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Abstract

We design in this work a discrete de Rham complex on manifolds. This complex, written in the framework of exterior calculus, is applicable on meshes on the manifold with generic elements, and has the same cohomology as the continuous de Rham complex. Notions of local (full and trimmed) polynomial spaces are developed, with compatibility requirements between polynomials on mesh entities of various dimensions. Explicit examples of polynomials spaces are presented. The discrete de Rham complex is then used to set up a scheme for the Maxwell equations on a 2D manifold without boundary, and we show that a natural discrete version of the constraint linking the electric field and the electric charge density is satisfied. Numerical examples are provided on the sphere and the torus, based on a bespoke analytical solution and mesh design on each manifold.

Key words. Discrete de Rham complex, manifolds, exterior calculus, Maxwell equations, numerical simulations

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1 Introduction

The goal of this paper is to design a discrete version of the de Rham complex on manifolds, which can be built on meshes made of generic elements and has the same cohomology as the continuous de Rham complex. This complex can be used to design schemes for partial differential equations (PDEs) which inherit some important stability properties from the continuous models (such as energy bounds or constraint preservation).

Let Ω be an n-dimensional compact Riemannian manifold. In the exterior calculus framework, the continuous de Rham complex on Ω is

$$\cdots \xrightarrow{\mathrm{d}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}} H\Lambda^{k+1}(\Omega) \xrightarrow{\mathrm{d}} \cdots, \tag{1}$$

where d is the exterior derivative and $H\Lambda^k(\Omega)$ is the space of L^2-integrable k-forms whose exterior derivative is also L^2-integrable. For $\Omega = \mathbb{R}^n$ with $n = 2, 3$, through vector proxies this complex is equivalent to the usual **grad**–rot complex (in 2D) or **grad**–**curl**–div complex (in 3D). The complex property, which reads $\mathrm{d} \circ \mathrm{d} = 0$, has important consequences for PDE models based on these operators, such

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as the preservation of the divergence constraint in Maxwell equations or the stability of magnetostatics problems [23, Section 2]. Designing discrete versions of this complex is essential to ensure that the resulting scheme also satisfy these properties.

Several approaches have been employed to achieve such a discretisation. One of the most successful one is the Finite Element Exterior Calculus (FEEC), which provides a generic framework for designing and analysing discrete complexes [2, 3]. Its design based on explicit locally polynomial and globally conforming functions however makes it only applicable to certain meshes – mostly made of triangles/tetrahedra or quadrangles/hexahedra. We also note the broader framework of Generalised Finite Element Systems [15], but whose analysis tools are currently limited to algebraic properties of the discrete complexes, and does not provide explicit means to construct fully computable complexes on generic meshes.

A recent new trend is that of polytopal methods, that is, numerical methods that are applicable on meshes made of generic polygons/polyhedra (polytopes), and have an arbitrary degree of accuracy. The Virtual Element Method (VEM) [6, 7] and Discrete De Rham (DDR) [20, 23] are two main examples of approaches to building *polytopal discrete complexes*. Such complexes have benefits over FEEC ones: the flexibility of polytopal meshes leads to seamless local mesh refinement (to better capture, e.g., stiff solutions or complex geometries), and their high-level approach (which does not require globally conforming basis functions) may lead to leaner constructions using systematic processes in their design, such as static condensation and serendipity; see [9, 11, 16, 19, 21] for examples of the efficiency of discrete polytopal complexes.

The question of stability of discretisations of PDEs also applies to models set on manifolds, and leads to the natural question of designing discrete version of the de Rham complex on such spaces. In the finite element setting, most studies have been done on embedded manifolds – typically, a surface in $\mathbb{R}^3$. An initial (extrinsic) approach is to approximate the surface by a piecewise linear surface, on which a triangulations and finite element spaces can be easily constructed, and to numerically approximate the PDE on that approximated surface, see, e.g., [17, 18] and [2] for FEEC (see also [25] for VEM). The drawback of this simple – although completely computable – approach is its practical limitation to low-order methods because the error committed by approximating the surface limits the accuracy of the resulting scheme (unless we have an explicit knowledge – which is rare – of the distance function in a tubular neighbourhood of the manifold). To mitigate this issue, a second (intrinsic) approach was developed which consists in defining triangulations and finite element spaces directly on the manifold, and to therefore discretise the PDE without the geometrical error created by approximating the manifold, see [4] and reference therein. The challenge is then to define a notion of piecewise polynomial functions on the triangulation of the manifold. One option to tackle this challenge is through the usage of an explicit piecewise linear manifold, on which finite element triangulations and spaces can be trivially defined (the notion of polynomials being obvious), and to transport these triangulations and spaces onto the physical manifold. In [4], it is required that this approximate manifold be defined on a close neighbourhood of the surface, such that each point of this neighbourhood has a unique orthogonal projection on the manifold; this is actually not a requirement, as the mere existence of an homeomorphism between the physical and piecewise linear manifolds is sufficient to transport all finite element spaces (H^1-conforming, but also $\mathbf{H}(\mathbf{curl})$- or $\mathbf{H}(\mathrm{div})$-conforming) through pull-back [30]. Another, strongly related, approach to construct intrinsic triangulation and finite element spaces is to assume the existence of a global explicit system of coordinates on the manifold; this system can then be used to represent the mesh and local polynomial spaces (and amounts to the above-mentioned homeomorphism and pull-back approach). This, however, has some limitations on the type of manifold that can be considered (a sphere, e.g., cannot be covered by a single chart). The reference [5] describes this approach in the context of VEM, and highlights that the usage of a single chart on a sphere generates a loss of accuracy on the sphere due to the singularity of this chart at a point (it is numerically shown that using two charts can mitigate this

issue, but the design of the method does not account for the usage of multiple charts).

In all cases, the existence of a global object to represent the manifold (piecewise linear manifold approximating and/or homeomorphic to the physical manifold) can be established, but the explicit description of this object remains elusive except in special cases, which mainly restricts global approaches to theoretical analysis (or requires to construct computable spaces that introduce an additional geometrical error, the control of which requiring additional properties on the global object – e.g., that the homeomorphism is close to an isometry); see the discussion in [30], which designs and analyses a FEEC on manifolds.

The reason for the need of a global approach mostly lies in the requirement for finite element spaces to be subspaces of the continuous spaces (conforming approximation). This is also visible in the abstract construction of [30, Section 3-4] where, although the design of the polynomial spaces is done locally through pullbacks, the continuity requirements of the *smooth triangulation* ultimately makes it non-computable (except for trivial case); practical usage then require the introduction of an approximation of the manifold.

Polytopal methods do not have such a requirement. Instead, their discrete spaces can be interpreted as spaces of vectors of polynomial functions on mesh entities of various dimensions (elements, but also faces, edges, vertices); these polynomial functions are completely unrelated with each other: for example, polynomials on the faces do not have to be the traces of polynomials in the elements, see [8, 10] for both DDR and VEM. As a consequence, the restrictions – even on flat spaces – imposed on the geometry of the mesh elements by finite element methods are lifted, and there is no issue in gluing local constructions to form a global space (no global property on the space need to be imposed). The trade-off is that notions of polynomials on faces and edges must be available. This is not in itself an issue even on manifolds, as local coordinates can be used for this, but when considering *complexes*, some compatibility conditions are necessary; in particular, traces on faces of polynomials in elements must be polynomial on faces (which imposes a compatibility of the local coordinates used to define polynomials). Moreover, *trimmed* local polynomial spaces play an important role in discrete complexes [2, 6, 10]; when going to polytopal complexes on manifolds, such notions of spaces must therefore also be defined, which asks questions on intrinsic Koszul operators in local coordinates and their compatibility conditions related to traces (traces of trimmed polynomial spaces should remain trimmed polynomial spaces of the same degree).

At this point, it is worthwhile to note other approaches to solving partial differential equations on manifolds that have been developed, primarily by the General Relativity community. The oldest approach is the Regge calculus [26, 33, 36], which is a lattice based method for numerically solving the Einstein equations. Related to the Regge calculus is the smooth lattice method developed in [14]. This method offer significant improvements over the Regge calculus and has been used to solve the Einstein equations on spacetime manifolds with compact spacial slices [12, 13]. In a different direction, an approach to solving partial differential equations, in particular the Einstein equations, on spacetime manifolds that is based on local discrete coordinate patches was developed in [35]. More recently, the authors of the articles [31, 32, 37] developed a multi-cube decomposition method for solving partial differential on compact manifolds and employed this method to construct numerical solutions of the Einstein constraint equations on various compact three-manifolds. Finally, we mention the article [24], which is most relevant to our results, where a discrete exterior calculus is employed to design a numerical scheme for solving the Einstein equations. This approach has been used to find solutions to the Einstein equations on spacetimes with compact spatial slices in [34]. All these methods, however, are limited to meshes with elements having a specific shape (tetrahedral or cubic).

In this work, we tackle the question of designing a discrete complex on generic (not necessarily embedded) manifolds, that supports meshes having generic element shapes. We define suitable notions of local (complete and trimmed) polynomial spaces on a generic mesh of Ω, tackling the issue of compat-

ibility of traces of polynomials. Like the abstract setting of [30], our construction relies on mappings between subsets of $\mathbb{R}^d$ and the manifold. However, our compatibility requirement between the mappings is much milder. This added flexibility enables the construction of computable mappings. Our construction works even when several charts are used to represent the manifold, and does not require a global homeomorphism between the manifold and a piecewise linear manifold; it is also purely local, and we provide practical spaces on various polygonal elements. We then use these local polynomial spaces to design, following the construction in [10], a discrete version of the de Rham complex of differential forms on the manifold. Some key properties this complex are stated, including the isomorphism of its cohomology with that of the de Rham complex. We then derive a 2+1 formulation of the Maxwell equations on 2D manifolds, and use the DDR complex to design a scheme for these equations; the preservation of constraint follows from the properties of the discrete complex. We numerically illustrate the behaviour of the scheme on manufactured solutions on the sphere and the torus, designing for this suitable meshes (based on two charts of each manifold) on which local polynomial spaces with compatibility conditions can be constructed.

The paper is organised as follows. In Section 2, we introduce the setting required to define the DDR complex on a manifold; in particular, we define a notion of polynomial spaces on a mesh of the manifold, in such a way that restrictions of polynomials on lower-dimensional mesh entities are also polynomials on these mesh entities. We construct trimmed polynomial spaces, that are then used in Section 3 to build the discrete spaces and operators of the discrete de Rham complex of differential forms on the manifold, following the design in [10]. We state the main properties (consistency, commutation, isomorphism of cohomologies) of this DDR complex. Section 4 then tackles an application of this construction: the electromagnetism equations on a manifold. Starting from the Maxwell equations written on a 3D Lorentzian manifold, we perform the 2+1 splitting to obtain a model written in time-and-space formulation in the language of differential forms. We then build a scheme for this model, use the properties of the DDR complex to show that this scheme preserves a discrete version of the constraint on the electric field, establish an energy estimate, and finally perform numerical tests on the 2-dimensional sphere and torus. Two appendices conclude the paper: Appendix A details a practical construction of polynomial spaces on a manifold mesh made of elements of various shapes, and Appendix B recalls important tensor calculus constructions used in the derivation of the electromagnetism equations.

2 Setting

2.1 Mesh

We let $\mathcal{M}_h$ be a mesh made of a collection of submanifolds of dimensions $d = 0, \dots, n$ that partition Ω. For such a d, the set $\Delta_d(\mathcal{M}_h)$ collects all the submanifolds in $\mathcal{M}_h$ of dimension d, which we also call d-cells, or cells when the dimension is not useful; "elements" refer to n-cells.

We assume the following properties (the first two explicitly re-state that $\mathcal{M}_h$ partitions Ω):

- The submanifolds cover Ω: $\bigcup_{d\in[0,n]} \bigcup_{f\in\Delta_d(\mathcal{M}_h)} f = \Omega$.
- The submanifolds are pairwise disjoint: $\forall f, f' \in \mathcal{M}_h, f \neq f' \implies f \cap f' = \emptyset$.
- Boundaries of the submanifolds are submanifolds: $\forall d \in [1,n], f \in \Delta_d(\mathcal{M}_h), d' \in [0, d-1], f' \in \Delta_{d'}(\mathcal{M}_h), f' \cap \overline{f} \neq \emptyset \implies f' \subset \partial f$.

The last property allows us to define the collection of cells on the boundary of f: $\Delta_{d'}(f) := \{f' \in \Delta_{d'}(\mathcal{M}_h) : f' \subset \partial f\}$.

Algebraic properties of the discrete de Rham complex designed in Section 3 do not require any mesh regularity assumptions. However, analytical properties (such as the consistency of the complex, see Section 3.3) do depend on the regularity of the mesh, a concept that we introduce now. In what follows, the notation $a \lesssim b$ stands for $a \leq Cb$ with a constant C which does not depend on the parameter h, the

dimension $d \leq n$, on the subcell $f \in \Delta_d(\mathcal{M}_h)$ or on the forms considered on f. The notation $a \approx b$ stands for $a \lesssim b$ and $b \lesssim a$. We say that a (flat) polytopal mesh M_h (as defined in [22, Definition 1.4]) is equivalent to $\mathcal{M}_h$ if:

- Their same-dimensional cells can be put in bijection: $\forall 0 \leq d \leq n$, there is a bijection $\Phi_d : \Delta_d(\mathcal{M}_h) \to \Delta_d(M_h)$.
- They are topologically equivalent: $\forall 1 \leq d \leq n, \forall f \in \Delta_d(\mathcal{M}_h), \Phi_{d-1}(\Delta_{d-1}(f)) = \Delta_{d-1}(\Phi_d(f))$.
- Their geometry are equivalent: $\forall 1 \leq d \leq n, \forall f \in \Delta_d(\mathcal{M}_h)$, there is a diffeomorphism $\phi_f : f \to \Phi_d(f)$ satisfying:

$$\|\nabla\phi_f\|_\infty \approx \|\det(\nabla\phi_f)\|_\infty^{\frac{1}{d}} \approx \|\det(\nabla\phi_f^{-1})\|_\infty^{-\frac{1}{d}} \approx \|\nabla\phi_f^{-1}\|_\infty^{-1}. \tag{2}$$

- The geometry is regular on boundaries: $\forall 1 \leq d \leq n, \forall f \in \Delta_d(\mathcal{M}_h), \forall f' \in \Delta_{d-1}(f)$,

$$\|\nabla\phi_f\|_\infty \approx \|\nabla\phi_{f'}\|_\infty. \tag{3}$$

Above, $\|\nabla\phi_f\|_\infty$ denotes the $L^\infty(f)$ norm of $\max_i |\partial_i\phi_f|$. Note that imposing, for example, that $\|\nabla\phi_f\|_\infty$ is comparable to $\|\det(\nabla\phi_f)\|_\infty^{\frac{1}{d}}$ amounts to enforcing a level of "isotropy" to f, and (2) is therefore an assumption on the shape regularity of f.

Definition 1 (Regular mesh sequence)**.** We say that a mesh sequence $(\mathcal{M}_h)_h$ is regular if there exists an equivalent polytopal mesh sequence $(M_h)_h$ which is regular in the sense of [22, Definition 1.9].

We define the characteristic size of a d-cell f as $h_f := |f|^{\frac{1}{d}}$. Noticing that

$$h_f \approx \|\nabla\phi_f\|_\infty^{-1} h_{\Phi_d(f)}, \tag{4}$$

we can translate the geometric bounds on regularity polytopal mesh sequences into our setting. We recall the main properties

- The size of neighbouring cells is comparable: $\forall 1 \leq d \leq n, \forall f \in \Delta_d(\mathcal{M}_h), \forall f' \in \Delta_{d-1}(f)$, $h_f \approx h_{f'}$.
- The number of sub-cells in an cell boundary is bounded: $\forall 1 \leq d \leq n, \forall f \in \Delta_d(\mathcal{M}_h), |\Delta_{d-1}(f)| \lesssim 1$.

2.2 Discrete polynomial spaces

To build a DDR complex on $\mathcal{M}_h$, we need a notion of "local exterior calculus polynomial space" on the submanifolds in $\mathcal{M}_h$. This is not an easy concept to define as these spaces must satisfy a range of compatibility properties with the exterior derivative, the trace, etc. In this section, we define the key axioms that these spaces must satisfy, and infer further notions such as trimmed polynomial spaces. The construction below mimics well-known properties of polynomial spaces, exterior derivative and Koszul operator on flat spaces; we however have to check carefully that the key minimal properties we require (the complex properties and Assumption 2) are indeed sufficient to deduce, in particular, the decomposition of polynomial spaces which justify the definition of trimmed spaces.

Let $0 \leq d \leq n$ and $f \in \Delta_d(\mathcal{M}_h)$. We consider spaces $(\mathcal{P}_r\Lambda^l(f))_{r\in\mathbb{Z},l\in\mathbb{Z}}$ such that $\mathcal{P}_r\Lambda^l(f) \subset C^1\Lambda^l(f)$, and that form a complex for the exterior derivative d in the following way:

$$\cdots \xrightarrow{\mathrm{d}} \mathcal{P}_r\Lambda^l(f) \xrightarrow{\mathrm{d}} \mathcal{P}_{r-1}\Lambda^{l+1}(f) \xrightarrow{\mathrm{d}} \cdots. \tag{5}$$

We assume the existence of a graded map κ_f (removing the index f when no confusion can arise), playing the role of a Koszul operator in our setting, such that the following "reverse" sequence forms a complex

$$\cdots \xrightarrow{\kappa} \mathcal{P}_r\Lambda^l(f) \xrightarrow{\kappa} \mathcal{P}_{r+1}\Lambda^{l-1}(f) \xrightarrow{\kappa} \cdots . \tag{6}$$

Assumption 2. We assume the following properties:

(A1) There is a graded decomposition of $\mathcal{P}_r\Lambda^l(f) := \bigoplus_{s\le r}\mathcal{H}_s\Lambda^l(f)$ such that, on each $\mathcal{H}_s\Lambda^l(f)$, $\mathrm{d}\kappa+\kappa\mathrm{d}$ acts as an homothetie: $\forall s,l, \exists\lambda_{s,l}\in\mathbb{R}$ such that $(\mathrm{d}\kappa+\kappa\mathrm{d})p = \lambda_{s,l}p$ for all $p\in\mathcal{H}_s\Lambda^l(f)$. For $s<0$, $\mathcal{H}_s\Lambda^l(f)=\{0\}$ and we fix $\lambda_{s,l}=\lambda_{s+l,0}$, with $\lambda_{s+l,0}=0$ if $s+l<0$.

(A2) The eigenvalues of $\mathrm{d}\kappa+\kappa\mathrm{d}$ identify the polynomial degree: for all $d>0$ and $0\le l\le d$ and all $s,s'\ge 0$, $\lambda_{s,l}=\lambda_{s',l} \implies s=s'$. Moreover, if $s\ge 0$ then $\lambda_{s,l}=0$ if and only if $(s,l)=(0,0)$.

(A3) $\mathrm{d}\mathcal{H}_s\Lambda^l(f)\neq\{0\}$ when $s>0$ and $l<d$.

Lemma 3. *For all $0\le l\le d$, $s\in\mathbb{Z}$, the eigenvalues are related by the relation $\lambda_{s,l}=\lambda_{s+l,0}$. Moreover $\mathrm{d}\mathcal{H}_{s+1}\Lambda^l(f)\subset\mathcal{H}_s\Lambda^{l+1}(f)$ and $\kappa\mathcal{H}_s\Lambda^{l+1}(f)\subset\mathcal{H}_{s+1}\Lambda^l(f)$*

Proof. The case $s<0$ being obvious by the choice in **(A1)**, we only need to consider $s\ge 0$. Let us first consider the case $l=0$ and let us prove by induction on $s\in\mathbb{N}$ that $\mathrm{d}\mathcal{H}_{s+1}\Lambda^0(f)\subset\mathcal{H}_s\Lambda^1(f)$ and $\lambda_{s,1}=\lambda_{s+1,0}$. Let us consider $s=0$. Using **(A1)** we have $\mathrm{d}\mathcal{H}_1\Lambda^0(f)\subset\mathrm{d}\mathcal{P}_1\Lambda^0(f)\subset\mathcal{P}_0\Lambda^1(f)=\mathcal{H}_0\Lambda^1(f)$, which proves the first property for the base case. For the second property, by **(A3)**, there is $p\in\mathcal{H}_1\Lambda^0(f)$ such that $\mathrm{d}p\neq 0$. Moreover $\kappa\mathrm{d}p=\lambda_{1,0}p-\mathrm{d}\kappa p$. Thus we have

$$(\mathrm{d}\kappa+\kappa\mathrm{d})\mathrm{d}p = \mathrm{d}\kappa\mathrm{d}p = \lambda_{1,0}\mathrm{d}p - \cancel{\mathrm{d}\mathrm{d}\kappa\mathrm{d}p},$$

and $\mathrm{d}p$ is an eigenvector of $\mathrm{d}\kappa+\kappa\mathrm{d}$ associated with the eigenvalue $\lambda_{1,0}$. We note that $\mathrm{d}p\in\mathrm{d}\mathcal{H}_1\Lambda^0(f)\subset\mathcal{H}_0\Lambda^1(f)$. Therefore, $\lambda_{0,1}=\lambda_{1,0}$ and the case $s=0$ is proved.

Let now $s\ge 1$ and assume that, for all $0\le k<s$, $\mathrm{d}\mathcal{H}_{k+1}\Lambda^0(f)\subset\mathcal{H}_k\Lambda^1(f)$ and $\lambda_{k,1}=\lambda_{k+1,0}$. We need to prove that $\lambda_{s,1}=\lambda_{s+1,0}$. Let $p\in\mathcal{H}_{s+1}\Lambda^0(f)$. If $\mathrm{d}p=0$ then clearly $\mathrm{d}p\in\mathcal{H}_s\Lambda^1(f)$. Otherwise, using the same arguments as for $s=0$ we note that $(\mathrm{d}\kappa+\kappa\mathrm{d})\mathrm{d}p=\lambda_{s+1,0}\mathrm{d}p$, and $\mathrm{d}p\in\mathcal{P}_s\Lambda^1(f)=\bigoplus_{k\le s}\mathcal{H}_k\Lambda^1(f)$. Since $\mathrm{d}p$ is an eigenvector of $\mathrm{d}\kappa+\kappa\mathrm{d}$ (as it is not equal to 0), by **(A2)** it belongs to an $\mathcal{H}_k\Lambda^1(f)$ for some k such that $\lambda_{k,1}=\lambda_{s+1,0}$. By induction hypothesis, if $k<s$ we have $\lambda_{k,1}=\lambda_{k+1,0}\neq\lambda_{s+1,0}$, see **(A2)**. Hence, $k=s$ and $\lambda_{s,1}=\lambda_{s+1,0}$. In passing, we have also proved that $\mathrm{d}p\in\mathcal{H}_s\Lambda^1(f)$, so that $\mathrm{d}\mathcal{H}_{s+1}\Lambda^0(f)\subset\mathcal{H}_s\Lambda^1(f)$, and the induction is therefore complete.

Another induction on l concludes the proof that $\lambda_{s,l}=\lambda_{s+l,0}$ for all s,l.

The fact that κ maps eigenspaces into eigenspaces also follows from **(A1)** and similar arguments. The details are left to the reader. □

Lemma 4. *The following mappings are one-to-one:*

$$\mathrm{d}\ :\ \kappa\mathcal{H}_s\Lambda^l(f)\to\mathrm{d}\mathcal{H}_{s+1}\Lambda^{l-1}(f)\subset\mathcal{H}_s\Lambda^l(f)$$
$$\kappa\ :\ \mathrm{d}\mathcal{H}_s\Lambda^l(f)\to\kappa\mathcal{H}_{s-1}\Lambda^{l+1}\subset\mathcal{H}_s\Lambda^l(f)\,.$$

Proof. We only need to prove that d is injective on $\kappa\mathcal{H}_s\Lambda^l(f)$, the other case being similar. The case $l=0$ is trivial since

$$\kappa\mathcal{H}_s\Lambda^0(f)=\kappa\mathcal{P}_s\Lambda^0(f)\subset\mathcal{P}_{s+1}\Lambda^{-1}(f)\subset C^1\Lambda^{-1}(f)=\{0\}. \tag{7}$$

We therefore assume that $l\ge 1$ and take $q\in\kappa\mathcal{H}_s\Lambda^l(f)$ such that $\mathrm{d}q=0$. We can write $q=\kappa p$ for some $p\in\mathcal{H}_s\Lambda^l(f)$. By **(A1)**, we have $(\mathrm{d}\kappa+\kappa\mathrm{d})p=\lambda_{s,l}p$. As $\mathrm{d}\kappa p=\mathrm{d}q=0$, we infer $\kappa\mathrm{d}p=\lambda_{s,l}p$ and $\lambda_{s,l}\kappa p=\kappa\kappa\mathrm{d}p=0$. By **(A2)**, $\lambda_{s,l}\neq 0$ (since $(s,l)\neq(0,0)$), and thus $q=\kappa p=0$.

To conclude the proof, we note that the property $\mathrm{d}\kappa\mathcal{H}_s\Lambda^l(f) \subset \mathrm{d}\mathcal{H}_{s+1}\Lambda^{l-1}(f)$ follows from Lemma 3 if $l \geq 1$, and from (7) if $l = 0$.

□

Lemma 5. *The following direct decompositions hold, for all $r \geq 0$:*

$$\mathcal{P}_r\Lambda^0(f) := \mathcal{P}_0\Lambda^0(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^{l+1}(f)$$
$$\mathcal{P}_r\Lambda^l(f) := \mathrm{d}\mathcal{P}_{r+1}\Lambda^{l-1}(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^{l+1}(f) \quad \text{if } l \geq 1$$

Proof. We first prove that

$$\mathcal{H}_s\Lambda^l(f) = \mathrm{d}\mathcal{H}_{s+1}\Lambda^{l-1}(f) \oplus \kappa\mathcal{H}_{s-1}\Lambda^{l+1}(f) \quad \text{if } (s,l) \neq (0,0). \tag{8}$$

For $p \in \mathcal{H}_s\Lambda^l(f)$, the assumption **(A1)** yields $(\mathrm{d}\kappa + \kappa\mathrm{d})p = \lambda_{s,l}p$ with $\lambda_{s,l} \neq 0$ since $(s,l) \neq (0,0)$. Hence,

$$p = \mathrm{d}(\lambda_{s,l}^{-1}\kappa p) + \kappa(\lambda_{s,l}^{-1}\mathrm{d}p).$$

Noticing that $\kappa p \in \mathcal{H}_{s+1}\Lambda^{l-1}(f)$, $\mathrm{d}p \in \mathcal{H}_{s-1}\Lambda^{l+1}(f)$, $\mathrm{d}\mathcal{H}_{s+1}\Lambda^{l-1}(f) \subset \mathcal{H}_s\Lambda^l(f)$ and $\kappa\mathcal{H}_{s-1}\Lambda^{l+1}(f) \subset \mathcal{H}_s\Lambda^l(f)$ by Lemma 3, this proves that $\mathcal{H}_s\Lambda^l(f) = \mathrm{d}\mathcal{H}_{s+1}\Lambda^{l-1}(f) + \kappa\mathcal{H}_{s-1}\Lambda^{l+1}(f)$. It remains to prove that this sum is direct. If $z \in \mathrm{d}\mathcal{H}_{s-1}\Lambda^{l+1}(f) \cap \kappa\mathcal{H}_{s+1}\Lambda^{l-1}(f)$ then there exists $p \in \mathcal{H}_{s-1}\Lambda^{l+1}(f)$ and $q \in \mathcal{H}_{s+1}\Lambda^{l-1}(f)$ such that $z = \kappa p = \mathrm{d}q$, then $\mathrm{d}\kappa p = \mathrm{d}\mathrm{d}q = 0$. Lemma 4 then implies $z = \kappa p = 0$ which concludes the proof of (8).

We now turn to the result in the lemma. If $l \neq 0$, we can take the direct sums of (8) to get

$$\begin{aligned}
\mathcal{P}_r\Lambda^l(f) &= \bigoplus_{s\leq r} \mathcal{H}_s\Lambda^l(f) \\
&= \bigoplus_{s\leq r} \mathrm{d}\mathcal{H}_{s+1}\Lambda^{l-1}(f) \oplus \kappa\mathcal{H}_{s-1}\Lambda^{l+1}(f) \\
&= \mathrm{d}\bigoplus_{s\leq r} \mathcal{H}_{s+1}\Lambda^{l-1}(f) \oplus \kappa\bigoplus_{s\leq r} \mathcal{H}_{s-1}\Lambda^{l+1}(f) \\
&= \mathrm{d}\mathcal{P}_{r+1}\Lambda^{l-1}(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^{l+1}(f).
\end{aligned}$$

If $l = 0$, we simply write

$$\mathcal{P}_r\Lambda^0(f) = \mathcal{H}_0\Lambda^0(f) \oplus \bigoplus_{1\leq s\leq r} \mathcal{H}_s\Lambda^0(f) = \mathcal{P}_0\Lambda^0(f) \oplus \bigoplus_{1\leq s\leq r} \mathcal{H}_s\Lambda^0(f)$$

and apply (8) with $s \in [1, r]$ and $l = 0$, noting that $\mathrm{d}\mathcal{H}_{s+1}\Lambda^{-1}(f) = \{0\}$. □

Definition 6. The trimmed polynomial spaces are defined as: for all $r \geq 0$,

$$\mathcal{P}_r^-\Lambda^0(f) := \mathcal{P}_r\Lambda^0(f)$$
$$\mathcal{P}_r^-\Lambda^l(f) := \mathrm{d}\mathcal{P}_r\Lambda^{l-1}(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^{l+1}(f) \quad \text{if } l \geq 1.$$

Remark 7. It follows directly from the direct decomposition of Lemma 5 that, for all $r, l, d \in \mathbb{Z}$, $f \in \Delta_d(\mathcal{M}_h)$,

$$\mathcal{P}_r\Lambda^l(f) \subset \mathcal{P}_{r+1}^-\Lambda^l(f) \subset \mathcal{P}_{r+1}\Lambda^l(f).$$

Assumption 8. We assume that traces of trimmed polynomials are trimmed polynomials, that is: for all $f \in \Delta_d(\mathcal{M}_h)$ and $d' \in [0, d]$,

$$\mathrm{tr}_{f'}\,\mathcal{P}_r^-\Lambda^l(f) \subset \mathcal{P}_r^-\Lambda^l(f') \qquad \forall f' \in \Delta_{d'}(f).$$

We show in Appendix A that local polynomial spaces satisfying Assumptions 2 and 8 can be designed and explicitly constructed if we suppose that Assumption 9 below holds. This assumption relates local charts of submanifolds and their boundaries.

For $f \in \Delta_d(\mathcal{M}_h)$ and $f' \in \Delta_{d-1}(f)$, we denote the inclusion of f' into $\overline{f}$ by $\mathfrak{I}_{f,f'} : f' \to \overline{f}$.

Assumption 9. For every $0 \le d \le n$ and $f \in \Delta_d(\mathcal{M}_h)$, there is a C^2-diffeomorphism I_f from a subset of $\mathbb{R}^d$ into $\overline{f}$. Let $J_f := (I_f)^{-1}$. We assume that, for all $f' \in \Delta_{d-1}(\mathcal{M}_h)$, the transformation $\mathcal{T}_{f,f'} := J_f \circ \mathfrak{I}_{f,f'} \circ I_{f'}$ from (a subset of) $\mathbb{R}^{d-1}$ to $\mathbb{R}^d$ is affine.

3 Discrete de Rham complex

3.1 Construction

Given a polynomial degree $r \ge 0$, the discrete counterpart $\underline{\boldsymbol{X}}^k_{r,h}$ of the space $H\Lambda^k(\Omega)$, $0 \le k \le n$, is defined as

$$\underline{\boldsymbol{X}}^k_{r,h} := \bigtimes_{d=k}^{n} \bigtimes_{f \in \Delta_d(\mathcal{M}_h)} \star^{-1}\mathcal{P}^-_r\Lambda^{d-k}(f). \tag{9}$$

A generic $\underline{\omega}_h \in \underline{\boldsymbol{X}}^k_{r,h}$ is denoted by $\underline{\omega}_h = (\omega_f)_{f\in\Delta_d(\mathcal{M}_h),\, d\in[k,n]}$ with $\omega_f \in \star^{-1}\mathcal{P}^-_r\Lambda^{d-k}(f)$ for all $f \in \mathcal{M}_h$. As commonly done in DDR constructions, we denote by $\underline{\omega}_f = (\omega_{f'})_{f'\in\Delta'_d(f),\, d'\in[k,d]}$ the restriction of $\underline{\omega}_h$ to $f \in \Delta_d(\mathcal{M}_h)$. The same convention (replacing an index h with f) will also be used to denote restrictions of operators.

The interpolator $\underline{\boldsymbol{I}}^k_{r,h} : C^0\Lambda^k(\overline{\Omega}) \to \underline{\boldsymbol{X}}^k_{r,h}$ is defined by projecting the traces on the polynomial spaces: for all $\omega \in C^0\Lambda^k(\overline{\Omega})$,

$$\underline{\boldsymbol{I}}^k_{r,h}\omega := (\star^{-1}\pi^{-,d-k}_{r,f} \star \mathrm{tr}_f\,\omega)_{f\in\Delta_d(\mathcal{M}_h),d\in[k,n]} \tag{10}$$

where $\pi^{-,d-k}_{r,f} : L^2\Lambda^{d-k}(f) \to \mathcal{P}^-_r\Lambda^{d-k}(f)$ is the L^2-orthogonal projector on the trimmed space.

The main difference with the flat case developed in [10] is that the Hodge star operator $\star$ is a general function, and does not preserve polynomials spaces. This leads us to consider a slightly different formulation, including $\star^{-1}$ in the definition (9) of the discrete space and adjusting accordingly the definitions of the interpolator above, and of the local discrete exterior derivative and discrete potential below.

Definition 10 (Local discrete exterior derivative and discrete potential)**.** Let $f \in \Delta_d(\mathcal{M}_h)$. The discrete exterior derivative $\mathrm{d}^k_{r,f} : \underline{\boldsymbol{X}}^k_{r,f} \to \star^{-1}\mathcal{P}_r\Lambda^{d-k-1}(f)$ and discrete potential $P^k_{r,f} : \underline{\boldsymbol{X}}^k_{r,f} \to \star^{-1}\mathcal{P}_r\Lambda^{d-k}(f)$ are defined inductively on the dimension of f as follows.

- If $d = k$:
$$P^k_{r,f}\underline{\omega}_f := \omega_f \in \star^{-1}\mathcal{P}_r\Lambda^0(f). \tag{11}$$
- If $d \ge k+1$:
  - $\forall \mu_f \in \mathcal{P}_r\Lambda^{d-k-1}(f)$,
$$\int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu_f := (-1)^{k+1}\int_f \omega_f \wedge \mathrm{d}\mu_f + \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_f \wedge \mathrm{tr}_{\partial f}\,\mu_f, \tag{12}$$
where we have denoted by $P^k_{r,\partial f}\underline{\omega}_f$ the piecewise function on ∂f obtained patching the functions $P^k_{r,f'}\underline{\omega}_{f'}$ for $f' \in \Delta_{d-1}(f)$.

$\diamond$ $\forall \mu_f \in \kappa\mathcal{P}_r\Lambda^{d-k}(f), \nu_f \in \kappa\mathcal{P}_{r-1}\Lambda^{d-k+1}(f)$,

$$(-1)^{k+1}\int_f P^k_{r,f}\underline{\omega}_f \wedge (\mathrm{d}\mu_f + \nu_f) := \int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu_f - \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu_f + (-1)^{k+1}\int_f \omega_f \wedge \nu_f. \tag{13}$$

The global discrete differential is then obtained projecting the local discrete differential on the spaces forming the component of the global discrete space (9): $\underline{\mathrm{d}}^k_{r,h} : \underline{\boldsymbol{X}}^k_{r,h} \to \underline{\boldsymbol{X}}^{k+1}_{r,h}$ is given by

$$\underline{\mathrm{d}}^k_{r,h}\underline{\omega}_h := (\star^{-1}\pi^{-,d-k-1}_{r,f} \star \mathrm{d}^k_{r,f}\underline{\omega}_f)_{f\in\Delta_d(\mathcal{M}_h),d\in[k+1,n]}.$$

The discrete DDR sequence on $\mathcal{M}_h$ is then

$$\cdots \xrightarrow{\underline{\mathrm{d}}^{k-1}_{r,h}} \underline{\boldsymbol{X}}^k_{r,h} \xrightarrow{\underline{\mathrm{d}}^k_{r,h}} \underline{\boldsymbol{X}}^{k+1}_{r,h} \xrightarrow{\underline{\mathrm{d}}^{k+1}_{r,h}} \cdots. \tag{14}$$

To design numerical schemes based on this discrete complex, we need to define discrete L^2-like inner products.

Definition 11 (Discrete L^2-like inner product)**.** For $\underline{\omega}_h, \underline{\mu}_h \in \underline{\boldsymbol{X}}^k_{r,h}$, we define the discrete L^2-like inner product as

$$\langle \underline{\omega}_h, \underline{\mu}_h \rangle := \sum_{f\in\Delta_n(\mathcal{M}_h)} \left(\int_f \omega_f \wedge \star\mu_f + \sum_{d=k}^{n-1} h_f^{n-d} \sum_{f'\in\Delta_d(f)} \int_{f'} (\omega_{f'} - \mathrm{tr}_{f'}\, P^k_{r,f}\underline{\omega}_f) \wedge \star(\mu_{f'} - \mathrm{tr}_{f'}\, P^k_{r,f}\underline{\mu}_f) \right).$$

3.2 Algebraic properties

The following results can be proved mimicking the proofs in [10].

Lemma 12 (Link between discrete exterior derivatives on submanifolds)**.** *For all $d \geq 2$, $f \in \Delta_d(\mathcal{M}_h)$, and $\underline{\omega}_f \in \underline{\boldsymbol{X}}^k_{r,f}$, it holds*

$$\int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mathrm{d}\alpha_f = (-1)^{k+1}\int_{\partial f} \mathrm{d}^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\alpha_f, \quad \forall \alpha_f \in \mathcal{P}^-_{r+1}\Lambda^{d-k-2}(f).$$

Lemma 13 (Complex property)**.** *For all $k \geq 1$, $d \geq k$, $f \in \Delta_d(\mathcal{M}_h)$, $\underline{\omega}_f \in \underline{\boldsymbol{X}}^k_{r,f}$, it holds*

$$\begin{aligned} P^k_{r,f}\underline{\mathrm{d}}^{k-1}_{r,f}\underline{\omega}_f &= \mathrm{d}^{k-1}_{r,f}\underline{\omega}_f, \\ \underline{\mathrm{d}}^k_{r,f}\underline{\mathrm{d}}^{k-1}_{r,f}\underline{\omega}_f &= 0. \end{aligned} \tag{15}$$

Lemma 14. *For all $d \geq 0$, $f \in \Delta_d(\mathcal{M}_h)$, and $\underline{\omega}_f \in \underline{\boldsymbol{X}}^k_{r,f}$, it holds*

$$\star^{-1}\pi^{-,d-k}_{r,f} \star P^k_{r,f}\underline{\omega}_f = \omega_f$$

Theorem 15 (Commutation property)**.** *For all $k \geq 0$, $d > k$, and $f \in \Delta_d(\mathcal{M}_h)$, it holds*

$$\underline{\mathrm{d}}^k_{r,f}(\underline{\boldsymbol{I}}^k_{r,f}\omega) = \underline{\boldsymbol{I}}^{k+1}_{r,f}(\mathrm{d}\omega) \quad \forall \omega \in C^1\Lambda^k(\overline{f}). \tag{16}$$

Theorem 16 (Isomorphism of cohomologies)**.** *The DDR sequence* (14) *is a complex that has the same cohomology as the continuous de Rham complex* (1)*.*

3.3 Consistency

In the usual construction of DDR, the initial consistency result for potentials and discrete differential are written as exact polynomial consistency. However, on manifolds, the Hodge-star of polynomials are not necessarily polynomials; for this reason it does not seem possible to obtain an exact polynomial consistency of the reconstructed polynomial and exterior derivatives (see Remark 20 for more details). Instead, we directly establish here the consistency in terms of approximation properties on smooth differential forms (this is, in the usual construction, seen as a consequence of polynomial consistency, but we establish it here independently). We also use the shorthand $\|\mu\|_X$ to denote the L^2-norm of $\mu \in L^2\Lambda^k(X)$.

As the DDR construction relies on the decompositions in Definition 6, we must ensure that the direct sum does not degenerate as h goes to zero. We present in Section A a construction of local polynomial spaces that satisfy the following assumption.

Assumption 17 (Local Poincaré inequality on Koszul complement, topological decomposition and discrete trace)**.** We assume the following bounds on the discrete spaces: For all $k \geq 0$, $d \geq k$ and $f \in \Delta_d(\mathcal{M}_h)$, we have

$$\|\mu\|_f \lesssim h_f \|\mathrm{d}\mu\|_f, \quad \forall \mu \in \kappa\mathcal{P}_{r-1}\Lambda^k(f), \tag{17}$$

$$\|\alpha\|_f + \|\beta\|_f \lesssim \|\alpha+\beta\|_f, \quad \forall \alpha \in \kappa\mathcal{P}_{r-1}\Lambda^{k+1}(f), \quad \forall \beta \in \mathrm{d}\mathcal{P}_{r+1}\Lambda^{k-1}(f), \tag{18}$$

$$\|\operatorname{tr}_{\partial f}\mu\|_{\partial f} \lesssim h_f^{-\frac{1}{2}}\|\mu\|_f, \quad \forall \mu \in \mathcal{P}_r\Lambda^k(f). \tag{19}$$

We assume that the discrete spaces have similar approximation properties as those of polynomials in Euclidean space (see [22, Theorem 1.45]).

Assumption 18 (Approximation properties)**.** For all for all $0 \leq k \leq d$, $f \in \Delta_d(\mathcal{M}_h)$ and $t \in [0, r]$

$$\|\star^{-1}\pi_{r,f}^{d-k}\star\omega - \omega\|_f \lesssim h_f^{t+1}|\omega|_{H^{t+1}\Lambda^k(f)} \quad \forall \omega \in H^{t+1}\Lambda^k(f), \tag{20}$$

where $\pi_{r,f}^{d-k}$ is the $L^2\Lambda^{d-k}(f)$-projection on $\mathcal{P}_k\Lambda^{d-k}(f)$.

To state the consistency result, we need a scaled norm that takes into account the regularity of differential forms and their traces on sub-dimensional cells. For $r \in \mathbb{N}$, we define the space

$$H^{r+1}\Lambda^k(f;\Delta) \coloneqq \left\{\omega \in H^{r+1}\Lambda^k(f) \,:\, \operatorname{tr}_{f'}\omega \in H^{r+1}\Lambda^k(f') \quad \forall f' \in \Delta_{d'}(f), \forall d' \in [k, d-1]\right\}$$

(note that the assumed regularity on f ensures the existence of traces on $f' \in \Delta_{d-1}(f)$, and the assumed regularity of these traces in turn ensure the existence of traces on $f'' \in \Delta_{d-2}(f)$, etc.), and endow it with the semi-norm

$$|\omega|_{r,f,\Delta} \coloneqq \sum_{d'=k}^{d} h_f^{\frac{d-d'}{2}} |\operatorname{tr}_{f'}\omega|_{H^{r+1}\Lambda^k(f')} \quad \forall \omega \in H^{r+1}\Lambda^k(f;\Delta). \tag{21}$$

Theorem 19 (Primal consistency)**.** *For all $k \geq 0$, $d \geq k$, $f \in \Delta_d(\mathcal{M}_h)$, it holds*

$$\|P_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega - \omega\|_f \lesssim h_f^{r+1}|\omega|_{r,f,\Delta} \quad \forall \omega \in H^{r+1}\Lambda^k(f;\Delta), \tag{22}$$

$$\|\mathrm{d}_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega - \mathrm{d}\omega\|_f \lesssim h_f^{r+1}|\mathrm{d}\omega|_{r,f,\Delta} \quad \forall \omega \in C^1\Lambda^k(\overline{f}) \text{ s.t. } \mathrm{d}\omega \in H^{r+1}\Lambda^{k+1}(f;\Delta). \tag{23}$$

Remark 20 (Polynomial consistency). Polynomial consistency is the property: $P_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega = \omega$ for all $\omega \in \mathcal{P}_r\Lambda^k(f)$. As shown in [10, Section 3.5], the initial step for proving this property for the DDR complex on the Euclidean space consists in considering $d = k$ and in writing $\star^{-1}\pi_{r,f}^{d-k}\star\omega = \star^{-1}\star\omega = \omega$;

the removal of the L^2-projector $\pi_{r,f}^{d-k}$ is justified since, when the metric is constant, $\star\omega$ is a polynomial form of the same degree as ω. However, in the context of DDR on a manifold, $\star\omega$ may no longer be polynomial of the same degree as ω since the $\star$ operator involves the coefficients of the (non-constant) metric. Hence, polynomial consistency for the DDR on a manifold does not seem readily accessible. This does not prevent the method, however, from having optimal consistency estimates on smooth forms, as demonstrated by Theorem 19.

Remark 21 (About the regularity requirement on ω). The space $H^{r+1}\Lambda^k(f;\Delta)$, with its assumed regularity of traces of functions on lower-dimensional cells, is the natural one to state the consistency estimates in Theorem 19. Classical spaces of differential forms can easily be embedded into $H^{r+1}\Lambda^k(f;\Delta)$.

For example, letting $C^{r+1}\Lambda^k(\overline{f})$ be the space of k-forms that are continuous over $\overline{f}$ along with all their derivatives up to order $r+1$, and considering the semi-norm

$$|\omega|_{C^{r+1}\Lambda^k(f)} = \max_{\alpha\in\mathbb{N}^d,\,|\alpha|=r+1} \|\partial^\alpha\omega\|_{L^\infty\Lambda^k(f)},$$

we have $C^{r+1}\Lambda^k(\overline{f}) \subset H^{r+1}\Lambda^k(f;\Delta)$ and

$$|\omega|_{r,f,\Delta} \lesssim |f|^{1/2}|\omega|_{C^{r+1}\Lambda^k(f)} \quad \forall\omega\in C^{r+1}\Lambda^k(\overline{f}).$$

Likewise, repeated uses of the continuous trace inequality

$$\|\operatorname{tr}_{\partial f'}\mu\|_{\partial f'} \lesssim h_{f'}^{-\frac12}\|\mu\|_{f'} + h_{f'}^{\frac12}|\mu|_{H^1\Lambda^k(f')}, \quad \forall\mu\in H^1\Lambda^k(f'), \quad \forall f'\in\Delta_{d'}(\mathcal{M}_h), \quad \forall d'\in[k,d]$$

show that $H^{r+1+d-k}\Lambda^k(f)\subset H^{r+1}\Lambda^k(f;\Delta)$ with

$$|\omega|_{r,f,\Delta} \lesssim \sum_{i=0}^{d-k} h_f^i|\omega|_{H^{r+1+i}\Lambda^k(f)} \quad \forall\omega\in H^{r+1+d-k}\Lambda^k(f).$$

Proof of Theorem 19. The proof if done by induction on d. If $d=k$, then both discrete and continuous differential vanish, so (23) is trivially satisfied. The definition (11) of $P_{r,f}^k$ combined with the assumed approximation property (20) of the L^2-projector on polynomial spaces readily gives (22).

Let us now take $d\geq k+1$ and assume that the result holds for $d-1$. Let $f\in\Delta_d(\mathcal{M}_h)$. We first prove a weaker (sub-optimal) version of (23), from which we deduce (22) which, finally, gives the optimal estimate (23) itself.

The definitions (12) of the discrete exterior derivative and (10) of the interpolator yield, for all $\mu_f\in\mathcal{P}_r\Lambda^{d-k-1}(f)$,

$$\begin{aligned}
\int_f \mathrm{d}_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k\omega\wedge\mu_f &= (-1)^{k+1}\int_f \star^{-1}\pi_{r,f}^{-,d-k}\star\omega\wedge\mathrm{d}\mu_f + \int_{\partial f} P_{r,\partial f}^k\underline{\boldsymbol{I}}_{r,\partial f}^k\omega\wedge\operatorname{tr}_{\partial f}\mu_f\\
&= (-1)^{k+1}\int_f \omega\wedge\mathrm{d}\mu_f + \int_{\partial f}\operatorname{tr}_{\partial f}\omega\wedge\operatorname{tr}_{\partial f}\mu_f\\
&\quad + \int_{\partial f}(P_{r,\partial f}^k\underline{\boldsymbol{I}}_{r,\partial f}^k\omega - \operatorname{tr}_{\partial f}\omega)\wedge\operatorname{tr}_{\partial f}\mu_f\\
&= \int_f \mathrm{d}\omega\wedge\mu_f + \int_{\partial f}(P_{r,\partial f}^k\underline{\boldsymbol{I}}_{r,\partial f}^k\omega - \operatorname{tr}_{\partial f}\omega)\wedge\operatorname{tr}_{\partial f}\mu_f
\end{aligned} \tag{24}$$

where, in the second equality, we have used the fact that $\mathrm{d}\mu_f\in\mathcal{P}_r^-\Lambda^{d-k}(f)$ together with the definition of the orthogonal projector $\pi_{r,f}^{-,d-k}$ to remove $\star^{-1}\pi_{r,f}^{-,d-k}\star$ (see [10, Lemma 1] for details), and we have used the Stokes formula in the conclusion. Subtracting

$$\int_f \mathrm{d}\omega\wedge\mu_f = \int_f \star^{-1}\pi_{r,f}^{d-k-1}\star\,\mathrm{d}\omega\wedge\mu_f$$

(the equality following from the definition of $\pi_{r,f}^{d-k-1}$ and the fact that $\mu_f \in \mathcal{P}_r\Lambda^{d-k-1}(f)$) to both sides of (24) and using a Cauchy-Schwartz inequality then gives

$$\begin{aligned}\int_f (\mathrm{d}_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega - \star^{-1}\pi_{r,f}^{d-k-1} \star \mathrm{d}\omega) \wedge \mu_f &\le \|P_{r,\partial f}^k \underline{\boldsymbol{I}}_{r,\partial f}^k \omega - \mathrm{tr}_{\partial f}\,\omega\|_{\partial f} \|\,\mathrm{tr}_{\partial f}\,\mu_f\|_{\partial f} \\ &\lesssim h_f^{r+1} |\,\mathrm{tr}_{\partial f}\,\omega|_{r,\partial f,\Delta} \|\,\mathrm{tr}_{\partial f}\,\mu_f\|_{\partial f} \\ &\lesssim h_f^r |\omega|_{r,f,\Delta} \|\mu_f\|_f, \end{aligned} \tag{25}$$

where we have used the induction hypothesis (22) on all $f' \in \partial f$ (the notation $|\cdot|_{r,\partial f,\Delta}$ stands for $\sum_{f' \in \Delta_{d-1}(f)} |\cdot|_{r,f',\Delta}$) and the conclusion follows from the discrete trace inequality (19) together with the definition (21) of the scaled norm, which gives

$$|\,\mathrm{tr}_{\partial f}\,\omega|_{r,\partial f,\Delta} \lesssim h_f^{-1/2} |\omega|_{r,f,\Delta}. \tag{26}$$

With $\mu_f = \mathrm{d}_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega - \pi_{r,f}^{d-k-1} \star \mathrm{d}\omega \in \mathcal{P}_r\Lambda^{d-k-1}(f)$, the left-hand side of (25) is equal to $\|\mathrm{d}_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega - \star^{-1}\pi_{r,f}^{d-k-1} \star \mathrm{d}\omega\|_f^2$. Simplifying, we therefore obtain

$$\|\mathrm{d}_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega - \star^{-1}\pi_{r,f}^{d-k-1} \star \mathrm{d}\omega\|_f \lesssim h_f^r |\omega|_{r,f,\Delta}.$$

Introducing $\pm \mathrm{d}\omega$ and using a triangle inequality together with the approximation property (20) (with $\mathrm{d}\omega$ instead of ω and $t = r-1$ if $r \ge 1$; if $r = 0$, we simply use the $L^2(f)$-boundedness of $\pi_{r,f}^{d-k-1}$), we infer the following sub-optimal version of (23):

$$\|\mathrm{d}_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega - \mathrm{d}\omega\|_f \lesssim h_f^r |\omega|_{r,f,\Delta}. \tag{27}$$

We can now prove (22). For all $\mu_f \in \kappa\mathcal{P}_r\Lambda^{d-k}(f)$ and $\nu_f \in \kappa\mathcal{P}_{r-1}\Lambda^{d-k+1}(f)$, we have from the definitions (13) of $P_{r,f}^k$ and (10) of $\underline{\boldsymbol{I}}_{r,f}^k$,

$$\begin{aligned}(-1)^{k+1} &\int_f P_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega \wedge (\mathrm{d}\mu_f + \nu_f) \\ &= \int_f \mathrm{d}_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega \wedge \mu_f - \int_{\partial f} P_{r,\partial f}^k \underline{\boldsymbol{I}}_{r,\partial f}^k \omega \wedge \mathrm{tr}_{\partial f}\,\mu_f + (-1)^{k+1} \int_f \star^{-1}\pi_{r,f}^{-,d-k} \star \omega \wedge \nu_f \\ &\overset{\nu_f \in \mathcal{P}_r^-\Lambda^{d-k}(f)}{=} \int_f \mathrm{d}\omega \wedge \mu_f - \int_{\partial f} \mathrm{tr}_{\partial f}\,\omega \wedge \mathrm{tr}_{\partial f}\,\mu_f + (-1)^{k+1} \int_f \omega \wedge \nu_f \\ &\quad - \int_{\partial f} (P_{r,\partial f}^k \underline{\boldsymbol{I}}_{r,\partial f}^k \omega - \mathrm{tr}_{\partial f}\,\omega) \wedge \mathrm{tr}_{\partial f}\,\mu_f + \int_f (\mathrm{d}_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega - \mathrm{d}\omega) \wedge \mu_f \\ &= (-1)^{k+1} \int_f \omega \wedge (\mathrm{d}\mu_f + \nu_f) - \int_{\partial f} (P_{r,\partial f}^k \underline{\boldsymbol{I}}_{r,\partial f}^k \omega - \mathrm{tr}_{\partial f}\,\omega) \wedge \mathrm{tr}_{\partial f}\,\mu_f \\ &\quad + \int_f (\mathrm{d}_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega - \mathrm{d}\omega) \wedge \mu_f, \end{aligned}$$

the conclusion following from the Stokes formula. A Cauchy-Schwartz inequality then gives

$$\begin{aligned}(-1)^{k+1} &\int_f (P_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega - \omega) \wedge (\mathrm{d}\mu_f + \nu_f) \\ &\le \|P_{r,\partial f}^k \underline{\boldsymbol{I}}_{r,\partial f}^k \omega - \mathrm{tr}_{\partial f}\,\omega\|_{\partial f} \|\,\mathrm{tr}_{\partial f}\,\mu_f\|_{\partial f} + \|\mathrm{d}_{r,f}^k \underline{\boldsymbol{I}}_{r,f}^k \omega - \mathrm{d}\omega\|_f \|\mu_f\|_f \\ &\lesssim h_f^{r+1} |\,\mathrm{tr}_{\partial f}\,\omega|_{r,\partial f,\Delta} \|\,\mathrm{tr}_{\partial f}\,\mu_f\|_{\partial f} + h_f^r |\omega|_{r,f,\Delta} \|\mu_f\|_f \\ &\lesssim h_f^r |\omega|_{r,f,\Delta} \|\mu_f\|_f, \end{aligned} \tag{28}$$

where we have used (22) for $d-1$ (induction hypothesis) together with (27) in the second inequality, and the discrete trace inequality (19) together with (26) in the conclusion. By Lemma 5, we can take μ_f and ν_f such that $\mathrm{d}\mu_f + \nu_f = (-1)^{k+1}(\star P^k_{r,f}\underline{I}^k_{r,f}\omega - \pi^{d-k}_{r,f}\star\omega)$, and (17) and (18) ensure that

$$\|\mu_f\|_f \lesssim h_f\|\mathrm{d}\mu_f\|_f \lesssim h_f\|\star P^k_{r,f}\underline{I}^k_{r,f}\omega - \pi^{d-k}_{r,f}\star\omega\|_f = h_f\|P^k_{r,f}\underline{I}^k_{r,f}\omega - \star^{-1}\pi^{d-k}_{r,f}\star\omega\|_f. \tag{29}$$

The bound (28) then gives

$$\begin{aligned}
&\|P^k_{r,f}\underline{I}^k_{r,f}\omega - \star^{-1}\pi^{d-k}_{r,f}\star\omega\|_f^2\\
&\quad= \int_f (P^k_{r,f}\underline{I}^k_{r,f}\omega - \omega)\wedge(\mathrm{d}\mu_f + \nu_f) + \int_f (\omega - \star^{-1}\pi^{d-k}_{r,f}\star\omega)\wedge(\mathrm{d}\mu_f + \nu_f)\\
&\quad\lesssim h_f^r|\omega|_{r,f,\Delta}\|\mu_f\|_f + \|\omega - \star^{-1}\pi^{d-k}_{r,f}\star\omega\|_f\|\mathrm{d}\mu_f + \nu_f\|_f\\
&\quad\overset{(29),(20)}{\lesssim} h_f^{r+1}|\omega|_{r,f,\Delta}\|P^k_{r,f}\underline{I}^k_{r,f}\omega - \star^{-1}\pi^{d-k}_{r,f}\star\omega\|_f
\end{aligned}$$

with the conclusion following from (29) and the approximation property (20) of $\pi^{d-k}_{r,f}$ with $t = r$. The bound (22) follows then by invoking (20) with $t = r$ and the triangle inequality

$$\|P^k_{r,f}\underline{I}^k_{r,f}\omega - \omega\|_f \le \|P^k_{r,f}\underline{I}^k_{r,f}\omega - \star^{-1}\pi^{d-k}_{r,f}\star\omega\|_f + \|\star^{-1}\pi^{d-k}_{r,f}\star\omega - \omega\|_f.$$

Finally, (23) follows by applying (22) to $\mathrm{d}\omega$ and $k+1$ instead of ω and k, and by using the commutation property (16) and the link (15) (also with $k+1$ instead of k) between potential reconstruction and discrete exterior derivative. □

4 Application

We present here a 2+1 model for Maxwell equations on a manifold written in the language of differential forms, and use the DDR complex to design a scheme for this model. We refer the reader to the notations recalled in Appendix B and used throughout this section.

4.1 Electromagnetism in $2+1$ dimensions

Following [27, 28], we foliate a 3-dimensional space-time manifold M by level sets of a time function t. Letting[1] $g = g_{\mu\nu}dx^\mu\otimes dx^\nu$ denote a Lorentzian metric on M, we perform a $2+1$ decomposition on g via

$$(g_{\mu\nu}) := \begin{pmatrix} -N^2 + |\beta|^2_\gamma & \beta_j \\ \beta_i & \gamma_{ij}\end{pmatrix},$$

where $\gamma = \gamma_{ij}dx^i\otimes dx^j$ is the induced metric on the t = constant spatial surfaces, N is the lapse, $\beta = \beta_i dx^i$ is the shift, and we use (x^i) to denote spatial coordinates on the spatial surfaces. We define a future pointing unit normal to the spatial surfaces by

$$n := (-N\mathrm{d}t)^\sharp, \tag{30}$$

and in the following, we distinguish geometric objects associated to the spatial surface with a tilde; for example, $\widetilde{\mathrm{vol}}$ and $\widetilde{\sharp}$ are the volume form and $\sharp$ operator associated with the spatial metric γ, respectively, on the spatial surfaces. Further, we introduce the vector field ∂_t via

$$\partial_t = Nn + \beta^{\widetilde{\sharp}},$$

[1] Throughout this section, we use lower case Greek letters, e.g. μ, ν, γ to label space-time coordinate indices that run over $0, 1, 2$ while lower case Latin letters, e.g. i, j, k, will label spatial coordinate indices that run over $1, 2$.

and note that for any adapted coordinate system (t, x^i), i.e. which satisfies $\mathcal{L}_{\partial_t} x^i = 0$, the vector field ∂_t will coincide with the coordinate vector field ∂_t associated to the coordinate system (t, x^i).

Letting $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ denote the electromagnetic field tensor, Maxwell's equations can be expressed in terms of F as [28, Section 18.2]

$$\mathrm{d}F = 0, \tag{31a}$$

$$\mathrm{d} \star F = \epsilon_0^{-1} \star \underline{j}, \tag{31b}$$

where $\underline{j}$ is the electric 3-current and ϵ_0 is the permittivity of the medium, supposed constant here. Following the usual convention in $3+1$ dimension, we define the electric field E and the magnetic field B by

$$E := -i_n F, \tag{32}$$

$$B := i_n(\star F). \tag{33}$$

Notice that, in $2+1$ dimensions, the magnetic field is a scalar field. Below, we will use $\widetilde{E}$ to denote the restriction of E to the spatial surfaces, that is, $\widetilde{E}(\tau) = \iota_\tau^* E$ where ι_τ is the inclusion map of the spatial surface $t = \tau$. If E is expressed in terms of the adapted coordinate (t, x^i) as $E = E_0 dt + E_i dx^i$, then $\widetilde{E} = E_i dx^i$ and $E = E_0 dt + \widetilde{E}$.

In the following lemmas, we show how the electromagnetic field tensor and its Hodge dual can be recovered from the electric and magnetic fields.

Lemma 22. *We can recover F from E and B by the relation*

$$F = n^\flat \wedge E + B\, i_n \mathrm{vol}. \tag{34}$$

Proof. Noting from (33) that $B = \frac{\sqrt{-\det g}}{2} \epsilon^{\alpha\beta}{}_\gamma F_{\alpha\beta} n^\gamma$, we can, with the help of (59), express $B\, i_n \mathrm{vol}$ as

$$\begin{aligned}
(B\, i_n \mathrm{vol})_{\mu\nu} &= \frac{-\det g}{2} \epsilon^{\alpha\beta}{}_\gamma F_{\alpha\beta} n^\gamma n^\lambda \epsilon_{\lambda\mu\nu} \\
&= -\frac{\det g}{2} \epsilon^{\alpha\beta\gamma} \epsilon_{\lambda\mu\nu} n_\gamma n^\lambda F_{\alpha\beta} \\
&\overset{(61)}{=} -\frac{1}{2} \delta^{\alpha\beta\gamma}_{\lambda\mu\nu} n_\gamma n^\lambda F_{\alpha\beta} \\
&= -\frac{1}{2} \left(\delta^\alpha_\lambda \delta^{\beta\gamma}_{\mu\nu} - \delta^\beta_\lambda \delta^{\alpha\gamma}_{\mu\nu} \right) n_\gamma n^\lambda F_{\alpha\beta} - \frac{1}{2} \delta^\gamma_\lambda \delta^{\alpha\beta}_{\mu\nu} n_\gamma n^\lambda F_{\alpha\beta}.
\end{aligned} \tag{35}$$

Next, from the definition (32) of E, we observe that $E_\alpha = -n^\beta F_{\beta\alpha} = \delta^\beta_\lambda n^\lambda F_{\alpha\beta}$. This allows us to express the following terms which appear in the last line of (35) in terms of $E \wedge n^\flat$ via the calculations:

$$\begin{aligned}
\delta^\alpha_\lambda n^\lambda F_{\alpha\beta} \delta^{\beta\gamma}_{\mu\nu} n_\gamma &= -E_\beta n_\gamma \delta^{\beta\gamma}_{\mu\nu} = -(E \wedge n^\flat)_{\mu\nu}, \\
\delta^\beta_\lambda n^\lambda F_{\alpha\beta} \delta^{\alpha\gamma}_{\mu\nu} n_\gamma &= E_\alpha n_\gamma \delta^{\alpha\gamma}_{\mu\nu} = (E \wedge n^\flat)_{\mu\nu},
\end{aligned} \tag{36}$$

where we have used (62) in the second equality in each line. Moreover, since $n^2 = -1$ and F antisymmetric, we have

$$\delta^\gamma_\lambda n_\gamma n^\lambda \delta^{\alpha\beta}_{\mu\nu} F_{\alpha\beta} = 2 n_\lambda n^\lambda F_{\mu\nu} = -2 F_{\mu\nu}. \tag{37}$$

Plugging (36) and (37) into (35) gives $B\, i_n \mathrm{vol} = E \wedge n^\flat + F$. □

Lemma 23. *The Hodge star of F is given by*

$$\star F = i_{\widetilde{E}^\sharp} i_n \mathrm{vol} - n^\flat B. \tag{38}$$

Proof. To establish the stated formula, we apply the Hodge star operator to each term of (34) individually. For the term involving E, we have:

$$\begin{aligned}(\star(n^\flat \wedge E))_\mu &\overset{(60),(62)}{=} \frac{\sqrt{-\det g}}{2}\epsilon^{\alpha\beta}{}_\mu \delta^{\alpha'\beta'}_{\alpha\beta} n_{\alpha'} E_{\beta'} \\ &= \frac{\sqrt{-\det g}}{2}\epsilon_{\alpha\beta\mu}(n^\alpha E^\beta - n^\beta E^\alpha) \\ &= \sqrt{-\det g}\epsilon_{\alpha\beta\mu} n^\alpha E^\beta \\ &= \left(i_{E^\sharp} i_n \mathrm{vol}\right)_\mu \overset{(65)}{=} \left(i_{\widetilde{E}^\sharp} i_n \mathrm{vol}\right)_\mu,\end{aligned}$$

while for the term involving B, we have:

$$\begin{aligned}(\star B\, i_n \mathrm{vol})_\mu &= \frac{\sqrt{-\det g}}{2}\epsilon^{\alpha\beta}{}_\mu B n^\lambda \sqrt{-\det g}\epsilon_{\lambda\alpha\beta} \\ &= -\frac{\det g}{2} g_{\mu\mu'}\epsilon^{\alpha\beta\mu'}\epsilon_{\alpha\beta\lambda} n^\lambda B \\ &= -\frac{1}{2} g_{\mu\mu'}\delta^{\alpha\beta\mu'}_{\alpha\beta\lambda} n^\lambda B \\ &= -\frac{1}{2} 2! g_{\mu\mu'}\delta^{\mu'}{}_\lambda n^\lambda B \\ &= -n_\mu B.\end{aligned}$$

Adding the above two expression yields the desired formula. □

In order to present Maxwell's equation in their simplest form, we will restrict our attention to foliations for which $\beta = 0$. In the next lemma, we derive $2+1$ decompositions for $\mathrm{d}F$ and $\mathrm{d}\star F$, which will be used below to perform a $2+1$ decomposition of Maxwell's equations.

Lemma 24. *If $\beta = 0$, then*

$$\mathrm{d}F = \mathrm{d}t \wedge \left[\widetilde{\mathrm{d}}(N\widetilde{E}) + \mathcal{L}_{\partial_t}(B\,\widetilde{\mathrm{vol}})\right], \tag{39}$$

$$\mathrm{d}\star F = \widetilde{\mathrm{d}}\widetilde{\star}\widetilde{E} + \mathrm{d}t \wedge \left[\mathcal{L}_{\partial_t}(\widetilde{\star}\widetilde{E}) - \widetilde{\mathrm{d}}(NB)\right]. \tag{40}$$

Proof. We infer from (63) with $\beta = 0$ that $i_n \mathrm{vol} = \widetilde{\mathrm{vol}}$. Replacing $n^\flat$ by its definition (30) in (34) gives

$$F = -N\mathrm{d}t \wedge E + B\,\widetilde{\mathrm{vol}} = -N\mathrm{d}t \wedge \widetilde{E} + B\,\widetilde{\mathrm{vol}}, \tag{41}$$

where in deriving the second equality, we used the fact that $E = \widetilde{E} + E_0 \mathrm{d}t$. Next, we apply the differential to each term in (41) to obtain:

$$\mathrm{d}(-\mathrm{d}t \wedge N\widetilde{E}) = \mathrm{d}t \wedge \mathrm{d}(N\widetilde{E}) = \mathrm{d}t \wedge \widetilde{\mathrm{d}}(N\widetilde{E}),$$

$$\mathrm{d}(B\,\widetilde{\mathrm{vol}}) = \partial_t B\, \mathrm{d}t \wedge \widetilde{\mathrm{vol}} + \cancel{\widetilde{\mathrm{d}}B \wedge \widetilde{\mathrm{vol}}} + B\, \mathrm{d}t \wedge \partial_t \widetilde{\mathrm{vol}} + \cancel{B\,\widetilde{\mathrm{d}}\widetilde{\mathrm{vol}}}.$$

Since $\partial_t(B\widetilde{\mathrm{vol}}) = \mathcal{L}_{\partial_t}(B\widetilde{\mathrm{vol}})$, (39) follow from adding the above two expressions.

Noting that

$$(i_{\widetilde{E}^\sharp}\widetilde{\mathrm{vol}})_j = \sqrt{\det\gamma}\widetilde{E}^i\epsilon_{ij} = \sqrt{\det\gamma}\epsilon^i{}_j\widetilde{E}_i = (\widetilde{\star}\widetilde{E})_j,$$

where the indices are raised with the spatial metric γ, we can express (38) as $\star F = \widetilde{\star}\widetilde{E} + BN\mathrm{d}t$. Formula (40) is then readily obtained from applying the differential to this expression and employing the identity $\mathrm{d}\widetilde{\star}\widetilde{E} = \widetilde{\mathrm{d}}\widetilde{\star}\widetilde{E} + dt \wedge \mathcal{L}_{\partial_t}(\widetilde{\star}\widetilde{E})$. □

The last $2+1$ decomposition that we will need is for the electric 3-current $\underline{j}$, which we will represent in terms the electric charge density $\rho := -\underline{j}(n)$, and electric current density $J := N(\underline{j} - n^\flat \rho)$.

Lemma 25. *The electric 3-current appearing in (31b) is decomposed as follows*

$$\star \underline{j} = \rho \, \widetilde{\mathrm{vol}} - \mathrm{d}t \wedge \widetilde{\star} \widetilde{J}. \tag{42}$$

Proof. From the formula (60) for the Hodge star operator, we have

$$\begin{aligned}(\star \underline{j})_{\mu\nu} &= \sqrt{-\det g}\, \epsilon^{\alpha}{}_{\mu\nu} \underline{j}_\alpha \\ &= \sqrt{-\det g}\, \left(\underline{j}^0 \delta^{12}_{\mu\nu} - (\delta^0_\mu \delta^{12}_{\alpha\nu} - \delta^0_\nu \delta^{12}_{\alpha\mu}) \underline{j}^\alpha \right),\end{aligned}$$

while we infer from (64) and the definition of ρ, recalling that we have set $\beta = 0$ for simplicity, that

$$\sqrt{-\det g}\, \underline{j}^0 = \frac{\sqrt{-\det g}}{N} \rho = \sqrt{\det \gamma}\, \rho,$$

and thus $\sqrt{-\det g}\, \underline{j}^0 \delta^{12}_{\mu\nu} = (\rho \widetilde{\mathrm{vol}})_{\mu\nu}$. Moreover, we have

$$\sqrt{-\det g}\, \delta^{12}_{\alpha\mu} \underline{j}^\alpha = N \underline{j}^\alpha \sqrt{\det \gamma}\, \epsilon_{\alpha\mu} = (i_{N \underline{j}^\sharp} \widetilde{\mathrm{vol}})_\mu,$$

and hence

$$\sqrt{-\det g}\, (\delta^0_\mu \delta^{12}_{\alpha\nu} - \delta^0_\nu \delta^{12}_{\alpha\mu}) \underline{j}^\alpha = (\mathrm{d}t \wedge i_{N \underline{j}^\sharp} \widetilde{\mathrm{vol}})_{\mu\nu}.$$

We also have

$$\mathrm{d}t \wedge i_{N \underline{j}^\sharp} \widetilde{\mathrm{vol}} = \mathrm{d}t \wedge i_{J^\sharp} \widetilde{\mathrm{vol}} + \mathrm{d}t \wedge i_{N \rho n} \widetilde{\mathrm{vol}} = \mathrm{d}t \wedge i_{J^\sharp} \widetilde{\mathrm{vol}},$$

where the last equality holds by (63) and the assumption $\beta = 0$, which show that $i_{N\rho n^\flat} \widetilde{\mathrm{vol}} = N \rho i_n i_n \mathrm{vol} = 0$. We also observe from (64) with $V = J$, noticing that $J(n) = N(\underline{j}(n) - n^2 \rho) = N(\underline{j}(n) - \underline{j}(n)) = 0$, that $J^\sharp = \widetilde{J}^{\widetilde{\sharp}}$, and consequently, that $i_{J^\sharp} \widetilde{\mathrm{vol}} = \widetilde{\star} \widetilde{J}$. Putting everything together, we deduce that $\star \underline{j} = \rho \, \widetilde{\mathrm{vol}} - \mathrm{d}t \wedge \widetilde{\star} \widetilde{J}$. □

We denote the spatial codifferential on k-forms by $\widetilde{\delta} := (-1)^k \widetilde{\star}^{-1} \widetilde{\mathrm{d}} \widetilde{\star}$. We now use the $2+1$ decompositions established above to derive a $2+1$ formulation of Maxwell's equations.

Theorem 26. *If $\beta = 0$ then the Maxwell equations (31) are equivalent to the system*

$$\begin{aligned}\widetilde{\mathrm{d}}(N \widetilde{E}) &= -\partial_t (B \widetilde{\mathrm{vol}}), \\ -\widetilde{\delta} \widetilde{E} &= \frac{\rho}{\epsilon_0}, \\ \widetilde{\delta}(N B \widetilde{\mathrm{vol}}) &= \epsilon_0^{-1} \underline{J} + \widetilde{\star}^{-1} \partial_t (\widetilde{\star} \widetilde{E}).\end{aligned} \tag{43}$$

Proof. The result follows directly from (39), (40) and (42) by noticing that $\widetilde{\star}^{-1} \widetilde{\mathrm{d}} \widetilde{\star} \widetilde{E} = -\widetilde{\delta} \widetilde{E}$, since $\widetilde{E}$ is a 1-form, and that $\widetilde{\star}^{-1} \widetilde{\mathrm{d}}(NB) = \widetilde{\star}^{-1} \widetilde{\mathrm{d}}(\widetilde{\star} \widetilde{\mathrm{vol}} N B) = \widetilde{\delta}(N B \widetilde{\mathrm{vol}})$. □

For the numerical tests, we will consider the case where the metric does not depend on the time coordinate, and where $N \equiv c$ is constant. The system (43) then reduces to the more familiar form

$$\widetilde{\mathrm{d}} \widetilde{E} = -\partial_t B', \tag{44a}$$

$$-\widetilde{\delta}\widetilde{E} = \frac{\rho}{\epsilon_0}, \tag{44b}$$

$$\widetilde{\delta}B' = \mu_0\widetilde{J} + \frac{1}{c^2}\partial_t\widetilde{E}, \tag{44c}$$

where $B' := \frac{1}{c}B\widetilde{\mathrm{vol}}$ and $\mu_0 := \frac{1}{c^2\epsilon_0}$ is the vacuum permeability. To simplify, we will also work in geometric units and take thus $c = \epsilon_0 = 1$.

We note that the following compatibility condition (from hereon assumed) on the source terms, following from the property $\widetilde{\delta}^2 = 0$:

$$\widetilde{\delta}\,\widetilde{J} = -\partial_t\widetilde{\delta}\,\widetilde{E} = \partial_t\rho. \tag{45}$$

We assume that the manifold Ω has no boundary. Denoting by $\langle\cdot,\cdot\rangle$ the L^2-inner product on spatial k-forms (for any k), recalling that $\widetilde{\delta}$ is the adjoint of $\widetilde{\mathrm{d}}$ for this inner product, and assuming that the constraint (44b) holds at time $t = 0$, a weak formulation of (44) is: find $(\widetilde{E}, \widetilde{B}) \in C^1([0,T];\Lambda^1(\Omega)) \times C^1([0,T];\Lambda^2(\Omega))$ such that, for all $(v^1, v^2) \in \Lambda^1(\Omega) \times \Lambda^2(\Omega)$,

$$\langle\widetilde{\mathrm{d}}\widetilde{E}, v^2\rangle = -\langle\partial_t B', v^2\rangle, \tag{46a}$$

$$\langle B', \widetilde{\mathrm{d}}v^1\rangle = \langle\widetilde{J}, v^1\rangle + \langle\partial_t\widetilde{E}, v^1\rangle. \tag{46b}$$

We note that the constraint (44b) has been dropped from this formulation as it can be recovered by selecting a generic (time-independent) $v^0 \in \Lambda^0(\Omega)$ and setting $v^1 = \widetilde{\mathrm{d}}v^0$ in (46b), to see that

$$\langle\partial_t\widetilde{\delta}\widetilde{E}, v^0\rangle = \langle\partial_t\widetilde{E}, \widetilde{\mathrm{d}}v^0\rangle = \langle B', \cancel{\widetilde{\mathrm{d}}^2 v^0}\rangle - \langle\widetilde{J}, \widetilde{\mathrm{d}}v^0\rangle = -\langle\widetilde{\delta}\widetilde{J}, v^0\rangle = -\langle\partial_t\rho, v^0\rangle,$$

showing that $\partial_t\langle\widetilde{\delta}\widetilde{E}+\rho, v^0\rangle = 0$ and thus, since we assumed that $\widetilde{\delta}\widetilde{E}+\rho = 0$ at $t = 0$, that $\langle\widetilde{\delta}\widetilde{E}+\rho, v^0\rangle = 0$ at all time.

4.2 Discrete scheme

We describe here a numerical scheme based on the discrete de Rham complex for the system (44). This scheme is of arbitrary order of accuracy, applicable to polygonal meshes on the chosen 2D manifold, and naturally preserves the constraint (44b) at the discrete level.

Recalling the definition of the discrete L^2-like inner product (Definition 11), the (semi-discrete) scheme is then built on the product space $X_h := \underline{\boldsymbol{X}}^1_{r,h} \times \underline{\boldsymbol{X}}^2_{r,h}$ and reads: find $(\underline{\boldsymbol{E}}_h, \underline{\boldsymbol{B}}'_h) \in C^1([0,T];X_h)$ such that for all $(\underline{\boldsymbol{v}}^1_h, \underline{v}^2_h) \in X_h$ and all $t \in (0,T)$,

$$\langle\underline{\mathrm{d}}^1_{r,h}\underline{\boldsymbol{E}}_h(t), \underline{v}^2_h\rangle = -\langle\partial_t\underline{\boldsymbol{B}}'_h(t), \underline{v}^2_h\rangle, \tag{47a}$$

$$\langle\underline{\boldsymbol{B}}'_h(t), \underline{\mathrm{d}}^1_{r,h}\underline{\boldsymbol{v}}^1_h\rangle = \langle\underline{\boldsymbol{I}}^1_{r,h}\widetilde{J}(t), \underline{\boldsymbol{v}}^1_h\rangle + \langle\partial_t\underline{\boldsymbol{E}}_h(t), \underline{\boldsymbol{v}}^1_h\rangle. \tag{47b}$$

We note that, in the context of the DDR complex, the discrete *spatial* exterior derivatives $\underline{\mathrm{d}}^k_{r,h}$ are denoted without a tilde.

Owing to the properties of the DDR complex, this scheme preserves a discrete version of the constraint.

Proposition 27 (Discrete constraint preservation)**.** *Let $\rho_h \in C^1([0,T];\underline{\boldsymbol{X}}^0_{r,h})$ be the discrete electric charge density defined by*

$$\langle\rho_h(t), \underline{v}^0_h\rangle = -\langle\underline{\boldsymbol{E}}_h(0), \underline{\mathrm{d}}^0_{r,h}\underline{v}^0_h\rangle + \int_0^t \langle\underline{\boldsymbol{I}}^1_{r,h}\widetilde{J}(s), \underline{\mathrm{d}}^0_{r,h}\underline{v}^0_h\rangle\, ds \qquad \forall t \in [0,T]\,, \forall\underline{v}^0_h \in \underline{\boldsymbol{X}}^0_{r,h}. \tag{48}$$

If $(\underline{\boldsymbol{E}}_h, \underline{\boldsymbol{B}}'_h)$ is a solution to (47), then

$$\langle\underline{\boldsymbol{E}}_h(t), \underline{\mathrm{d}}^0_{r,h}\underline{v}^0_h\rangle = -\langle\rho_h(t), \underline{v}^0_h\rangle \qquad \forall t \in [0,T]\,, \forall\underline{v}^0_h \in \underline{\boldsymbol{X}}^0_{r,h}. \tag{49}$$

Remark 28 (On the discrete constraint preservation). By the Riesz representation theorem, (48) uniquely defines the discrete electric charge density $\rho_h(t) \in \underline{\boldsymbol{X}}^0_{r,h}$. This definition only depends on the data of the model: initial value of the electric field, and electric current density and, given the relation (45), provides a consistent discrete version of the continuous electric charge density.

Letting $\underline{\delta}^1_{r,h}$ be the adjoint of $\underline{\mathrm{d}}^1_{r,h}$, the relation (49) can be recast as $\underline{\delta}^1_{r,h}\underline{\boldsymbol{E}}_h(t) = -\rho_h(t)$, which is a coherent discrete version of the continuous constraint (44b) (recall that $\epsilon_0 = 1$ here).

Proof. Let $\underline{v}^0_h \in \underline{\boldsymbol{X}}^0_{r,h}$ and set $\underline{\boldsymbol{v}}^1_h = \underline{\mathrm{d}}^0_{r,h}\underline{v}^0_h$ in (47b). Since $\underline{\mathrm{d}}^1_{r,h}\underline{\mathrm{d}}^0_{r,h} = 0$ by complex property of the DDR sequence, we obtain

$$\langle \partial_t \underline{\boldsymbol{E}}_h(t), \underline{\mathrm{d}}^0_{r,h}\underline{v}^0_h\rangle = -\langle \underline{\boldsymbol{I}}^1_{r,h}\widetilde{J}(t), \underline{\mathrm{d}}^0_{r,h}\underline{v}^0_h\rangle = -\langle \partial_t \rho_h(t), \underline{v}^0_h\rangle,$$

where the second equality follows differentiating the definition (48) of ρ_h with respect to t. Since $\underline{v}^0_h$ does not depend on t, this shows that the derivative of $C(t) := \langle \underline{\boldsymbol{E}}_h(t), \underline{\mathrm{d}}^0_{r,h}\underline{v}^0_h\rangle + \langle \rho_h(t), \underline{v}^0_h\rangle$ vanishes. Since $C(0) = 0$ by (48), we infer that $C = 0$ on $[0,T]$, which proves (49). □

Lemma 29 (Energy preservation)**.** *In the absence of a current density, the solution* $(\underline{\boldsymbol{E}}_h, \underline{\boldsymbol{B}}'_h)$ *of* (47) *satisfies*

$$\langle \partial_t \underline{\boldsymbol{E}}_h, \underline{\boldsymbol{E}}_h\rangle + \langle \partial_t \underline{\boldsymbol{B}}'_h, \underline{\boldsymbol{B}}'_h\rangle = 0. \tag{50}$$

Proof. The result readily follows evaluating (47a) and (47b) with $\underline{\boldsymbol{v}}^1_h = \underline{\boldsymbol{E}}_h$ and $\underline{v}^2_h = \underline{\boldsymbol{B}}'_h$. □

Remark 30 (Discrete energy preservation). When using a time discretisation scheme preserving quadratic invariants, such as the Crank–Nicolson time stepping, a discrete version of (50) can be established and shows that the total energy of the system $\langle \underline{\boldsymbol{E}}_h, \underline{\boldsymbol{E}}_h\rangle + \langle \underline{\boldsymbol{B}'}_h, \underline{\boldsymbol{B}'}_h\rangle$ remains constant in time. Dissipative time-stepping, such as the implicit Euler method, lead to a decrease of the total energy. In both cases, the preservation or decrease of energy establishes the stability and well-posedness of the scheme for this linear model.

4.3 Test case

We verify the scheme on a manufactured solution, using a Crank–Nicolson time-stepping for the time discretisation. We consider an unit sphere with an atlas consisting of the north and south stereographic projections, and endowed with the induced metric from $\mathbb{R}^3$. In each map, this induced metric is given by

$$\gamma_{ij} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda := \frac{4}{(1+X^2+Y^2)^2},$$

where X, Y are the coordinates in the map. Writing $\widetilde{E} = E_X \mathrm{d}X + E_Y \mathrm{d}Y$, we have

$$\widetilde{\mathrm{d}}\widetilde{E} = \frac{1}{\lambda}(\partial_X E_Y - \partial_Y E_X)\widetilde{\mathrm{vol}}, \quad -\widetilde{\delta}\widetilde{E} = \frac{1}{\lambda}(\partial_X E_X + \partial_Y E_Y), \quad \widetilde{\delta}(B\widetilde{\mathrm{vol}}) = \partial_Y B \mathrm{d}X - \partial_X B \mathrm{d}Y.$$

The solution in the north map is given by

$$B' = \big[(X^2+Y^2-1)\cos(t) + X^2 + Y^2 + 1 - 2X\sin(t)\big]\mathrm{d}X \wedge \mathrm{d}Y,$$

$$\begin{aligned} E &= Y\big((2-X^2-Y^2)\sin(t) - 2X\cos(t)\big)/4\mathrm{d}X \\ &\quad + X\big((X^2+Y^2-2)\sin(t) + (3X^2+Y^2-3)\cos(t)\big)/4\mathrm{d}Y, \\ \rho &= 0, \\ J &= \big[Y\big(3(X^2+Y^2)^2/2(1+\cos(t)) + (X^2+Y^2)(3+5/4\cos(t)) + 1/2 - \cos(t)\big) \\ &\quad - XY(2X^2+2Y^2+5/2)\sin(t)\big]\mathrm{d}X \end{aligned}$$

$$+\Big[-X\big(3(X^2+Y^2)^2/2(1+\cos(t))+(X^2+Y^2)(3+5/4\cos(t))+1/2-\cos(t)\big)$$
$$+(10X^4+12X^2Y^2+15X^2+2Y^4+5Y^2-1)/4\sin(t)\Big]\mathrm{d}Y$$

and in the south map by

$$\begin{aligned}
B' &= \big[(X^2+Y^2-1)\cos(t)-X^2-Y^2-1+2X\sin(t)\big]\mathrm{d}X\wedge\mathrm{d}Y,\\
E &= Y\big((2-X^2-Y^2)\sin(t)+2X\cos(t)\big)/4\mathrm{d}X\\
&\quad +X\big((X^2+Y^2-2)\sin(t)-(3X^2+Y^2-3)\cos(t)\big)/4\mathrm{d}Y,\\
\rho &= 0,\\
J &= \Big[-Y\big(3(X^2+Y^2)^2/2(1-\cos(t))+(X^2+Y^2)(3-5/4\cos(t))+1/2+\cos(t)\big)\\
&\quad +XY(2X^2+2Y^2+5/2)\sin(t)\Big]\mathrm{d}X\\
&\quad +\Big[X\big(3(X^2+Y^2)^2/2(1-\cos(t))+(X^2+Y^2)(3-5/4\cos(t))+1/2+\cos(t)\big)\\
&\quad -(10X^4+12X^2Y^2+15X^2+2Y^4+5Y^2-1)/4\sin(t)\Big]\mathrm{d}Y.
\end{aligned}$$

We have also verified the scheme on a flat torus, in order to compare our results with a case featuring a trivial metric. The reference solution is given in coordinates $(X,Y)\in[0,1]^2$ by

$$\begin{aligned}
B' &= (2+d_2(X,Y,t))\mathrm{d}X\wedge\mathrm{d}Y,\\
E &= d_2(X,0.5,t)\mathrm{d}Y,\\
\rho &= 0,\\
J &= (2Y-1)\mathrm{d}X,
\end{aligned}$$

where $d_2(X,Y,t)=\|(X,Y)-(t,0.5)\|^2$ is the euclidean distance on the torus.

4.4 Building the mesh

Meshing manifolds while preserving the requirements of A is not trivial, and to our knowledge, there is no available software producing the required data. Therefore, we had to build the meshes for the test cases manually. The meshing of the flat torus is a Cartesian grid on $[0,1)^2$ using four charts to correctly match the boundary. The meshing of the sphere is more interesting, and is described in this section.

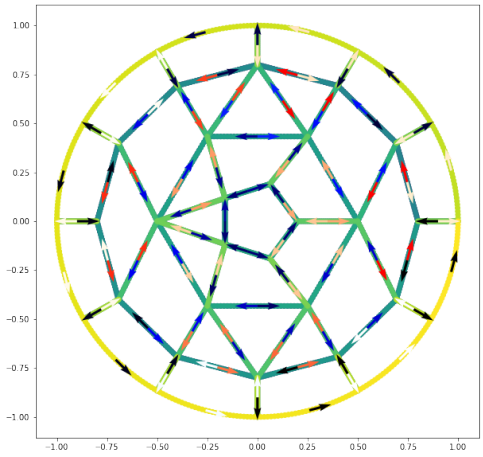
We restrict the two charts of the atlas to the north and south hemispheres; these charts do not overlap on an open set, as their transition occurs on a closed 1-dimensional interface (the equator), but that is sufficient to build a mesh and the associated polynomial spaces for the entire sphere. Each chart maps an hemisphere into the unit disk of $\mathbb{R}^2$, and coincide on the equator (that they map onto the unit circle). We therefore only have to discretise this unit disk to get the mesh on the whole sphere. In order to correctly discretise to boundary circle, we chose to use one layer of curved cell, and then to map the inside of the disk using arbitrary flat cells.

We devised an automated mesh generation algorithm. Starting from a chosen radius r_s, we cut the outer circle into $\lfloor 2\pi(1-r_s)/r_s\rfloor$ segment, each of the same length $\Delta\alpha$. Then, we used the construction A.2.2 with

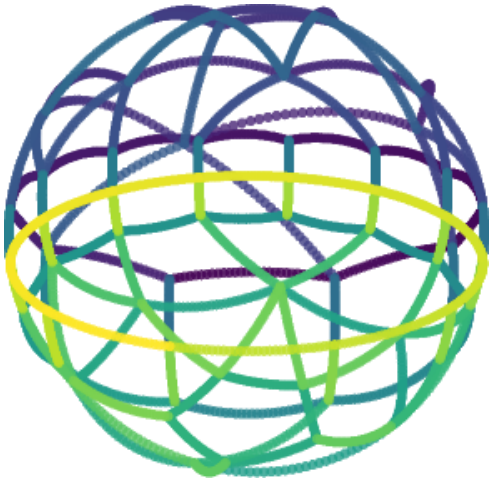
$$I_1(t)=R_{\alpha-\frac{\pi-\Delta\alpha}{2}}\begin{pmatrix}\frac{2tr}{\sqrt{1+(2tr)^2}}\\ \frac{1}{\sqrt{1+(2tr)^2}}\end{pmatrix},\quad t\in\left[-\frac{1}{2},\frac{1}{2}\right],$$

where $r=\frac{\cos(\frac{\pi-\Delta\alpha}{2})}{\sin(\frac{\pi-\Delta\alpha}{2})}$, α is the angle between E_4 and the line $x=0$ in the plane, and $R_{\alpha-\frac{\pi-\Delta\alpha}{2}}$ the rotation of angle $\alpha-\frac{\pi-\Delta\alpha}{2}$. The connected cell ensuring the transition between the curved boundary and flat interior is the cone given by the mapping

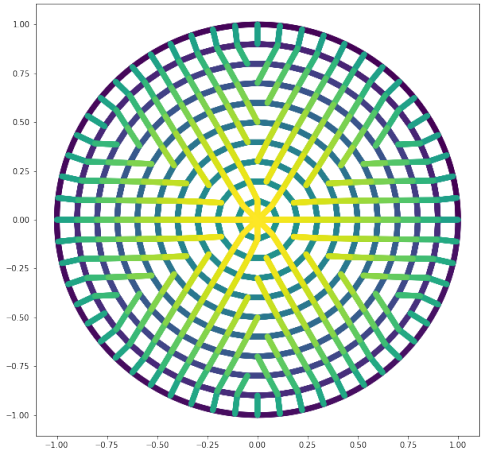
$$I_f(t,p)=R_{\alpha-\frac{\pi-\Delta\alpha}{2}}\begin{pmatrix}2t\frac{x_B}{y_B}\frac{p+(1-p)r}{\sqrt{1+(2t\frac{x_B}{y_B})^2}}\\ \frac{p+(1-p)r}{\sqrt{1+(2t\frac{x_B}{y_B})^2}}\end{pmatrix},\quad t\in\left[-\frac{1}{2},\frac{1}{2}\right],p\in[0,1],$$



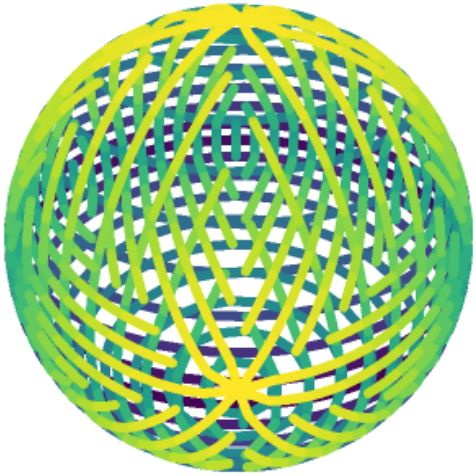
(a) Mesh used to discretise one chart with its orientation.



(b) Embedding of the 1-skeleton of the mesh into $\mathbb{R}^3$.



(c) Finer meshing of the disk.



(d) Embedding of the finer mesh into $\mathbb{R}^3$.

Figure 1: Visualization of the mesh

where

$$x_T = \cos(\frac{\pi - \Delta\alpha}{2}), \quad x_B = 0.8\cos(\frac{\pi - \Delta\alpha}{2}),$$
$$y_T = \sin(\frac{\pi - \Delta\alpha}{2}), \quad y_B = \sin(\frac{\pi - \Delta\alpha}{2}) \quad \text{and} \quad r = \sqrt{y_B^2 + (2tx_B)^2}.$$

Then, we divide the interior with concentric circles of radius $1 - ir_s$, for $1 \leq i \leq \lfloor 1/r_s \rfloor$, and connect the vertices of these circle to their nearest neighbor. The resulting mapping is given in Figure 1.

4.5 Results

We present here the results of the numerical tests. The scheme has been implemented in the Manicore C++ framework (see https://mlhanot.github.io/Manicore/), using linear algebra facilities from the Eigen3 library (see https://eigen.tuxfamily.org). Although the framework has been reimplemented from scratch, its design borrows from the HArDCore C++ framework implementing the DDR complexes on flat spaces (see https://github.com/jdroniou/HArDCore). From classical results

on flat spaces, we expect to see a convergence in space of order $r+1$ for the L^2 norm of E and B. In order to neglect the error originated from the time discretisation, we first used a time step of $\Delta t = 10^{-5}$. Then we compared the results with ones obtained using a larger time step of $\Delta t = 10^{-3}$ and saw only a negligible difference (a relative difference of order 10^{-6}), showing that the error from the space discretisation fully dominate. Therefore, we conducted all the tests using this larger time step. The simulation are computed for $0 \leq t \leq 2\pi$, and the error is the L^2 error over time and space

$$\sqrt{\int_{t=0}^{2\pi} \langle \underline{\boldsymbol{I}}^k_{r,h} E - \underline{E}_h, \underline{\boldsymbol{I}}^k_{r,h} E - \underline{E}_h \rangle}.$$

The error on E and B is given in Figure 2. We do not expect and have not noticed a convergence of the error on $\mathrm{d}E$, since the model (and thus the scheme) does not have any natural stability with respect to this quantity (the energy estimate is only on the L^2-norms of E and B). Because the proposed exact solution is not smooth, we do no necessarily expect an optimal convergence rate of order $r+1$. On the sphere, we however note a rate of convergence of order 1 when $r = 0$ and order 2 when $r \geq 1$. For the torus, the rates seem to stagnate at 1. However, in both cases, the error is reduced by almost an order of convergence when going from the lowest-order case to the case $r \geq 1$; such a phenomenon (improvement of the ratio cost/accuracy when using a slightly higher-order method, even when the solution is not smooth) has already been qualitatively observed in several other numerical schemes for different models [1, 29].

A Construction of local polynomial spaces on manifolds

A.1 Generic procedure

The notion of polynomial depends on the choice of coordinates. In general, we cannot cover a manifold with a single chart. Therefore, if we wish to define polynomial spaces on a manifold, the definition need to be coherent with (some) change of coordinates. We only need our polynomial spaces to be locally defined on d-cells, but we also need to ensure that restrictions of polynomials to the boundary of d-cells are polynomials on $(d-1)$-cells (of the same or lower degree), for all $d = 1, \dots, n$.

In this appendix, we show that Assumption 9 is sufficient in principle to construct suitable local polynomial spaces, and we detail in Section A.2 how to practically construct such spaces in dimension $n = 2$ and for various cell shapes.

In the following, we denote by $\mathcal{P}_r\Lambda^l(\mathbb{R}^d)$, the set of l-forms with coefficients that are polynomial on $\mathbb{R}^d$ of degree at most r, and $\mathcal{H}_r\Lambda^l(\mathbb{R}^d)$ the corresponding space of homogeneous polynomial forms of degree r. We also define X_d the identity vector field on $\mathbb{R}^d$, that is, $X_d(\boldsymbol{x}) = \boldsymbol{x}$, for all $\boldsymbol{x} \in \mathbb{R}^d$. We will often take the pullback of these object with functions that are not surjective on $\mathbb{R}^d$ (which is not an issue).

Definition 31. Under Assumption 9, we define the polynomial space on f by:

$$\mathcal{H}_r\Lambda^l(f) := J_f^* \mathcal{H}_r\Lambda^l(\mathbb{R}^d), \tag{51}$$

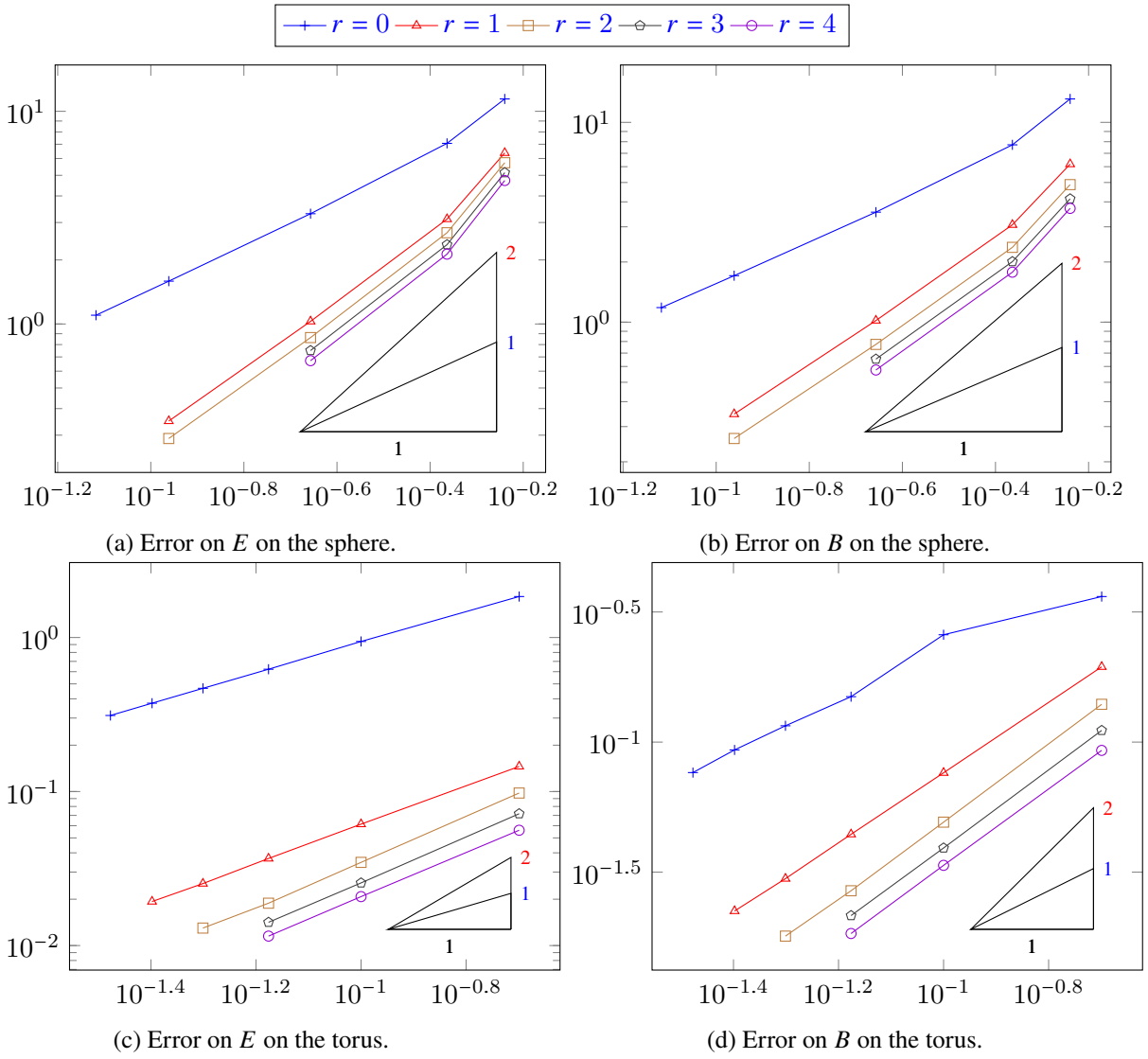
$$\mathcal{P}_r\Lambda^l(f) := \bigoplus_{s \leq r} \mathcal{H}_r\Lambda^l(f), \tag{52}$$

$$\kappa_f := i_{X_f}, \text{ with } X_f := (I_f)_* X_d. \tag{53}$$

In particular, we also have $\mathcal{P}_r\Lambda^l(f) = J_f^* \mathcal{P}_r\Lambda^l(\mathbb{R}^d)$.

Let us check that this construction satisfies Assumption 2 and 8. Moreover, if we assume that the diffeomorphisms J_f are regular, then this construction also satisfies Assumption 17.

Lemma 32. *Assumption 2 holds.*



(a) Error on E on the sphere.

(b) Error on B on the sphere.

(c) Error on E on the torus.

(d) Error on B on the torus.

Figure 2: Absolute error L^2 integrated over time and space vs. mesh size.

Proof. We first prove that, for these polynomial spaces, (5) and (6) are well-defined complexes. Since the exterior derivative d commutes with pullbacks, (51) and Lemma 3 give, for any (s, l),

$$\mathrm{d}\mathcal{H}_s\Lambda^l(f) = \mathrm{d}J_f^*\mathcal{H}_s\Lambda^l(\mathbb{R}^d) = J_f^*\mathrm{d}\mathcal{H}_s\Lambda^l(\mathbb{R}^d) \subset J_f^*\mathcal{H}_{s-1}\Lambda^{l+1}(\mathbb{R}^d) = \mathcal{H}_{s-1}\Lambda^{l+1}(f).$$

By taking the sum over $s \leq r$ on both sides and recalling (52), we infer that the spaces $(\mathcal{P}_r\Lambda^l(f))_{r\in\mathbb{Z},l\in\mathbb{Z}}$ form a complex for d.

If g is a diffeomorphism and v a vector field, then it holds that

$$i_{(g^{-1})_*v}g^* = g^*i_v. \tag{54}$$

Applying this result to a generic $J_f^*\alpha \in \mathcal{P}_r\Lambda^l(f)$, with $\alpha \in \mathcal{P}_r\Lambda^l(\mathbb{R}^d)$, and recalling the definition (53) of κ_f gives

$$\kappa_f J_f^*\alpha = i_{(I_f)_*X_d}(J_f^*\alpha) = J_f^*(i_{X_d}\alpha) \in J_f^*\mathcal{P}_{r+1}\Lambda^{l-1}(\mathbb{R}^d) = \mathcal{P}_{r+1}\Lambda^{l-1}(f). \tag{55}$$

This shows that the spaces $(\mathcal{P}_r\Lambda^l(f))_{r\in\mathbb{Z},l\in\mathbb{Z}}$ form a complex for κ_f.

Let us show that the graded decomposition $\mathcal{P}_r\Lambda^l(f) = \bigoplus_{s\leq r}\mathcal{H}_r\Lambda^l(f)$ satisfies Assumption **(A1)**. For any $p \in \mathcal{H}_s\Lambda^l(f)$, there is $\alpha \in \mathcal{H}_s\Lambda^l(\mathbb{R}^d)$ such that $J_f^*\alpha = p$, and we have

$$(\mathrm{d}\kappa_f + \kappa_f\mathrm{d})p = (\mathrm{d}\kappa_f + \kappa_f\mathrm{d})J_f^*\alpha = J_f^*(\mathrm{d}i_{X_d} + i_{X_d}\mathrm{d})\alpha = J_f^*(\lambda_{s,l}\,\alpha) = \lambda_{s,l}p,$$

where we have used (55) (with α and $\mathrm{d}\alpha$) together with the commutation of pull-back and exterior derivative to write the second equality, and the fact that $\mathcal{H}_s\Lambda^l(\mathbb{R}^d)$ is the eigenspace associated with the eigenvalue $\lambda_{s,l} := s + l$ for $\mathrm{d}i_{X_d} + i_{X_d}\mathrm{d}$ for the second equality. Hence, $\mathcal{H}_s\Lambda^l(f)$ is an eigenspace of $\mathrm{d}\kappa_f + \kappa_f\mathrm{d}$ for this eigenvalue $\lambda_{s,l}$.

As we just saw, the eigenvalues $(\lambda_{s,l})_{s,l}$ are the same as the eigenvalues on the flat space $\mathbb{R}^d$, for which we know that Assumption **(A2)** holds.

Finally, let (s, l) with $s > 0$ and $l < d$. There is $\alpha \in \mathcal{H}_s\Lambda^l(\mathbb{R}^d)$ such that $\mathrm{d}\alpha \neq 0$. Moreover, it holds that $\mathrm{d}J_f^*\alpha = J_f^*\mathrm{d}\alpha$, and because J_f is an isomorphism, $J_f^*\mathrm{d}\alpha \neq 0$. Thus, Assumption **(A3)** holds. □

Lemma 33. *For all $f' \in \Delta_{d-1}(f)$, it holds $\mathrm{tr}_{f'}\,\mathcal{P}_r\Lambda^l(f) \subset \mathcal{P}_r\Lambda^l(f')$.*

Proof. A polynomial form $\eta \in \mathcal{P}_r\Lambda^l(f)$ is written as $\eta = J_f^*\alpha$ for some $\alpha \in \mathcal{P}_r\Lambda^l(\mathbb{R}^d)$. Then, noting that

$$J_f \circ \mathfrak{I}_{f,f'} = J_f \circ \mathfrak{I}_{f,f'} \circ I_{f'} \circ J_{f'} = \mathcal{T}_{f,f'} \circ J_{f'}, \tag{56}$$

$$\mathfrak{I}_{f,f'}^*\eta = \mathfrak{I}_{f,f'}^*J_f^*\alpha = J_{f'}^*\left(\mathcal{T}_{f,f'}^*\alpha\right).$$

By assumption $\mathcal{T}_{f,f'}$ is affine, so $\mathrm{D}_{x_{f'}}\mathcal{T}_{f,f'}$ is constant. Therefore $\mathcal{T}_{f,f'}^*\alpha$ is polynomial of degree at most r, and belongs in $\mathcal{P}_r\Lambda^l(\mathbb{R}^{d-1})$. Hence, $\mathfrak{I}_{f,f'}^*J_f^*\alpha \in J_{f'}^*\mathcal{P}_r\Lambda^l(\mathbb{R}^{d-1})$. □

Lemma 34. *Assumption 8 holds.*

Proof. Let $f' \in \Delta_{d-1}(f)$. We need to prove that $\mathrm{tr}_{f'}\,\mathcal{P}_r^-\Lambda^l(f) \subset \mathcal{P}_r^-\Lambda^l(f')$. Since $\mathcal{P}_r^-\Lambda^l(f) = \mathcal{P}_{r-1}\Lambda^l(f) + \kappa_f\mathcal{P}_{r-1}\Lambda^{l+1}(f)$ by Lemma 5 and Definition 6, and $\mathrm{tr}_{f'}\,\mathcal{P}_{r-1}\Lambda^l(f) \subset \mathcal{P}_{r-1}\Lambda^l(f')$ by Lemma 33, we only need to show that $\mathrm{tr}_{f'}\,\kappa_f\mathcal{P}_{r-1}\Lambda^{l+1}(f) \subset \mathcal{P}_{r-1}\Lambda^l(f') + \kappa_{f'}\mathcal{P}_{r-1}\Lambda^{l+1}(f')$.

By assumption $\mathcal{T}_{f,f'}$ is affine, so there is a constant $C \in \mathbb{R}^d$, such that for all $\boldsymbol{x} \in J_{f'}(f') \subset \mathbb{R}^{d-1}$,

$$\mathcal{T}_{f,f'}(\boldsymbol{x}) = \mathrm{D}_{\boldsymbol{x}}(\mathcal{T}_{f,f'})\boldsymbol{x} + C,$$
$$X_d(\mathcal{T}_{f,f'}(\boldsymbol{x})) = \mathrm{D}_{\boldsymbol{x}}(\mathcal{T}_{f,f'})X_{d-1}(\boldsymbol{x}) + C.$$

For any $\alpha \in \mathcal{P}_{r-1}\Lambda^{l+1}(\mathbb{R}^d)$, any point $x_{f'} \in f'$, and any l-uplet of vectors $(v,\dots) \in (T_{x_{f'}}f')^l$, by (54), (55) and (56), we have

$$\begin{aligned}
(\mathfrak{I}^*_{f,f'} i_{X_f}(J_f^*\alpha))_{x_{f'}}(v,\dots) &= (\underbrace{\mathfrak{I}^*_{f,f'}J_f^*}_{=J_{f'}^*\mathcal{T}^*_{f,f'}}(i_{X_d}\alpha))_{x_{f'}}(v,\dots)\\
&= \alpha_{\mathcal{T}_{f,f'}(J_{f'}(x_{f'}))}(X_d(\mathcal{T}_{f,f'}(J_{f'}(x_{f'}))), \mathrm{D}_{J_{f'}(x_{f'})}(\mathcal{T}_{f,f'})\mathrm{D}_{x_{f'}}J_{f'}v,\dots)\\
&= \alpha_{\mathcal{T}_{f,f'}(J_{f'}(x_{f'}))}(\mathrm{D}_{x_{f'}}(\mathcal{T}_{f,f'})X_{d-1}(J_{f'}(x_{f'})), \mathrm{D}_{J_{f'}(x_{f'})}(\mathcal{T}_{f,f'})\mathrm{D}_{x_{f'}}J_{f'}v,\dots)\\
&\quad + \alpha_{\mathcal{T}_{f,f'}(J_{f'}(x_{f'}))}(C, \mathrm{D}_{J_{f'}(x_{f'})}(\mathcal{T}_{f,f'})\mathrm{D}_{x_{f'}}J_{f'}v,\dots)\\
&= (\mathcal{T}^*_{f,f'}\alpha)_{J_{f'}(x_{f'})}(X_{d-1}(J_{f'}(x_{f'})), \mathrm{D}_{x_{f'}}J_{f'}v,\dots) + J^*_{f'}\mathcal{T}^*_{f,f'}i_C\alpha_{x_{f'}}(v,\dots)\\
&= J^*_{f'}i_{X_{d-1}}I^*_{f'}\mathfrak{I}^*_{f,f'}J_f^*\alpha_{x_{f'}}(v,\dots) + \mathfrak{I}^*_{f,f'}J_f^*i_C\alpha_{x_{f'}}(v,\dots)\\
&= \underbrace{i_{X_{f'}}}_{=\kappa_{f'}}\mathfrak{I}^*_{f,f'}J_f^*\alpha_{x_{f'}}(v,\dots) + \mathfrak{I}^*_{f,f'}J_f^*i_C\alpha_{x_{f'}}(v,\dots).
\end{aligned}$$

Hence,

$$\mathrm{tr}_{f'}\, J_f^*\alpha = \kappa_{f'}\,\mathrm{tr}_{f'}\, J_f^*\alpha + \mathrm{tr}_{f'}\, J_f^* i_C\alpha.$$

Since C is a constant, we have $i_C\alpha \in \mathcal{P}_{r-1}\Lambda^l(\mathbb{R}^d)$, and Lemma 33 yields $\mathrm{tr}_{f'}\, J_f^* i_C\alpha \in \mathcal{P}_{r-1}\Lambda^l(f')$ and $\mathrm{tr}_{f'}\, J_f^*\alpha \in \mathcal{P}_{r-1}\Lambda^{l+1}(f')$. □

Lemma 35. *If the diffeomorphisms J_f satisfy (2), then Assumption 17 holds.*

Proof. The main idea is to use the general bound $\|\phi^* u\|_f \le \|\nabla\phi\|_\infty^k \|\det(\nabla\phi^{-1})\|_\infty^{\frac{1}{2}}\|u\|_f$ for $u \in L^2\Lambda^k(f)$. We infer from the regularity assumption (2) that, for all $1 \le d \le n$, all $f \in \Delta_d(\mathcal{M}_h)$, and all $p \in \mathcal{P}_r\Lambda^k(\mathbb{R}^d)$, we have

$$\|J_f^* p\|_f \approx \|\nabla J_f\|_\infty^{k-\frac{d}{2}} \|p\|_{J_f(f)}. \tag{57}$$

Moreover, (55) and the fact that the pullback commutes with the differential give, for all $t, \ell \in \mathbb{N}$

$$\kappa\mathcal{P}_t\Lambda^\ell(f) = J_f^*\kappa\mathcal{P}_t\Lambda^\ell(\mathbb{R}^d), \quad \mathrm{d}\mathcal{P}_t\Lambda^\ell(f) = J_f^*\mathrm{d}\mathcal{P}_t\Lambda^\ell(\mathbb{R}^d). \tag{58}$$

Let us now prove (17). For all $\mu \in \kappa\mathcal{P}_{r-1}\Lambda^k(f)$, the relation (58) shows that $\mu = J_f^* p$ and $\mathrm{d}\mu = J_f^*\mathrm{d}p$ for some $p \in \kappa\mathcal{P}_{r-1}\Lambda^k(\mathbb{R}^d)$, and since (17) is valid in $\mathbb{R}^d$ (see, e.g., [20, Lemma 9] for $d = 3$ in vector proxies), we therefore have

$$\begin{aligned}
\|\mu\|_f &\overset{(57)}{\approx} \|\nabla J_f\|_\infty^{k-\frac{d}{2}} \|p\|_{J_f(f)}\\
&\lesssim \|\nabla J_f\|_\infty^{k-\frac{d}{2}} h_{J_f(f)} \|\mathrm{d}p\|_{J_f(f)}\\
&\overset{(4)}{\approx} \|\nabla J_f\|_\infty^{k-\frac{d}{2}+1} h_f \|\mathrm{d}p\|_{J_f(f)}\\
&\overset{(57)}{\approx} h_f \|\mathrm{d}\mu\|_f.
\end{aligned}$$

We now turn to (18). For all $\mu = J_f^* p \in \kappa\mathcal{P}_{r-1}\Lambda^{k+1}(f)$ and $\nu = J_f^* q \in \mathrm{d}\mathcal{P}_r\Lambda^{k-1}(f)$ with p, q in the corresponding spaces on $\mathbb{R}^d$ instead of f, we have

$$\|\mu\|_f + \|\nu\|_f \overset{(57)}{\approx} \|\nabla J_f\|_\infty^{k-\frac{d}{2}}(\|p\|_{J_f(f)} + \|q\|_{J_f(f)}) \approx \|\nabla J_f\|_\infty^{k-\frac{d}{2}} \|p+q\|_{J_f(f)} \overset{(57)}{\approx} \|\mu+\nu\|_f,$$

where the second equality follows from the fact that (18) is valid on $f = \mathbb{R}^d$, see [20, Lemma 2].

Finally, we have to prove (19), which is done in a similar way. For all $\mu = J_f^* p \in \mathcal{P}_r\Lambda^k(f)$ and all $f' \in \partial f$, we have, using the discrete trace inequality in $\mathbb{R}^d$,

$$\begin{aligned}
\|\operatorname{tr}_{f'} \mu\|_{f'} &\overset{(57)}{\approx} \|\nabla J_{f'}\|_\infty^{k-\frac{d-1}{2}} \|\operatorname{tr}_{f'} p\|_{J_{f'}(f')} \\
&\lesssim \|\nabla J_{f'}\|_\infty^{k-\frac{d-1}{2}} h_{J_{f'}(f')}^{-\frac{1}{2}} \|p\|_{J_f(f)} \\
&\overset{(4)}{\lesssim} \|\nabla J_{f'}\|_\infty^{k-\frac{d-1}{2}-\frac{1}{2}} h_{f'}^{-\frac{1}{2}} \|p\|_{J_f(f)} \\
&\overset{(3),(57)}{\lesssim} h_f^{-\frac{1}{2}} \|\mu\|_f. \quad \square
\end{aligned}$$

A.2 Constructions in dimension $n = 2$

In this section, we provide a few examples of construction that satisfy Assumption 9 for various topologies. They can be used where a curved boundary is required and combined with flat cells elsewhere.

The idea is to apply a hierarchical construction of the spaces, starting from 1-cells (the edges) and defining the space on a face $f \in \Delta_2(\mathcal{M}_h)$ from those of $\Delta_1(f)$.

When a cell is flat in a chart, we can simply restrict this chart to get a suitable I_f verifying Assumption 9. Therefore, our method is really a generalization of the polynomials on $\mathbb{R}^d$, and we only need to create more exotic spaces on curved cells. While there is no fully generic way to construct I_f, we give below some possible constructions in several cases.

In practice, we work in coordinates, and therefore start from a given chart $J_f^0 : \overline{f} \to \mathbb{R}^2$. When we mention the injection of an edge $E \in \Delta_1(f)$, we are actually speaking about the composition $J_f^0 \circ \mathfrak{I}_{f,E} \circ I_E$. Likewise, we will not directly build maps from f to $\mathbb{R}^2$, but rather some diffeomorphism J_f between two subsets of $\mathbb{R}^2$. The actual map that verifies Assumption 9 is $(J_f \circ J_f^0)^{-1}$.

A.2.1 Triangle with two curved edges

Let $f \in \Delta_2(\mathcal{M}_h)$ be a face with 3 edges. Assume that, in a given chart on f, the injection of the edges are given by

$$\begin{aligned}
I_1(t) &= (h_1(t), g_1(t)) \\
I_2(t) &= (h_2(t), g_2(t)) \\
I_3(t) &= (0, t),
\end{aligned}$$

where $t \in [0,1]$ and h_i, g_i are C^2. See Figure 3 for an illustration (in which each I_r corresponds to E_r).

Moreover assume that, for $i \in \{1,2\}$, $h_i(0) = 0$, $h_i(1) = 1$, and h_i is strictly increasing, and that $g_1(0) = 0$, $g_1(1) = 0$, $g_2(0) = 1$, and $g_2(1) = 0$. Let $G_i(x) := g_i(h_i^{-1}(x))$, $i \in \{1,2\}$.

Lemma 36. *If the function*

$$J_f(x,y) := \frac{1}{G_2(x) - G_1(x)} \begin{pmatrix} (y - G_1(x))h_2^{-1}(x) + (G_2(x) - y)h_1^{-1}(x) \\ (y - G_1(x))(1 - h_2^{-1}(x)) \end{pmatrix}$$

is a C^2 diffeomorphism, then it verifies Assumption 9.

Proof. We need to check the compatibility with the injections. Notice that, by definition, $G_i(h_i(t)) = g_i(t)$. For E_1, we have

$$J_f(I_1(t)) = J_f(h_1(t), g_1(t))$$

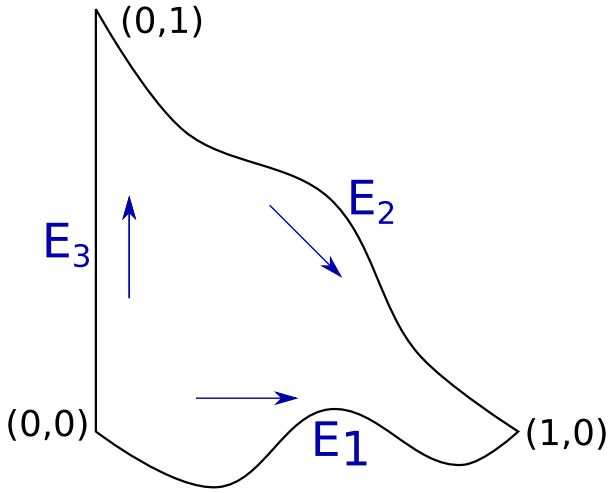


Figure 3: f with three edges, one of which being straight in a chosen chart.

$$
\begin{aligned}
&= \frac{1}{G_2(h_1(t)) - g_1(t)} \begin{pmatrix} (g_1(t) - g_1(t))h_2^{-1}(h_1(t)) + (G_2(h_1(t)) - g_1(t))t \\ (g_1(t) - g_1(t))(1 - h_2^{-1}(h_1(t)) \end{pmatrix} \\
&= \frac{1}{G_2(h_1(t)) - g_1(t)} \begin{pmatrix} (G_2(h_1(t)) - g_1(t))t \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} t \\ 0 \end{pmatrix}.
\end{aligned}
$$

For E_2, we have

$$
\begin{aligned}
J_f(I_2(t)) &= J_f(h_2(t), g_2(t)) \\
&= \frac{1}{g_2(t) - G_1(h_2(t))} \begin{pmatrix} (g_2(t) - G_1(h_2(t)))t + (g_2(t) - g_2(t))h_1^{-1}(h_2(t)) \\ (g_2(t) - G_1(h_2(t)))(1 - t) \end{pmatrix} \\
&= \begin{pmatrix} t \\ 1 - t \end{pmatrix}.
\end{aligned}
$$

Finally, for E_3, noticing that $h_i^{-1}(0) = 0$, we have $G_1(0) = 0$ and $G_2(0) = 1$ and thus

$$
\begin{aligned}
J_f(I_3(t)) &= J_f(0, t) \\
&= \frac{1}{G_2(0) - G_1(0)} \begin{pmatrix} (t - G_1(0))h_2^{-1}(0) + (G_2(0) - t)h_1^{-1}(0) \\ (t - G_1(0))(1 - h_2^{-1}(0)) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ t \end{pmatrix}.
\end{aligned}
$$

□

A.2.2 Cone section with a curved edge

Let $f \in \Delta_2(\mathcal{M}_h)$ be a face with 4 edges. Assume that, when expressed in polar coordinates (r, θ) on the chart of f, the injections of the edges are given by

$$
\begin{aligned}
I_1(t) &= (g_1(t), h(t)), \\
I_2(t) &= (t + 1, \theta_1), \\
I_3(t) &= (g_3(t), h(t)),
\end{aligned}
$$

$$I_4(t) = (t(a-1)+1, \theta_2),$$

where $t \in [0,1]$, and the functions h, g_1, g_3 are C^2 and satisfy

$$\begin{aligned} h(0) &= \theta_1, \quad h(1) = \theta_2, \\ g_1(0) &= 2, \quad g_1(1) = a, \\ g_3(0) &= 1, \quad g_3(1) = 1. \end{aligned}$$

See Figure 4 for an illustration.

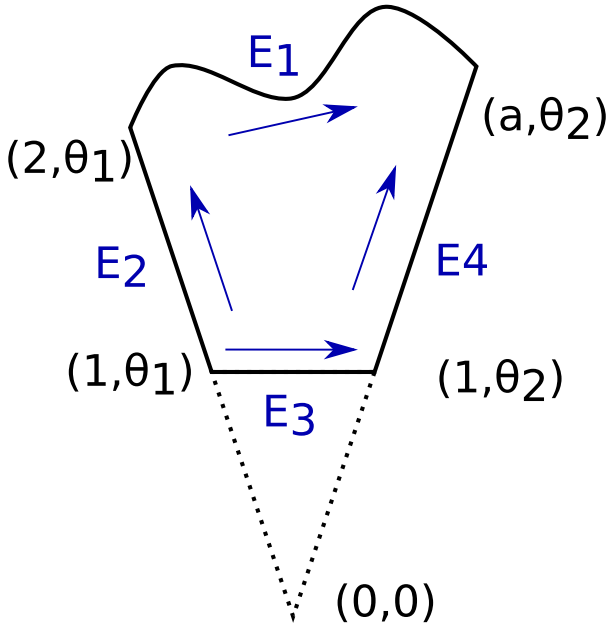


Figure 4: f with four edges, corresponding to a section of cone with a curved edge.

Moreover, assume that h is strictly increasing with a derivative bounded from below, and that $g_1 - g_3$ is strictly positive (meaning that E_1 and E_3 do not intersect).

Lemma 37. *The function $I_f : [0,1]^2 \to \overline{f}$ defined by*

$$I_f(t,p) := \begin{pmatrix} pg_1(t) + (1-p)g_3(t) \\ h(t) \end{pmatrix},$$

satisfies Assumption 9.

Proof. Since h is a C^2-diffeomorphism and $g_1 - g_3 > 0$, it can easily be verified that I_f is a C^2-isomorphism with inverse

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} \frac{a - g_3(h^{-1}(b))}{g_1(h^{-1}(b)) - g_3(h^{-1}(b))} \\ h^{-1}(b) \end{pmatrix}.$$

In order to show the compatibility with the trace, we exhibit some explicit affine function T_i, such that $I_f \circ T_i = I_i$. This will give $J_f \circ I_i = T_i$. For this, we simply readily check that

$$I_f(t,0) = \begin{pmatrix} g_3(t) \\ h(t) \end{pmatrix} = I_3(t),$$

$$
\begin{aligned}
I_f(t,1) &= \begin{pmatrix} g_1(t) \\ h(t) \end{pmatrix} = I_1(t), \\
I_f(0,p) &= \begin{pmatrix} p(g_1(0) - g_3(0)) + g_3(0) \\ h(0) \end{pmatrix} = \begin{pmatrix} p+1 \\ \theta_1 \end{pmatrix} = I_2(t), \\
I_f(1,p) &= \begin{pmatrix} p(g_1(1) - g_3(1)) + g_3(1) \\ h(1) \end{pmatrix} = \begin{pmatrix} p(a-1)+1 \\ \theta_2 \end{pmatrix} = I_4(t).
\end{aligned}
$$

so that $T_1(t,p) = (t,0)$, $T_2(t,p) = (t,1)$, $T_3(t,p) = (0,p)$ and $T_4(t,p) = (1,p)$. □

A.2.3 Quadrilateral with four curved edges

Let $f \in \Delta_2(\mathcal{M}_h)$ be a face with 4 edges. Assume that, in a given chart on f, the injections of the edges are given by

$$
I_i(t) = (h_i(t), g_i(t)), \quad i \in \{1,2,3,4\},
$$

where $t \in [0,1]$. Moreover assume that the functions h_i, g_i are C^2 and that

$$
\begin{aligned}
(h_1(0), g_1(0)) &= (0,0), \quad (h_1(1), g_1(1)) = (1,0), \\
(h_2(0), g_2(0)) &= (1,0), \quad (h_2(1), g_2(1)) = (a,b), \\
(h_3(0), g_3(0)) &= (0,1), \quad (h_3(1), g_3(1)) = (a,b), \\
(h_4(0), g_4(0)) &= (0,0), \quad (h_4(1), g_4(1)) = (0,1).
\end{aligned}
$$

This situation is illustrated in Figure 5. Let

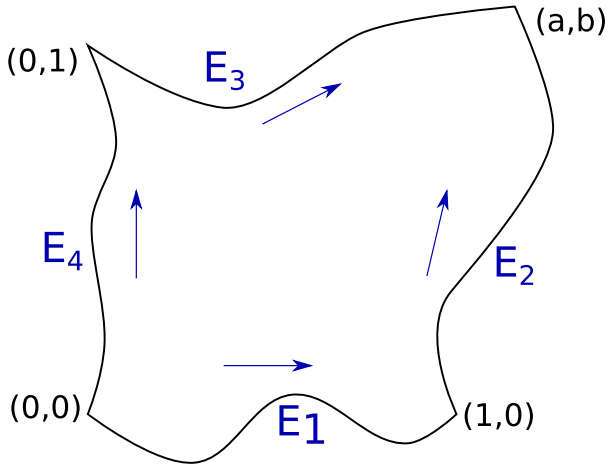


Figure 5: f with four curved edges.

$$
I_f(t,p) := \begin{pmatrix} \big(p\frac{h_3(t)}{a} + (1-p)h_1(t)\big)h_2(p) + \big(1 - (p\frac{h_3(t)}{a} + (1-p)h_1(t))\big)h_4(p) \\ \big(t\frac{g_2(p)}{b} + (1-t)g_4(p)\big)g_3(t) + \big(1 - (t\frac{g_2(p)}{b} + (1-t)g_4(p))\big)g_1(t) \end{pmatrix}.
$$

Lemma 38. *If I_f is bijective and $\det(\mathrm{D}I_f) \neq 0$ then the function $J_f := I_f^{-1}$ verifies Assumption 9.*

Proof. Since J_f is defined as the inverse of I_f, we will exhibit some explicit affine functions $\mathbb{R} \to \mathbb{R}^2$

that give I_i when composed with I_f. For E_1, we have

$$I_f(t,0) = \begin{pmatrix} \big(0\frac{h_3(t)}{a} + (1-0)h_1(t)\big)\overbrace{h_2(0)}^{=1} + \big(1 - (0\frac{h_3(t)}{a} + (1-0)h_1(t))\big)\overbrace{h_4(0)}^{=0} \\ \big(t\underbrace{\frac{g_2(0)}{b}}_{=0} + (1-t)\underbrace{g_4(0)}_{=0}\big)g_3(t) + \big(1 - (t\underbrace{\frac{g_2(0)}{b}}_{=0} + (1-t)\underbrace{g_4(0)}_{=0})\big)g_1(t) \end{pmatrix} = \begin{pmatrix} h_1(t) \\ g_1(t) \end{pmatrix}.$$

For E_2, we have

$$I_f(1,t) = \begin{pmatrix} \big(t\overbrace{\frac{h_3(1)}{a}}^{=1} + (1-t)\overbrace{h_1(1)}^{=1}\big)h_2(t) + \big(1 - (t\overbrace{\frac{h_3(1)}{a}}^{=1} + (1-t)\overbrace{h_1(1)}^{=1}\big)h_4(t) \\ \big(\frac{g_2(t)}{b} + (1-1)g_4(t)\big)\underbrace{g_3(1)}_{=b} + \big(1 - (\frac{g_2(t)}{b} + (1-1)g_4(t))\big)\underbrace{g_1(1)}_{=0} \end{pmatrix} = \begin{pmatrix} h_2(t) \\ g_2(t) \end{pmatrix}.$$

For E_3, we have

$$I_f(t,1) = \begin{pmatrix} \big(\frac{h_3(t)}{a} + (1-1)h_1(t)\big)\overbrace{h_2(1)}^{=a} + \big(1 - (\frac{h_3(t)}{a} + (1-1)h_1(t))\big)\overbrace{h_4(1)}^{=0} \\ \big(t\underbrace{\frac{g_2(1)}{b}}_{=1} + (1-t)\underbrace{g_4(1)}_{=1}\big)g_3(t) + \big(1 - (t\underbrace{\frac{g_2(1)}{b}}_{=1} + (1-t)\underbrace{g_4(1)}_{=1})\big)g_1(t) \end{pmatrix} = \begin{pmatrix} h_3(t) \\ g_3(t) \end{pmatrix}.$$

For E_4, we have

$$I_f(0,t) = \begin{pmatrix} \big(t\overbrace{\frac{h_3(0)}{a}}^{=0} + (1-t)\overbrace{h_1(0)}^{=0}\big)h_2(t) + \big(1 - (t\overbrace{\frac{h_3(0)}{a}}^{=0} + (1-t)\overbrace{h_1(0)}^{=0})\big)h_4(t) \\ \big(0\frac{g_2(t)}{b} + (1-0)g_4(t)\big)\underbrace{g_3(0)}_{=1} + \big(1 - (0\frac{g_2(t)}{b} + (1-0)g_4(t))\big)\underbrace{g_1(0)}_{=0} \end{pmatrix} = \begin{pmatrix} h_4(t) \\ g_4(t) \end{pmatrix}.$$

□

B Tensor calculus identities

Let $g = (g_{\mu\nu})$ denote the metric on the tangent bundle of an n-dimensional Lorentzian manifold, and let $(g^{\mu\nu}) := (g_{\mu\nu})^{-1}$. Assuming $\det g < 0$, the volume form is given in coordinates by

$$(\mathrm{vol})_{\alpha_1,\ldots,\alpha_n} = \sqrt{-\det g}\,\epsilon_{\alpha_1,\ldots,\alpha_n} \tag{59}$$

where $\epsilon_{i_1\ldots i_n}$ denotes the Levi–Civita symbol. Given a 1-form $\alpha := \alpha_\mu \mathrm{d}x^\mu$ and a vector field $b := b^\mu \partial_\mu$, we define the sharp $\sharp$ and flat $\flat$ operator to raise and lower indices by:

$$(\alpha^\sharp)^\nu := g^{\nu\mu}\alpha_\mu, \qquad (b^\flat)_\nu := g_{\nu\mu}b^\mu.$$

We use the Hodge star operator $\star$ to identify k-forms with $(n-k)$-forms. In index notation, it is defined point wise on the linear basis of k-forms by

$$\star(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}) := \frac{\sqrt{-\det g}}{(n-k)!}\epsilon^{i_1\ldots i_k}{}_{j_{k+1}\ldots j_n}\mathrm{d}x^{j_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{j_n} \tag{60}$$

where

$$\epsilon^{i_1\dots i_k}{}_{j_{k+1}\dots j_n} := g^{i_1 j_1}\dots g^{i_k j_k}\epsilon_{j_1\dots j_n}.$$

In general, the product of two ϵ is given by

$$\epsilon^{i_1\dots i_n}\epsilon_{j_1\dots j_n} = \det(g^{-1})\delta^{i_1\dots i_n}_{j_1\dots j_n}, \tag{61}$$

where $\delta^{i_1\dots i_n}_{j_1\dots j_n}$ is the generalized Kronecker delta (equal to 1 if $i_1\dots i_n$ is an even permutation of $j_1\dots j_n$, -1 if it is an odd permutation and 0 otherwise). The generalized Kronecker delta can also be used to compute wedge products in index notation. The wedge product of two 1-forms $\alpha = \alpha_\mu \mathrm{d}x^\mu$ and $\beta = \beta_\nu \mathrm{d}x^\nu$ is given by

$$\alpha\wedge\beta = \frac{1}{2}\delta^{\mu'\nu'}_{\mu\nu}\alpha_{\mu'}\beta_{\nu'}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu. \tag{62}$$

Specialising to the case of $2+1$ dimensions, let

$$(g_{\mu\nu}) := \begin{pmatrix} -N^2+|\beta|^2_\gamma & \beta_j \\ \beta_i & \gamma_{ij}\end{pmatrix},$$

for some 2 by 2 symmetric and positive definite matrix γ, and let

$$n := (-N\mathrm{d}t)^\sharp.$$

The inverse matrix of g is given by

$$g^{-1} = (g^{\mu\nu}) = \begin{pmatrix} -\frac{1}{N^2} & \frac{\beta_j}{N^2} \\ \frac{\beta_i}{N^2} & \gamma^{ij} - \frac{\beta_i\beta_j}{N^2}\end{pmatrix},$$

and $n = \frac{1}{N}\partial_t - \frac{1}{N}\beta^i\partial_i$. The determinant of g and γ are related by $\sqrt{-\det g} = N\sqrt{\det\gamma}$.

Lemma 39. *The contraction of the volume form with n is given by*

$$i_n\mathrm{vol} = \widetilde{\mathrm{vol}} + \mathrm{d}t\wedge i_{\widetilde{\beta^\sharp}}\widetilde{\mathrm{vol}} \tag{63}$$

where $\widetilde{\mathrm{vol}}$ *is the spatial volume form (associated with* γ*).*

Proof. An explicit computation gives

$$i_n\ \mathrm{vol} = i_n\sqrt{-\det g}\ \mathrm{d}t\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 = \frac{\sqrt{-\det g}}{N}\left(\mathrm{d}x^1\wedge\mathrm{d}x^2 + \mathrm{d}t\wedge\left[\beta^1\mathrm{d}x^2 - \beta^2\mathrm{d}x^1\right]\right).$$

□

Lemma 40. *For any* 1*-form V, denoting by* $\widetilde{V}$ *the restriction of V on a time slice, we have*

$$V^\sharp = \widetilde{V}^{\widetilde{\sharp}} - nV(n). \tag{64}$$

We readily infer the following corollary from (64) and the identity $i_n i_n = 0$:

$$i_{V^\sharp}i_n = i_{\widetilde{V}^{\widetilde{\sharp}}}i_n. \tag{65}$$

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