



HAL
open science

Vorticity Leray- α model for Navier-Stokes equations with viscosity depending on the distance to the wall

Guillaume Leloup

► **To cite this version:**

Guillaume Leloup. Vorticity Leray- α model for Navier-Stokes equations with viscosity depending on the distance to the wall. 2024. hal-04424189

HAL Id: hal-04424189

<https://hal.science/hal-04424189>

Preprint submitted on 29 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

Vorticity Leray- α model for Navier-Stokes equations with viscosity depending on the distance to the wall.

Guillaume LELOUP

Univ Rennes, IRMAR, UMR CNRS 6625, and Odyssey Team, INRIA Rennes,
France

E-mail: guillaume.leloup@univ-rennes.fr

Abstract

We introduce a vorticity Leray- α model with eddy viscosity depending on $d(x, \partial\Omega)^\eta$ where $\partial\Omega$ is the boundary of the domain and $\eta \in]0; 1[$. We prove that this system admits fairly regular weak solutions converging when α goes to 0 to the solution of a reference system.

Key words : Fluid mechanics, Navier-Stokes equations, Leray- α models, Vorticity.

2010 MSC : 35Q30, 76D03, 76D05, 35A35, 35B33.

1 Introduction

Let Ω be a \mathcal{C}^2 open bounded subset of \mathbb{R}^3 . According to turbulence modelling, we introduce in this paper the following vorticity Leray- α model

$$(1.1) \quad \bar{\omega} \times \mathbf{u} - \operatorname{div}(\varrho^\eta \tilde{\nu} \nabla \mathbf{u}) + \lambda \mathbf{u} + \nabla \tilde{p} = \mathbf{f} \quad \text{in } \Omega,$$

$$(1.2) \quad \operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega,$$

$$(1.3) \quad \boldsymbol{\omega} = \operatorname{curl}(\mathbf{u}) \quad \text{in } \Omega,$$

$$(1.4) \quad -\alpha^2 \Delta \bar{\omega} + \bar{\omega} = P(\boldsymbol{\omega}) \quad \text{in } \mathbb{R}^3,$$

$$(1.5) \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

where \mathbf{u} models the long time average of the flow, $\tilde{p} = p - \frac{|\mathbf{u}|^2}{2}$ is the modified mean pressure, and $\mathbf{f} \in L^2(\Omega)^3$ is a given source term. The linear term $\lambda \mathbf{u}$ with $\lambda > 0$ is a damping term.

The eddy viscosity is of the form $\nu_{\text{turb}} = \varrho^\eta \tilde{\nu}$, where $0 < \eta < 1$ and $\tilde{\nu}$ is a continuous function satisfying

$$(1.6) \quad \forall t \in \mathbb{R}, \quad 0 < \nu_m \leq \tilde{\nu}(t) \leq \nu_M,$$

where ν_m and ν_M are constants.

The function ϱ refers to the Prandtl mixing length (see [20]), which appears like a regular function behaving like the distance to the boundary $d(\cdot, \partial\Omega)$. More precisely, we consider $\varrho : \Omega \rightarrow \mathbb{R}_+$ such that

$$(1.7) \quad \lim_{\substack{d(x, \partial\Omega) \rightarrow 0 \\ x \in \Omega}} \frac{\varrho(x)}{d(x, \partial\Omega)} = 1 \quad \text{and} \quad \forall n > 0, \quad \varrho_n = \inf_{\substack{d(x, \partial\Omega) \geq \frac{1}{n} \\ x \in \Omega}} \varrho(x) > 0.$$

The operator P in (1.4) is the extension by 0, which means

$$(1.8) \quad P(\boldsymbol{\omega}) : x \mapsto \begin{cases} \boldsymbol{\omega}(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Equation (1.4) defining $\bar{\boldsymbol{\omega}}$ is the Helmholtz equation (studied in detail for instance in [7, 13]), so that (1.1)-(1.5) is a kind of Leray- α model, which is a LES model.

In this paper we show that (1.1)-(1.5) with $\eta < 1$ admits sufficiently regular weak solutions satisfying the energy balance (3.2) (Theorem 3.1) that converge when α goes to zero to a solution of the following system

$$(1.9) \quad \boldsymbol{\omega} \times \mathbf{u} - \operatorname{div}(\varrho^\eta \tilde{\nu} \nabla \mathbf{u}) + \lambda \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega,$$

$$(1.10) \quad \operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega,$$

$$(1.11) \quad \boldsymbol{\omega} = \operatorname{curl}(\mathbf{u}) \quad \text{in } \Omega,$$

$$(1.12) \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega.$$

As previously shown in [1], (1.9)-(1.12) admits weak solutions for $0 \leq \eta < 1$, but the energy balance (3.2) can be satisfied if and only if $0 \leq \eta < 1/5$. Consequently, the convergence stated in Theorem 3.2 is weak when $1/5 \leq \eta < 1$ and strong if $0 \leq \eta < 1/5$.

Leray- α models were first introduced by J. Leray in [15, 16], and have been studied in detail in [3, 4, 6, 8, 9, 18, 19]. In this kind of models, the convection term is given by $(\bar{\mathbf{u}} \cdot \nabla) \mathbf{u}$.

By the way, it is possible to apply the filter to both velocities, so that the convection term becomes $\overline{(\mathbf{u} \cdot \nabla) \mathbf{u}}$. This is called Bardina-type models. These have been studied for instance in [3, 5, 8, 11, 17]. In particular, it is shown that solutions of the α -models are regular and converge to the Leray solution of NSE, most of time with periodic boundary conditions. A comparison between these different models is made for instance in [3, 6, 8, 18].

In order to ensure coercivity results, a free-divergence assumption is required for $\bar{\mathbf{u}}$. Consequently, the standard Leray- α model adapted to (1.1) should be like

$$(1.13) \quad (\bar{\mathbf{u}} \cdot \nabla) \mathbf{u} - \operatorname{div}(\varrho^\eta \tilde{\nu} \nabla \mathbf{u}) + \lambda \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega,$$

$$(1.14) \quad -\alpha^2 \Delta \bar{\mathbf{u}} + \bar{\mathbf{u}} + \nabla q = \mathbf{u} \quad \text{in } \Omega,$$

$$(1.15) \quad \operatorname{div}(\mathbf{u}) = \operatorname{div}(\bar{\mathbf{u}}) = 0 \quad \text{in } \Omega,$$

$$(1.16) \quad \mathbf{u} = 0 \quad \text{on } \Gamma.$$

Unfortunately, this model does not provide sufficient results to ensure the existence of suitable solutions. Indeed, we work in weighted function spaces, which only provides a $W^{1,p}$ regularity for \mathbf{u} , with $1 \leq p < 2$. This regularity is not enough, because of the Lagrange multiplier in (1.14).

This is why the vorticity has been introduced. This quantity appears in [8, 19] to introduce an α -model called NS- α . In this model, the vorticity term $\text{curl}(\mathbf{u}) \times \mathbf{u}$ is replaced by $\text{curl}(\bar{\mathbf{u}}) \times \mathbf{u}$.

Here we consider another version of this model, similar to the $NS - \bar{\omega}$ model introduced by W.J. Layton and al. in [12, 14], where the vorticity term of (1.9)-(1.12) becomes $\text{curl}(\bar{\mathbf{u}}) \times \mathbf{u}$. A theoretical study is made in [12], for a system of NSE with constant viscosity. This idea is relevant given that there is no free-divergence assumption required to prove coercivity in the calculations.

The paper is organized as follows. In section 2, we focus on the vorticity Helmholtz equation with a L^p source term. The goal is proving some convergence results that are applied later.

In section 3, we start by showing the existence of a distributional solution of (1.1)-(1.5) using a fixed-point strategy, and then we prove the convergence to a solution of (1.9)-(1.12) when α goes to zero. The value of η has an influence on the sense of the convergence.

I would like to express my gratitude to my PhD advisor Professor Roger LEWANDOWSKI to have suggested this problem and guided me during this work.

2 Vorticity Leray- α model

This section starts by defining the function spaces in which solutions of (1.9)-(1.12) are searched. As already done in [1], some Sobolev regularity is given on these spaces.

Then, the solution of the Helmholtz vorticity equation is written with a convolution kernel, to ease the calculations in order to find results on $\bar{\omega}$ starting from equivalent statements on ω .

2.1 Functions spaces and estimates

We first define a space of smooth functions

$$(2.1) \quad \mathcal{V} = \{\mathbf{v} \in \mathcal{D}(\Omega)^3 / \text{div}(\mathbf{v}) = 0\},$$

and V_η is the closure of this space for the norm

$$(2.2) \quad \|\mathbf{u}\|_{V_\eta} = \left(\|\mathbf{u}\|_{0,2,\Omega}^2 + \|\varrho^{\eta/2} \nabla \mathbf{u}\|_{0,2,\Omega}^2 \right)^{1/2}.$$

We start by stating the regularity of V_η functions. Indeed, looking at the norm (2.2), we understand that such functions are not likely to be H^1 if $\eta > 0$.

We recall two embedding theorems. The first is from [1, 2] brings a H^s estimate, with $1/2 < s \leq 1$.

Theorem 2.1. *Let $\eta \in [0; 1[$. The continuous embedding*

$$(2.3) \quad V_\eta \hookrightarrow H^{1-\frac{\eta}{2}}(\Omega)^3$$

stands, which implies the Sobolev compact embedding

$$(2.4) \quad V_\eta \hookrightarrow L^r(\Omega)^3, \quad \text{where } 1 \leq r < \frac{6}{1+\eta}.$$

The second result from [10] brings a $W^{1,p}$ estimate with $1 \leq p < 2$.

Theorem 2.2. *Let $\eta \in [0; 1[$. The continuous embedding*

$$(2.5) \quad V_\eta \hookrightarrow W^{1,p}(\Omega)^3$$

stands for $p \in \left[1; \frac{2}{1+\eta}\right]$.

Remark 2.1. *The compact Sobolev embedding given by Theorem 2.2 is weaker than this given by Theorem 2.1. Indeed, $\frac{6}{1+3\eta}$ is lower than $\frac{6}{1+\eta}$.*

Remark 2.2. *These two results also give an estimate on the vorticity.*

- *If $\mathbf{u} \in H^{1-\frac{\eta}{2}}(\Omega)^3$, then $\boldsymbol{\omega} = \text{curl}(\mathbf{u}) \in H^{-\frac{\eta}{2}}(\Omega)^3$.*
- *If $\mathbf{u} \in W^{1,p}(\Omega)^3$, then $\boldsymbol{\omega} = \text{curl}(\mathbf{u}) \in L^p(\Omega)^3$.*

2.2 Green kernel for the Helmholtz problem

Until the end of this section, we simply write $\boldsymbol{\omega}$ instead of $P(\boldsymbol{\omega})$. This means

$$(2.6) \quad \boldsymbol{\omega} \in L^p(\mathbb{R}^3)^3 \quad \text{with} \quad 1 \leq p < \frac{2}{1+\eta} \quad \text{and} \quad 0 \leq \eta \leq 1.$$

We focus on the following Helmholtz equation, where $\alpha > 0$ is a constant parameter, supposed to be near 0.

$$(2.7) \quad -\alpha^2 \Delta \bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\omega}} = \boldsymbol{\omega}.$$

Theorem 2.3 (Green function). *Let $\boldsymbol{\omega} \in L^p(\mathbb{R}^3)^3$ defined in (2.6), and $\bar{\boldsymbol{\omega}}$ satisfying (2.7). Then*

$$(2.8) \quad \bar{\boldsymbol{\omega}}(x) = K_\alpha * \boldsymbol{\omega}(x)$$

for almost every $x \in \mathbb{R}^3$, where K_α is the scalar function given by

$$(2.9) \quad \begin{aligned} K_\alpha : \mathbb{R}^3 &\rightarrow \mathbb{R} \\ x &\mapsto \frac{e^{-|x|/\alpha}}{4\pi\alpha^2|x|}. \end{aligned}$$

This result is already proved in [17].

Lemma 2.1. *The function K_α given by (2.9) is in $L^q(\mathbb{R}^3, \mathbb{R}_+)$ for $q \in [1; 3[$ and*

$$(2.10) \quad \int_{\mathbb{R}^3} K_\alpha(x)^q dx = \Gamma(3-q) \frac{(4\pi\alpha^3)^{1-q}}{q^{3-q}},$$

where Γ refers to the Euler function: $\Gamma(t) = \int_0^{+\infty} s^{t-1} e^{-s} ds$, for any $t > 0$.

Remark 2.3. *When $q = 1$, we obtain in particular*

$$(2.11) \quad \int_{\mathbb{R}^3} K_\alpha(x) dx = 1.$$

Proof. We use a change of variables which transforms the classical cartesian coordinates of \mathbb{R}^3 into spherical coordinates $(r, \theta, \varphi) \in \mathbb{R}_+ \times [0; \pi] \times [0; 2\pi]$, so that the 3D Lebesgue measure is written

$$(2.12) \quad d\lambda(y) = r^2 \sin(\theta) dr d\theta d\varphi.$$

The Fubini-Tonelli theorem then yields

$$(2.13) \quad \begin{aligned} \iiint_{\mathbb{R}^3} K_\alpha(y)^q dy &= \int_{r=0}^{+\infty} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \frac{e^{-qr/\alpha}}{(4\pi)^q \alpha^{2q} r^q} r^2 \sin(\theta) dr d\theta d\varphi \\ &= \frac{(4\pi)^{1-q}}{\alpha^{2q}} \int_0^{+\infty} r^{2-q} e^{-qr/\alpha} dr. \end{aligned}$$

The change of variables $s = \frac{qr}{\alpha}$ yields

$$(2.14) \quad \int_0^{+\infty} r^{2-q} e^{-qr/\alpha} dr = \left(\frac{\alpha}{q}\right)^{3-q} \int_0^{+\infty} s^{2-q} e^{-s} ds.$$

This integral is defined if and only if $2 - q > -1$, which means $q < 3$. When this condition is satisfied, (2.13) yields

$$(2.15) \quad \int_{\mathbb{R}^3} K_\alpha(y)^q dy = \frac{(4\pi)^{1-q}}{\alpha^{2q}} \left(\frac{\alpha}{q}\right)^{3-q} \int_0^{+\infty} s^{2-q} e^{-s} ds,$$

which corresponds to (2.10). □

Since the vector function $\bar{\omega}$ is defined as a convolution, the Young inequality gives some regularity results for $\bar{\omega}$. Indeed, the extension by 0 preserves the L^p norms.

Lemma 2.2 (L^γ -regularity for $\bar{\omega}$). *Let $\omega \in L^p(\mathbb{R}^3)^3$ with p defined by (2.6), and $K_\alpha \in L^q(\mathbb{R}^3)$ with $q \in [1; 3[$.*

- If $0 \leq \eta < 1/3$, $\bar{\omega} \in L^\gamma(\mathbb{R}^3)^3$ with $1 \leq \gamma \leq +\infty$.
- If $\eta = 1/3$, $\bar{\omega} \in L^\gamma(\mathbb{R}^3)^3$ with $1 \leq \gamma < +\infty$.
- If $1/3 < \eta \leq 1$, $\bar{\omega} \in L^\gamma(\mathbb{R}^3)^3$ with $1 \leq \gamma < \frac{6}{3\eta - 1}$.

Proof. Since $K_\alpha \in L^q(\mathbb{R}^3)$ and $\omega \in L^p(\mathbb{R}^3)^3$, the Young inequality yields

$$(2.16) \quad \bar{\omega} \in L^\gamma(\mathbb{R}^3)^3 \quad \text{with} \quad 1 + \frac{1}{\gamma} = \frac{1}{p} + \frac{1}{q}.$$

The assumptions on p and q yield

$$(2.17) \quad \frac{1 + \eta}{2} + \frac{1}{3} < \frac{1}{p} + \frac{1}{q} \leq 2,$$

which means

$$(2.18) \quad \frac{\eta}{2} - \frac{1}{6} < \frac{1}{\gamma} \leq 1.$$

We see that $\frac{\eta}{2} - \frac{1}{6}$ is always strictly lower than 1, and it is positive if and only if $\eta > 1/3$. Consequently, when $\eta \leq 1/3$ we directly obtain the result. And if $\eta > 1/3$, (2.18) yields

$$(2.19) \quad 1 \leq \gamma < \frac{6}{3\eta - 1}.$$

Hence the result. □

2.3 Main convergence results

The two next results are the main statements of this paper. First, we consider the case where α is fixed. The following lemma will be used to find solutions of the approximated system (1.1)-(1.5), by a fixed-point method.

Lemma 2.3 (Convergence when α is fixed). *Let $(\omega_n)_{n \in \mathbb{N}}$ be a sequence of $L^p(\mathbb{R}^3)^3$ functions ($1 \leq p < +\infty$) that weakly converges to $\omega \in L^p(\mathbb{R}^3)^3$. Let $(\bar{\omega}_n)_{n \in \mathbb{N}}$ be the sequence of the solutions of (2.7) with source term ω_n , and $\bar{\omega}$ the solution of (2.7) with source term ω .*

Then $(\bar{\omega}_n)_{n \in \mathbb{N}}$ weakly converges in $L^p(\mathbb{R}^3)^3$ to $\bar{\omega}$.

Proof. Let $\phi \in L^{p'}(\mathbb{R}^3)$, the Fubini theorem yields

$$(2.20) \quad \begin{aligned} \int_{\mathbb{R}^3} \bar{\omega}_n(x) \phi(x) dx &= \iint_{x, y \in \mathbb{R}^3} \omega_n(y) K_\alpha(x-y) \phi(x) dx dy \\ &= \int_{y \in \mathbb{R}^3} \omega_n(y) \left(\int_{x \in \mathbb{R}^3} K_\alpha(x-y) \phi(x) dx \right) dy. \end{aligned}$$

Since $\phi \in L^{p'}(\mathbb{R}^3)$ and $K_\alpha \in L^1(\mathbb{R}^3)$, the function $[y \mapsto \int_{\mathbb{R}^3} K_\alpha(x-y) \phi(x) dx]$ is in $L^{p'}(\mathbb{R}^3)$. Hence the weak convergence of $(\omega_n)_{n \in \mathbb{N}}$ yields the result. \square

The following theorem gives the convergence of the solutions $\bar{\omega}$ of (2.7) when α goes to zero. This will be used later to show the convergence of a sequence of solutions of (1.1)-(1.5) to a solution of (1.9)-(1.12).

Theorem 2.4. *Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\alpha_n \xrightarrow{n \rightarrow +\infty} 0$, and p be defined by (2.6). Let $(\omega_{\alpha_n})_{n \in \mathbb{N}}$ be a sequence of $L^p(\mathbb{R}^3)^3$ functions that weakly converges to $\omega \in L^p(\mathbb{R}^3)^3$.*

Let $(\bar{\omega}_{\alpha_n})_{n \in \mathbb{N}}$ satisfy for all $n \in \mathbb{N}$

$$(2.21) \quad -\alpha_n^2 \Delta \bar{\omega}_{\alpha_n} + \bar{\omega}_{\alpha_n} = \omega_{\alpha_n}.$$

Then $(\bar{\omega}_{\alpha_n})_{n \in \mathbb{N}}$ weakly converges to ω in $L^p(\mathbb{R}^3)^3$.

Proof. Let $\phi \in \mathcal{D}(\mathbb{R}^3)$, by the Fubini theorem

$$(2.22) \quad \begin{aligned} \int_{\mathbb{R}^3} \bar{\omega}_{\alpha_n}(x) \phi(x) dx &= \iint_{x, y \in \mathbb{R}^3} \omega_{\alpha_n}(y) K_{\alpha_n}(x-y) \phi(x) dx dy \\ &= \int_{y \in \mathbb{R}^3} \omega_{\alpha_n}(y) \left(\int_{x \in \mathbb{R}^3} K_{\alpha_n}(x-y) \phi(x) dx \right) dy. \end{aligned}$$

The change of variables $z = \frac{x-y}{\alpha_n}$ in the middle integral yields

$$(2.23) \quad \int_{\mathbb{R}^3} K_{\alpha_n}(x-y) \phi(x) dx = \int_{\mathbb{R}^3} \frac{e^{-|x-y|/\alpha_n}}{4\pi\alpha_n^2|x-y|} \phi(x) dx = \int_{\mathbb{R}^3} \frac{e^{-|z|}}{4\pi|z|} \phi(y + \alpha_n z) dz.$$

Then (2.22) becomes

$$(2.24) \quad \int_{\mathbb{R}^3} \bar{\omega}_{\alpha_n}(x) \phi(x) dx = \int_{z \in \mathbb{R}^3} K_1(z) \left(\int_{y \in \mathbb{R}^3} \omega_{\alpha_n}(y) \phi(y + \alpha_n z) dy \right) dz.$$

We now aim at showing the following convergence for any fixed $z \in \mathbb{R}^3$

$$(2.25) \quad \int_{\mathbb{R}^3} \omega_{\alpha_n}(y) \phi(y + \alpha_n z) dy \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^3} \omega(y) \phi(y) dy.$$

To do so, we decompose the difference

$$(2.26) \quad \int_{\mathbb{R}^3} \omega_{\alpha_n}(y) \phi(y + \alpha_n z) dy - \int_{\mathbb{R}^3} \omega(y) \phi(y) dy = \int_{\mathbb{R}^3} (\omega_{\alpha_n}(y) - \omega(y)) \phi(y) dy \\ + \int_{\mathbb{R}^3} \omega_{\alpha_n}(y) (\phi(y + \alpha_n z) - \phi(y)) dy.$$

On one hand, the weak convergence of $(\omega_{\alpha_n})_{n \in \mathbb{N}}$ yields

$$(2.27) \quad \int_{\mathbb{R}^3} (\omega_{\alpha_n}(y) - \omega(y)) \phi(y) dy \xrightarrow{n \rightarrow +\infty} 0.$$

On the other hand, the Hölder inequality yields

$$(2.28) \quad \int_{\mathbb{R}^3} \omega_{\alpha_n}(y) (\phi(y + \alpha_n z) - \phi(y)) dy \leq \|\omega_{\alpha_n}\|_p \left(\int_{\mathbb{R}^3} |\phi(y + \alpha_n z) - \phi(y)|^{p'} dy \right)^{1/p'} \\ \xrightarrow{n \rightarrow +\infty} 0,$$

by the Lebesgue theorem. Hence (2.25).

Then the following domination stands:

$$(2.29) \quad \left| K_1(z) \left(\int_{y \in \mathbb{R}^3} \omega_{\alpha_n}(y) \phi(y + \alpha_n z) dy \right) \right| \leq K_1(z) \|\omega_{\alpha_n}\|_p \|\phi\|_{p'},$$

which is $L^1(\mathbb{R}^3)$.

Therefore by the Lebesgue theorem, (2.25) and (2.29)

$$(2.30) \quad \int_{\mathbb{R}^3} \bar{\omega}_{\alpha_n}(x) \phi(x) dx \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^3} \omega(y) \phi(y) dy.$$

The density of $\mathcal{D}(\mathbb{R}^3)$ in $L^{p'}(\mathbb{R}^3)$ yields the result. □

3 Solutions of the Leray- α approximated system

This section aims at proving the existence of solutions of (1.9)-(1.12), in three steps. First, we solve a linearized system by the Lax-Milgram theorem. Such a solution is unique, and can be written $T\mathbf{u}$. Then the existence of solutions of (1.1)-(1.5) is shown by using the Leray-Schauder fixed-point theorem on the operator T .

We then obtain a sequence $(\mathbf{u}_{\alpha_n})_{n \in \mathbb{N}}$ of solutions, corresponding to a sequence of parameters $(\alpha_n)_{n \in \mathbb{N}}$ converging to 0. Finally, Theorem 2.4 provides the convergence to a solution of (1.9)-(1.12).

3.1 Variational formulation

The function $\bar{\omega}$ in (1.1)-(1.5) is obtained this way.

- We consider the extension by 0 of $\omega \in L^p(\Omega)^3$, still named the same,
- $\bar{\omega}$ satisfies (2.7), so that $\bar{\omega} \in L^\gamma(\mathbb{R}^3)^3$, where γ is given by Corollary 2.2.
- we take the restriction of $\bar{\omega}$ to Ω , still named the same.

We write the variational formulation of system (1.1)-(1.5)

$$(3.1) \quad \begin{cases} \mathbf{u} \in V_\eta, \\ \forall \mathbf{v} \in V_\eta, \quad \int_\Omega (\bar{\boldsymbol{\omega}}(\mathbf{u}) \times \mathbf{u}) \cdot \mathbf{v} + \int_\Omega \varrho^\eta \tilde{\nu} \nabla \mathbf{u} : \nabla \mathbf{v} + \lambda \int_\Omega \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}. \end{cases}$$

Taking $\mathbf{v} = \mathbf{u}$ in (3.1) yields the following energy balance

$$(3.2) \quad \int_\Omega \varrho^\eta \tilde{\nu} |\nabla \mathbf{u}|^2 + \lambda \int_\Omega |\mathbf{u}|^2 = \int_\Omega \mathbf{f} \cdot \mathbf{u}.$$

These two equations have a sense if and only if the integrals are defined. The condition is given by the following lemma.

Lemma 3.1 (Critical exponents). *Let $\mathbf{f} \in L^2(\Omega)^3$ and $\lambda > 0$. For any $\eta \in]0; 1[$, all the integrals in (3.1) are well defined.*

Proof. The definition of V_η directly yields the three last integrals in (3.1) are finite. In order to show that $\int_\Omega (\bar{\boldsymbol{\omega}} \times \mathbf{u}) \cdot \mathbf{v}$ is finite for $\mathbf{u}, \mathbf{v} \in V_\eta$, we have to distinguish the two cases of Lemma 2.2.

If $\eta \leq 1/3$, $\bar{\boldsymbol{\omega}} \in L^\infty(\Omega)^3$ and $\mathbf{u}, \mathbf{v} \in L^r(\Omega)^3$ for $1 \leq r < \frac{6}{1+\eta}$. The Hölder inequality then yields

$$(3.3) \quad \int_\Omega |(\bar{\boldsymbol{\omega}} \times \mathbf{u}) \cdot \mathbf{v}| \leq \|\bar{\boldsymbol{\omega}}\|_{0,\infty,\Omega} \|\mathbf{u}\|_{0,r,\Omega} \|\mathbf{v}\|_{0,r,\Omega}, \quad \text{if } \frac{2}{r} = 1.$$

This condition can be fulfilled if and only if $\frac{6}{1+\eta} > 2$, which means $\eta < 2$. This is the case, since $\eta \leq 1/3$.

If, $\eta > 1/3$, $\bar{\boldsymbol{\omega}} \in L^s(\Omega)^3$ with $1 \leq s < \frac{6}{3\eta-1}$. The functions \mathbf{u} and \mathbf{v} have the same regularity as before. The Hölder inequality yields

$$(3.4) \quad \int_\Omega |(\bar{\boldsymbol{\omega}} \times \mathbf{u}) \cdot \mathbf{v}| \leq \|\bar{\boldsymbol{\omega}}\|_{0,s,\Omega} \|\mathbf{u}\|_{0,r,\Omega} \|\mathbf{v}\|_{0,r,\Omega}, \quad \text{if } \frac{1}{s} + \frac{2}{r} = 1.$$

This equality of exponents can be fulfilled if and only if $\frac{3\eta-1}{6} + \frac{1+\eta}{3} < 1$, which is equivalent to $\eta < 1$. \square

Now we can initiate the method explained at the beginning of this section.

3.2 Solutions for the vorticity Leray- α system

The variational problem (3.1) is linearized by considering a vorticity field \mathbf{w} which does not depend on \mathbf{u} .

$$(3.5) \quad \begin{cases} \mathbf{u} \in V_\eta, \\ \forall \mathbf{v} \in V_\eta, \quad \int_\Omega (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} + \int_\Omega \varrho^\eta \tilde{\nu} \nabla \mathbf{u} : \nabla \mathbf{v} + \lambda \int_\Omega \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}, \end{cases}$$

Lemma 3.2 (Linearized problem). *Let $\eta \in [0; 1[$, $\mathbf{f} \in L^2(\Omega)$, $\lambda > 0$ and $\mathbf{w} \in L^s(\Omega)$ for*

$$(3.6) \quad \begin{cases} 1 \leq s \leq +\infty & \text{if } \eta < 1/3, \\ 1 \leq s < +\infty & \text{if } \eta = 1/3, \\ 1 \leq s < \frac{6}{3\eta-1} & \text{if } \eta > 1/3. \end{cases}$$

The variational problem (3.5) admits a unique solution.

Proof. The Lax-Milgram Theorem is applied to the bilinear form $B(\mathbf{u}, \mathbf{v}) = A(\mathbf{u}, \mathbf{v}) + b_{\mathbf{w}}(\mathbf{u}, \mathbf{v})$, where A and $b_{\mathbf{w}}$ are defined by

$$(3.7) \quad A(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla \mathbf{u} : \nabla \mathbf{v} + \lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v},$$

$$(3.8) \quad b_{\mathbf{w}}(\mathbf{u}, \mathbf{v}) = \int_{\Omega} (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}.$$

First, we write

$$(3.9) \quad |A(\mathbf{u}, \mathbf{v})| \leq (\lambda + \nu_M) \|\mathbf{u}\|_{V_\eta} \|\mathbf{v}\|_{V_\eta}.$$

Let $r \in \left[1; \frac{6}{1+\eta}\right]$, we have

$$(3.10) \quad |b_{\mathbf{w}}(\mathbf{u}, \mathbf{v})| \leq c \|\mathbf{w}\|_{0,s,\Omega} \|\mathbf{u}\|_{0,r,\Omega} \|\mathbf{v}\|_{0,r,\Omega} \leq C \|\mathbf{u}\|_{V_\eta} \|\mathbf{v}\|_{V_\eta}.$$

Inequalities (3.9) and (3.10) yield the continuity of the bilinear form B on V_η . Let us show it is coercive, by considering any $\mathbf{v} \in V_\eta$.

$$(3.11) \quad B(\mathbf{v}, \mathbf{v}) = \int_{\Omega} \varrho^\eta \tilde{\nu} |\nabla \mathbf{v}|^2 + \lambda \int_{\Omega} |\mathbf{v}|^2 \geq \min(\nu_m, \lambda) \|\mathbf{v}\|_{V_\eta}^2.$$

Finally, the linear form $\ell : \mathbf{v} \mapsto \int_{\Omega} \mathbf{f} \cdot \mathbf{v}$ is continuous on V_η , since $\mathbf{f} \in L^2(\Omega)^3$.

This concludes the proof. \square

Lemma 3.2 yields the existence of an operator T associating to any function $\mathbf{u} \in V_\eta$ the solution of the linearized problem where $\mathbf{w} = \bar{\omega}(\mathbf{u})$, which means

$$(3.12) \quad \forall \mathbf{v} \in V_\eta, \quad \int_{\Omega} (\bar{\omega}(\mathbf{u}) \times T\mathbf{u}) \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla T\mathbf{u} : \nabla \mathbf{v} + \lambda \int_{\Omega} T\mathbf{u} \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

Replacing \mathbf{v} by $T\mathbf{u}$ in (3.12) yields

$$(3.13) \quad \int_{\Omega} \varrho^\eta \tilde{\nu} |\nabla T\mathbf{u}|^2 + \lambda \int_{\Omega} |T\mathbf{u}|^2 = \int_{\Omega} \mathbf{f} \cdot T\mathbf{u},$$

that yields the bound

$$(3.14) \quad \|T\mathbf{u}\|_{V_\eta} \leq \frac{\|\mathbf{f}\|_{0,2,\Omega}}{\min(\nu_m, \lambda)} =: M.$$

Then we use the Leray-Schauder fixed point theorem to prove the following result.

Theorem 3.1. *Let $\mathbf{f} \in L^2(\Omega)^3$ and $\lambda > 0$. The operator T defined by (3.12) has a fixed point in the closed ball $\bar{B}_{V_\eta}(0, M)$ simply named B . This means that (3.1) admits a solution in B .*

Proof. The fixed point theorem will be applied on B , which is a nonempty closed bounded convex subset of V_η . The continuity of T and compactness of $T(B)$ need to be proved. To do so, we consider a sequence $(\mathbf{u}_n)_{n \in \mathbb{N}}$ of functions in B that weakly converges to a certain \mathbf{u} in V_η . The goal is showing that $(T\mathbf{u}_n)_{n \in \mathbb{N}}$ strongly converges to $T\mathbf{u}$ in V_η . We recall that $T\mathbf{u}_n$ is defined by

$$(3.15) \quad \forall \mathbf{v} \in V_\eta, \quad \int_{\Omega} (\bar{\omega}(\mathbf{u}_n) \times T\mathbf{u}_n) \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla T\mathbf{u}_n : \nabla \mathbf{v} + \lambda \int_{\Omega} T\mathbf{u}_n \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

Since $(T\mathbf{u}_n)_{n \in \mathbb{N}}$ is bounded in V_η , a subsequence still named the same way weakly converges to a certain φ in V_η . The first step of the proof consists in showing that $\varphi = T\mathbf{u}$.

The weak convergence and the compact Sobolev embedding (2.4) directly yield for any $\mathbf{v} \in V_\eta$

$$(3.16) \quad \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla T\mathbf{u}_n : \nabla \mathbf{v} \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla \varphi : \nabla \mathbf{v},$$

$$(3.17) \quad \lambda \int_{\Omega} T\mathbf{u}_n \cdot \mathbf{v} \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \varphi \cdot \mathbf{v}.$$

For the vorticity term, we write the decomposition

$$(3.18) \quad \int_{\Omega} (\bar{\omega}(\mathbf{u}_n) \times T\mathbf{u}_n) \cdot \mathbf{v} - \int_{\Omega} (\bar{\omega}(\mathbf{u}) \times \varphi) \cdot \mathbf{v} = \int_{\Omega} ((\bar{\omega}(\mathbf{u}_n) - \bar{\omega}(\mathbf{u})) \times \varphi) \cdot \mathbf{v} \\ + \int_{\Omega} (\bar{\omega}(\mathbf{u}_n) \times (T\mathbf{u}_n - \varphi)) \cdot \mathbf{v}.$$

On one hand, the assumption and Theorem 2.2 imply $\mathbf{u}_n \xrightarrow{n \rightarrow +\infty} \mathbf{u}$ in $W^{1,p}(\Omega)^3$ for

$$(3.19) \quad 1 \leq p < \frac{2}{1+\eta}, \quad \text{with } 0 \leq \eta < 1.$$

This yields $\text{curl}(\mathbf{u}_n) \xrightarrow{n \rightarrow +\infty} \text{curl}(\mathbf{u})$ in $L^p(\Omega)^3$, and the extensions by 0 satisfy

$$(3.20) \quad \omega(\mathbf{u}_n) \xrightarrow{n \rightarrow +\infty} \omega(\mathbf{u}) \quad \text{in } L^p(\mathbb{R}^3)^3,$$

In particular, $(\omega(\mathbf{u}_n))_{n \in \mathbb{N}}$ is bounded in $L^p(\Omega)$, with the same values of p . Therefore, Lemma 2.2 yields

$$(3.21) \quad \|\bar{\omega}(\mathbf{u}_n)\|_{0,s,\Omega} \leq \|K_\alpha\|_{0,q,\Omega} \|\omega(\mathbf{u}_n)\|_{0,p,\Omega},$$

with s given by (3.6) and $1 \leq q < 3$. Thus, $(\bar{\omega}(\mathbf{u}_n))_{n \in \mathbb{N}}$ is bounded in $L^s(\Omega)$. We obtain

$$(3.22) \quad \left| \int_{\Omega} (\bar{\omega}(\mathbf{u}_n) \times (T\mathbf{u}_n - \varphi)) \cdot \mathbf{v} \right| \leq c_1 \|\bar{\omega}(\mathbf{u}_n)\|_{0,s,\Omega} \|T\mathbf{u}_n - \varphi\|_{0,r,\Omega} \|\mathbf{v}\|_{0,r,\Omega} \xrightarrow{n \rightarrow +\infty} 0,$$

where r is given by (2.4).

On the other hand, convergence (3.20) and Lemma 2.3 yield

$$(3.23) \quad \bar{\omega}(\mathbf{u}_n) \xrightarrow{n \rightarrow +\infty} \bar{\omega}(\mathbf{u}) \quad \text{in } L^p(\mathbb{R}^3)^3.$$

And the restrictions weakly converge in $L^p(\Omega)^3$. This yields

$$(3.24) \quad \int_{\Omega} ((\bar{\omega}(\mathbf{u}_n) - \bar{\omega}(\mathbf{u})) \times \varphi) \cdot \mathbf{v} \xrightarrow{n \rightarrow +\infty} 0.$$

To sum up, convergences (3.16), (3.17), (3.22) and (3.24) yield

$$(3.25) \quad \forall \mathbf{v} \in V_\eta, \quad \int_{\Omega} (\bar{\omega}(\mathbf{u}) \times \boldsymbol{\varphi}) \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla \boldsymbol{\varphi} : \nabla \mathbf{v} + \lambda \int_{\Omega} \boldsymbol{\varphi} \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

That means $\boldsymbol{\varphi} = T\mathbf{u}$.

Now the strong convergence has to be proved. Taking $\mathbf{v} = T\mathbf{u}_n$ in (3.15) and $\mathbf{v} = T\mathbf{u}$ in (3.12) yields

$$(3.26) \quad \int_{\Omega} \varrho^\eta \tilde{\nu} |\nabla T\mathbf{u}_n|^2 + \lambda \int_{\Omega} |T\mathbf{u}_n|^2 = \int_{\Omega} \mathbf{f} \cdot T\mathbf{u}_n,$$

$$(3.27) \quad \int_{\Omega} \varrho^\eta \tilde{\nu} |\nabla T\mathbf{u}|^2 + \lambda \int_{\Omega} |T\mathbf{u}|^2 = \int_{\Omega} \mathbf{f} \cdot T\mathbf{u}.$$

What was done before directly yields

$$(3.28) \quad \lambda \int_{\Omega} |T\mathbf{u}_n|^2 \xrightarrow{n \rightarrow +\infty} \lambda \int_{\Omega} |T\mathbf{u}|^2,$$

$$(3.29) \quad \int_{\Omega} \mathbf{f} \cdot T\mathbf{u}_n \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \mathbf{f} \cdot T\mathbf{u},$$

so that we deduce

$$(3.30) \quad \int_{\Omega} \varrho^\eta \tilde{\nu} |\nabla T\mathbf{u}_n|^2 \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \varrho^\eta \tilde{\nu} |\nabla T\mathbf{u}|^2.$$

The weak convergence and (3.30) yield the strong convergence $T\mathbf{u}_n \xrightarrow{n \rightarrow +\infty} T\mathbf{u}$ in V_η for the norm

$$(3.31) \quad N_\nu(\mathbf{u}) := \left(\|\mathbf{u}\|_{0,2,\Omega}^2 + \|\varrho^{\eta/2} \tilde{\nu}^{1/2} \nabla \mathbf{u}\|_{0,2,\Omega}^2 \right)^{1/2}.$$

Given that this norm and $\|\cdot\|_{V_\eta}$ are equivalent, we obtain the strong convergence $T\mathbf{u}_n \xrightarrow{n \rightarrow +\infty} T\mathbf{u}$ in V_η , which concludes the proof. \square

3.3 Convergence to the solutions of the original problem

In this subsection, we consider a sequence of parameters $(\alpha_n)_{n \in \mathbb{N}}$ converging to 0, and a sequence $(\mathbf{u}_{\alpha_n})_{n \in \mathbb{N}}$ of solutions of (1.1)-(1.5) with the parameter α_n in (1.4). We prove that this sequence converges to a solution \mathbf{u} of (1.9)-(1.12).

Theorem 3.2. *Let $\eta \in [0; 1[$, $\lambda > 0$ and $\mathbf{f} \in L^2(\Omega)^3$. Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\alpha_n \xrightarrow{n \rightarrow +\infty} 0$, and $(\mathbf{u}_{\alpha_n})_{n \in \mathbb{N}}$ a sequence of functions satisfying for any $\mathbf{v} \in V_\eta$*

$$(3.32) \quad \int_{\Omega} (\bar{\omega}(\mathbf{u}_{\alpha_n}) \times \mathbf{u}_{\alpha_n}) \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla \mathbf{u}_{\alpha_n} : \nabla \mathbf{v} + \lambda \int_{\Omega} \mathbf{u}_{\alpha_n} \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v},$$

where $\bar{\omega}(\mathbf{u}_{\alpha_n})$ is obtained with the method detailed in section 3, with equation

$$(3.33) \quad -\alpha_n^2 \Delta \bar{\omega}(\mathbf{u}_{\alpha_n}) + \bar{\omega}(\mathbf{u}_{\alpha_n}) = \boldsymbol{\omega}(\mathbf{u}_{\alpha_n}).$$

- If $0 \leq \eta < 1/5$, a subsequence of $(\mathbf{u}_{\alpha_n})_{n \in \mathbb{N}}$ strongly converges in V_η to a function \mathbf{u} satisfying for any $\mathbf{v} \in V_\eta$

$$(3.34) \quad \int_{\Omega} (\boldsymbol{\omega}(\mathbf{u}) \times \mathbf{u}) \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla \mathbf{u} : \nabla \mathbf{v} + \lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

- If $1/5 \leq \eta < 1$, a subsequence of $(\mathbf{u}_{\alpha_n})_{n \in \mathbb{N}}$ converges at the distribution sense to a function $\mathbf{u} \in V_\eta$ satisfying (3.34) for any $\mathbf{v} \in \mathcal{V}$.

Proof. Let $\mathbf{u}_{\alpha_n} \in V_\eta$ satisfying (3.32). Taking $\mathbf{v} = \mathbf{u}_{\alpha_n}$ directly yields $\|\mathbf{u}_{\alpha_n}\|_{V_\eta} \leq M$, where $M > 0$ is defined in (3.14).

Therefore $(\mathbf{u}_{\alpha_n})_{n \in \mathbb{N}}$ is bounded, so there is a subsequence still named the same way which weakly converges in V_η to a certain function $\mathbf{u} \in V_\eta$. As in the proof of Theorem 3.1, we directly obtain for any $\mathbf{v} \in V_\eta$

$$(3.35) \quad \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla \mathbf{u}_{\alpha_n} : \nabla \mathbf{v} \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \varrho^\eta \tilde{\nu} \nabla \mathbf{u} : \nabla \mathbf{v},$$

$$(3.36) \quad \lambda \int_{\Omega} \mathbf{u}_{\alpha_n} \cdot \mathbf{v} \xrightarrow{n \rightarrow +\infty} \lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v}.$$

Thus, we only need to focus on the transport term, and we write a decomposition

$$(3.37) \quad \int_{\Omega} (\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) \times \mathbf{u}_{\alpha_n}) \cdot \mathbf{v} - \int_{\Omega} (\boldsymbol{\omega}(\mathbf{u}) \times \mathbf{u}) \cdot \mathbf{v} = \int_{\Omega} ((\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) - \boldsymbol{\omega}(\mathbf{u})) \times \mathbf{u}) \cdot \mathbf{v} \\ + \int_{\Omega} (\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) \times (\mathbf{u}_{\alpha_n} - \mathbf{u})) \cdot \mathbf{v}.$$

Case 1 ($0 \leq \eta < 1/5$) : we consider any test function $\mathbf{v} \in V_\eta$.

On one hand, $\mathbf{u}_{\alpha_n} \xrightarrow[n \rightarrow +\infty]{V_\eta} \mathbf{u}$ which implies $\boldsymbol{\omega}(\mathbf{u}_{\alpha_n}) \xrightarrow[n \rightarrow +\infty]{L^p} \boldsymbol{\omega}(\mathbf{u})$. Thus, Theorem 2.4 yields $\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) \xrightarrow[n \rightarrow +\infty]{L^p} \boldsymbol{\omega}(\mathbf{u})$. We deduce

$$(3.38) \quad \int_{\Omega} ((\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) - \boldsymbol{\omega}(\mathbf{u})) \times \mathbf{u}) \cdot \mathbf{v} \xrightarrow{n \rightarrow +\infty} 0.$$

On the other hand, taking s given by (3.6), we have

$$(3.39) \quad \left| \int_{\Omega} (\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) \times (\mathbf{u}_{\alpha_n} - \mathbf{u})) \cdot \mathbf{v} \right| \leq \|\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n})\|_{0,s,\Omega} \|\mathbf{u}_{\alpha_n} - \mathbf{u}\|_{0,r,\Omega} \|\mathbf{v}\|_{0,r,\Omega} \xrightarrow{n \rightarrow +\infty} 0,$$

with the same arguments as in Theorem 3.1.

Consequently, (3.37) yields

$$(3.40) \quad \int_{\Omega} (\bar{\boldsymbol{\omega}}(\mathbf{u}_{\alpha_n}) \times \mathbf{u}_{\alpha_n}) \cdot \mathbf{v} \xrightarrow{n \rightarrow +\infty} \int_{\Omega} (\boldsymbol{\omega}(\mathbf{u}) \times \mathbf{u}) \cdot \mathbf{v}.$$

Passing to the limit in (3.32) shows that \mathbf{u} satisfies (3.34).

Now the strong convergence needs to be proved. Taking $\mathbf{v} = \mathbf{u}_{\alpha_n}$ in (3.32) and $\mathbf{v} = \mathbf{u}$ in (3.34) yields

$$(3.41) \quad \int_{\Omega} \varrho^\eta \tilde{\nu} |\nabla \mathbf{u}_{\alpha_n}|^2 + \lambda \int_{\Omega} |\mathbf{u}_{\alpha_n}|^2 = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}_{\alpha_n},$$

$$(3.42) \quad \int_{\Omega} \varrho^\eta \tilde{\nu} |\nabla \mathbf{u}|^2 + \lambda \int_{\Omega} |\mathbf{u}|^2 = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}.$$

We conclude with the same method as used in the proof of Theorem 3.1.

Case 2 ($1/5 \leq \eta < 1$) : we consider a test function $\mathbf{v} \in \mathcal{V}$.

The proof is the same as before, but \mathbf{u} and \mathbf{u}_{α_n} cannot be taken as test functions, so that the convergence is not strong. □

References

- [1] Chérif Amrouche, Guillaume Leloup, and Roger Lewandowski. The model involving the distance to the wall. part 1: the relaxed case. *Preprint on Hal*, 2023.
- [2] Cherif Amrouche and Mohand Moussaoui. The dirichlet and neumann problems in lipschitz and in $C^{1,1}$ domains. abstract. *arXiv e-prints*, pages arXiv–2204, 2022.
- [3] Luigi C Berselli, Argus A Dunca, Lewandowski Roger, Dinh Duong Nguyen, et al. Modeling error of alpha-models of turbulence on a two-dimensional torus. *Discrete and continuous dynamical systems. Series B*, 26(9):4613–4643, 2021.
- [4] Luigi C Berselli and Roger Lewandowski. Convergence of approximate deconvolution models to the mean navier–stokes equations. In *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*, volume 29, pages 171–198. Elsevier, 2012.
- [5] Yanping Cao, Evelyn M Lunasin, and Edriss S Titi. Global well-posedness of the three-dimensional viscous and inviscid simplified bardina turbulence models. *Communications in Mathematical Sciences*, 4(4):823–848, 2006.
- [6] Alexey Cheskidov, Darryl D Holm, Eric Olson, and Edriss S Titi. On a leray– α model of turbulence. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 461(2055):629–649, 2005.
- [7] Ciprian Foias, Darryl D Holm, and Edriss S Titi. The navier–stokes-alpha model of fluid turbulence. *Physica D: Nonlinear Phenomena*, 152:505–519, 2001.
- [8] JD Gibbon and DD Holm. Estimates for the lans- α , leray- α and bardina models in terms of a navier-stokes reynolds number. *Indiana University Mathematics Journal*, pages 2761–2773, 2008.
- [9] Alexei A Ilyin, Evelyn M Lunasin, and Edriss S Titi. A modified-leray- α subgrid scale model of turbulence. *Nonlinearity*, 19(4):879, 2006.
- [10] A. Kufner. *Weighted Sobolev spaces*. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985. Translated from the Czech.
- [11] W Layton and Roger Lewandowski. On a well-posed turbulence model. *Discrete and continuous dynamical systems series B*, 6(1):111, 2006.
- [12] William Layton. Existence of smooth attractors for the navier–stokes-omega model of turbulence. *Journal of mathematical analysis and applications*, 366(1):81–89, 2010.
- [13] William Layton and Roger Lewandowski. Residual stress of approximate deconvolution models of turbulence. *Journal of Turbulence*, (7):N46, 2006.

- [14] William Layton, Carolina C Manica, Monika Neda, and Leo G Rebholz. Numerical analysis and computational comparisons of the ns-alpha and ns-omega regularizations. *Computer Methods in Applied Mechanics and Engineering*, 199(13-16):916–931, 2010.
- [15] Jean Leray. Essai sur les mouvements plans d’un liquide visqueux que limitent des parois. *Journal de Mathématiques pures et appliquées*, 13:331–418, 1934.
- [16] Jean Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta mathematica*, 63:193–248, 1934.
- [17] Roger Lewandowski and Luigi C Berselli. On the bardina’s model in the whole space. *Journal of Mathematical Fluid Mechanics*, 20:1335–1351, 2018.
- [18] Evelyn Lunasin, Susan Kurien, and Edriss S Titi. Spectral scaling of the leray- α model for two-dimensional turbulence. *Journal of Physics A: Mathematical and Theoretical*, 41(34):344014, 2008.
- [19] Eric Olson and Edriss S Titi. Viscosity versus vorticity stretching: global well-posedness for a family of navier–stokes-alpha-like models. *Nonlinear Analysis: Theory, Methods & Applications*, 66(11):2427–2458, 2007.
- [20] Ludwig Prandtl. Mutual influence of wings and propeller. Technical report, 1921.