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# Geometrical reasoning in the intersection between historical original sources and the dynamic geometry environment GeoGebra

Marianne Thomsen<sup>1</sup> and Uffe Thomas Jankvist<sup>2</sup>

<sup>1</sup>University College Copenhagen, Denmark; [mart@kp.dk](mailto:mart@kp.dk)

<sup>2</sup>Danish School of Education, Aarhus University, Denmark

*This paper addresses how an interplay between original sources and digital tools can support students' possibilities to develop the constructive facet of the reasoning competency among 7<sup>th</sup> grade students. This is done through a use of the distinction between epistemic and pragmatic mediations. The case described takes its outset in a pair of students' work with (part of) Euclid's proposition 6, Book IV: To inscribe a square in a given circle. Firstly, the study shows how students' work with an original source can be seen as a springboard to create their own mathematical arguments, while working with GeoGebra. Secondly, it provides insight to how the above-mentioned theoretical distinction can be seen as a way to highlight some of the opportunities and challenges when working with an original source (with a geometrical content) in interplay with GeoGebra.*

*Keywords: Original sources, GeoGebra, reasoning competency.*

## Background

The research literature on students' work in the interplay between the history of mathematics and digital technologies, e.g., dynamic geometry systems (DGS), computer algebra systems (CAS), is limited (Thomsen et al., 2022). In this paper, we focus on mathematical reasoning in relation to the interplay between original sources and use of DT. Chorlay (2015) describes different ways of working with the combination of original sources and the use of DT to support upper secondary students reasoning. Based on an empirical example from undergraduate mathematics education, Jankvist et al. (2019) show how working with the interplay between an original source and CAS can contribute to qualifying students' work with proofs. This too can indeed support students' development of what Niss and Højgaard (2019) define as the receptive and constructive facets of the reasoning competency (Thomsen, 2021). The receptive facet concerns students' ability to understand others' mathematical arguments, reasoning, and proofs, while the constructive facet concerns their own actual production of such. It may seem obvious that working with original sources can activate and develop the receptive facet of students' reasoning competency. Still, we argue that there are, as of yet non-realized, potentials in constructing teaching and learning situations with a particular focus on supporting students' possibilities to develop the constructive facet of their reasoning competency, while working with the interplay between original sources and DT (in this case GeoGebra). Moreover, that this can be done with young students, grades 5 to 7 (Thomsen, 2022). One lens that has shown applicable in addressing students' possibilities to activate, exercise and develop their reasoning competency is Misfeldt's and Jankvist's (2018) distinction between pragmatic, justificational and epistemic mediation, drawing among others on Artigue (2002) and Trouche (2005b), when working with DT (e.g., Thomsen & Jankvist, 2020). In this paper, we focus on DT as pragmatic and epistemic mediators—the latter, which has already proven itself as a useful construct for addressing the interplay in question (Jankvist & Geraniou, 2021). We ask the following research question:

How can students' work with tasks in an interplay between an original source and GeoGebra, designed so that GeoGebra alters between acting as a pragmatic and an epistemic mediator, support the students' possibilities to develop the constructive facet of the reasoning competency?

We first explain the dual character of the reasoning competency (Niss & Højgaard, 2019) as well as the distinction between pragmatic and epistemic mediation. In this paper, we mainly focus on Trouche's (2005a; 2005b) interpretation of instrumental genesis.

## **Theoretical constructs and outset**

In their recent paper, Niss and Højgaard (2019) revisit the Danish KOM framework, originally published in Danish in 2002, and its eight mathematical competencies, one of which is on reasoning:

The core of the mathematical reasoning competency is to analyse or produce arguments (i.e., chains of statements linked by inferences) put forward in oral or written form to justify mathematical claims. (p. 16)

All eight competencies have a dual nature, i.e., they possess both a constructive and a receptive facet. Among other things the constructive facet of the reasoning competency focuses on the individual's ability to devise "a mathematical proof of a statement", whereas the receptive facet focuses on the ability to follow and assess "an alleged mathematical proof" (p. 19). Hence, even though we are particularly interested in the constructive part of the reasoning competency, we must see this as related to the receptive facet.

Trouche's (2005a) interpretation of instrumental genesis in relation to DT in mathematics education relies on Vergnaud's and Piaget's description of schemes as well as Rabardel's (1995) description of mediation and instrumental genesis. According to Monaghan (2007), Trouche understands techniques as "sets of gestures realized in the execution of a task on an instrument; instrumented techniques are techniques involving artefacts and are thus the gestures of instrumented action schemes" (p. 65). The techniques are the signs of a subject's, i.e., an individual's, actions and thoughts in interaction with an artifact, e.g. a digital tool. Within this understanding, the instrumental genesis can be described as a dialectic relation between tool and scheme.

According to Trouche, schemes have three functions related to the subject: 1) a pragmatic function—to do something; 2) a heuristic function—to anticipate and plan actions; and 3) an epistemic function—to understand something (Monaghan, 2007). From an empirical point of view, it is important being aware that it can be difficult to distinguish between pragmatic and epistemic mediations in given situations. This not least in cases where, e.g., the visible feedback from a DGE, such as GeoGebra, are in the form of figures that may seem convincing or almost self-explanatory (Thomsen & Jankvist, 2020). Here, previous studies by the first author show that working with the interplay between original sources and GeoGebra can contribute to a special constructive interactivity between a pragmatic and epistemic mediation (Thomsen, 2022). Students can use various functions in GeoGebra to support their ability to follow others' reasoning in a receptive manner, as seen in the original source. GeoGebra can serve as a pragmatic mediator, allowing students to use specific examples to support their understanding of mathematical arguments or proofs without fully


comprehending them. Additionally, GeoGebra can function as an epistemic mediator, providing students with feedback and functions to help them comprehend others' reasoning.

Moreover, students can also use GeoGebra to construct their own arguments based on the functions available. In this case, GeoGebra can serve as a pragmatic mediator as students refer to visual feedback on the screen without further questioning, or as an epistemic mediator where students understand and build up arguments and chains of reasoning. The process of using GeoGebra may involve switching between pragmatic and epistemic mediation, as well as between the receptive and constructive facets of reasoning competency. Therefore, it is important to focus on the constructive facet of reasoning competency when using GeoGebra to work with the interplay between original sources and the software. In the following, we will try to unfold this perspective by presenting and analyzing an empirical example of a pair of students' work.

### Empirical example and analysis

The first author carried out the empirical study, which took place in a classroom of 7<sup>th</sup> grade students. A teacher and 22 students participated in the study. The study consisted in 11 lessons (of 45 minutes) over a period of three weeks. The students worked with excerpts from Euclid's *Elements*, more precisely the first five postulates from Book I, the title of proposition 34 from Book I, as well as proposition 6 from Book IV: *To inscribe a square in a given circle*. We used Eibe's (1897) translation of the *Elements* into Danish.<sup>1</sup> The students worked gradually with the proposition, and it was also divided into a construction part and a proof part (inspired by Olsen and Thomsen, 2017). Related to each part of the text, the students were asked questions related to their work with that part of the text in GeoGebra. Finally, the students were asked to make screencasts of their dialogues and use of GeoGebra related to their work with the tasks and they also handed-in written answers.

The empirical example takes its outset in the students' work with a part of Euclid's proposition 6, Book IV, and accompanying questions. No measurement tools or the grid in GeoGebra were applied.

Example of text and questions in Danish	Translated into English
<p>Euklids sætning 6, bog IV – Bevisdelen</p> <p>Prøv at beskriv med jeres egne ord, hvad der er Euklid skriver.</p> <p>Hvorfor gælder det?</p> <p>1. Da nu <math>BE</math> er lig <math>ED</math>, thi <math>E</math> er Centrum, og da <math>EA</math> er fælles og vinkelret, saa er Grundlinje <math>AB</math> lig Grundlinje <math>AD</math>.</p>  <p>Kan I blive overbevist om og overbevise andre om, at grundlinje <math>AB</math> er lig med grundlinje <math>AD</math> på en anden måde I GeoGebra?</p>	<p>Euclid's proposition 6, Book IV – Proof part. With your own words, try to describe what Euclid writes. Why is it true?</p> <p>Then, since <math>BE</math> equals <math>ED</math>, for <math>E</math> is the center, and <math>EA</math> is common and at right angles, therefore the base <math>AB</math> equals the base <math>AD</math>.</p> <p>Can you convince yourself, and convince others, that baseline <math>AB</math> is equal to baseline <math>AD</math> in a different way in GeoGebra?</p>

<sup>1</sup> In this paper, we use Joyce's (1998) translation into English.

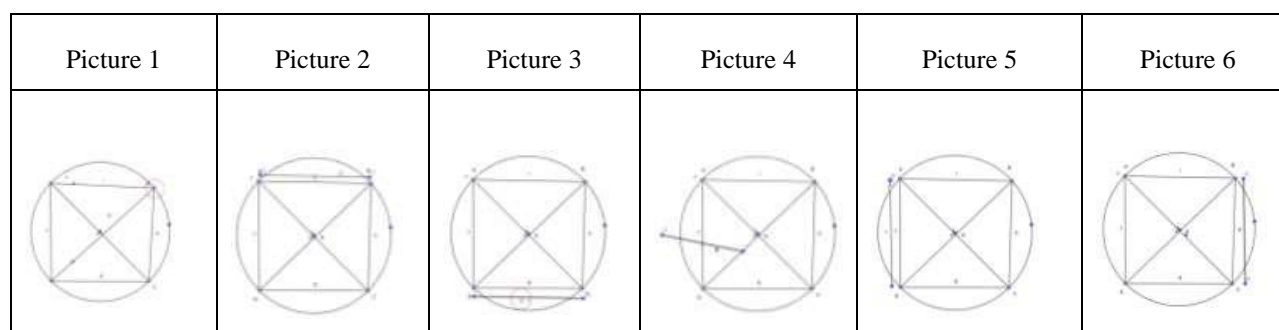
<p>2. Af samme Grund er ogsaa hver af Linierne <math>BC</math> og <math>CD</math> lig hver af Linierne <math>AB</math> og <math>AD</math>. Altsaa er Firsiden <math>ABCD</math> ligesidet.</p>	<p>For the same reason each of the straight lines <math>BC</math> and <math>CD</math> also equals each of the straight lines <math>AB</math> and <math>AD</math>. Therefore, the quadrilateral <math>ABCD</math> is equilateral.</p>
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**Figure 1: The tasks related to a bit of Euclid's proposition 6, Book IV**

The empirical example concerns a pair of students' work with the text and the related questions. It consists of two excerpts from their screencast. Student 1 begins reading the question aloud and continues reading the last bit of the text in the original source:

Student 1: Can you convince yourself, and convince others, that baseline  $AB$  is equal to baseline  $AD$  in a different way in GeoGebra? Hmm ... (reads the next bit of the source), For the same reason each of the straight lines  $BC$  and  $CD$  also equals each of the straight lines  $AB$  and  $AD$ .

Figures 2 and 3 are examples of screen captures from the students' screencast. References are made to the various pictures in their conversation along the way.



**Figure 2: Screen captures of the students' work with GeoGebra**

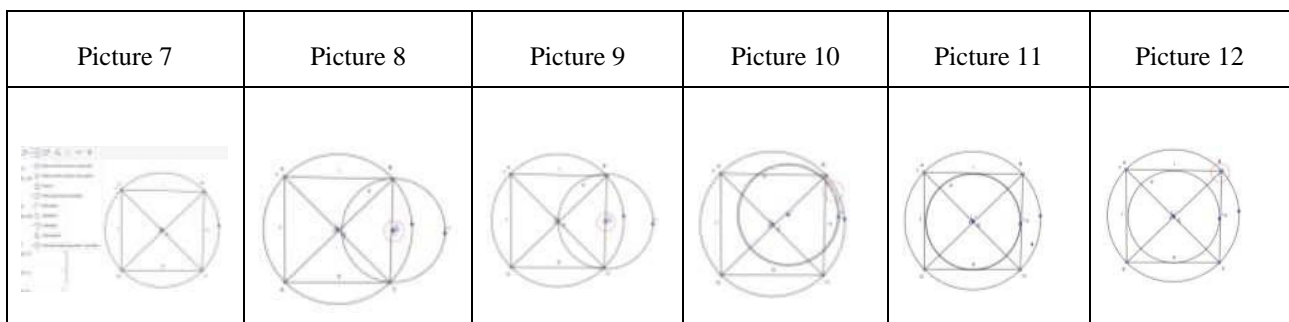
Student 1 drags at the point in the upper right corner of the figure, without noticing that it is now no longer a square (see Figure 2, picture 1).

Student 2: You can see that the lines are equal.  
Student 1: Yes, but how?  
Student 2: You can see that  $AB$  and  $AD$  are equal.  
Student 1: If you take a line segment and do like this (picture 2).  
Student 2: And then take them down (picture 3). Can you turn it somehow?

The students continue moving the line segment around, and in picture 4, it seems as if they shorten the line segment. Afterwards, they lengthen it again (picture 5), and pushes the line segment to the right side of the four-side (picture 6). Here the students use the visual feedback from GeoGebra to construct an argument. One might say that it seems as if it is a pragmatic mediation we witness here, since the student use the functions in GeoGebra to show that the lengths of the four-side are equal. It is interesting to notice that they are not very precise in their construction of the length of their line segment, and that they change it along the way. They actually do notice this, but they do not seem to pay much attention to it. On the one hand, this can be seen as an indication of the students using it as

an overall argument, where it is implicit that it is possible to do so in GeoGebra. This may be regarded as an epistemic mediation, even though the mathematical reasons are not explained. In fact, it may serve as an example of the distinction between pragmatic and epistemic mediations not always being completely sharp in the analyses of students' work in a DGE. It did seem as if the students themselves found this to be too easy, so they continued.

- Student 1: But what else could we do?
- Student 2: I don't know. Because we are not allowed to do what we would like to do. We cannot use the grid to say that it is 6 long.
- Student 1: Hmm, what about the angles? You know, in a way, no, it does not have anything to do with the length (they look at different functions in GeoGebra, see picture 7). Ohh, I have an idea.
- Student 2: Okay.
- Student 1: If we take a circle and then (...) make it as big as this (picture 8).
- Student 2: Ohh, I know what you mean, put it into the middle.
- Student 1: Like this, then it is as long as this one, right. So, if we (enlarge the circle so the periphery touches the center, picture 9). I cannot see, I cannot get it to ... (picture 10).
- Student 2: Can I just say something? I have an idea. Put it into the middle (the center). Because if it can fill it all inside the middle, then they are of equal length in each side (picture 11).
- Student 1: Oh, yes.
- Student 2: If it was further down it could not reach it all the way round.
- Student 1: So, ours is not quite equal.
- Student 2: You moved it a bit in the beginning. So, now they are almost of equal length (picture 12).
- Student 1: Now we have figured it out.



**Figure 3: Other screen captures from the students' work with GeoGebra**

In the dialogue above, the students use the receptive facet of their reasoning competency to follow each other's ideas. They also make use of the constructive facet when they give new ideas based upon the solutions they see while they are working with GeoGebra. This becomes clear when Student 1 says, "If we take a circle and then (...) make it as big as this", and Student 2 replies "Ohh, I know what you mean, put it into the middle." This too may be considered an indication of a beginning epistemic mediation since the students can make use of the visual argument for any square. Still, they are aware that they are not allowed to argue by means of the DGE grid, which may be considered a sort of consciousness of being able to follow different routes to conviction.

Now, had they used the grid and had become convinced by means of counting grid units, then surely a pragmatic mediation would have been in play, since conviction would have relied on using the function in GeoGebra without any deeper reasoning about why the length of the sides were the same

or not. Hypothetically speaking, had the students gone even further and argued by using the DGE as an authority, e.g. by referring to GeoGebra's 'polygon functionality', then we would have witnessed what Misfeldt and Jankvist (2018) refer to as a techno-authoritarian proof-scheme, inspired by Harel's and Sowder's (2007) external conviction proof schemes. However, this did not happen. Hence, it appears that the students were indeed searching for an explanation. We may get some insight into this by looking at the written answers.

To the first question, where they had to explain in their own words what Euclid meant, they wrote:

*BE is as long as DE, E is center, so AE is probably also as long and therefore AB is as long as AD.*

In their written answer as to how they would use GeoGebra in a different way in order to become convinced themselves or to convince others, they wrote:

*If you take a circle and make it as broad as one of the lines, and then put it into the middle, then if the lines are equals, and the middle of the circle is at the center, then there would not be any space between the lines, and they are of equal length.*

If we compare the two answers, we notice some sort of alignment. This because the students seem to use their knowledge that radius has the same length in the circle, and that this is the reason why the lengths of all the sides are equal. Still, in their written answer they miss the point that the diagonals have to be perpendicular. However, as part of their oral argumentation, Student 1 says, "Hmm, what about the angles? You know in a way, no, it does not have anything to do with the length". Here they could have investigated this further, which had in fact been quite easy in GeoGebra. Still, they did provide an answer to the question posed to them.

We may characterize their constructive reasoning while using GeoGebra as a sign of a beginning visible epistemic mediation. Surely, it goes further than a pragmatic mediation. The students seem to be aware of the general aspect of the convincing part of using a circle to illustrate that the baselines have the same lengths. In their written answer, they also explicate that the circle at first must be as "broad" as one of the lines. Of course, this be seen as a kind of measurement and hence a pragmatic mediation. But the students continue to work and move focus to the middle of the square, or the center of the first circle. One might see this as a way of making a chain of arguments and as an interaction from a pragmatic mediation to a more epistemic mediation. In the oral dialogue, they also comment that they had to move the point back, so the figure had equal length of the baseline. In this case, they did not doubt their visible reasoning based on the GeoGebra figure. The students' dialogue can be characterized as a sign of them going back and forward between using the constructive and receptive facets of their reasoning competency, and between pragmatic and epistemic mediations taking place.

## **Concluding discussion**

From the above, it seems clear that students' work with the interplay between original sources and GeoGebra can indeed support and qualify their constructive reasoning through a switching back and forth between pragmatic and epistemic mediations (see also, Thomsen, 2022). The students' work with the interplay seems to have the potential to put them in a mathematical situation, where they had to use the receptive facet of their reasoning competency, when they had to explain the content of an excerpt from the source in their own words. In addition, they used the receptive facet of their

reasoning competency to understand and try to grasp each other's ideas, this also based on an alternation between a pragmatic and an epistemic mediation while working with the arguments in GeoGebra. This part of the task required the students to formulate their own (chains of) arguments, i.e., to use the constructive facet of their reasoning competency. It seems clear that GeoGebra can support students' possibilities in this regard, not least due to the DGE's immediate visual feedback.

In the empirical example provided in this paper, the students seemed to reach the same point of understanding. However, they did not include the perpendicular angles anywhere in the figure as part of their argumentation. They mentioned the angles in their dialogue about becoming convinced and convince others, but then rejected it without further investigations. We do not know, if they came to think of angles because they had read about it in the original source, i.e., "EA is common and at right angles". Earlier in the teaching sequence, the students had constructed the figure by reading the beginning of the construction part of Euclid's proposition 6, Book IV. Among other things it says here, "Draw two diameters  $AC$  and  $BD$  of the circle  $ABCD$  at right angles to one another". This may be the reason why they brought the angles into play as an idea.

The distinction between pragmatic and epistemic mediation has proven to be a useful analytical distinction in relation to students' work with reasoning and proving in an interplay between original sources and DT. This not least when looking into the two facets of students' reasoning competency. We hypothesize that something similar may indeed be the case for a focus on other of the mathematical competences from the KOM framework (Niss & Højgaard, 2019), when focus is on the interplay between original sources and DT.

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