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Vorticity Leray- α model for Navier-Stokes equations with viscosity depending on the distance to the wall.

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Abstract

We introduce a vorticity Leray- α model with eddy viscosity depending on $d(x, \partial\Omega)^\eta$ where $\partial\Omega$ is the boundary of the domain and $\eta \in]0; 1[$. We prove that this system admits fairly regular weak solutions converging when α goes to 0 to the solution of a reference system.

Key words : Fluid mechanics, Navier-Stokes equations, Leray- α models, Vorticity.

2010 MSC : 35Q30, 76D03, 76D05, 35A35, 35B33.

1 Introduction

Let Ω be a \mathcal{C}^2 open bounded subset of \mathbb{R}^3 . According to turbulence modelling, we introduce in this paper the following vorticity Leray- α model

$$(1.1) \quad \bar{\omega} \times \mathbf{u} - \operatorname{div}(\varrho^\eta \tilde{\nu} \nabla \mathbf{u}) + \lambda \mathbf{u} + \nabla \tilde{p} = \mathbf{f} \quad \text{in } \Omega,$$

$$(1.2) \quad \operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega,$$

$$(1.3) \quad \boldsymbol{\omega} = \operatorname{curl}(\mathbf{u}) \quad \text{in } \Omega,$$

$$(1.4) \quad -\alpha^2 \Delta \bar{\omega} + \bar{\omega} = P(\boldsymbol{\omega}) \quad \text{in } \mathbb{R}^3,$$

$$(1.5) \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

where \mathbf{u} models the long time average of the flow, $\tilde{p} = p - \frac{|\mathbf{u}|^2}{2}$ is the modified mean pressure, and $\mathbf{f} \in L^2(\Omega)^3$ is a given source term. The linear term $\lambda \mathbf{u}$ with $\lambda > 0$ is a damping term.

The eddy viscosity is of the form $\nu_{\text{turb}} = \varrho^\eta \tilde{\nu}$, where $0 < \eta < 1$ and $\tilde{\nu}$ is a continuous function satisfying

$$(1.6) \quad \forall t \in \mathbb{R}, \quad 0 < \nu_m \leq \tilde{\nu}(t) \leq \nu_M,$$

where ν_m and ν_M are constants.

The function ϱ refers to the Prandtl mixing length (see [20]), which appears like a regular function behaving like the distance to the boundary $d(\cdot, \partial\Omega)$. More precisely, we consider $\varrho : \Omega \rightarrow \mathbb{R}_+$ such that

$$(1.7) \quad \lim_{\substack{d(x, \partial\Omega) \rightarrow 0 \\ x \in \Omega}} \frac{\varrho(x)}{d(x, \partial\Omega)} = 1 \quad \text{and} \quad \forall n > 0, \quad \varrho_n = \inf_{\substack{d(x, \partial\Omega) \geq \frac{1}{n} \\ x \in \Omega}} \varrho(x) > 0.$$

The operator P in (1.4) is the extension by 0, which means

$$(1.8) \quad P(\boldsymbol{\omega}) : x \mapsto \begin{cases} \boldsymbol{\omega}(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Equation (1.4) defining $\bar{\boldsymbol{\omega}}$ is the Helmholtz equation (studied in detail for instance in [7, 13]), so that (1.1)-(1.5) is a kind of Leray- α model, which is a LES model.

In this paper we show that (1.1)-(1.5) with $\eta < 1$ admits sufficiently regular weak solutions satisfying the energy balance (3.2) (Theorem 3.1) that converge when α goes to zero to a solution of the following system

$$(1.9) \quad \boldsymbol{\omega} \times \mathbf{u} - \operatorname{div}(\varrho^\eta \tilde{\nu} \nabla \mathbf{u}) + \lambda \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega,$$

$$(1.10) \quad \operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega,$$

$$(1.11) \quad \boldsymbol{\omega} = \operatorname{curl}(\mathbf{u}) \quad \text{in } \Omega,$$

$$(1.12) \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega.$$

As previously shown in [1], (1.9)-(1.12) admits weak solutions for $0 \leq \eta < 1$, but the energy balance (3.2) can be satisfied if and only if $0 \leq \eta < 1/5$. Consequently, the convergence stated in Theorem 3.2 is weak when $1/5 \leq \eta < 1$ and strong if $0 \leq \eta < 1/5$.

Leray- α models were first introduced by J. Leray in [15, 16], and have been studied in detail in [3, 4, 6, 8, 9, 18, 19]. In this kind of models, the convection term is given by $(\bar{\mathbf{u}} \cdot \nabla) \mathbf{u}$.

By the way, it is possible to apply the filter to both velocities, so that the convection term becomes $\overline{(\mathbf{u} \cdot \nabla) \mathbf{u}}$. This is called Bardina-type models. These have been studied for instance in [3, 5, 8, 11, 17]. In particular, it is shown that solutions of the α -models are regular and converge to the Leray solution of NSE, most of time with periodic boundary conditions. A comparison between these different models is made for instance in [3, 6, 8, 18].

In order to ensure coercivity results, a free-divergence assumption is required for $\bar{\mathbf{u}}$. Consequently, the standard Leray- α model adapted to (1.1) should be like

$$(1.13) \quad (\bar{\mathbf{u}} \cdot \nabla) \mathbf{u} - \operatorname{div}(\varrho^\eta \tilde{\nu} \nabla \mathbf{u}) + \lambda \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega,$$

$$(1.14) \quad -\alpha^2 \Delta \bar{\mathbf{u}} + \bar{\mathbf{u}} + \nabla q = \mathbf{u} \quad \text{in } \Omega,$$

$$(1.15) \quad \operatorname{div}(\mathbf{u}) = \operatorname{div}(\bar{\mathbf{u}}) = 0 \quad \text{in } \Omega,$$

$$(1.16) \quad \mathbf{u} = 0 \quad \text{on } \Gamma.$$

Unfortunately, this model does not provide sufficient results to ensure the existence of suitable solutions. Indeed, we work in weighted function spaces, which only provides a $W^{1,p}$ regularity for \mathbf{u} , with $1 \leq p < 2$. This regularity is not enough, because of the Lagrange multiplier in (1.14).

This is why the vorticity has been introduced. This quantity appears in [8, 19] to introduce an α -model called NS- α . In this model, the vorticity term $\text{curl}(\mathbf{u}) \times \mathbf{u}$ is replaced by $\text{curl}(\bar{\mathbf{u}}) \times \mathbf{u}$.

Here we consider another version of this model, similar to the $NS - \bar{\omega}$ model introduced by W.J. Layton and al. in [12, 14], where the vorticity term of (1.9)-(1.12) becomes $\text{curl}(\bar{\mathbf{u}}) \times \mathbf{u}$. A theoretical study is made in [12], for a system of NSE with constant viscosity. This idea is relevant given that there is no free-divergence assumption required to prove coercivity in the calculations.

The paper is organized as follows. In section 2, we focus on the vorticity Helmholtz equation with a L^p source term. The goal is proving some convergence results that are applied later.

In section 3, we start by showing the existence of a distributional solution of (1.1)-(1.5) using a fixed-point strategy, and then we prove the convergence to a solution of (1.9)-(1.12) when α goes to zero. The value of η has an influence on the sense of the convergence.

I would like to express my gratitude to my PhD advisor Professor Roger LEWANDOWSKI to have suggested this problem and guided me during this work.

2 Vorticity Leray- α model

This section starts by defining the function spaces in which solutions of (1.9)-(1.12) are searched. As already done in [1], some Sobolev regularity is given on these spaces.

Then, the solution of the Helmholtz vorticity equation is written with a convolution kernel, to ease the calculations in order to find results on $\bar{\omega}$ starting from equivalent statements on ω .

2.1 Functions spaces and estimates

We first define a space of smooth functions

$$(2.1) \quad \mathcal{V} = \{\mathbf{v} \in \mathcal{D}(\Omega)^3 / \text{div}(\mathbf{v}) = 0\},$$

and V_η is the closure of this space for the norm

$$(2.2) \quad \|\mathbf{u}\|_{V_\eta} = \left(\|\mathbf{u}\|_{0,2,\Omega}^2 + \|\varrho^{\eta/2} \nabla \mathbf{u}\|_{0,2,\Omega}^2 \right)^{1/2}.$$

We start by stating the regularity of V_η functions. Indeed, looking at the norm (2.2), we understand that such functions are not likely to be H^1 if $\eta > 0$.

We recall two embedding theorems. The first is from [1, 2] brings a H^s estimate, with $1/2 < s \leq 1$.

Theorem 2.1. *Let $\eta \in [0; 1[$. The continuous embedding*

$$(2.3) \quad V_\eta \hookrightarrow H^{1-\frac{\eta}{2}}(\Omega)^3$$

stands, which implies the Sobolev compact embedding

$$(2.4) \quad V_\eta \hookrightarrow L^r(\Omega)^3, \quad \text{where } 1 \leq r < \frac{6}{1+\eta}.$$

The second result from [10] brings a $W^{1,p}$ estimate with $1 \leq p < 2$.

Theorem 2.2. *Let $\eta \in [0; 1[$. The continuous embedding*

$$(2.5) \quad V_\eta \hookrightarrow W^{1,p}(\Omega)^3$$

stands for $p \in \left[1; \frac{2}{1+\eta}\right]$.

Remark 2.1. *The compact Sobolev embedding given by Theorem 2.2 is weaker than this given by Theorem 2.1. Indeed, $\frac{6}{1+3\eta}$ is lower than $\frac{6}{1+\eta}$.*

Remark 2.2. *These two results also give an estimate on the vorticity.*

- *If $\mathbf{u} \in H^{1-\frac{\eta}{2}}(\Omega)^3$, then $\boldsymbol{\omega} = \text{curl}(\mathbf{u}) \in H^{-\frac{\eta}{2}}(\Omega)^3$.*
- *If $\mathbf{u} \in W^{1,p}(\Omega)^3$, then $\boldsymbol{\omega} = \text{curl}(\mathbf{u}) \in L^p(\Omega)^3$.*

2.2 Green kernel for the Helmholtz problem

Until the end of this section, we simply write $\boldsymbol{\omega}$ instead of $P(\boldsymbol{\omega})$. This means

$$(2.6) \quad \boldsymbol{\omega} \in L^p(\mathbb{R}^3)^3 \quad \text{with} \quad 1 \leq p < \frac{2}{1+\eta} \quad \text{and} \quad 0 \leq \eta \leq 1.$$

We focus on the following Helmholtz equation, where $\alpha > 0$ is a constant parameter, supposed to be near 0.

$$(2.7) \quad -\alpha^2 \Delta \bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\omega}} = \boldsymbol{\omega}.$$

Theorem 2.3 (Green function). *Let $\boldsymbol{\omega} \in L^p(\mathbb{R}^3)^3$ defined in (2.6), and $\bar{\boldsymbol{\omega}}$ satisfying (2.7). Then*

$$(2.8) \quad \bar{\boldsymbol{\omega}}(x) = K_\alpha * \boldsymbol{\omega}(x)$$

for almost every $x \in \mathbb{R}^3$, where K_α is the scalar function given by

$$(2.9) \quad \begin{aligned} K_\alpha : \mathbb{R}^3 &\rightarrow \mathbb{R} \\ x &\mapsto \frac{e^{-|x|/\alpha}}{4\pi\alpha^2|x|}. \end{aligned}$$

This result is already proved in [17].

Lemma 2.1. *The function K_α given by (2.9) is in $L^q(\mathbb{R}^3, \mathbb{R}_+)$ for $q \in [1; 3[$ and*

$$(2.10) \quad \int_{\mathbb{R}^3} K_\alpha(x)^q dx = \Gamma(3-q) \frac{(4\pi\alpha^3)^{1-q}}{q^{3-q}},$$

where Γ refers to the Euler function: $\Gamma(t) = \int_0^{+\infty} s^{t-1} e^{-s} ds$, for any $t > 0$.

Remark 2.3. *When $q = 1$, we obtain in particular*

$$(2.11) \quad \int_{\mathbb{R}^3} K_\alpha(x) dx = 1.$$

Proof. We use a change of variables which transforms the classical cartesian coordinates of \mathbb{R}^3 into spherical coordinates $(r, \theta, \varphi) \in \mathbb{R}_+ \times [0; \pi] \times [0; 2\pi]$, so that the 3D Lebesgue measure is written

$$(2.12) \quad d\lambda(y) = r^2 \sin(\theta) dr d\theta d\varphi.$$

The Fubini-Tonelli theorem then yields

$$(2.13) \quad \begin{aligned} \iiint_{\mathbb{R}^3} K_\alpha(y)^q dy &= \int_{r=0}^{+\infty} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \frac{e^{-qr/\alpha}}{(4\pi)^q \alpha^{2q} r^q} r^2 \sin(\theta) dr d\theta d\varphi \\ &= \frac{(4\pi)^{1-q}}{\alpha^{2q}} \int_0^{+\infty} r^{2-q} e^{-qr/\alpha} dr. \end{aligned}$$

The change of variables $s = \frac{qr}{\alpha}$ yields

$$(2.14) \quad \int_0^{+\infty} r^{2-q} e^{-qr/\alpha} dr = \left(\frac{\alpha}{q}\right)^{3-q} \int_0^{+\infty} s^{2-q} e^{-s} ds.$$

This integral is defined if and only if $2 - q > -1$, which means $q < 3$. When this condition is satisfied, (2.13) yields

$$(2.15) \quad \int_{\mathbb{R}^3} K_\alpha(y)^q dy = \frac{(4\pi)^{1-q}}{\alpha^{2q}} \left(\frac{\alpha}{q}\right)^{3-q} \int_0^{+\infty} s^{2-q} e^{-s} ds,$$

which corresponds to (2.10). □

Since the vector function $\bar{\omega}$ is defined as a convolution, the Young inequality gives some regularity results for $\bar{\omega}$. Indeed, the extension by 0 preserves the L^p norms.

Lemma 2.2 (L^γ -regularity for $\bar{\omega}$). *Let $\omega \in L^p(\mathbb{R}^3)^3$ with p defined by (2.6), and $K_\alpha \in L^q(\mathbb{R}^3)$ with $q \in [1; 3[$.*

- If $0 \leq \eta < 1/3$, $\bar{\omega} \in L^\gamma(\mathbb{R}^3)^3$ with $1 \leq \gamma \leq +\infty$.
- If $\eta = 1/3$, $\bar{\omega} \in L^\gamma(\mathbb{R}^3)^3$ with $1 \leq \gamma < +\infty$.
- If $1/3 < \eta \leq 1$, $\bar{\omega} \in L^\gamma(\mathbb{R}^3)^3$ with $1 \leq \gamma < \frac{6}{3\eta - 1}$.

Proof. Since $K_\alpha \in L^q(\mathbb{R}^3)$ and $\omega \in L^p(\mathbb{R}^3)^3$, the Young inequality yields

$$(2.16) \quad \bar{\omega} \in L^\gamma(\mathbb{R}^3)^3 \quad \text{with} \quad 1 + \frac{1}{\gamma} = \frac{1}{p} + \frac{1}{q}.$$

The assumptions on p and q yield

$$(2.17) \quad \frac{1 + \eta}{2} + \frac{1}{3} < \frac{1}{p} + \frac{1}{q} \leq 2,$$

which means

$$(2.18) \quad \frac{\eta}{2} - \frac{1}{6} < \frac{1}{\gamma} \leq 1.$$

We see that $\frac{\eta}{2} - \frac{1}{6}$ is always strictly lower than 1, and it is positive if and only if $\eta > 1/3$. Consequently, when $\eta \leq 1/3$ we directly obtain the result. And if $\eta > 1/3$, (2.18) yields

$$(2.19) \quad 1 \leq \gamma < \frac{6}{3\eta - 1}.$$

Hence the result. □

2.3 Main convergence results

The two next results are the main statements of this paper. First, we consider the case where α is fixed. The following lemma will be used to find solutions of the approximated system (1.1)-(1.5), by a fixed-point method.

Lemma 2.3 (Convergence when α is fixed). *Let $(\omega_n)_{n \in \mathbb{N}}$ be a sequence of $L^p(\mathbb{R}^3)^3$ functions ($1 \leq p < +\infty$) that weakly converges to $\omega \in L^p(\mathbb{R}^3)^3$. Let $(\bar{\omega}_n)_{n \in \mathbb{N}}$ be the sequence of the solutions of (2.7) with source term ω_n , and $\bar{\omega}$ the solution of (2.7) with source term ω .*

Then $(\bar{\omega}_n)_{n \in \mathbb{N}}$ weakly converges in $L^p(\mathbb{R}^3)^3$ to $\bar{\omega}$.

Proof. Let $\phi \in L^{p'}(\mathbb{R}^3)$, the Fubini theorem yields

$$(2.20) \quad \begin{aligned} \int_{\mathbb{R}^3} \bar{\omega}_n(x) \phi(x) dx &= \iint_{x, y \in \mathbb{R}^3} \omega_n(y) K_\alpha(x-y) \phi(x) dx dy \\ &= \int_{y \in \mathbb{R}^3} \omega_n(y) \left(\int_{x \in \mathbb{R}^3} K_\alpha(x-y) \phi(x) dx \right) dy. \end{aligned}$$

Since $\phi \in L^{p'}(\mathbb{R}^3)$ and $K_\alpha \in L^1(\mathbb{R}^3)$, the function $[y \mapsto \int_{\mathbb{R}^3} K_\alpha(x-y) \phi(x) dx]$ is in $L^{p'}(\mathbb{R}^3)$. Hence the weak convergence of $(\omega_n)_{n \in \mathbb{N}}$ yields the result. \square

The following theorem gives the convergence of the solutions $\bar{\omega}$ of (2.7) when α goes to zero. This will be used later to show the convergence of a sequence of solutions of (1.1)-(1.5) to a solution of (1.9)-(1.12).

Theorem 2.4. *Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\alpha_n \xrightarrow{n \rightarrow +\infty} 0$, and p be defined by (2.6). Let $(\omega_{\alpha_n})_{n \in \mathbb{N}}$ be a sequence of $L^p(\mathbb{R}^3)^3$ functions that weakly converges to $\omega \in L^p(\mathbb{R}^3)^3$.*

Let $(\bar{\omega}_{\alpha_n})_{n \in \mathbb{N}}$ satisfy for all $n \in \mathbb{N}$

$$(2.21) \quad -\alpha_n^2 \Delta \bar{\omega}_{\alpha_n} + \bar{\omega}_{\alpha_n} = \omega_{\alpha_n}.$$

Then $(\bar{\omega}_{\alpha_n})_{n \in \mathbb{N}}$ weakly converges to ω in $L^p(\mathbb{R}^3)^3$.

Proof. Let $\phi \in \mathcal{D}(\mathbb{R}^3)$, by the Fubini theorem

$$(2.22) \quad \begin{aligned} \int_{\mathbb{R}^3} \bar{\omega}_{\alpha_n}(x) \phi(x) dx &= \iint_{x, y \in \mathbb{R}^3} \omega_{\alpha_n}(y) K_{\alpha_n}(x-y) \phi(x) dx dy \\ &= \int_{y \in \mathbb{R}^3} \omega_{\alpha_n}(y) \left(\int_{x \in \mathbb{R}^3} K_{\alpha_n}(x-y) \phi(x) dx \right) dy. \end{aligned}$$

The change of variables $z = \frac{x-y}{\alpha_n}$ in the middle integral yields

$$(2.23) \quad \int_{\mathbb{R}^3} K_{\alpha_n}(x-y) \phi(x) dx = \int_{\mathbb{R}^3} \frac{e^{-|x-y|/\alpha_n}}{4\pi\alpha_n^2|x-y|} \phi(x) dx = \int_{\mathbb{R}^3} \frac{e^{-|z|}}{4\pi|z|} \phi(y + \alpha_n z) dz.$$

Then (2.22) becomes

$$(2.24) \quad \int_{\mathbb{R}^3} \bar{\omega}_{\alpha_n}(x) \phi(x) dx = \int_{z \in \mathbb{R}^3} K_1(z) \left(\int_{y \in \mathbb{R}^3} \omega_{\alpha_n}(y) \phi(y + \alpha_n z) dy \right) dz.$$

We now aim at showing the following convergence for any fixed $z \in \mathbb{R}^3$

$$(2.25) \quad \int_{\mathbb{R}^3} \omega_{\alpha_n}(y) \phi(y + \alpha_n z) dy \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^3} \omega(y) \phi(y) dy.$$

To do so, we decompose the difference

$$(2.26) \quad \int_{\mathbb{R}^3} \omega_{\alpha_n}(y) \phi(y + \alpha_n z) dy - \int_{\mathbb{R}^3} \omega(y) \phi(y) dy = \int_{\mathbb{R}^3} (\omega_{\alpha_n}(y) - \omega(y)) \phi(y) dy \\ + \int_{\mathbb{R}^3} \omega_{\alpha_n}(y) (\phi(y + \alpha_n z) - \phi(y)) dy.$$

On one hand, the weak convergence of $(\omega_{\alpha_n})_{n \in \mathbb{N}}$ yields

$$(2.27) \quad \int_{\mathbb{R}^3} (\omega_{\alpha_n}(y) - \omega(y)) \phi(y) dy \xrightarrow{n \rightarrow +\infty} 0.$$

On the other hand, the Hölder inequality yields

$$(2.28) \quad \int_{\mathbb{R}^3} \omega_{\alpha_n}(y) (\phi(y + \alpha_n z) - \phi(y)) dy \leq \|\omega_{\alpha_n}\|_p \left(\int_{\mathbb{R}^3} |\phi(y + \alpha_n z) - \phi(y)|^{p'} dy \right)^{1/p'} \\ \xrightarrow{n \rightarrow +\infty} 0,$$

by the Lebesgue theorem. Hence (2.25).

Then the following domination stands:

$$(2.29) \quad \left| K_1(z) \left(\int_{y \in \mathbb{R}^3} \omega_{\alpha_n}(y) \phi(y + \alpha_n z) dy \right) \right| \leq K_1(z) \|\omega_{\alpha_n}\|_p \|\phi\|_{p'},$$

which is $L^1(\mathbb{R}^3)$.

Therefore by the Lebesgue theorem, (2.25) and (2.29)

$$(2.30) \quad \int_{\mathbb{R}^3} \bar{\omega}_{\alpha_n}(x) \phi(x) dx \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^3} \omega(y) \phi(y) dy.$$

The density of $\mathcal{D}(\mathbb{R}^3)$ in $L^{p'}(\mathbb{R}^3)$ yields the result. □

3 Solutions of the Leray- α approximated system

This section aims at proving the existence of solutions of (1.9)-(1.12), in three steps. First, we solve a linearized system by the Lax-Milgram theorem. Such a solution is unique, and can be written $T\mathbf{u}$. Then the existence of solutions of (1.1)-(1.5) is shown by using the Leray-Schauder fixed-point theorem on the operator T .

We then obtain a sequence $(\mathbf{u}_{\alpha_n})_{n \in \mathbb{N}}$ of solutions, corresponding to a sequence of parameters $(\alpha_n)_{n \in \mathbb{N}}$ converging to 0. Finally, Theorem 2.4 provides the convergence to a solution of (1.9)-(1.12).

3.1 Variational formulation

The function $\bar{\omega}$ in (1.1)-(1.5) is obtained this way.

- We consider the extension by 0 of $\omega \in L^p(\Omega)^3$, still named the same,
- $\bar{\omega}$ satisfies (2.7), so that $\bar{\omega} \in L^\gamma(\mathbb{R}^3)^3$, where γ is given by Corollary 2.2.
- we take the restriction of $\bar{\omega}$ to Ω , still named the same.

We write the variational formulation of system (1.1)-(1.5)

$$(3.1) \quad \begin{cases} \mathbf{u} \in V_\eta, \\ \forall \mathbf{v} \in V_\eta, \quad \int_\Omega (\bar{\boldsymbol{\omega}}(\mathbf{u}) \times \mathbf{u}) \cdot \mathbf{v} + \int_\Omega \varrho^\eta \tilde{\nu} \nabla \mathbf{u} : \nabla \mathbf{v} + \lambda \int_\Omega \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}. \end{cases}$$

Taking $\mathbf{v} = \mathbf{u}$ in (3.1) yields the following energy balance

$$(3.2) \quad \int_\Omega \varrho^\eta \tilde{\nu} |\nabla \mathbf{u}|^2 + \lambda \int_\Omega |\mathbf{u}|^2 = \int_\Omega \mathbf{f} \cdot \mathbf{u}.$$

These two equations have a sense if and only if the integrals are defined. The condition is given by the following lemma.

Lemma 3.1 (Critical exponents). *Let $\mathbf{f} \in L^2(\Omega)^3$ and $\lambda > 0$. For any $\eta \in]0; 1[$, all the integrals in (3.1) are well defined.*

Proof. The definition of V_η directly yields the three last integrals in (3.1) are finite. In order to show that $\int_\Omega (\bar{\boldsymbol{\omega}} \times \mathbf{u}) \cdot \mathbf{v}$ is finite for $\mathbf{u}, \mathbf{v} \in V_\eta$, we have to distinguish the two cases of Lemma 2.2.

If $\eta \leq 1/3$, $\bar{\boldsymbol{\omega}} \in L^\infty(\Omega)^3$ and $\mathbf{u}, \mathbf{v} \in L^r(\Omega)^3$ for $1 \leq r < \frac{6}{1+\eta}$. The Hölder inequality then yields

$$(3.3) \quad \int_\Omega |(\bar{\boldsymbol{\omega}} \times \mathbf{u}) \cdot \mathbf{v}| \leq \|\bar{\boldsymbol{\omega}}\|_{0,\infty,\Omega} \|\mathbf{u}\|_{0,r,\Omega} \|\mathbf{v}\|_{0,r,\Omega}, \quad \text{if } \frac{2}{r} = 1.$$

This condition can be fulfilled if and only if $\frac{6}{1+\eta} > 2$, which means $\eta < 2$. This is the case, since $\eta \leq 1/3$.

If, $\eta > 1/3$, $\bar{\boldsymbol{\omega}} \in L^s(\Omega)^3$ with $1 \leq s < \frac{6}{3\eta-1}$. The functions \mathbf{u} and \mathbf{v} have the same regularity as before. The Hölder inequality yields

$$(3.4) \quad \int_\Omega |(\bar{\boldsymbol{\omega}} \times \mathbf{u}) \cdot \mathbf{v}| \leq \|\bar{\boldsymbol{\omega}}\|_{0,s,\Omega} \|\mathbf{u}\|_{0,r,\Omega} \|\mathbf{v}\|_{0,r,\Omega}, \quad \text{if } \frac{1}{s} + \frac{2}{r} = 1.$$

This equality of exponents can be fulfilled if and only if $\frac{3\eta-1}{6} + \frac{1+\eta}{3} < 1$, which is equivalent to $\eta < 1$. \square

Now we can initiate the method explained at the beginning of this section.

3.2 Solutions for the vorticity Leray- α system

The variational problem (3.1) is linearized by considering a vorticity field \mathbf{w} which does not depend on \mathbf{u} .

$$(3.5) \quad \begin{cases} \mathbf{u} \in V_\eta, \\ \forall \mathbf{v} \in V_\eta, \quad \int_\Omega (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} + \int_\Omega \varrho^\eta \tilde{\nu} \nabla \mathbf{u} : \nabla \mathbf{v} + \lambda \int_\Omega \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}, \end{cases}$$

Lemma 3.2 (Linearized problem). *Let $\eta \in [0; 1[$, $\mathbf{f} \in L^2(\Omega)$, $\lambda > 0$ and $\mathbf{w} \in L^s(\Omega)$ for*

$$(3.6) \quad \begin{cases} 1 \leq s \leq +\infty & \text{if } \eta < 1/3, \\ 1 \leq s < +\infty & \text{if } \eta = 1/3, \\ 1 \leq s < \frac{6}{3\eta-1} & \text{if } \eta > 1/3. \end{cases}$$

The variational problem (3.5) admits a unique solution.

Proof. The Lax-Milgram Theorem is applied to the bilinear form $B(\mathbf{u}, \mathbf{v}) = A(\mathbf{u}, \mathbf{v}) + b_{\mathbf{w}}(\mathbf{u}, \mathbf{v})$, where A and $b_{\mathbf{w}}$ are defined by

$$(3.7) \quad A(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \varrho^{\eta} \tilde{\nu} \nabla \mathbf{u} : \nabla \mathbf{v} + \lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v},$$

$$(3.8) \quad b_{\mathbf{w}}(\mathbf{u}, \mathbf{v}) = \int_{\Omega} (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}.$$

First, we write

$$(3.9) \quad |A(\mathbf{u}, \mathbf{v})| \leq (\lambda + \nu_M) \|\mathbf{u}\|_{V_{\eta}} \|\mathbf{v}\|_{V_{\eta}}.$$

Let $r \in \left[1; \frac{6}{1+\eta}\right]$, we have

$$(3.10) \quad |b_{\mathbf{w}}(\mathbf{u}, \mathbf{v})| \leq c \|\mathbf{w}\|_{0,s,\Omega} \|\mathbf{u}\|_{0,r,\Omega} \|\mathbf{v}\|_{0,r,\Omega} \leq C \|\mathbf{u}\|_{V_{\eta}} \|\mathbf{v}\|_{V_{\eta}}.$$

Inequalities (3.9) and (3.10) yield the continuity of the bilinear form B on V_{η} . Let us show it is coercive, by considering any $\mathbf{v} \in V_{\eta}$.

$$(3.11) \quad B(\mathbf{v}, \mathbf{v}) = \int_{\Omega} \varrho^{\eta} \tilde{\nu} |\nabla \mathbf{v}|^2 + \lambda \int_{\Omega} |\mathbf{v}|^2 \geq \min(\nu_m, \lambda) \|\mathbf{v}\|_{V_{\eta}}^2.$$

Finally, the linear form $\ell : \mathbf{v} \mapsto \int_{\Omega} \mathbf{f} \cdot \mathbf{v}$ is continuous on V_{η} , since $\mathbf{f} \in L^2(\Omega)^3$.

This concludes the proof. \square

Lemma 3.2 yields the existence of an operator T associating to any function $\mathbf{u} \in V_{\eta}$ the solution of the linearized problem where $\mathbf{w} = \bar{\omega}(\mathbf{u})$, which means

$$(3.12) \quad \forall \mathbf{v} \in V_{\eta}, \quad \int_{\Omega} (\bar{\omega}(\mathbf{u}) \times T\mathbf{u}) \cdot \mathbf{v} + \int_{\Omega} \varrho^{\eta} \tilde{\nu} \nabla T\mathbf{u} : \nabla \mathbf{v} + \lambda \int_{\Omega} T\mathbf{u} \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

Replacing \mathbf{v} by $T\mathbf{u}$ in (3.12) yields

$$(3.13) \quad \int_{\Omega} \varrho^{\eta} \tilde{\nu} |\nabla T\mathbf{u}|^2 + \lambda \int_{\Omega} |T\mathbf{u}|^2 = \int_{\Omega} \mathbf{f} \cdot T\mathbf{u},$$

that yields the bound

$$(3.14) \quad \|T\mathbf{u}\|_{V_{\eta}} \leq \frac{\|\mathbf{f}\|_{0,2,\Omega}}{\min(\nu_m, \lambda)} =: M.$$

Then we use the Leray-Schauder fixed point theorem to prove the following result.

Theorem 3.1. *Let $\mathbf{f} \in L^2(\Omega)^3$ and $\lambda > 0$. The operator T defined by (3.12) has a fixed point in the closed ball $\bar{B}_{V_{\eta}}(0, M)$ simply named B . This means that (3.1) admits a solution in B .*

Proof. The fixed point theorem will be applied on B , which is a nonempty closed bounded convex subset of V_η . The continuity of T and compactness of $T(B)$ need to be proved. To do so, we consider a sequence $(\mathbf{u}_n)_{n \in \mathbb{N}}$ of functions in B that weakly converges to a certain \mathbf{u} in V_η . The goal is showing that $(T\mathbf{u}_n)_{n \in \mathbb{N}}$ strongly converges to $T\mathbf{u}$ in V_η . We recall that $T\mathbf{u}_n$ is defined by

$$(3.15) \quad \forall \mathbf{v} \in V_\eta, \quad \int_{\Omega} (\bar{\omega}(\mathbf{u}_n) \times T\mathbf{u}_n) \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla T\mathbf{u}_n : \nabla \mathbf{v} + \lambda \int_{\Omega} T\mathbf{u}_n \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

Since $(T\mathbf{u}_n)_{n \in \mathbb{N}}$ is bounded in V_η , a subsequence still named the same way weakly converges to a certain φ in V_η . The first step of the proof consists in showing that $\varphi = T\mathbf{u}$.

The weak convergence and the compact Sobolev embedding (2.4) directly yield for any $\mathbf{v} \in V_\eta$

$$(3.16) \quad \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla T\mathbf{u}_n : \nabla \mathbf{v} \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla \varphi : \nabla \mathbf{v},$$

$$(3.17) \quad \lambda \int_{\Omega} T\mathbf{u}_n \cdot \mathbf{v} \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \varphi \cdot \mathbf{v}.$$

For the vorticity term, we write the decomposition

$$(3.18) \quad \int_{\Omega} (\bar{\omega}(\mathbf{u}_n) \times T\mathbf{u}_n) \cdot \mathbf{v} - \int_{\Omega} (\bar{\omega}(\mathbf{u}) \times \varphi) \cdot \mathbf{v} = \int_{\Omega} ((\bar{\omega}(\mathbf{u}_n) - \bar{\omega}(\mathbf{u})) \times \varphi) \cdot \mathbf{v} \\ + \int_{\Omega} (\bar{\omega}(\mathbf{u}_n) \times (T\mathbf{u}_n - \varphi)) \cdot \mathbf{v}.$$

On one hand, the assumption and Theorem 2.2 imply $\mathbf{u}_n \xrightarrow{n \rightarrow +\infty} \mathbf{u}$ in $W^{1,p}(\Omega)^3$ for

$$(3.19) \quad 1 \leq p < \frac{2}{1+\eta}, \quad \text{with } 0 \leq \eta < 1.$$

This yields $\text{curl}(\mathbf{u}_n) \xrightarrow{n \rightarrow +\infty} \text{curl}(\mathbf{u})$ in $L^p(\Omega)^3$, and the extensions by 0 satisfy

$$(3.20) \quad \omega(\mathbf{u}_n) \xrightarrow{n \rightarrow +\infty} \omega(\mathbf{u}) \quad \text{in } L^p(\mathbb{R}^3)^3,$$

In particular, $(\omega(\mathbf{u}_n))_{n \in \mathbb{N}}$ is bounded in $L^p(\Omega)$, with the same values of p . Therefore, Lemma 2.2 yields

$$(3.21) \quad \|\bar{\omega}(\mathbf{u}_n)\|_{0,s,\Omega} \leq \|K_\alpha\|_{0,q,\Omega} \|\omega(\mathbf{u}_n)\|_{0,p,\Omega},$$

with s given by (3.6) and $1 \leq q < 3$. Thus, $(\bar{\omega}(\mathbf{u}_n))_{n \in \mathbb{N}}$ is bounded in $L^s(\Omega)$. We obtain

$$(3.22) \quad \left| \int_{\Omega} (\bar{\omega}(\mathbf{u}_n) \times (T\mathbf{u}_n - \varphi)) \cdot \mathbf{v} \right| \leq c_1 \|\bar{\omega}(\mathbf{u}_n)\|_{0,s,\Omega} \|T\mathbf{u}_n - \varphi\|_{0,r,\Omega} \|\mathbf{v}\|_{0,r,\Omega} \xrightarrow{n \rightarrow +\infty} 0,$$

where r is given by (2.4).

On the other hand, convergence (3.20) and Lemma 2.3 yield

$$(3.23) \quad \bar{\omega}(\mathbf{u}_n) \xrightarrow{n \rightarrow +\infty} \bar{\omega}(\mathbf{u}) \quad \text{in } L^p(\mathbb{R}^3)^3.$$

And the restrictions weakly converge in $L^p(\Omega)^3$. This yields

$$(3.24) \quad \int_{\Omega} ((\bar{\omega}(\mathbf{u}_n) - \bar{\omega}(\mathbf{u})) \times \varphi) \cdot \mathbf{v} \xrightarrow{n \rightarrow +\infty} 0.$$

To sum up, convergences (3.16), (3.17), (3.22) and (3.24) yield

$$(3.25) \quad \forall \mathbf{v} \in V_\eta, \quad \int_\Omega (\bar{\omega}(\mathbf{u}) \times \boldsymbol{\varphi}) \cdot \mathbf{v} + \int_\Omega \varrho^\eta \tilde{\nu} \nabla \boldsymbol{\varphi} : \nabla \mathbf{v} + \lambda \int_\Omega \boldsymbol{\varphi} \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}.$$

That means $\boldsymbol{\varphi} = T\mathbf{u}$.

Now the strong convergence has to be proved. Taking $\mathbf{v} = T\mathbf{u}_n$ in (3.15) and $\mathbf{v} = T\mathbf{u}$ in (3.12) yields

$$(3.26) \quad \int_\Omega \varrho^\eta \tilde{\nu} |\nabla T\mathbf{u}_n|^2 + \lambda \int_\Omega |T\mathbf{u}_n|^2 = \int_\Omega \mathbf{f} \cdot T\mathbf{u}_n,$$

$$(3.27) \quad \int_\Omega \varrho^\eta \tilde{\nu} |\nabla T\mathbf{u}|^2 + \lambda \int_\Omega |T\mathbf{u}|^2 = \int_\Omega \mathbf{f} \cdot T\mathbf{u}.$$

What was done before directly yields

$$(3.28) \quad \lambda \int_\Omega |T\mathbf{u}_n|^2 \xrightarrow{n \rightarrow +\infty} \lambda \int_\Omega |T\mathbf{u}|^2,$$

$$(3.29) \quad \int_\Omega \mathbf{f} \cdot T\mathbf{u}_n \xrightarrow{n \rightarrow +\infty} \int_\Omega \mathbf{f} \cdot T\mathbf{u},$$

so that we deduce

$$(3.30) \quad \int_\Omega \varrho^\eta \tilde{\nu} |\nabla T\mathbf{u}_n|^2 \xrightarrow{n \rightarrow +\infty} \int_\Omega \varrho^\eta \tilde{\nu} |\nabla T\mathbf{u}|^2.$$

The weak convergence and (3.30) yield the strong convergence $T\mathbf{u}_n \xrightarrow{n \rightarrow +\infty} T\mathbf{u}$ in V_η for the norm

$$(3.31) \quad N_\nu(\mathbf{u}) := \left(\|\mathbf{u}\|_{0,2,\Omega}^2 + \|\varrho^{\eta/2} \tilde{\nu}^{1/2} \nabla \mathbf{u}\|_{0,2,\Omega}^2 \right)^{1/2}.$$

Given that this norm and $\|\cdot\|_{V_\eta}$ are equivalent, we obtain the strong convergence $T\mathbf{u}_n \xrightarrow{n \rightarrow +\infty} T\mathbf{u}$ in V_η , which concludes the proof. \square

3.3 Convergence to the solutions of the original problem

In this subsection, we consider a sequence of parameters $(\alpha_n)_{n \in \mathbb{N}}$ converging to 0, and a sequence $(\mathbf{u}_{\alpha_n})_{n \in \mathbb{N}}$ of solutions of (1.1)-(1.5) with the parameter α_n in (1.4). We prove that this sequence converges to a solution \mathbf{u} of (1.9)-(1.12).

Theorem 3.2. *Let $\eta \in [0; 1[$, $\lambda > 0$ and $\mathbf{f} \in L^2(\Omega)^3$. Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\alpha_n \xrightarrow{n \rightarrow +\infty} 0$, and $(\mathbf{u}_{\alpha_n})_{n \in \mathbb{N}}$ a sequence of functions satisfying for any $\mathbf{v} \in V_\eta$*

$$(3.32) \quad \int_\Omega (\bar{\omega}(\mathbf{u}_{\alpha_n}) \times \mathbf{u}_{\alpha_n}) \cdot \mathbf{v} + \int_\Omega \varrho^\eta \tilde{\nu} \nabla \mathbf{u}_{\alpha_n} : \nabla \mathbf{v} + \lambda \int_\Omega \mathbf{u}_{\alpha_n} \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v},$$

where $\bar{\omega}(\mathbf{u}_{\alpha_n})$ is obtained with the method detailed in section 3, with equation

$$(3.33) \quad -\alpha_n^2 \Delta \bar{\omega}(\mathbf{u}_{\alpha_n}) + \bar{\omega}(\mathbf{u}_{\alpha_n}) = \boldsymbol{\omega}(\mathbf{u}_{\alpha_n}).$$

- If $0 \leq \eta < 1/5$, a subsequence of $(\mathbf{u}_{\alpha_n})_{n \in \mathbb{N}}$ strongly converges in V_η to a function \mathbf{u} satisfying for any $\mathbf{v} \in V_\eta$

$$(3.34) \quad \int_{\Omega} (\boldsymbol{\omega}(\mathbf{u}) \times \mathbf{u}) \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla \mathbf{u} : \nabla \mathbf{v} + \lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

- If $1/5 \leq \eta < 1$, a subsequence of $(\mathbf{u}_{\alpha_n})_{n \in \mathbb{N}}$ converges at the distribution sense to a function $\mathbf{u} \in V_\eta$ satisfying (3.34) for any $\mathbf{v} \in \mathcal{V}$.

Proof. Let $\mathbf{u}_{\alpha_n} \in V_\eta$ satisfying (3.32). Taking $\mathbf{v} = \mathbf{u}_{\alpha_n}$ directly yields $\|\mathbf{u}_{\alpha_n}\|_{V_\eta} \leq M$, where $M > 0$ is defined in (3.14).

Therefore $(\mathbf{u}_{\alpha_n})_{n \in \mathbb{N}}$ is bounded, so there is a subsequence still named the same way which weakly converges in V_η to a certain function $\mathbf{u} \in V_\eta$. As in the proof of Theorem 3.1, we directly obtain for any $\mathbf{v} \in V_\eta$

$$(3.35) \quad \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla \mathbf{u}_{\alpha_n} : \nabla \mathbf{v} \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla \mathbf{u} : \nabla \mathbf{v},$$

$$(3.36) \quad \lambda \int_{\Omega} \mathbf{u}_{\alpha_n} \cdot \mathbf{v} \xrightarrow{n \rightarrow +\infty} \lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v}.$$

Thus, we only need to focus on the transport term, and we write a decomposition

$$(3.37) \quad \int_{\Omega} (\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) \times \mathbf{u}_{\alpha_n}) \cdot \mathbf{v} - \int_{\Omega} (\boldsymbol{\omega}(\mathbf{u}) \times \mathbf{u}) \cdot \mathbf{v} = \int_{\Omega} ((\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) - \boldsymbol{\omega}(\mathbf{u})) \times \mathbf{u}) \cdot \mathbf{v} \\ + \int_{\Omega} (\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) \times (\mathbf{u}_{\alpha_n} - \mathbf{u})) \cdot \mathbf{v}.$$

Case 1 ($0 \leq \eta < 1/5$) : we consider any test function $\mathbf{v} \in V_\eta$.

On one hand, $\mathbf{u}_{\alpha_n} \xrightarrow[n \rightarrow +\infty]{V_\eta} \mathbf{u}$ which implies $\boldsymbol{\omega}(\mathbf{u}_{\alpha_n}) \xrightarrow[n \rightarrow +\infty]{L^p} \boldsymbol{\omega}(\mathbf{u})$. Thus, Theorem 2.4 yields $\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) \xrightarrow[n \rightarrow +\infty]{L^p} \boldsymbol{\omega}(\mathbf{u})$. We deduce

$$(3.38) \quad \int_{\Omega} ((\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) - \boldsymbol{\omega}(\mathbf{u})) \times \mathbf{u}) \cdot \mathbf{v} \xrightarrow{n \rightarrow +\infty} 0.$$

On the other hand, taking s given by (3.6), we have

$$(3.39) \quad \left| \int_{\Omega} (\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) \times (\mathbf{u}_{\alpha_n} - \mathbf{u})) \cdot \mathbf{v} \right| \leq \|\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n})\|_{0,s,\Omega} \|\mathbf{u}_{\alpha_n} - \mathbf{u}\|_{0,r,\Omega} \|\mathbf{v}\|_{0,r,\Omega} \xrightarrow{n \rightarrow +\infty} 0,$$

with the same arguments as in Theorem 3.1.

Consequently, (3.37) yields

$$(3.40) \quad \int_{\Omega} (\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) \times \mathbf{u}_{\alpha_n}) \cdot \mathbf{v} \xrightarrow{n \rightarrow +\infty} \int_{\Omega} (\boldsymbol{\omega}(\mathbf{u}) \times \mathbf{u}) \cdot \mathbf{v}.$$

Passing to the limit in (3.32) shows that \mathbf{u} satisfies (3.34).

Now the strong convergence needs to be proved. Taking $\mathbf{v} = \mathbf{u}_{\alpha_n}$ in (3.32) and $\mathbf{v} = \mathbf{u}$ in (3.34) yields

$$(3.41) \quad \int_{\Omega} \varrho^\eta \tilde{\nu} |\nabla \mathbf{u}_{\alpha_n}|^2 + \lambda \int_{\Omega} |\mathbf{u}_{\alpha_n}|^2 = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}_{\alpha_n},$$

$$(3.42) \quad \int_{\Omega} \varrho^\eta \tilde{\nu} |\nabla \mathbf{u}|^2 + \lambda \int_{\Omega} |\mathbf{u}|^2 = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}.$$

We conclude with the same method as used in the proof of Theorem 3.1.

Case 2 ($1/5 \leq \eta < 1$) : we consider a test function $\mathbf{v} \in \mathcal{V}$.

The proof is the same as before, but \mathbf{u} and \mathbf{u}_{α_n} cannot be taken as test functions, so that the convergence is not strong. □

References

- [1] Chérif Amrouche, Guillaume Leloup, and Roger Lewandowski. The model involving the distance to the wall. part 1: the relaxed case. *Preprint on Hal*, 2023.
- [2] Cherif Amrouche and Mohand Moussaoui. The dirichlet and neumann problems in lipschitz and in $C^{1,1}$ domains. abstract. *arXiv e-prints*, pages arXiv–2204, 2022.
- [3] Luigi C Berselli, Argus A Dunca, Lewandowski Roger, Dinh Duong Nguyen, et al. Modeling error of alpha-models of turbulence on a two-dimensional torus. *Discrete and continuous dynamical systems. Series B*, 26(9):4613–4643, 2021.
- [4] Luigi C Berselli and Roger Lewandowski. Convergence of approximate deconvolution models to the mean navier–stokes equations. In *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*, volume 29, pages 171–198. Elsevier, 2012.
- [5] Yanping Cao, Evelyn M Lunasin, and Edriss S Titi. Global well-posedness of the three-dimensional viscous and inviscid simplified bardina turbulence models. *Communications in Mathematical Sciences*, 4(4):823–848, 2006.
- [6] Alexey Cheskidov, Darryl D Holm, Eric Olson, and Edriss S Titi. On a leray- α model of turbulence. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 461(2055):629–649, 2005.
- [7] Ciprian Foias, Darryl D Holm, and Edriss S Titi. The navier–stokes-alpha model of fluid turbulence. *Physica D: Nonlinear Phenomena*, 152:505–519, 2001.
- [8] JD Gibbon and DD Holm. Estimates for the lans- α , leray- α and bardina models in terms of a navier-stokes reynolds number. *Indiana University Mathematics Journal*, pages 2761–2773, 2008.
- [9] Alexei A Ilyin, Evelyn M Lunasin, and Edriss S Titi. A modified-leray- α subgrid scale model of turbulence. *Nonlinearity*, 19(4):879, 2006.
- [10] A. Kufner. *Weighted Sobolev spaces*. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985. Translated from the Czech.
- [11] W Layton and Roger Lewandowski. On a well-posed turbulence model. *Discrete and continuous dynamical systems series B*, 6(1):111, 2006.
- [12] William Layton. Existence of smooth attractors for the navier–stokes-omega model of turbulence. *Journal of mathematical analysis and applications*, 366(1):81–89, 2010.
- [13] William Layton and Roger Lewandowski. Residual stress of approximate deconvolution models of turbulence. *Journal of Turbulence*, (7):N46, 2006.

- [14] William Layton, Carolina C Manica, Monika Neda, and Leo G Rebholz. Numerical analysis and computational comparisons of the ns-alpha and ns-omega regularizations. *Computer Methods in Applied Mechanics and Engineering*, 199(13-16):916–931, 2010.
- [15] Jean Leray. Essai sur les mouvements plans d’un liquide visqueux que limitent des parois. *Journal de Mathématiques pures et appliquées*, 13:331–418, 1934.
- [16] Jean Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta mathematica*, 63:193–248, 1934.
- [17] Roger Lewandowski and Luigi C Berselli. On the bardina’s model in the whole space. *Journal of Mathematical Fluid Mechanics*, 20:1335–1351, 2018.
- [18] Evelyn Lunasin, Susan Kurien, and Edriss S Titi. Spectral scaling of the leray- α model for two-dimensional turbulence. *Journal of Physics A: Mathematical and Theoretical*, 41(34):344014, 2008.
- [19] Eric Olson and Edriss S Titi. Viscosity versus vorticity stretching: global well-posedness for a family of navier–stokes-alpha-like models. *Nonlinear Analysis: Theory, Methods & Applications*, 66(11):2427–2458, 2007.
- [20] Ludwig Prandtl. Mutual influence of wings and propeller. Technical report, 1921.