdot x + \int_{-\infty}^{+\infty} \left( \varphi(x+y) - \varphi(x) - \varphi'(x)y \right) \frac{\eta\alpha}{2} \frac{1}{|y|^{1+\alpha}} dy \\ &= -\frac{1}{\alpha} \varphi'(x) \cdot x + \eta L\varphi(x) \end{aligned}$$



**Remark 8.** We point out here the coefficient  $\eta$  that was not present or rather equal to one in the Pareto case. We can put this coefficient here, up to a modification of the Lévy measure of the stable process, and this saves a renormalisation by some coefficient  $\sigma$  in Nourdin [7].

Let  $Y$  be a random variable with distribution in  $\mathcal{F}^{\alpha,\eta}$  and  $Z$  be a random variable with symmetric Pareto distribution and  $F_Y, F_Z$  their distribution function. Then  $F_Y - \eta F_Z$  is a function of finite bounded variation associated to a signed measure  $\mu$  such that

$$\begin{aligned} |\mu| &\leq \mathbb{P}_Y + \mathbb{P}_Z \leq (1 + \|\varepsilon\|_\infty) \mathbb{P}_Z, \\ \mu(]t, +\infty[) &= \frac{\varepsilon(t)}{t^\alpha} \quad \forall t > 1, \\ \mu(]-\infty, t]) &= \frac{\varepsilon(t)}{t^\alpha}. \end{aligned}$$

With these notations in hand, since  $Y$  is a centered random variable

$$\begin{aligned} \mathbf{E}[\mathcal{L}(G)(Y)] &= -\frac{1}{\alpha} \int_{\mathbb{R}} y [G'(y) - G'(0)] [\eta d\mathbb{P}_Z(y) + (1-\eta)d\mu(y)] \\ &\quad + \eta \int_{\mathbb{R}} L(G)(y) d\mathbb{P}_Y(y). \end{aligned}$$

Using Lemma 8 for the integral with respect to  $d\mathbb{P}_Z$  we obtain

$$\begin{aligned} \mathbf{E}[\mathcal{L}(G)(Y)] &= -\frac{\alpha}{2} \eta [G(1) + G(-1) - 2G(0)] \\ &\quad - \frac{1}{\alpha} \eta \int_{|u| \leq 1} [G(u) - G(0) - uG'(0)] \frac{du}{|u|^{\alpha+1}} \\ &\quad + \eta \int_{\mathbb{R}} [L(G)(y) - L(G)(0)] d\mathbb{P}_Y(y) \\ &\quad - \frac{1}{\alpha} \int_{\mathbb{R}} y [G'(y) - G'(0)] (1-\eta) d\mu(y). \end{aligned}$$

Using Lemma 10 the three first terms of the right member are bounded by  $C_{\alpha,\eta} \|G''\|_\infty$  where  $C_{\alpha,\eta}$  depends only on  $\alpha$  and  $\eta$ .

For the last term, notice how when  $\eta = 1$ , then term is not present, and the result follows from the Pareto case. When  $\eta < 1$ , we shall use the following Lemma to bound the last term by for some  $A > 1$  to be chosen later.

$$\left| \int_{\mathbb{R}} y [G'(y) - G'(0)] d\mu(y) \right| \leq C_{\alpha,\beta,\eta} \left[ A^{\beta-\alpha} \|G'\|_{\beta,\text{Hö}l} + A^{2-\alpha} \|G''\|_\infty \right]$$

Thus this achieves the proof of Proposition 11 under Lemma 12.  $\square$

**Lemma 12.** Under the assumptions of Proposition 11, there exists a constant  $C_{\alpha,\beta,\eta}$  such that for all  $G \in \mathcal{F}^{\alpha,\beta}$  and  $A > 1$

$$\left| \int_{\mathbb{R}} y [G'(y) - G'(0)] d\mu(y) \right| \leq C_{\alpha,\beta,\eta} \left[ A^{\beta-\alpha} \|G'\|_{\beta-1,\text{Hö}l} + A^{2-\alpha} \|G''\|_\infty \right].$$

*Proof of Lemma 12* ∴ Let  $A > 1$  We split the integral in fourth terms :

$$\begin{aligned} \int_{\mathbb{R}} y [G'(y) - G'(0)] d\mu(y) &= \int_{\{|y| \leq 1\}} y [G'(y) - G'(0)] d\mu(y) \\ &\quad + \int_{\{|y| \geq A\}} y [G'(y) - G'(0)] d\mu(y) \\ &\quad + \int_1^A y [G'(y) - G'(0)] d\mu(y) + \int_{-A}^{-1} y [G'(y) - G'(0)] d\mu(y). \end{aligned}$$

The first one is bounded using Taylor expansion by

$$\left| \int_{\{|y| \leq 1\}} y [G'(y) - G'(0)] d\mu(y) \right| \leq \|G''\|_{\infty} |\mu(\mathbb{R})| \leq 2\|G''\|_{\infty}.$$

Using the fact that  $G'$  is  $\beta-1$  Hölder continuous and that  $|\mu| \leq (1 + \|\varepsilon\|_{\infty})\mathbb{P}_Z$  (an  $\alpha$  Pareto distribution) the second term is bounded by

$$\begin{aligned} \left| \int_{\{|y| \geq A\}} y [G'(y) - G'(0)] d\mu(y) \right| &\leq \|G'\|_{\beta-1, \text{Hö}} (1 + \|\varepsilon\|_{\infty}) \int_{\{|y| \geq A\}} |y|^{\beta} d\mathbb{P}_Z(y) \\ &\leq \|G'\|_{\beta-1, \text{Hö}} (1 + \|\varepsilon\|_{\infty}) \frac{\alpha}{\beta - \alpha}. \end{aligned}$$

For the third one, we perform an integration by part

$$\begin{aligned} \int_1^A y [G'(y) - G'(0)] d\mu(y) &= [-y [G'(y) - G'(0)] \mu(]y, +\infty[)]_1^A \\ &\quad + \int_1^A ([G'(y) - G'(0)] + yG''(y)) \mu(]y, \infty[) dy. \end{aligned}$$

Thus since

$$|\mu(]y, +\infty[)| \leq \frac{(1 + \|\varepsilon\|_{\infty})}{2} \frac{1}{y^{\alpha}}$$

it is bounded by

$$\left| \int_1^A y [G'(y) - G'(0)] d\mu(y) \right| \leq 2 \left[ 1 + \frac{\alpha}{2 - \alpha} \right] \|G''\|_{\infty} (1 + \|\varepsilon\|_{\infty}) [1 + A^{2-\alpha}].$$

The integral on  $] - A, -1[$  term is bounded in the same spirit. This achieves the proof of Lemma 12, and thus the proof of Proposition 11 is complete.  $\square$

Let us now return to the general case. Recall that the goal is to control

$$\mathbf{E}[F(\mathbf{Y})] = \mathbf{E} \left[ \phi_p \left( \frac{Y_1 + \dots + Y_n}{n^{1/\alpha}} \right) \right],$$

where  $Y_i$ 's are in the normal domain of attraction. Hence, we need to control:

$$E = \sum_{i=1}^n \mathbf{E} \left[ -\frac{1}{\alpha} \partial_{y_i} G(\mathbf{Y}) Y_i + \eta \int_{\mathbb{R}} \left( G(\mathbf{Y} + u \mathbf{e}_i) - G(\mathbf{Y}) - \nabla G(\mathbf{Y}) \cdot u \right) \frac{1}{|u|^{\alpha+1}} du \right].$$

Keeping the notations as in the previous paragraph, we can reduce this calculation to the one dimensional case by conditioning. Let  $G = P_t F$  and for  $i = 1, \dots, n$ :

$$G_{\setminus Y_i}(y) = G(Y_1, \dots, Y_{i-1}, y, Y_{i+1}, \dots, Y_n).$$

We need to control:

$$E = \sum_{i=1}^n \mathbf{E} \left[ -\frac{1}{\alpha} G'_{\setminus Y_i}(Y_i) Y_i + \eta \int_{\mathbb{R}} \left( G_{\setminus Y_i}(Y_i + u) - G_{\setminus Y_i}(Y_i) - G'_{\setminus Y_i}(Y_i) \cdot u \right) \frac{1}{|u|^{\alpha+1}} du \right].$$

Now, we can plug the one-dimensional result from Proposition 11:

$$E \leq c_{\alpha, \beta, \eta} \sum_{i=1}^n \mathbf{E} \left[ \|G''_{\setminus Y_i}\|_{\infty} + (1 - \eta)(1 + \|\varepsilon\|_{\infty}) \left\{ A^{\beta - \alpha} \|G'_{\setminus Y_i}\|_{\beta-1, \text{Höl}} + A^{2 - \alpha} \|G''_{\setminus Y_i}\|_{\infty} \right\} \right].$$

Finally, we recall that  $\|G''\|_{\infty} = \frac{1}{n^{2/\alpha}} e^{-2t/\alpha} \|\phi''\|_{\infty}$  and

$$\|G'\|_{\beta-1, \text{Höl}} \leq C e^{-t/\alpha} \frac{1}{n^{\beta/\alpha}} \|\phi_p\|_{\infty}.$$

This power in  $n$  prompts us to chose  $A = n^{\delta}$ , where  $\delta < \frac{1}{\alpha}$ , so that

$$n \times A^{\beta - \alpha} \|G'_{\setminus Y_i}\|_{\beta-1, \text{Höl}} \xrightarrow{n \rightarrow +\infty} 0.$$

A similar calculation leads to the same restriction over  $\delta$  for the contribution of  $A^{2 - \alpha} \|G''_{\setminus Y_i}\|_{\infty}$ . This choice of  $A$  lets us finally control the difference by something bounded in  $n$ . As for the integral in time, the extra  $e^{-t/\alpha}$  factor allows us to integrate from 0 to  $+\infty$ , thus the moment condition holds.

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