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Abstract: This paper is devoted to the study of evolution problems involving fractional flow and time and state dependent maximal monotone operator which is absolutely continuous in variation with respect to the Vladimirov's pseudo distance. In a first part, we solve a second order problem and give an application to sweeping process. In a second part, we study a class of fractional order problem driven by a time and state dependent maximal monotone operator with a Lipschitz perturbation in a separable Hilbert space. In the last part, we establish a Filippov theorem and a relaxation variant for fractional differential inclusion in a separable Banach space. In every part, some variants and applications are presented.

Keywords: fractional differential inclusion; maximal monotone operator; Riemann–Liouville integral; absolutely continuous in variation; Vladimirov pseudo-distance

MSC: 34H05; 34K35; 47H10; 28A25; 28B20; 28C20

1. Introduction

In recent decades, fractional equations and inclusions have proven to be interesting tools in the modeling of many physical or economic phenomena. In addition, there has been a significant development in fractional differential theory and applications in recent years [1–7]. In the case of the **sole inclusion**, $D^{\alpha}u(t) \in F(t,u(t))$, one can find an important piece of literature. For examples, in following papers, study is made with different boundary conditions [8–12], with use of the non-compactness measure [13,14], with use of contraction principle in the space of selections of the set valued map instead in the space of solutions [15], with compactness conditions [16] or inclusions with infinite delay [17]. To the best of our acknowledge, a very few study is available in the fractional order differential inclusion **coupled** with a time and state dependent maximal monotone operator ([18] with subdifferential operators).

The main objective of the present work is to develop the existence theory for a **coupled system** of evolution inclusion driven by fractional differential equation and time and state dependent maximal monotone operators. The developments of the article are as follows.

At first, we investigate a second order problem governed a time and state dependent maximal monotone operator with Lipschitz perturbation in a separable Hilbert space E (The second order is in the state variable x).

$$(1.1)\begin{cases} x(t) = x_0 + \int_0^t u(s)ds, t \in [0,T] \\ u(t) \in D(A_{t,x(t)}), t \in [0,T] \\ -\dot{u}(t) \in A_{t,x(t)}u(t) + f(t,x(t),u(t)) \quad a.e. \end{cases}$$

Secondly, we investigate a class of fractional order problem driven by a time and state dependent maximal monotone operator with Lipschitz perturbation in E of the form

$$(1.2)\begin{cases} D^{\alpha}h(t) + \lambda D^{\alpha-1}h(t) = u(t), t \in [0,1] \\ I_{0^+}^{\beta}h(t)\,|_{t=0} := \lim_{t\to 0}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}h(s)ds = 0, \quad h(1) = I_{0^+}^{\gamma}h(1) = \int\limits_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)}h(s)ds \\ -\dot{u}(t) \in A_{t,h(t)}u(t) + f(t,h(t),u(t)) \quad a.e. \end{cases}$$

where $\alpha \in]1,2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0, \gamma > 0$ are given constants, D^{α} is the standard Riemann–Liouville fractional derivative , Γ is the gamma function, $(t,x) \to A_{(t,x)} : D(A_{(t,x)}) \to 2^E$ is a maximal monotone operator with domain $D(A_{(t,x)})$ and $f : [0,1] \times E \times E \to E$ is a single valued Lipschitz perturbation w.r.t $y \in E$.

Thirdly, we finish the paper with a Fillipov theorem and relaxation theorem for fractional differential inclusion in a separable Banach space E

$$(\mathcal{P}_F)\begin{cases} D^{\alpha}u(t) + \lambda D^{\alpha-1}u(t) \in F(t,u(t)), a.e.\ t \in [0,1] \\ I_{0^+}^{\beta}u(t)\,|_{t=0} = 0, \quad u(1) = I_{0^+}^{\gamma}u(1) \end{cases}$$

and

$$(\mathcal{P}_{\overline{co}F})\begin{cases} D^{\alpha}u(t) + \lambda D^{\alpha-1}u(t) \in \overline{co}F(t,u(t)), a.e.\ t \in [0,1] \\ I_{0^+}^{\beta}u(t)\,|_{t=0} = 0, \quad u(1) = I_{0^+}^{\gamma}u(1) \end{cases}$$

where F is closed valued $\mathcal{L}(I) \times \mathcal{B}(E)$-measurable and Lipschitz w.r.t $x \in E$.

Within the framework of studies concerning coupled systems of evolution inclusion driven by fractional differential equation and time and state dependent maximal monotone operator, our results are fairly general and new and give further insight into the characteristics of both evolution inclusion and fractional order boundary value problems.

2. Notations and Preliminaries

In the whole paper, $I := [0,T]$ $(T > 0)$ is an interval of $\mathbb{R}$ and E is a separable Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. $\overline{B}_E$ denotes the unit closed ball of E and $r\overline{B}_E$ its closed ball of center 0 and radius $r > 0$. We denote by $\mathcal{L}(I)$ the sigma algebra on I, $\lambda := dt$ the Lebesgue measure and $\mathcal{B}(E)$ the Borel sigma algebra on E. If μ is a positive measure on I, we will denote by $L^p(I,E,\mu)$ $p \in [1,+\infty[$, (resp. $p = +\infty$), the Banach space of classes of measurable functions $u : I \to E$ such that $t \mapsto \|u(t)\|^p$ is μ-integrable (resp. u is μ-essentially bounded), equipped with its classical norm $\| \cdot \|_p$ (resp. $\| \cdot \|_{\infty}$). We denote by $\mathcal{C}(I,E)$ the Banach space of all continuous mappings $u : I \to E$, endowed with the sup norm.

The excess between closed subsets C_1 and C_2 of E is defined by $e(C_1,C_2) := \sup_{x\in C_1} d(x,C_2)$, and the Hausdorff distance between them is given by

$$d_H(C_1,C_2) := \max\Big\{e(C_1,C_2), e(C_2,C_1)\Big\}.$$

The support function of $S \subset E$ is defined by: $\delta^*(a,S) := \sup_{x\in S}\langle a,x\rangle$, $\forall a \in E$.

If X is a Banach space and X^* its topological dual, we denote by $\sigma(X,X^*)$ the weak topology on X, and by $\sigma(X^*,X)$ the weak* topology on X^*.

Let $A : E \rightrightarrows E$ be a set-valued map. We denote by $D(A)$, $R(A)$ and $Gr(A)$ its domain, range and graph. We say that A is monotone, if $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$ whenever $x_i \in D(A)$, and $y_i \in A(x_i)$, $i = 1, 2$. In addition, we say that A is a maximal monotone operator of E, if its graph could not be contained properly in the graph of any other monotone operator. By Minty's Theorem, A is maximal monotone iff $R(I_E + A) = E$.

If A is a maximal monotone operator of E, then, for every $x \in D(A)$, $A(x)$ is nonempty closed and convex. We denote the projection of the origin on the set $A(x)$ by $A^0(x)$.

Let $\lambda > 0$; then, the resolvent and the Yosida approximation of A are the well-known operators defined respectively by $J_\lambda^A = (I_E + \lambda A)^{-1}$ and $A_\lambda = \frac{1}{\lambda}(I_E - J_\lambda^A)$. These operators are single-valued and defined on all of E, and we have $J_\lambda^A(x) \in D(A)$, for all $x \in E$. For more details about the theory of maximal monotone operators, we refer the reader to [5,19,20].

Let $A : D(A) \subset E \to 2^E$ and $B : D(B) \subset E \to 2^E$ be two maximal monotone operators, then we denote by $dis(A, B)$ the pseudo-distance between A and B defined by

$$dis(A, B) = \sup \left\{ \frac{\langle y - y', x' - x \rangle}{1 + \|y\| + \|y'\|} : \ x \in D(A), \ y \in Ax, \ x' \in D(B), \ y' \in Bx' \right\}. \tag{1}$$

This pseudo-distance due to Vladimiro [21] is particularly well suited to the study of operators (see its use in [22]) and also, in the sweeping process, for its links with the Hausdorff distance in convex analysis. Indeed, if $N_{C(t,x)}$ is the normal cone of the closed convex set $C(t, x)$, we have

$$dis(N_{C(t,x)}, N_{C(s,y)}) = d_H(C(t, x), C(s, y)).$$

This property will be used in this paper.

For the proof of our main theorems, we will need some elementary lemmas taken from reference [23].

Lemma 1. *Let A be a maximal monotone operator of E. If $x \in \overline{D(A))}$ and $y \in E$ are such that*

$$\langle A^0(z) - y, z - x \rangle \geq 0 \ \ \forall z \in D(A),$$

then $x \in D(A)$ and $y \in A(x)$.

Lemma 2. *Let A_n $(n \in \mathbb{N})$, A be maximal monotone operators of E such that $dis(A_n, A) \to 0$. Suppose also that $x_n \in D(A_n)$ with $x_n \to x$ and $y_n \in A_n(x_n)$ with $y_n \to y$ weakly for some $x, y \in E$. Then, $x \in D(A)$ and $y \in A(x)$.*

Lemma 3. *Let A, B be maximal monotone operators of E. Then,*
(1) for $\lambda > 0$ and $x \in D(A)$

$$\|x - J_\lambda^B(x)\| \leq \lambda \|A^0(x)\| + dis(A, B) + \sqrt{\lambda \big(1 + \|A^0(x)\|\big) dis(A, B)}.$$

(2) For $\lambda > 0$ and $x, x' \in E$

$$\|J_\lambda^A(x) - J_\lambda^A(x')\| \leq \|x - x'\|.$$

Lemma 4. *Let A_n $(n \in \mathbb{N})$, A be maximal monotone operators of E such that $dis(A_n, A) \to 0$ and $\|A_n^0(x)\| \leq c(1 + \|x\|)$ for some $c > 0$, all $n \in \mathbb{N}$ and $x \in D(A_n)$. Then, for every $z \in D(A)$, there exists a sequence (ζ_n) such that*

$$\zeta_n \in D(A_n), \quad \zeta_n \to z \ \text{ and } \ A_n^0(\zeta_n) \to A^0(z). \tag{2}$$

3. On Second Order Problem Driven by a Time and State Dependent Maximal Operator

Let $I = [0, T]$ and let E be a separable Hilbert space. In this part, we are interested in solving the problem (1.1).

Lemma 5. *Let $(t, x) \to A_{(t,x)} : D(A_{(t,x)}) \to 2^E$ a maximal monotone operator satisfying:*
(H_1) $\|A^0_{(t,x)}y\| \leq c(1 + \|x\| + \|y\|)$ for all $(t, x, y) \in I \times E \times D(A_{(t,x)})$, for some positive constant c,
(H_2) $dis(A_{(t,x)}, A_{(\tau,y)}) \leq a(t) - a(\tau) + r\|x - y\|$, for all $0 \leq \tau \leq t \leq T$ and for all $(x, y) \in E \times E$, where r is a positive number, $a : I \to [0, +\infty[$ is nondecreasing absolutely continuous on I with $\dot{a} \in L^2$, shortly $a \in W^{1,2}(I)$.
Then, the following hold:

Fact $\mathcal{I}$*: For any absolutely continuous $x \in W^{1,2}_E(I)$ and for any $u_0 \in D(A_{(0,x(0))})$, the problem*

$$\begin{cases} -\dot{u}(t) \in A_{(t,x(t))}u(t), \text{ a.e. } t \in I \\ u(t) \in D(A_{(t,x(t))}), \forall t \in I \\ u(0) = u_0 \in D(A_{(0,x(0))}) \end{cases}$$

has a unique absolutely continuous solution with $\|\dot{u}(t)\| \leq K(1 + \dot{\beta}(t))$ where $\beta(t) = \int_0^t [\dot{a}(s) + r\|\dot{x}(s))\|]ds$, $\forall t \in I$ and K is a positive constant depending on $\|u_0\|, c, T, x$ and β.

Fact $\mathcal{J}$*: Assume that*
(H_3) $(t, x, y) \to J^{A_{(t,x)}}_\lambda(y)$ is $\mathcal{L}(I) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$-measurable.
Then, the composition operator $\mathcal{A}_x : D(\mathcal{A}_x) \subset L^2(I, E, dt) \to 2^{L^2(I,E,dt)}$ defined by

$$\mathcal{A}_x u = \{v \in L^2(I, E, dt) : v(t) \in A_{(t,x(t))}u(t) \text{ a.e. } t \in I\}$$

for each $u \in D(\mathcal{A}_x)$ where
$D(\mathcal{A}_x) := \{u \in L^2(I, E, dt) : u(t) \in D(A_{(t,x(t))})$ a.e. $t \in I$, for which $\exists\, y \in L^2(I, E, dt) : y(t) \in A_{(t,x(t))}u(t)$, a.e. $t \in I\}$
is maximal monotone. Consequently, the graph of $\mathcal{A}_x : D(\mathcal{A}_x) \subset L^2(I, E, dt) \to 2^{L^2(I,E,dt)}$ is strongly-weakly sequentially closed in $L^2(I, E, dt) \times L^2(I, E, dt)$.

Proof. **Fact** $\mathcal{I}$. The mapping $B_t = A_{(t,x(t))}$ is a time dependent absolutely continuous in variation maximal monotone operator: For all $0 \leq \tau \leq t \leq T$, we have by (H_2)

$$\begin{cases} dis(B_t, B_\tau) = dis(A_{(t,x(t))}, A_{(\tau,x(\tau))}) \\ \leq |a(t) - a(\tau)| + r||x(t) - x(\tau)|| \\ \leq \int_\tau^t \dot{a}(s)ds + r\int_\tau^t \|\dot{x}(s)\|ds \\ = \beta(t) - \beta(\tau) \end{cases}$$

where $\beta(t) = \int_0^t [\dot{a}(s) + r\|\dot{x}(s)\|]ds$, $\forall t \in I$. Furthermore, by (H_1), we have

$$\begin{cases} \|B^0_t y\| = \|A^0_{(t,x(t))}y\| \leq c(1 + \|x(t)\| + \|y\|) \\ \leq c_1(1 + \|y\|) \end{cases}$$

for all $y \in D(A_{(t,x(t))})$, where c_1 is a positive generic constant. Consequently, by [22] (Theorem 3.5), for every $u_0 \in D(B_0)$, a unique absolutely continuous mapping $u : I \to E$ exists satisfying

$$\begin{cases} -\dot{u}(t) \in B_t u(t) = A_{(t,x(t))}u(t), \ a.e.\ t \in I \\ u(t) \in D(B_t) = D(A_{(t,x(t))}), \forall t \in I \\ u(0) = u_0 \in D(B_0) = D(A_{(0,x(0))}) \end{cases}$$

with $\|\dot{u}(t)\| \leq K(1 + \dot{\beta}(t))$, where $\beta(t) = \int_0^t [\dot{a}(s) + r\|\dot{x}(s))\|]ds$, $\forall t \in I$ and K is a positive constant depending on $\|u_0\|, c, T, \beta$.

Fact $\mathcal{J}$**.** Taking account $\mathcal{J}$, it is clear that $D(\mathcal{A}_x)$ is nonempty and $\mathcal{A}_x$ is well defined. It is easy to see that $\mathcal{A}_x$ is monotone. Let us prove that $\mathcal{A}_x$is maximal monotone. We have to check that $R(I_{L^2(I,E,dt)} + \lambda\mathcal{A}_x) = L^2(I, E, dt)$ for each $\lambda > 0$. Let $g \in L^2(I, E, dt)$. Then, from $(H3)$ $t \mapsto v(t) = J_\lambda^{A_{(t,x(t))}} g(t) = g(t) - \lambda A_\lambda^{A_{(t,x(t))}} g(t)$ is measurable. Set

$$h(t) = \lambda A_\lambda^{A_{(t,x(t))}} g(t) = \lambda A_\lambda^{A_{(t,x(t))}} g(t) - \lambda A_\lambda^{A_{(t,x(t))}} u(t) + \lambda A_\lambda^{A_{(t,x(t))}} u(t)$$

where u denotes the absolutely continuous solution to $-\frac{du}{dt}(t) \in A_{(t,x(t))}u(t)$ using **Fact** $\mathcal{I}$. Then, h is measurable with

$$\|h(t)\| \leq 2\|g(t) - u(t)\| + \lambda \|A_\lambda^{A_{(t,x(t))}} u(t)\|$$

by noting that $A_\lambda^{A_{(t,x(t))}}$ is $\frac{2}{\lambda}$-Lipschitz and so we deduce that $h \in L^2(I, E, dt)$ because $g \in L^2(I, E, dt)$ and $t \mapsto A_\lambda^{A_{(t,x(t))}} u(t) \in L^\infty(I, E, dt)$ using $(H1)$. This proves that $v \in L^2(I, E, dt)$ and $g \in v + \lambda\mathcal{A}_x v$ so that $R(I_{L^2(I,E;dt)} + \lambda\mathcal{A}_x) = L^2(I, E, dt)$. □

Here is a useful application.

Corollary 1. *With hypotheses and notation of the preceding lemma, let (v_n) and (u_n) be two sequences in $L^2(I, E, dt)$ such that $v_n(t) \in A_{(t,x(t))}u_n(t)$ a.e for all $n \in \mathbf{N}$. If $v_n \to v$ weakly in $L^2(I, E, dt)$ and $u_n \to u$ strongly in $L^2(I, E, dt)$, then $v(t) \in A_{(t,x(t))}u(t)$ a.e.*

Theorem 1. *Let $I = [0, T]$. Let $(t, x) \to A_{(t,x)} : D(A_{(t,x)}) \to 2^E$ a maximal monotone operator satisfying:*
(H_1) $\|A^0_{(t,x)}y\| \leq c(1 + \|x\| + \|y\|)$ for all $(t, x, y) \in I \times E \times D(A_{(t,x)})$, for some positive constant c,
(H_2) $dis(A_{(t,x)}, A_{(\tau,y)}) \leq a(t) - a(\tau) + r\|x - y\|$, for all $0 \leq \tau \leq t \leq T$ and for all $(x, y) \in E \times E$, where r is a positive number, $a : I \to [0, +\infty[$ is nondecreasing absolutely continuous on I with $\dot{a} \in L^2(I, \mathbb{R}, dt)$,
(H_3) $D(A_{(t,x)})$ is boundedly-compactly measurable in the sense, for any bounded set $B \subset E$, there is a measurable compact valued integrably bounded mapping $\Psi_B : I \to E$ such that $D(A_{(t,x)}) \subset \Psi_B(t) \subset \gamma(t)\overline{B}_E$ for all $(t, x) \in I \times B$ where $\gamma \in L^2(I, \mathbb{R}, dt)$.

Then, for any $(x_0, u_0) \in E \times D(A_{(0,x_0)})$, there exist an absolutely continuous $x : I \to E$ and an absolutely continuous $u : I \to E$ such that

$$\begin{cases} x(t) = x_0 + \int_0^t u(s)ds, \quad \forall t \in I \\ x(0) = x_0, u(0) = u_0 \in D(A_{(0,x_0)}) \\ -\dot{u}(t) \in A_{(t,x(t))}u(t) \quad \text{a.e. } t \in I \\ u(t) \in D(A_{(t,x(t))}), \forall t \in I \end{cases}$$

Proof. Let us consider the closed convex subset $\mathcal{X}_\gamma$ in the Banach space $\mathcal{C}_E(I)$ defined by

$$\mathcal{X}_\gamma : \{h \in W^{1,2}(I, E) : h(t) = x_0 + \int_0^t \dot{h}(s)ds, ||\dot{h}(s)|| \leq \gamma(s) \, a.e., \gamma \in L^2(I, \mathbf{R}, dt)\}.$$

Then, $\mathcal{X}_\gamma$ is equi-absolutely continuous. By the fact that $\mathcal{J}$, for each $h \in X_\gamma$, there is a unique $W^{1,2}(I,E)$ mapping $u_h : I \to E$, which is the $W^{1,2}(I,E)$ solution to the inclusion

$$\begin{cases} -\dot{u}_h(t) \in A_{(t,h(t))}u_h(t) \quad \text{a.e. } t \in I \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in I \\ u_h(0) = u_0 \in D(A_{(0,h(0))}) = D(A_{(0,x_0)}), \end{cases}$$

with $\|\dot{u}_h(t)\| \leq K(1+\dot{\beta}(t))$, where $\beta(t) = \int_0^t [\dot{a}(s) + \gamma(s)]ds$, $\forall t \in I$ and K is a positive constant depending on $||u_0||, c, T, \beta$. We refer to [22] (Theorem 3.5) for details of the estimate of the velocity. Now, for each $h \in \mathcal{X}_\gamma$, let us consider the mapping

$$\Phi(h)(t) := x_0 + \int_0^t u_h(s)ds, \ t \in I.$$

As $u_h(s) \in D(A_{(s,h(s))}) \subset \bigcup_{x \in \mathcal{X}_\gamma(s)} D(A_{(s,x)}) \subset \Psi_\gamma(s) \subset \gamma(s)\overline{B}_E$ for all $s \in [0,T]$, where $\Psi_\gamma : I \to E$ is a compact valued measurable mapping given by condition (H_3). It is clear that $\Phi(h) \in \mathcal{X}_\gamma$. Our aim is to prove the existence theorem by applying some ideas developed in [24] via a generalized fixed point theorem [25] (Theorem 4.3), [26] (Lemma 1). Nevertheless, this needs a careful look using the estimation of the absolutely continuous solution given above. For this purpose, we first claim that $\Phi : \mathcal{X}_\gamma \to \mathcal{X}_\gamma$ is continuous and, for any $h \in \mathcal{X}_\gamma$ and for any $t \in I$, the inclusion holds

$$\Phi(h)(t) \in u_0 + \int_0^t \overline{co}\Psi_\gamma(s)ds.$$

Since $s \mapsto \overline{co}\Psi_\gamma(s)$ is a convex compact valued and integrably bounded multifunction, the second member is convex compact valued [27] so that $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E(I)$. Now, we check that Φ is continuous. It is sufficient to show that, if (h_n) converges uniformly to h in $\mathcal{X}_\gamma$, then the AC solution u_{h_n} associated with h_n

$$\begin{cases} u_{h_n}(0) \in D(A_{(0,h_n(0))}) \\ u_{h_n}(t) \in D(A_{(t,h_n(t))}), \forall t \in I \\ -\dot{u}_{h_n}(t) \in A_{(t,h_n(t))}u_{h_n}(t) \quad \text{a.e. } t \in I \end{cases}$$

uniformly converges to the AC solution u_h associated with h

$$\begin{cases} u_h(0) = u_0 \in D(A_{(0,h(0))}) \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in I \\ -\dot{u}_h(t) \in A_{(t,h(t))}u_h(t) \quad \text{a.e. } t \in I \end{cases}$$

As (u_{h_n}) is equi-absolutely continuous with the estimate $||\dot{u}_{h_n}(t)|| \leq K(1+\dot{\beta}(t))$ a.e for all $n \in \mathbf{N}$, we may assume that (u_{h_n}) converges uniformly to a AC mapping u and $(\frac{du_{h_n}}{dt})$ converges weakly in $L^2_E(I,dt)$ to $w \in L^2_E(I,dt)$ with $\|w(t)\| \leq K(1+\dot{\beta}(t))$ *a.e.* $t \in I$ so that

$$\text{weak-}\lim_n u_{h_n} = \text{weak-}\lim_n u_{h_n}(0) + \text{weak-}\lim_n \int_I \frac{du_{h_n}}{dt}$$

$$= u(0) + \int_I w\,dt := z(t), \ t \in I$$

By identifying the limits, we get $u(t) = z(t) = u(0) + \int_I w\,dt$, $t \in I$ with $u(0) =$ weak-$\lim_n u_{h_n}(0) = \lim_n u_{h_n}(0)$ and $\frac{du}{dt} = w$. As $u_{h_n}(t) \in D(A_{(t,h_n(t))}), \forall t \in I$ and $u_{h_n}(t) \to u(t)$, $A^0_{(t,h_n(t))}u_{h_n}(t)$ is bounded using (H_1) for every $t \in [0,T]$ and

$$dis(A_{(t,h_n(t))}, A_{(t,h(t))}) \leq r\|h_n(t) - h(t)\| \to 0$$

when $n \to \infty$ by (H_2), from Lemma 2, we deduce that $u(t) \in D(A_{(t,h(t))}), \forall t \in I$. Now, we are going to check that u satisfies the inclusion

$$-\frac{du}{dt}(t) \in A_{(t,h(t))}u(t) \quad \text{a.e. } t \in I$$

As $\frac{du_{h_n}}{dt} \to \frac{du}{dt}$ weakly in $L^2(I,E,dt)$, we may assume that $(\frac{du_{h_n}}{dt})$ Komlos converges to $\frac{du}{dt}$. There is a dt-negligible set N such that for $t \in I \setminus N$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \frac{du_{h_j}}{dt}(t) = \frac{du}{dt}(t). \tag{3}$$

$$-\frac{du_{h_n}}{dt}(t) \in A_{(t,h_n(t))}u_n(t). \tag{4}$$

Let $\eta \in D(A_{(t,h(t))})$.

Using Lemma 4, there is a sequence (η_n) such that $\eta_n \in D(A_{(t,h_n(t))}), \eta_n \to \eta$ and $A^0_{(t,h_n(t))}\eta_n \to A^0_{(t,h(t))}\eta$. From (4), by monotonicity,

$$\langle \frac{du_{h_n}}{dt}, u_{h_n}(t) - \eta_n\rangle \leq \langle A^0_{(t,h_n(t))}\eta_n, \eta_n - u_{h_n}(t)\rangle. \tag{5}$$

From

$$\langle \frac{du_{h_n}}{dt}(t), u(t) - \eta\rangle = \langle \frac{du_{h_n}}{dt}(t), u_{h_n}(t) - \eta_n\rangle + \langle \frac{du_{h_n}}{dt}(t), u(t) - u_{h_n}(t) - (\eta - \eta_n)\rangle,$$

let us write

$$\frac{1}{n}\sum_{j=1}^{n} \langle \frac{du_{h_j}}{dt}(t), u(t) - \eta\rangle = \frac{1}{n}\sum_{j=1}^{n} \langle \frac{du_{h_j}}{dt}(t), u_{h_j}(t) - \eta_j\rangle + \frac{1}{n}\sum_{j=1}^{n} \langle \frac{du_{h_j}}{dt}(t), u(t) - u_{h_j}(t)\rangle$$

$$+ \sum_{j=1}^{n} \langle \frac{du_{h_j}}{dt}(t), \eta_j - \eta\rangle,$$

so that

$$\frac{1}{n}\sum_{j=1}^{n} \langle \frac{du_{h_j}}{dt}(t), u(t) - \eta\rangle \leq \frac{1}{n}\sum_{j=1}^{n} \langle A^0_{(t,h_j(t))}\eta_j, \eta_j - u_{h_j}(t)\rangle + K(1 + \dot{\beta}(t))\frac{1}{n}\sum_{j=1}^{n} \|u(t) - u_{h_j}(t))\|.$$

$$+K(1 + \dot{\beta}(t))\frac{1}{n}\sum_{j=1}^{n} \|\eta_j - \eta\|.$$

Passing to the limit using (3) when $n \to \infty$, this last inequality gives immediately

$$\langle \frac{du}{dt}(t), u(t) - \eta\rangle \leq \langle A^0_{(t,h(t))}\eta, \eta - u(t)\rangle \text{ a.e.}$$

As a consequence, by Lemma 1, we get $-\frac{du}{dt}(t) \in A_{(t,h(t))}u(t)$ a.e. with $u(0) \in D(A_{(0,h(0))})$ so that, by uniqueness, $u = u_h$.
Now, let us check that $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous. Let $h_n \to h$. We have

$$\Phi(h_n)(t) - \Phi(h)(t) = \int_0^t u_{h_n}(s)ds - \int_0^t u_h(s)ds = \int_0^t [u_{h_n}(s) - u_h(s)]ds$$

As $||u_{h_n}(.) - u_h(.)|| \to 0$ pointwisely and is uniformly bounded, we conclude that

$$\sup_{t\in I} ||\Phi(h_n)(t) - \Phi(h)(t)|| \leq \sup_{t\in I} \int_0^t ||u_{h_n}(.) - u_h(.)||ds \to 0$$

so that $\Phi(h_n) - \Phi(h) \to 0$ in $\mathcal{C}_E(I)$. Since $\Phi : \mathcal{X}_\gamma \to \mathcal{X}_\gamma$ is continuous and $\Phi(\mathcal{X}_\gamma)$ is relatively compact in $\mathcal{C}_E(I)$, by [25] (Theorem 4.3), [26] (Lemma 1), Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}_\gamma$ that means

$$h(t) = \Phi(h)(t) = x_0 + \int_0^t u_h(s)ds,\ t \in I,$$
$$\begin{cases} u_h(t) \in D(A_{(t,h(t))}) \\ -\dfrac{du_h}{dt}(t) \in A_{(t,h(t))}u_h(t)\ dt\text{-a.e.} \end{cases}$$

the proof is complete. □

There is a direct application to sweeping process.

Corollary 2. *Let $C : I \times E \to E$ be a convex compact valued mapping satisfying*
(i) $C(t,x) \subset \gamma(t)\overline{B}_E, \forall (t,x) \in I \times E$, where $\gamma \in L^2(I,\mathbb{R},dt)$,
(ii) $d_H(C(s,x),C(t,y)) \leq a(t) - a(\tau) + r||x-y||$, for all $0 \leq \tau \leq t \leq 1$ and for all $(x,y) \in E \times E$, where r is a positive number, $a : I \to [0,+\infty[$ is nondecreasing absolutely continuous on I with $\dot{a} \in L^2(I,\mathbb{R},dt)$,
(iii) For any $t \in I$, for any bounded set $B \subset E$, $C(t,B)$ is relatively compact.
Then, for any $(x_0,u_0) \in E \times C(0,x_0)$, there exist an absolutely continuous $x : I \to E$ and and absolutely continuous $u : I \to E$ such that

$$\begin{cases} x(t) = x_0 + \int_0^t u(s)ds, \quad \forall t \in I \\ x(0) = x_0, u(0) = u_0 \in C(0,x_0) \\ -\dot{u}(t) \in N_{C(t,x(t))}u(t) \quad \text{a.e. } t \in I \\ u(t) \in C(t,x(t)),\ \forall t \in I \end{cases}$$

Proof. It is easy to apply Theorem 1 with $A_{(t,x(t))} = N_{C(t,x(t))}$ □

Now, we proceed to the Lipschitz perturbation of the preceding theorem.

Theorem 2. *Let $I = [0,T]$. Let $(t,x) \to A_{(t,x)} : D(A_{(t,x)}) \to 2^E$ be a maximal monotone operator satisfying:*
(H_1) $\|A^0_{(t,x)}y\| \leq c(1 + \|x\| + \|y\|)$ for all $(t,x,y) \in I \times E \times D(A_{(t,x)})$, for some positive constant c,
(H_2) $dis(A_{(t,x)}, A_{(\tau,y)}) \leq a(t) - a(\tau) + r\|x-y\|$, for all $0 \leq \tau \leq t \leq T$ and for all $(x,y) \in E \times E$, where r is a positive number, $a : I \to [0,+\infty[$ is nondecreasing absolutely continuous on I with $\dot{a} \in L^2(I,\mathbb{R},dt)$,
(H_3) $D(A_{(t,x)})$ is boundedly-compactly measurable in the sense, for any bounded set $B \subset E$, there is a measurable compact valued integrably bounded mapping $\Psi_B : I \to E$ such that $D(A_{(t,x)}) \subset \Psi_B(t) \subset \gamma(t)\overline{B}_E$ for all $(t,x) \in I \times B$, where $\gamma \in L^2(I,\mathbb{R},dt)$.
Let $f : I \times E \times E \to E$ such that
(i) $f(.,x,y)$ is Lebesgue measurable on I for all $(x,y) \in E \times E$
(ii)$f(t,.,.)$ is continuous on $E \times E$,
(iii) $||f(t,x,y)|| \leq M$ for all $(t,x,y) \in I \times E \times E$,
(iv) $||f(t,x,y) - f(t,x,z)|| \leq M||y-z||$, for all $(t,x,y,z) \in I \times E \times E \times E$

for some positive constant M.
Then, for any $(x_0, u_0) \in E \times D(A_{(0,x_0)})$*, there exists an absolutely continuous* $x : I \to E$ *and an absolutely continuous* $u : I \to E$ *such that*

$$\begin{cases} x(t) = x_0 + \int_0^t u(s)ds, \quad \forall t \in I \\ x(0) = x_0, u(0) = u_0 \in D(A_{(0,x_0)}) \\ -\dot{u}(t) \in A_{(t,x(t))}u(t) + f(t, x(t), u(t)) \quad \text{a.e. } t \in I \\ u(t) \in D(A_{(t,x(t))}), \forall t \in I \end{cases}$$

Proof. Let us consider the closed convex subset $\mathcal{X}_\gamma$ in the Banach space $\mathcal{C}_E(I)$ defined by

$$\mathcal{X}_\gamma : \{h \in W^{1,2}(I, E) : h(t) = x_0 + \int_0^t \dot{h}(s)ds, ||\dot{h}(s)|| \leq \gamma(s) \, a.e., \gamma \in L^2(I, \mathbf{R}, dt)\}.$$

Then, $\mathcal{X}_\gamma$ is equi-absolutely continuous. By fact $\mathcal{J}$, for each $h \in X_\gamma$, there is a unique $W^{1,2}(I, E)$ mapping $u_h : I \to E$, which is the $W^{1,2}(I, E)$ solution to the inclusion

$$\begin{cases} -\dot{u}_h(t) \in A_{(t,h(t))}u_h(t) + f(t, h(t), u_h(t)) \quad \text{a.e. } t \in I \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in I \\ u_h(0) = u_0 \in D(A_{(0,h(0))}) = D(A_{(0,x_0)}), \end{cases}$$

with $||\dot{u}_h(t)|| \leq K(1 + \dot{\beta}(t)) + M(K + 1) = \eta(t)$ where $\beta(t) = \int_0^t [\dot{a}(s) + \gamma(s)]ds$, $\forall t \in I$ and K is a positive constant depending on $||u_0||, c, T, \beta$. We refer to (Theorem 3.5) for details of the estimate of the velocity. Now, for each $h \in \mathcal{X}_\gamma$, let us consider the mapping

$$\Phi(h)(t) := x_0 + \int_0^t u_h(s)ds, \ t \in I.$$

As $u_h(s) \in D(A_{(s,h(s))}) \subset \bigcup_{x \in \mathcal{X}_\gamma(s)} D(A_{(s,x)}) \subset \Psi_\gamma(s) \subset \gamma(s)\overline{B}_E$ for all $s \in [0, T]$, where $\Psi_\gamma : I \to E$ is a compact valued measurable mapping given by condition (H_3). It is clear that $\Phi(h) \in \mathcal{X}_\gamma$. Our aim is to prove the existence theorem by applying some ideas developed in Castaing et al. [24] via the same generalized fixed point theorem already used [25,26]. Nevertheless, this needs a careful look using the estimation of the absolutely continuous solution given above. For this purpose, we first claim that $\Phi : \mathcal{X}_\gamma \to \mathcal{X}_\gamma$ is continuous, and, for any $h \in \mathcal{X}_\gamma$ and for any $t \in I$, the inclusion holds

$$\Phi(h)(t) \in u_0 + \int_0^t \overline{co}\Psi_\gamma(s)ds.$$

Since $s \mapsto \overline{co}\Psi_\gamma(s)$ is a convex compact valued and integrably bounded multifunction, the second member is convex compact valued [27] so that $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E(I)$. Now, we check that Φ is continuous. It is sufficient to show that, if (h_n) converges uniformly to h in $\mathcal{X}_\gamma$, then the AC solution u_{h_n} associated with h_n

$$\begin{cases} u_{h_n}(0) \in D(A_{(0,h_n(0))}) \\ u_{h_n}(t) \in D(A_{(t,h_n(t))}), \forall t \in I \\ -\dot{u}_{h_n}(t) \in A_{(t,h_n(t))}u_{h_n}(t) + f(t, h_n(t), u_{h_n}(t)), \quad \text{a.e. } t \in I \end{cases}$$

uniformly converges to the AC solution u_h associated with h

$$\begin{cases} u_h(0) \in D(A_{(0,h(0))}) \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in I \\ -\dot{u}_h(t) \in A_{(t,h(t))}u_h(t) + f(t, h(t), u_h(t)) \quad \text{a.e. } t \in I \end{cases}$$

As (u_{h_n}) is equi-absolutely continuous with the estimate $||\dot{u}_{h_n}(t)|| \leq K(1+\dot{\beta}(t)) + (K+1)M = \psi(t)$ a.e for all $n \in \mathbf{N}$, we may assume that (u_{h_n}) converges uniformly to a AC mapping u and $(\frac{du_{h_n}}{dt})$ converges weakly in $L^2_E(I,dt)$ to $w \in L^2_E(I,dt)$ with $||w(t)|| \leq K(1+\dot{\beta}(t)) + (K+1)M$ *a.e.* $t \in I$ so that

$$\text{weak-}\lim_n u_{h_n} = \text{weak-}\lim_n u_{h_n}(0) + \text{weak-}\lim_n \int_{[0,t]} \frac{du_{h_n}}{dt}$$

$$= u(0) + \int_{[0,t]} w\, dt := z(t),\ t \in I$$

By identifying the limits, we get
$u(t) = z(t) = u(0) + \int_{[0,t]} w\, dt$, $t \in I$ with $u(0) = \text{weak-}\lim_n u_{h_n}(0) = \lim_n u_{h_n}(0)$ and $\frac{du}{dt} = w$. As $u_{h_n}(t) \in D(A_{(t,h_n(t))}), \forall t \in I$ and $u_{h_n}(t) \to u(t)$, $A^0_{(t,h_n(t))}u_{h_n}(t)$ is bounded using (H_1) for every $t \in I$ and

$$dis(A_{(t,h_n(t)}, A_{(t,h(t))} \leq r||h_n(t) - h(t)|| \to 0$$

when $n \to \infty$ by (H_2), from Lemma 2, we deduce that $u(t) \in D(A_{(t,h(t))}), \forall t \in I$.

Now, we are going to check that u satisfies the inclusion

$$-\frac{du}{dt}(t) \in A_{(t,h(t))}u(t) + f(t,h(t),u_h(t)) \quad \text{a.e. } t \in I$$

As $\dot{u}_{h_n} \to \dot{u}$ weakly in $L^2_H([0,1])$, $\dot{u}_{h_n} \to \dot{u}$ Komlos. Note that $f(t,h_n(t),u_{h_n}(t)) \to f(t,h(t),u(t))$ weakly in $L^2_E([0,1])$. Thus, $z_n(t) := f(t,h_n(t),u_{h_n}(t)) \to z(t) := f(t,h(t),u(t))$ Komlos. Hence, $\dot{u}_{h_n}(t) + f(t,h_n(t),u_{h_n}(t) \to \dot{u}(t) + f(t,h(t),u(t))$ Komlos. Apply Lemma 4 to $A_{(t,h_n(t))}$ and $A_{(t,h(t))}$ to find a sequence (η_n) such that $\eta_n \in D(A_{(t,h_n(t))})$, $\eta_n \to \eta$, $A^0_{(t,h_n(t)}\eta_n \to A^0_{(t,h(t))}u(t)$. From

$$-\dot{u}_{h_n}(t) \in A_{(t,h_n(t))}u_{h_n}(t) + f(t,h_n(t),u_{h_n}(t))$$

by monotonicity

$$\langle \frac{du_{h_n}}{dt} + z_n(t), u_{h_n}(t) - \eta_n \rangle \leq A^0_{(t,h_n(t))}\eta_n, \eta_n - u_{h_n}(t) \rangle.$$

From

$$\langle \frac{du_{h_n}}{dt}(t) + z_n(t), u(t) - \eta \rangle = \langle \frac{du_{h_n}}{dt}(t) + z_n(t), u_{h_n}(t) - \eta_n \rangle$$

$$+ \langle \frac{du_{h_n}}{dt}(t) + z_n(t), u(t) - u_{h_n}(t) - (\eta - \eta_n) \rangle,$$

let us write

$$\frac{1}{n}\sum_{j=1}^n \langle \frac{du_{h_j}}{dt}(t) + z_j(t), u(t) - \eta \rangle = \frac{1}{n}\sum_{j=1}^n \langle \frac{du_{h_j}}{dt}(t) + z_j(t), u_{h_j}(t) - \eta_j \rangle$$

$$+ \frac{1}{n}\sum_{j=1}^n \langle \frac{du_{h_j}}{dt}(t) + z_j(t), u(t) - u_{h_j}(t) \rangle$$

$$+ \sum_{j=1}^n \langle \frac{du_{h_j}}{dt}(t) + z_j(t), \eta_j - \eta \rangle,$$

so that

$$\frac{1}{n}\sum_{j=1}^n \langle \frac{du_{h_j}}{dt}(t) + z_j(t), u(t) - \eta \rangle \leq \frac{1}{n}\sum_{j=1}^n \langle A^0_{(t,h_j(t))}\eta_j, \eta_j - u_{h_j}(t) \rangle + (\psi(t) + M)\frac{1}{n}\sum_{j=1}^n \|v(t) - u_{h_j}(t))\|.$$

$$+(\psi(t)+M)\frac{1}{n}\sum_{j=1}^{n}\|\eta_j-\eta\|.$$

Passing to the limit using (3) when $n\to\infty$, this last inequality gives immediately

$$\langle \frac{du}{dt}(t)+z(t),u(t)-\eta\rangle \le \langle A^0_{(t,h(t))}\eta,\eta-u(t)\rangle \text{ a.e.}$$

As a consequence, by Lemma 1, we get $-\frac{du}{dt}(t)\in A_{(t,h(t))}u(t)+z(t)$ a.e. with $u(t)\in D(A_{(t,h(t))})$ for all $t\in[0,1]$ so that, by uniqueness, $u=u_h$.
Since $h_n\to h$, we have

$$\Phi(h_n)(t)-\Phi(h)(t)=\int_0^1 u_{h_n}(s)ds-\int_0^1 u_h(s)ds$$

$$=\int_0^1[u_{h_n}(s)-u_h(s)]ds$$

$$\le\int_0^1\|u_{h_n}(s)-u_h(s)\|ds$$

As $\|u_{h_n}(\cdot)-u_h(\cdot)\|\to 0$ uniformly, we conclude that

$$\sup_{t\in[0,1]}\|\Phi(h_n)(t)-\Phi(h)(t)\|\le\int_0^1\|u_{h_n}(\cdot)-u_h(\cdot)\|ds\to 0$$

so that $\Phi(h_n)\to\Phi(h)$ in $\mathcal{C}_E([0,1])$. Since $\Phi:\mathcal{X}_\gamma\to\mathcal{X}_\gamma$ is continuous and $\Phi(\mathcal{X}_\gamma)$ is relatively compact in $\mathcal{C}_E(I)$, by [25,26] Φ has a fixed point, say $h=\Phi(h)\in\mathcal{X}_\gamma$ that means

$$h(t)=\Phi(h)(t)=x_0+\int_0^t u_h(s)ds,\ t\in I,$$

$$\begin{cases} u_h(t)\in D(A_{(t,h(t))}) \\ -\dfrac{du_h}{dt}(t)\in A_{(t,h(t))}u_h(t)+f(t,h(t),u_h(t))\ dt\text{-a.e.}\end{cases}$$

The proof is complete. □

4. Towards a Fractional Order of Evolution Inclusion with a Time and State Dependent Maximal Monotone Operator

Now, $I=[0,1]$ and we investigate a class of boundary value problem governed by a fractional differential inclusion (FDI) in a separable Hilbert space E coupled with an evolution inclusion governed by a time and stated dependent maximal monotone operator:

$$D^\alpha h(t)+\lambda D^{\alpha-1}h(t)=u(t),t\in I, \tag{6}$$

$$I^\beta_{0^+}h(t)\,|_{t=0}:=\lim_{t\to 0}\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}h(s)ds=0,\quad h(1)=I^\gamma_{0^+}h(1)=\int_0^1\frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)}h(s)ds, \tag{7}$$

$$-\frac{du}{dt}(t)\in A_{(t,h(t))}u(t)\text{ a.e. }t\in I. \tag{8}$$

where $\alpha\in]1,2]$, $\beta\in[0,2-\alpha]$, $\lambda\ge 0,\gamma>0$ are given constants, D^α is the standard Riemann–Liouville fractional derivative, and Γ is the gamma function.

4.1. Fractional Calculus

For the convenience of the reader, we begin with a few reminders of the concepts that will be used in the rest of the paper.

Definition 1 (Fractional Bochner integral)**.** *Let E be a separable Banach space. Let $f : I = [0,1] \to E$. The fractional Bochner-integral of order $\alpha > 0$ of the function f is defined by*

$$I_{a^+}^{\alpha} f(t) := \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \ t > a.$$

In the above definition, the sign "$\int$" denotes the classical Bochner integral.

Lemma 6 ([10])**.** *Let $f \in L^1([0,1], E, dt)$. We have*

(i) If $\alpha \in]0,1[$ then $I^{\alpha} f$ exists almost everywhere on I and $I^{\alpha} f \in L^1(I, E, dt)$.
(ii) If $\alpha \in [1, \infty)$, then $I^{\alpha} f \in C_E(I)$.

Definition 2. *Let E be a separable Banach space. Let $f \in L^1(I, E, dt)$. We define the Riemann–Liouville fractional derivative of order $\alpha > 0$ of f by*

$$D^{\alpha} f(t) := D_{0^+}^{\alpha} f(t) = \frac{d^n}{dt^n} I_{0^+}^{n-\alpha} f(t) = \frac{d^n}{dt^n} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(s)ds,$$

where $n = [\alpha] + 1$.

In the case $E \equiv \mathbf{R}$, we have the following well-known results.

Lemma 7 ([1,3])**.** *Let $\alpha > 0$. The general solution of the fractional differential equation $D^{\alpha} x(t) = 0$ is given by*

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \tag{9}$$

where $c_i \in \mathbf{R}$, $i = 1, 2, \ldots, N$ (N is the smallest integer greater than or equal to α).

Remark 1. Since $D_{0^+}^{\alpha} I_{0^+}^{\alpha} v(t) = v(t)$, for every $v \in C(I)$, $D_{0^+}^{\alpha} [I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) - x(t)] = 0$ and, by Lemma 7, it follows that

$$x(t) = I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) + c_1 t^{\alpha-1} + \cdots + c_N t^{\alpha-N}, \tag{10}$$

for some $c_i \in \mathbf{R}$, $i = 1, 2, \ldots, N$.

We denote by $W_{B,E}^{\alpha,1}(I)$ the space of all continuous functions in $C_E(I)$ such that their Riemann–Liouville fractional derivative of order $\alpha - 1$ are continuous and their Riemann–Liouville fractional derivative of order α are Bochner integrable.

4.2. Green Function and Its Properties

Let $\alpha \in]1,2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0$, $\gamma > 0$ and $G : [0,1] \times [0,1] \to \mathbf{R}$ be a function defined by

$$G(t,s) = \varphi(s) I_{0^+}^{\alpha-1}(\exp(-\lambda t)) + \begin{cases} \exp(\lambda s) I_{s^+}^{\alpha-1}(\exp(-\lambda t)), & 0 \leq s \leq t \leq 1, \\ 0, & 0 \leq t \leq s \leq 1, \end{cases} \tag{11}$$

where

$$\varphi(s) = \frac{\exp(\lambda s)}{\mu_0}\left[\left(I_{s^+}^{\alpha-1+\gamma}(\exp(-\lambda t))\right)(1) - \left(I_{s^+}^{\alpha-1}(\exp(-\lambda t))\right)(1)\right] \tag{12}$$

with

$$\mu_0 = \left(I_{0^+}^{\alpha-1}(\exp(-\lambda t))\right)(1) - \left(I_{0^+}^{\alpha-1+\gamma}(\exp(-\lambda t))\right)(1). \tag{13}$$

We recall and summarize a useful result ([28]).

Lemma 8. *Let E be a separable Banach space. Let G be the function defined by (11)–(13).*

(i) *$G(\cdot,\cdot)$ satisfies the following estimate*

$$|G(t,s)| \leq \frac{1}{\Gamma(\alpha)}\left(\frac{1+\Gamma(\gamma+1)}{|\mu_0|\Gamma(\alpha)\Gamma(\gamma+1)}+1\right) = M_G.$$

(ii) *If $u \in W_{B,E}^{\alpha,1}([0,1])$ satisfying boundary conditions (7), then*

$$u(t) = \int_0^1 G(t,s)\left(D^{\alpha}u(s) + \lambda D^{\alpha-1}u(s)\right)ds \quad \text{for every } t \in [0,1].$$

(iii) *Let $f \in L_E^1([0,1])$ and let $u_f : [0,1] \to E$ be the function defined by*

$$u_f(t) := \int_0^1 G(t,s)f(s)ds \quad \text{for} \quad t \in [0,1].$$

Then,

$$I_{0^+}^{\beta}u_f(t)\,|_{t=0} = 0 \quad \text{and} \quad u_f(1) = \left(I_{0^+}^{\gamma}u_f\right)(1).$$

Moreover $u_f \in W_{B,E}^{\alpha,1}([0,1])$ and we have

$$\left(D^{\alpha-1}u_f\right)(t) = \int_0^t \exp(-\lambda(t-s))f(s)ds + \exp(-\lambda t)\int_0^1 \varphi(s)f(s)ds \quad \text{for } t \in [0,1], \tag{14}$$

$$\left(D^{\alpha}u_f\right)(t) + \lambda\left(D^{\alpha-1}u_f\right)(t) = f(t) \quad \text{for all} \quad t \in [0,1]. \tag{15}$$

Remark 2. From Lemma 8, we can claim that, if

$$u_f(t) = \int_0^1 G(t,s)f(s)ds, \quad f \in L_E^1([0,1]),$$

then, for all $t \in [0,1]$,

$$\left\|u_f(t)\right\| \leq M_G\|f\|_{L_E^1([0,1])} \quad \text{and} \quad \left\|D^{\alpha-1}u_f(t)\right\| \leq M_G\|f\|_{L_E^1([0,1])}, \tag{16}$$

Indeed, by Lemma 8(i), it suffices to prove that $\left\|D^{\alpha-1}u_f(t)\right\| \leq M_G\|f\|_{L_E^1([0,1])}$.
It follows from (14) that

$$\left\|D^{\alpha-1}u_f(t)\right\| \leq \int_0^1 (1+|\varphi(s)|)|f(s)|ds.$$

This, by an increase of φ (See [28] (2.9)), gives

$$\left\|D^{\alpha-1}u_f(t)\right\| \leq \Gamma(\alpha)M_G\|f\|_{L_E^1([0,1])}$$

and, since $\alpha \in [1,2]$, implies our conclusion.

4.3. Topological Structure of the Solution Set

From Lemma 8, we summarize a crucial fact.

Lemma 9. *Let E be a separable Banach space. Let $f \in L^1(I,E,dt)$. Then, the boundary value problem*

$$\begin{cases} D^{\alpha}u(t) + \lambda D^{\alpha-1}u(t) = f(t), \quad t \in I \\ I_{0^+}^{\beta}u(t)\,|_{t=0} = 0, \quad u(1) = I_{0^+}^{\gamma}u(1) \end{cases}$$

has a unique $W_{B,E}^{\alpha,1}(I)$-solution defined by

$$u(t) = \int_0^1 G(t,s)f(s)ds,\ t \in I.$$

Theorem 3. *Let E be a separable Banach space. Let $X : I \to E$ be a convex compact valued measurable multifunction such that $X(t) \subset \gamma \overline{B}_E$ for all $t \in I$, where γ is a positive constant and S_X^1 be the set of all measurable selections of X. Then, the $W_{B,E}^{\alpha,1}(I)$-solutions set of problem*

$$\begin{cases} D^{\alpha}u(t) + \lambda D^{\alpha-1}u(t) = f(t), f \in S_X^1,\ a.e.\ t \in I \\ I_{0^+}^{\beta}u(t)\,|_{t=0} = 0, \quad u(1) = I_{0^+}^{\gamma}u(1) \end{cases} \tag{17}$$

is compact in $C_E(I)$.

Proof. By virtue of Lemma 6, the $W_{B,E}^{\alpha,1}([0,1])$-solutions set $\mathcal{X}$ to the above inclusion is characterized by

$$\mathcal{X} = \{u_f : I \to E,\ u_f(t) = \int_0^1 G(t,s)f(s)ds,\ f \in S_X^1,\ t \in I\}$$

Claim: $\mathcal{X}$ is bounded, convex, equicontinuous and **compact** in $C_E(I)$.

From definition of the Green function G, it is not difficult to show that $\{u_f : f \in S_X^1\}$ is bounded, equicontinuous in $C_E(I)$. Indeed, let $\left(u_{f_n}\right)$ be a sequence in $\mathcal{X}$. We note that, for each $n \in \mathbb{N}$, we have $u_{f_n} \in W_{B,E}^{\alpha,1}(I)$, and

$$u_{f_n}(t) = \int_0^1 G(t,s)f_n(s)ds, \quad t \in I,$$

with

- $I_{0^+}^{\beta}u_{f_n}(t)|_{t=0} = 0, \quad u_{f_n}(1) = I_{0^+}^{\gamma}u(1),$
- $\left(D^{\alpha-1}u_{f_n}\right)(t) = \int_0^t \exp(-\lambda(t-s))f_n(s)ds + \exp(-\lambda t)\int_0^1 \varphi(s)f_n(s)ds, \quad t \in I,$
- $\left(D^{\alpha}u_{f_n}\right)(t) + \lambda\left(D^{\alpha-1}u_{f_n}\right)(t) = f_n(t), t \in I.$

For $t_1, t_2 \in I$, $t_1 < t_2$, we have

$$\begin{aligned}
u_{f_n}(t_2) - u_{f_n}(t_1) &= \int_0^1 G(t,s)(f_n(t_2,s) - f_n(t_1,s))ds \\
&= \int_0^1 \varphi(s) f_n(s) ds \left(\int_0^{t_2} \frac{e^{-\lambda\tau}}{\Gamma(\alpha-1)} (t_2-\tau)^{\alpha-2} d\tau - \int_0^{t_1} \frac{e^{-\lambda\tau}}{\Gamma(\alpha-1)} (t_1-\tau)^{\alpha-2} d\tau \right) \\
&+ \int_0^{t_2} e^{\lambda s} \left(\int_s^{t_2} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} e^{-\lambda\tau} d\tau \right) f(s) ds - \int_0^{t_1} e^{\lambda s} \left(\int_s^{t_1} \frac{e^{-\lambda\tau}}{\Gamma(\alpha-1)} (t_1-\tau)^{\alpha-2} d\tau \right) f(s) ds \\
&= \int_0^1 \phi(s) f(s) ds \left[\int_0^{t_1} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2} - (t_1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau + \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \right] \\
&+ \int_0^{t_1} e^{\lambda s} \left(\int_s^{t_1} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2} - (t_1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \right) f(s) ds \\
&+ \int_0^{t_1} e^{\lambda s} \left(\int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \right) f(s) ds + \int_{t_1}^{t_2} e^{\lambda s} \left(\int_s^{t_2} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} e^{-\lambda\tau} d\tau \right) f(s) ds.
\end{aligned}$$

Then, we get

$$\begin{aligned}
\|u_{f_n}(t_2) - u_{f_n}(t_1)\| &\leq \int_0^1 \left(|\varphi(s)| + e^{\lambda s} \right) |X(s)| ds \int_0^{t_1} e^{-\lambda\tau} \frac{(t_1-\tau)^{\alpha-2} - (t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \\
&+ \int_0^1 \left(|\varphi(s)| + e^{\lambda s} \right) |X(s)| ds \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \\
&+ \int_{t_1}^{t_2} e^{\lambda s} |X(s)| ds \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau.
\end{aligned}$$

It is easy to obtain, after an integration by part, that

$$\int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau = e^{-\lambda t_1} \frac{(t_2-t_1)^{\alpha-2}}{\Gamma(\alpha)} + \lambda \int_{t_1}^{t_2} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau \leq \frac{1+\lambda}{\Gamma(\alpha)} (t_2-t_1)^{\alpha-1}$$

and

$$\begin{aligned}
\int_0^{t_1} e^{-\lambda\tau} \frac{(t_1-\tau)^{\alpha-2} - (t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau &\leq \int_0^{t_1} \frac{(t_1-\tau)^{\alpha-2} - (t_2-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \\
&= \frac{(t_2-t_1)^{\alpha-1} + t_1^{\alpha-1} - t_2^{\alpha-1}}{\Gamma(\alpha)}
\end{aligned}$$

Using the inequality that $|a^p - b^p| \leq |a-b|^p$ for all $a, b \geq 0$ and $0 < p \leq 1$, we yield

$$\int_0^{t_1} e^{-\lambda\tau} \frac{(t_2-\tau)^{\alpha-2} - (t_1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} d\tau \leq \frac{2}{\Gamma(\alpha)} (t_2-t_1)^{\alpha-1}$$

Then, since $\alpha \in]1,2]$, we can increase $\|u_{f_n}(t_2) - u_{f_n}(t_1)\|$ by

$$\|u_{f_n}(t_2) - u_{f_n}(t_1)\| \leq K |t_2 - t_1|^{\alpha-1}$$

with $K = \int_0^1 \left[(3+\lambda)|\phi(s)| + (4+2\lambda)e^{\lambda s} \right] |X(s)| ds$ This shows that $\left\{ u_{f_n} : n \in \mathbf{N} \right\}$ is equicontinuous in $C_E(I)$. Moreover, for each $t \in I$, the set $\left\{ u_{f_n}(t) : n \in \mathbf{N} \right\}$ is contained in the convex compact set $\int_0^1 G(t,s) X(s) ds$ [27,29] so that $\mathcal{X}$ is relatively compact in $C_E(I)$ as claimed. Thus, we can assume that

$$\lim_{n \to \infty} u_{f_n} = u_\infty \in C_E(I)$$

As S^1_X is $\sigma(L^1_E, L^\infty_{E^*})$-compact, e.g., [29], we may assume that (f_n) $\sigma(L^1_E, L^\infty_{E^*})$-converges to $f_\infty \in S^1_X$, so that u_{f_n} weakly converges to u_{f_∞} in $C_E(I)$ where $u_{f_\infty}(t) = \int_0^1 G(t,s) f_\infty(s) ds$ and so, for every $t \in I$,

$$u_\infty(t) = w\text{-}\lim_{n\to\infty} u_{f_n}(t) = w\text{-}\lim_{n\to\infty} \int_0^1 G(t,s) f_n(s) ds = \int_0^1 G(t,s) f_\infty(s) ds = u_{f_\infty}(t),$$

and

$$\begin{aligned} w\text{-}\lim_{n\to\infty} \left(D^{\alpha-1} u_{f_n}\right)(t) &= w\text{-}\lim_{n\to\infty} \left[\int_0^t \exp(-\lambda(t-s)) f_n(s) ds + \exp(-\lambda t) \int_0^1 \varphi(s) f_n(s) ds\right] \\ &= \int_0^t \exp(-\lambda(t-s)) f_\infty(s) ds + \exp(-\lambda t) \int_0^1 \varphi(s) f_\infty(s) ds \\ &= \left(D^{\alpha-1} u_{f_\infty}\right)(t), \quad t \in I. \end{aligned}$$

This means $u_\infty \in \mathcal{X}$, and the proof of the theorem is complete. □

Remark 3. *In the course of the proof of Theorem 3, we have proven the continuous dependence of the mappings $f \mapsto u_f$ and $f \mapsto D^{\alpha-1} u_f$ on the convex $\sigma(L^1_E, L^\infty_{E^*})$-compact set S^1_X. This fact has some importance in further applications.*

Theorem 4. *Let $I = [0,1]$. Let $(t,x) \to A_{(t,x)} : D(A_{(t,x)}) \to 2^E$ a maximal monotone operator satisfying:*
(H_1) $||A^0_{(t,x)} y|| \le c(1 + ||x|| + ||y||)$ for all $(t,x,y) \in I \times E \times D(A_{(t,x)})$, for some positive constant c,
(H_2) $dis(A_{(t,x)}, A_{(\tau,y)}) \le a(t) - a(\tau) + r||x-y||$, for all $0 \le \tau \le t \le 1$ and for all $(x,y) \in E \times E$, where r is a positive number, $a : I \to [0, +\infty[$ is nondecreasing absolutely continuous on I with $\dot{a} \in L^2(I, \mathbb{R}, dt)$,
(H_3) $D(A_{(t,x)}) \subset X(t) \subset \gamma \overline{B}_E$ for all $(t,x) \in I \times E$, where $X : I \to E$ is a convex compact valued measurable mapping and γ is a positive number.
Then, there is a $W^{\alpha,1}_{B,E}(I)$ mapping $x : I \to E$ and an absolutely continuous mapping $u : I \to E$ satisfying

$$\begin{cases} D^\alpha x(t) + \lambda D^{\alpha-1} x(t) = u(t), \ t \in I \\ I^\beta_{0^+} x(t)\,|_{t=0} = 0, \quad x(1) = I^\gamma_{0^+} x(1) \\ u(t) \in D(A_{(t,x(t))}) \\ -\frac{du}{dt}(t) \in A_{(t,x(t))} u(t) \quad \text{a.e. } t \in I. \end{cases}$$

Proof. Let us consider the convex compact subset $\mathcal{X}$ in the Banach space $\mathcal{C}_E(I)$ defined by

$$\mathcal{X} := \{u_f : I \to E : u_f(t) = \int_0^1 G(t,s) f(s) ds, \ f \in S^1_X, \ t \in I\}$$

We note that $\mathcal{X}$ is convex compact and equi-Lipschitz. Cf the proof of Theorem 3. Now, for each $h \in \mathcal{X}$, let us consider the unique absolutely continuous solution u_h to

$$\begin{cases} -\dot{u}_h(t) \in A_{(t,h(t))} u_h(t) \quad \text{a.e. } t \in I \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in I \\ u_h(0) = u_0 \in D(A_{(0,h(0))}) \end{cases}$$

For each h, let us set

$$\Phi(h)(t) = \int_0^1 G(t,s) u_h(s) ds, \ t \in I$$

Since $u_h(s) \in D(A_{(s,h(s))}) \subset X(s)$, then it is clear that $\Phi(h) \in \mathcal{X}$.
Now, we check that Φ is continuous. It is sufficient to show that, if (h_n) converges uniformly to h in $\mathcal{X}$, then the absolutely continuous solution u_{h_n} associated with h_n

$$\begin{cases} u_{h_n}(0) = u_0^n \in D(A_{(0,h_n(0))}) \\ u_{h_n}(t) \in D(A_{(t,h_n(t))}), \forall t \in I \\ -\dot{u}_{h_n}(t) \in A_{(t,h_n(t))}u_{h_n}(t) \quad \text{a.e. } t \in I \end{cases}$$

uniformly converges to the absolutely solution u_h associated with h

$$\begin{cases} u_h(0) = u_0 \in D(A_{(0,h(0))}) \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in [0,T] \\ -\dot{u}_h(t) \in A_{(t,h(t))}u_h(t) \quad \text{a.e. } t \in [0,T] \end{cases}$$

This fact is ensured by repeating the proof of Theorem 1. Since $h_n \to h$, we have

$$\Phi(h_n)(t) - \Phi(h)(t) = \int_0^1 G(t,s)u_{h_n}(s)ds - \int_0^1 G(t,s)u_h(s)ds$$

$$= \int_0^1 G(t,s)[u_{h_n}(s) - u_h(s)]ds$$

$$\leq \int_0^1 M_G||u_{h_n}(s) - u_h(s)||ds$$

As $||u_{h_n}(\cdot) - u_h(\cdot)|| \to 0$ uniformly, we conclude that

$$\sup_{t\in I} ||\Phi(h_n)(t) - \Phi(h)(t)|| \leq \int_0^1 M_G||u_{h_n}(\cdot) - u_h(\cdot)||ds \to 0$$

so that $\Phi(h_n) \to \Phi(h)$ in $\mathcal{C}_E(I)$. Since $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous, Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that

$$h(t) = \Phi(h)(t) = \int_0^1 G(t,s)u_h(s)ds,$$

with

$$\begin{cases} u_h(0) \in D(A_{(0,h(0))}) \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in I \\ -\dot{u}_h(t) \in A_{(t,h(t))}u_h(t) \quad \text{a.e. } t \in I \end{cases}$$

Coming back to Lemma 9 and applying the above notations, this means that we have just shown that there exists a mapping $h \in W_E^{\alpha,\infty}(I)$ satisfying

$$\begin{cases} D^\alpha h(t) + \lambda D^{\alpha-1}h(t) = u_h(t), \\ I_{0^+}^\beta h(t)\,|_{t=0} = 0, \quad h(1) = I_{0^+}^\gamma h(1) \\ u_h(0) \in D(A_{(0,h(0))}) \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in I \\ -\dot{u}_h(t) \in A_{(t,h(t))}u_h(t) \quad \text{a.e. } t \in I \end{cases}$$

□

Now, we present an extension of the preceding theorem dealing with a Lipschitz perturbation.

Theorem 5. *Let $I = [0,1]$. Let $(t,x) \to A_{(t,x)} : D(A_{(t,x)}) \to 2^E$ a maximal monotone operator satisfying:*
(H_1) $||A^0_{(t,x)}y|| \leq c(1 + ||x|| + ||y||)$ for all $(t,x,y) \in I \times E \times D(A_{(t,x)})$, for some positive constant c,
(H_2) $dis(A_{(t,x)}, A_{(\tau,y)}) \leq a(t) - a(\tau) + r||x-y||$, for all $0 \leq \tau \leq t \leq 1$ and for all $(x,y) \in E \times E$, where r is a positive number, $a : I \to [0, +\infty[$ is nondecreasing absolutely continuous on I with $\dot{a} \in L^2(I, \mathbb{R}, dt)$,
(H_3) $D(A_{(t,x)}) \subset X(t) \subset \gamma \overline{B}_E$ for all $(t,x) \in I \times E$, where $X : I \to E$ is a convex compact valued measurable mapping and γ is a positive number.
Let $f : I \times E \times E \to E$ such that

(i) $f(.,x,y)$ *is Lebesgue measurable on I for all $(x,y) \in E \times E$*
(ii) $f(t,.,.)$ *is continuous on $E \times E$,*
(iii) $||f(t,x,y)|| \leq M$ *for all $(t,x,y) \in I \times E \times E$,*
(iv) $||f(t,x,y) - f(t,x,z)|| \leq M||y-z||$, *for all $(t,x,y,z) \in I \times E \times E \times E$*

for some positive constant M.
Then, there is a $W^{\alpha,1}_{B,E}(I)$ mapping $x : I \to E$ and an absolutely continuous mapping $v : I \to E$ satisfying

$$\begin{cases} D^\alpha x(t) + \lambda D^{\alpha-1}x(t) = v(t), \ t \in I \\ I^\beta_{0^+}x(t)\,|_{t=0} = 0, \quad x(1) = I^\gamma_{0^+}x(1) \\ v(t) \in D(A_{(t,x(t))}), \ t \in I \\ -\frac{dv}{dt}(t) \in A_{(t,x(t))}v(t) + f(t,x(t),v(t)) \quad \text{a.e. } t \in I. \end{cases}$$

Proof. Let us consider the convex compact subset $\mathcal{X}$ in the Banach space $\mathcal{C}_E(I)$ defined by

$$\mathcal{X} := \{u_f : I \to E : u_f(t) = \int_0^1 G(t,s)f(s)ds, \ f \in S^1_X, \ t \in I\}$$

We note that $\mathcal{X}$ is convex compact and equi-Lipschitz. Cf the proof of Theorem 3. Now, for each $h \in \mathcal{X}$, let us consider the unique absolutely continuous solution u_h to

$$\begin{cases} -\dot{u}_h(t) \in A_{(t,h(t))}u_h(t) + f(t,h(t),u_h(t)) \quad \text{a.e. } t \in I \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in I \\ u_h(0) = u_0 \in D(A_{(0,h(0))}) \end{cases}$$

Existence and uniqueness of absolutely solution u_h are ensured by the fact that the operator $B_h(t) = A_{(t,h(t))}$ is a time dependent maximal monotone operator absolutely continuous in variation (See Lemma 5), and the mapping $f_h(t,x) := f(t,h(t),y)$ is measurable with $t \in I$ and Lipschitz with $y \in E$. Furthermore, we have **the estimate** $||\dot{u}_h(t)|| \leq \psi(t)$ a.e for all $h \in \mathcal{X}$ where $\psi \in L^2(I)$ by the consideration given in Lemma 5 and the estimate of velocity given in ([22], Theorem 1). For each h, let us set

$$\Phi(h)(t) = \int_0^1 G(t,s)u_h(s)ds, \ t \in I.$$

Since $u_h(s) \in D(A_{(s,h(s))}) \subset X(s)$, then it is clear that $\Phi(h) \in \mathcal{X}$.

Now, we check that Φ is continuous. It is sufficient to show that, if (h_n) converges uniformly to h in $\mathcal{X}$, then the absolutely continuous solution u_{h_n} associated with h_n

$$\begin{cases} u_{h_n}(0) = u^n_0 \in D(A_{(0,h_n(0))}) \\ u_{h_n}(t) \in D(A_{(t,h_n(t))}), \forall t \in I \\ -\dot{u}_{h_n}(t) \in A_{(t,h_n(t))}u_{h_n}(t) + f(t,h_n(t),u_{h_n}(t)) \quad \text{a.e. } t \in I \end{cases}$$

uniformly converges to the absolutely solution u_h associated with h

$$\begin{cases} u_h(0) \in D(A_{(0,h(0))}) \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in I \\ -\dot{u}_h(t) \in A_{(t,h(t))}u_h(t) + f(t,h(t),u_h(t)) \quad \text{a.e. } t \in I \end{cases}$$

This **need careful look**. We note that u_{h_n} is equicontinuous with $||\dot{u}_{h_n}(t)|| \leq \psi(t)$ for almost all $t \in I$ and for all $n \in N$ where $\psi \in L^2$ and $u_{h_n}(t) \in D(A_{(t,h_n(t))}) \subset X(t)$ for all $t \in I$ and for all $n \in N$. Thus, by extracting subsequence, we may assume that $u_{h_n}(t) \to v(t) = v(0) + \int_0^t \dot{v}(s)ds$ with $\dot{v} \in L^2_E(I)$ for all $t \in I$ and $\dot{u}_{h_n} \to \dot{v}$ weakly in $L^2_E(I)$. Let us check that $v(t) \in D(A_{(t,h(t))})$ for all $t \in I$. We have $dis(A_{(t,h_n(t)}, A_{(t,h(t))}) \leq r||h_n(t) - h(t)|| \to 0$. It is clear that $(y_n = A^0_{(t,h_n(t)}u_{h_n}(t))$ is bounded and hence relatively weakly compact. By applying Lemma 2 to $u_{h_n}(t) \to v(t)$ and to a convergence subsequence of (y_n) using $u_{h_n}(t) \in X(t) \subset \gamma\overline{B}_E$ to show that $v(t) \in D(A_{(t,h(t))})$. As $\dot{u}_{h_n} \to \dot{v}$ weakly in $L^2_E(I)$, $\dot{u}_{h_n} \to \dot{v}$ Komlos. Note that $f(t,h_n(t),u_{h_n}(t)) \to f(t,h(t),u_h(t))$ weakly in $L^2_E(I)$. Thus, $z_n(t) := f(t,h_n(t),u_{h_n}(t)) \to z(t) := f(t,h(t),v(t))$ Komlos. Hence, $\dot{u}_{h_n}(t) + f(t,h_n(t),u_{h_n}(t) \to \dot{v}(t) + f(t,h(t),v(t))$ Komlos. Apply Lemma 4 to $A_{(t,h_n(t))}$ and $A_{(t,h(t))}$ to find a sequence (η_n) such that such that $\eta_n \in D(A_{(t,h_n(t))})$, $\eta_n \to \eta$, $A^0_{(t,h_n(t)}\eta_n \to A^0_{(t,h(t))}v(t)$ From

$$-\dot{u}_{h_n}(t) \in A_{(t,h_n(t))}u_{h_n}(t) + f(t,h_n(t),u_{h_n}(t))(^{**})$$

by monotonicity

$$\langle \frac{du_{h_n}}{dt} + z_n(t), u_{h_n}(t) - \eta_n \rangle \leq A^0_{(t,h_n(t))}\eta_n, \eta_n - u_{h_n}(t) \rangle .(^{***})$$

From

$$\langle \frac{du_{h_n}}{dt}(t) + z_n(t), v(t) - \eta \rangle = \langle \frac{du_{h_n}}{dt}(t) + z_n(t), u_{h_n}(t) - \eta_n \rangle$$

$$+ \langle \frac{du_{h_n}}{dt}(t) + z_n(t), v(t) - u_{h_n}(t) - (\eta - \eta_n) \rangle,$$

let us write

$$\frac{1}{n}\sum_{j=1}^{n} \langle \frac{du_{h_j}}{dt}(t) + z_j(t), v(t) - \eta \rangle =$$

$$\frac{1}{n}\sum_{j=1}^{n} \langle \frac{du_{h_j}}{dt}(t) + z_j(t), u_{h_j}(t) - \eta_j \rangle + \frac{1}{n}\sum_{j=1}^{n} \langle \frac{du_{h_j}}{dt}(t) + z_j(t), v(t) - u_{h_j}(t) \rangle$$

$$+ \sum_{j=1}^{n} \langle \frac{du_{h_j}}{dt}(t) + z_j(t), \eta_j - \eta \rangle,$$

so that

$$\frac{1}{n}\sum_{j=1}^{n} \langle \frac{du_{h_j}}{dt}(t) + z_j(t), v(t) - \eta \rangle \leq \frac{1}{n}\sum_{j=1}^{n} \langle A^0_{(t,h_j(t))}\eta_j, \eta_j - u_{h_j}(t) \rangle + (\psi(t) + M)\frac{1}{n}\sum_{j=1}^{n} \|v(t) - u_{h_j}(t))\|.$$

$$+ (\psi(t) + M)\frac{1}{n}\sum_{j=1}^{n} \|\eta_j - \eta\|.$$

Passing to the limit using (5) when $n \to \infty$, this last inequality gives immediately

$$\langle \frac{dv}{dt}(t) + z(t), v(t) - \eta \rangle \leq \langle A^0_{(t,h(t))}\eta, \eta - v(t) \rangle \text{ a.e.}$$

As a consequence, by Lemma 1, we get
$-\frac{dv}{dt}(t) \in A_{(t,h(t))}v(t) + z(t)$ a.e. with $v(t) \in D(A_{(t,h(t))})$ for all $t \in I$ so that, by uniqueness, $v = u_h$. Since $h_n \to h$, we have

$$\Phi(h_n)(t) - \Phi(h)(t) = \int_0^1 G(t,s)u_{h_n}(s)ds - \int_0^1 G(t,s)u_h(s)ds$$

$$= \int_0^1 G(t,s)[u_{h_n}(s) - u_h(s)]ds$$

$$\leq \int_0^1 M_G||u_{h_n}(s) - u_h(s)||ds$$

As $||u_{h_n}(\cdot) - u_h(\cdot)|| \to 0$ uniformly, we conclude that

$$\sup_{t\in I}||\Phi(h_n)(t) - \Phi(h)(t)|| \leq \int_0^1 M_G||u_{h_n}(\cdot) - u_h(\cdot)||ds \to 0$$

so that $\Phi(h_n) \to \Phi(h)$ in $\mathcal{C}_E(I)$. Since $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous, Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that

$$h(t) = \Phi(h)(t) = \int_0^1 G(t,s)u_h(s)ds,$$

with

$$\begin{cases} u_h(0) \in D(A_{(0,h(0))}) \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in I \\ -\dot{u}_h(t) \in A_{(t,h(t))}u_h(t) + f(t,h(t),u_h(t)) \quad \text{a.e. } t \in I \end{cases}$$

Coming back to Lemma 9 and applying the above notations, this means that we have just shown that there exists a mapping $h \in W^{\alpha,\infty}_{B,E}(I)$ satisfying

$$\begin{cases} D^\alpha h(t) + \lambda D^{\alpha-1}h(t) = u_h(t), \\ I^\beta_{0^+}h(t)\,|_{t=0} = 0, \quad h(1) = I^\gamma_{0^+}h(1) \\ u_h(0) \in D(A_{(0,h(0))}) \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in I \\ -\dot{u}_h(t) \in A_{(t,h(t))}u_h(t) + f(t,h(t),u_h(t)) \quad \text{a.e. } t \in I \end{cases}$$

□

We finish the paper by investigating a fractional order to a sweeping process [30,31].

We begin recall the existence of absolutely continuous solution to a class of sweeping process [18,32].

Theorem 6. *Let $f : [0,T] \to E$ be a continuous mapping such that $||f(t)|| \leq \beta$ for all $t \in [0,T]$, let $v : [0,T] \to \mathbf{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : [0,T] \to E$ be a convex weakly compact valued mapping such that $d_H(C(t),C(\tau)) \leq |v(t) - v(\tau)|$ for all $t,\tau \in [0,T]$. Let $A : E \to E$ be a linear continuous coercive symmetric operator and let $B : E \to E$ be a linear continuous compact operator. Then, for any $u_0 \in E$, the evolution inclusion*

$$f(t) + Bu(t) - A\frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t))$$

$$u(0) = u_0$$

admits a unique $W^{1,\infty}_E([0,T])$ solution $u : [0,T] \to E$.

Theorem 7. *Let $f : I \times E \to E$ be a bounded continuous mapping such that $||f(t,x)|| \leq M$ for all $(t,x) \in I \times E$, for some positive constant M, let $v : I \to \mathbf{R}^+$ be a positive nondecreasing continuous function with $v(0) = 0$. Let $C : I \to E$ be a convex compact valued mapping such that $d_H(C(t), C(\tau)) \leq |v(t) - v(\tau)|$ for all $t, \tau \in I$. Let $A : E \to E$ be a linear continuous coercive symmetric operator and let $B : E \to E$ be a linear continuous compact operator.*
Then, for any $u_0 \in E$, there exists a $W^{\alpha,1}_{B,E}(I)$ mapping $x : I \to E$ and an absolutely continuous mapping $u : I \to E$ satisfying

$$\begin{cases} u(0) = u_0 \in E \\ D^\alpha x(t) + \lambda D^{\alpha-1} x(t) = u(t), \ t \in I \\ I^\beta_{0^+} x(t)\,|_{t=0} = 0, \quad x(1) = I^\gamma_{0^+} x(1) \\ f(t, x(t)) + Bu(t) - A\frac{du}{dt}(t)) \in N_{C(t)}(\frac{du}{dt}(t)), \text{ a.e. } t \in I \end{cases}$$

Proof. By Theorem 6 and the assumptions on f, for any bounded continuous mapping $h : I \to E$, there is a unique absolutely continuous solution v_h to the inclusion

$$\begin{cases} v_h(0) = u_0 \in E \\ f(t, h(t)) + Bv_h(t) - A\frac{dv_h}{dt}(t)) \in N_{C(t)}(\frac{dv_h}{dt}(t)), \text{ a.e. } t \in I \end{cases}$$

with $\frac{dv_h}{dt}(t) \in C(t)$ a.e. so that $v_h(t) = u_0 + \int_0^t \frac{dv_h}{ds}(s)ds \in u_0 + \int_0^t C(s)ds, \forall t \in I$. By our assumption, C is scalarly upper semicontinuous convex compact valued integrably bounded: $C(t) \subset \rho\overline{B}_E, \forall t \in I$, hence, by [33], $t \mapsto \Psi(t) := u_0 + \int_0^t C(s)ds$ is a scalarly upper semicontinuous convex **compact** valued integrably bounded mapping with $\Psi(t) := u_0 + \int_0^t C(s)ds \subset u_0 + \rho\overline{B}_E, \forall t \in I$. Let us consider the closed convex subset $\mathcal{X}$ in the Banach space $\mathcal{C}_E(I)$ defined by

$$\mathcal{X} := \{u_f : I \to E : u_f(t) = \int_0^1 G(t,s)f(s)ds, \ f \in S^1_{u_0+\rho\overline{B}_E}, \ t \in I\},$$

where $S^1_{u_0+\rho\overline{B}_E}$ denotes the set of all integrable selections of the convex weakly compact valued constant multifunction $u_0 + \rho\overline{B}_E$. Now, for each $h \in \mathcal{X}$, let us consider the mapping defined by

$$\Phi(h)(t) := \int_0^t G(t,s)v_h(s)ds,$$

for $t \in I$. Then, it is clear that $\Phi(h) \in \mathcal{X}$. Since $u_0 + \int_0^t C(s)ds$ is a convex compact, $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E(I)$ by virtue of Theorem 3 using the compactness of $\Psi(t)$. Now, we check that Φ is continuous. It is sufficient to show that, if (h_n) uniformly converges to h in $\mathcal{X}$, then the absolutely continuous solution v_{h_n} associated with h_n

$$\begin{cases} v_{h_n}(0) = u_0 \in E \\ f(t, h_n(t)) + Bv_{h_n}(t) - A\frac{dv_{h_n}}{dt}(t)) \in N_{C(t)}(\frac{dv_{h_n}}{dt}(t)), \text{ a.e. } t \in I \end{cases}$$

uniformly converges to the absolutely continuous solution v_h associated with h

$$\begin{cases} v_h(0) = u_0 \in E \\ f(t, h(t)) + Bv_h(t) - A\frac{dv_h}{dt}(t)) \in N_{C(t)}(\frac{dv_h}{dt}(t)), \text{ a.e. } t \in I \end{cases}$$

As (v_{h_n}) is equi-absolutely continuous with $v_{h_n}t) \in u_0 + \int_0^t C(s)ds, \forall t \in I$, we may assume that (v_{h_n}) uniformly converges to an absolutely continuous mapping z.

Since $v_{h_n}(t) = u_0 + \int_{]0,t]} \frac{dv_{h_n}}{ds}(s)ds$, $t \in I$ and $\frac{dv_{h_n}}{ds}(s) \in C(s)$, *a.e.* $s \in I$, we may assume that $(\frac{dv_{h_n}}{dt})$ weakly converges in $L^1_E(I)$ to $w \in L^1_E(I)$ with $w(t) \in C(t)$, $t \in I$ so that

$$\lim_n v_{h_n}(t) = u_0 + \int_0^t w(s)ds := u(t),\ t \in I.$$

By identifying the limits, we get

$$u(t) = z(t) = u_0 + \int_0^t w(s)ds$$

with $\dot{u} = w$. Therefore, by applying the arguments in the variational limit result in [34], we get

$$f(t, h(t)) + Bu(t) - A\frac{du}{dt}(t)) \in N_{C(t)}(\frac{du}{dt}(t)),\ \text{a.e. } t \in I$$

with $u(0) = u_0 \in E$, so that, by uniqueness, $u = v_h$. Since $h_n \to h$, we have

$$\Phi(h_n)(t) - \Phi(h)(t) = \int_0^1 G(t,s)v_{h_n}(s)ds - \int_0^1 G(t,s)v_h(s)ds$$

$$= \int_0^1 G(t,s)[v_{h_n}(s) - v_h(s)]ds$$

$$\leq \int_0^1 M_G||v_{h_n}(s) - v_h(s)||ds$$

As $||v_{h_n}(\cdot) - v_h(\cdot)|| \to 0$ uniformly, we conclude that

$$\sup_{t\in I} ||\Phi(h_n)(t) - \Phi(h)(t)|| \leq \int_0^1 M_G||v_{h_n}(\cdot) - v_h(\cdot)||ds \to 0$$

so that $\Phi(h_n) \to \Phi(h)$ in $\mathcal{C}_E(I)$. Since $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous and $\Phi(\mathcal{X})$ is relatively compact in $\mathcal{C}_E(I)$, by [25,26] Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that

$$h(t) = \Phi(h)(t) = \int_0^1 G(t,s)v_h(s)ds,$$

with

$$\begin{cases} v_h(0) = u_0 \in E \\ D^\alpha h(t) + \lambda D^{\alpha-1}h(t) = v_h(t),\ t \in I \\ I^\beta_{0^+}h(t)\,|_{t=0} = 0, \quad h(1) = I^\gamma_{0^+}h(1) \\ f(t, h(t)) + Bv_h(t) - A\frac{dv_h}{dt}(t)) \in N_{C(t)}(\frac{dv_h}{dt}(t)),\ \text{a.e. } t \in I \end{cases}$$

The proof is complete. □

Theorem 8. *Theorems 6 and 7 results are inspired by some ideas in [18]. At this point, some variants are available, mainly when the second member is a time dependent subdifferential operator [35], namely, for any $u_0 \in E$, there exists a $W^{\alpha,1}_{B,E}(I)$ mapping $x : I \to E$ and an absolutely continuous mapping $u : I \to E$ satisfying*

$$\begin{cases} u(0) = u_0 \in E \\ D^\alpha x(t) + \lambda D^{\alpha-1}x(t) = u(t),\ t \in I \\ I^\beta_{0^+}x(t)\,|_{t=0} = 0, \quad x(1) = I^\gamma_{0^+}x(1) \\ f(t, x(t)) + Bu(t) - A\frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t)),\ \text{a.e. } t \in I \end{cases}$$

5. On a Fillipov Theorem

We end this section with a Fillipov theorem and a relaxation theorem for the fractional differential inclusion

$$\begin{cases} D^{\alpha}u(t) + \lambda D^{\alpha-1}u(t) \in F(t,u(t)), a.e.\ t \in I \\ I_{0^+}^{\beta}u(t)\,|_{t=0} = 0, \quad u(1) = I_{0^+}^{\gamma}u(1) \end{cases}$$

where $F : I \times E \to E$ is a closed valued Lipschitz mapping w.r.t.o $x \in E$.

Theorem 9. *Assume that E is a separable Banach space. Let $F : I \times E \to E$ be a closed valued $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable mapping such that*
$(\mathcal{H}_1)$: $d_H(F(t,x),F(t,y)) \leq l(t)||x-y||$ for all t,x,y where $l \in L^1_{\mathbf{R}}(I))$ such that $\rho := M_G||l||_{L^1_{\mathbf{R}}(I)} < 1$.

Assume further that

$(\mathcal{H}_2)$: there exists $g \in L^1_E(I)$ such that $d(g(t),F(t,u_g(t))) < \frac{l(t)}{\sum_{n=1}^{\infty} n\rho^{n-1}}$ where $u_g(t) = \int_0^1 G(t,s)g(s)ds, \forall t \in I$.

Then, the fractional differential inclusion

$$\begin{cases} D^{\alpha}u(t) + \lambda D^{\alpha-1}u(t) \in F(t,u(t)), a.e.\ t \in I \\ I_{0^+}^{\beta}u(t)\,|_{t=0} = 0, \quad u(1) = I_{0^+}^{\gamma}u(1) \end{cases}$$

has at least a $W^{\alpha,1}_{B,E}(I)$-solution $u : I \to E$.

Proof. We use the ideas in the proof of Theorem 4.3 in [36], Remark 2 and Lemma 9.

It is worth mentioning that the series $\Lambda := \sum_{n=1}^{\infty} n\rho^{n-1}$ is convergent. Indeed, we have

$$\lim_{n\to\infty} \frac{(n+1)\rho^n}{n\rho^{n-1}} = \lim_{n\to\infty} \frac{n+1}{n}\rho = \rho < 1.$$

Thus, by d'Alembert's ratio test, the series $\sum_{n=1}^{\infty} n\rho^{n-1}$ is convergent

Step 1. We shall construct inductively sequence $\{f_n(\cdot)\}_{n=1}^{\infty}$ where $f_1 = g$ such that the following conditions are fulfilled, for all $n \geq 1$,

$$f_n \in L^1_E(I) \quad \text{and} \quad f_{n+1}(t) \in F(t,u_{f_n}(t)), t \in I, \tag{18}$$

$$\|f_{n+1}(t) - f_n(t)\| \leq (n+1)\rho^{n-1}l(t)\Lambda^{-1}, \tag{19}$$

$$\left\|u_{f_{n+1}}(t) - u_{f_n}(t)\right\| = \left\|\int_0^1 G(t,s)[f_{n+1}(s) - f_n(s)]ds\right\| \leq (n+1)\rho^n\Lambda^{-1}, \tag{20}$$

for all $t \in I$. We note that the passage from (18) to (19) is obtained, thanks to (16) of Remark 2, with

$$\left\|u_{f_{n+1}}(t) - u_{f_n}(t)\right\| = \left\|\int_0^1 G(t,s)[f_{n+1}(s) - f_n(s)]ds\right\| \leq M_G\left\|f_{n+1}(t) - f_n(t)\right\|$$

By $(\mathcal{H}_2)$, we have $d(f_1(t),F(t,u_{f_1}(t))) < l(t)\Lambda^{-1}$, $t \in I$. Let us consider the multifunction $\Sigma_1 : I \to c(E)$ defined by

$$\Sigma_1(t) = \left\{v \in F(t,u_{f_1}(t)) : \|v - f_1(t)\| \leq 2l(t)\Lambda^{-1}\right\}.$$

Clearly, Σ_1 is Lebesgue measurable with nonempty closed values. In view of the existence theorem of measurable selections (see [29]), there is a measurable function $f_2 : I \to E$ such that $f_2(t) \in \Sigma_1(t)$ for all $t \in I$. This yields

$$f_2(t) \in F(t,u_{f_1}(t)), \quad \|f_2(t) - f_1(t)\| \leq 2l(t)\Lambda^{-1},$$

for all $t \in I$. Thus, it is easy to see that $f_2 \in L_E^1(I)$ and

$$\left\| u_{f_2}(t) - u_{f_1}(t) \right\| = \left\| \int_0^1 G(t,s)[f_2(s) - f_1(s)]ds \right\| \leq 2\rho\Lambda^{-1},$$

for all $t \in I$.

- Suppose that we have constructed integrable functions $f_1, f_2, \ldots, f_n$ such that

$$f_{i+1}(t) \in F(t, u_{f_i}(t)),\ t \in I,$$

$$\|f_{i+1}(t) - f_i(t)\| \leq (i+1)\rho^{i-1} l(t)\Lambda^{-1},$$

for all $i = 1, 2, \ldots, n-1$. Then,

$$\left\| u_{f_{i+1}}(t) - u_{f_i}(t) \right\| = \left\| \int_0^1 G(t,s)[f_{i+1}(s) - f_i(s)]ds \right\| \leq (i+1)\rho^i \Lambda^{-1},$$

for $i = 1, 2, \ldots, n-1$.

- The function f_{n+1} is constructed as follows. We have

$$\begin{aligned} d\left(f_n(t), F\left(t, u_{f_n}(t)\right)\right) &\leq d_H\left(F(t, u_{f_{n-1}}(t)), F(t, u_{f_n}(t))\right) \\ &\leq l(t)\left\| u_{f_n}(t) - u_{f_{n-1}}(t) \right\| \\ &\leq n\rho^{n-1} l(t)\Lambda^{-1}. \end{aligned}$$

The multifunction $\Sigma_n : I \to c(E)$, defined by

$$\Sigma_n(t) = \left\{ v \in F\left(t, u_n(t)\right) : \|v - f_n(t)\| \leq (n+1)\rho^{n-1} l(t)\varepsilon\Lambda^{-1} \right\},$$

is Lebesgue measurable with nonempty closed values. Thus, there exists a measurable function f_{n+1} such that

$$f_{n+1}(t) \in F\left(t, u_{f_n}(t)\right), \quad \|f_{n+1}(t) - f_n(t)\| \leq (n+1)\rho^{n-1} l(t)\Lambda^{-1},$$

for all $t \in I$. Then, it is clear that, for all $t \in I$,

$$\left\| u_{f_{n+1}}(t) - u_{f_n}(t) \right\| = \left\| \int_0^1 G(t,s)[f_{n+1}(s) - f_n(s)]ds \right\| \leq (n+1)\rho^n \Lambda^{-1},$$

Thus, such a sequence $\{f_n\}_{n=1}^{\infty}$ with the required properties exists.

Step 2. It follows that, for all $n \geq 1$, we have

$$\|f_{n+1} - f_n\|_{L_E^1(I)} = \int_0^1 \|f_{n+1}(t) - f_n(t)\|\, dt \leq (n+1)\rho^{n-1} \|l\|_{L^1_{\mathbf{R}^+}(I)} \Lambda^{-1}. \tag{21}$$

On the other hand, by $\rho < 1$ the series $\sum_{n=1}^{\infty}(n+1)\rho^{n-1}$ is convergent (using d'Alembert's ratio test). Now, we assert that $\{f_n(\cdot)\}_{n=1}^{\infty}$ is a Cauchy sequence in $L_E^1(I)$. Indeed, using (10), for $n, m \in \mathbf{N}$ such that $m > n$, we have the estimate

$$\begin{aligned}\|f_m - f_n\|_{L_E^1(I)} &\leq \|f_{n+1} - f_n\|_{L_E^1(I)} + \|f_{n+2} - f_{n+1}\|_{L_E^1(I)} + \cdots + \|f_m - f_{m-1}\|_{L_E^1(I)} \\ &\leq \left[(n+1)\rho^{n-1} + (n+2)\rho^n + \cdots + m\rho^{m-2}\right] \|l\|_{L_{\mathbf{R}^+}^1(I)} \Lambda^{-1} \\ &\leq \left(\sum_{k=n}^{\infty}(k+1)\rho^{k-1}\right) \|l\|_{L_{\mathbf{R}^+}^1(I)} \Lambda^{-1}\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we see that $\|f_m - f_n\|_{L_E^1(I)}$ goes to 0 when m, n goes to ∞. Since the normed space $L_E^1(I)$ is complete, (f_n) norm converges to an element $f \in L_E^1(I)$. By the properties of our Green function and the definition of u_{f_n}, we conclude that u_{f_n} pointwise converge with respect to the norm topology to u_f

$$u_f(t) = \int_0^1 G(t,s)f(s)ds, \forall t \in I.$$

Now, we claim that $f(t) \in F(t, u_f(t))$, *a.e.* $t \in I$. Let us write

$$\begin{aligned}d(f(t), F(t, u_f(t)) \leq &\left|d(f(t), F(t, u_f(t))) - d(f_n(t), F(t, u_f(t))\right| \\ &+ d(f_n(t), F(t, u_f(t)).\end{aligned} \tag{22}$$

On the other hand,

$$\left|d(f(t), F(t, u_f(t)) - d(f_n(t), F(t, u_f(t)))\right| \leq \|f(t) - f_n(t)\|, \tag{23}$$

and, by $f_n(t) \in F(t, u_{f_{n-1}}(t))$, $t \in I$, we have

$$\begin{aligned}d(f_n(t), F(t, u_f(t)) &\leq d_H\left(F(t, u_{f_{n-1}}(t)), F(t, u_f(t))\right) \\ &\leq l(t)\left\|u_{f_{n-1}}(t) - u_f(t)\right\|.\end{aligned} \tag{24}$$

Since $(f_n)_{n\in\mathbf{N}}$ norm converges to $f \in L_E^1(I)$, we may, by extracting subsequences, assume that $||f_n(t) - f(t)||_E \to 0$ a.e. Now, passing to the limit when $n \to \infty$ in the preceding inequality, we get

$$d(f(t), F(t, u_f(t))) = 0 \quad a.e. t \in I$$

This implies that $f(t) \in F(t, u_f(t))$, *a.e.t* $\in I$ because F is closed valued. Thus, by Lemma 9, we have shown that u_f is a solution of the problem

$$\begin{cases} D^{\alpha}u_f(t) + \lambda D^{\alpha-1}u_f(t) \in F(t, u_f(t)), a.e.\ t \in I \\ I_{0^+}^{\beta}u_f(t)\,|_{t=0} = 0, \quad u_f(1) = I_{0^+}^{\gamma}u_f(1) \end{cases}$$

The proof of theorem is complete.

□

A relaxation theorem is available using the machinery developed in [36] Theorem 4.2 and Lemma 9.

Theorem 10. ***Relaxation*** *Assume that E is a separable Banach space. Let $F : I \times E \to E$ be a closed valued $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable mapping such that*

$(\mathcal{H}_1)$: $d_H(F(t,x),F(t,y)) \leq l(t)||x-y||$ *for all* t,x,y *where* $l \in L^1_{\mathbf{R}}(I))$ *such that* $\rho := M_G||l||_{L^1_{\mathbf{R}}(I)} < 1$.
Assume further that
$(\mathcal{H}_2)$: *there exists* $g \in L^1_E(I)$ *such that* $d(g(t),F(t,u_g(t))) < \frac{l(t)}{\sum_{n=1}^{\infty} n\rho^{n-1}}$ *where* $u_g(t) = \int_0^1 G(t,s)g(s)ds, \forall t \in I$.
$(\mathcal{H}_3)$: $d(0,F(t,x)) < c(t)(1+||x||), \forall (t,x) \in I \times E$ *where* c *is a positive integrable function.*
Then, the following holds:
(a)

$$(\mathcal{P}_F)\begin{cases} D^{\alpha}u(t) + \lambda D^{\alpha-1}u(t) \in F(t,u(t)), a.e.\ t \in I \\ I^{\beta}_{0^+}u(t)\,|_{t=0} = 0, \quad u(1) = I^{\gamma}_{0^+}u(1) \end{cases}$$

and

$$(\mathcal{P}_{\overline{co}F})\begin{cases} D^{\alpha}u(t) + \lambda D^{\alpha-1}u(t) \in \overline{co}F(t,u(t)), a.e.\ t \in I \\ I^{\beta}_{0^+}u(t)\,|_{t=0} = 0, \quad u(1) = I^{\gamma}_{0^+}u(1) \end{cases}$$

have at least a solution in $W^{\alpha,1}_{B,E}(I)$.
(b) Let $f_0 \in L^1_E(I))$ *such that*

$$f_0(t) \in \overline{co}F(t,u_{f_0}(t))$$

$$u_{f_0}(t) = \int_0^1 G(t,s)f_0(s)ds, \forall t \in I$$

Then, for every $\varepsilon > 0$, *there exists* $f \in L^1_E(I)$ *such that*

$$f(t) \in F(t,u_f(t)), \quad a.e.$$

$$u_f(t) = \int_0^1 G(t,s)f(s)ds, \forall t \in I$$

and

$$\sup_{t\in I} ||u_f(t) - u_{f_0}(t)|| \leq \varepsilon.$$

Proof. We will proceed in several steps.

Step 1. (a) follows from Theorem 9 applied to both F and $\overline{co}F$ taking account of $(\mathcal{H}_1)-(\mathcal{H}_2)$. Let $u_{f_0}(\cdot)$ be a $W^{\alpha,1}_{B,E}(I)$-solution of the problem $(\mathcal{P}_{\overline{co}F})$ that is, $u_{f_0} \in \mathcal{S}_{\mathcal{P}_{\overline{co}F}}$

$$f_0(t) \in \overline{co}F(t,u_{f_0}(t)),\ a.e.\, t \in I, \tag{25}$$

$$u_{f_0}(t) := \int_0^1 G(t,s)f_0(s)ds, \forall t \in I \tag{26}$$

Let S^1_F and $S^1_{\overline{co}F}$ denote the set of all $L^1_E(I)$-selections of the set valued mappings $t \to F(t,u_{f_0}(t))$ and $t \to \overline{co}F(t,u_{f_0}(t))$ By $(\mathcal{H}_3)$, the multifunction $t \mapsto F(t,u_{f_0}(t))$ is closed valued and integrable:

$$d(0,F(t,u_{f_0}(t)) < c(t)(1+||u_{f_0}(t)||)$$

so that S^1_F is non empty. Then, according to Hiai–Umegaki [37], $S^1_{\overline{co}F} = \overline{co}S^1_F$ where $\overline{co}$ is taken in $L^1_E(I)$. This equality along with $f_0(t) \in \overline{co}F(t,u_{f_0}(t)),\ a.e.\, t \in I$ yields $f_0 \in \overline{co}S^1_F$. Let $\varepsilon > 0$. There exists $g_\varepsilon \in L^1_E(I)$ such that $g_\varepsilon \in coS^1_F$ and $||f_0 - g_\varepsilon||_{L^1_E(I)} \leq \frac{1}{2}\varepsilon\Lambda^{-1}M_G^{-1}$ so that

$$||u_{f_0}(t) - u_{g_\varepsilon}(t)|| < \frac{1}{2}\varepsilon\Lambda^{-1}.$$

As $g_\varepsilon \in coS_F^1$, then $g_\varepsilon = \sum_{i=1}^n \lambda_i f_I$ with $f_i \in L_E^1(I)$, $f_i(t) \in F(t, u_{f_0}(t))$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$. Let $\Phi(t) := \{f_i(t : 1 \leq i \leq n\}$, then $\Phi(t)$ is a compact valued integrably bounded mapping with $|\Phi(t)| \leq r(t) := \sup_{1\leq i\leq n} |f_i(t)|$. Then, from [38], there exists

$$h_1 \in L_E^1(I), h_1(t) \in \Phi(t) \subset F(t, u_{f_0}(t))), \forall t \in I$$

such that

$$\sup_{0\leq t<\tau\leq 1} || \int_t^\tau [h_1(s) - g_\varepsilon(s)]ds|| \leq \frac{1}{2}\varepsilon M_G^{-1}\Delta^{-1}$$

so that

$$||u_{h_1}(t) - u_{g_\varepsilon}(t)|| = || \int_0^1 G(t,s)[h_1(s) - g_\varepsilon(s)ds|| \leq M_G|| \int_0^1 [h_1(s) - g_\varepsilon(s)ds|| \leq \frac{1}{2}\varepsilon\Delta^{-1}.$$

Consequently,

$$(*) \qquad ||u_{h_1}(t) - u_{f_0}(t)|| \leq \varepsilon\Delta^{-1}.$$

Step 2. We shall construct inductively sequence $\{h_n(\cdot)\}_{n=1}^\infty$ such that the following conditions are fulfilled, for all $n \geq 1$,

$$h_n \in L_E^1(I) \quad \text{and} \quad h_{n+1}(t) \in F(t, u_{h_n}(t)), t \in I, \tag{27}$$

$$\|h_{n+1}(t) - h_n(t)\| \leq (n+1)\rho^{n-1}l(t)\varepsilon\Lambda^{-1}, \tag{28}$$

$$\left\|u_{h_{n+1}}(t) - u_{h_n}(t)\right\| = \left\| \int_0^1 G(t,s)[h_{n+1}(s) - h_n(s)]ds\right\| \leq (n+1)\rho^n\varepsilon\Lambda^{-1}, t \in I \tag{29}$$

- The multifunction $F(\cdot, u_{h_1}(\cdot))$ is Lebesgue-measurable and

$$d_H\left(F(t, u_{h_1}(t)), F(t, u_{f_0}(t))\right) \leq l(t)\left\|u_{h_1}(t) - u_{f_0}(t)\right\|$$

This implies that, for $t \in I$,

$$d_H\left(F(t, u_{h_1}(t)), F(t, u_{f_0}(t))\right) \leq l(t)\varepsilon\Lambda^{-1},$$

As $h_1(t) \in F(t, u_{f_0}(t))$, we have $d(h_1(t), F(t, u_{h_1}(t)) \leq l(t)\varepsilon\Lambda^{-1}$, $t \in I$. Let us consider the multifunction $\Sigma_1 : I \to c(E)$ defined by

$$\Sigma_1(t) = \left\{v \in F(t, u_{h_1}(t)) : \|v - h_1(t)\| \leq 2l(t)\varepsilon\Lambda^{-1}\right\}.$$

Clearly, Σ_1 is Lebesgue measurable with nonempty closed values. In view of the existence theorem of measurable selections (see [29]), there is a measurable function $h_2 : I \to E$ such that $h_2(t) \in \Sigma_1(t)$ for all $t \in I$. This yields

$$h_2(t) \in F(t, u_{h_1}(t)), \quad \|h_2(t) - h_1(t)\| \leq 2l(t)\varepsilon\Lambda^{-1},$$

for all $t \in I$. Thus, it is easy to see that $h_2 \in L_E^1(I)$ and

$$\left\|u_{h_2}(t) - u_{h_1}(t)\right\| = \left\| \int_0^1 G(t,s)[h_2(s) - h_1(s)]ds\right\| \leq 2\rho\varepsilon\Lambda^{-1},$$

for all $t \in I$.

- Suppose that we have constructed integrable functions $h_1, h_2, \ldots, h_n$ such that

$$h_{i+1}(t) \in F(t, u_{h_i}(t)), \ a.e.t \in I,$$

$$\|h_{i+1}(t) - h_i(t)\| \leq (i+1)\rho^{i-1} l(t)\varepsilon\Lambda^{-1},$$

for all $i = 1, 2, \ldots, n-1$. Then,

$$\left\|u_{h_{i+1}}(t) - u_{h_i}(t)\right\| = \left\|\int_0^1 G(t,s)[h_{i+1}(s) - h_i(s)]ds\right\| \leq (i+1)\rho^i \varepsilon\Lambda^{-1},$$

for $i = 1, 2, \ldots, n-1$.

- The function h_{n+1} is constructed as follows. We have

$$d(h_n(t), F(t, u_{h_n}(t))) \leq d_H(F(t, u_{h_{n-1}}(t)), F(t, u_{h_n}(t)))$$

$$\leq l(t)\|u_{h_n}(t) - u_{h_{n-1}}(t)\| \leq n\rho^{n-1} l(t)\varepsilon\Lambda^{-1}$$

The multifunction $\Sigma_n : I \to c(E)$, defined by

$$\Sigma_n(t) = \left\{v \in F\left(t, u_{h_n}(t)\right) : \|v - h_n(t)\| \leq (n+1)\rho^{n-1} l(t)\varepsilon\Lambda^{-1}\right\},$$

is Lebesgue measurable with nonempty closed values. Thus, there exists a measurable function h_{n+1} such that

$$h_{n+1}(t) \in F\left(t, u_{h_n}(t)\right), \quad \|h_{n+1}(t) - h_n(t)\| \leq (n+1)\rho^{n-1} l(t)\varepsilon\Lambda^{-1},$$

for all $t \in I$. Then, it is clear that, for all $t \in I$,

$$\left\|u_{h_{n+1}}(t) - u_{h_n}(t)\right\| = \left\|\int_0^1 G(t,s)[h_{n+1}(s) - h_n(s)]ds\right\| \leq (n+1)\rho^n \varepsilon\Lambda^{-1},$$

Thus, a sequence $\{h_n\}_{n=1}^{\infty}$ satisfying (27)–(29)exists.

Step 3. It follows from (28) that, for all $n \geq 1$, we have

$$\|h_{n+1} - h_n\|_{L_E^1(I)} = \int_0^1 \|h_{n+1}(t) - h_n(t)\|\, dt \leq (n+1)\rho^{n-1} \|l\|_{L^1_{\mathbf{R}^+}(I)} \varepsilon\Lambda^{-1}. \tag{30}$$

On the other hand, by $\rho < 1$, the series $\sum_{n=1}^{\infty}(n+1)\rho^{n-1}$ is convergent (using d'Alembert's ratio test). Now, we assert that $\{h_n(\cdot)\}_{n=1}^{\infty}$ is a Cauchy sequence in $L_E^1(I)$. Indeed, using (30), for $n, m \in \mathbf{N}$, such that $m > n$, we have the estimate

$$\begin{aligned}
\|h_m - h_n\|_{L_E^1(I)} &\leq \|h_{n+1} - h_n\|_{L_E^1(I)} + \|h_{n+2} - h_{n+1}\|_{L_E^1(I)} + \cdots + \|h_m - h_{m-1}\|_{L_E^1(I)} \\
&\leq \left[(n+1)\rho^{n-1} + (n+2)\rho^n + \cdots + m\rho^{m-2}\right] \|\mathfrak{l}\|_{L^1_{\mathbf{R}^+}(I)} \varepsilon\Lambda^{-1} \\
&\leq \left(\sum_{k=n}^{\infty}(k+1)\rho^{k-1}\right) \|l\|_{L^1_{\mathbf{R}^+}(I)} \varepsilon\Lambda^{-1}
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we see that $\|h_m - h_n\|_{L_E^1(I)}$ goes to 0 when m, n goes to ∞. Since the normed space $L_E^1(I)$ is complete, (h_n) norm converges to an element $f \in L_E^1(I)$. By the properties of our Green function and the definition of u_{h_n} , we conclude that u_{h_n} pointwise converges with respect to the norm topology to u_f where

$$u_f(t) = \int_0^1 G(t,s) f(s) ds.$$

Moreover, from (29), we deduce that

$$\left\|u_{h_n}(t) - u_{f_0}(t)\right\| \leq ||u_{h_1}(t) - u_{f_0}(t)|| + ||u_{h_2}(t) - u_{h_1}(t)|| + \ldots + ||u_{h_n}(t) - u_{h_{n-1}}(t)||$$

$$\leq \left(\sum_{j=1}^{n} j\rho^{j-1}\right) \varepsilon\Lambda^{-1}$$

for all $t \in I$. Recall that $\Lambda = \sum_{n=1}^{\infty} n\rho^{n-1}$. Thus, by letting $n \to \infty$ in the last inequality, we get

$$\left\|u_f - u_{f_0}\right\|_{C_E(I)} = \max_{t\in I} \left\|u_f(t) - u_{f_0}(t)\right\| \leq \varepsilon.$$

Now, we claim that $f(t) \in F(t, u_f(t))$, *a.e.* $t \in I$. Let us write

$$\begin{aligned} d(f(t), F(t, u_f(t)) \leq &\left|d(f(t), F(t, u_f(t)) - d(h_n(t), F(t, u_f(t))\right| \\ &+ d(h_n(t), F(t, u_f(t)). \end{aligned} \tag{31}$$

On the other hand,

$$\left|d(f(t), F(t, u_f(t)) - d(h_n(t), F(t, u_f(t)))\right| \leq \|f(t) - h_n(t)\|, \tag{32}$$

and, by $h_n(t) \in F(t, u_{h_{n-1}}(t))$, $t \in I$, we have

$$\begin{aligned} d(h_n(t), F(t, u_f(t)) &\leq d_H\left(F(t, u_{h_{n-1}}(t)), F(t, u_f(t))\right) \\ &\leq l(t)\left\|u_{h_{n-1}}(t) - u_f(t)\right\| \end{aligned} \tag{33}$$

Since $(h_n)_{n\in\mathbf{N}}$ norm converges to $f \in L^1_E(I)$ we may, by extracting subsequences, assume that $||h_n(t) - f(t)||_E \to 0$ a.e. Now, passing to the limit when $n \to \infty$ in (31)–(33), we get

$$d(f(t), F(t, u_f(t)) = 0 \quad a.e. t \in I$$

This implies that $f(t) \in F(t, u_f(t))$, *a.e.* $t \in I$ because F is closed valued. Hence, u_f is a solution of the problem $(\mathcal{P}_F)$, satisfying the required density property. The proof of theorem is complete. □

6. Conclusions

In the context of separable Hilbert space, our algorithm and tools are fairly general and they allow for treating several variants of system of fractional differential inclusion coupled with a time and state dependent maximal monotone operators with Lipschitz perturbation, in particular the second order solution of evolution inclusion governed time and state dependent maximal monotone operators with Lipschitz perturbation. Our results contain novelties. Nevertheless, there are several issues—for instance, the existence of solutions for the case of closed unbounded Lipschitz perturbation that is needed in the optimal control.

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